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Large Deviations and Ruin Probabilities for Solutions to Stochastic Recurrence Equations with Heavy-Tailed Innovations

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LARGE DEVIATIONS AND RUIN PROBABILITIES FOR SOLUTIONS TO STOCHASTIC RECURRENCE EQUATIONS WITH HEAVY-TAILED INNOVATIONS

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Abstract. In this paper we consider the stochastic recurrence equation $Y_t = A_t Y_{t-1} + B_t$ for an iid sequence of pairs (A_t, B_t) of non-negative random variables, where we assume that B_t is regularly varying with index $\kappa > 0$ and $EA_t^\kappa < 1$. We show that the stationary solution (Y_t) to this equation has regularly varying finite-dimensional distributions with index κ. This implies that the partial sums $S_n = Y_1 + \cdots + Y_n$ of this process are regularly varying. In particular, the relation $P(S_n > x) \sim c_1 \, n \, P(Y_1 > x)$ as $x \to \infty$ holds for some constant $c_1 > 0$. For $\kappa > 1$, we also study the large deviation probabilities $P(S_n - ES_n > x)$, $x \ge x_n$, for some sequence $x_n \to \infty$ whose growth depends on the heaviness of the tail of the distribution of Y_1. We show that the relation $P(S_n - ES_n > x) \sim c_2 \, n \, P(Y_1 > x)$ holds uniformly for $x \ge x_n$ and some constant $c_2 > 0$. Then we apply the large deviation results to derive bounds for the ruin probability $\psi(u) = P(\sup_{n\ge 1}((S_n - ES_n) - \mu\, n) > u)$ for any $\mu > 0$. We show that $\psi(u) \sim c_3 \, u \, P(Y_1 > u) \, \mu^{-1}(\kappa - 1)^{-1}$ for some constant $c_3 > 0$. In contrast to the case of iid regularly varying Y_t's, when the above results hold with $c_1 = c_2 = c_3 = 1$, the constants c_1, c_2 and c_3 are different from 1.

1. Introduction

The stochastic recurrence equation

$$Y_t = A_t \, Y_{t-1} + B_t \,, \quad t \in \mathbb{Z} \,, \tag{1.1}$$

and its stationary solution have attracted quite a few attention over the last years. Here $((A_t, B_t))$ is an iid sequence of pairs of non-negative random variables A_t and B_t. (In what follows, we write $A, B, Y, \ldots$, for generic elements of the stationary sequences (A_t), (B_t), (Y_t), etc. We also write c for any positive constant whose value is not of interest.)

Major applications of stochastic recurrence equations are in financial time series analysis. For example, the squares of the GARCH process can be embedded in a stochastic recurrence equation of type (1.1); we refer to Section 8.4 in Embrechts et al. [14] for an introduction to stochastic recurrence equations and Basrak et al. [1] and Mikosch [20] for recent surveys on the mathematics of GARCH models, their properties and relation with stochastic recurrence equations. The stochastic recurrence equation approach has also proved useful for the estimation of GARCH and related models; see Straumann and Mikosch [34, 35, 24]. In a financial or insurance context, the stochastic recurrence equation (1.1) has natural interpretations. For example, B_t can be considered as annual payment and A_t as a discount factor. The value Y_t is then the aggregated value of past discounted payments. In a life insurance context, (Y_t) is referred to as a perpetuity; see for example Dufresne [13]. Stochastic recurrence equations have also been used to describe evolutions in biology; see Baxendale and Khasminskii [2] and the references therein.

It will be convenient to use the notation

$$\Pi_{s,t} = \begin{cases} A_s \cdots A_t \,, & s \le t \,, \\ 1 \,, & s > t \,, \end{cases} \qquad \Pi_t = \Pi_{1,t} \,.$$

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It is well known (Bougerol and Picard [5]) that (1.1) has a unique strictly stationary ergodic causal solution (Y_t) (i.e., Y_t is a function only of $(A_s, B_s), s \le t$) if and only if

$$-\infty \le E\log A < 0 \quad \text{and} \quad E\log^+ B < \infty\,. \tag{1.2}$$

In what follows, we always assume this condition to be satisfied. The stationary solution has representation

$$Y_t = \sum_{i=-\infty}^{t} \Pi_{i+1,t}\, B_i = B_t + \sum_{i=-\infty}^{t-1} \Pi_{i+1,t}\, B_i\,, \quad t \in \mathbb{Z}\,. \tag{1.3}$$

We say that any non-negative random variable Z and its distribution are regularly varying with index κ if its right tail is of the form

$$P(Z > x) = \frac{L(x)}{x^\kappa}\,, \quad x > 0\,,$$

for some $\kappa \ge 0$ and a slowly varying function L. A result of Kesten [19] shows that the stationary solution to the stochastic recurrence equation (1.1) has regularly varying distribution, under quite general conditions on A and B. We cite this benchmark result for comparison with the results we obtain in this paper.

Theorem 1.1. (Kesten [19]) *Assume that the following conditions hold:*
- *For some* $\epsilon > 0$, $EA^\epsilon < 1$.
- *The set*

$$\{\log(a_n \cdots a_1):\ n \ge 1,\ a_n \cdots a_1 > 0 \text{ and } a_n, \ldots, a_1 \in \text{the support of } P_A\}$$

generates a dense group in $\mathbb{R}$ *with respect to summation and the Euclidean topology. Here* P_A *denotes the distribution of* A.
- *There exists* $\kappa_0 > 0$ *such that*

$$EA^{\kappa_0} \ge 1\,, \tag{1.4}$$

and $E\left(A^{\kappa_0}\log^+ A\right) < \infty$.

Then the following statements hold:
1. *There exists a unique solution* $\kappa \in (0, \kappa_0]$ *to the equation*
$$EA^\kappa = 1\,.$$
2. *If* $EB^\kappa < \infty$, *there exists a unique strictly stationary ergodic causal solution* (Y_t) *to the stochastic recurrence equation* (1.1) *with representation* (1.3).
3. *If* $EB^\kappa < \infty$, *then* Y *is regularly varying with index* $\kappa > 0$. *In particular, there exists* $c > 0$ *such that*
$$P(Y > x) \sim c\, x^{-\kappa}\,, \quad x \to \infty\,.$$

Condition (1.4) is crucial. Goldie and Grübel [16] show that $P(Y > x)$ can decay exponentially fast to zero if (1.4) is not satisfied. Notice that (1.4) ensures that the support of A is spread out sufficiently far.

The set-up of this paper is different from the one in Kesten's Theorem 1.1. The latter result is surprising insofar that a light-tailed distribution of A (such as the exponential or the truncated normal distribution) can cause the stationary solution (Y_t) to (1.1) to have a marginal distribution with Pareto-like tails. In this paper we consider the case when B is regularly varying with index κ and A has lighter right tail than B. In this case the conditions of Kesten's theorem are not met. In particular, we always assume that $EA^\kappa < 1$. The marginal distribution of the stationary solution (Y_t) turns out to be regularly varying with the same index κ as the innovations B_t.

It is the objective of this paper to study the interplay of the regular variation of Y and the particular dependence structure of the Y_t's with respect to the partial sums

$$S_n = Y_1 + \cdots + Y_n\,, \quad n \ge 1\,.$$

Due to (multivariate) regular variation of the finite-dimensional distributions of (Y_t), S_n is regularly varying with index κ, and we establish the precise tail asymptotics for $P(S_n > x)$ for fixed n and as $x \to \infty$. We will see that, in contrast to iid regularly varying random variables Y_t (cf. Lemma 1.3.1 in [14]), the relation

$$\lim_{x\to\infty} \frac{P(S_n > x)}{n\,P(Y > x)} = 1\,, \quad n \ge 2\,,$$

does not hold for the stationary solution (Y_t) to (1.1), neither under the conditions of Kesten's theorem nor under the conditions imposed in this paper; see Section 3.3. We will show in Proposition 3.2 that

$$\lim_{x\to\infty} \frac{P(S_n > x)}{P(Y > x)} = E\left(\sum_{i=1}^{n} \Pi_i\right)^{\kappa} + (1 - EA^{\kappa}) \sum_{t=0}^{n-1} E\left(\sum_{i=0}^{t} \Pi_i\right)^{\kappa}, \quad n \geq 2. \tag{1.5}$$

A question which is closely related to (1.5) concerns the *large deviations* of the partial sum process (S_n). In this case, one is interested in the asymptotic behavior of the tail $P(S_n > x_n)$ for real sequences (x_n) increasing to infinity sufficiently fast. Classical results (see for example A.V. Nagaev [26], S.V. Nagaev [27], Cline and Hsing [7]; cf. the surveys in Section 8.6 in [14] and Mikosch and Nagaev [21]) say that for iid (Y_t) and thresholds $x_n \to \infty$, the relation

$$\begin{aligned} P(S_n > x_n) &\sim n\, P(Y > x_n) \\ &\sim P(\max(Y_1, \ldots, Y_n) > x_n) \end{aligned} \tag{1.6}$$

holds. For reasons of comparison, we quote a general large deviation result for iid random variables.

Theorem 1.2. *Assume that $B > 0$ is regularly varying with index $\kappa > 0$.*

(1) (A.V. Nagaev [26]; cf. S.V. Nagaev [27]) *Assume that $\kappa > 2$. Then*

$$P\left(\sum_{t=1}^{n} (B_t - EB) > x\right) = \overline{\Phi}(x/\sqrt{n})\,(1 + o(1)) + n\, P(B > x)\,(1 + o(1)),$$

as $n \to \infty$ and uniformly for $x \geq \sqrt{n}$, where $\overline{\Phi} = 1 - \Phi$ is the right tail of the standard normal distribution function Φ. In particular,

$$P\left(\sum_{t=1}^{n} (B_t - EB) > x\right) = \overline{\Phi}(x/\sqrt{n})\,(1 + o(1))$$

uniformly for $\sqrt{n} \leq x \leq \sqrt{a\, n \log n}$ and $a < \kappa - 2$, and

$$P\left(\sum_{t=1}^{n} (B_t - EB) > x\right) = n\, P(B > x)\,(1 + o(1))$$

uniformly for $x \geq \sqrt{a\, n \log n}$ and $a > \kappa - 2$.

(2) (Cline, Hsing [7]) *Assume that $\kappa \in (1, 2)$. Then*

$$P\left(\sum_{t=1}^{n} (B_t - EB) > x\right) = n\, P(B > x)\,(1 + o(1)), \tag{1.7}$$

as $n \to \infty$ and uniformly for $x \geq a_n c_n$, where (a_n) satisfies $n\, P(B > a_n) \sim 1$ and (c_n) is any sequence satisfying $c_n \to \infty$.

The uniformity of these large deviation results refers to the fact that the error bounds hold uniformly for the indicated x-regions. For example, in the case $\kappa \in (1, 2)$, (1.7) means that

$$\lim_{n\to\infty} \sup_{x \geq x_n} \left| \frac{P\left(\sum_{t=1}^{n} (B_t - EB) > x\right)}{n\, P(B > x)} - 1 \right| = 0 \tag{1.8}$$

where $x_n = a_n\, c_n$.

We will show in Theorem 4.2 that the following analog to Theorem 1.2 holds, under the more restrictive condition that (A_t) and (B_t) are independent:

$$\lim_{n\to\infty} \sup_{x \geq x_n} \left| \frac{P(S_n - ES_n > x)}{n\, P(Y > x)} - (1 - EA^{\kappa})\, E\left(\sum_{i=0}^{\infty} \Pi_i\right)^{\kappa} \right| = 0. \tag{1.9}$$

The question about large deviations is closely related to the *ruin probability* of the random walk (S_n). Given that $EY < \infty$, this is the probability

$$\psi(u) = P\left(\sup_{n \geq 1} [(S_n - ES_n) - \mu\, n] > u\right), \quad u, \mu > 0.$$

It is one of the very well studied objects of applied probability theory, starting with classical work by Cramér in the 1930s. For iid regularly varying and, more generally, subexponential Y_t's the asymptotic behavior of $\psi(u)$ as $u \to \infty$ was studied by various authors; see Chapter 1 in Embrechts et al. [14]. The following benchmark result is classical in the context of ruin for heavy-tailed distributions. We cite it here for comparison with the results of this paper.

Theorem 1.3. *Assume that B is regularly varying with index $\kappa > 1$. Then for any $\mu > 0$,*

$$P\left(\sup_{n\ge 1}\left(\sum_{t=1}^n (B_t - EB) - \mu\, n\right) > u\right) \sim \frac{1}{\mu}\,\frac{1}{\kappa-1} u\, P(B > u)\,, \quad u \to \infty\,.$$

In Theorem 4.9 we prove an analogous result for (Y_t):

$$\psi(u) \sim \frac{1}{\mu}\,\frac{1}{\kappa-1}\,(1 - EA^\kappa)\, E\left(\sum_{i=0}^\infty \Pi_i\right)^\kappa u\, P(Y > u)\,. \tag{1.10}$$

The results of this paper are derived by applications of the *heavy-tailed large deviations heuristics*. In the case of iid Y_t's this means that a large deviation of the random walk S_n from its mean ES_n must be due to exactly one unusually large value Y_t, whereas the Y_s's for $s \neq t$ are small compared to Y_t. We refer to Samorodnitsky [32] for a review on these heuristics which can be exploited in the context of various applied probability models. For dependent Y_t's, as considered in this paper, the large deviations heuristics has to be combined with the understanding of the dependence structure of the random walk S_n exceeding high thresholds. In the proof of the ruin probability result it turns out that the ruin probability of the random walk (S_n) behaves very much like the ruin probability of the random walk $\sum_{t=1}^n B_t C_t$, where $C_t = \sum_{i=t}^\infty \Pi_{t+1,i}$, $t \in \mathbb{Z}$. This is another stationary sequence, but under the conditions of this paper its marginal distributions have tails less heavy than (B_t). Since we assume independence of (A_t) and (B_t), hence of (C_t) and (B_t), in Section 4.2, it is likely that a large value of S_n is now caused by a large value $B_t C_t$, which in turn is caused by a large value of B_t. We make this intuition precise by showing (1.10).

The results (1.5) on the tail of S_n for fixed n, (1.9) on the large deviations of (S_n) and (1.10) on the ruin probability of (S_n) and their analogs for iid Y_t's illustrate some crucial differences between the behavior of a random walk with dependent and independent heavy-tailed step sizes far away from the origin. The constants on the right-hand sides of (1.5), (1.9) and (1.10), which differ from those in the case of iid regularly varying Y_t's, can be considered as alternative measures of the extremal clustering behavior of the Y_t's. Similar results were obtained only for a few classes of stationary processes (Y_t). Those include results by Mikosch and Samorodnitsky [22, 23] on large deviations and ruin for random walks with step sizes which constitute a linear process with regularly varying innovations or a stationary ergodic stable process, and by Davis and Hsing [8] on large deviations for random walks with infinite variance regularly varying step sizes. So far the known results do not allow one to draw a general picture which would allow one to classify stationary sequences of regularly varying random variables Y_t with respect to their extremal behavior of the random walk with negative drift $((S_n - ES_n) - \mu n)$. The cited results and also those of the present paper are steps in the search for appropriate measures of extremal dependence in a stationary sequence by studying the behavior of suitable functionals acting on the sequence.

