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Performance analysis with truncated heavy–tailed distributions

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Abstract

This paper deals with queues and insurance risk processes where a generic service time, resp. generic claim, has the form $U \wedge K$ for some r.v. U with distribution B which is heavy–tailed, say Pareto or Weibull, and a typically large K, say much larger than $\mathbb{E}U$. We study the compound Poisson ruin probability $\psi(u)$ or, equivalently, the tail $\mathbb{P}(W > u)$ of the $M/G/1$ steady–state waiting time W. In the first part of the paper, we present numerical values of $\psi(u)$ for different values of K by using the classical Siegmund algorithm as well as a more recent algorithm designed for heavy–tailed claims/service times, and compare the results to different approximations of $\psi(u)$ in order to figure out the threshold between the light–tailed regime and the heavy–tailed regime. In the second part, we investigate the asymptotics as $K \to \infty$ of the asymptotic exponential decay rate $\gamma = \gamma^{(K)}$ in a more general truncated Lévy process setting, and give a discussion of some of the implications for the approximations.

Keywords adjustment coefficient, Cramér–Lundberg approximation, insurance risk, $M/G/1$ queue, Pollaczeck–Khinchine formula, rare event, ruin probability, simulation, tail asymptotics, regular variation.

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1 Introduction

This paper deals with queues and insurance risk processes where a generic service time, resp. generic claim, has the form $U \wedge K$ for some r.v. U with a heavy–tailed distribution, say Pareto or Weibull, and a typically large K, say much larger than $\mathbb{E}U$.

An earlier study in this setting and some motivation has been given by Jelenković [12] in the queueing context. In insurance risk, Hipp [11] noted that many reinsurance contracts (the simplest of which is stop–loss) will lead to an actual claim of this type. Certainly in this area, there will most often be an upper limit to a claim, and the extensive use of heavy–tailed modeling is then oftened defended by an argument roughly saying that working with a heavy tail gives a better approximation than a light one in the relevant range. An empirical investigation of this was the original stimulus for the present study.

We will work in the framework of the compound Poisson ruin probability $\psi(u)$ or, equivalently, the tail $\mathbb{P}(W > u)$ of the $M/G/1$ steady–state waiting time W (to justify 'equivalently': $\psi(u) = \mathbb{P}(W > u)$ holds provided the Poisson rates are the same, say β, the claim size distribution, say B, of the risk process is the same as the service distribution of the queue and the rate of premium inflow for the risk process is 1; see [3], p. 399). Then light–tailed assumptions predict (we will use use the customary notation $f(u) \sim g(u)$ as $u \to \infty$ to denote $f(u)/g(u) \to 1,\ u \to \infty$)

$$\psi(u) = \mathbb{P}(W > u) \ \sim\ C\mathrm{e}^{-\gamma u}, \quad u \to \infty, \tag{1}$$

where $\gamma > 0$ is the solution of $\beta\big(\mathbb{E}\mathrm{e}^{\gamma U} - 1\big) = \gamma$ and $C = (1-\rho)/\big(\beta\mathbb{E}[U\mathrm{e}^{\gamma U}] - 1\big)$, whereas with heavy tails one has

$$\psi(u) = \mathbb{P}(W > u) \ \sim\ \frac{\rho}{1-\rho}\overline{B}_I(u), \quad u \to \infty, \tag{2}$$

where B_I is the integrated tail distribution with density $\mathbb{P}(U > x)/\mathbb{E}U$. When $B = B^{(K)}$ is the distribution of $U \wedge K$, we will write $\psi_K(u)$, W_K etc.

In our truncated heavy–tailed setting, the conditions for (1) are certainly fulfilled so that (1) will be superior to (2) for sufficiently large u. On the other hand, with a large K one would expect (2) to be better for small u The key issue is to quantify this range. Our numerical results will show what came as a surprise at least to us, that it is virtually always better to work with (1) and that it provides an excellent approximation even for small u and large K. We also perform a comparison with an adaptation of an approximation of [12] which deals with the behaviour of $\psi_K(u) = \mathbb{P}(W_K > u)$ when u and K both go to ∞ at roughly the same rate (see Appendix B for more details). Finally, since the exact values of $\psi_K(u) = \mathbb{P}(W_K > u)$ have to be found by simulation, the study also gives an opportunity to check out whether simulation algorithms initially worked out for either light or heavy tails have the better performance in the truncated heavy–tailed regime.

In addition to these empirical points, the paper also investigates a more mathematical issue; the asymptotics of the exponential decay rate (often denoted adjustment coefficient in insurance risk) $\gamma = \gamma^{(K)}$ as $K \to \infty$, and we give a discussion of some of the implications for the approximations of $\psi_K(u)$. To this end, we choose to work in the following setting, which is more general than the compound Poisson framework: We let $\gamma = \gamma^{(K)}$ be the root of the Lundberg equation $\kappa^{(K)}(\alpha) = 0$, where $\kappa^{(K)}$ is the Lévy exponent of the process $\left\{X_t^{(K)} = J_t^{(K)} + Y_t\right\}$ where $\left\{J_t^{(K)}\right\}$ is a pure jump process with $J_t^{(K)} - J_{t-}^{(K)} = (J_t - J_{t-}) \wedge K$ for some jump process $\{J_t\}$ with Lévy measure being regularly varying at ∞ and $\{Y_t\}$ is a Lévy process with finite mean and no upwards jumps. The investigation of the asymptotic behaviour of γ is in Section 3. Finally, in Section 4 and Appendix C, we briefly consider a subexponential example different from regular variation, a heavy–tailed Weibull B.

2 Numerical examples

We considered the stable $M/G/1$ queue with service time distribution B being Pareto with indices $\alpha = 3/2$ (often argued to be a typical value with finite mean but infinite variance) as well as $\alpha = 5/2$. The traffic intensity was $\rho = 0.4$ and $\rho = 0.8$ and we considered $K = 10, 100$ and 1000, giving a totality of 12 different sets of parameter combinations.

Note that in this case stability means

$$1 > \rho = \beta\Big(\int_0^K \frac{\alpha x}{(x+1)^{\alpha+1}}\,\mathrm{d}x + K\int_K^\infty \frac{\alpha}{(x+1)^{\alpha+1}}\,\mathrm{d}x\Big)\,.$$

To get numerical values of $\psi(u) = \mathbb{P}(W > u)$, we used simulation. As is well known ([6], [10]), the evaluation of small probabilities by simulation requires some sophistication. Up to recently, most algorithms developed required light tails. The classical one, which we use here, is due to Siegmund and amounts to importance sampling where β is changed to to $\beta^{(\gamma)} = \beta + \gamma$ and $B^{(K)}(\,\mathrm{d}x)$ to

$$B^{(K;\gamma)}(\,\mathrm{d}x) \;=\; \frac{\mathrm{e}^{\gamma x}}{\mathbb{E}\mathrm{e}^{\gamma X^{(K)}}}B^{(K)}(\,\mathrm{d}x) \tag{3}$$

where $\gamma = \gamma^{(K)}$ has to be found numerically as the solution of $\beta(\hat{B}^{(K)}(\gamma) - 1) = \gamma$. See [2], pp. 287–290.

The algorithms for heavy tails of the literature have a different form and all rely on the Pollaczeck–Khinchine formula

$$\psi(u) = \mathbb{P}(W > u) \;=\; \mathbb{P}(Y_1 + \cdots + Y_N > u) \tag{4}$$

where the Y_i are i.i.d. with distribution B_I and N is an independent geometric r.v. with parameter ρ. We use the one from [5] (documented there to be superior to earlier ones from [4] or [14] in the traditional genuinely heavy–tailed setting) where the estimator of $\mathbb{P}(Y_1 + \cdots + Y_n > u)$ is

$$n\overline{B}_I\big(M_{n-1} \vee (u - S_{n-1})\big) \tag{5}$$

where $M_n = \max(Y_1, \ldots, Y_n)$ and $S_n = Y_1 + \cdots + Y_n$. Thus, in our Pareto example the Y_i have density

$$f_Y(u) = \frac{\alpha - 1}{(u+1)^\alpha[1 - 1/(K+1)^{\alpha-1}]}I(0 \le u < K)$$

where $I(\cdot)$ is the indicator function.

Figures 1–12 give the waiting time tail, estimated by the two simulation algorithms for all 12 combinations of α, ρ, K, as well as the three types of approximations outlined in Section 1. The details of the simulations are straightforward except that it is not trivial to perform efficient sampling from the tilted measure in (3) when $B^{(K)}$ is truncated Pareto. We present our solution to this problem in Appendix A.

It is somewhat ambiguous how to implement the heavy tail approximation (2), more precisely whether to use the untruncated Pareto distribution B or the truncated version $B^{(K)}$. We have included both possibilities; in the figures, "Heavytailappr" refers to the truncated case whereas "w.t." is short for without truncation. "CL" stands for the Cramér–Lundberg approximation, see (1), and "Jelenković" to the adaptation of the approximation of [12] given in Theorem B.2 in Appendix B.

