General Probabilistic Framework of Randomness

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Abstract

We introduce a new mathematical framework for the probabilistic description of an experiment upon a system of any type in terms of initial information representing this system. Based on the notions of an information state, an information state space and a generalized observable, this general framework covers the description of a wide range of experimental situations including those where, with respect to a system, an experiment is perturbing.

We prove that, to any experiment upon a system, there corresponds a unique generalized observable on a system initial information state space, which defines the probability distribution of outcomes under this experiment. We specify the case where the initial information on a system provides "no knowledge" for the description of an experiment.

Incorporating in a uniform way the basic notions of conventional probability theory and the non-commutativity aspects and the basic notions of quantum measurement theory, our framework clarifies the principle difference between Kolmogorov's model in probability theory and the statistical model of quantum theory. Both models are included into our framework as particular cases. We show that the phenomenon of "reduction" of a system initial information state is inherent, in general, to any non-destructive experiment and upon a system of any type.

Based on our general framework, we introduce the probabilistic model for the description of non-destructive experiments upon a quantum system and prove that positive bounded linear mappings on the Banach space of trace class operators, arising in the description of experiments upon a quantum system, are completely positive.

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1 Introduction

The problem of the relation between the statistical model of quantum theory and conventional probability theory is a point of intensive discussions, beginning from von Neumann's axioms [1] in quantum measurement theory and Kolmogorov's axioms [2] in the theory of probability.

In the physical literature on quantum physics one can find statements on the peculiarities of "quantum" probabilities and "quantum" events. In the mathematical physics literature, the structure of conventional probability theory is often referred to as Kolmogorov's model or as classical probability, and it is argued that the Kolmogorov model is embedded as a particular case into the so-called "non-commutative probability theory" - the algebraic framework based on the structure of the statistical model of quantum theory.

However, since the algebraic framework does not inherit the structure of probability theory and does not cover the description of all possible general probabilistic situations, this framework cannot be considered to represent an extension of conventional probability theory. Moreover, the algebraic framework cannot also, in principle, incorporate the developments of quantum measurement theory.

Many attempts have been also made to include the statistical model of quantum theory into the formalism of conventional probability theory. However, all these attempts cannot be constructive, since random variables (classical observables) in Kolmogorov's model represent only non-perturbing experiments and, in general, this is not true for an experiment upon a quantum system.

In the present paper we formulate the basics of a new general framework for the probabilistic description of an experiment upon a system in terms of initial information representing this system. Our mathematical setting is most general and covers, in particular, those probabilistic situations where a system is represented initially by a set of maximally available "bits" of information and a probability distribution of possible "bits".

We introduce the notions of an information state, an information state space and a generalized observable and prove that any experiment upon a system is represented on an initial information state space of this system by a uniquely defined generalized observable. We discuss the situation where an initial information state space of a system provides "no knowledge" for the description of an experiment.

Our framework incorporates in a uniform way the basic notions of conventional probability theory, the non-commutativity aspects and the basic notions of quantum measurement theory. This allows us to clarify the principle difference between Kolmogorov's model and the statistical model of quantum theory. Both models are included into our framework as particular cases.

We introduce the concept of a complete information description of a non-destructive experiment upon a system and show that the phenomenon of "reduction" of an information state is, in general, inherent to any non-destructive experiment and upon a system of any type. In the most general settings, this phenomenon is induced by a "renormalization" of the information on a system, conditioned upon the recorded outcome under a single trial, and by the "dynamical" change of a system information state in the course of a perturbing experiment. A "reduction" of a mixed Kolmogorov's probability state occurs, in particular, even under a non-perturbing classical measurement. Since, as we establish in this paper, the probabilistic model of quantum theory represents a special model of our general framework, the well-known von Neumann quantum "state collapse", postulated in [1], and its further generalizations represent particular cases of this general phenomenon.

In case where an information space of a system has a Banach space based structure, we define, in the most general settings, the notions of a mean information state, a conditional posterior mean information state and the concept of a complete statistical description of a non-destructive experiment. We show that, for the definite class of non-destructive experiments upon a system of this type, the change from an initial mean information state to a conditional posterior mean information state is given by the notion of a mean information state instrument, which we introduce in this paper.

We formulate the probabilistic model for the description of a non-destructive experiment upon a quantum system. This model specifies not only the probability distribution of outcomes but also the conditional probability distribution of posterior pure quantum state outcomes following a single experimental trial.

Based on our framework, we prove that positive bounded linear mappings on the Banach space of trace class operators, arising in the description of non-destructive experiments upon a quantum system, are completely positive. We note that, under the operational approach to the description of quantum measurements, the complete positivity is always introduced axiomatically, rather than actually proved as in the present paper.

The basics of the quantum stochastic approach to the description of quantum measurements, formulated in [17-19], correspond to the general probabilistic framework, introduced in this paper

2 Description of experiments

Consider an experiment upon a system of any type. Let the experimental situation be specified by a "complex of conditions, which allows of any number of repetitions" [2] and, under each trial, let an experimental outcome ω be of any kind. We denote by Ω the set of all outcomes ω , and by \mathcal{F}_{Ω} a set of subsets of Ω , which includes \emptyset and Ω and represents mathematically possible questions on an outcome ω , being posed under this experiment. Namely, each subset $B \in \mathcal{F}_{\Omega}$ represents the event that an outcome ω belongs to B is recorded. The pair $(\Omega, \mathcal{F}_{\Omega})$ is called an outcome space.

Suppose that an experimental situation is such that, under numerous identical trials, the limit of relative frequencies of the occurrence of an event $\omega \in B \in \mathcal{F}_{\Omega}$ exists (up to a measurement error) and defines a non-negative number $\operatorname{Prob}\{\omega \in B\} \leq 1$, called the probability, or the chance, that, under a single trial of this experiment, the event $\omega \in B$ occurs. In this case we say that this experimental situation admits the probabilistic description and, for specificity, call such experimental situations *normal*.

For any $B \in \mathcal{F}_{\Omega}$, denote $P(B) := \operatorname{Prob}\{\omega \in B\}$. The family $\mathbb{P} = \{P(B) : B \in \mathcal{F}_{\Omega}\}$ of probabilities of all events is called an outcome probability law of this experimental situation. Clearly, $P(\emptyset) = 0$, $P(\Omega) = 1$, and $P(B \cup B') = P(B) + P(B')$, whenever $B \cap B' = \emptyset$.

According to Kolmogorov's axioms [2] in the theory of probability:

(i) \mathcal{F}_{Ω} is a σ -algebra on Ω , so that $(\Omega, \mathcal{F}_{\Omega})$ is a measurable space;

(ii) an outcome probability law $\mathbb P$ is represented by a normalized $\sigma\text{-additive positive real valued measure}$

$$\mu: \mathcal{F}_{\Omega} \to [0,1], \quad \mu(\Omega) = 1,$$

such that $P(B) = \mu(B), \forall B \in \mathcal{F}_{\Omega}$.

A triple $(\Omega, \mathcal{F}_{\Omega}, \mu)$ represents a positive measure space¹. In conventional probability theory² a normalized σ -additive positive real valued measure μ and a positive measure space $(\Omega, \mathcal{F}_{\Omega}, \mu)$ are called a probability measure and a probability space, respectively.

Otherwise expressed:

To any normal experimental situation upon a system S of any type, there corresponds the unique probability space $(\Omega, \mathcal{F}_{\Omega}, \mu)$, where the measurable space $(\Omega, \mathcal{F}_{\Omega})$ represents a space of outcomes and the normalized σ -additive positive real valued measure μ represents an outcome probability law \mathbb{P} of this experimental situation.

The above axioms are crucial and, to our knowledge, are valid for all models, introduced in different concrete sciences, to describe normal experimental situations.

These axioms are valid, in particular, in the quantum case where, under any generalized quantum measurement with outcomes in a measurable space $(\Omega, \mathcal{F}_{\Omega})$, the outcome probability law is given³ by a normalized σ -additive positive real valued measure

$$\mu(B) = \operatorname{tr}\{\rho M(B)\}, \quad \forall B \in \mathcal{F}_{\Omega},$$

expressed via a density operator ρ , tr{ ρ } = 1, on a separable complex Hilbert space \mathcal{H} and a normalized σ -additive measure $M(\cdot)$ on $(\Omega, \mathcal{F}_{\Omega})$, with values M(B), $B \in \mathcal{F}_{\Omega}$, $M(\Omega) = I_{\mathcal{H}}$, that are non-negative bounded linear operators on \mathcal{H} .

3 Probabilistic framework

In this section we introduce, in the most general settings, the representation of the outcome probability law of a normal experimental situation upon a system in terms of information representing this system before an experiment. This allows to formalize the probabilistic description of all experimental situations upon a system, in particular, those on which the initial information on a system provides "no knowledge" and also those which perturb a system.

All experimental situations, discussed in this paper, are hypothetical.

3.1 General settings

Let S be a system of any type. In the most general settings, we express the information, representing S, by a positive measure space

 $(\Theta, \mathcal{F}_{\Theta}, \pi)$

where Θ is a set, \mathcal{F}_{Θ} is a σ -algebra of subsets of Θ and π is a normalized σ -additive positive real valued measure on a measurable space $(\Theta, \mathcal{F}_{\Theta})$. The mathematical structure of a measurable space $(\Theta, \mathcal{F}_{\Theta})$ is not specified. In particular, we do not, in general, presume any linear or convex linear structure of a set Θ .

This mathematical setting is most general and covers, in particular, those situations where each element of Θ is interpreted as a maximally available "bit" of information on a system and a measure π represents a probability distribution of possible $\theta \in \Theta$.

¹On notions of measure theory, see, for example, [22,23].

²On the notions of conventional probability theory, see, for example, [10].

³On the main notions of quantum measurement theory, see, for example, [8,9,11-21].

Consider the description of an experiment \mathcal{E} , with outcomes in a measurable space $(\Omega, \mathcal{F}_{\Omega})$, upon a system \mathcal{S} . Let, before an experiment, a system \mathcal{S} be described by any of the positive measure spaces in

$$\{(\Theta, \mathcal{F}_{\Theta}, \pi) : \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}\},\tag{1}$$

where $\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$ is the convex linear set of all normalized σ -additive positive real valued measures on $(\Theta, \mathcal{F}_{\Theta})$.

For any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, denote by $\mathcal{E} + \mathcal{S}(\pi)$ the experimental situation where \mathcal{E} is carried out upon \mathcal{S} , represented initially by $(\Theta, \mathcal{F}_{\Theta}, \pi)$. If all experimental situations $\mathcal{E} + \mathcal{S}(\pi), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, are normal we call the experiment \mathcal{E} upon \mathcal{S} normal. We consider further the description of only normal experiments \mathcal{E} upon \mathcal{S} and, therefore, suppress the term "normal".

According to the consideration in section 2, let a normalized σ -additive positive real valued measure

$$\mu_{\mathcal{E}}(\cdot; (\Theta, \mathcal{F}_{\Theta}, \pi)) : \mathcal{F}_{\Omega} \to [0, 1]$$

represent on $(\Omega, \mathcal{F}_{\Omega})$ the outcome probability law of an experimental situation $\mathcal{E} + \mathcal{S}(\pi), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$. For short, we further use the notation $\mu_{\mathcal{E}}(\cdot; \pi) := \mu_{\mathcal{E}}(\cdot; (\Theta, \mathcal{F}_{\Theta}, \pi))$.

The mapping

$$\mu_{\mathcal{E}}(\cdot;\cdot): \mathcal{F}_{\Omega} \times \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})} \to [0,1]$$
(2)

describes all experimental situations $\mathcal{E} + \mathcal{S}(\pi)$, $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, that is, an experiment \mathcal{E} upon a system \mathcal{S} .

Thus, to an experiment \mathcal{E} upon a system \mathcal{S} , represented initially by the information, expressed by (1), there corresponds the unique mapping (2).

However, the converse statement is not true and the same mapping (2) may correspond to a variety of experiments upon S.

In general, the initial information on \mathcal{S} may be such that

$$\mu_{\mathcal{E}}(B;\pi) = \mu_{\mathcal{E}}(B), \quad \forall B \in \mathcal{F}_{\Omega}, \quad \forall \pi \in \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}, \tag{3}$$

and this relation implies that the initial information on S, represented by (1), is not relevant for the description of an experiment \mathcal{E} upon S.

In this case, we say that the initial information on S provides "no knowledge" for the description of an experiment \mathcal{E} upon S.

If, however, the initial information on S provides "the knowledge" on an experiment \mathcal{E} upon S then, in general, the randomness may be caused by⁴:

(i) the uncertainty, encoded in a measure space $(\Theta, \mathcal{F}_{\Theta}, \pi)$, where, in the most general settings, two mathematical objects, elements of a set Θ and a measure π , may be responsible for this;

(ii) by a probabilistic set-up of an experiment \mathcal{E} itself.

Notice that in case where an experiment is carried out upon a physical microsystem, even if all macroscopic parameters of the experimental set-up are defined with certainty, this may not be true for parameters, characterizing the microscopic environment of the experimental device. The latter is due to the fact that, in the most general case, we can not specify definitely either a physical state of this microscopic environment or its interaction with the observed microsystem.

In view of the informational context of a probability distribution $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$ in a measure space $(\Theta, \mathcal{F}_{\Theta}, \pi)$, representing the initial information on a system \mathcal{S} , it is natural to assume:

Convention. For a system S, any positive measure space $(\Theta, \mathcal{F}_{\Theta}, \tilde{\pi})$ with a finite σ -additive positive real valued measure $\tilde{\pi}$ on $(\Theta, \mathcal{F}_{\Theta})$, satisfying the relation

$$\widetilde{\pi}(\cdot)/\widetilde{\pi}(\Theta) = \pi(\cdot),\tag{4}$$

represents the same information on S as $(\Theta, \mathcal{F}_{\Theta}, \pi)$.

 $^{^{4}}$ On the discussion of possible types of uncertainties, see [7].

Axiom (Statistical)⁵. For an experiment \mathcal{E} with outcomes in $(\Omega, \mathcal{F}_{\Omega})$ upon a system \mathcal{S} , the mapping $\mu_{\mathcal{E}}(\cdot; \cdot)$ is such that

$$\mu_{\mathcal{E}}(B;\pi) = \alpha_1 \mu_{\mathcal{E}}(B;\pi_1) + \alpha_2 \mu_{\mathcal{E}}(B;\pi_2), \quad \forall B \in \mathcal{F}_{\Omega},$$
(5)

for any measures $\pi, \pi_1, \pi_2 \in \mathcal{V}_{\cdot(\Theta, \mathcal{F}_{\Theta})}$, satisfying the relation

$$\pi = \alpha_1 \pi_1 + \alpha_2 \pi_2, \quad \alpha_1, \alpha_2 \ge 0, \quad \alpha_1 + \alpha_2 = 1.$$

Remark 1 The statistical axiom is true even if the initial information provides "no knowledge" for the description of an experiment (see (3)).