The paper is organized as follows. In Section 2 we give conditions under which the stationary solution (Y_t) to the stochastic recurrence equation (1.1) has regularly varying finite-dimensional distributions. In Section 3 we consider applications of this property to the weak convergence of related point processes, the central limit theorem of (S_n) and the partial maxima of (Y_t). In Section 4.1 we study the large deviations of (S_n) and in Section 4.2 we give our main result on the asymptotic behavior of the ruin probability $\psi(u)$. Since the proofs of the main results are quite technical, we postpone them to particular sections at the end of the paper. The proof of Theorem 4.2 will be given in Section 5 and the one of Theorem 4.9 in Section 6.

2. Regular variation of the solution to the stochastic recurrence equation

2.1. **Preliminaries.** We start with some auxiliary results in order to establish regular variation of Y. In what follows, we write $\overline{F}(x) = 1 - F(x)$ for the right tail of any distribution function F.

Lemma 2.1. (Davis and Resnick [11]) *Let F be a distribution function concentrated on $(0,\infty)$. Assume $Z_1,\dots,Z_n$ are non-negative random variables satisfying*

$$\lim_{x\to\infty}\frac{P(Z_i>x)}{\overline{F}(x)}=c_i \tag{2.1}$$

for some non-negative finite values c_i, where $F(x)=P(Z_1\le x)$, and

$$\lim_{x\to\infty}\frac{P(Z_i>x\,,Z_j>x)}{\overline{F}(x)}=0\,,\quad i\neq j\,. \tag{2.2}$$

Then

$$\lim_{x\to\infty}\frac{P(Z_1+\cdots+Z_n>x)}{\overline{F}(x)}=c_1+\cdots+c_n\,.$$

We will frequently make use of the following elementary property which was proved by Breiman [6] in a special case. We refer to it as *Breiman's result* and prove a uniform version of it for further use.

Lemma 2.2. (Breiman [6]) *Let ξ,η be non-negative non-degenerate random variables such that ξ is regularly varying with index $\kappa>0$ and $E\eta^{\kappa+\varepsilon}<\infty$ for some $\varepsilon>0$. Then for any sequence $x_n\to\infty$,*

$$\lim_{n\to\infty}\sup_{x\ge x_n}\left|\frac{P(\xi\,\eta>x)}{P(\xi>x)}-E\eta^\kappa\right|=0\,.$$

This means that the product $\xi\,\eta$ inherits regular variation from ξ.

Proof. Fix $M>0$. Then

$$\begin{aligned}\Delta(x)&=\frac{P(\xi\,\eta>x)}{P(\xi>x)}-E\eta^\kappa\\ &=\int_{[0,M]}\left[\frac{P(\xi\,y>x)}{P(\xi>x)}-y^\kappa\right]dP(\eta\le y)\\ &\quad-E\eta^\kappa I_{(M,\infty)}(\eta)+\int_{(M,\infty)}\frac{P(\xi\,y>x)}{P(\xi>x)}\,dP(\eta\le y)\\ &=\Delta_1(x)-\Delta_2+\Delta_3(x)\,.\end{aligned}$$

Obviously,

$$\lim_{M\to\infty}\Delta_2=0\,.$$

Moreover, the uniform convergence theorem for regularly varying functions (see Bingham et al. [4]) implies that for every fixed $M>0$,

$$\begin{aligned}\sup_{x\ge x_n}|\Delta_1(x)|&\le\sup_{x\ge x_n}\int_{[0,M]}\left|\frac{P(\xi\,y>x)}{P(\xi>x)}-y^\kappa\right|dP(\eta\le y)\\ &\le\sup_{x\ge x_n}\sup_{y\le M}\left|\frac{P(\xi\,y>x)}{P(\xi>x)}-y^\kappa\right|\to0\,.\end{aligned}$$

An application of the Potter bounds for regularly varying functions (see Bingham et al. [4], p. 25) yields for $x,x/y\ge x_0$, for sufficiently large $x_0>0$ and all $y>M>1$ that

$$\frac{P(\xi>x/y)}{P(\xi>x)}\le y^{\kappa+\varepsilon}\,.$$

Hence

$$\begin{aligned}\sup_{x\ge x_n}|\Delta_3(x)|&\le\sup_{x\ge x_n}\int_{M<y\le x/x_0}y^{\kappa+\varepsilon}\,dP(\eta\le y)+\sup_{x\ge x_n}\frac{P(\eta>x/x_0)}{P(\xi>x)}\\ &\to0\end{aligned}$$

by first letting $n\to\infty$ and then $M\to\infty$, since $E\eta^{\kappa+\varepsilon}<\infty$. This proves the lemma. □

We now turn to the stochastic recurrence equation (1.1). After n iterations we obtain

$$Y_n = \Pi_n \, Y_0 + \sum_{t=1}^{n} \Pi_{t+1,n} \, B_t \,. \tag{2.3}$$

As in Section 1, we assume that $((A_t, B_t))$ is an iid sequence of pairs of non-negative random variables A_t and B_t. In addition, suppose that B is regularly varying with index $\kappa > 0$ and $EA^{\kappa+\delta} < \infty$ for some $\delta > 0$. Then Breiman's result (Lemma 2.2) applies:

$$\frac{P(\Pi_{i-1} B_i > x)}{P(B > x)} \sim (EA^\kappa)^{i-1} \quad \text{as } x \to \infty \,. \tag{2.4}$$

The following result will be crucial for the property of regular variation of the finite-dimensional distributions of the stationary solution (Y_n) to (1.1). For its formulation we assume that $Y_0 = c$ in (2.3) for some constant c. We use the same notation (Y_n) in this case, slightly abusing notation since (Y_n) is then not the stationary solution to (1.1).

Proposition 2.3. *Assume B is regularly varying with index $\kappa > 0$ and $EA^{\kappa+\delta} < \infty$ for some $\delta > 0$. Then the following relation holds for fixed $n \geq 1$ and Y_n defined in (2.3) with $Y_0 = c$:*

$$P(Y_n > x) \sim P(B > x) \sum_{i=0}^{n-1} (EA^\kappa)^i \,, \quad \textit{as } x \to \infty.$$

Proof. We write

$$Z_0 = \Pi_n \, c \,, \quad Z_t = \Pi_{t-1} \, B_t \,, \quad t = 1, \ldots, n \,.$$

Observe that

$$Y_n = \Pi_n \, c + \sum_{t=1}^{n} \Pi_{t+1,n} \, B_t \stackrel{d}{=} \Pi_n \, c + \sum_{t=1}^{n} \Pi_{t-1} \, B_t = \sum_{t=0}^{n} Z_t \,.$$

We have for $1 \leq i < j \leq n$,

$$P(Z_i > x \,, Z_j > x) \quad \leq \quad P(\Pi_{i-1} \, \min(B_i, \Pi_{i,j-1} \, B_j) > x) \,.$$

Since $EA^{\kappa+\delta} < \infty$ and B is regularly varying with index κ, we can find a function $g(x) \to \infty$ such that $g(x)/x \to 0$, and $P(\max(A_i, \Pi_i) > g(x)) = o(P(B > x))$. Hence, for $i < j$,

$$\begin{aligned}
\frac{P(Z_i > x \,, Z_j > x)}{P(B > x)} &\leq \frac{P(\Pi_{i-1} \, \min(B_i, \Pi_{i,j-1} \, B_j) > x \,, \max(A_i, \Pi_i) > g(x))}{P(B > x)} \\
&\quad + \frac{P(\Pi_{i-1} \, \min(B_i, \Pi_{i,j-1} \, B_j) > x \,, \max(A_i, \Pi_i) \leq g(x))}{P(B > x)} \\
&\leq \frac{P(\max(A_i, \Pi_i) > g(x))}{P(B > x)} + \frac{P(\Pi_{i-1} \, B_i > x \,, \Pi_{i+1,j-1} \, B_j > x/g(x))}{P(B > x)} \\
&= o(1) + (EA^\kappa)^{j-2} \, P(B > x/g(x)) \, (1 + o(1)) \to 0 \,.
\end{aligned}$$

In the last step we made multiple use of Breiman's result and the independence of $\Pi_{i-1} \, B_i$ and $\Pi_{i+1,j-1} \, B_j$. By Markov's inequality, we also have for $1 \leq i \leq n$,

$$\frac{P(Z_0 > x \,, Z_i > x)}{P(B > x)} \leq \frac{P(Z_0 > x)}{P(B > x)} \leq c^n \, \frac{(EA^{\kappa+\delta})^n \, x^{-\kappa-\delta}}{P(B > x)} \to 0 \,.$$

Hence we are in the framework of Lemma 2.1 with $c_0 = 0$ and $c_i = (EA^\kappa)^{i-1}$, $i = 1, \ldots, n$; see (2.4). This proves the proposition. $\square$

2.2. **Univariate regular variation of Y.** In this section we prove regular variation of the marginal distribution of the stationary solution to the stochastic recurrence equation (1.1). From Proposition 2.3 and the representation (1.3) of the stationary solution (Y_t) we conclude that

$$\liminf_{x\to\infty}\frac{P(Y>x)}{P(B>x)}\geq\lim_{x\to\infty}\frac{P\left(\sum_{i=1}^n\Pi_{i-1}B_i>x\right)}{P(B>x)}=\sum_{i=0}^{n-1}(EA^\kappa)^i\,. \tag{2.5}$$

Letting $n\to\infty$ yields a lower bound for $P(Y>x)$. We want to show that

$$P(Y>x)\sim P(B>x)\sum_{i=0}^{\infty}(EA^\kappa)^i\,,\quad x\to\infty\,, \tag{2.6}$$

under the conditions that B is regularly varying with index $\kappa>0$ and $EA^\kappa<1$. Obviously, only if the latter condition holds relation (2.6) is meaningful. This also means that the conditions of Kesten's Theorem 1.1 cannot be satisfied. In that case, the index of regular variation κ of Y satisfies $EA^\kappa=1$ and $EB^\kappa<\infty$. Since in our case B is assumed to be regularly varying with index κ, the moment condition on B is not necessarily met either.

Proposition 2.4. *Assume that B is regularly varying with index $\kappa>0$, $EA^{\kappa+\delta}<\infty$ for some $\delta>0$ and $EA^\kappa<1$. Then a unique strictly stationary solution (Y_t) to the stochastic recurrence equation* (1.1) *exists and satisfies*

$$P(Y>x)\sim P(B>x)\,(1-EA^\kappa)^{-1}\,.$$

The result and its proof are in the spirit of Resnick [29], pages 229–230, who treats the case of infinite moving averages with regularly varying innovations, Davis and Resnick [11], who treat the bilinear process with regularly varying innovations, and to Resnick and Willekens [30]. The latter authors consider a stochastic recurrence equation of type (1.1) with regularly varying B, but (A_i) and (B_i) are independent in their case. The latter condition is often not satisfied in applications. For example, the squares of an ARCH(1) process satisfy an equation of type (1.1) with A_t and B_t dependent for every t; see for example Embrechts et al. [14], Section 8.4.

Proof. The function

$$g(h)=EA^h$$

satisfies $g(0)=1$, $g(\kappa)<1$ and it is continuous and convex in $[0,\kappa]$. Therefore $g'(0+)=E\log A<0$. Moreover, since $EB^\gamma<\infty$ for $\gamma<\kappa$, both conditions in (1.2) are satisfied and, hence, a unique stationary solution (Y_t) to (1.1) exists.

Since we have already proved the lower bound in (2.5), we focus on the upper bound. We make use of the representation

$$Y\stackrel{d}{=}\sum_{i=1}^{\infty}\Pi_{i-1}B_i\,,$$

which follows from (1.3). We have for small $\varepsilon>0$,

$$P(Y>x)\leq P\left(\sum_{i=1}^n\Pi_{i-1}B_i>(1-\varepsilon)x\right)+P\left(\sum_{i=n+1}^{\infty}\Pi_{i-1}B_i>\varepsilon x\right)=I_1(x)+I_2(x)\,.$$

From Proposition 2.3 with $c=0$ and regular variation of B we have

$$I_1(x)\sim(1-\varepsilon)^{-\kappa}P(B>x)\sum_{i=0}^{n-1}(EA^\kappa)^i\,, \tag{2.7}$$

Hence

$$\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\limsup_{x\to\infty}\frac{I_1(x)}{P(B>x)}\leq(1-EA^\kappa)^{-1}\,.$$

It suffices to show that

$$\lim_{n\to\infty}\limsup_{x\to\infty}\frac{I_2(x)}{P(B>x)}=0\,.$$

We have

$$
\begin{aligned}
I_2(x) &\le \sum_{i=n+1}^{\infty} P(\Pi_{i-1}\,B_i > x) + P\left(\sum_{i=n+1}^{\infty} \Pi_{i-1} B_i I_{[0,x]}(\Pi_{i-1}\,B_i) > \varepsilon\, x\right)\\
&= I_{21}(x) + I_{22}(x)\,.
\end{aligned}
$$

First assume that $\kappa < 1$. Write $p = \kappa + \delta$. Since the function $g(h)$ is continuous, $g(\kappa) < 1$ and $g(p) < \infty$ for small δ, we may assume without loss of generality that $p < 1$ and $g(p+\delta) < 1$. Applying Markov's inequality, we have

$$
I_{22}(x) \le (\varepsilon x)^{-p} \sum_{i=n+1}^{\infty} E(\Pi_{i-1}\,B_i)^p\, I_{[0,x]}(\Pi_{i-1}\,B_i)\,.
$$

Moreover,

$$
\begin{aligned}
J_i(x) &= \frac{E(\Pi_{i-1}\,B_i)^p\, I_{[0,x]}(\Pi_{i-1}\,B_i)}{x^p\, P(B > x)}\\
&= \int_0^{\infty} a^p\, \frac{EB^p I_{[0,x/a]}(B)}{EB^p I_{[0,x]}(B)}\, dP(\Pi_{i-1} \le a)\, \frac{EB^p I_{[0,x]}(B)}{x^p\, P(B > x)}\,.
\end{aligned}
$$