The conclusions we draw from the figures go in two directions, the quality of the approximations and the reliability of the simulation algorithms, and are as follows:

- The two simulation algorithms produce similar estimates when $u \in [0, K]$. For larger u, (5) has a skewness problem because even if the algorithm is unbiased in the usual sense, most of the estimates it produces will be zero because $\overline{B}_I^{(K)}(y) = 0$ for $y > K$. When comparing the two algorithms we note that each iteration in the Siegmund algorithm requires roughly 10 times the CPU time needed for each iteration in (5) when $u \in [0, K]$. This must be accounted for in the comparison and we use 5000 iterations for each estimate in the Siegmund case and 50000 iterations in (5). In Table 1 we present the relative error defined as the halfwidth of the 95% confidence interval divided by the point estimate for certain combinations of K, α, u, ρ for the Siegmund algorithm and (5). We conclude that for small u, $u < K$, say, the algorithm (5) performs better (in terms of lower variance) than the Siegmund algorithm, which on the other hand is superior as soon as $u \geq K$ and produces very reliable estimates. The reason why so few '+' are seen in the plots is simply that we use a logarithmic scale and ignore all estimates equal to zero.

- The results provided by the heavy–tailed approximation without truncation are not very convincing. In contrast, for $u > K$ the values are off the true ones (the Siegmund estimates) by orders of magnitude no matter the values of K, α, u, ρ. The best results are in Figure 7, but even there, the overall fit in $[0, K]$ is hardly better than the Cramér–Lundberg algorithm.

- The heavy–tailed approximation with truncation has the problem of dropping off to 0 near $u = K$, as is also clear from the expression. Again, one may as well use the Cramér–Lundberg approximation in $[0, K]$.

- The Jelenković approximation improves as K grows, as it should in view of Theorem B.2, and also as α grows (we believe that our figures appearing more jagged than the ones in [12] is due largely to our smaller values of α). It is better than the heavy–tailed approximations in terms of giving the correct order of magnitude, but can still not compete with the Cramér–Lundberg approximation.

ρ	α	K	u	Siegmund	(5)
0.4	3/2	1000	500	0.03	0.0072
			1000	0.0453	0.0416
			2000	0.038	0.6223
0.4	3/2	100	50	0.0218	0.0087
			100	0.0267	0.0287
			200	0.0245	0.23
0.4	3/2	10	5	0.0143	0.0083
			10	0.0163	0.0205
			20	0.0157	0.0772
0.8	3/2	1000	500	0.0179	0.0119
			1000	0.0193	0.0243
			2000	0.0188	0.1204
0.8	3/2	100	50	0.01	0.011
			100	0.0101	0.0174
			200	0.0099	0.0428
0.8	3/2	10	5	0.0048	0.0088
			10	0.005	0.0121
			20	0.005	0.0204
0.4	5/2	1000	500	0.0949	0.0057
			1000	0.1439	0.1323
			2000	0.1922	—
0.4	5/2	100	50	0.0479	0.0079
			100	0.0561	0.0453
			200	0.0559	1.1548
0.4	5/2	10	5	0.0216	0.0116
			10	0.0229	0.0295
			20	0.0218	0.2215
0.8	5/2	1000	500	0.0698	0.0091
			1000	0.0687	0.0382
			2000	0.0916	—
0.8	5/2	100	50	0.0312	0.0208
			100	0.0257	0.0359
			200	0.027	0.5018
0.8	5/2	10	5	0.008	0.0114
			10	0.0077	0.0173
			20	0.0077	0.0379

Table 1: Comparison of relative errors

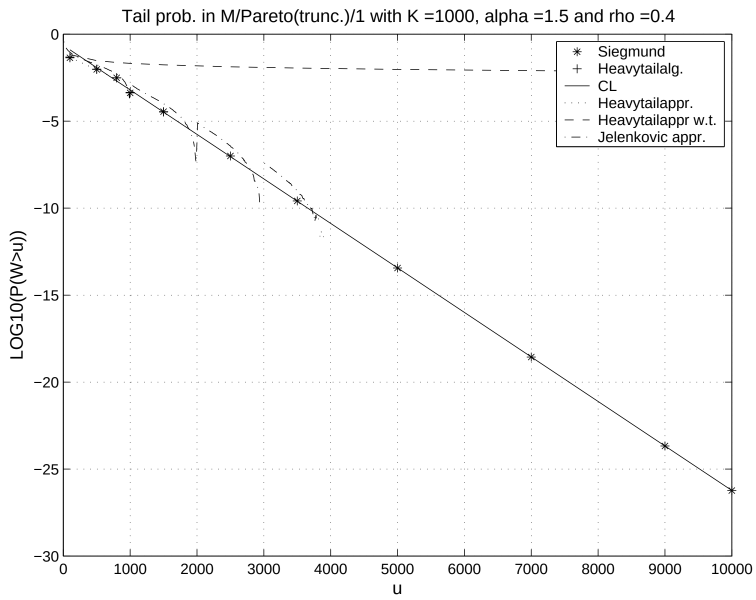

Figure 1: $\alpha = 1.5$, $\rho = 0.4$ and $K = 1000$.

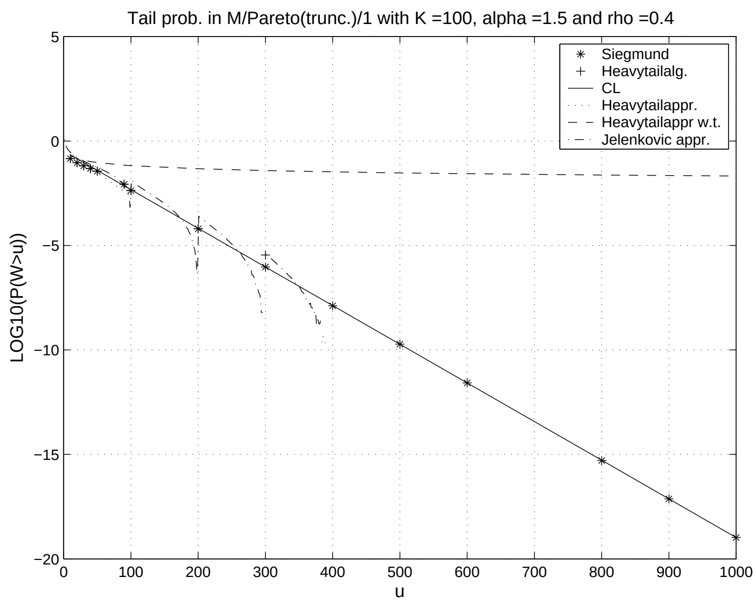

Figure 2: $\alpha = 1.5$, $\rho = 0.4$ and $K = 100$.

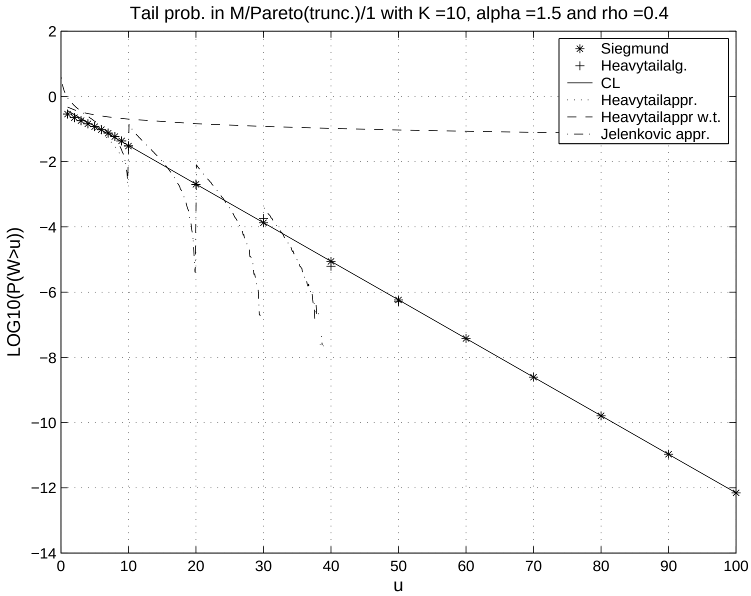

Figure 3: $\alpha = 1.5$, $\rho = 0.4$ and $K = 10$.

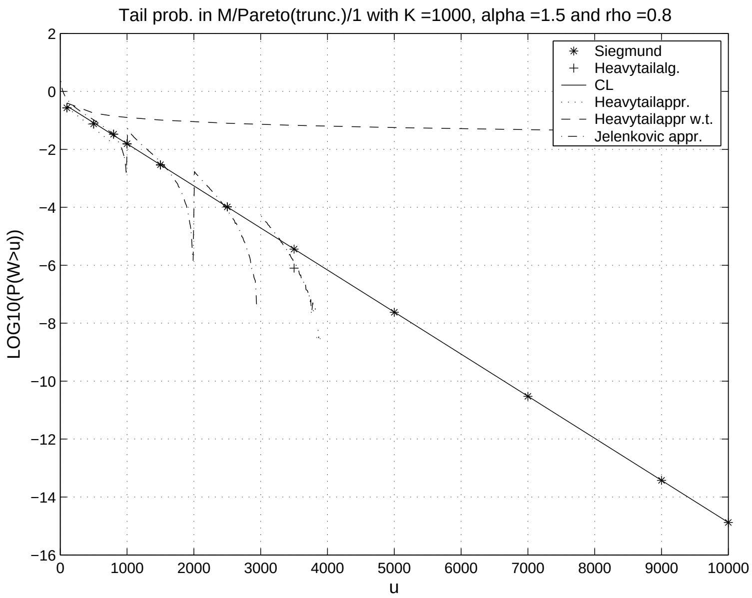

Figure 4: $\alpha = 1.5$, $\rho = 0.8$ and $K = 1000$.