Introduce the following notations.

(a) for any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, we denote by $[\pi]$ the equivalence class of all finite σ -additive positive real valued measures $\tilde{\pi}$ on $(\Theta, \mathcal{F}_{\Theta})$ equivalent to π , due to (4), and by

$$(\Theta, \mathcal{F}_{\Theta}, [\pi]) := \{ (\Theta, \mathcal{F}_{\Theta}, \widetilde{\pi}) : \widetilde{\pi} \in [\pi] \}$$

$$(6)$$

the set of all positive measure spaces, representing, due to the convention, the same information on S as $(\Theta, \mathcal{F}_{\Theta}, \pi)$;

(b) we denote by $[\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]$ the set

$$[\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}] := \{ [\pi] : \pi \in \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})} \}$$

$$\tag{7}$$

of all equivalence classes $[\pi]$;

(c) for any measurable space $(\Lambda, \mathcal{F}_{\Lambda})$, we denote by $\mathcal{J}_{(\Lambda, \mathcal{F}_{\Lambda})}$ the linear space of all σ -additive bounded real valued measures on $(\Lambda, \mathcal{F}_{\Lambda})$ and by $\mathcal{J}_{(\Lambda, \mathcal{F}_{\Lambda})}^{(+)} \subset \mathcal{J}_{(\Lambda, \mathcal{F}_{\Lambda})}$ the set of all finite σ -additive non-negative real valued measures on $(\Lambda, \mathcal{F}_{\Lambda})$. Endowed with the norm

$$||\nu||_{\mathcal{J}_{(\Lambda,\mathcal{F}_{\Lambda})}} = \sup_{B\in\mathcal{F}_{\Lambda}}|\nu(B)|, \quad \forall \nu\in\mathcal{J}_{(\Lambda,\mathcal{F}_{\Lambda})},$$

the normed linear space $\mathcal{J}_{(\Lambda,\mathcal{F}_{\Lambda})}$ is Banach⁶.

Due to the convention, the information on S, represented by $(\Theta, \mathcal{F}_{\Theta}, \pi)$, is equivalently described by any element of the equivalence class $(\Theta, \mathcal{F}_{\Theta}, [\pi])$. The latter implies that the mapping (2) can be uniquely extended to all of $\mathcal{J}_{(\Theta, \mathcal{F}_{\Theta})}^{(+)} \setminus \{0\}$, with the property

$$\mu_{\mathcal{E}}(\cdot; \widetilde{\pi}_1) = \mu_{\mathcal{E}}(\cdot; \widetilde{\pi}_2), \quad \forall \widetilde{\pi}_1, \widetilde{\pi}_2 \in [\pi], \quad \forall \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}.$$

Here "0" denotes the zero valued measure on $(\Theta, \mathcal{F}_{\Theta})$.

Hence, to an experiment \mathcal{E} , with outcomes in $(\Omega, \mathcal{F}_{\Omega})$, upon a system \mathcal{S} , there corresponds the unique mapping

$$\mu_{\mathcal{E}}(\cdot;\cdot): \mathcal{F}_{\Omega} \times (\mathcal{J}_{(\Theta,\mathcal{F}_{\Theta})}^{(+)} \setminus \{0\}) \to [0,1],$$
(8)

where, for each $\tilde{\pi} \in \mathcal{J}^{(+)}_{(\Theta,\mathcal{F}_{\Theta})} \setminus \{0\}$, the mapping $\mu_{\mathcal{E}}(\cdot; \tilde{\pi})$ is a normalized σ -additive positive real valued measure on $(\Omega, \mathcal{F}_{\Omega})$ and

$$\mu_{\mathcal{E}}(\cdot;\alpha\widetilde{\pi}) = \mu_{\mathcal{E}}(\cdot;\widetilde{\pi}), \quad \forall \alpha > 0.$$
(9)

Furthermore, due to the statistical axiom and (9), the mapping $\mu_{\mathcal{E}}(\cdot; \cdot)$ must satisfy the condition

$$\mu_{\mathcal{E}}(B;\tilde{\pi}) = \alpha_1 \mu_{\mathcal{E}}(B;\tilde{\pi}_1) + \alpha_2 \mu_{\mathcal{E}}(B;\tilde{\pi}_2), \tag{10}$$

for any $\tilde{\pi} \in [\pi]$, $\tilde{\pi}_1 \in [\pi_1]$, $\tilde{\pi}_2 \in [\pi_2]$, such that

 $\pi=\alpha_1\pi_1+\alpha_2\pi_2,\quad \alpha_1,\alpha_2\geq 0, \ \alpha_1+\alpha_2=1.$

 $^{{}^{5}}$ This axiom is similar to the statistical axioms in [4,9,11,16], introduced, however, under different settings. 6 See [22], section 3.7.4.

3.2 Information state spaces

Based on our considerations in section 3.1, we introduce new mathematical notions and prove the corresponding statements.

3.2.1 Information states

Let $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ be an equivalence class, defined by (6).

Definition 1 (Information state) We call an equivalence class $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ an information state.

If a σ -algebra \mathcal{F}_{Θ} is trivial, that is, $\mathcal{F}_{\Theta} = \{\emptyset, \Theta\}$, then there exists only one measure $\pi_0 \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, with $\pi_0(\Theta) = 1$, $\pi_0(\emptyset) = 0$. We call the corresponding information state $(\Theta, \mathcal{F}_{\Theta}, [\pi_0])$ trivial.

If a σ -algebra \mathcal{F}_{Θ} contains all atom subsets $\{\theta\}$ of Θ , we call an information state *pure* if π is a Dirac measure on $(\Theta, \mathcal{F}_{\Theta})$ and *mixed*, otherwise.

We say that an information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ is *induced* by an information state $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$ if a measure π is subordinated to a measure π' , that is,

$$\pi(F) = \int_{\Theta'} \Phi(F;\theta') \pi'(d\theta'), \quad \forall F \in \mathcal{F}_{\Theta},$$
(11)

where $\Phi(\cdot; \cdot) : \mathcal{F}_{\Theta} \times \Theta' \to [0, 1]$ is a mapping⁷ such that:

(i) for each $\theta' \in \Theta'$, the mapping $\Phi(\cdot; \theta')$ is a normalized σ -additive positive real valued measure on $(\Theta, \mathcal{F}_{\Theta})$;

(ii) for each $F \in \mathcal{F}_{\Theta}$, the function $\Phi(F; \cdot) : \Theta' \to [0, 1]$ is $\mathcal{F}_{\Theta'}$ -measurable.

If, in particular,

$$\Phi(F;\theta') = \chi_{\phi^{-1}(F)}(\theta'), \quad \forall \theta' \in \Theta', \quad \forall F \in \mathcal{F}_{\Theta},$$

where $\chi_{F'}(\cdot)$ is an indicator function of a subset $F' \in \mathcal{F}_{\Theta'}$, a function $\phi : \Theta' \to \Theta$ is $\mathcal{F}_{\Theta'}/\mathcal{F}_{\Theta}$ measurable and $\phi^{-1}(F) \in \mathcal{F}_{\Theta'}$ is the preimage of a subset $F \in \mathcal{F}_{\Theta}$, then

$$\pi(F) = \pi'(\phi^{-1}(F)), \quad \forall F \in \mathcal{F}_{\Theta}$$

In this case, we say that an information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ is an ϕ -image of the information state $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$ and, respectively, $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$ is an ϕ -preimage of $(\Theta, \mathcal{F}_{\Theta}, [\pi])$. We denote this ϕ -image subordination of measures and states by

$$\pi(F) = (\pi' \circ \phi^{-1})(F) := \pi'(\phi^{-1}(F)), \quad \forall F \in \mathcal{F}_{\Theta},$$

$$(\Theta, \mathcal{F}_{\Theta}, [\pi]) = \phi[(\Theta', \mathcal{F}_{\Theta'}, [\pi'])].$$
(12)

In view of the notation (7), denote by

$$(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]) := \{(\Theta, \mathcal{F}_{\Theta}, [\pi]) : \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}\}$$
(13)

the set of all information states on $(\Theta, \mathcal{F}_{\Theta})$.

Definition 2 We call a set $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ an information state space.

We say that an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is trivial if \mathcal{F}_{Θ} is trivial. In this case, $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ consists of only one trivial information state $(\Theta, \mathcal{F}_{\Theta}, [\pi_0])$.

If each state in $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is induced by a state in $(\Theta', \mathcal{F}_{\Theta'}, [\mathcal{V}_{(\Theta', \mathcal{F}_{\Theta'})}])$ then we say that an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is *induced* by $(\Theta', \mathcal{F}_{\Theta'}, [\mathcal{V}_{(\Theta', \mathcal{F}_{\Theta'})}])$.

In particular, if measurable spaces $(\Theta, \mathcal{F}_{\Theta})$ and $(\Theta', \mathcal{F}_{\Theta'})$ are isomorphic, that is, there exists a bijection $f : \Theta' \to \Theta$ such that functions f and f^{-1} are, respectively, $\mathcal{F}_{\Theta'}/\mathcal{F}_{\Theta}$ and $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta'}$ measurable, then, to each state $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$ in $(\Theta', \mathcal{F}_{\Theta'}, [\mathcal{V}_{(\Theta', \mathcal{F}_{\Theta'})}])$, there is put into one-to-one correspondence the f-image state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ in $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$, with $\pi = \pi' \circ f^{-1}$, and vice versa.

⁷Called in probability theory as a Markov kernel from $(\Theta', \mathcal{F}_{\Theta'})$ to $(\Theta, \mathcal{F}_{\Theta})$.

3.2.2 Statistical information states

Let $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ be an information state space.

Definition 3 Let \mathfrak{R} be a set. For a mapping $\Phi : (\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]) \to \mathfrak{R}$, we call values $\eta_{\Phi} = \Phi((\Theta, \mathcal{F}_{\Theta}, [\pi])), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}, as \Phi$ -statistical information states on $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

On $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ there exists a variety of different types of statistical information states.

Consider, in particular, the situation where V is a Banach space and a mapping $\varphi : \Theta \to V$ is measurable with respect to the σ -algebra \mathcal{F}_{Θ} and the σ -algebra \mathcal{B}_{V} of Borel subsets of V, and also bounded, that is, there exists some C > 0 such that $||\varphi(\theta)||_{V} \leq C, \forall \theta \in \Theta$.

The mapping

$$(\Theta, \mathcal{F}_{\Theta}, [\pi]) \mapsto \eta_{mean(\varphi)}((\Theta, \mathcal{F}_{\Theta}, [\pi])) := \int_{\Theta} \varphi(\theta) \pi(d\theta) \in \mathbf{V}, \quad \forall \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})},$$

is well defined and $||\eta_{mean(\varphi)}||_{V} \leq C$. For short, we denote

$$\eta_{mean(\varphi)}(\pi) := \eta_{mean(\varphi)}((\Theta, \mathcal{F}_{\Theta}, [\pi]))$$

and refer to $\eta_{mean(\varphi)}(\pi)$ as a φ -mean information state on $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

The set $\mathfrak{R}_{mean(\varphi)} \subset V$ of all φ -mean information states is convex linear and bounded.

Denote by $(\Theta_{\varphi}, \mathcal{B}_{\Theta_{\varphi}})$ a measurable space where $\Theta_{\varphi} = \varphi(\Theta) \subset V$ and $\mathcal{B}_{\Theta_{\varphi}}$ is the trace on Θ_{φ} of the Borel σ -algebra \mathcal{B}_{V} on V. We have $||\theta_{\varphi}||_{V} \leq C, \forall \theta_{\varphi} \in \Theta_{\varphi}$, and, for any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$,

$$\eta_{mean(\varphi)}(\pi) = \int_{\Theta_{\varphi}} \theta_{\varphi} \pi_{\varphi}(d\theta_{\varphi})$$

where $\pi_{\varphi}(F) = \pi(\varphi^{-1}(F)), \forall F \in \mathcal{B}_{\Theta_{\varphi}}.$

In view of this representation, we further restrict our consideration in section 4.5 of the statistical description of experiments only to the case where

$$\Theta = \{ \theta \in \mathcal{V} : ||\theta||_{\mathcal{V}} \le C, \text{ for some } C > 0 \},$$

$$\mathcal{F}_{\Theta} \supset \mathcal{B}_{\Theta}$$
(14)

For this case we introduce the special type of statistical information states.

Definition 4 (Mean information state) For an information state space, with a measurable space $(\Theta, \mathcal{F}_{\Theta})$ satisfying (14), we call the values of the linear mapping

$$\pi \mapsto \eta_{mean}(\pi) = \int_{\Theta} \theta \pi(d\theta) \in \mathbf{V}, \quad \forall \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})},$$

as mean information states on $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

We say that the convex linear set $\mathfrak{R}_{mean}(\Theta)$ of all mean information states on $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ represents a *mean information state space*.

We say that a mean information state is *pure* if it is represented by an extreme element of $\mathfrak{R}_{mean}(\Theta)$ and *mixed*, otherwise.

In case of (14), all atom subsets $\{\theta\}$ of Θ belong to \mathcal{F}_{Θ} , the set of all extreme elements of $\mathfrak{R}_{mean}(\Theta)$ is included in Θ , and to each pure mean information state there corresponds a unique pure information state. However, the converse is not, in general, true and a pure information state may correspond to a mixed mean information state. Moreover, a mixed mean information state can be, in general, induced by a variety of information states.

3.3 Positive mapping valued measures

Let $(\Lambda, \mathcal{F}_{\Lambda})$ and $(\Theta, \mathcal{F}_{\Theta})$ be any measurable spaces. Consider a mapping

$$(\widetilde{\mathcal{K}}(\cdot))(\cdot): \quad \mathcal{F}_{\Lambda} \times [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}] \to [0, 1],$$
(15)

such that, for each equivalence class $[\pi] \in [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]$, the mapping

$$(\widetilde{\mathcal{K}}(\cdot))([\pi]): \mathcal{F}_{\Lambda} \to [0,1], \qquad (\widetilde{\mathcal{K}}(\Lambda))([\pi]) = 1,$$

is a normalized σ -additive positive real valued measure on $(\Lambda, \mathcal{F}_{\Lambda})$.