An application of Karamata's theorem yields for some positive constant c,

$$
\frac{EB^p I_{[0,x]}(B)}{x^p\, P(B > x)} \le c\,, \quad x > 0\,. \tag{2.8}
$$

Since $EB^p I_{[0,x]}(B)$ is regularly varying with index $\delta = p - \kappa$, the Potter bounds (Bingham et al. [4], p. 25) for regularly varying functions ensure that for any $a \in (0,1)$, $\gamma > 0$, and for sufficiently large x,

$$
\frac{EB^p I_{[0,x/a]}(B)}{E[B^p I_{[0,x]}(B)]} \le a^{-\delta-\gamma}\,.
$$

This implies for large x, $\gamma \in (0,\kappa)$ and for all i,

$$
\begin{aligned}
J_i(x) &\le c\left[\int_0^1 a^{p-\delta-\gamma}\, dP(\Pi_{i-1} \le a) + \int_1^{\infty} a^p\, dP(\Pi_{i-1} \le a)\right]\\
&\le c\,[(EA^{\kappa-\gamma})^{i-1} + (EA^p)^{i-1}]\\
&\le c\, r^{i-1}\,,
\end{aligned}
$$

for some $r \in (0,1)$. Indeed, since $g(p+\delta) < 1$, the convexity of the function $g(h)$ and $g(0) = 1$ imply that $\max(g(\kappa-\gamma), g(p)) \le r < 1$. We conclude that

$$
\lim_{n\to\infty} \limsup_{x\to\infty} \frac{I_{22}(x)}{P(B > x)} \le \varepsilon^{-p}\, c \lim_{n\to\infty} \sum_{i=n+1}^{\infty} r^{i-1} = 0\,. \tag{2.9}
$$

It remains to consider $I_{21}(x)$. We start by observing that

$$
P(\Pi_{i-1} B_i > x) = P(\Pi_{i-1} I_{[0,1]}(\Pi_{i-1}) B_i > x) + P(\Pi_{i-1} I_{(1,\infty)}(\Pi_{i-1}) B_i > x)\,.
$$

An application of the Potter bounds with $\gamma < \kappa$ yields

$$
\begin{aligned}
\frac{P(\Pi_{i-1} I_{[0,1]}(\Pi_{i-1}) B_i > x)}{P(B > x)} &= \frac{E[P(\Pi_{i-1} I_{[0,1]}(\Pi_{i-1}) B_i > x \mid \Pi_{i-1})]}{P(B > x)}\\
&\le c\, E(\Pi_{i-1})^{\kappa-\gamma} I_{[0,1]}(\Pi_{i-1})\\
&\le c\, r^{i-1}\,.
\end{aligned}
$$

By Markov's inequality and (2.8),

$$\begin{aligned} P(\Pi_{i-1} I_{(1,\infty)}(\Pi_{i-1}) B_i > x) &\le P(\Pi_{i-1} I_{(1,\infty)}(\Pi_{i-1}) B_i I_{[0,x]}(B_i) > x) + P(\Pi_{i-1} > 1)\, P(B > x) \\ &\le x^{-p} E(\Pi_{i-1})^p I_{(1,\infty)}(\Pi_{i-1})\, EB^p I_{[0,x]}(B) + (EA^p)^{i-1}\, P(B > x) \\ &\le c\, r^{i-1}\, P(B > x)\,, \end{aligned}$$

for some positive $c > 0$. Finally, we conclude that

$$\lim_{n\to\infty} \limsup_{x\to\infty} \frac{I_{21}(x)}{P(B > x)} \le c \lim_{n\to\infty} \sum_{i=n+1}^{\infty} r^{i-1} = 0\,.$$

This proves the theorem for $\kappa < 1$.

Now we turn to the case $\kappa \ge 1$. As before, choose $p = \kappa + \delta > \kappa$ such that $EA^{p+\delta} < 1$. Then $I_1(x)$ and $I_{21}(x)$ can be treated in the same way as for $\kappa < 1$. As to $I_{22}(x)$, an application of the Markov and triangle inequalities yields

$$\begin{aligned} \frac{I_{22}(x)}{P(B > x)} &\le \frac{(\varepsilon x)^{-p} E\left(\sum_{i=n+1}^{\infty} \Pi_{i-1} B_i I_{[0,x]}(\Pi_{i-1}\, B_i)\right)^p}{P(B > x)} \\ &\le \varepsilon^{-p} \left(\sum_{i=n+1}^{\infty} J_i^{1/p}(x) \right)^p . \end{aligned}$$

Following the lines of proof above, $J_i(x) \le c\, r^{i-1}$ for sufficiently large x and all i, and therefore an estimate similar to (2.9) holds. This concludes the proof. □

2.3. **Regular variation of the finite-dimensional distributions of** (Y_t)**.** In what follows, we assume that the conditions of Proposition 2.4 are satisfied. The latter result states that the marginal distribution of the stationary sequence (Y_n) is regularly varying with the same index κ as the innovations B_t. It is the aim of this section to extend this result to the finite-dimensional distributions of the process (Y_t).

For this reason, we introduce the notion of regular variation for an m-dimensional random vector: the vector $\mathbf{Y} \in \mathbb{R}^m$ is regularly varying with index $\kappa > 0$ if there exists a Radon measure μ on the Borel σ-field $\mathcal{B}$ of $[0,\infty]^m \backslash \{\mathbf{0}\}$ such that

$$n\, P(a_n^{-1}\, \mathbf{Y} \in \cdot) \stackrel{v}{\to} \mu\,.$$

Here the sequence (a_n) satisfies $P(|\mathbf{Y}| > a_n) \sim n^{-1}$, $\stackrel{v}{\to}$ denotes vague convergence in $\mathcal{B}$, and μ is a measure with the property $\mu(t\cdot) = t^{-\kappa}\mu(\cdot)$ for all $t > 0$; see Resnick [29] for an introduction to regular variation, related point process convergence and vague convergence. An equivalent way to characterize the limiting measure μ is via a presentation in spherical coordinates. This means that for every fixed $t > 0$ and (a_n) as above,

$$n\, P(|\mathbf{Y}| > t\, a_n\,, \mathbf{Y}/|\mathbf{Y}| \in \cdot) \stackrel{v}{\to} t^{-\kappa}\, P(\mathbf{\Theta} \in \cdot)\,,$$

where $|\cdot|$ is any fixed norm, $\stackrel{v}{\to}$ refers to vague convergence on the Borel σ-field of the unit sphere $\mathbb{S}^{d-1}$ corresponding to this norm and $\mathbf{\Theta}$ is a vector with values in $\mathbb{S}^{d-1}$. Its distribution is referred to as the *spectral distribution* of $\mathbf{Y}$.

For fixed $m \geq 1$, we have

$$\begin{aligned}
\mathbf{Y}_m &= (Y_1,\ldots,Y_m)' \\
&= (\Pi_1,\Pi_2,\ldots,\Pi_m)'\,Y_0 + \Big(B_1\,,B_2+A_2B_1\,,\ldots\,,B_m+\sum_{i=1}^{m-1}\Pi_{i+1,m}\,B_i\Big)' \\
&= \begin{pmatrix}\Pi_1\\ \Pi_2\\ \Pi_3\\ \vdots\\ \Pi_{m-1}\\ \Pi_m\end{pmatrix} Y_0 + \begin{pmatrix}1\\ \Pi_{2,2}\\ \Pi_{2,3}\\ \vdots\\ \Pi_{2,m-1}\\ \Pi_{2,m}\end{pmatrix} B_1 + \begin{pmatrix}0\\ 1\\ \Pi_{3,3}\\ \vdots\\ \Pi_{3,m-1}\\ \Pi_{3,m}\end{pmatrix} B_2 + \cdots + \begin{pmatrix}0\\ 0\\ 0\\ \vdots\\ 0\\ 1\end{pmatrix} B_m \\
&=: \mathbf{A}_0\,Y_0 + \mathbf{A}_1\,B_1 + \cdots + \mathbf{A}_m\;B_m\,.
\end{aligned}$$

Notice that $\mathbf{A}_0$ and Y_0 are independent, and so are $\mathbf{A}_i$ and B_i for every i. Since $E|\mathbf{A}_i|^{\kappa+\delta}<\infty$ for some $\delta>0$ and $Y_0,B_1,\ldots,B_m$ are independent and regularly varying with index κ, a multivariate version of Breiman's result (cf. Basrak et al. [1], Resnick and Willekens [30]) applies to conclude that each of the vectors $\mathbf{A}_0Y_0$, $\mathbf{A}_1B_1,\ldots,\mathbf{A}_mB_m$ is regularly varying with index κ with corresponding limiting measures $\mu_0,\ldots,\mu_m$. We mention that the normalizing sequences for these vectors are of the same size since, by the one-dimensional Breiman result and Proposition 2.4, as $x\to\infty$,

$$\begin{aligned}
P(|\mathbf{A}_0\,Y_0|>x) &\sim E|\mathbf{A}_0|^\kappa\,P(Y_0>x)\sim E|\mathbf{A}_0|^\kappa\,(1-EA^\kappa)^{-1}\,P(B>x)\,,\\
P(|\mathbf{A}_i\,B_i|>x) &\sim E|\mathbf{A}_i|^\kappa\,P(B>x)\,,\quad i=1,\ldots,m\,.
\end{aligned}$$

We choose one normalizing sequence (a_n) for all vectors such that $n\,P(|\mathbf{A}_0|Y_0>a_n)\sim 1$. We can characterize μ_i via its spectral distribution. Indeed, by Breiman's result we have for any Borel set $S\subset\mathbb{S}^{d-1}$ whose boundary has mean zero with respect to the spectral distribution,

$$n\,P(|\mathbf{A}_i|\,B_i>t\,a_n\,,\mathbf{A}_i/|\mathbf{A}_i|\in S)\sim t^{-\kappa}\,n\,P(B>a_n)\,E[|\mathbf{A}_i|^\kappa\,I_S(\mathbf{A}_i/|\mathbf{A}_i|)]\,,$$

and therefore the spectral distribution of $\mathbf{A}_i\,Y_i$ for these sets S is given by

$$\frac{E[|\mathbf{A}_i|^\kappa\,I_S(\mathbf{A}_i/|\mathbf{A}_i|)]}{E|\mathbf{A}_i|^\kappa}\,.$$

Adapting the proof of Lemma 2.1 in Davis and Resnick [11] to the multivariate case, it follows that $\mathbf{Y}_m$ is regularly varying with index κ and limiting measure

$$\mu(d\mathbf{x}) = \mu_0(d\mathbf{x})+c_1\,\mu_1(d\mathbf{x})+\cdots+c_m\,\mu_m(d\mathbf{x})\,, \tag{2.10}$$

where

$$c_i=\frac{E|\mathbf{A}_i|^\kappa}{E|\mathbf{A}_0|^\kappa}\,(1-EA^\kappa)$$

provided that the following relations holds for any Borel sets $C_1,C_2\subset[0,\infty]^m\backslash\{\mathbf{0}\}$ which are bounded away from zero:

$$n\,P(a_n^{-1}\mathbf{A}_iB_i\in C_1\,,a_n^{-1}\mathbf{A}_jB_j\in C_2)\to 0\,,\quad 0\le i<j\le m\,,$$

where we write $B_0 = Y_0$ for the sake of simplicity. Since C_1 and C_2 are bounded away from zero, there exists $M > 0$ such that $|\mathbf{x}| > M$ for all $\mathbf{x} \in C_1, C_2$. Therefore for $i < j$ and any $\gamma > 0$,

$$
\begin{aligned}
& \big\{a_n^{-1}\mathbf{A}_i B_i \in C_1\,,a_n^{-1}\mathbf{A}_j B_j \in C_2\big\} \subset \{|\mathbf{A}_i|\,B_i > a_n\,M\,,|\mathbf{A}_j|\,B_j > a_n\,M\}\\
\subset\;& \{\gamma\,B_i > M\,a_n\,,\gamma\,B_j > M\,a_n\}\\
& \cup\big\{\gamma\,B_i > M\,a_n\,,|\mathbf{A}_j| I_{(\gamma,\infty)}(|\mathbf{A}_j|)\,B_j > M\,a_n\big\}\\
& \cup\big\{\gamma\,B_j > M\,a_n\,,|\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)\,B_i > M\,a_n\big\}\\
& \cup\big\{|\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)\,B_i > M\,a_n\,,|\mathbf{A}_j| I_{(\gamma,\infty)}(|\mathbf{A}_j|)\,B_j > M\,a_n\big\}\\
=\;& D_1 \cup \cdots \cup D_4\,.
\end{aligned}
$$

By definition of (a_n) and the independence of B_i and B_j, it follows immediately that $n\,P(D_1) \to 0$. A similar approach applies to D_2 since B_i is independent of $B_j\mathbf{A}_j$ and, by Breiman's result,

$$
n\,P(D_2) \quad\sim\quad n\,P(\gamma B_i > M\,a_n)\,E|\mathbf{A}_j|^\kappa I_{(\gamma,\infty)}(|\mathbf{A}_j|)\,P(B_j > M\,a_n) \to 0\,.
$$

Similarly,

$$
\begin{aligned}
n\,P(D_3) &\le n\,P\big(|\mathbf{A}_i| I_{(\gamma,\infty)}(|\mathbf{A}_i|)\,B_i > M\,a_n\big)\\
&\sim n\,E|\mathbf{A}_i|^\kappa I_{(\gamma,\infty)}(|\mathbf{A}_i|)\,P(B_i > M\,a_n)\,,
\end{aligned}
$$

and by Lebesgue dominated convergence,

$$
\lim_{\gamma\to\infty}\limsup_{n\to\infty} n\,P(D_3) = 0\,.
$$

The relation $nP(D_4) \to 0$ can be proved in the same way.

We summarize our findings.

Proposition 2.5. *If the conditions of Proposition 2.4 hold, then the finite-dimensional distributions of the stationary solution* (Y_t) *to the stochastic recurrence equation* (1.1) *are regularly varying with index* κ *and limiting measure given in* (2.10).

3. Some applications of the regular variation property

In this section we consider some applications of the property of regular variation of the solution (Y_t) to the stochastic recurrence equation (1.1). In particular, we are interested in functionals of the Y_t's and their limit behavior. The results include the central limit theorem for the partial sums of the sequence (Y_t) and limit theory for its partial maxima.

3.1. A remark about the strong mixing property of (Y_t).

Recall that a stationary ergodic sequence (Y_t) is said to be *strongly mixing* if

$$
\alpha_k = \sup_{A\in\sigma(Y_s\,,s\le t)\,,B\in\sigma(Y_s\,,s\ge k)} |P(A\cap B) - P(A)\,P(B)| \to 0\,,
$$

and it is said to be *strongly mixing with geometric rate* if there are constants $r \in (0,1)$ and $c > 0$ such that $\alpha_k \le c\,r^k$ for all $k \ge 1$; see Rosenblatt [31], cf. Doukhan [12]. Under general conditions, the latter property is satisfied for the stationary solution (Y_t) of the stochastic recurrence equation (1.3). This follows from work by Mokkadem [25]; cf. also Basrak et al. [1], Theorem 2.8.

Proposition 3.1. *Assume* $EA^\varepsilon < 1$, $EB^\varepsilon < \infty$ *for some* $\varepsilon > 0$ *and* A *has an a.e. positive Lebesgue density on its support* $[0, x_0]$, *for some* $x_0 \le \infty$. *Then the stochastic recurrence equation* (1.1) *has a stationary ergodic solution* (Y_t) *which is strongly mixing with geometric rate.*

3.2. **The central limit theorem.** If the assumptions of Proposition 2.4 hold and (Y_t) is μ-irreducible, then we may conclude from Propositions 2.4, 2.5 and 3.1 that there exists a unique stationary solution (Y_t) to the stochastic recurrence equation (1.1) which is strongly mixing with geometric rate and which has regularly varying finite-dimensional distributions with index $\kappa > 0$.