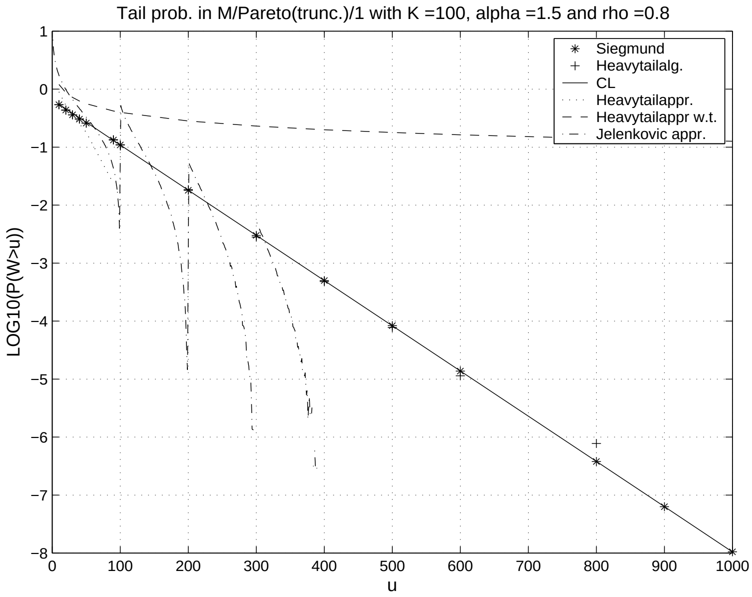

Figure 5: $\alpha = 1.5$, $\rho = 0.8$ and $K = 100$.

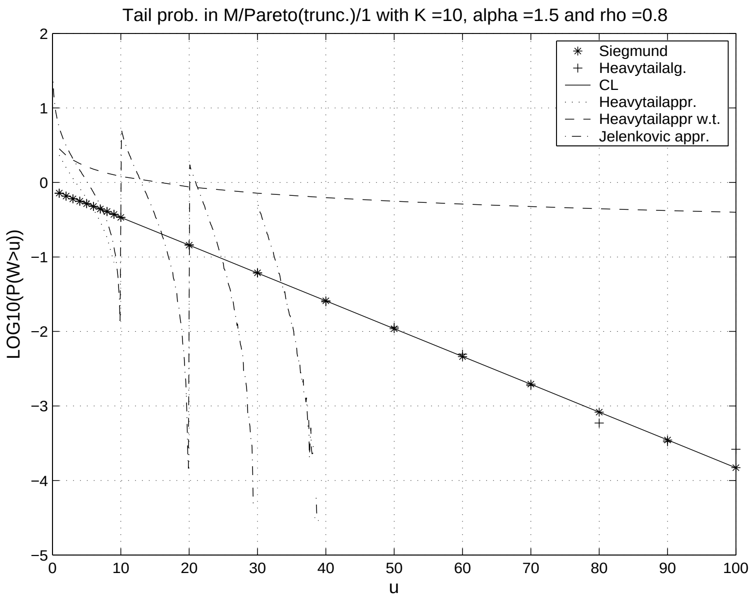

Figure 6: $\alpha = 1.5$, $\rho = 0.8$ and $K = 10$.

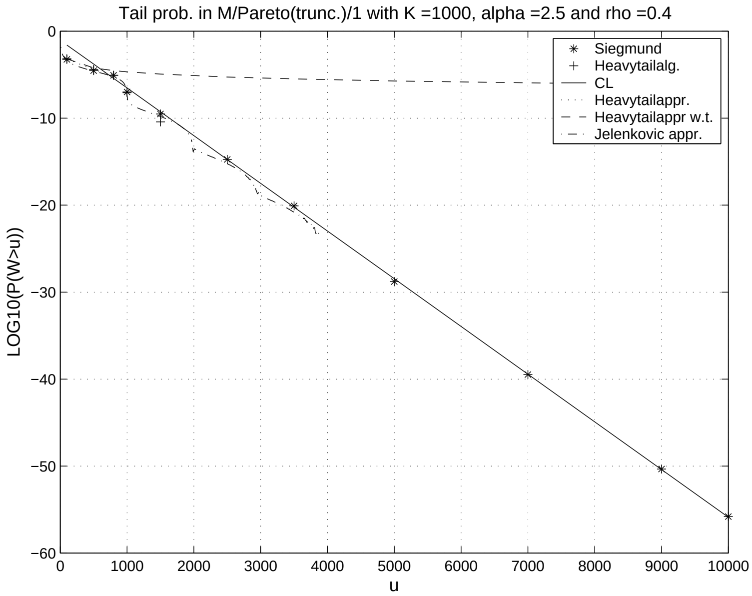

Figure 7: $\alpha = 2.5$, $\rho = 0.4$ and $K = 1000$.

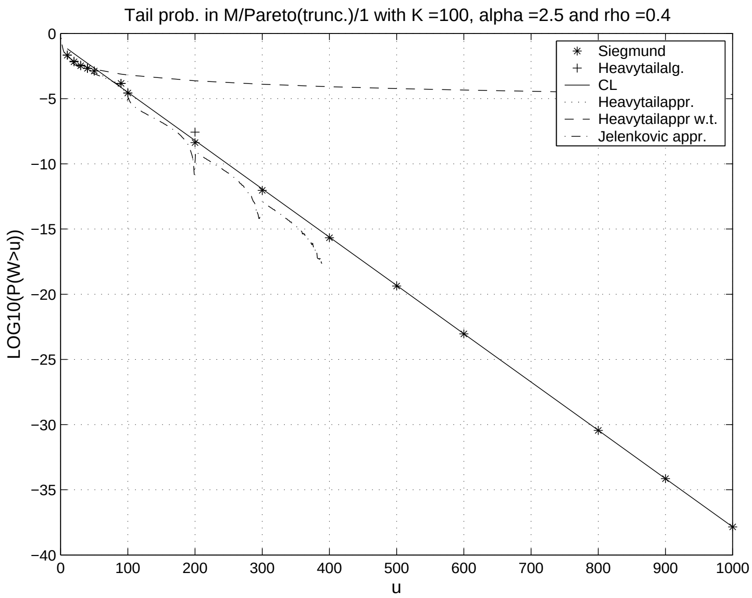

Figure 8: $\alpha = 2.5$, $\rho = 0.4$ and $K = 100$.

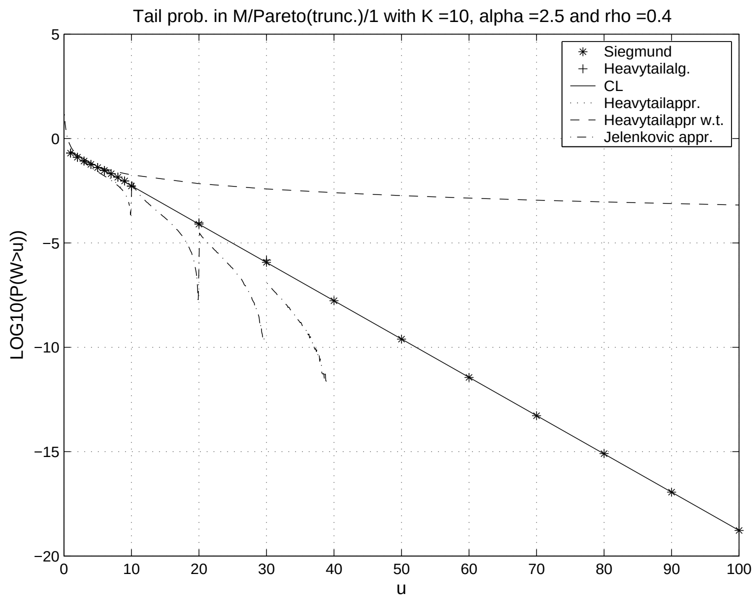

Figure 9: $\alpha = 2.5$, $\rho = 0.4$ and $K = 10$.

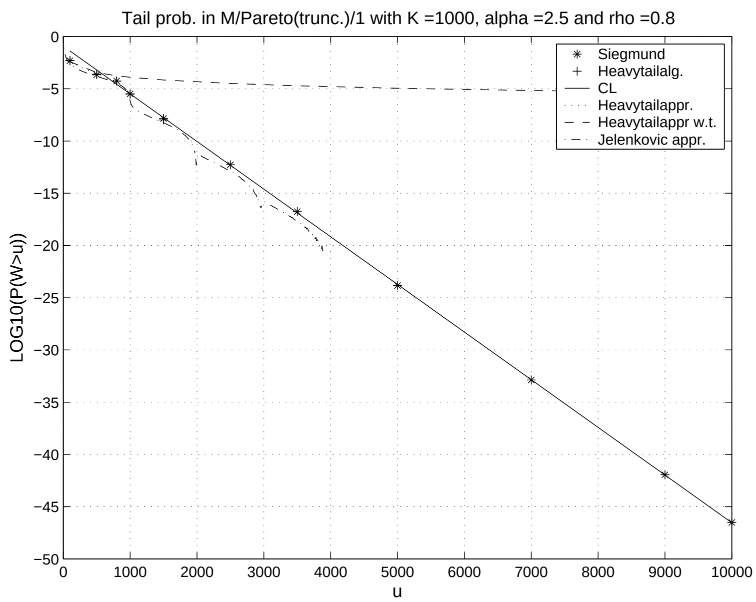

Figure 10: $\alpha = 2.5$, $\rho = 0.8$ and $K = 1000$.