For a set X, denote by $\mathfrak{B}(X)$ the Banach space of all bounded complex valued functions $\Psi: X \to \mathbb{C}$. Recall that, in $\mathfrak{B}(X)$, the norm is defined by

$$||\Psi||_{\mathfrak{B}(X)} = \sup_{x \in X} |\Psi(x)|, \quad \forall \Psi \in \mathfrak{B}(X).$$

Denote by $\mathfrak{B}_+(X) \subset \mathfrak{B}(X)$ the set of all non-negative real valued bounded functions on X. For any $\Psi_1, \Psi_2 \in \mathfrak{B}(X)$, we write $\Psi_1 \leq \Psi_2$ if $(\Psi_2 - \Psi_1) \in \mathfrak{B}_+(X)$. Let $\mathcal{I}_{\mathfrak{B}(X)} \in \mathfrak{B}_+(X)$ be the positive function

$$\mathcal{I}_{\mathfrak{B}(X)}(x) = 1, \quad \forall x \in X.$$

We can now say that, in (15), the mapping $\widetilde{\mathcal{K}}$ is a normalized σ -additive⁸ measure on $(\Lambda, \mathcal{F}_{\Lambda})$ with values $\widetilde{\mathcal{K}}(B), B \in \mathcal{F}_{\Lambda}$, that are non-negative real valued bounded functions on the set $[\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]$, that is,

$$\widetilde{\mathcal{K}}(B) \in \mathfrak{B}_+([\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}]), \quad \forall B \in \mathcal{F}_{\Lambda}$$

There is the one-to-one correspondence between $\widetilde{\mathcal{K}}$ and the mapping

$$\mathcal{K}(\cdot;\cdot): \mathcal{F}_{\Lambda} \times (\mathcal{J}^{(+)}_{(\Theta,\mathcal{F}_{\Theta})}) \setminus \{0\}) \to [0,1],$$

defined by the relation

$$\mathcal{K}(\cdot;\pi'\} := (\widetilde{\mathcal{K}}(\cdot))([\pi]), \quad \forall \pi' \in [\pi], \quad \forall \pi \in \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}.$$

We formulate the following theorem.

Theorem 1 Let $(\Lambda, \mathcal{F}_{\Lambda})$ and $(\Theta, \mathcal{F}_{\Theta})$ be any measurable spaces. To a mapping

$$\widetilde{\mathcal{K}}: \mathcal{F}_{\Lambda} \to \mathfrak{B}_{+}([\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}]), \quad \widetilde{\mathcal{K}}(\Lambda) = \mathcal{I}_{\mathfrak{B}([\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}])},$$

satisfying the conditions:

(i) for each $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, the mapping $(\widetilde{\mathcal{K}}(\cdot))([\pi]) : \mathcal{F}_{\Lambda} \to [0, 1], (\widetilde{\mathcal{K}}(\Lambda))([\pi]) = 1$, is a normalized σ -additive positive real valued measure on $(\Lambda, \mathcal{F}_{\Lambda})$; (ii) $(\widetilde{\mathcal{K}}(B))([\pi]) = \alpha_1(\widetilde{\mathcal{K}}(B))([\pi_1]) + \alpha_2(\widetilde{\mathcal{K}}(B))([\pi_2]), \forall B \in \mathcal{F}_{\Lambda}$ for any $\pi, \pi_1, \pi_2 \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}, \pi = \alpha_1 \pi_1 + \alpha_2 \pi_2, \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$;

there exists a unique normalized σ -additive measure

$$\Pi: \mathcal{F}_{\Lambda} \to \mathfrak{B}_{+}(\Theta), \quad \Pi(\Lambda) = \mathcal{I}_{\mathfrak{B}(\Theta)},$$

with $\Pi(B): \Theta \to [0,1]$ being \mathcal{F}_{Θ} -measurable for each $B \in \mathcal{F}_{\Lambda}$; such that

$$((\widetilde{\mathcal{K}}(B))([\pi]) = \int_{\Theta} (\Pi(B))(\theta) \pi(d\theta),$$

for all $B \in \mathcal{F}_{\Lambda}, \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$.

⁸Here the σ -additivity is understood in the following sense: for any $\pi \in \mathcal{V}_{(\Lambda, \mathcal{F}_{\Lambda})}$, a normalized positive real valued measure $(\tilde{\mathcal{K}}(\cdot))([\pi])$ is σ -additive.

3.4 Generalized observables

Introduce the following general concept.

3.4.1 Definition, properties

Let $(\Theta, \mathcal{F}_{\Theta})$ and $(\Lambda, \mathcal{F}_{\Lambda})$ be any measurable spaces.

Definition 5 (Generalized observable) We call a normalized σ -additive measure

$$\Pi: \mathcal{F}_{\Lambda} \to \mathfrak{B}_{+}(\Theta), \quad \Pi(\Lambda) = \mathcal{I}_{\mathfrak{B}(\Theta)},$$

where, for each $B \in \mathcal{F}_{\Lambda}$, the function $\Pi(B)$ is \mathcal{F}_{Θ} -measurable, a generalized observable, with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$ and on $(\Theta, \mathcal{F}_{\Theta})$.

In the terminology of conventional probability theory, for each $B \in \mathcal{F}_{\Lambda}$, the \mathcal{F}_{Θ} -measurable function $\Pi(B) : \Theta \to [0, 1]$ is a random variable on $(\Theta, \mathcal{F}_{\Theta})$.

The set of all generalized observables on $(\Theta, \mathcal{F}_{\Theta})$, with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, is convex linear. For any generalized observable Π , we have $\Pi(\emptyset) = 0$ and

$$0 \le \Pi(B_1) \le \Pi(B_2) \le \mathcal{I}_{\mathfrak{B}(\Theta)},$$

for any $B_1 \subseteq B_2 \subseteq \Lambda$.

Since an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is defined uniquely by a measurable space $(\Theta, \mathcal{F}_{\Theta})$, we further equivalently refer to Π as a generalized observable on an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

Definition 6 We call a generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$ trivial if it has the form

$$\Pi(B) = \nu(B)\mathcal{I}_{\mathfrak{B}(\Theta)}, \quad \forall B \in \mathcal{F}_{\Lambda},$$

with some $\nu \in \mathcal{V}_{(\Lambda, \mathcal{F}_{\Lambda})}$.

If a σ -algebra \mathcal{F}_{Θ} is trivial then any \mathcal{F}_{Θ} -measurable function $\Theta \to [0, 1]$ is constant and, in this case, on $(\Theta, \mathcal{F}_{\Theta})$ there exist only trivial generalized observables.

If \mathcal{F}_{Λ} is trivial then, in $\mathcal{V}_{(\Lambda,\mathcal{F}_{\Lambda})}$, there is only measure π_0 , with $\pi_0(\Lambda) = 1$, $\pi_0(\emptyset) = 0$. In this case, on $(\Theta, \mathcal{F}_{\Theta})$ there exists only one generalized observable $\Pi_0 = \pi_0 \mathcal{I}_{\mathfrak{B}(\Theta)}$ with the outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, and it is also trivial.

Remark 2 Due to definition 5, to any mapping $\widetilde{\mathcal{K}}$, specified in theorem 1, there corresponds a unique generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$. If $\widetilde{\mathcal{K}}$ has the form

$$(\widetilde{\mathcal{K}}(\cdot))([\pi]) = k(\cdot), \quad \forall \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}.$$

with $k \in \mathcal{V}_{(\Lambda,\mathcal{F}_{\Lambda})}$, then the corresponding generalized observable Π is given by $\Pi(B) = k(B)\mathcal{I}_{\mathfrak{B}(\Theta)}$, $\forall B \in \mathcal{F}_{\Lambda}$, and is trivial.

Denote by $\mathcal{G}_{(\Theta,\mathcal{F}_{\Theta})}$ the set of all non-trivial generalized observables on an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}])$. From our above discussion it follows that $\mathcal{G}_{(\Theta,\mathcal{F}_{\Theta})} = \emptyset$, whenever $\mathcal{F}_{\Lambda} = \{\emptyset, \Lambda\}$, or $\mathcal{F}_{\Theta} = \{\emptyset, \Theta\}$. The latter relation implies that on a trivial information state space all generalized observables are trivial.

For our further needs in section 3.5, we specify a special type of a generalized observable.

Definition 7 We say that a generalized observable, with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$ and on $(\Theta, \mathcal{F}_{\Theta})$, represents an **observable** and, for specificity, we denote an observable by E, if, for any $B_0 \in \mathcal{F}_{\Lambda}$, $E(B_0) \neq 0$, there exists an element $\theta_0 \in \Theta$ such that:

$$\begin{aligned} &(\mathcal{E}(B))(\theta_0) = 1, \quad B \supseteq B_0, \quad \forall B \in \mathcal{F}_{\Lambda}, \\ &(\mathcal{E}(B))(\theta_0) = 0, \quad B \cap B_0 = \emptyset, \quad \forall B \in \mathcal{F}_{\Lambda}. \end{aligned}$$

Let Π_1 and Π_2 be generalized observables on $(\Theta, \mathcal{F}_{\Theta})$, with outcome spaces $(\Lambda_1, \mathcal{F}_{\Lambda_1})$ and $(\Lambda_2, \mathcal{F}_{\Lambda_2})$, respectively.

To any generalized observables Π_1 and Π_2 , there exists the unique generalized observable $\Pi_1 \times \Pi_2$ on $(\Theta, \mathcal{F}_{\Theta})$, with the product outcome space $(\Lambda_1 \times \Lambda_2, \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2})$, such that

$$((\Pi_1 \times \Pi_2)(B_1 \times B_2))(\theta) = (\Pi_1(B_1))(\theta)(\Pi_2(B_2))(\theta),$$

for all $\theta \in \Theta$, $B_1 \in \mathcal{F}_{\Lambda_1}$, $B_2 \in \mathcal{F}_{\Lambda_2}$.

We call $\Pi_1 \times \Pi_2$ the product generalized observable of Π_1 and Π_2 .

We say that a generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$, with the outcome product space $(\Lambda_1 \times \Lambda_2, \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2})$, is a *joint generalized observable* of Π_1 and Π_2 if the latter are the marginal measures of Π , that is:

$$\Pi_1(B_1) = \Pi(B_1 \times \Lambda_2), \quad \forall B_1 \in \mathcal{F}_{\Lambda_1}, \\ \Pi_2(B_2) = \Pi(\Lambda_1 \times B_2), \quad \forall B_2 \in \mathcal{F}_{\Lambda_2}.$$

In particular, the product generalized observable $\Pi_1 \times \Pi_2$ represents a joint generalized observable of Π_1 and Π_2 .

However, if generalized observables Π_1 and Π_2 on $(\Theta, \mathcal{F}_{\Theta})$ belong to some definite class then the product generalized observable may not belong to this class, and, hence, in this class there may not exist a joint generalized observable of Π_1 and Π_2 .

Let, for example, a measurable space $(\Theta, \mathcal{F}_{\Theta})$ has the property (14) and both Π_1 and Π_2 belong to the class $\mathcal{G}_{(\Theta, \mathcal{F}_{\Theta})}^{(lin)}$ of non-trivial generalized observables, represented by measures with values that are non-negative continuous *linear* functionals on V. Then the product generalized observable $\Pi_1 \times \Pi_2$ does not belong to the class $\mathcal{G}_{(\Theta, \mathcal{F}_{\Theta})}^{(lin)}$.

3.4.2 Convolution of generalized observables

Consider measurable spaces $(\Theta, \mathcal{F}_{\Theta})$ and $(\Theta', \mathcal{F}_{\Theta'})$, and let Π be a generalized observable on $(\Theta, \mathcal{F}_{\Theta})$, with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, and S be a generalized observable on $(\Theta', \mathcal{F}_{\Theta'})$, with an outcome space $(\Theta, \mathcal{F}_{\Theta})$.

Definition 8 We call a generalized observable on $(\Theta', \mathcal{F}_{\Theta'})$, defined by the relation

$$(\Pi(B) * \mathbf{S})(\theta') := \int_{\Theta} (\Pi(B))(\theta)(\mathbf{S}(d\theta))(\theta'), \quad \forall B \in \mathcal{F}_{\Lambda}, \quad \forall \theta' \in \Theta',$$
(16)

the **convolution** of a generalized observable Π , with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$ and on $(\Theta, \mathcal{F}_{\Theta})$, and a generalized observable S, with the outcome space $(\Theta, \mathcal{F}_{\Theta})$ and on $(\Theta', \mathcal{F}_{\Theta'})$

If $\phi: \Theta' \to \Theta$ is an $\mathcal{F}_{\Theta'}/\mathcal{F}_{\Theta}$ -measurable function and

$$(\widetilde{\mathbf{S}}(F))(\theta') = \chi_{\phi^{-1}(F)}(\theta'), \quad \forall \theta' \in \Theta', \ \forall F \in \mathcal{F}_{\Theta},$$

then

$$(\Pi(B) * \widetilde{\mathbf{S}})(\theta') = (\Pi(B))(\phi(\theta')) = (\Pi(B) \circ \phi)(\theta')$$

for all $B \in \mathcal{F}_{\Lambda}, \theta' \in \Theta'$.

Definition 9 We say that a generalized observable on $(\Theta', \mathcal{F}_{\Theta'})$, defined by the relation

$$\Pi_{\phi^{-1}}(B) := \Pi(B) \circ \phi, \qquad \forall B \in \mathcal{F}_{\Lambda}, \tag{17}$$

is the ϕ -preimage on $(\Theta', \mathcal{F}_{\Theta'})$ of a generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$.

For a trivial generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$, its preimage $\Pi_{\phi^{-1}}$ on $(\Theta', \mathcal{F}_{\Theta'})$ is also trivial. Let a generalized observable Π be itself an *f*-preimage of some generalized observable $\widetilde{\Pi}$ on $(\widetilde{\Theta}, \mathcal{F}_{\widetilde{\Theta}})$, that is, $\Pi = \widetilde{\Pi}_{f^{-1}}$, for an $\mathcal{F}_{\Theta}/\mathcal{F}_{\widetilde{\Theta}}$ -measurable function $f : \Theta \to \widetilde{\Theta}$, then

$$\Pi_{\phi^{-1}} = \Pi_{g^{-1}}, \qquad g = f \circ \phi$$

and $\Pi_{\phi^{-1}}$ is the *g*-preimage on $(\Theta', \mathcal{F}_{\Theta'})$ of the generalized observable $\widetilde{\Pi}$ on $(\widetilde{\Theta}, \mathcal{F}_{\widetilde{\Theta}})$.

Proposition 1 If $\Pi_{\phi^{-1}}$ on $(\Theta', \mathcal{F}_{\Theta'})$ is an observable then Π on $(\Theta, \mathcal{F}_{\Theta})$ is also an observable.

Proposition 2 Let an $\mathcal{F}_{\Theta'}/\mathcal{F}_{\Theta}$ -measurable function $\phi: \Theta' \to \Theta$ be surjective. Then: (i) Π on $(\Theta, \mathcal{F}_{\Theta})$ is trivial iff $\Pi_{\phi^{-1}}$ on $(\Theta', \mathcal{F}_{\Theta'})$ is trivial; (ii) Π is an observable on $(\Theta, \mathcal{F}_{\Theta})$ iff $\Pi_{\phi^{-1}}$ is an observable on $(\Theta', \mathcal{F}_{\Theta'})$.