If $\kappa > 2$, a standard central limit theorem for stationary ergodic martingale difference sequences applies to (Y_t) and no further mixing condition is needed. Indeed, we have

$$n^{-1/2}\,(S_n - ES_n) = n^{-1/2}\sum_{t=1}^{n}[(A_t - EA)\,Y_{t-1} + (B_t - EB)] + n^{-1/2}\,EA\sum_{t=1}^{n}(Y_{t-1} - EY)\,.$$

Hence

$$n^{-1/2}\,(S_n - ES_n) = n^{-1/2}\,(1 - EA)^{-1}\sum_{t=1}^{n}[(A_t - EA)\,Y_{t-1} + (B_t - EB)] + o_P(1)\,.$$

The sequence $[(A_t - EA)Y_{t-1} + (B_t - EB)]$ is a stationary ergodic martingale difference sequence with respect to the filtration $\mathcal{F}_t = \sigma((A_x, B_x)\,, x \le t)$. Therefore the central limit theorem from Billingsley [3], Chapter 23, applies:

$$n^{-1/2}\,(S_n - ES_n) \stackrel{d}{\to} N(0, \sigma_Y^2)\,,$$

where $\sigma_Y^2 = \mathrm{var}(Y)$. Notice that $EA < 1$ since $EA^\kappa < 1$, $\kappa > 2$ and $g(h) = EA^h$ is a convex function.

If $\kappa < 2$, infinite variance limits may occur for (S_n); see Davis and Hsing [8] and Davis and Mikosch [9]. The proof relies on a point process argument for the lagged vectors $\mathbf{Y}_t(m) = (Y_t, \ldots, Y_{t+m})'$ which is identical to the proof of Theorem 2.10 in Basrak et al. [1] and requires regular variation of the finite-dimensional distributions and the strong mixing condition for (Y_t) with geometric rate. It implies weak convergence of the point processes

$$N_n = \sum_{t=1}^{n}\varepsilon_{\mathbf{Y}_t(m)/a_n} \stackrel{d}{\to} N\,. \tag{3.1}$$

The limiting Poisson point process N is described in [1] and (a_n) is a sequence satisfying $n\,P(Y > a_n) \sim 1$.

The convergence result (3.1) and the arguments in [8, 9, 1] imply the weak convergence of the partial sums, sample autocovariances, sample autocorrelations and the partial maxima of the sequence (Y_t). For details we refer to the mentioned literature. For example, if $\kappa \in (0,2)\backslash\{1\}$,

$$a_n^{-1}\,(S_n - b_n) \stackrel{d}{\to} Z_\kappa\,,$$

where Z_κ is a totally skewed to the right infinite variance κ-stable random variable, $b_n = ES_n$ for $\kappa > 1$ and $b_n = 0$ for $\kappa < 1$. (We refer to Samorodnitsky and Taqqu [33] for an encyclopedic treatment of stable distributions and processes.) The proof of the weak convergence of the sample autocovariances and sample autocorrelations is identical to the one treated in Basrak et al. [1] for solutions to the stochastic recurrence equation (1.1).

Moreover,

$$a_n^{-1}\max(Y_1\,, \ldots\,, Y_n) \stackrel{d}{\to} R_\kappa(\theta)\,,$$

where $P(R_\kappa \le x) = \mathrm{e}^{\,-x^{-\kappa}}$, $x > 0$, is the Fréchet distribution function with shape parameter κ, $P(R_\kappa(\theta) \le x) = [P(R_\kappa \le x)]^\theta$ and $\theta \in (0,1)$ is the extremal index of the sequence (Y_t). See Embrechts et al. [14] for an introduction to extreme value theory and in particular Section 8.1, where the notion of extremal index is treated. Extreme value theory for the solution (Y_t) to (1.1) under the conditions of Kesten's Theorem 1.1 was studied in de Haan et al. [17]. In their Theorem 2.1 they calculate

$$\theta = \int_1^\infty P\left(\max_{j\ge 1}\prod_{i=1}^{j}A_i \le y^{-1}\right)\,\kappa\,y^{\kappa-1}\,dy\,.$$

We mention that the same proof as in [17] (with $n^{1/\kappa}$ replaced by (a_n) as above) applies under the conditions of Proposition 2.4, when Kesten's result does not apply. Indeed, an inspection of their proof shows that it only requires the structure of the stochastic recurrence equation (1.1), the definition of (a_n), the regular variation of (Y_t) and the existence of some $h > 0$ such that $EA^h < 1$.

The definition of the extremal index θ implies that for $x_n \geq a_n$,

$$P(\max(Y_1,\ldots,Y_n) > x_n) \sim \theta\, n\, P(Y > x_n)\,.$$

This is in contrast to iid Y_t's, where this relation holds with $\theta = 1$. In the iid case we also know that $P(S_n - ES_n > x_n) \sim P(\max(Y_1,\ldots,Y_n) > x_n)$ for suitable sequences (x_n) with $x_n \to \infty$. The various results proved in this paper, including Proposition 3.2 and Theorem 4.2, show that the exceedances of the random walk (S_n) and of the partial maxima $(\max(Y_1,\ldots,Y_n))$ above high thresholds have different asymptotic behavior which is also different from the case of iid Y_t's.

3.3. **Regular variation of sums.** In what follows we study the tail behavior of the sums

$$S_n = Y_1 + \cdots + Y_n$$

for fixed $n \geq 1$ under the assumptions of Proposition 2.5. It follows from Proposition 2.5 that all linear combinations of the lagged vector $\mathbf{Y}_m$ are regularly varying with index κ. In particular, S_n is regularly varying with index κ. In this section we give a precise description of the tail asymptotics of $P(S_n > x)$ for fixed n as $x \to \infty$.

We have

$$S_n = \sum_{i=1}^n \left(\Pi_i\, Y_0 + \sum_{t=1}^i \Pi_{t+1,i} B_t \right) = Y_0 \sum_{i=1}^n \Pi_i + \sum_{t=1}^n B_t \sum_{i=t}^n \Pi_{t+1,i}\,. \tag{3.2}$$

Write

$$Z_0 = Y_0 \sum_{i=1}^n \Pi_i \quad \text{and} \quad Z_t = B_t \sum_{i=t}^n \Pi_{t+1,i}\,, \quad t = 1,\ldots,n\,.$$

Notice that Y_0 is independent of $\sum_{i=1}^n \Pi_i$ and B_t is independent of $\sum_{i=t}^n \Pi_{t+1,i}$. Now an argument similar to the one in the proof of Proposition 2.3 shows that for $0 \leq s < t \leq n$,

$$\frac{P(Z_t > x\,, Z_s > x)}{P(Z_0 > x)} \to 0\,, \quad x \to \infty\,.$$

Also notice that the same result holds if $Y_0 = c$ is a constant initial value. An application of Lemma 2.1 yields the following result.

Proposition 3.2. *Assume that the conditions of Proposition 2.4 hold. If (Y_n) is the stationary solution to the stochastic recurrence equation* (1.1) *then*

$$\lim_{x\to\infty} \frac{P\,(S_n > x)}{P(B > x)} = (1 - EA^\kappa)^{-1}\, E\left(\sum_{i=1}^n \Pi_i\right)^\kappa + \sum_{t=0}^{n-1} E\left(\sum_{i=0}^t \Pi_i\right)^\kappa\,. \tag{3.3}$$

If (Y_n) satisfies the stochastic recurrence equation (1.1) *with $Y_0 = c$ for some constant c, then*

$$\lim_{x\to\infty} \frac{P\,(S_n > x)}{P(B > x)} = \sum_{t=0}^{n-1} E\left(\sum_{i=0}^t \Pi_i\right)^\kappa\,.$$

For comparison, assume for the moment that $(\widetilde{Y}_t)$ is an iid sequence $\widetilde{Y}_1 \stackrel{d}{=} Y$ and Y has the stationary distribution given by (1.3). Then for every fixed $n \geq 1$,

$$\lim_{x\to\infty} \frac{P\left(\widetilde{Y}_1 + \cdots + \widetilde{Y}_n > x\right)}{n\, P(Y > x)} = 1\,. \tag{3.4}$$

This is the subexponential property of a regularly varying distribution; see Embrechts et al. [14], Section 1.3.2 and Appendix A3 for an extensive discussion of subexponential distributions. Property (3.4) does not remain valid for dependent stationary sequences with regularly varying finite-dimensional distributions. This was shown in Mikosch and Samorodnitsky [22] for the case of linear processes. In that case the limiting constant in (3.4) is in general different from 1 and depends on the coefficients of the linear process. Proposition 3.2

shows that a similar behavior can be expected for other non-linear stationary processes. In particular, by Proposition 3.2 relation (3.3) can be re-written in the form

$$\lim_{x\to\infty} \frac{P(S_n > x)}{P(Y > x)} = E\left(\sum_{i=1}^{n} \Pi_i\right)^{\kappa} + (1 - EA^{\kappa}) \sum_{t=0}^{n-1} E\left(\sum_{i=0}^{t} \Pi_i\right)^{\kappa} . \tag{3.5}$$

It is interesting to observe that a similar relationship holds if the (A_t, B_t)'s satisfy the conditions of Kesten's Theorem 1.1. In that case the condition $EA^{\kappa} = 1$ is needed for regular variation of the stationary solution (Y_t) to the stochastic recurrence equation (1.1) with index $\kappa > 0$. Assume in addition that $EB^{\kappa+\delta}$ and $EA^{\kappa+\delta}$ are finite for some $\delta > 0$. Then we may conclude from the representation (3.2), regular variation of Y_0 and Breiman's result that

$$\lim_{x\to\infty} \frac{P(S_n > x)}{P(Y > x)} = E\left(\sum_{i=1}^{n} \Pi_i\right)^{\kappa} .$$

In a sense, this is the limiting result in (3.5) for $EA^{\kappa} = 1$.

4. Large deviations and ruin probabilities

4.1. **Results on large deviations.** In this subsection we couple the increase of x with n to obtain probabilities of large deviations of the type

$$P(S_n - ES_n > x) \sim n\, EC^{\kappa}\, P(B > x) , \quad \text{uniformly for } x \geq x_n$$

and appropriate sequences $x_n \to \infty$. Here C is a generic element of the stationary sequence

$$C_t = \sum_{i=t}^{\infty} \Pi_{t+1,i} , \quad t \in \mathbb{Z} . \tag{4.1}$$

We start with an auxiliary result, where we collect some useful properties of this sequence.

Lemma 4.1. *Assume that (A_t) is an iid sequence and $EA^{\kappa} < 1$ for some $\kappa > 0$.*

1. *The sequence (C_t) defined in (4.1) is well defined and strictly stationary.*
2. *The random variable C has finite pth moment if and only if $EA^p < \infty$ for $p > 0$.*
3. *The sequences (C_t) and (D_t) given by (4.2) have the same finite-dimensional distributions. If A has an a.e. positive Lebesgue density on the support $[0, x_0]$ for some $x_0 \leq \infty$, then (D_t) is strongly mixing with geometric rate.*

Proof. (1) The sequence (C_t) has the same distribution as the sequence

$$D_t = \sum_{i=-\infty}^{t} \Pi_{i+1,t} , \quad t \in \mathbb{Z} . \tag{4.2}$$

The latter satisfies the stochastic recurrence equation

$$D_t = 1 + A_t \sum_{i=-\infty}^{t-1} \Pi_{i+1,t-1} = 1 + A_t\, D_{t-1} , \quad t \in \mathbb{Z} , \tag{4.3}$$

It constitutes a unique strictly stationary sequence if and only if $E \log A < 0$, see (1.2), which is satisfied if $EA^{\kappa} < 1$ for some $\kappa > 0$.

(2) From (4.3), the independence of D_{t-1} and A_t and the stationarity of (D_t) we conclude that D_t has finite pth moment if and only if A_t has. Since $D \stackrel{d}{=} C$, the statement follows.

(3) follows from Proposition 3.1. □

In the following result we assume in addition that the sequences (A_t) and (B_t), hence (C_t) and (B_t), are independent. Although we conjecture that this assumption can be avoided, we need the independence at various technical steps in the proof.

Theorem 4.2. *Assume that (A_t) and (B_t) are independent iid sequences of non-negative random variables, B is regularly varying with index $\kappa > 1$, $EA^\kappa < 1$ and $EA^{2\kappa} < \infty$. Consider a sequence of positive numbers such that $n\,P(B > x_n) \to 0$ and for every $c > 0$,*

$$\lim_{n\to\infty} \sup_{x\ge x_n} \left| \frac{P\left(\mathrm{var}(BI_{[0,x]}(B)) \sum_{t=1}^n C_t^2 > c\,x^2/\log x\right) + P\left(|\sum_{t=1}^n (C_t - EC)| > c\,x\right)}{n\,PB(> x)} \right| = 0\,. \tag{4.4}$$

Then the large deviation relations

$$\lim_{n\to\infty} \sup_{x\ge x_n} \left| \frac{P(S_n - ES_n > x)}{n\,P(B > x)} - EC^\kappa \right| = 0\,, \tag{4.5}$$

$$\lim_{n\to\infty} \sup_{x\ge x_n} \frac{P(S_n - ES_n \le -x)}{n\,P(B > x)} = 0 \tag{4.6}$$

are satisfied.

The proof of the theorem is rather technical and therefore postponed until Section 5.

Remark 4.3. The validation of (4.4) is in general difficult. Sufficient conditions for (4.4) can be verified by assuming certain mixing conditions on (C_t); see Lemma 4.6 below and Lemma 4.1(3).

Remark 4.4. Theorem 4.5 is applicable for finite or infinite variance sequences (B_t). The infinite variance comes into the picture in condition (4.4). For $\kappa > 2$, $\mathrm{var}(BI_{[0,x]}(B)) \to c$ for some finite $c > 0$. Hence condition (4.4) can be formulated without $\mathrm{var}(BI_{[0,x]}(B))$. If $\kappa < 2$ or $\kappa = 2$ and $\mathrm{var}(B) = \infty$, $\mathrm{var}(BI_{[0,x]}(B)) \to \infty$. In particular, for $\kappa = 2$, $\mathrm{var}(BI_{[0,x]}(B))$ is a slowly varying function which increases to infinity. If $\kappa \in (1, 2)$, an application of Karamata's theorem yields for some $c > 0$, $\mathrm{var}(BI_{[0,x]}(B)) \sim c\,x^2\,P(B > x) \to \infty$.