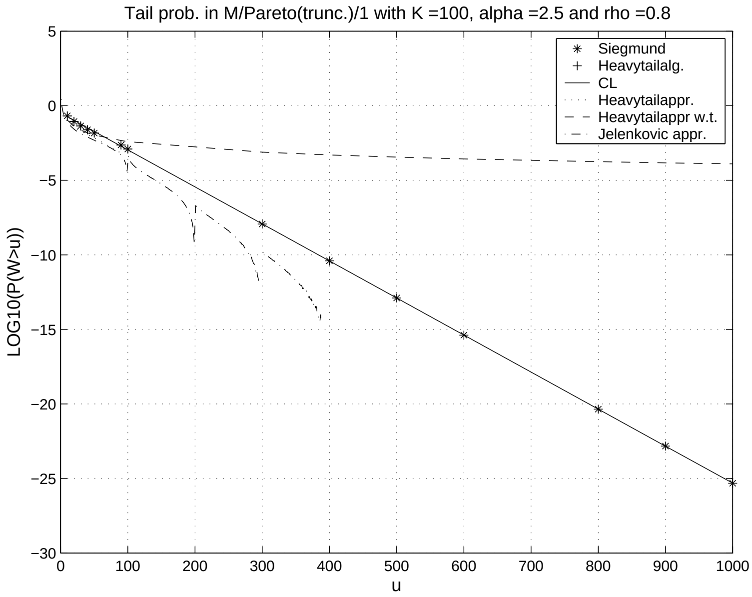

Figure 11: $\alpha = 2.5$, $\rho = 0.8$ and $K = 100$.

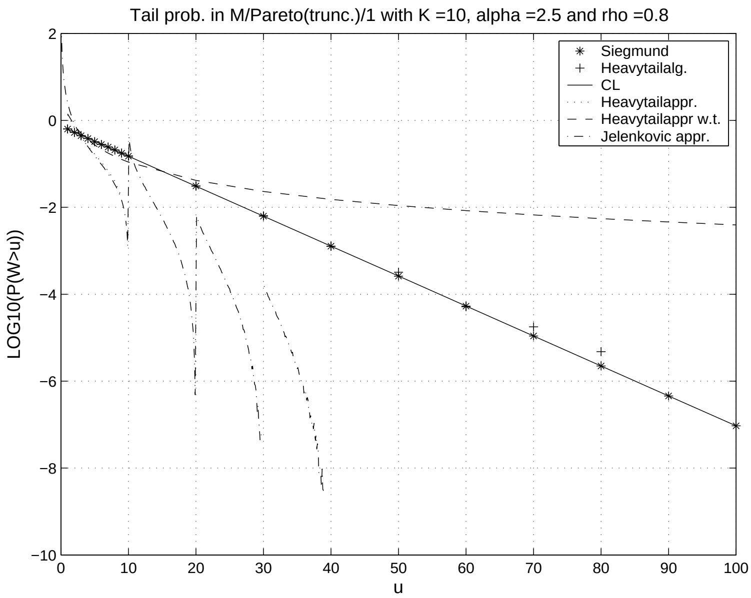

Figure 12: $\alpha = 2.5$, $\rho = 0.8$ and $K = 10$.

3 Exponential decay rates and truncation

Let $\{J_t\}$ be a pure jump Lévy process with Lévy measure ν satisfying

$$\int_{-\infty}^{\infty} |x|\nu(\,\mathrm{d}x) < \infty,\ \overline{\nu}(x) = \int_x^{\infty} \nu(\,\mathrm{d}u) = L(x)/x^{\alpha},$$

for some function $L(x)$ which is slowly varying at infinity ($\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$) and $\alpha > 1$. For each K, we consider the jump process $\left\{J_t^{(K)}\right\}$ with Lévy measure $\nu^{(K)}$ given by

$$\nu^{(K)}(\,\mathrm{d}x) = \begin{cases} \nu(\,\mathrm{d}x) & \text{if } x < K \\ \overline{\nu}(K) & \text{if } x = K \\ 0 & \text{otherwise.} \end{cases}$$

Further, we let $\{Y_t\}$ be a Lévy process with representation

$$Y_t = \mu t + \sigma B_t + Y_t^j$$

where $\{B_t\}$ is standard Brownian motion and $\left\{Y_t^j\right\}$ is a jump process with finite mean and without upwards jumps, i.e. the Lévy measure of $\left\{Y_t^j\right\}$, ξ, satisfies

$$\xi(\mathbb{R}^+) = 0,\ \int_{-\infty}^{0} x\xi(\,\mathrm{d}x) < \infty.$$

For each K we define the process $\left\{X_t^{(K)}\right\}$ by $X_t^{(K)} = J_t^{(K)} + Y_t$. The Lévy exponent is then

$$\begin{aligned}\kappa^{(K)}(s) \;=\; \log \mathbb{E} \exp\left\{sX_1^{(K)}\right\} \;&=\; \Big(\int_{-\infty}^{K} \big(\mathrm{e}^{sx} - 1\big)\nu(\,\mathrm{d}x) + \big(\mathrm{e}^{sK} - 1\big)\overline{\nu}(K)\Big) \\ &\qquad + s\mu + \sigma^2 s^2/2 + \int_{-\infty}^{0} (\mathrm{e}^{sx} - 1)\xi(\,\mathrm{d}x),\end{aligned}$$

which we, in obvious notation, write as

$$\big(A(s) + B(s)\big) + s\mu + \sigma^2 s^2/2 + \int_{-\infty}^{0} (\mathrm{e}^{sx} - 1)\xi(\,\mathrm{d}x),$$

and we define $\gamma^{(K)}$ as the solution of the Lundberg equation $\kappa^{(K)}\big(\gamma^{(K)}\big) = 0$. We will further assume that $\mathbb{E}J_1 + \mathbb{E}Y_1 < 0$ which implies that $\mathbb{E}X_1^{(K)} < 0$ for all K and hence that $\gamma^{(K)} > 0$. We have:

Theorem 3.1 *Under the stated assumptions on regular variation and negative drift, it holds as $K \to \infty$ and ν, ξ fixed that*

$$\gamma^{(K)} \;=\; \frac{1}{K}\Big[(\alpha - 1)\log K + \log\log K - \log L(K) + \log\big[-(\mathbb{E}J_1 + \mathbb{E}Y_1)(\alpha - 1)\big] + \mathrm{o}(1)\Big]. \tag{6}$$

For convenience, we state a slightly less general version of this result, adapted to the present study. Let F be a distribution on $(0,\infty)$ with density of the form $f(x) = \alpha L(x)/x^{\alpha+1}$ for some $\alpha > 1$ and some slowly varying function $L(x)$ (then the tail $\overline{F}(x)$ is of order $L(x)/x^\alpha$, see (17)). Let μ_F denote the mean. Consider for each K a compound Poisson process $\{X_t^{(K)}\}$ with a drift term such that the jumps have distribution F truncated at K. That is,

$$X_t^{(K)} \;=\; \sum_{i=1}^{N_t} X_i \wedge K \;-\; \theta t$$

where the X_i are i.i.d. with common distribution F and $\{N_t\}$ is an independent Poisson process, with rate say β. The Lévy exponent is then

$$\begin{aligned}\kappa^{(K)}(s) &= \log \mathbb{E} \exp\Big\{sX_1^{(K)}\Big\}\\ &= \beta\Big(\int_0^K \big(\mathrm{e}^{sx}-1\big)f(x)\,\mathrm{d}x + \big(\mathrm{e}^{sK}-1\big)\overline{F}(K)\Big) \;-\; s\theta.\end{aligned}$$

In this case, where we assume that $\beta\mu_F - \theta < 0$ so that $\gamma^{(K)} > 0$ for all K, we have:

Corollary 3.1 *For the compound Poisson process case, under the stated assumptions on regular variation and negative drift, it holds as $K \to \infty$ and β fixed that*

$$\gamma^{(K)} \;=\; \frac{1}{K}\Big[(\alpha-1)\log K + \log\log K - \log L(K) + \log\big[(\theta-\beta\mu_F)(\alpha-1)\big] + \mathrm{o}(1)\Big]. \quad (7)$$

Proof. In the compound Poisson case, $\mathbb{E}J_1 = \beta\mu_F$ and $\mathbb{E}Y_1 = -\theta$. Now we just apply Theorem 3.1. □

In view of Corollary 3.1, a tempting conclusion is that (1) and (2) are almost the same for $u = K$ so that $u = K$ would appear the natural value where to switch from the heavy–tailed to the light–tailed approximation which could also be intuitively appealing. More precisely, ignoring constants the ratio of logarithms at $u = K$ is

$$\frac{-\gamma^{(K)}K}{\log \overline{B}_I(K)} \;\sim\; \frac{-(\alpha-1)\log K}{-(\alpha-1)\log K} \;=\; 1.$$

I.e., the dominant factors in both approximations should have the common value $K^{-(\alpha-1)}$. The numerical results in Section 2 show, however, that (1) and (2) are quite far apart at $u = K$. The flaw in the argument just given is that we have ignored that $C = C_K$ varies with K. More precisely, we will show below that the following result holds:

Proposition 3.1 $C_K \;\sim\; \dfrac{1}{(\alpha-1)\log K}$, *$K \to \infty$.*

Turning to the proof of Theorem 3.1, we need the following lemma:

Lemma 3.1 *Let $\gamma = \gamma(K)$ be positive constants such that $\gamma(K)K = c\log K + \mathrm{o}(\log K)$, $K \to \infty$, where c is a positive constant. Then, under the stated assumption on slow variation of $L(x)$,*

$$A(\gamma) = \gamma\mathbb{E}J_1 + \mathrm{o}\Big(\frac{L(K)\mathrm{e}^{\gamma K}}{K^\alpha}\Big).$$

Proof. Define

$$T(\gamma) = A(\gamma)/\gamma = \int_{-\infty}^{K} \frac{\mathrm{e}^{\gamma x} - 1}{\gamma} \nu(\,\mathrm{d}x)$$

which we write as

$$T = \int_{-\infty}^{1/\gamma} \frac{\mathrm{e}^{\gamma x} - 1}{\gamma x} x\nu(\,\mathrm{d}x) + \int_{1/\gamma}^{K} \frac{\mathrm{e}^{\gamma x} - 1}{\gamma} \nu(\,\mathrm{d}x). \tag{8}$$

We now observe that on $-\infty < x < 1/\gamma$, $\left|\frac{\mathrm{e}^{\gamma x}-1}{\gamma x}\right| < \mathrm{e} - 1$, and thus dominated convergence together with $\mathbb{E}J_1 < \infty$ and $\lim_{t\to 0} \frac{\mathrm{e}^t - 1}{t} = 1$ implies

$$\int_{-\infty}^{1/\gamma} \frac{\mathrm{e}^{\gamma x} - 1}{\gamma x} x\nu(\,\mathrm{d}x) \to \mathbb{E}J_1, \;\; \gamma \to 0$$

Let $0 < r < 1$, and pick $\delta \in (1-r, 1)$. If we perform partial integration and make the change of variables $y = \gamma x$ in the second integral of (8) we get, if we let $z = \gamma K$,

$$\begin{aligned} \int_{1/\gamma}^{K} \frac{\mathrm{e}^{\gamma x} - 1}{\gamma} \nu(\,\mathrm{d}x) &= -\overline{\nu}(K)\frac{\mathrm{e}^{z} - 1}{\gamma} + \overline{\nu}(1/\gamma)\frac{\mathrm{e} - 1}{\gamma} + \int_{1/\gamma}^{K} \mathrm{e}^{\gamma x}\overline{\nu}(x)\,\mathrm{d}x \\ &= -\overline{\nu}(K)\frac{\mathrm{e}^{z} - 1}{\gamma} + \overline{\nu}(1/\gamma)\frac{\mathrm{e} - 1}{\gamma} + \gamma^{\alpha-1}\int_{1}^{z} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y \\ &= -\overline{\nu}(K)\frac{\mathrm{e}^{z} - 1}{\gamma} + \overline{\nu}(1/\gamma)\frac{\mathrm{e} - 1}{\gamma} + \gamma^{\alpha-1}\int_{1}^{(1-\delta)z} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y \\ &\qquad + \gamma^{\alpha-1}\int_{(1-\delta)z}^{z} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y. \end{aligned} \tag{9}$$

Let $0 < \epsilon_1 < (r-1+\delta)c$. We now pick $1 < R(\epsilon_1) < \infty$ such that $L(x) \leq x^{\epsilon_1}$ if $x \geq R(\epsilon_1)$. This is possible because $L(x)$ is slowly varying, see e.g. Theorem A.3.3 p. 566 in [8] or Lemma 2, p. 277 in [9]. We then obtain

$$\int_{1}^{(1-\delta)z} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y \leq \int_{1}^{R(\epsilon_1)} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y + \int_{R(\epsilon_1)}^{(1-\delta)z} \mathrm{e}^{y}\frac{(y/\gamma)^{\epsilon_1}}{y^{\alpha}}\,\mathrm{d}y.$$

(Since $z \to \infty$, we may (and shall) assume that $(1-\delta)z > R(\epsilon_1)$ so that all splittings of integrals above make sense.) The slow variation of $L(x)$ implies that $L(tx)/L(x) \to 1$, $x \to \infty$ uniformly on each compact interval, and thus in particular on $[1, R(\epsilon_1)]$. This gives us that

$$\int_{1}^{R(\epsilon_1)} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y \sim L(1/\gamma)\int_{1}^{R(\epsilon_1)} \frac{\mathrm{e}^{y}}{y^{\alpha}}\,\mathrm{d}y,$$

because, for any $\varepsilon > 0$ and sufficiently small γ,

$$\begin{aligned} (1-\varepsilon)L(1/\gamma)\int_{1}^{R(\epsilon_1)} \frac{\mathrm{e}^{y}}{y^{\alpha}}\,\mathrm{d}y &< \int_{1}^{R(\epsilon_1)} \mathrm{e}^{y}\frac{L(y/\gamma)}{y^{\alpha}}\,\mathrm{d}y \\ &< (1+\varepsilon)L(1/\gamma)\int_{1}^{R(\epsilon_1)} \frac{\mathrm{e}^{y}}{y^{\alpha}}\,\mathrm{d}y. \end{aligned} \tag{10}$$

$L(1/\gamma)$ is $\mathrm{o}(\mathrm{e}^{r\gamma K})$, since, for any $0 < \epsilon_2 < cr$, it may be bounded by $(1/\gamma)^{\epsilon_2}$ for small γ and $\mathrm{e}^{r\gamma K}$ has K^{rc} as dominant factor. For the part $\int_{R(\epsilon_1)}^{(1-\delta)z} \mathrm{e}^y \frac{(y/\gamma)^{\epsilon_1}}{y^\alpha}\,\mathrm{d}y$, we use that

$$\int_{R(\epsilon_1)}^{(1-\delta)z} \mathrm{e}^y \frac{(y/\gamma)^{\epsilon_1}}{y^\alpha}\,\mathrm{d}y \sim \frac{1}{\gamma^{\epsilon_1}} \frac{\mathrm{e}^{(1-\delta)z}}{((1-\delta)z)^{\alpha-\epsilon_1}}, \tag{11}$$

which follows directly from

$$\int_a^z \frac{\mathrm{e}^y}{y^\beta}\,\mathrm{d}y \sim \frac{\mathrm{e}^z}{z^\beta},\ \ z \to \infty \tag{12}$$

(to see this, substitute $t = \mathrm{e}^y$ in (12) and use Karamata's theorem, e.g. [7], Proposition 1.5.8., p. 26). The assumption $\gamma K \sim c \log K$ together with (11) gives

$$\int_{R(\epsilon_1)}^{(1-\delta)z} \mathrm{e}^y \frac{(y/\gamma)^{\epsilon_1}}{y^\alpha}\,\mathrm{d}y \sim C \frac{\mathrm{e}^{(1-\delta)z} K^{\epsilon_1}}{(\log K)^\alpha},$$

where C is a constant. $0 < \epsilon_1 < (r-1+\delta)c$ now implies $\int_{R(\epsilon_1)}^{(1-\delta)z} \mathrm{e}^y \frac{(y/\gamma)^{\epsilon_1}}{y^\alpha}\,\mathrm{d}y = \mathrm{o}(\mathrm{e}^{r\gamma K})$. Using the slow variation of $L(x)$, the arguments leading to (10) and (12), we get that

$$\int_{(1-\delta)z}^{z} \mathrm{e}^y \frac{L(y/\gamma)}{y^\alpha}\,\mathrm{d}y \sim \frac{L(z/\gamma)\mathrm{e}^z}{z^\alpha} = \frac{L(K)\mathrm{e}^{\gamma K}}{(\gamma K)^\alpha}.$$

From (9) it now follows that $A(\gamma) = \gamma \mathbb{E}J_1 + \mathrm{o}(L(K)\mathrm{e}^{\gamma K}/K^\alpha)$, since $-\overline{\nu}(K)(\mathrm{e}^{\gamma K}-1) \sim -L(K)\mathrm{e}^{\gamma K}/K^\alpha$ and $\overline{\nu}(1/\gamma) = \mathrm{o}(L(K)\mathrm{e}^{\gamma K}/K^\alpha)$. The last claim follows from

$$\lim_{K\to\infty} \left| \frac{\overline{\nu}(1/\gamma)K^\alpha}{L(K)\mathrm{e}^{\gamma K}} \right| = \lim_{K\to\infty} \left| \frac{L(1/\gamma)\gamma^\alpha K^\alpha}{L(K)\mathrm{e}^{\gamma K}} \right| \leq \lim_{K\to\infty} \left| \frac{c^\alpha K^{c/2} (\log K)^\alpha}{D\mathrm{e}^{\gamma K}} \right| = 0$$

where $D > 0$ is a constant such that $\lim_{K\to\infty} |L(K)| \geq D$. □

Proof of Theorem 3.1. Let $\epsilon > 0$. For simplicity of notation, we omit K and ϵ in much what follows and let

$$\gamma_\pm \;=\; \frac{1}{K}\Big[(\alpha-1)\log K + \log\log K - \log L(K) + \log\big[-(\mathbb{E}J_1 + \mathbb{E}Y_1)(\alpha-1)\big] \pm \epsilon\Big],$$