3.4.3 Functional subordination

Consider a generalized observable $\Pi : \mathcal{F}_{\Lambda} \to \mathfrak{B}^{(+)}(\Theta), \Pi(\Lambda) = \mathcal{I}_{\mathfrak{B}(\Theta)}, \text{ on } (\Theta, \mathcal{F}_{\Theta})$ with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$.

Let $(\widetilde{\Lambda}, \mathcal{F}_{\widetilde{\Lambda}})$ be a measurable space and $\varphi : \Lambda \to \widetilde{\Lambda}$ be an $\mathcal{F}_{\Lambda}/\mathcal{F}_{\widetilde{\Lambda}}$ -measurable function.

Definition 10 We say that on $(\Theta, \mathcal{F}_{\Theta})$ a generalized observable Π , with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, is φ -functionally subordinated to a generalized observable Π , with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, and denote this by $\Pi = \Pi \circ \varphi^{-1}$, if

$$(\Pi \circ \varphi^{-1})(B) := \Pi(\varphi^{-1}(B)), \tag{18}$$

for all $\widetilde{B} \in \mathcal{F}_{\widetilde{\Lambda}}$.

Consider an ϕ -preimage $(\Pi \circ \varphi^{-1})_{\phi^{-1}}$ on $(\Theta', \mathcal{F}_{\Theta'})$ of a generalized observable $\Pi \circ \varphi^{-1}$ on $(\Theta, \mathcal{F}_{\Theta})$. Due to (17) and (18), we have

$$(\Pi \circ \varphi^{-1})_{\phi^{-1}}(\widetilde{B}) = (\Pi \circ \varphi^{-1})(\widetilde{B}) \circ \phi = \Pi(\varphi^{-1}(\widetilde{B})) \circ \phi$$
$$= \Pi_{\phi^{-1}}(\varphi^{-1}(\widetilde{B})) = (\Pi_{\phi^{-1}} \circ \varphi^{-1})(\widetilde{B}),$$

for all $\widetilde{B} \in \mathcal{F}_{\widetilde{\Lambda}}$

Conclusion 1 An ϕ -preimage subordination preserves a φ -functional subordination.

3.5 Outcome probability laws

In view of the notions, introduced in sections 3.2-3.4, and theorem 1, let us come back to the general settings of section 3.1.

Consider an experiment \mathcal{E} with outcomes in $(\Omega, \mathcal{F}_{\Omega})$ upon a system \mathcal{S} of any type. Suppose that, before this experiment, a system \mathcal{S} is represented by an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

Let $\mu_{\mathcal{E}}(\cdot; \cdot)$ be the mapping, specified by (8), (9), (10). For each $\pi \subset \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, the mapping $\mu_{\mathcal{E}}(\cdot; \pi)$ defines the outcome probability law of the experimental situation $\mathcal{E} + \mathcal{S}(\pi)$. The mapping

$$\widetilde{\mu}_{\mathcal{E}}: \mathcal{F}_{\Omega} \to \mathfrak{B}_+([\mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})}]),$$

associated to $\mu_{\mathcal{E}}(\cdot; \cdot)$ by the relation

$$(\widetilde{\mu}_{\mathcal{E}}(\cdot))([\pi]) := \mu_{\mathcal{E}}(\cdot,\pi), \quad \forall \pi \subset \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})},$$

satisfies the conditions of theorem 1. Hence, we have the following proposition.

Theorem 2 (Representation theorem) To an experiment \mathcal{E} , with an outcome space $(\Omega, \mathcal{F}_{\Omega})$, upon a system \mathcal{S} represented initially by an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$, there corresponds a unique generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$ such that the outcome probability law of each experimental situation $\mathcal{E} + \mathcal{S}(\pi), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, is given by

$$\mu_{\mathcal{E}}(B;\pi) = \int_{\Theta} (\Pi(B))(\theta)\pi(d\theta), \tag{19}$$

for all $B \in \mathcal{F}_{\Omega}$.

Remark 3 Clearly, the converse statement is not true and, to a generalized observable Π on $(\Theta, \mathcal{F}_{\Theta})$ there may, in general, correspond a variety of experiments upon S, with the same probability distribution of outcomes. We further denote by $[\Pi]$ the equivalence class of all experiments upon S, represented on $(\Theta, \mathcal{F}_{\Theta})$ by the same non-trivial generalized observable Π .

In the terminology of section 3.1, the relation (19) implies that, for each experimental situation $\mathcal{E} + \mathcal{S}(\pi), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, the probability space

 $(\Omega, \mathcal{F}_{\Omega}, \mu_{\mathcal{E}}(\cdot; \pi))$

is *induced* by an initial information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$.

3.5.1 "No knowledge"

Notice that if:

(i) an initial information state space is trivial, then on this space there exist only trivial generalized observables and, in this case, any experiment upon S is represented on this space by a trivial generalized observable;

(ii) in an outcome space $(\Omega, \mathcal{F}_{\Omega})$, the σ -algebra \mathcal{F}_{Ω} is trivial, then there exists only one normalized positive scalar measure π_0 in $\mathcal{V}_{(\Omega, \mathcal{F}_{\Omega})}$. In this case, in (19), $\mu_{\mathcal{E}}(\cdot; \pi) = \pi_0(\cdot)$, $\forall \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$ and the generalized observable on $(\Theta, \mathcal{F}_{\Theta})$, corresponding to this experiment, is given by $\Pi = \pi_0 \mathcal{I}_{\mathfrak{B}(\Theta)}$, and is also trivial.

(iii) $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is a non-trivial initial information state space of \mathcal{S} and

$$\mu_{\mathcal{E}}(\cdot;\pi) = \mu_{\mathcal{E}}(\cdot), \quad \forall \pi \in \mathcal{V}_{(\Theta,\mathcal{F}_{\Theta})},$$

then, in (19), the corresponding generalized observable is trivial.

Due to our discussion in section 3.1, in all above cases the initial information on \mathcal{S} , represented by $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$, provides "no knowledge" for the description of an experiment \mathcal{E} .

Proposition 3 ("No knowledge") An information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ of S provides "no knowledge" on an experiment \mathcal{E} iff the generalized observable Π , representing on $(\Theta, \mathcal{F}_{\Theta})$ this experiment, is trivial.

From this proposition it follows that a trivial information state space provides "no knowledge" on experiments upon S. A non-trivial information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ provides the knowledge only on the class of experiments, represented on $(\Theta, \mathcal{F}_{\Theta})$ by non-trivial generalized observables, that is, those in $\mathcal{G}_{(\Theta, \mathcal{F}_{\Theta})}$.

If an information state space of S is obvious from the context, we further refer to Π as a generalized observable of S.

3.5.2 Informational equivalence

Consider the situation where an initial information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ of \mathcal{S} is *induced*, due to (11), by an information state $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$. In terms of generalized observables, the relation (11) implies

that there exists a generalized observable S, with the outcome space $(\Theta, \mathcal{F}_{\Theta})$ and on $(\Theta', \mathcal{F}_{\Theta'})$, such that

$$\pi(F) = \int_{\Theta'} (\mathbf{S}(F))(\theta') \pi'(d\theta'), \quad \forall F \in \mathcal{F}_{\Theta}.$$
 (20)

Let \mathcal{E} be an experiment upon \mathcal{S} , represented on $(\Theta, \mathcal{F}_{\Theta})$ by a non-trivial generalized observable II. Consider the probability distribution of an experimental situation $\mathcal{E} + \mathcal{S}(\pi)$, with π being defined by (20). From (20) and (19) it follows that

$$\mu_{\mathcal{E}}(B;\pi) = \int_{\Theta} (\Pi(B))(\theta)\pi(d\theta) = \int_{\Theta'} (\Pi'(B))(\theta')\pi'(d\theta'), \qquad \forall B \in \mathcal{F}_{\Omega},$$
(21)

where Π' is a generalized observable

$$(\Pi'(B))(\theta') = (\Pi(B) * S)(\theta'), \quad \forall B \in \mathcal{F}_{\Omega},$$
(22)

generated on $(\Theta', \mathcal{F}_{\Theta'})$ by generalized observables Π on $(\Theta', \mathcal{F}_{\Theta'})$ and S on $(\Theta', \mathcal{F}_{\Theta'})$ and representing the *convolution* of these generalized observables.

The relation (21) shows that if an information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$ is induced by an information state $(\Theta', \mathcal{F}_{\Theta'}, [\pi'])$ then, to an experimental situation $\mathcal{E}_{\Pi} + \mathcal{S}(\pi)$, there corresponds the experimental situation $\mathcal{E}'_{\Pi'} + \mathcal{S}(\pi')$, represented by the generated generalized observable Π' on $(\Theta', \mathcal{F}_{\Theta'})$, defined by (22). In view of (21), we denote by

$$\mathcal{E}_{\Pi} + \mathcal{S}(\pi) \sim \mathcal{E}'_{\Pi'} + \mathcal{S}(\pi')$$

the informational equivalence of these experimental situations.

3.5.3 Deterministic set-up

Let a system S be initially represented by an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

To separate the cases where an experiment \mathcal{E} upon a system \mathcal{S} may have a probabilistic setup (see point (ii) in section 3.1), consider the situation where all atom subsets $\{\theta\}$ of Θ belong to \mathcal{F}_{Θ} and, for, at least, one pure information state ($\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}]$) of \mathcal{S} , the predictions are formulated in the language of "yes - no" statements. Since a set-up of an experiment does not depend on an initial information state of \mathcal{S} , the considered experiment has a deterministic set-up⁹.

Due to definition 7, this experiment is represented on $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ by an observable E and from (19) it follows that the outcome probability law of each experimental situation $\mathcal{E} + \mathcal{S}(\pi)$, $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, is given by

$$\mu(B;\pi) = \int_{\Theta} (\mathcal{E}(B))(\theta)\pi(d\theta), \quad \forall B \in \mathcal{F}_{\Lambda}.$$

Thus, on a system information state space an observable represents an experiment with a deterministic set-up.

In general, a non-trivial observable represents an experiment \mathcal{E} upon \mathcal{S} as a whole and can not be referred to some property of \mathcal{S} , existing before this experiment. However, the latter is true for the special type of observables, which we introduce in section 4.3.1 and call *beables*.

Example 1 Let $\varphi : \Theta \to \Omega$ be an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Omega}$ -measurable function. Consider on $(\Theta, \mathcal{F}_{\Theta})$ an observable

$$(\mathbf{E}(B))(\theta) = \chi_{\varphi^{-1}(B)}(\theta), \quad \forall \theta \in \Theta, \quad \forall B \in \mathcal{F}_{\Omega},$$
(23)

with an outcome space $(\Omega, \mathcal{F}_{\Omega})$. For each experimental situation $\mathcal{E} + \mathcal{S}(\pi)$, $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, the probability distribution of outcomes in $(\Omega, \mathcal{F}_{\Omega})$ is given by

$$\mu(B;\pi) = \pi(\varphi^{-1}(B)) = (\pi \circ \varphi^{-1})(B), \quad \forall B \in \mathcal{F}_{\Omega},$$
(24)

 $^{^{9}}$ See also the discussion in [9,11].

and, hence, the probability space $(\Omega, \mathcal{F}_{\Omega}, \mu(\cdot; \pi))$ is the φ -image of the initial information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$, that is:

$$(\Omega, \mathcal{F}_{\Omega}, \mu(\cdot; \pi)) = \varphi[(\Theta, \mathcal{F}_{\Theta}, [\pi])].$$

In the frame of conventional probability theory, the probability distribution (24) is called an image probability law while an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Omega}$ -measurable mapping φ is called a random variable on $(\Omega, \mathcal{F}_{\Omega})$. It is assumed that the image law predicts the probability distribution of outcomes under a nonperturbing (by device) experiment, with a deterministic set-up, representing a classical "errorless" measurement¹⁰ of the property φ of S, existing objectively before this measurement.

Let σ -algebras \mathcal{F}_{Θ} and \mathcal{F}_{Ω} contain all atom subsets. In this case, under an experiment, described by (23), to each initial pure information state $(\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}]), \theta_0 \in \Theta$, the outcome $\omega_0 = \varphi(\theta_0)$ is predicted with certainty. However, since, immediately after this experiment, the description of the system in terms of information states is not specified, one can not claim that the observable (23) represents a non-perturbing experiment on the measurement of a property φ of S. To specify when this is really the case, we introduce in section 4.3 the notion of a non-perturbing experiment upon a system S.

4 Non-destructive experiments

According to section 3.5.1, an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ provides the knowledge on the description of only such class of experiments upon \mathcal{S} which are represented on $(\Theta, \mathcal{F}_{\Theta})$ by non-trivial generalized observables, that is, the generalized observables in $\mathcal{G}_{(\Theta, \mathcal{F}_{\Theta})}$.

In this section we introduce, in the most general settings, the concepts of a complete information description and a complete statistical description of a non-destructive experiment. We define the notion of a non-perturbing experiment and discuss the phenomenon of "reduction" of an information state.

4.1 Extended generalized observables

Let the initial information on a system S be described by $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}])$, with a nonempty set $\mathcal{G}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$.

Consider the description of an experiment upon S such that, immediately after this experiment, the system S exists. We call such experiments *non-destructive*.

Let a non-destructive experiment \mathcal{E} , with outcomes in $(\Omega, \mathcal{F}_{\Omega})$, upon \mathcal{S} be represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by a non-trivial generalized observable. Suppose that, immediately after this non-destructive experiment, a system \mathcal{S} is characterized in terms of an output information state space

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\mathcal{V}_{(\Theta_{out}, \mathcal{F}_{\Theta_{out}})}]),$$

with a non-empty set $\mathcal{G}_{(\Theta_{out},\mathcal{F}_{\Theta_{out}})}$, and, in general, different from the initial information state space of \mathcal{S} .

Each trial of an experimental situation $\mathcal{E} + \mathcal{S}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, results in a outcome $\omega \in \Omega$ and a posterior system \mathcal{S} , represented by a posterior \mathcal{S} -outcome θ_{out} in $(\Theta_{out}, \mathcal{F}_{\Theta_{out}})$. Under a non-destructive experiment \mathcal{E} upon \mathcal{S} , the pair (ω, θ_{out}) represents a compound outcome in the extended outcome space

$$(\Omega \times \Theta_{out}, \mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}}).$$

Denote by

$$\nu(\cdot; \pi_{in}): \ \mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}} \to [0, 1]$$
(25)

the probability distribution of compound outcomes (ω, θ_{out}) in the extended outcome space $(\Omega \times \Theta_{out}, \mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}})$. Notice that $\nu(\cdot; \pi'_{in}) = \nu(\cdot; \pi_{in}), \forall \pi'_{in} \in [\pi_{in}]$.