Remark 4.5. The literature on large deviations for sums of stationary heavy-tailed random variables is rather sparse. The case of linear processes $Y_t = \sum_{j=-\infty}^{\infty} \varphi_j Z_{t-j}$ for iid regularly varying sequences (Z_t) was treated in Mikosch and Samorodnitsky [22]. In this case, the limit of $(P(S_n - ES_n > x)/(nP(Y > x))$ is approximated uniformly for $x \ge c\,n$, any positive c. The limit depends in a complicated way on the coefficients φ_j and on the coefficient of regular variation. Davis and Hsing [8] seems to be the only reference, where large deviation results were proved for general regularly varying stationary sequences, assuming certain mixing conditions and $\kappa < 2$. They exploit point process convergence results and express the limit of the sequence $(P(S_n > x_n)/(nP(Y > x_n))$ in terms of the limiting point process, which is difficult to interpret. Unfortunately, their approach seems to work only in the case of infinite variance random variables.

We continue by giving some sufficient conditions for the validity of the relation (4.4).

Lemma 4.6. *Assume A has an a.e. positive Lebesgue density on its support $[0, x_0]$ for some $x_0 \le \infty$, B is regularly varying with index κ and $EA^\kappa < 1$ for some $\kappa > 1$.*

(1) *Assume*

$$\sup_{x\ge x_n} n^{(\kappa+\gamma)/2-1} \left(x/\sqrt{\mathrm{var}(BI_{[0,x]}(B))\log x}\right)^{-(\gamma+\kappa)} /P(B > x) \to 0 \tag{4.7}$$

and $EA^{\kappa+\gamma} < \infty$ for some γ such that $\kappa + \gamma > 2$. Then relation (4.4) holds.

(2) *Assume that $C \le c$ a.s. for some constant $c > 0$ and for some $d, \varepsilon > 0$,*

$$\sup_{x\ge x_n} \frac{\mathrm{e}^{-d\,(x/\sqrt{n})^2/(\log x\,\mathrm{var}(BI_{[0,x]}(B)))}}{n\,P(B > x)} + \sup_{x\ge x_n} \frac{\mathrm{e}^{-(x\,/\sqrt{n})\,n^{-\varepsilon}}}{n\,P(B > x)} \to 0\,. \tag{4.8}$$

Then relation (4.4) holds.

Remark 4.7. In particular, (4.4) holds for (x_n) with (4.8) if $A \le c_0$ for some constant $c_0 < 1$ and B is regularly varying with index $\kappa > 1$. Indeed, then $EA^d < 1$ for all $d > 0$ and $C \le \sum_{i=0}^{\infty} c_0^i = (1 - c_0)^{-1}$.

Remark 4.8. We discuss the conditions on the x-regions where (4.4) holds. If $\kappa > 2$, $\mathrm{var}(B) < \infty$. Writing $P(B > x) = x^{-\kappa} L(x)$ for some slowly varying function L, (4.7) is satisfied if

$$\left[n^{(\kappa+\gamma)/2-1}\, x_n^{-\gamma}\right] \sup_{x\ge x_n} [(\log x)^{(\kappa+\gamma)/2}/L(x)] \to 0\,. \tag{4.9}$$

Since $(\log x)^{(\kappa+\gamma)/2}/L(x) \le x^{\varepsilon}$, for every $\varepsilon > 0$ and sufficiently large x, (4.9) holds if $x_n = n^{0.5+\delta}$ with $\delta > \gamma^{-1}(\kappa/2-1)$. This δ can be chosen the closer to zero the more moments of A exist, i.e., the larger γ can be chosen. These growth rates are comparable to the case of iid Y_t's for $\kappa > 2$, see Theorem 1.2, where one could choose $x_n = c\sqrt{n\log n}$ for some constant $c > 0$. Such precise results are hard to derive in the case of dependent Y_t's.

If $\kappa \in (1,2)$, a similar remark applies. Then x_n can be chosen of the order $n^{(1/\kappa)+\delta}$ for some $\delta > 0$ which is in agreement with the order of magnitude of (x_n) for iid sequences, see again Theorem 1.2.

Notice that, under the above conditions, $x_n = cn$ can be chosen in most cases of interest for $\kappa > 1$.

Proof. By Lemma 4.1(3), the sequence $(D_t) \stackrel{d}{=} (C_t)$ is strongly mixing with geometric rate and so is $(N_t D_t)$, where the iid standard normal sequence (N_t) is assumed to be independent of (D_t). This follows by standard results on strong mixing; see for example Doukhan [12].

By Markov's inequality, for every $y > 0$ and $\gamma > 0$ such that $EA^{\kappa+\gamma} < \infty$,

$$\begin{aligned} P\left(\sum_{t=1}^n C_t^2 > y\right) &\le y^{-(\gamma+\kappa)/2} E\left(\sum_{t=1}^n C_t^2\right)^{(\kappa+\gamma)/2} \\ &= y^{-(\gamma+\kappa)/2}\, E\left|\sum_{t=1}^n D_t\, N_t\right|^{\kappa+\gamma} /E|N|^{(\kappa+\gamma)/2} \\ &\le c\,(n/y)^{(\gamma+\kappa)/2}\,. \end{aligned} \tag{4.10}$$

In the last step we applied a moment estimate for sums of strongly mixing random variables with geometric rate and used the fact that $\gamma+\kappa > 2$; see Doukhan [12], p. 31. Applying (4.10) for $x > 1$, $d > 0$, we obtain

$$\frac{P\left(\mathrm{var}(BI_{[0,x]}(B))\sum_{t=1}^n C_t^2 > d\,x^2/\log x\right)}{n\,P(B>x)} \le c\,\frac{\left(x/\sqrt{\log x\,\mathrm{var}(BI_{[0,x]}(B))}\right)^{-(\gamma+\kappa)} n^{(\kappa+\gamma)/2}}{n\,P(B>x)}\,,$$

and the right-hand side converges to zero uniformly for $x \ge x_n$, by virtue of assumption (4.7).

Similarly, if $C \le c$ a.s., applying an exponential Markov inequality for $h > 0$,

$$\begin{aligned} P\left(\sum_{t=1}^n C_t^2 > y\right) &\le \mathrm{e}^{-(h^2/2)\,(y/n)} E\mathrm{e}^{(h^2/2)\,n^{-1}\sum_{t=1}^n C_t^2} \\ &= \mathrm{e}^{-(h^2/2)\,(y/n)} E\mathrm{e}^{hn^{-1/2}\sum_{t=1}^n D_t N_t}\,. \end{aligned}$$

The central limit theorem for strongly mixing random variables with geometric rate (see Ibragimov and Linnik [18]) yields

$$n^{-1/2}\sum_{t=1}^n D_t N_t \stackrel{d}{\to} N(0,\sigma^2)\,,$$

where $\sigma^2 = \mathrm{var}(D)$. Moreover,

$$E\mathrm{e}^{(h^2/2)\,n^{-1}\sum_{t=1}^n C_t^2} \le E\mathrm{e}^{(h^2/2)\,C_1^2} < \infty\,.$$

Applying a domination argument, the central limit theorem and assumption (4.8) prove that

$$\frac{P\left(\mathrm{var}(BI_{[0,x]}(B))\sum_{t=1}^n C_t^2 > d\,x^2/\log x\right)}{n\,P(B>x)} \le c\,\frac{\mathrm{e}^{-(h^2/2)d\,(x/\sqrt{n})^2/(\log x\,\mathrm{var}(BI_{[0,x]}(B)))}}{n\,P(B>x)} \to 0\,.$$

The estimates for $P(\sum_{t=1}^n (C_t - EC) > x)$ can be derived in a similar fashion. If $EA^{\kappa+\gamma} < \infty$ we have

$$P\left(\left|\sum_{t=1}^n (C_t - EC)\right| > x\right) \leq x^{-(\kappa+\gamma)}\, E\left|\sum_{t=1}^n (C_t - EC)\right|^{\kappa+\gamma}$$
$$\leq c\, x^{-(\kappa+\gamma)}\, n^{(\kappa+\gamma)/2}\,.$$

Now assume $C \leq c$ a.s. Since (C_t) is strongly mixing with geometric rate the following exponential bound holds (see Doukhan [12], p. 34). For any $\varepsilon < 0.5$ there exists a constant $h > 0$ such that

$$P\left(\sum_{t=1}^n (C_t - EC) > x\right) \leq \mathrm{e}^{-h\,(x\,/\sqrt{n})\, n^{-\varepsilon}}\,.$$

This concludes the proof. □

4.2. **Results on ruin probabilities.** In this subsection we study the ruin probability

$$\psi(u) = P\left(\sup_{n\geq 0}\left((S_n - ES_n) - \mu\, n\right) > u\right)\,,$$

when the initial capital $u \to \infty$ and $\mu > 0$. Here (Y_t) is the unique stationary ergodic solution to (1.3), (A_t) and (B_t) are independent and satisfy the conditions of Theorem 4.2. In particular, we assume that $\kappa > 1$. Then $EB < \infty$ and $EA < 1$ since $EA^\kappa < 1$. In particular, $EY = EB\,(1 - EA)^{-1} = EB\, EC$ is well defined. This choice and the strong law of large numbers ensure that the random walk $((S_n - ES_n) - \mu\, n)_{n\geq 0}$ has a negative drift.

Theorem 4.9. *Assume that the conditions of Theorem 4.2 hold, that $\kappa > 1$ and $x_n = c\,n$ is a possible threshold sequence for every $c > 0$. Moreover, assume there exists $\gamma > \kappa$ such that $EC^{\kappa+\gamma} < \infty$. Assume that (C_t) is strongly mixing with geometric rate. Then we have for any $\mu > 0$,*

$$\lim_{u\to\infty} \frac{\psi(u)}{u\, P(B > u)} = EC^\kappa\, \frac{1}{\mu}\, \frac{1}{\kappa - 1}\,. \tag{4.11}$$

We postpone the proof of Theorem 4.9 to Section 6.

Remark 4.10. The assumption that Theorem 4.2 holds for $x_n = cn$ is not really a strong restriction. Indeed, we discussed in Remark 4.8 that this condition is satisfied under very mild conditions.

Remark 4.11. This result is similar to the case of iid Y_t's; see Theorem 1.3 above. To compare with the latter one, we mention that (4.11) can be reformulated by using Proposition 2.4:

$$\lim_{u\to\infty} \frac{\psi(u)}{u\, P(Y > u)} = (1 - EA^\kappa)\, EC^\kappa\, \frac{1}{\mu}\, \frac{1}{\kappa - 1}\,.$$

5. Proof of Theorem 4.2

We will make use of the decomposition

$$S_n = Y_0 \sum_{i=1}^n \Pi_i + \sum_{t=1}^n B_t \sum_{i=t}^\infty \Pi_{t+1,i} - \sum_{t=1}^n B_t \sum_{i=n+1}^\infty \Pi_{t+1,i} \tag{5.1}$$
$$= S_{n,1} + S_{n,2} - S_{n,3}\,.$$

Proof of (4.5). We start with an upper bound. Observe that for small $\varepsilon > 0$,

$$P(S_n - ES_n > x)$$
$$\leq P\left(S_{n,1} - ES_{n,1} > x\,\varepsilon/2\right) + P\left(S_{n,2} - ES_{n,2} > x\,(1-\varepsilon)\,\right) + P\left(-S_{n,3} + ES_{n,3} > x\,\varepsilon/2\right)$$
$$= I_1(x) + I_2(x) + I_3(x)\,.$$

We bound the I_j's in a series of lemmas.

Lemma 5.1. *We have*

$$\limsup_{n\to\infty}\sup_{x\ge x_n}\frac{I_j(x)}{n\,P(B>x)}=0\,,\quad j=1,3\,.$$

Proof of Lemma 5.1. We start with I_1. The random variable Y_0 is regularly varying with index κ, by virtue of Proposition 2.4, and independent of (Π_i). Moreover,

$$\sum_{i=1}^n \Pi_i\uparrow\sum_{i=1}^\infty \Pi_i\stackrel{d}{=}C-1\,.$$

We also see that

$$ES_{n,1}=EY\sum_{i=1}^n(EA)^i\uparrow\frac{EY\,EA}{1-EA}=c'\,.$$

The expectation EA is smaller than one since $EA^\kappa<1$ for some $\kappa>1$ and $g(h)=EA^h$ is a convex function; see the discussion in the proof of Proposition 2.4. An application of Breiman's result (Lemma 2.2) and Proposition 2.4 yield that for independent C,Y,

$$\begin{aligned}\sup_{x\ge x_n}\frac{I_1(x)}{n\,P(B>x)}&\le\sup_{x\ge x_n}\frac{P(|S_{n,1}-ES_{n,1}|>\varepsilon x/2)}{n\,P(B>x)}\\&\le\sup_{x\ge x_n}\frac{P(Y\,(C-1)>\varepsilon x/2-c')}{n\,P(B>x)}\\&\le c\sup_{x\ge x_n}\frac{P(Y>\varepsilon x/2)\,E(C-1)^\kappa}{n\,P(B>x)}\to0\,.\end{aligned}\tag{5.2}$$

For Breiman's result one needs that $EC^{\kappa+\delta}<\infty$ for some $\delta>0$. This condition is satisfied since $EA^{2\kappa}<\infty$, by virtue of Lemma 4.1(2).