$\gamma = \gamma^{(K)}$ etc. Since $\gamma_\pm K \to \infty$, we get

$$\begin{aligned} B(\gamma\pm) \sim \frac{L(K)}{K^\alpha}\mathrm{e}^{\gamma_\pm K} &= -\frac{L(K)}{K^\alpha} \frac{K^{\alpha-1}\log K(\alpha-1)\big(\mathbb{E}J_1 + \mathbb{E}Y_1\big)\mathrm{e}^{\pm\epsilon}}{L(K)} \\ &= -\frac{(\alpha-1)\log K\big(\mathbb{E}J_1 + \mathbb{E}Y_1\big)\mathrm{e}^{\pm\epsilon}}{K} \\ &\sim -\gamma_\pm\big(\mathbb{E}J_1 + \mathbb{E}Y_1\big)\mathrm{e}^{\pm\epsilon}. \end{aligned}$$

To complete the proof, it suffices to show that $\kappa(\gamma_-) < 0 < \kappa(\gamma_+)$ for all large K, because this implies $\gamma \in (\gamma_-, \gamma_+)$ and, since ϵ is arbitrary, the assertion (6). The proofs are identical for γ_+ and γ_-, so we treat only γ_+. The preceeding lemma tells us that $A(\gamma_+) \sim \mathbb{E}J_1\gamma_+$. Thus

$$\begin{aligned}
\kappa(\gamma_+) &= A(\gamma_+) + B(\gamma_+) + \mu\gamma_+ + \sigma^2\gamma_+^2/2 + \int_{-\infty}^0 (\mathrm{e}^{\gamma_+ x} - 1)\xi(\,\mathrm{d}x) \\
&= \gamma_+\Big(A(\gamma_+)/\gamma_+ + B(\gamma_+)/\gamma_+ + \mu + \sigma^2\gamma_+/2 + \int_{-\infty}^0 \frac{\mathrm{e}^{\gamma_+ x} - 1}{\gamma_+}\xi(\,\mathrm{d}x)\Big) \\
&\sim \gamma_+\big(\mathbb{E}J_1 - \big(\mathbb{E}J_1 + \mathbb{E}Y_1\big)\mathrm{e}^{\epsilon} + \mathbb{E}Y_1\big) \\
&= \gamma_+(1 - \mathrm{e}^{\epsilon})\big(\mathbb{E}J_1 + \mathbb{E}Y_1\big) > 0
\end{aligned}$$

where we used $\mathbb{E}Y_1 < \infty$, the fact that $\lim_{t\to 0}(\mathrm{e}^t - 1)/t = 1$ and dominated convergence to get

$$\int_{-\infty}^0 \frac{\mathrm{e}^{\gamma_+ x} - 1}{\gamma_+}\xi(\,\mathrm{d}x) \to \int_{-\infty}^0 x\xi(\,\mathrm{d}x).$$

This completes the proof. □

Proof of Proposition 3.1. For simplicity, we suppress K and let $U = X_1 \wedge K$, $J_t = J_t^{(K)}$ etc. Since $C = C_K = (1-\rho)/\big(\beta\mathbb{E}[U\mathrm{e}^{\gamma U}] - 1\big) = (1-\rho)/\big(\mathbb{E}[J_1\mathrm{e}^{\gamma J_1}] - 1\big)$, we seek the asymptotic behaviour of $\mathbb{E}[J_1\mathrm{e}^{\gamma J_1}]$ with $\gamma = \gamma^{(K)}$ as in Corollary 3.1 with $\theta = 1$. To this end, we write

$$\mathbb{E}[J_1\mathrm{e}^{\gamma J_1}] = \int_0^K u\mathrm{e}^{\gamma u}\frac{\alpha L(u)}{(u+1)^{\alpha+1}}\,\mathrm{d}u + K\mathrm{e}^{\gamma K}\int_K^\infty \frac{\alpha L(u)}{(u+1)^{\alpha+1}}\,\mathrm{d}u. \tag{13}$$

We express (13) as $\mathbb{E}[J_1\mathrm{e}^{\gamma J_1}] = a + b$. It is immediate from Karamata's theorem that

$$b \sim K\mathrm{e}^{\gamma K}L(K)/(K+1)^\alpha \sim \mathrm{e}^{\gamma K}L(K)/K^{\alpha-1} \sim (\alpha-1)(1-\beta\mu_F)\log K,$$

so we turn to the part a. By using the same arguments as in the proof of Lemma 3.1, we find that for some $0 < \delta < 1$ we get, where $z = \gamma K$,

$$\begin{aligned}
a &= \int_0^{1/\gamma} u\mathrm{e}^{\gamma u}\frac{\alpha L(u)}{(u+1)^{\alpha+1}} + \alpha\gamma^{\alpha-1}\left[\int_1^{(1-\delta)z} \mathrm{e}^y\frac{L(y/\gamma)}{y^\alpha}\,\mathrm{d}y + \int_{(1-\delta)z}^z \mathrm{e}^y\frac{L(y/\gamma)}{y^\alpha}\,\mathrm{d}y\right] \\
&\sim \beta\mu_F + \alpha\gamma^{\alpha-1}\int_{(1-\delta)z}^z \mathrm{e}^y\frac{L(y/\gamma)}{y^\alpha}\,\mathrm{d}y \sim \beta\mu_F + \alpha\gamma^{\alpha-1}L(K)\frac{\mathrm{e}^z}{z^\alpha} \\
&\to \beta\mu_F + \alpha(1-\beta\mu_F),
\end{aligned}$$

and a is therefore of smaller order than b and thus asymptotically immaterial. The claim now follows if we note that $\rho \to \beta\mu_F$. □

4 A Weibull example

The regularly varying case is without question the main example of heavy tails, but also other cases like log–normal or Weibull distributions have received attention. For the sake of completeness, we briefly include a Weibull example, $\mathbb{P}(U > x) = \mathrm{e}^{-x^\beta}$. For heavy tails, one needs $0 < \beta < 1$ and we took $\beta = 1/2$ (motivated in part by the fact that this is the only case where we can establish the asymptotics of $\gamma^{(K)}$, see Appendix C). As in the Pareto case, the simulations are straightforward and we take $\rho = 0.4$ and $\rho = 0.8$ and $K = 10, 100$ and 1000. Note that we may use the same methodology as in Algorithm 1 to generate random numbers from the distribution with density

$$f(u) = \mathrm{e}^{\gamma u} \frac{\mathrm{e}^{-\sqrt{u}}}{2\sqrt{u}\mathbb{E}\mathrm{e}^{\gamma X^{(K)}}}$$

(in the third step we now just simulate a Weibull r.v.). Relative errors of the two different algorithms for different parameter combinations are in Table 2. As in the Pareto case we used 5000 iterations for each simulation in the Siegmund algorithm and 50000 iterations in (5). We see that in this case the Siegmund algorithm performs better than (5) even when $u \in [0, K]$ for some parameter combinations.

ρ	β	K	u	Siegmund	(5)
0.4	1/2	1000	500	0.2948	0.025
			1000	0.0698	0.0371
			2000	0.0961	—
0.4	1/2	100	50	0.0308	0.0195
			100	0.0299	0.0671
			200	0.0294	1.1773
0.4	1/2	10	5	0.0126	0.0077
			10	0.0146	0.0188
			20	0.0137	0.0635
0.8	1/2	1000	500	0.0612	0.3046
			1000	0.037	0.1494
			2000	0.0445	—
0.8	1/2	100	50	0.0105	0.0167
			100	0.0103	0.0341
			200	0.0102	0.1279
0.8	1/2	10	5	0.004	0.0082
			10	0.0042	0.0113
			20	0.0042	0.0182

Table 2: Comparison of relative errors

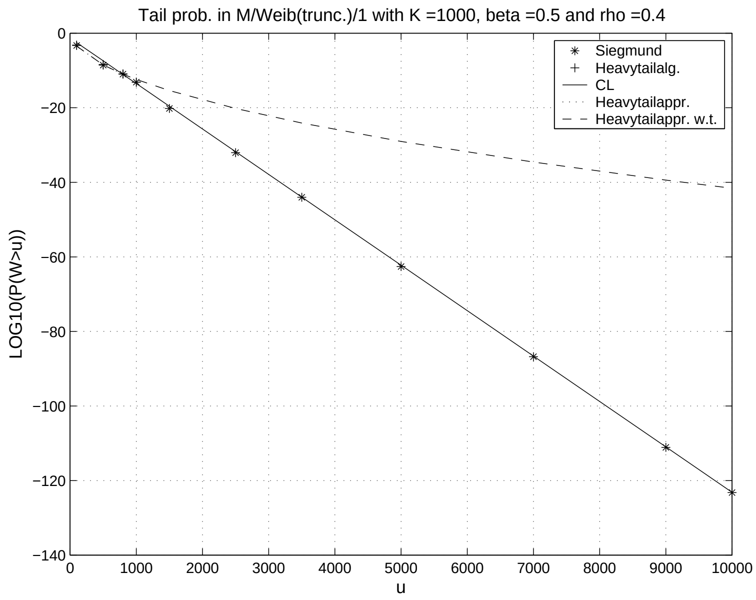

Figure 13: $\beta = 1/2$, $\rho = 0.4$ and $K = 1000$.