It is also natural to assume that, for the probability distribution $\nu(\cdot; \cdot)$, the statistical axiom, section 3.1, is valid

 $^{^{10}}$ See, for example, the discussion in [11].

Consider the mapping $(\tilde{\nu}(\cdot))(\cdot)$, associated to the mapping (25) by the relation

$$(\widetilde{\nu}(\cdot))([\pi_{in}]) := \nu(\cdot; \pi_{in}), \qquad \forall \pi_{in} \subset \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}.$$

Then

$$(\widetilde{\nu}(\cdot))(\cdot): (\mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}}) \times [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}] \to [0, 1]$$

satisfies the conditions of theorem 1 and, hence, to $\tilde{\nu}$, there corresponds the uniquely defined generalized observable

$$f: \mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}} \to \mathfrak{B}_+(\Theta_{in}),$$

with the outcome space $(\Omega \times \Theta_{out}, \mathcal{F}_{\Omega} \otimes \mathcal{F}_{\Theta_{out}})$ and on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, such that

$$\nu(B;\pi_{in}) = \int_{\Theta_{in}} (\Upsilon(B))(\theta_{in})\pi_{in}(d\theta_{in}),$$

for all $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}, B \in \mathcal{F}_{\Omega}.$

The marginal generalized observables

$$M_{\Upsilon}(B) := \Upsilon(B \times \Theta_{out}), \quad \forall B \in \mathcal{F}_{\Omega}, \\ S_{\Upsilon}(F_{out}) := \Upsilon(\Omega \times F_{out}), \quad \forall F_{out} \in \mathcal{F}_{\Theta_{out}}$$

define, for each $\mathcal{E} + \mathcal{S}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, the probability distribution

$$\mu_{\Upsilon}(B;\pi_{in}) := \mu_{M_{\Upsilon}}(B;\pi_{in}) = \int_{\Theta_{in}} (M_{\Upsilon}(B))(\theta_{in})\pi_{in}(d\theta_{in}), \quad \forall B \in \mathcal{F}_{\Omega},$$
(26)

of outcomes ω in $(\Omega, \mathcal{F}_{\Omega})$ and the unconditional probability distribution

$$\tau_{\Upsilon}(F_{out};\pi_{in}) := \tau_{S_{\Upsilon}}(F_{out};\pi_{in}) = \int_{\Theta_{in}} (S_{\Upsilon}(F_{out}))(\theta_{in})\pi_{in}(d\theta_{in}), \quad \forall F_{out} \in \mathcal{F}_{\Theta_{out}},$$
(27)

of posterior S-outcomes θ_{out} in $(\Theta_{out}, \mathcal{F}_{\Theta_{out}})$, in case where outcomes ω in $(\Omega, \mathcal{F}_{\Omega})$ are ignored completely.

For specificity, we further refer to the generalized observables Υ , M_{Υ} , S_{Υ} , as an *extended*, an *outcome* and a *system generalized observable* on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, respectively. Since, in our settings, an outcome generalized observable $M_{\Upsilon} \in \mathcal{G}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, representing \mathcal{E} , is non-trivial, the extended generalized observable Υ is also non-trivial, that is, $\Upsilon \in \mathcal{G}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$.

4.2 Complete information description

For a subset $B \in \mathcal{F}_{\Omega}$, $\mu_{\Upsilon}(B; \pi_{in}) \neq 0$, the conditional measure

$$\pi_{\Upsilon}^{out}(F_{out}|B;\pi_{in}) := \frac{\int_{\Theta_{in}} (\Upsilon(B \times F_{out}))(\theta_{in})\pi_{in}(d\theta_{in})}{\mu_{\Upsilon}(B;\pi_{in})}$$
(28)

defines the probability that, immediately after a single experimental trial $\mathcal{E} + \mathcal{S}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, where only the event that the outcome $\omega \in B$ has been recorded, the posterior \mathcal{S} -outcome θ_{out} belongs to a subset $F_{out} \in \mathcal{F}_{\Theta_{out}}$.

Hence, immediately after this single trial, the information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\Upsilon}(\cdot|B; \pi_{in})])$$

represents the conditional posterior information state¹¹ of S in the output information state space

 $(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\mathcal{V}_{(\Theta_{out}, \mathcal{F}_{\Theta_{out}})}]).$

 $^{^{11}}$ This terminology is introduced here in view of the similar terminology, used for the description of experiments upon quantum systems (see, for example, [17-19]).

The unconditional posterior information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi_{\Upsilon}^{out}(\cdot | \Omega; \pi_{in})]) = (\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\tau_{\Upsilon}(\cdot; \pi_{in})])$$

corresponds to the situation where outcomes in $(\Omega, \mathcal{F}_{\Omega})$ are ignored completely and only posterior \mathcal{S} -outcomes are considered.

Definition 11 Under the complete information description of a non-destructive experiment \mathcal{E} upon a system \mathcal{S} , we mean the knowledge, for each $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, of the outcome probability law $\mu_{\mathcal{E}}(\cdot; \pi_{in})$ and the family

$$\{(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi_{\Upsilon}^{out}(\cdot | B; \pi_{in})]) : B \in \mathcal{F}_{\Omega}\}$$

of all conditional posterior information states of S in $(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\mathcal{V}_{(\Theta_{out}, \mathcal{F}_{\Theta_{out}})}])$.

Denoting in (28)

$$((\mathcal{M}^{pr}_{\Upsilon}(B))(\pi_{in}))(F_{out}) := \int_{\Theta_{in}} (\Upsilon(B \times F_{out}))(\theta_{in})\pi_{in}(d\theta_{in}),$$

we derive

$$\pi_{\Upsilon}^{out}(\cdot|B;\pi_{in})) = \frac{((\mathcal{M}_{\Upsilon}^{pr}(B))(\pi_{in}))(\cdot)}{\mu_{\Upsilon}(B;\pi_{in})}$$

Definition 12 We call the mapping

$$(\mathcal{M}^{pr}_{\Upsilon}(\cdot))(\cdot):\mathcal{F}_{\Omega}\times\mathcal{J}_{(\Theta_{in},\mathcal{F}_{\Theta_{in}})}\to\mathcal{J}_{(\Theta_{out},\mathcal{F}_{\Theta_{out}})},$$

defined to an extended generalized observable Υ by the relation

$$((\mathcal{M}^{pr}_{\Upsilon}(B))(\nu_{in}))(F_{out}) := \int_{\Theta_{in}} (\Upsilon(B \times F_{out}))(\theta_{in})\nu_{in}(d\theta_{in}),$$
(29)

for all $B \in \mathcal{F}_{\Omega}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$, $\nu_{in} \in \mathcal{J}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, an information state instrument.

For an information state instrument, the mapping $\mathcal{M}^{pr}_{\Upsilon}(\cdot)$ is a σ -additive measure on $(\Omega, \mathcal{F}_{\Omega})$ with values $\mathcal{M}^{pr}_{\Upsilon}(B), B \in \mathcal{F}_{\Omega}$, that are positive bounded linear operators $\mathcal{J}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})} \to \mathcal{J}_{(\Theta_{out}, \mathcal{F}_{\Theta_{out}})}$ and, for each normalized measure $\nu_{in}(\Lambda) = 1$, the measure $(\mathcal{M}^{pr}_{\Upsilon}(\Omega))(\nu_{in})$ on $(\Theta_{out}, \mathcal{F}_{\Theta_{out}})$ is also normalized:

$$((\mathcal{M}^{pr}_{\Upsilon}(\Omega))(\nu_{in}))(\Theta_{out}) = 1$$

The concept of an information state instrument, defined by (29), coincides with the notion of an instrument, introduced in [6] in case where the latter is specified for the case of Kolmogorov's model [6, section 4].

From (26)-(29) it follows that, for each experimental situation $\mathcal{E} + \mathcal{S}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, the information state instrument $\mathcal{M}_{\Upsilon}^{pr}$ defines by the relations

$$\pi_{\Upsilon}^{out}(F_{out}|B;\pi_{in})) = \frac{((\mathcal{M}_{\Upsilon}^{pr}(B))(\pi_{in}))(F_{out})}{\mu_{\Upsilon}(B;\pi_{in})}, \quad \mu_{\Upsilon}(B;\pi_{in}) \neq 0,$$
(30)
$$\mu_{\Upsilon}(B;\pi_{in}) = ((\mathcal{M}_{\Upsilon}^{pr}(B))(\pi_{in}))(\Theta_{out}),$$

$$\tau_{\Upsilon}(F_{out};\pi_{in}) = ((\mathcal{M}_{\Upsilon}^{pr}(\Omega))(\pi_{in}))(F_{out}),$$

valid for all $F_{out} \in \mathcal{F}_{\Theta_{out}}, B \in \mathcal{F}_{\Omega}$, respectively, the probability distribution of outcomes in $(\Omega, \mathcal{F}_{\Omega})$, the conditional posterior information states and the unconditional probability distribution of posterior \mathcal{S} -outcomes in $(\Theta_{out}, \mathcal{F}_{\Theta_{out}})$.

Conclusion 2 For a non-destructive experiment \mathcal{E} upon \mathcal{S} , represented on an initial information state space of \mathcal{S} by a non-trivial extended generalized observable Υ , the complete information description is given by the notion of an information state instrument $\mathcal{M}^{pr}_{\Upsilon}$.

Example 2 Consider a non-destructive experiment \mathcal{E} with outcomes in $(\Omega, \mathcal{F}_{\Omega})$ upon \mathcal{S} , which is represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by the extended observable

$$(\mathcal{E}_{(\varphi,g)}(B \times F_{out}))(\theta_{in}) = \chi_{\varphi^{-1}(B) \cap g^{-1}(F_{out})}(\theta_{in}), \tag{31}$$

 $\forall B \in \mathcal{F}_{\Omega}, F_{out} \in \mathcal{F}_{\Theta_{out}}.$ Here functions $\varphi : \Theta_{in} \to \Omega$ and $g : \Theta_{in} \to \Theta_{out}$ are, respectively, $\mathcal{F}_{\Theta_{in}}/\mathcal{F}_{\Omega}$ and $\mathcal{F}_{\Theta_{in}}/\mathcal{F}_{\Theta_{out}}$ measurable.

To the observable (31), the information state instrument, defined, in general, by (29), has the form

$$((\mathcal{M}_{\mathcal{E}(\varphi,g)}^{pr}(B))(\pi_{in}))(F_{out}) = \pi_{in}(g^{-1}(F_{out}) \cap \varphi^{-1}(B)),$$
(32)

for all $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$, $B \in \mathcal{F}_{\Omega}$. From (30) it follows that, for each experimental situation $\mathcal{E} + \mathcal{S}(\pi_{in})$, $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, the probability distribution of outcomes is given by the image probability distribution

$$\mu_{\mathcal{E}_{(\varphi,g)}}(B;\pi_{in}) = \pi_{in}(\varphi^{-1}(B)), \quad \forall B \in \mathcal{F}_{\Omega},$$
(33)

while, for each $B \in \mathcal{F}_{\Omega}$, $\pi_{in}(\varphi^{-1}(B)) \neq 0$, the conditional posterior information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\mathrm{E}_{(\varphi,g)}}(\cdot|B;\pi_{in})])$$

is represented by

$$\pi_{\mathcal{E}_{(\varphi,g)}}^{out}(F|B;\pi_{in}) = \frac{\pi_{in}(\varphi^{-1}(B) \cap g^{-1}(F_{out}))}{\pi_{in}(\varphi^{-1}(B))}, \quad \forall F_{out} \in \mathcal{F}_{\Theta_{out}}.$$
(34)

4.2.1**Reduction of an information state**

In the most general settings, for any $B \in \mathcal{F}_{\Omega}$, the conditional change

$$(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\pi_{in}]) \mapsto (\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\Upsilon}(\cdot | B; \pi_{in})])$$

described, due to (30), by the notion of an information state instrument $\mathcal{M}^{pr}_{\Upsilon}$, represents the phenomenon of "reduction" of an initial information state.

From our presentation it follows that this phenomenon is inherent, in general, to any nondestructive experiment and upon a system \mathcal{S} of any type, described in terms of information state spaces.

As we discuss below in section 4.3, in the most general settings, a reduction of an information state is induced by:

(i) the "renormalization" of information on a system, conditioned upon the recorded event under a single experimental trial;

(ii) the "dynamical" change of an information state of a system in the course of a perturbing experiment.

Since, as we establish in section 6, the probabilistic model of quantum theory represents a special model of our general framework, the well-known von Neumann quantum "state collapse", postulated in [1], and its further generalizations (see in [3,13,17,18]), represent particular cases of this general phenomenon.

4.3Non-perturbing experiments

Consider a non-destructive experiment \mathcal{E} upon a system \mathcal{S} , represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by a nontrivial extended generalized observable Υ .

Introduce the following concept.

Definition 13 We call a non-trivial extended generalized observable Υ on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ non-perturbing and, for specificity, denote it by $\Upsilon^{(np)}$ if there exist:

(i) a measurable space $(\Theta, \mathcal{F}_{\Theta})$; (ii) an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta_{in}}$ -measurable function $\phi_{in} : \Theta \to \Theta_{in}$; (iii) an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta_{out}}$ -measurable function $\phi_{out} : \Theta \to \Theta_{out}$; such that the ϕ_{in} -preimage $\Upsilon_{\phi_{in}^{-1}}^{(np)}$ on $(\Theta, \mathcal{F}_{\Theta})$ has the form:

$$(\Upsilon^{(np)}_{\phi^{-1}_{in}}(B \times F_{out}))(\theta) = (\mathcal{M}(B))(\theta)\chi_{\phi^{-1}_{out}(F_{out})}(\theta),$$
(35)

for all $\theta \in \Theta$, $B \in \mathcal{F}_{\Omega}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$, where $M : \mathcal{F}_{\Omega} \to \mathfrak{B}_{+}(\Theta)$, $M(\Omega) = \mathcal{I}_{\mathfrak{B}(\Theta)}$, is an outcome generalized observable on $(\Theta, \mathcal{F}_{\Theta})$.

To see why we call $\Upsilon^{(np)}$ non-perturbing, consider an experiment \mathcal{E} upon \mathcal{S} , represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by a non-perturbing generalized observable $\Upsilon^{(np)}$.