Now we turn to I_3. We have

$$S_{n,3}=\sum_{t=1}^n B_t\Pi_{t+1,n+1}\sum_{i=n+1}^\infty\Pi_{n+2,i}\stackrel{d}{=}A_0\,C_{n+1}\sum_{t=1}^n B_t\Pi_{t-1}\tag{5.3}$$

$$\stackrel{d}{\to}A\,C\sum_{t=1}^\infty B_t\Pi_{t-1}=A\,C\,Y'\,,\tag{5.4}$$

where Y',A,C are independent and $Y\stackrel{d}{=}Y'$. Similar arguments as for I_1 show that

$$\sup_{x\ge x_n}\frac{I_3(x)}{n\,P(B>x)}\le\sup_{x\ge x_n}\frac{P(|S_{n,3}-ES_{n,3}|>x\varepsilon/2)}{n\,P(B>x)}\to0\,.\tag{5.5}$$

This proves the lemma. □

Lemma 5.2. *We have*

$$\lim_{\varepsilon\downarrow0}\limsup_{n\to\infty}\sup_{x\ge x_n}\left(\frac{I_2(x)}{n\,P(B>x)}-EC^\kappa\right)\le0\,.$$

Proof of Lemma 5.2. Write for any $\delta>0$,

$$\begin{aligned}Q_{n,1}(\delta)&=\bigcup_{1\le t<s\le n}\{B_t>\delta x,B_s>\delta x\}\,,\\Q_{n,2}(\delta)&=\left\{\max_{t\le n}B_t\le\delta x\right\}\,,\\Q_{n,3}(\delta)&=\bigcup_{t=1}^n\{B_t>\delta x\,,B_s\le\delta x\,,1\le s\ne t\le n\}\,.\end{aligned}$$

Then

$$\begin{aligned}
&\frac{I_2(x)}{n\,P(B>x)}\\
&= \frac{P\left(\{S_{n,2}-ES_{n,2}>x\,(1-\varepsilon)\}\cap Q_{n,1}(\delta)\right)}{n\,P(B>x)}+\frac{P\left(\{S_{n,2}-ES_{n,2}>x\,(1-\varepsilon)\}\cap Q_{n,2}(\delta)\right)}{n\,P(B>x)}\\
&\quad+\frac{P\left(\{S_{n,2}-ES_{n,2}>x\,(1-\varepsilon)\}\cap Q_{n,3}(\delta)\right)}{n\,P(B>x)}\\
&= I_{2,1}(x)+I_{2,2}(x)+I_{2,3}(x)\,,
\end{aligned}$$

Obviously, for any $\delta>0$,

$$\lim_{n\to\infty}\sup_{x\ge x_n} I_{2,1}(x)=0\,.$$

Writing for any $t\in\mathbb{Z}$, $x>0$,

$$B_{t,x}=B_t I_{[0,x]}(B_t)\,,$$

we obtain

$$\sup_{x\ge x_n} I_{2,2}(x) \;\le\; \sup_{x\ge x_n}\frac{P\left(\sum_{t=1}^n (B_{t,\delta x}C_t-EB_{1,\delta x}\,EC)>(1-\varepsilon)x\right)}{n\,P(B>x)}\,.$$

Notice that

$$\sup_{x\ge x_n} I_{2,2}(x)\le \sup_{x\ge x_n}\frac{P(E_1)}{n\,P(B>x)}+\sup_{x\ge x_n}\frac{P(E_2)}{n\,P(B>x)}\,,$$

where

$$\begin{aligned}
E_1 &= \left\{\sum_{t=1}^n (B_{t,\delta x}-EB_{1,\delta x})\,C_t>0.5(1-\varepsilon)x\right\}\,,\\
E_2 &= \left\{EB_{1,\delta x}\sum_{t=1}^n (C_t-EC)>0.5(1-\varepsilon)x\right\}\,.
\end{aligned}$$

Conditioning on (C_t) and using the Fuk-Nagaev inequality (p. 78 in Petrov [28]), we have with $EC^{2\kappa}<\infty$,

$$\begin{aligned}
&E[P(E_1\mid (C_t))]\\
&\le c\,E\left((0.5(1-\varepsilon)x)^{-2\kappa}\sum_{t=1}^n C_t^{2\kappa}+\exp\left\{-c(0.5(1-\varepsilon)x)^2\left[\mathrm{var}(B_{1,\delta x})\sum_{t=1}^n C_t^2\right]^{-1}\right\}\right)\\
&\le c\,x^{-2\kappa}\,n\,EC^{2\kappa}\\
&\quad+c\,E\left(\exp\left\{-c(0.5(1-\varepsilon)x)^2\left[\mathrm{var}(B_{1,\delta x})\sum_{t=1}^n C_t^2\right]^{-1}\right\}I_{\left\{\mathrm{var}(B_{1,\delta x})\sum_{t=1}^n C_t^2\le d\,x^2/\log x\right\}}\right)
\end{aligned}$$

$$+P\left(\mathrm{var}(B_{1,\delta x})\sum_{t=1}^n C_t^2>d\,x^2/\log x\right) \tag{5.6}$$

$$= J_1(x)+J_2(x)+J_3(x)\,,$$

where $d>0$ is chosen small enough such that $d'=d'(d)=c[0.5(1-\varepsilon)]^2/d$ is large enough implying

$$\sup_{x\ge x_n}\frac{J_2(x)}{n\,P(B>x)}\le\sup_{x\ge x_n}\frac{\mathrm{e}^{-c\,(0.5(1-\varepsilon))^2\log x/d}}{n\,P(B>x)}=\sup_{x\ge x_n}\frac{x^{-d'}}{n\,P(B>x)}\to 0\,.$$

We also have

$$\sup_{x\ge x_n}\frac{J_1(x)}{n\,P(B>x)}\to 0\,.$$

The relations

$$\sup_{x\ge x_n}\frac{J_3(x)}{n\,P(B>x)}\to 0\quad\text{and}\quad\sup_{x\ge x_n}\frac{P(E_2)}{n\,P(B>x)}\to 0\,,$$

follow by assumption (4.4). Collecting the above estimates, we proved for every δ,

$$\lim_{n\to\infty}\sup_{x\ge x_n}I_{2,2}(x)=0\,.$$

Thus it remains to show that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{x\ge x_n}\left(I_{2,3}(x)-EC^\kappa\right)\le 0\,. \tag{5.7}$$

We have

$$\begin{aligned}
&I_{2,3}(x)\\
\le\;&\sum_{t=1}^n\frac{P\Big(B_tC_t+\sum_{s=1,s\ne t}^n(B_s\,C_s-EB\,EC)>x\,(1-\varepsilon)\,,B_t>\;\delta\,x\,,\max_{1\le s\le n\,,s\ne t}B_s\le\delta x\Big)}{n\,P(B>x)}\\
\le\;&\sum_{t=1}^n\frac{P\left(B_t\,\min(C_t,\delta^{-1}(1-2\varepsilon))>(1-2\varepsilon)\,x\right)}{n\,P(B>x)}\\
&+\sum_{t=1}^n\frac{P\Big(\sum_{s=1,s\ne t}^n(B_s\,C_s-EB\,EC)>x\,\varepsilon\,,B_t>\;\delta\,x\,,\max_{1\le s\le n\,,s\ne t}B_s\le\delta x\Big)}{n\,P(B>x)}\\
\le\;&\frac{P\left(B_1\,\min(C_1,\delta^{-1}(1-2\varepsilon))>(1-2\varepsilon)\,x\right)}{P(B>x)}\\
&+\sum_{t=1}^n\frac{P\Big(\sum_{s=1,s\ne t}^n(B_s\,C_s-EB\,EC)>x\,\varepsilon\,,\max_{1\le s\le n\,,s\ne t}B_s\le\delta x\Big)\;P(B>\delta\,x)}{n\,P(B>x)}\\
=\;&L_1(x)+L_2(x)\,.
\end{aligned}$$

By Breiman's result,

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}\limsup_{n\to\infty}$$
$$\sup_{x\ge x_n}\left[\left(L_1(x)-\frac{E\left[(\min(C,\delta^{-1}(1-2\varepsilon)))^\kappa\right]}{(1-2\varepsilon)^\kappa}\right)+\left(\frac{E\left[(\min(C,\delta^{-1}(1-2\varepsilon)))^\kappa\right]}{(1-2\varepsilon)^\kappa}-EC^\kappa\right)\right]=0\,.$$

Similar calculations as for $I_{2,2}(x)$ yield that for every δ,ε,

$$\lim_{n\to\infty}\sup_{x\ge x_n}L_2(x)=0\,.$$

We conclude that (5.7) holds. This finishes the proof of the lemma. □

Lemmas 5.1 and 5.2 prove that

$$\limsup_{n\to\infty}\sup_{x\ge x_n}\left(\frac{P\left(S_n-ES_n>x\right)}{n\,P(B>x)}-EC^\kappa\right)\le 0\,. \tag{5.8}$$

We conclude the proof of (4.5) with the bound

$$\limsup_{n\to\infty}\sup_{x\ge x_n}\left(EC^\kappa-\frac{P\left(S_n-ES_n>x\right)}{n\,P(B>x)}\right)\le 0\,. \tag{5.9}$$

Arguing as for (5.8), we see that for any $\delta > 0$, uniformly for $x \ge x_n$,

$$\begin{aligned} \frac{P(S_n - ES_n > x)}{nP(B>x)} &\sim \frac{P(\{S_{n,2} - ES_{n,2} > x\} \cap Q_{n,2}(\delta))}{nP(B>x)} + \frac{P(\{S_{n,2} - ES_{n,2} > x\} \cap Q_{n,3}(\delta))}{nP(B>x)} \\ &= K_1(x) + K_2(x)\,. \end{aligned}$$

It follows by analogous arguments as for $I_{2,2}(x)$ that

$$\sup_{x \ge x_n} \frac{K_1(x)}{n\,P(B>x)} \to 0\,.$$

Write for $\varepsilon > 0$,

$$L_t = \{B_t \min(C_t, \delta^{-1}(1+\varepsilon)) > (1+\varepsilon)\,x\}\,, \quad t \in \mathbb{Z}\,.$$

As regards $K_2(x)$, we have

$$\begin{aligned} K_2(x) &= \sum_{t=1}^n \frac{P\left(B_tC_t + \sum_{s=1,s\neq t}^n B_sC_s > x + nEBEC\,, B_t > \delta x\,, \max_{s\le n, s\neq t} B_s \le \delta x\right)}{n\,P(B>x)} \\ &\ge [P(B_1 \le \delta x)]^{n-1} \sum_{t=1}^n \frac{P(L_t)}{n\,P(B>x)} \\ &\quad - \sum_{t=1}^n \frac{P\left(\left\{\sum_{s=1,s\neq t}^n (B_s\,C_s - EBEC) < -\varepsilon x + EBEC\right\} \cap L_t \cap \{\max_{s\le n,s\neq t} B_s \le \delta x\}\right)}{n\,P(B>x)} \\ &= K_{2,1}(x) - K_{2,2}(x)\,. \end{aligned}$$

Since $n\,P(B > \delta x_n) \to 0$ we have

$$\sup_{x\ge x_n} |[P(B \le \delta x)]^{n-1} - 1| \to 0\,.$$

Therefore and by regular variation of B,

$$\sup_{x \ge x_n} \big((1+\varepsilon)^{-\kappa}\, E[\min(C, \delta^{-1}(1+\varepsilon))]^\kappa - K_{2,1}(x)\big) \to 0\,. \tag{5.10}$$

Write

$$T_{n,t} = \left\{ \sum_{s=1,s\neq t}^n (B_{s,\delta x} C_s - EBEC) \le -\varepsilon x + EBEC \right\}\,.$$

As regards $K_{2,2}(x)$, we have for $0 < m < M < \infty$,

$$\begin{aligned} n\,P(B>x)\,K_{2,2}(x) &\le \sum_{t=1}^n P\left(T_{n,t} \cap L_t \cap \{C_t \le m\}\right) + \sum_{t=1}^n P\left(T_{n,t} \cap L_t \cap \{C_t > M\}\right) \\ &\quad + \sum_{t=1}^n P\left(T_{n,t} \cap L_t \cap \{C_t \in (m, M]\}\right) \\ &= K_{2,2,1}(x) + K_{2,2,2}(x) + K_{2,2,3}(x)\,. \end{aligned}$$

Then for small $\delta > 0$, by the uniform convergence theorem for regularly varying functions,

$$\begin{aligned} \lim_{m\to 0} \lim_{n\to\infty} \sup_{x\ge x_n} \frac{K_{2,2,1}(x)}{n\,P(B>x)} &\le \lim_{m\to 0} \lim_{n\to\infty} \sup_{x\ge x_n} \frac{P(m\,B > (1+\varepsilon)\,x)}{P(B>x)} \\ &= \lim_{m\to 0} m^\kappa\,(1+\varepsilon)^{-\kappa} = 0\,. \end{aligned}$$

Moreover, by Breiman's result and Lebesgue dominated convergence,

$$\begin{aligned}\lim_{M\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{K_{2,2,2}(x)}{n\,P(B>x)} &= \lim_{M\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(CI_{\{C>M\}}\,B>(1+\varepsilon)\,x)}{P(B>x)}\\ &= \lim_{M\to\infty}E(C^\kappa I_{(M,\infty)}(C))\,(1+\varepsilon)^{-\kappa}\\ &= 0\,.\end{aligned}$$

Finally, using the same method of proof as for $I_{2,2}(x)$.

$$\begin{aligned}\sup_{x\ge x_n}\frac{K_{2,2,3}(x)}{n\,P(B>x)} &\le \sup_{x\ge x_n} n^{-1}\,c\sum_{t=1}^n P(T_{n,t})\\ &\le \sup_{x\ge x_n} P\left(\sum_{s=1}^n (B_{s,\delta x}C_s-EBEC)\le -\varepsilon\,x/2\right)+\sup_{x\ge x_n}P(B_{1,\delta x}C>x\,\varepsilon/2)\\ &\to 0\,.\end{aligned}$$

Taking the above bounds and, in particular, (5.10) into account, we conclude that

$$\lim_{n\to\infty}\sup_{x\ge x_n}\big((1+\varepsilon)^{-\kappa}\,E[\min(C,\delta^{-1}(1+\varepsilon))]^\kappa-K_2(x)\big)=0\,,$$

and letting $\delta\downarrow 0$, $\varepsilon\downarrow 0$, (5.9) follows.

The proof of relation (4.5) is now complete.

Proof of (4.6) The proof is similar to the one for (4.5). It follows from relations (5.2) and (5.5) that it suffices to show

$$\sup_{x\ge x_n}\frac{P(S_{n,2}-ES_{n,2}\le -x\,r)}{n\,P(B>x)}\to 0$$

for any $r>0$. We proceed similarly as for $I_2(x)$ and use the same notation. Then for any $\delta>0$,

$$\sup_{x\ge x_n}\frac{P(\{S_{n,2}-ES_{n,2}\le -x\,r\}\cap Q_{n,1}(\delta))}{n\,P(B>x)}\le \sup_{x\ge x_n}\frac{P(Q_{n,1}(\delta))}{n\,P(B>x)}\to 0\,.$$

Moreover, by the uniform convergence theorem for regularly varying functions,

$$\begin{aligned}\lim_{\delta\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(\{S_{n,2}-ES_{n,2}\le -x\,r\}\cap Q_{n,3}(\delta))}{n\,P(B>x)} &\le \lim_{\delta\to\infty}\lim_{n\to\infty}\sup_{x\ge x_n}\frac{P(B>\delta x)}{P(B>x)}\\ &= \lim_{\delta\to\infty}\delta^{-\kappa}=0\,.\end{aligned}$$

Finally, uniformly for $x\ge x_n$, sufficiently large n,

$$\begin{aligned}\Sigma &= \frac{P(\{S_{n,2}-ES_{n,2}\le -x\,r\}\cap Q_{n,2}(\delta))}{n\,P(B>x)}\\ &\le \frac{P\left(\sum_{t=1}^n(B_{t,\delta x}\,C_t-EB_{1,\delta x}\,EC)\le -x\,r+n\,EC\,E(BI_{(\delta x,\infty)}(B))\right)}{n\,P(B>x)}\\ &\le \frac{P\left(\sum_{t=1}^n(B_{t,\delta x}\,C_t-EB_{1,\delta x}\,EC)\le -x\,r/2\right)}{n\,P(B>x)}\,.\end{aligned}$$

Here we used the fact that, by Karamata's theorem, since $x\ge x_n$ and $n\,P(B>x_n)\to 0$,

$$n\,EC\,E(BI_{(\delta x,\infty)}(B)\le c\,n\,x\,P(B>x)\le c\,n\,x\,P(B>x_n)=o(x)\,.$$

Hence

$$\begin{aligned}\Sigma &\le \frac{P\left(\sum_{t=1}^n(B_{t,\delta x}-EB_{1,\delta x})\,C_t\le -x\,r/4\right)}{n\,P(B>x)}+\frac{P\left(EB_{1,\delta x}\,\sum_{t=1}^n(C_t-EC)\le -x\,r/4\right)}{n\,P(B>x)}\\ &= \Sigma_1(x)+\Sigma_2(x)\,.\end{aligned}$$

The relation $\sup_{x\ge x_n}\Sigma_2(x)\to 0$ follows from assumption (4.4). The relation $\sup_{x\ge x_n}\Sigma_1(x)\to 0$ follows by another application of the Fuk-Nagaev inequality in the same way as for $P(E_1)$ in combination with assumption (4.4). □

6. Proof of Theorem 4.9

We will use the notation

$$T_0=0\,,\quad T_n=(Y_1-EY)+\cdots+(Y_n-EY)\,,\quad n\ge 1\,.$$

Proof of the upper bound. First, we show the relation

$$\limsup_{u\to\infty}\frac{\psi(u)}{u\,P(B>u)}\le EC^\kappa\,\frac{1}{\mu}\,\frac{1}{\kappa-1}\,,\tag{6.1}$$

by a series of auxiliary results. Before we proceed with them we give some intuition on the steps of the proof.