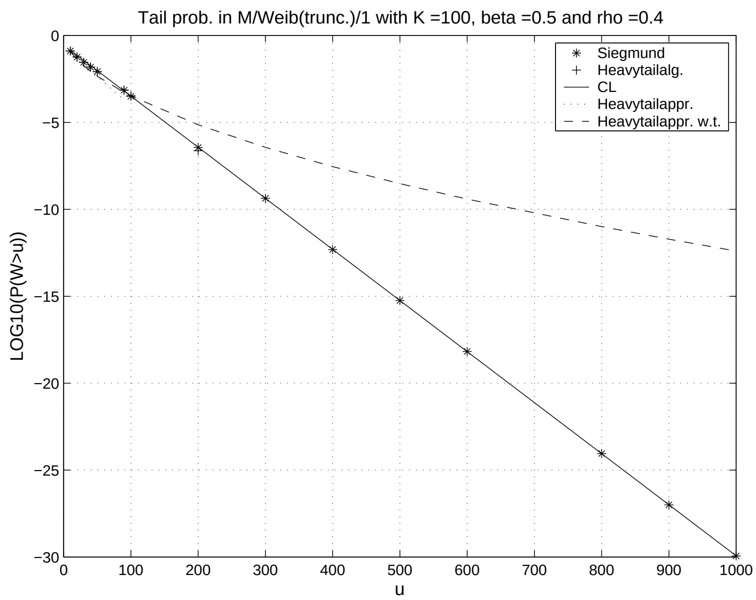

Figure 14: $\beta = 1/2$, $\rho = 0.4$ and $K = 100$.

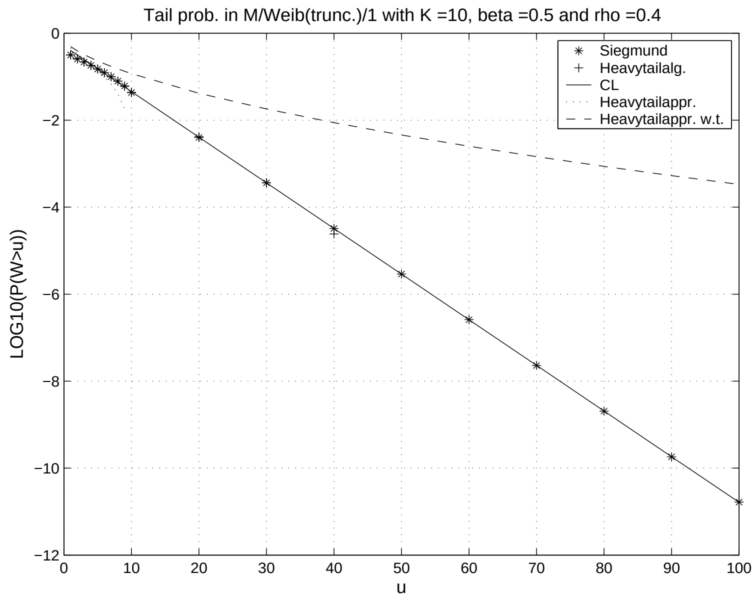

Figure 15: $\beta = 1/2$, $\rho = 0.4$ and $K = 10$.

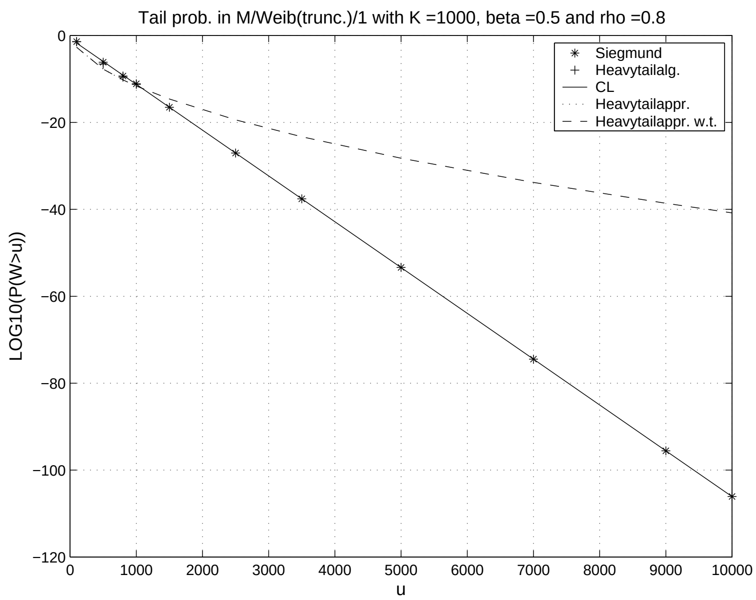

Figure 16: $\beta = 1/2$, $\rho = 0.8$ and $K = 1000$.

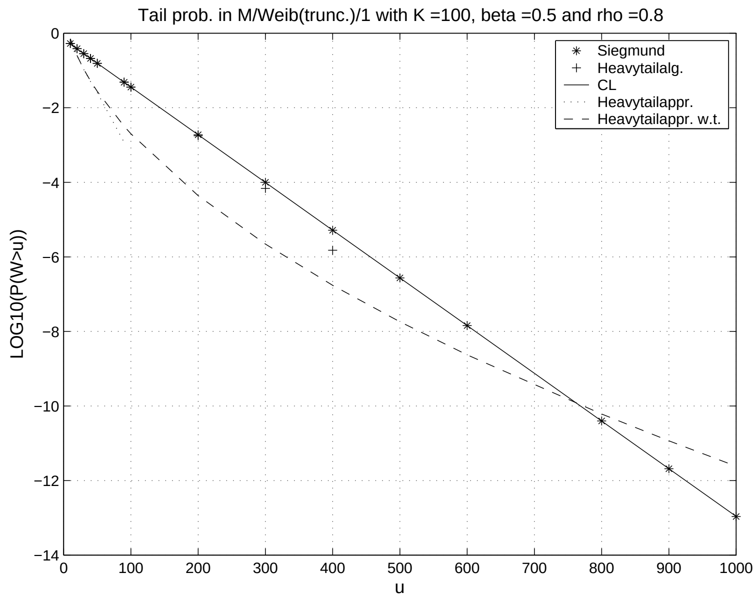

Figure 17: $\beta = 1/2$, $\rho = 0.8$ and $K = 100$.

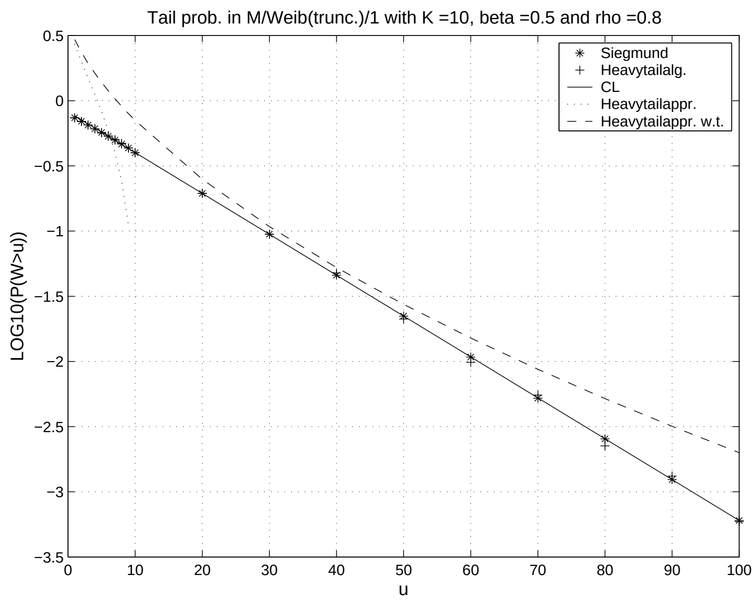

Figure 18: $\beta = 1/2$, $\rho = 0.8$ and $K = 10$.

A Generating r.v.'s from $B^{(K;\gamma)}$

Algorithm 1 The following acceptance/rejection algorithm produces random numbers from the distribution defined by (3).

1) Divide the interval $[0, K)$ into the subintervals $J_i = [x_{i-1}, x_i)$, $i = 1, \ldots, N$ where $x_j = jK/N$.

2) For each subinterval J_i, compute

$$p_i = \int_{J_i} B^{(K;\gamma)}(\,\mathrm{d}x).$$

This is done numerically.

3) A r.v. X with the desired distribution is obtained by first choosing interval J_i with probability p_i and taking $X = K$ with probability $1 - \sum_{j=1}^{N} p_j$. For the given interval, simulate a Pareto r.v. conditioned to belong to J_i, x (easily done by using the standard inversion algorithm), which is accepted with probability $\mathrm{e}^{\gamma(x-x_i)}$.

That this gives a r.v. with the desired density follows since we in the third step pick J_i with probability $B^{(K;\gamma)}(J_i)$ and then sample from the conditional distribution $\mu_i(\cdot) = B^{(K;\gamma)}(\cdot \cap J_i)/B^{(K;\gamma)}(J_i)$. The minimal acceptance probability in step 3 is

$$\mathrm{e}^{-\gamma K/N} \sim \mathrm{e}^{-(\alpha-1)\log K/N} = K^{-(\alpha-1)/N},\ K \to \infty \tag{14}$$

in view of Corollary 3.1. □

We observe that for $\alpha = 3/2$, $N = 100$ and an extreme case like $K = 10^9$, the asymptotic expression in (14) suggests that the minimal acceptance probability, p, is in the vicinity of 90%. For smaller K, p is obviously larger.