Let $\mathcal{E} + \mathcal{S}(\pi_{in})$ be an experimental situation where the initial information state $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\pi_{in}])$ is the ϕ_{in} -image of an information state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$, that is¹²:

$$(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\pi_{in}]) = \phi_{in}[(\Theta, \mathcal{F}_{\Theta}, [\pi])],$$
$$\pi_{in} = \pi \circ \phi_{in}^{-1}.$$

From (26) it follows that, for any $\mathcal{E} + \mathcal{S}(\pi \circ \phi_{in}^{-1}), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, the probability distribution of outcomes is given by

$$\mu_{\Upsilon^{(np)}}(B;\pi_{in}) = \int_{\Theta_{in}} (\Upsilon^{(np)}(B \times \Theta_{out}))(\theta_{in})\pi_{in}(d\theta_{in})$$

$$= \int_{\Theta} (\Upsilon^{(np)}_{\phi_{in}^{-1}}(B \times \Theta_{out}))(\theta)\pi(d\theta)$$

$$= \int_{\Theta} \mathcal{M}(B)(\theta)\pi(d\theta) = \mu_{\mathcal{M}}(B;\pi), \qquad \forall B \in \mathcal{F}_{\Omega},$$
(36)

while, for each $B \in \mathcal{F}_{\Omega}$, $\mu_{\Upsilon^{(np)}}(B;\pi_{in}) \neq 0$, the conditional posterior information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\Upsilon^{(np)}}(\cdot|B; \pi \circ \phi_{in}^{-1})])$$

is represented by

$$\pi_{\Upsilon^{(np)}}^{out}(F_{out}|B;\pi\circ\phi_{in}^{-1}) = \frac{\int_{\Theta_{in}}(\Upsilon^{(np)}(B\times F_{out}))(\theta_{in})(\pi\circ\phi_{in}^{-1})(d\theta_{in})}{\mu_{\Upsilon^{(np)}}(B;\pi_{in})}$$

$$= \frac{\int_{\phi_{out}^{-1}(F_{out})}(\mathcal{M}(B))(\theta)\pi(d\theta)}{\int_{\Theta}(\mathcal{M}(B))(\theta)\pi(d\theta)}, \quad \forall F_{out} \in \mathcal{F}_{\Theta_{out}}.$$

$$(37)$$

From (37) it follows that, for any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}, B \in \mathcal{F}_{\Omega}, \mu_{M}(B; \pi) \neq 0$, we have

$$\pi^{out}_{\Upsilon^{(np)}}(F_{out}|B;\pi\circ\phi_{in}^{-1}) = \pi^{out}_{\Upsilon^{(np)}_{in}}(\phi^{-1}_{out}(F_{out})|B;\pi), \qquad \forall F_{out}\in\mathcal{F}_{\Theta_{out}},\tag{38}$$

where

$$\pi^{out}_{\Upsilon^{(np)}_{\phi_{in}^{-1}}}(F|B;\pi) := \frac{\int_F(\mathcal{M}(B))(\theta)\pi(d\theta)}{\int_{\Theta}(\mathcal{M}(B))(\theta)\pi(d\theta)}, \quad \forall F \in \mathcal{F}_{\Theta}.$$

Hence, for any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}, B \in \mathcal{F}_{\Omega}, \mu_{\mathrm{M}}(B; \pi) \neq 0$, the conditional posterior information state is given by

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\Upsilon^{(np)}}(\cdot|B; \pi \circ \phi_{in}^{-1})]) = \phi_{out}[(\Theta, \mathcal{F}_{\Theta}, [\pi^{out}_{\Upsilon^{(np)}_{\phi_{in}^{-1}}}(\cdot|B; \pi)])].$$
(39)

 $^{^{12}\}mathrm{See}$ section 3.2.1.

In particular, the unconditional posterior information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\Upsilon^{(np)}}(\cdot | \Omega; \pi \circ \phi_{in}^{-1})]) = \phi_{out}[(\Theta, \mathcal{F}_{\Theta}, [\pi])]$$

is the ϕ_{out} -image of the initial ϕ_{in} -preimage state.

Let an initial and an output information state spaces coincide:

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\mathcal{V}_{(\Theta_{out}, \mathcal{F}_{\Theta_{out}})}]) = (\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}]),$$

then $\phi_{in} = \phi_{out} = \phi$ and we derive

$$\pi_{\Upsilon^{(np)}}^{out}(F_{out}|\Omega;\pi_{in})]) = \pi_{in}(F_{out}), \quad \forall F_{out} \in \mathcal{F}_{\Theta_{in}}.$$
(40)

that is, in this case, the unconditional posterior information state coincides with the initial one.

Suppose that \mathcal{F}_{Θ} contains all atom subsets of Θ and let an initial information state of \mathcal{S} in $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}}}])$ be an ϕ_{in} -image of the pure information state

$$(\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}]), \tag{41}$$

with an arbitrary $\theta_0 \in \Theta$. Then from (37) it follows that, for any $B \in \mathcal{F}_{\Omega}$, $\mu_M(B; \pi) \neq 0$,

$$\pi^{out}_{\Upsilon^{(np)}}(F_{out}|B;\delta_{\theta_0}\circ\phi_{in}^{-1}) = \chi_{\phi^{-1}_{out}(F_{out})}(\theta_0), \quad \forall F_{out}\in\mathcal{F}_{\Theta_{out}}.$$
(42)

This relation implies that, under any experimental situation $\mathcal{E}_{\Upsilon(np)} + \mathcal{S}(\delta_{\theta_0} \circ \phi_{in}^{-1}), \ \theta_0 \in \Theta$, a conditional posterior information state, following a single experimental trial, does not depend on an event $B \in \mathcal{F}_{\Omega}$, recorded under this trial, and represents the ϕ_{out} -image of the initial preimage pure information state (41) in $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$.

Conclusion 3 Under any single experimental trial of an experiment, represented by a non-perturbing generalized observable, a preimage pure information state on $(\Theta, \mathcal{F}_{\Theta})$ is not perturbed. We call this experiment non-perturbing.

Notice that a non-perturbing experiment may have, in general, a probabilistic set-up and represent, for example, a classical measurement with errors [10].

4.3.1 Beables

Analyze now the special case of a non-perturbing generalized observable $\widetilde{\Upsilon}^{(np)}$ on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, for which the ϕ_{in} -preimage (35) has the special form:

$$(\widetilde{\Upsilon}^{(np)}_{\phi^{-1}_{in}}(B \times F_{out}))(\theta) = \chi_{\varphi^{-1}(B) \cap \phi^{-1}_{out}(F_{out})}(\theta),$$
(43)

for all $\theta \in \Theta$, $B \in \mathcal{F}_{\Omega}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$. Here, in addition to the specifications, introduced in definition 13, a function $\varphi : \Theta \to \Omega$ is $\mathcal{F}_{\Theta}/\mathcal{F}_{\Omega}$ -measurable.

From definition 7 and proposition 1 it follows that $\widetilde{\Upsilon}^{(np)}$ is an observable on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ and, hence, describes an experiment with a deterministic set-up.

Due to (17), (29), for each experimental situation

$$\mathcal{E}_{\widetilde{\Upsilon}^{(np)}} + \mathcal{S}(\pi \circ \phi_{in}^{-1}), \quad \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$$

the value $(\mathcal{M}^{pr}_{\tilde{\boldsymbol{\gamma}}^{(np)}}(\cdot))(\pi \circ \phi_{in}^{-1})$ of the information state instrument is given by

$$((\mathcal{M}^{pr}_{\widetilde{\Upsilon}^{(np)}}(B))(\pi \circ \phi_{in}^{-1}))(F_{out}) = \pi(\varphi^{-1}(B) \cap \phi_{out}^{-1}(F_{out})),$$

for all $B \in \mathcal{F}_{\Omega}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$.

From (30), as well as from (36), (37), it follows that the outcome probability law of each experimental situation $\mathcal{E}_{\tilde{\Upsilon}^{(np)}} + \mathcal{S}(\pi \circ \phi_{in}^{-1}), \pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, is given by the *image probability distribution*

$$\mu_{\widetilde{\Upsilon}^{(n_p)}}(B;\pi) = \pi(\varphi^{-1}(B)), \quad \forall B \in \mathcal{F}_{\Omega},$$
(44)

while, for each $B \in \mathcal{F}_{\Omega}$, $\pi(\varphi^{-1}(B)) \neq 0$, the conditional posterior state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\tilde{\Upsilon}^{(np)}}(\cdot | B; \pi \circ \phi_{in}^{-1})])$$

is represented by

$$\pi^{out}_{\widetilde{\Upsilon}(np)}(F_{out}|B;\pi\circ\phi_{in}^{-1}) = \frac{\pi(\varphi^{-1}(B)\cap\phi_{out}^{-1}(F_{out}))}{\pi(\varphi^{-1}(B))}, \quad \forall F_{out}\in\mathcal{F}_{\Theta_{out}}.$$
(45)

Suppose that all atom subsets $\{\omega\}$ of Ω belong to \mathcal{F}_{Ω} .

Let an initial information state be an ϕ_{in} -image of a pure information state $(\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}])$. In this case, the relations (44) and (45) imply that, for any experimental situation

$$\mathcal{E}_{\widetilde{\Upsilon}(np)} + \mathcal{S}(\delta_{\theta_0} \circ \phi_{in}^{-1}), \quad \theta_0 \in \Theta,$$

the outcome $\omega_0 = \varphi(\theta_0)$ is predicted with certainty while the corresponding conditional posterior information state

$$(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\pi^{out}_{\widetilde{\Upsilon}(np)}(\cdot|B; \delta_{\theta_0} \circ \phi_{in}^{-1})] = \phi_{out}[(\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}])], \quad B \ni \omega_0, \quad \forall B \in \mathcal{F}_{\Omega},$$

is the ϕ_{out} -image of an initial ϕ_{in} -preimage state $(\Theta, \mathcal{F}_{\Theta}, [\delta_{\theta_0}])$. Summing up, we introduce the following definition.

Definition 14 (Beable) We call an extended observable on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, with an outcome space $(\Omega, \mathcal{F}_{\Omega})$, a **beable** and, for specificity, denote it by $E^{(be)}$ if there exist:

(i) a measurable space $(\Theta, \mathcal{F}_{\Theta})$;

(ii) an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta_{in}}$ -measurable function $\phi_{in}: \Theta \to \Theta_{in};$ (iii) an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta_{out}}$ -measurable function $\phi_{out}: \Theta \to \Theta_{out};$ (iiii) an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Omega}$ measurable function $\varphi: \Theta \to \Omega;$ such that $(\Sigma^{(be)}(B \times E_{e^{-}}))(0)$

$$(\mathbf{E}_{\phi_{in}^{-1}}^{(be)}(B \times F_{out}))(\theta) = \chi_{\varphi^{-1}(B) \cap \phi_{out}^{-1}(F_{out})}(\theta), \tag{46}$$

for all $\theta \in \Theta$, $B \in \mathcal{F}_{\Omega}$, $F_{out} \in \mathcal{F}_{\Theta_{out}}$.

From this definition it follows that, for the outcome beable $M^{(be)}(B) := E^{(be)}(B \times \Theta_{out})$, the ϕ_{in} -preimage on $(\Theta, \mathcal{F}_{\Theta})$ is given by

$$(\mathbf{M}_{\phi_{in}^{-1}}^{(be)}(B))(\theta) = \chi_{\varphi^{-1}(B)}(\theta), \quad \forall B \in \mathcal{F}_{\Omega}, \quad \forall \theta \in \Theta.$$

A beable describes a non-perturbing experiment, with a deterministic set-up, on an "errorless" classical measurement¹³ of a property φ of S on $(\Theta, \mathcal{F}_{\Theta})$, existing objectively before this measurement.

Under an experiment, represented by a beable, the randomness is caused only by to the uncertainty, encoded in a preimage initial probability distribution $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$.

From the above definition and (45) it follows that, under an experiment, represented by a beable $E^{(be)}$, a ϕ_{in} -preimage conditional posterior information state in $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is given by

$$\pi_{\mathbf{E}_{\phi_{nit}}^{(be)}}^{out}(F_{out}|B;\pi_{in}) = \frac{\pi(\varphi^{-1}(B) \cap F_{out})}{\pi(\varphi^{-1}(B))},\tag{47}$$

¹³In the quantum case the term "measurement" is understood in a broader sense and means any experiment upon a quantum system, admitting the probabilistic description.

for any $\pi \in \mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}$, and, hence, is, in general, different from an initial preimage state $(\Theta, \mathcal{F}_{\Theta}, [\pi])$.

In the frame of Kolmogorov's model, only experiments, represented by beables, are considered. Example 3 and the formulae (34), (45), (47) indicate the main general reasons of the phenomenon of "reduction" of an initial information state, discussed in 4.2.1.

Specifically, due to (47), the reduction of a preimage mixed information state is inherent even to a classical "errorless" measurement and represents the "renormalization" of the initial information on S, conditional on the recorded event under a single experimental trial.

The formula (34) indicates that, in general, a "reduction" of an information state may be caused not only by the renormalization of our information on S, acquired during an experiment, but also by the "dynamical" change of a system state in case where, with respect to a system, an experiment is perturbing.

4.4 Equivalence classes of experiments

We call experiments upon a system S, represented initially by $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}])$, statistically equivalent and denote by [M] the corresponding equivalence class, if these experiments are represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by the same outcome generalized observable M of S. For all experiments in [M], the outcome probability laws on $(\Omega, \mathcal{F}_{\Omega})$ are identical. In general, an outcome generalized observable M may, however, correspond to different extended generalized observables Υ on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$.

Under all non-destructive experiments upon S, represented by the same system generalized observable S on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, the unconditional posterior information on S is represented by the same unconditional posterior information state $(\Theta_{out}, \mathcal{F}_{\Theta_{out}}, [\tau(\cdot; \pi_{in})])$. We denote the corresponding equivalence class of such experiments by [S].

We say that non-destructive experiments upon S are completely information equivalent if they are represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by the same extended generalized observable Υ . We denote the corresponding equivalence class by $[\Upsilon]$. For all experiments in $[\Upsilon]$, the outcome probability laws on $(\Omega, \mathcal{F}_{\Omega})$ and the families of conditional posterior information states are identical.

Clearly, $[\Upsilon] \subseteq [S_{\Upsilon}] \cap [M_{\Upsilon}].$

4.5 Complete statistical description

In this section we consider the case where an initial and an output information state spaces satisfy the condition (14). For specificity, we denote this system by S_V .

Let \mathcal{E} be a non-destructive experiment upon \mathcal{S}_{V} , represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by a non-trivial extended generalized Υ .

Denote by V_{in} , V_{out} the corresponding initial and output Banach spaces in (14) and by \mathfrak{R}_{mean}^{in} and $\mathfrak{R}_{mean}^{out}$ the corresponding initial and output mean information state spaces.