- In Lemmas 6.1 and 6.2 we show that the event $\{\sup_{n\le u/M}(T_n-\mu n)>u\}$ does not contribute to the order of $\psi(u)$ for sufficiently large u and M.
- In Lemma 6.3 we show that the order of $\psi(u)$ is essentially determined by the event $D(u)=\{\sup_{n\ge u/M}(\sum_{t=[u/M]}^n(B_t-EB)C_t-\mu n)>u\}$.
- In Lemma 6.4 we show that it is unlikely that $D(u)$ is caused by more than one large value $B_t>\theta t$ for any $\theta>0$.
- In Lemma 6.5 we show that it is unlikely that $D(u)$ occurs if all B_t's in the sum $\sum_{t=[u/M]}^n(B_t-EB)C_t$ are bounded by $\theta(t+u)$.
- In Lemma 6.6 we finally show that $D(u)$ is essentially caused by exactly one unusually large value $B_t>\delta(\mu t+u)$, whereas all other values B_s, $s\ne t$, are of smaller order. This lemma also gives the desired upper bound (6.1) of $\psi(u)$.

Lemma 6.1. *For any $\mu>0$,*

$$\lim_{M\to\infty}\limsup_{u\to\infty}\frac{P\Big(\sup_{n\le u/M}(T_n-\mu\,n)>u\Big)}{u\,P(B>u)}=0\,.$$

Proof. We have

$$P\Big(\sup_{n\le u/M}(T_n-\mu\,n)>u\Big)\quad\le\quad P\big(T_{[u/M]}>u-EY\,[u/M]\big)\,.\tag{6.2}$$

For sufficiently large M, $(1-EY/M)>0$. Then an application of the large deviation result of Theorem 4.2 yields that the right-hand side in (6.2) is of the order

$$\sim c\,[u/M]\,(1-EY/M)^{-\kappa}P(B>u)\,,\quad u\to\infty\,.$$

The latter estimate implies the statement of the lemma by letting $M\to\infty$. □

Lemma 6.2. *We have for any $\mu>0$,*

$$\lim_{M\to\infty}\limsup_{u\to\infty}\frac{P\Big(\sup_{n\ge u/M}(T_{[u/M]}-\mu\,n)>u\Big)}{u\,P(B>u)}=0\,.$$

Proof. We have by virtue of the large deviation results,

$$\begin{aligned}\frac{P\Big(\sup_{n\ge u/M}(T_{[u/M]}-\mu\,n)>u\Big)}{u\,P(B>u)}&\le\frac{P\big(T_{[u/M]}>\ u+\mu\,[u/M]\big)}{u\,P(B>u)}\\&\sim c\,\frac{[u/M]\,P(B>u\,(1+\mu/M))}{u\,P(B>u)}\,,\quad u\to\infty\\&\sim c\,M^{-1}\,(1+\mu/M)^{-\kappa}\to 0\,,\quad M\to\infty\,.\end{aligned}$$

□

In the light of the two lemmas, it suffices to bound the probability

$$J(u) = P\left(\sup_{n\ge u/M} [(T_n - T_{[u/M]}) - (1-\varepsilon)\,\mu\, n] > (1-\varepsilon)\,u\right)$$

for fixed $M>0$ and any small $\varepsilon>0$. By (5.1) and by virtue of Breiman's result, for large u,

$$\begin{aligned}
J(u) &\le P\left(Y_0\sum_{i=1}^\infty \Pi_i + \sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (B_t\,C_t - EB\,EC) - (1-\varepsilon)\,\mu\,n\right) > (1-2\varepsilon)\,u\right)\\
&\le P\left(Y_0\sum_{i=1}^\infty \Pi_i > \varepsilon u\right) + P\left(\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (B_t\,C_t - EB\,EC) - (1-\varepsilon)\,\mu\,n\right) > (1-3\,\varepsilon)u\right)\\
&\sim \varepsilon^{-\kappa}P(Y>u)E(C-1)^\kappa + P\left(\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (B_tC_t - EB\,EC) - (1-\varepsilon)\,\mu\,n\right) > (1-3\,\varepsilon)u\right)\\
&\le c\,P(Y>u) + P\left(\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (B_t - EB)C_t - (1-\varepsilon/2)\,\mu\,n\right) > (1-4\,\varepsilon)u\right)\\
&\quad + P\left(\sup_{n\ge u/M}\left(EB\sum_{t=[u/M]+1}^n (C_t - EC) - \varepsilon\,\mu\,n/2\right) > \varepsilon\,u\right)\\
&= J_1(u) + J_2(u) + J_3(u)\,.
\end{aligned}$$

We show that $J_3(u) = o(uP(Y>u))$.

Lemma 6.3. *Assume (C_t) is strongly mixing with geometric rate and $EC^{\kappa+\gamma}<\infty$ for some $\gamma>\kappa$. Then for any $M,\mu>0$,*

$$\lim_{u\to\infty}\frac{P\left(\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (C_t-EC) - \mu\,n\right) > u\right)}{u\,P(B>u)} = 0$$

Proof. We have by Markov's inequality,

$$\begin{aligned}
&P\left(\sup_{n\ge u/M}\left(\sum_{t=[u/M]+1}^n (C_t - EC) - \mu\,n\right) > u\right)\\
&\le \sum_{n=[u/M]}^\infty P\left(\sum_{t=[u/M]+1}^n (C_t - EC) > \mu\,n + u\right)\\
&\le \sum_{n=[u/M]}^\infty (\mu\,n+u)^{-(\kappa+\gamma)} E\left|\sum_{t=[u/M]+1}^n (C_t - EC)\right|^{\kappa+\gamma}\\
&\le c\sum_{n=[u/M]}^\infty (n+u)^{-(\kappa+\gamma)}\,n^{(\kappa+\gamma)/2}\,. \qquad (6.3)
\end{aligned}$$

In the last step we applied the moment estimate

$$E\left|n^{-1/2}\sum_{t=1}^n (C_t - EC)\right|^{\kappa+\gamma} \le c\,,$$

which is valid for strongly mixing sequences with geometric rate if $\gamma > \kappa$ and $EC^{\gamma+\kappa} < \infty$, see e.g. Doukhan [12], p. 31. An application of Karamata's theorem shows that (6.3) is of the order

$$\sim c\, u^{1-(\kappa+\gamma)/2} = o(u\, P(B > u))\,,$$

for $\gamma > \kappa$. □

Thus it remains to estimate $J_2(u)$. We proceed by a series of lemmas.

Lemma 6.4. *For every $\theta > 0$,*

$$P(B_t > \theta t \text{ for at least two } t \geq u) = o(u\, P(B > u))\,.$$

Proof. We have by Karamata's theorem,

$$\begin{aligned} P(B_t > \theta\, t \text{ for at least two } t \geq u) &\leq \sum_{t=[u]}^{\infty} P(B_t > \theta\, t\,, B_j > \theta\, j \text{ for some } j \neq t) \\ &\leq \sum_{t=[u]}^{\infty} P(B > \theta\, t) \sum_{j=[u], j\neq t}^{\infty} P(B > \theta\, j) \\ &\sim c\, [u\, P(B > u)]^2\,, \end{aligned}$$

from which the statement of the lemma follows. □

Lemma 6.5. *Assume (C_t) is strongly mixing with geometric rate, $EC^{\kappa+\gamma} < \infty$ for some $\gamma > \kappa$. Then for every $M, \mu, \theta > 0$,*

$$\widetilde{J}(u) = P(A_u) = o(u\, P(B > u))\,,$$

where

$$\begin{aligned} A_u &= \bigcup_{n \geq [u/M]} \Big\{ \sum_{t=[u/M]+1}^{n} (B_t - EB)\, C_t > (1-4\varepsilon)(\mu\, n + u)\,, \\ &\qquad B_j \leq \theta\,(j+u) \text{ for all } j = [u/M]+1, \ldots, n \Big\}\,. \end{aligned}$$

Proof. We have

$$\begin{aligned} \widetilde{J}(u) &\leq \sum_{n=[u/M]}^{\infty} P\left(\sum_{t=[u/M]+1}^{n} (B_t - EB)\, C_t > (1-4\varepsilon)(\mu\, n + u)\,, \max_{j=[u/M]+1,\ldots,n} B_j \leq \theta\,(n+u) \right) \\ &\leq \sum_{n=[u/M]}^{\infty} P\left(\sum_{t=[u/M]+1}^{n} (B_{t,\theta\,(n+u)} - EB_{1,\theta\,(n+u)})\, C_t > (1-4\varepsilon)(\mu\, n + u) \right)\,, \end{aligned}$$

where

$$B_{t,x} = B_t I_{[0,x]}(B_t)\,, \quad x > 0\,.$$

Analogously to (5.6), an application of the Fuk-Nagaev inequality, conditionally on (C_t), yields for $d > 0$ and $d' = d'(d) > 0$,

$$\begin{aligned} \widetilde{J}(u) &\leq c \sum_{n=[u/M]}^{\infty} n\,(n+u)^{-2\kappa} + c \sum_{n=[u/M]}^{\infty} (n+u)^{-d'} \\ &\quad + c \sum_{n=[u/M]}^{\infty} P\left(\mathrm{var}(B_{1,\theta\,(n+u)}) \sum_{t=1}^{n} C_t^2 > d\,[(1-4\varepsilon)(\mu\, n+u)]^2 / \log((1-4\varepsilon)(\mu\, n + u)) \right) \\ &= \widetilde{J}_1(u) + \widetilde{J}_2(u) + \widetilde{J}_3(u)\,. \end{aligned}$$

Choosing $d > 0$ sufficiently small such that d' becomes sufficiently large, an application of Karamata's theorem yields

$$\widetilde{J}_1(u) \le c\,u^{2-2\kappa} = o(u\,P(B > u)) \quad \text{and} \quad \widetilde{J}_2(u) \le c\,u^{1-d'} = o(u\,P(B > u))\,.$$

An application of (4.10) yields for $\gamma > \kappa$,

$$\widetilde{J}_3(u) \quad \le \quad c \sum_{n=[u/M]}^{\infty} n^{(\kappa+\gamma)/2} \left(\frac{(\mu\,n+u)^2}{\log(\mu\,n+u)\,\mathrm{var}(B_{1,\theta\,(n+u)})} \right)^{-(\kappa+\gamma)/2} . \tag{6.4}$$

If $\kappa \ge 2$, $\mathrm{var}(B_{1,x})$ is slowly varying and if $\kappa \in (1,2)$, $\mathrm{var}(B_{1,x}) \sim c\,x^2\,P(B > x)$. This follows by Karamata's theorem. These facts and (6.4) ensure that $\widetilde{J}_3(u) = o(u\,P(B > u))$. This proves the lemma. □

Finally, we bound $J_2(u)$ and obtain the desired upper bound (6.1) in the theorem.

Lemma 6.6. *The following result holds:*

$$\lim_{\varepsilon\downarrow 0}\limsup_{u\to\infty} \frac{J_2(u)}{u\,P(B > u)} \le EC^\kappa\,\frac{1}{\mu}\frac{1}{\kappa-1}\,.$$

Proof. By virtue of Lemmas 6.4 and 6.5,

$$\begin{aligned} &\limsup_{u\to\infty} \frac{J_2(u)}{u\,P(B > u)} \\ &\le \quad \limsup_{u\to\infty} \frac{P\left(\bigcup_{n\ge u/M}\left\{\sum_{t=[u/M]+1}^{n}(B_t - EB)\,C_t > (1-4\varepsilon)\,(\mu\,n+u)\right\} \cap A_\delta\right)}{u\,P(B > u)}\,, \end{aligned}$$

where for any $\delta > 0$,

$$A_\delta \quad = \quad \bigcup_{t=[u/M]}^{\infty} \{B_t > \delta\,(\mu\,t+u)\,, B_s \le \delta\,(\mu\,s+u) \text{ for all } s \ge [u/M],\ s \ne t\}\,.$$

Hence

$$\begin{aligned} &\limsup_{u\to\infty} \frac{J_2(u)}{u\,P(B > u)} \\ &\le \quad \limsup_{u\to\infty} \frac{\sum_{t=1}^{\infty} P(B_1 \min(C_1, \delta^{-1}(1-5\varepsilon)) > (1-5\varepsilon)\,(\mu\,t+u))}{u\,P(B > u)} \\ &\quad + \limsup_{u\to\infty} \sum_{t=[u/M]}^{\infty} \frac{P(B > \delta\,(\mu t+u))}{u\,P(B > u)} \times \\ &\quad \times P\Big(\bigcup_{t>n\ge[u/M]} \Big\{ \sum_{s=[u/M]+1}^{n} (B_s - EB)\,C_s > (1-4\varepsilon)(\mu n+u) \Big\} \cup \\ &\quad \cup \bigcup_{n\ge t} \Big\{ \sum_{s=[u/M]+1\,,s\ne t}^{n} (B_s - EB)\,C_s > \varepsilon\,(\mu\,n+u) \Big\} \cap \{B_s \le \delta\,(\mu s+u)\,, \text{all } s\ne t\} \Big) \\ &= \quad \limsup_{u\to\infty} K_1(u) + \limsup_{u\to\infty} K_2(u)\,. \end{aligned}$$

Similar arguments as for $\widetilde{J}(u)$ above show that

$$K_2(u) \quad = \quad o(1) \sum_{t=[u/M]}^{\infty} \frac{P(B > \delta(\mu t+u))}{u\,P(B > u)} = o(1)\,.$$

An application of Breiman's result and Karamata's theorem yields

$$K_1(u) \quad \sim \quad (1-5\,\varepsilon)^{-\kappa}\,E[\min(C_1, \delta^{-1}(1-5\varepsilon))]^\kappa\,\frac{1}{\mu}\frac{1}{\kappa-1}\,.$$

Noticing that

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}(1-5\,\varepsilon)^{-\kappa}\,E[\min(C_1,\delta^{-1}(1-5\varepsilon))]^{\kappa}=EC^{\kappa}\,,$$

the lemma is proved. □

Proof of the lower bound. Now we want to prove that

$$\liminf_{u\to\infty}\frac{\psi(u)}{u\,P(B>u)}\geq EC^{\kappa}\,\frac{1}{\mu}\frac{1}{\kappa-1}\,. \tag{6.5}$$

Again, we proceed by a series of auxiliary results. We start with a short outline of the steps in the proof.