B The Jelenković approximation

Let $Z, Z_1, Z_2, \ldots$ be a sequence of non–negative i.i.d. random variables with density $f(x) = L(x)/x^\alpha$, $\alpha > 1$ with L slowly varying and $Z^K, Z_1^K, Z_2^K, \ldots$ i.i.d. with density

$$f^K(x) = f(x)/\mathbb{P}\big(0 \le Z \le K\big),\ 0 \le x \le K. \tag{15}$$

We quote the following theorem from Jelenković [12]:

Theorem B.1 *Let $R_n^K = \sum_{i=1}^{n} Z_i^K$, $n \ge 1$. If $f(x) = L(x)/x^\alpha$, $\alpha > 1$, then for any constant $C > 0$, fixed $k = 0, 1, \ldots$, fixed $0 < \delta < 1$, and uniformly for all $k+1 \le n \le C\log K$,*

$$\mathbb{P}\big(R_n^K \ge (k+\delta)K\big) \sim \binom{n}{k+1} h_k(\delta) \frac{L(K)^{k+1}}{K^{(k+1)(\alpha-1)}}, \tag{16}$$

as $K \to \infty$, where

$$h_k(\delta) = \int\limits_{\substack{0 \le x_i \le 1,\, 1 \le i \le k+1 \\ x_1 + \cdots + x_{k+1} \ge k+\delta}} x_1^{-\alpha} \cdots x_{k+1}^{-\alpha} \, \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1}.$$

We use this result to obtain an approximation of $\psi_K(u) = \mathbb{P}\big(W_K > u\big)$. To this end, recall the Pollaczeck–Khinchine formula

$$\psi_K(u) = \mathbb{P}\big(W_K > u\big) \;=\; \mathbb{P}\big(Y_1^K + \cdots + Y_N^K > u\big)$$

where the Y_i^K are i.i.d. with distribution $B_I^{(K)}$ and N is an independent geometric r.v. with parameter ρ. It holds that

$$B_I^{(K)}(\,\mathrm{d}x) = \begin{cases} \mathbb{P}\big(U > x\big)\,\mathrm{d}x / \mathbb{E}\big(U \wedge K\big) & \text{if } x \le K \\ 0 & \text{if } x > K, \end{cases}$$

and

$$\mathbb{P}\big(U > x\big) = \int_x^\infty \alpha \frac{L(u)}{u^{\alpha+1}}\,\mathrm{d}u \sim L(x) \int_x^\infty \frac{\alpha}{u^{\alpha+1}}\,\mathrm{d}u = \frac{L(x)}{x^\alpha},\; x \to \infty, \qquad (17)$$

by Karamata's theorem. Thus the density of $B_I^{(K)}$ is of the form described in (15), and we have the following result:

Theorem B.2 *Suppose that $\rho < 1$ is fixed throughout. As $K \to \infty$ it holds, for each fixed $k = 0, 1, \ldots$ and fixed $0 < \delta < 1$ that*

$$\mathbb{P}\big(W_K > (k+\delta)K\big) \sim h_k(\delta) \left(\frac{\rho L(K)}{(1-\rho)K^{(\alpha-1)}} \right)^{k+1}.$$

Proof. The details are very close to the proof of Theorem 2 in [12] and therefore omitted. □

Note that $h_k(\delta)$ is difficult to express analytically for $k > 0$. However, the following observation makes it easy to numerically evaluate $h_k(\delta)$:

$$h_k(\delta) = \mathbb{E}\left[U_1^{-\alpha} \cdots U_{k+1}^{-\alpha};\, \sum_{i=1}^{k+1} U_i \ge k+\delta \right]$$

where $U_1, \ldots, U_{k+1}$ are independent r.v.'s uniformly distributed on $[0,1]$, so we may estimate $h_k(\delta)$ by using standard Monte Carlo integration.

C The asymptotics of $\gamma^{(K)}$ in the Weibull example

Theorem C.1 *Let $\gamma = \gamma^{(K)}$ be the root of the Lundberg equation for a compound Poisson process with intensity β, jump distribution F truncated at K and drift term $-\theta$. If $\overline{F}(u) = \mathrm{e}^{-\sqrt{u}}$ and $\theta > 2\beta$, then it holds as $K \to \infty$ that*

$$\gamma^{(K)} \;=\; \frac{1}{K}\Big[\sqrt{K} - \frac{\log K}{2} + \log\Big[\frac{\theta - 2\beta}{2}\Big] + \mathrm{o}(1) \Big].$$

Proof. Let $\kappa^{(K)}(s)$, μ_F etc. be defined as in Section 3. For simplicity, we omit K in much what follows. Then

$$\begin{aligned}\kappa(s) = \kappa^{(K)}(s) &= \beta \int_0^K (\mathrm{e}^{sx}-1)\frac{\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x + \beta(\mathrm{e}^{sK}-1)\mathrm{e}^{-\sqrt{K}} - \theta s \\ &= A(s)+B(s)-\theta s.\end{aligned}$$

For $\epsilon > 0$ we now define

$$\gamma_\pm = \frac{1}{K}\Big[\sqrt{K} - \frac{\log K}{2} + \log\Big[\frac{\theta-2\beta}{2}\Big] \pm \epsilon\Big]$$

and show that $\kappa(\gamma_-) < 0 < \kappa(\gamma_+)$. Let

$$T(\gamma_+) = A(\gamma_+)/\gamma_+ = \beta\int_0^{1/\gamma_+} \frac{\mathrm{e}^{\gamma_+ x}-1}{\gamma_+}\frac{\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x + \int_{1/\gamma_+}^K \frac{\mathrm{e}^{\gamma_+ x}-1}{\gamma_+}\frac{\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x. \qquad (18)$$

By dominated convergence and the fact that $\lim_{t\to 0}(\mathrm{e}^t-1)/t = 1$ we get that the first part of (18) tends to $\beta\mu_F = 2\beta$. In the second part we make the change of variables $z = \sqrt{\gamma_+ x}$ and get that

$$\begin{aligned}\int_{1/\gamma_+}^K \frac{\mathrm{e}^{\gamma_+ x}-1}{\gamma_+}\frac{\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x &\sim \frac{1}{\gamma_+}\int_{1/\gamma_+}^K \mathrm{e}^{\gamma_+ x}\frac{\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x}}\,\mathrm{d}x \\ &= \frac{1}{{\gamma_+}^{3/2}}\mathrm{e}^{-1/4\gamma_+}\int_1^{\sqrt{\gamma_+ K}} \mathrm{e}^{(z-1/2\sqrt{\gamma_+})^2}\,\mathrm{d}z \equiv I.\end{aligned}$$

We may write I as

$$I = \frac{1}{{\gamma_+}^{3/2}}\mathrm{e}^{-1/4\gamma_+}\int_{1-1/2\sqrt{\gamma_+}}^{\sqrt{\gamma_+ K}-1/2\sqrt{\gamma_+}} \mathrm{e}^{z^2}\,\mathrm{d}z$$

and it is easy to see that $\sqrt{\gamma_+ K} - 1/2\sqrt{\gamma_+} \sim K^{1/4}/2 \to \infty$ and we get

$$\begin{aligned}I &\sim \frac{1}{{\gamma_+}^{3/2}}\mathrm{e}^{-1/4\gamma_+}\Bigg(\frac{1}{2(1/2\sqrt{\gamma_+}-1)}\mathrm{e}^{(1/2\sqrt{\gamma_+}-1)^2} \\ &\qquad + \frac{1}{2(\sqrt{\gamma_+ K}-1/2\sqrt{\gamma_+})}\mathrm{e}^{(\sqrt{\gamma_+ K}-1/2\sqrt{\gamma_+})^2}\Bigg) \\ &\sim \frac{1}{{\gamma_+}^{3/2}}\Bigg(\sqrt{\gamma_+}\mathrm{e}^{1-1/\sqrt{\gamma_+}} + \frac{1}{2(\sqrt{\gamma_+ K}-1/2\sqrt{\gamma_+})}\mathrm{e}^{\gamma_+ K-\sqrt{K}}\Bigg) \\ &\sim K^{3/4}\Bigg(\frac{1}{K^{1/4}}\mathrm{e}^{1-K^{1/4}+\mathrm{O}(K^{-1/4}\log K)} + \frac{1}{K^{1/4}}\mathrm{e}^{-\log K/2+\log((\theta-2\beta)/2)+\epsilon}\Bigg)\end{aligned}$$

where we used that

$$\int_a^t \mathrm{e}^{y^2}\,\mathrm{d}y \sim \frac{\mathrm{e}^{t^2}}{2t}, \quad t\to\infty,$$

$\sqrt{\gamma_+ K} - 1/2\sqrt{\gamma_+} \sim K^{1/4}/2$ and $\gamma_+ \sim 1/\sqrt{K}$. Now it follows that $I \to e^{\epsilon}(\theta - 2\beta)/2$, $K \to \infty$ and thus $A(\gamma_+) \sim \gamma_+(2\beta + e^{\epsilon}(\theta - 2\beta)/2)$. It is easily seen that $B(\gamma_+) \sim \gamma_+ e^{\epsilon}(\theta - 2\beta)/2$. We now get that

$$\begin{aligned}\kappa(\gamma_+) &= A(\gamma_+) + B(\gamma_+) - \theta\gamma_+ \sim \gamma_+(2\beta + e^{\epsilon}(\theta - 2\beta)/2 + e^{\epsilon}(\theta - 2\beta)/2 - \theta) \\ &= \gamma_+(2\beta + e^{\epsilon}(\theta - 2\beta) - \theta) = \gamma_+((e^{\epsilon} - 1)(\theta - 2\beta)) > 0.\end{aligned}$$

In an identical way we show that $\kappa(\gamma_-) < 0$. This concludes the proof. □

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