Due to definition 4, to an initial information state $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\pi_{in}])$, the *initial mean informa*tion state is defined by

$$\eta_{in}(\pi_{in}) := \eta_{mean}(\pi_{in}) = \int_{\Theta_{in}} \theta_{in} \pi_{in}(d\theta_{in}) \in \mathfrak{R}_{mean}^{in}.$$
(48)

Immediately after a single trial of an experimental situation $\mathcal{E} + \mathcal{S}_{\mathrm{V}}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})},$ where an event $\omega \in B \in \mathcal{F}_{\Omega}$ has been recorded, the output mean information state

$$\eta^{out}(B;\pi_{in}) := \eta_{mean}(\pi^{out}_{\Upsilon}(\cdot|B;\pi_{in})) = \int_{\Theta_{out}} \theta_{out}\pi^{out}_{\Upsilon}(d\theta_{out}|B;\pi_{in}) \in \mathfrak{R}^{out}_{mean}$$
(49)

represents the *conditional statistical average of posterior* S-outcomes, defined by the conditional probability distribution (28).

Definition 15 We call a conditional statistical average of posterior S-outcomes (49) a conditional posterior mean information state of S_V following a single experimental trial where an event $\omega \in B \in \mathcal{F}_{\Omega}$ has been recorded. Due to (30) and (49), for each $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$ and each $B \in \mathcal{F}_{\Omega}$, $\mu_{\Upsilon}(B, \pi_{in}) \neq 0$, the conditional posterior mean information state is defined by

$$\eta^{out}(B;\pi_{in}) = \frac{\int_{\Theta_{out}} \theta_{out}((\mathcal{M}^{pr}_{\Upsilon}(B))(\pi_{in}))(d\theta_{out})}{\mu_{\Upsilon}(B;\pi_{in})}$$
(50)

via the notion of the information state instrument $\mathcal{M}^{pr}_{\Upsilon}$, corresponding to this experiment.

Definition 16 Under the complete statistical description of a non-destructive experiment \mathcal{E} upon a system \mathcal{S}_{V} , we mean the knowledge, for each initial information state $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\pi_{in}])$, of the outcome probability law $\mu(\cdot; \pi_{in})$ and the family $\{\eta^{out}(B; \pi_{in}) : B \in \mathcal{F}_{\Omega}\}$ of all conditional posterior mean information states.

To different initial information states, represented by measures $\pi_{in} \neq \pi'_{in}$, but inducing the same initial mean information states $\eta_{in}(\pi_{in}) = \eta'_{in}(\pi'_{in})$, the conditional posterior mean information states, conditioned by the same recorded event $B \in \mathcal{F}_{\Omega}$, are, in general, different.

In general, for a non-destructive experiment upon $S_{\rm V}$, the complete statistical description is less informative than the complete information description. The latter is due to the fact that the knowledge of only the initial mean information state $\eta_{in}(\pi_{in})$ does not allow to make predictions upon all experiments on $S_{\rm V}$, described by generalized observables in $\mathcal{G}_{(\Theta_{in},\mathcal{F}_{\Theta_{in}})}$.

4.5.1 Mean information state instrument

Recall that any complex valued function on a linear space is called a functional. Denote by V_{in}^* the Banach space of all continuous linear functionals on V_{in} . Suppose that on V_{in} there exists a *continuous linear* functional $\mathfrak{I}_{in} \in V_{in}^*$ such that

$$\mathfrak{I}_{in}\{\theta_{in}\} = \mathcal{I}(\theta_{in}) = 1, \qquad \forall \theta_{in} \in \Theta_{in} \subset \mathcal{V}_{in}.$$
(51)

In this case, any σ -additive¹⁴ measure

$$\Pi: \mathcal{F}_{\Lambda} \to \mathcal{V}_{in}^* \tag{52}$$

on $(\Lambda, \mathcal{F}_{\Lambda})$, satisfying the relations

$$\Pi(\Lambda) = \mathfrak{I}_{in}, \quad (\Pi(D))(\eta_{in}) \ge 0, \qquad \forall \eta_{in} \in \mathfrak{R}_{mean}^{in}, \quad \forall D \in \mathcal{F}_{\Lambda},$$
(53)

defines a generalized observable (extended, or outcome, or system) of a system $S_{\rm V}$.

For short, we refer to this generalized observable as a *linear* generalized observable and denote it by Π_{lin} .

The information state instrument, corresponding, due to (29), to an extended *linear* generalized observable Υ_{lin} , has the form

$$((\mathcal{M}^{pr}_{\Upsilon_{lin}}(B))(\pi_{in}))(F) = \int_{\Theta_{in}} (\Upsilon_{lin}(B \times F))(\theta_{in})\pi_{in}(d\theta_{in}) = (\Upsilon_{lin}(B \times F))(\eta_{in}),$$

for all $\pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, $B \in \mathcal{F}_{\Omega}$, $F \in \mathcal{F}_{\Theta_{out}}$, and, hence, depends only on the initial mean information state $\eta_{in} = \eta_{mean}(\pi_{in})$ but not on an initial information state, represented by π_{in} .

From (30), (50) it follows that, to each experimental situation $\mathcal{E}_{\Upsilon_{lin}} + \mathcal{S}(\pi_{in}), \pi_{in} \in \mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}$, the probability distribution of outcomes in $(\Omega, \mathcal{F}_{\Omega})$

$$\mu_{\Upsilon_{lin}}(B;\pi_{in}) = (\Upsilon_{lin}(B \times \Theta_{out}))(\eta_{in}) = (\mathcal{M}_{\Upsilon_{lin}}(B))(\eta_{in}) := \widetilde{\mu}_{\Upsilon_{lin}}(B,\eta_{in}), \quad \forall B \in \mathcal{F}_{\Omega}$$

and the conditional posterior mean information states

$$\eta^{out}(B;\pi_{in}) = \frac{\int_{\Theta_{out}} \theta_{out}(\Upsilon_{lin}(B \times d\theta_{out}))(\eta_{in})}{\mu_{\Upsilon_{lin}}(B,\eta_{in})} := \tilde{\eta}^{out}(B;\eta_{in}), \quad \forall B \in \mathcal{F}_{\Omega}$$
(54)

¹⁴Here the convergence in the σ -additivity condition is in the strong operator topology in V_{in}^* .

depend only on the initial mean information state η_{in} .

Denote in (54)

$$(\mathcal{M}_{\Upsilon_{lin}}^{st}(B))(\eta_{in}) := \int_{\Theta_{out}} \theta_{out}(\Upsilon_{lin}(B \times d\theta_{out}))(\eta_{in}) \in \mathcal{V}_{out},$$
(55)

 $\forall \eta_{in} \in \mathfrak{R}_{mean}^{in}, \forall B \in \mathcal{F}_{\Omega}.$

Definition 17 We call the mapping $(\mathcal{M}^{st}_{\Upsilon_{lin}}(\cdot))(\cdot) : \mathcal{F}_{\Omega} \times \mathfrak{R}^{in}_{mean} \to V_{out}$, defined by an extended linear generalized observable Υ_{lin} through the relation

$$(\mathcal{M}_{\Upsilon_{lin}}^{st}(B))(\eta_{in}) := \int_{\Theta_{out}} \theta_{out}(\Upsilon_{lin}(B \times d\theta_{out}))(\eta_{in}), \tag{56}$$

for all $B \in \mathcal{F}_{\Omega}$, $\eta_{in} \in \mathfrak{R}_{mean}^{in}$, a mean information state instrument.

For each $\eta_{in} \in \mathfrak{R}_{mean}^{in}$, the mapping $(\mathcal{M}_{\mathfrak{r}_{lin}}^{(st)}(\cdot))(\eta_{in})$ is a σ -additive measure on $(\Omega, \mathcal{F}_{\Omega})$ with values in V_{out} .

Remark 4 To a linear extended generalized observable Υ_{lin} of S_V , there corresponds a unique mean information state instrument though not vice versa.

Remark 5 Notice that the notion of a mean information state instrument appears: (i) only for a system S_V , with the definite type (14) of measurable spaces $(\Theta_{in}, \mathcal{F}_{in})$ and $(\Theta_{out}, \mathcal{F}_{out})$; (ii) only in case where on a Banach space V_{in} there exists a continuous linear functional, satisfying (51);

(iii) only to a non-destructive experiment \mathcal{E} upon \mathcal{S}_{V} , represented by a linear generalized observable Υ_{lin} .

From (54) and (56) it follows that, under an non-destructive experiment $\mathcal{E} \in [\Upsilon_{lin}]$, to any initial mean information state η_{in} of \mathcal{S} and any $B \in \mathcal{F}_{\Omega}, \tilde{\mu}_{\Upsilon_{lin}}(B; \eta_{in}) \neq 0$, the conditional posterior mean information state depends only on η_{in} and is given by

$$\widetilde{\eta}^{out}(B;\eta_{in}) = \frac{(\mathcal{M}_{\Upsilon_{lin}}^{st}(B))(\eta_{in})}{\widetilde{\mu}_{\Upsilon_{lin}}(B;\eta_{in})}.$$
(57)

In the most general settings, the conditional change

$$\eta_{in} \mapsto \widetilde{\eta}^{out}(B;\eta_{in})$$

represents the phenomenon of "reduction" of a mean information state, induced by the reduction of an information state of $S_{\rm V}$ (see section 4.2.1).

If, further, on the Banach space V_{out} there also exists a *continuous linear* functional \mathfrak{J}_{out} : $V_{out} \to \mathbb{C}$ such that

$$\mathfrak{I}_{out}\{\eta_{out}\} = 1, \quad \forall \eta_{out} \in \mathfrak{R}_{mean}^{out},$$

then

$$\widetilde{\mu}_{\Upsilon_{lin}}(B;\eta_{in}) = \mathfrak{J}_{out}\{(\mathcal{M}_{\Upsilon_{lin}}^{st}(B))(\eta_{in})\}, \quad \forall \eta_{in} \in \mathfrak{R}_{mean}^{out}, \quad \forall B \in \mathcal{F}_{\Omega},$$
(58)

and in this case the mean information state instrument $\mathcal{M}_{\Upsilon_{lin}}^{st}$ gives the complete statistical description of the corresponding equivalence class $[\Upsilon_{lin}]$ of experiments upon \mathcal{S}_{V} although not the complete information description¹⁵.

 $^{^{15}}$ Under the description of quantum measurements, this point was first indicated in [17], where we, in particular, introduced the concept of stochastic realizations of a quantum instrument. Different equivalence classes of stochastic realizations of the same quantum instrument represent different experiments.

5 Probabilistic and statistical models

As we discussed in section 4.3.1, in Kolmogorov's model only the description of non-perturbing experiments is considered and the random character of predictions is caused only by the uncertainty, encoded in a probability distribution π_{in} . The uncertainty, encoded in elements of a set Θ_{in} , as well as the description of perturbing experiments, are not analyzed.

Definition 18 We call a pair $(\mathcal{U}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}}^{ext})$ where:

(i) \mathcal{U}_S is a specified family of initial and output information state spaces of S; (ii) \mathcal{G}_S^{ext} is a specified family of non-trivial extended generalized observables on initial information state spaces in \mathcal{U}_S ;

a probabilistic model for the description of non-destructive experiments upon S.

A probabilistic model gives the complete information description of all non-destructive experiments upon S, represented by extended generalized observable in \mathcal{G}_{S}^{ext} .

Definition 19 We call a pair $(\mathcal{U}_S, \mathcal{G}_s^{outcome})$ where:

(i) U_S is a specified family of initial information state spaces of S; (ii) $\mathcal{G}_S^{outcome}$ is a specified family of non-trivial outcome generalized observables on initial information state spaces in U_S ;

a statistical model for the description of experiments upon S.

In case of a non-destructive experiment, the concept of a statistical model is less informative and gives predictions only on outcome probability laws. We call this type of description of an experiment *statistical*.

In natural sciences, for a system of a concrete type, the specification of a family $\mathcal{U}_{\mathcal{S}}$ of information state spaces and a family $\mathcal{G}_{\mathcal{S}}$ of allowed non-trivial generalized observables on these information state spaces must be based on the fundamental laws, describing this concrete type of a system.

As the following proposition shows, in general, there is a correspondence between families $\mathcal{U}_{\mathcal{S}}$ and $\mathcal{G}_{\mathcal{S}}$.

Consider a system $\widetilde{\mathcal{S}}_{V}$, specified by the following conditions:

(i) all possible information state spaces in a family $\mathcal{U}_{\tilde{S}_{V}}$ satisfy the condition (14);

(ii) on each Banach space V in (14), corresponding to an information state space in $\mathcal{U}_{\tilde{S}_{V}}$, there exists a continuous linear functional $\mathfrak{J}_{V}\{\cdot\}$;

(iii) for each Banach space V in (14), a mean information state space \Re_{mean} is such that any element $u \in V$ admits a representation $u = \beta_1 \eta_1 - \beta_2 \eta_2$, with $||u||_V = \inf(\beta_1 + \beta_2)$, where $\eta_1, \eta_2 \in \Re_{mean}, \beta_1, \beta_2 \geq 0$. The latter relation implies, in particular, that, for this system, in (14), $C \geq 1$.

Proposition 4 If, under a non-destructive experiment \mathcal{E} upon $\widetilde{\mathcal{S}}_{V}$, for each $\pi_{in} \in \mathcal{V}_{(\Theta_{in},\mathcal{F}_{\Theta_{in}})}$, a probability distribution $\gamma(\cdot; \pi_{in})$ (extended, or outcome, or system) depends only on the corresponding initial mean state $\eta_{in}(\pi_{in})$, but not on π_{in} , that is, $\gamma(\cdot; \pi_{in}) = \widetilde{\gamma}(\cdot; \eta_{in})$, for each $\pi_{in} \in \mathcal{V}_{(\Theta_{in},\mathcal{F}_{\Theta_{in}})}$, $\eta_{in}(\pi_{in}) = \eta_{in}$, then

$$\widetilde{\gamma}(\cdot;\alpha_1\eta_{in}^{(1)}+\alpha_2\eta_{in}^{(2)})=\alpha_1\widetilde{\gamma}(\cdot;\eta_{in}^{(1)})+\alpha_2\widetilde{\gamma}(\cdot;\eta_{in}^{(2)}),$$

for any $\eta_{in}^{(1)}, \eta_{in}^{(2)} \in \mathfrak{R}_{mean}^{in}, \alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$, and this situation is possible iff \mathcal{E} is represented on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$ by a linear generalized observable Π_{lin} and

$$\widetilde{\gamma}(\cdot;\eta_{in}) = (\Pi_{lin}(\cdot))(\eta_{in}), \quad \forall \eta_{in} \in \mathfrak{R}_{mean}^{in}$$

Remark 6 If, under all really performed experiments upon a concrete system \tilde{S}_V , probability distributions of outcomes depend only on initial mean information states, then, due to this proposition, for the description of experiments upon \tilde{S}_V , only linear generalized observables Π_{lin} (extended, or outcome, or system) of \tilde{S}_V are allowed.