- In Lemmas 6.7 and 6.8 we show that the order of $\psi(u)$ is essentially determined by the event $\widetilde{D}(u)=\{\sup_{n\geq u/M}(\sum_{t=[u/M]+1}^{n}(B_tC_t-EBEC)-\mu n)>u\}$.
- In Lemma 6.9 we complete the lower bound (6.5) of $\psi(u)$ by first showing that $\widetilde{D}(u)$ is essentially determined by the event $D(u)=\{\sup_{n\geq u/M}(\sum_{t=[u/M]+1}^{n}(B_t-EB)C_t-\mu n)>u\}$. The probability of $D(u)$ is bounded from below by intersecting $D(u)$ with the union of the events $\{B_t>\delta(\mu t+u)\,,B_s\leq\delta(\mu t+u)\,,\text{for all } s\neq t\ \}$, i.e., B_t is unusually large, whereas all the other B_s's are smaller.

Lemma 6.7. *For every* $\varepsilon,M,\mu>0$,

$$\psi(u)\geq L_1(u)+o(u\,P(B>u))\,,$$

where

$$L_1(u)=P\left(\sup_{n\geq[u/M]}\big(T_n-T_{[u/M]}-\mu\,n\big)>u\,(1+\varepsilon)\right)\,.$$

Proof. We have

$$\begin{aligned}\psi(u)&\geq P\left(\sup_{n\geq[u/M]}\big(T_n-T_{[u/M]}-\mu\,n\big)+T_{[u/M]}>u\right)\\&\geq P\left(\sup_{n\geq[u/M]}\big(T_n-T_{[u/M]}-\mu\,n\big)>(1+\varepsilon)\,u\,,T_{[u/M]}\geq-\varepsilon\,u\right)\\&\geq P\left(\sup_{n\geq[u/M]}\big(T_n-T_{[u/M]}-\mu\,n\big)>(1+\varepsilon)\,u\right)-P\left(T_{[u/M]}\leq-\varepsilon\,u\right)\,,\end{aligned}$$

but by (4.6),

$$P\left(T_{[u/M]}\leq-\varepsilon\,u\right)=o(u\,P(B>u))\,.$$

This concludes the proof. □

Lemma 6.8. *We have for any* $\varepsilon,\mu,M>0$, $k\geq 1$ *and some* $c>0$,

$$L_1(u)\geq P\left(\sup_{n\geq[u/M]}\left(\sum_{t=[u/M]+1}^{n-k}(B_t\,C_t-EB\,EC)-(1+\varepsilon)\,\mu\,n\right)>(1+3\varepsilon)\,u\right)-c\,(EA^{\kappa})^k\,u\,P(B>u)\,.$$

Proof. Using the decomposition (5.1) and writing

$$\begin{aligned}R_1(k,u)&=\sup_{n\geq[u/M]}\left(\sum_{t=[u/M]+1}^{n-k}(B_t\,C_t-EB\,EC)-(1+\varepsilon)\,\mu\,n\right)\,,\\R_2(k,u)&=\sup_{n\geq[u/M]}\left(\sum_{t=1}^{n-k}B_t\sum_{i=n+1}^{\infty}\Pi_{t+1,i}-\varepsilon\mu n\right)\,,\end{aligned}$$

we have for large u,

$$\begin{aligned} L_1(u) &\geq P(R_1(k,u) - R_2(k,u) > (1+2\,\varepsilon)\,n) \\ &\geq P(R_1(k,u) > (1+3\varepsilon)\,u\,, -R_2(k,u) > -\varepsilon\,u) \\ &\geq P(R_1(k,u) > (1+3\varepsilon)\,u) - P(R_2(k,u) \geq \varepsilon\,u) \\ &= L_2(u) - L_3(u)\,. \end{aligned}$$

We show that

$$L_3(u) \leq c\,(EA^\kappa)^k\; u\,P(B>u)\,.$$

We have for $k \geq 1$,

$$\begin{aligned} L_3(u) &\leq P\Big(\sup_{n\geq [u/M]} \left(\sum_{t=1}^{[u/M]} B_t \Pi_{t+1,n+1}\,C_{n+1} - \varepsilon\mu n/2\right) \geq \varepsilon\,u/2\Big) \\ &\quad + P\Big(\sup_{n\geq [u/M]} \left(\sum_{t=[u/M]+1}^{n-k} B_t \Pi_{t+1,n+1}\,C_{n+1} - \varepsilon\mu n/2\right) \geq \varepsilon\,u/2\Big) \\ &= L_{3,1}(u) + L_{3,2}(u)\,. \end{aligned}$$

Then, by (5.3) and Markov's inequality, for $0<\delta<1$,

$$\begin{aligned} L_{3,1}(u) &\leq \sum_{n=[u/M]}^{\infty} P\Big(\sum_{t=1}^{[u/M]} B_t \Pi_{t+1,[u/M]} \Pi_{[u/M]+1,n+1}\,C_{n+1} > (\varepsilon/2)(\mu n+u)\Big) \\ &\leq \sum_{n=[u/M]}^{\infty} P\Big(Y_0\,\Pi_{[u/M]+1,n+1}\,C_{n+1} > (\varepsilon/2)(\mu n+u)\Big) \\ &\leq c \sum_{n=[u/M]}^{\infty} (EA^{\kappa-\delta})^{n-[u/M]}\,(n+u)^{-\kappa+\delta} \\ &\leq c\,u^{-\kappa+\delta} = o(u\,P(B>u))\,. \end{aligned}$$

Moreover, by (5.3) and Breiman's result,

$$\begin{aligned} L_{3,2}(u) &\leq \sum_{n=[u/M]}^{\infty} P\Big(\sum_{t=[u/M]+1}^{n-k} B_t \Pi_{t+1,n-k} \Pi_{n-k+1,n+1}\,C_{n+1} > (\varepsilon/2)(\mu n+u)\Big) \\ &\leq \sum_{n=[u/M]}^{\infty} P\Big(Y_0\,\Pi_{n-k+1,n+1}\,C_{n+1} > (\varepsilon/2)(\mu n+u)\Big) \\ &\leq c\,(EA^\kappa)^k\,u\,P(B>u)\,. \end{aligned}$$

□

Next we bound L_2.

Lemma 6.9. *We have for every $k \geq 1$,*

$$\lim_{\varepsilon\downarrow 0} \liminf_{u\to\infty} \frac{L_2(u)}{u\,P(B>u)} \geq EC^\kappa\,\frac{1}{\mu}\,\frac{1}{\kappa-1}\,.$$

Proof. Writing

$$\begin{aligned}
R_1(k,u) &= \sup_{n\ge[u/M]}\left(\sum_{t=[u/M]+1}^{n-k}(B_t-EB)\,C_t-(1+2\varepsilon)\,\mu\,n\right),\\
R_2(k,u) &= \inf_{u\ge[u/M]}\left(EB\sum_{t=[u/M]+1}^{n-k}(C_t-EC)+\varepsilon\mu n\right)
\end{aligned}$$

we have

$$\begin{aligned}
L_2(u) &\ge P(R_1(k,u)+R_2(k,u)>(1+3\varepsilon)\,u)\\
&\ge P(R_1(k,u)>(1+4\varepsilon)\,u\,,R_2(k,u)>-\varepsilon\,u)\\
&\ge P(R_1(k,u)>(1+4\varepsilon)\,u)-P(R_2(k,u)\le-\varepsilon\,u)\\
&= L_4(u)-L_5(u)\,.
\end{aligned}$$

Lemma 6.3 and its proof show that

$$L_5(u)=o(u\,P(B>u))\,.$$

Now we turn to L_4. Writing

$$\begin{aligned}
D_t(\delta,u) &= \{B_s\le\delta(\mu\,s+u)\text{ for all }s\in[[u/M],\infty)\backslash\{t\}\}\,,\\
E_t(\delta,u) &= \{B_t\min(C_t,\delta^{-1}(1+5\varepsilon))>(1+5\varepsilon)(\mu t+u)\}\,,
\end{aligned}$$

we have for small $\delta>0$,

$$\begin{aligned}
L_4(u) \ge\ & \sum_{t=[u/M]}^{\infty}P\Big(\{B_t>\delta(\mu t+u)\}\cap D_t(\delta,u)\cap\\
&\Big\{\sup_{n\ge t}\left(\sum_{r=[u/M]+1}^{n-k}(B_r-EB)\,C_r-(1+4\varepsilon)\,\mu\,n\right)>(1+4\varepsilon)\,u\ \Big\}\Big)\\
\ge\ & \sum_{t=[u/M]}^{\infty}P\Big(E_t(\delta,u)\cap D_t(\delta,u)\Big)-\sum_{t=[u/M]}^{\infty}P\Big(E_t(\delta,u)\cap D_t(\delta,u)\cap\\
&\Big\{\sup_{n\ge t}\Big(\sum_{r=[u/M]+1}^{n-k}(B_r-EB)\,C_r-(1+4\varepsilon)\,\mu\,n\Big)\le(1+4\varepsilon)\,u\Big\}\Big)\\
\ge\ & \sum_{t=[u/M]}^{\infty}P(E_1(\delta,u))\,P\Big(B_s\le\delta(\mu\,s+u)\text{ for all }s\ge[u/M]\Big)-\sum_{t=[u/M]}^{\infty}P\Big(E_t(\delta,u)\cap D_t(u,\delta)\cap\\
&\Big\{\sup_{n\ge t}\Big(\sum_{r=[u/M]+1,r\ne t}^{n-k}(B_r-EB)\,C_r-(1+4\varepsilon)\,\mu\,n\Big)\le(1+4\varepsilon)\,u-(1+5\varepsilon)(\mu t+u)+EB\,C_t\Big\}\Big)\\
=\ & L_{4,1}(u)-L_{4,2}(u)\,.
\end{aligned}$$

By Breiman's result and Karamata's theorem, as $u \to \infty$,

$$\begin{aligned}
L_{4,1}(u) &\sim \sum_{t=[u/M]}^{\infty} \Big[(1+5\varepsilon)^{-\kappa} E[\min(C_1, \delta^{-1}(1+5\varepsilon))]^{\kappa}\, P(B > \mu t + u)\Big] \times \\
&\qquad \times P\Big(B_s \le \delta(\mu s + u) \text{ for all } s \ge [u/M]\Big) \\
&\ge (1+6\varepsilon)^{-\kappa}\, E[\min(C_1, \delta^{-1}(1+5\varepsilon))]^{\kappa} \sum_{t=[u/M]}^{\infty} P(B > \mu t + u)\Big] \\
&\sim (1+6\varepsilon)^{-\kappa}\, E[\min(C_1, \delta^{-1}(1+5\varepsilon))]^{\kappa} \frac{1}{\mu} \frac{1}{\kappa - 1}\, P(B > u)\,.
\end{aligned}$$

We conclude that

$$\lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \liminf_{u \to \infty} \frac{I_{4,1}(u)}{u\, P(B > u)} \quad \ge \quad EC^{\kappa} \frac{1}{\mu} \frac{1}{\kappa - 1}\,.$$

As regards $L_{4,2}(u)$, we have

$$\begin{aligned}
L_{4,2}(u) &\le c \sum_{t=[u/M]}^{\infty} P(B > t + u) \times \\
&\qquad \times P\Big(\sup_{n \ge t} \Big(\sum_{r=[u/M]+1, r \ne t}^{n-k} (B_r - EB)\, C_r - (1+4\varepsilon)\, \mu\, n\Big) \le -\varepsilon\, u - (1+5\varepsilon)\mu t + EBC_t\Big) \\
&\le c \sum_{t=[u/M]}^{\infty} P(B > t + u) \times \\
&\qquad \times P\Big(\sup_{n \ge t} \Big(\sum_{r=[u/M]+1, r \ne t}^{n-k} (B_r - EB)\, C_r - (1+4\varepsilon)\, \mu\, n\Big) \le -\varepsilon\, u - (1+5\varepsilon)\mu t + EB\, M\Big) \\
&\qquad + c \sum_{t=[u/M]}^{\infty} P(B > t + u)\, P(C > M) \\
&= I_{4,2,1}(u) + I_{4,2,2}(u)\,.
\end{aligned}$$

We have

$$\lim_{M \to \infty} \limsup_{u \to \infty} \frac{I_{4,2,2}(u)}{u\, P(B > u)} \le c \lim_{M \to \infty} P(C > M) = 0\,.$$

Observe that for large u

$$\begin{aligned}
&P\Big(\sup_{n \ge t} \Big(\sum_{r=[u/M]+1, r \ne t}^{n-k} (B_r - EB)\, C_r - (1+4\varepsilon)\, \mu\, n\Big) \le -\varepsilon\, u - (1+5\varepsilon)\mu t + EB\, M\Big) \\
&\le P\Big(\sum_{r=[u/M]+1, r \ne t}^{[u/M]+u-k} (B_r - EB)\, C_r - (1+4\varepsilon)\, \mu\, ([u/M] + u)\Big) \le -\varepsilon\, u\Big)\,.
\end{aligned}$$

Now an argument similar to the one for Theorem 4.2 shows that

$$I_{4,2,1}(u) = o(u\, P(B > u))\,.$$

This proves the lemma. □

Now a combination of the above lemmas shows that the lower bound (6.5) holds. Indeed, we have for any $k \geq 1$,

$$\liminf_{u\to\infty} \frac{\psi(u)}{u\,P(B>u)} \geq \liminf_{u\to\infty} \frac{L_1(u)}{u\,P(B>u)} \geq \liminf_{u\to\infty} \frac{L_2(u)}{u\,P(B>u)} - c\,(EA^\kappa)^k\,.$$

Now, observing that $EA^\kappa < 1$, let $k \to \infty$, $\varepsilon \downarrow 0$. This proves the theorem. □

Extensions. A careful study of the proofs in the previous sections shows that the particular structure of the sequence (Y_t) was inessential for the proofs. Indeed, we made extensive use of the fact that the random walk (S_n) can be approximated by the random walk $\widetilde{S}_n = \sum_{t=1}^n B_t C_t$. It is not difficult to see that the results of Theorems 4.2 and 4.9 remain valid if S_n is replaced by $\widetilde{S}_n$ and the following conditions on any stationary sequence (C_t) hold: (B_t) is independent of (C_t), (C_t) is strongly mixing with geometric rate, $EC^{\kappa+\gamma} < \infty$ for some $\gamma > \kappa$ and (4.4) holds. Moreover, the assertion of Lemma 4.6 remains valid. A stationary sequence $X_t = B_t C_t$ for (B_t) and (C_t) independent is called a *stochastic volatility model* in the econometrics literature; see Davis and Mikosch [10] for some theory and further references.

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