Let, for a system $\widetilde{\mathcal{S}}_{V}$, only linear generalized observables be allowed and

$$(\mathcal{U}_{\widetilde{\mathcal{S}}_{\mathrm{V}}}, \mathcal{G}^{ext\&lin}_{\widetilde{\mathcal{S}}_{\mathrm{V}}})$$

be a probabilistic model for the description of non-destructive experiments upon a system \mathcal{S}_{V} .

If we are interested only in the *complete statistical description* (section 4.5), we can equivalently replace $\mathcal{U}_{\widetilde{S}_{V}}$ by the family $\mathfrak{R}_{mean}(\mathcal{U}_{\widetilde{S}_{V}})$ of all mean information state spaces, corresponding to the information state spaces in $\mathcal{U}_{\widetilde{S}_{V}}$. We call the pair

$$(\mathfrak{R}_{mean}(\mathcal{U}_{\widetilde{\mathcal{S}}_{\mathcal{V}}}), \mathcal{G}_{\widetilde{\mathcal{S}}_{\mathcal{V}}}^{ext\&lin}),$$

the reduced probabilistic model.

For the reduced probabilistic model, the complete statistical description is represented by the notion of a mean information state instrument, introduced, in the most general settings, in section 4.5.1.

Further, the statistical description of experiments upon a system $\hat{\mathcal{S}}_{V}$, in the frame of the statistical model $(\mathcal{U}_{\tilde{\mathcal{S}}_{V}}, \mathcal{G}_{\tilde{\mathcal{S}}_{V}}^{outcome\&lin})$ coincides with the statistical description in the frame of the corresponding reduced statistical model.

5.1 Reducible models

Let $(\mathcal{U}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}})$ and $(\widetilde{\mathcal{U}}_{\mathcal{S}}, \widetilde{\mathcal{G}}_{\mathcal{S}})$ be two different (probabilistic, or statistical) models for the description of experiments upon a system $\widetilde{\mathcal{S}}$.

We say that a model $(\mathcal{U}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}})$ is reducible to a model $(\mathcal{U}_{\mathcal{S}}, \mathcal{G}_{\mathcal{S}})$ if:

(i) any information state space in $\widetilde{\mathcal{U}}_{\mathcal{S}}$ is induced by an information state space in $\mathcal{U}_{\mathcal{S}}$;

(ii) any generalized observable in $\widetilde{\mathcal{G}}_{\mathcal{S}}$ and any system generalized observable in $\mathcal{G}_{\mathcal{S}}$, generate by (22) a generalized observable in $\mathcal{G}_{\mathcal{S}}$.

5.2 Kolmogorov's model

From our presentation in section 4.3 it follows that, in our framework, Kolmogorov's probabilistic model for the description of experiments upon S can be specified as the pair

$$(\mathcal{U}_{\mathcal{S}},\mathcal{G}^{be}_{\mathcal{S}}),$$

for which there exists an information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ such that all observables in $\mathcal{G}_{\mathcal{S}}^{be}$ are beables with the corresponding preimage observables (46), defined on $(\Theta, \mathcal{F}_{\Theta})$.

In conventional probability theory, the existence of an "underlying" information state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ is postulated.

In case of Kolmogorov's model, we call $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}])$ as a probability state space and any information state in this space as a probability state.

Kolmogorov's statistical model can be specified as the pair $(\mathcal{U}_{\mathcal{S}}, \mathcal{G}^{outcome\&be}_{\mathcal{S}})$, with $\mathcal{G}^{outcome\&be}_{\mathcal{S}}$ representing the family of outcome beables with preimages on $(\Theta, \mathcal{F}_{\Theta})$.

6 Generalized quantum measurements

In quantum theory, a system is described in terms of a separable complex Hilbert space \mathcal{H} , in general, infinite dimensional.

Denote by $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$, the Banach spaces of bounded linear operators and trace class operators on \mathcal{H} , respectively. Let $\mathcal{L}^{(+)}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ and $\mathcal{T}^{(+)}(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ be the sets of all non-negative operators in the corresponding Banach spaces.

As we mentioned, in the most general settings in section 5, for the description of non-destructive experiments upon a system, we distinguish between a statistical model and a probabilistic model.

With respect to a system, a probabilistic model is more detailed and includes the specification of a system conditional posterior probability state following each trial of this experiment.

In this section we introduce the probabilistic model, the reduced probabilistic model and the statistical model for the description of experiments upon a quantum system and prove the corresponding statements.

Under a generalized quantum measurement, we further mean any experiment upon a quantum system which admits the probabilistic description and results in imprints in the classical world of any most general possible nature.

6.1 Quantum probabilistic model

6.1.1 Information states, mean information states

Consider a quantum system S_q , described in terms of \mathcal{H} . Denote by $\mathcal{O}_{\mathcal{H}} = \{\psi \in \mathcal{H} : ||\psi||_{\mathcal{H}} = 1\}$ the unit sphere in \mathcal{H} . Let

$$\mathcal{P}_1(\mathcal{H}) = \{ p \in \mathcal{T}^{(+)}(\mathcal{H}) : p = |\psi\rangle\langle\psi|, \ \forall\psi\in\mathcal{O}_{\mathcal{H}} \}$$

be the set of all one-dimensional projections¹⁶ on \mathcal{H} and $\mathcal{B}_{\mathcal{P}_1(\mathcal{H})}$ be the trace on $\mathcal{P}_1(\mathcal{H})$ of the Borel σ -algebra on $\mathcal{T}(\mathcal{H})$. For our further consideration, we also denote by $R : \mathcal{O}_{\mathcal{H}} \to \mathcal{P}_1(\mathcal{H})$ the surjective mapping $\psi \mapsto p = R(\psi) := |\psi\rangle\langle\psi|$.

Definition 20 For a quantum system, described in terms of \mathcal{H} , introduce an information state by

$$(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})}, [\pi])$$

for any $\pi \in \mathcal{V}_{(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})}$, and an information state space by

$$(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})}, [\mathcal{V}_{(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})}]).$$

In the quantum case, the measurable space $(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})$ satisfies the condition (14), with V being the Banach space $\mathcal{T}_s(\mathcal{H})$ of self-adjoint trace class operators on \mathcal{H} .

For any $\pi \in \mathcal{V}_{(\mathcal{P}_1(\mathcal{H}),\mathcal{B}_{\mathcal{P}_1}(\mathcal{H}))}$, a quantum mean information state is given by

$$\eta_{mean}(\pi) = \int_{\mathcal{P}_1(\mathcal{H})} p\pi(dp) = \int_{\mathcal{O}_{\mathcal{H}}} |\psi\rangle \langle \psi| \ \pi'(d\psi) = \rho,$$

and represents a density operator ρ on \mathcal{H} . Here, to each $\pi \in \mathcal{V}_{(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})}$, the measure π' is defined uniquely by the relation

$$\pi'(R^{-1}(F)) = \pi(F), \qquad \forall F \in \mathcal{B}_{\mathcal{P}_1(\mathcal{H})},$$

and is a normalized σ -additive positive real valued measure on the σ -algebra $\mathcal{F}_{\mathcal{O}_{\mathcal{H}}} = \{R^{-1}(F) : F \in \mathcal{B}_{\mathcal{P}_1(\mathcal{H})}\}$ on $\mathcal{O}_{\mathcal{H}}$.

The convex linear set \Re_{mean} of all quantum mean information states coincides with the set

$$\mathcal{R}_{\mathcal{H}} = \{\rho \in \mathcal{T}^{(+)}(\mathcal{H}) : ||\rho||_{\mathcal{T}(\mathcal{H})} = \operatorname{tr}\{\rho\} = 1\}$$

of all density operators on \mathcal{H} .

Since the σ -algebra $\mathcal{B}_{\mathcal{P}_1(\mathcal{H})}$ contains all atom subsets $\{p\}$ of $\mathcal{P}_1(\mathcal{H})$ and $\mathcal{P}_1(\mathcal{H})$ is the set of all extreme elements in $\mathcal{R}_{\mathcal{H}}$, to each pure information state of a quantum system there corresponds the unique pure mean information state and vice versa.

According to the terminology, used in quantum theory, we further refer to a quantum mean information state, represented by a density operator ρ on \mathcal{H} , as a quantum state, pure or mixed.

We further denote by

$$\mathcal{U}_{\mathcal{K}} = (\mathcal{P}_1(\mathcal{K}), \mathcal{B}_{\mathcal{P}_1(\mathcal{K})}, [\mathcal{V}_{(\mathcal{P}_1(\mathcal{K}), \mathcal{B}_{\mathcal{P}_1(\mathcal{K})})}])$$
(59)

a possible information state space of a quantum system S_q . Here \mathcal{K} is a separable complex Hilbert space.

¹⁶Called also as pure density operators.

6.1.2 Generalized observables

Recall that a *linear* functional $\Phi(\cdot) : \mathcal{T}(\mathcal{H}) \to \mathbb{C}$ is called non-negative if $\Phi(T) \geq 0$ whenever $T \in \mathcal{T}^{(+)}(\mathcal{H})$. Any non-negative linear functional on $\mathcal{T}(\mathcal{H})$ is bounded (equivalently, continuous), that is, there exists some C>0 such that $|\Phi(T)| \leq C||T||_{\mathcal{T}(\mathcal{H})}$.

Denote by $\mathcal{T}^*(\mathcal{H})$ the Banach space of bounded linear functionals on $\mathcal{T}(\mathcal{H})$ and by $\mathcal{T}^*_+(\mathcal{H}) \subset \mathcal{T}^*(\mathcal{H})$ the set of all non-negative bounded linear functionals on $\mathcal{T}(\mathcal{H})$.

In the quantum case, the conditions (i)-(iii), specified in section 4.5.1, are valid. In particular, the mapping $\operatorname{tr}\{\cdot\}: \mathcal{T}(\mathcal{H}) \to \mathbb{C}$ represents a unique positive continuous linear functional $\mathfrak{J}\{\cdot\}$ on $\mathcal{T}(\mathcal{H})$, satisfying (51). Hence, all items, introduced in section 4.5.1 in the most general settings, are applicable to the quantum case.

Any σ -additive measure

$$\Pi_{lin}^{(q)}: \mathcal{F}_{\Lambda} \to \mathcal{T}_{+}^{*}(\mathcal{H}), \quad (\Pi_{lin}^{(q)}(\Lambda))(T) = \operatorname{tr}\{T\}, \quad \forall T \in \mathcal{T}(\mathcal{H}),$$

on a measurable space $(\Lambda, \mathcal{F}_{\Lambda})$, with values $\Pi_{lin}^{(q)}(B)$, $B \in \mathcal{F}_{\Lambda}$, that are non-negative continuous linear functionals on $\mathcal{T}(\mathcal{H})$, satisfies conditions (53), and, hence, represents a linear generalized observable of a quantum system.

Due to the linear isometric isomorphism between $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}^*(\mathcal{H})$, to each non-negative continuous linear functional $\Pi_{lin}^{(q)}(B)$, $B \in \mathcal{F}_{\Lambda}$, there corresponds the uniquely defined non-negative bounded linear operator $A_{\Pi_{lin}}(B)$, $B \in \mathcal{F}_{\Lambda}$, and vice versa, such that

$$(\Pi_{lin}^{(q)}(B))(T) = \operatorname{tr}\{TA_{\Pi_{lin}^{(q)}}(B)\}, \quad \forall T \in \mathcal{T}(\mathcal{H}).$$

Since $\Pi_{lin}^{(q)}: \mathcal{F}_{\Lambda} \to \mathcal{T}_{+}^{*}(\mathcal{H})$ is a σ -additive measure and

$$\operatorname{tr}\{\rho A_{\Pi^{(q)}}(\Lambda)\} = 1, \quad \forall \rho \in \mathcal{R}_{\mathcal{H}},$$

each $A_{\Pi^{(q)}}(B) \in \mathcal{L}^{(+)}(\mathcal{H}), B \in \mathcal{F}_{\Lambda}$, represents a value of the normalized positive operator valued measure^{*lin*}

$$A_{\Pi_{lin}^{(q)}}: \mathcal{F}_{\Lambda} \to \mathcal{L}^{(+)}(\mathcal{H}), \quad A_{\Pi_{lin}^{(q)}}(\Lambda) = I_{\mathcal{H}},$$

on $(\Lambda, \mathcal{F}_{\Lambda})^{17}$, which is σ -additive in the strong operator topology in $\mathcal{L}(\mathcal{H})$.

This implies that, to each linear generalized observable $\Pi_{lin}^{(q)}$ (extended, or outcome, or system) of a quantum system, with an outcome space $(\Lambda, \mathcal{F}_{\Lambda})$, there corresponds the unique (extended, or outcome, or system) normalized σ -additive positive operator valued measure $A_{\Pi_{lin}^q}$ on $(\Lambda, \mathcal{F}_{\Lambda})$ and vice versa.

A projection valued measure P on $(\Lambda, \mathcal{F}_{\Lambda})$ represents a quantum measurement with a deterministic set-up.

Furthermore, real experimental situations show that, under any quantum measurement, the outcome probability law $\mu(B; \rho_{in})$ on $(\Omega, \mathcal{F}_{\Omega})$ depends only on an initial quantum state ρ_{in} of \mathcal{S}_q but not on an initial probability distribution π_{in} on $(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})$.

Due to proposition 4, this implies that, in the quantum case, only linear outcome generalized observables $M_{lin}^{(q)}$ are allowed and

$$\mu(B;\rho_{in}) = (\mathcal{M}_{lin}^{(q)}(B))(\rho_{in}) = \operatorname{tr}\{\rho_{in}M(B)\},\tag{60}$$

for all $\rho_{in} \in \mathcal{R}_{\mathcal{H}}, B \in \mathcal{F}_{\Omega}$. A normalized positive operator valued measure

$$M: \mathcal{F}_{\Omega} \to \mathcal{L}^{(+)}(\mathcal{H}), \qquad M(\Omega) = I_{\mathcal{H}},$$

is usually called in quantum measurement theory as a *probability operator valued measure* or a *POV measure*, for short.

¹⁷Measures Π_{lin}^q and $A_{\Pi_{lin}^q}$ are bounded in the sense that, $||\Pi^q((B)||_{\mathcal{T}^*(\mathcal{H})} \leq 1$ and $||A_{\Pi_{lin}}(B)||_{\mathcal{L}(\mathcal{H})} \leq 1$, for all $B \in \mathcal{F}_{\Lambda}$.

Furthermore, in quantum measurement theory¹⁸ it is postulated that, under a non-destructive quantum measurement, with outcomes in $(\Omega, \mathcal{F}_{\Omega})$, for any $B \in \mathcal{F}_{\Omega}$, the conditional change of an initial quantum state

$$\rho_{in} \mapsto \rho^{out}(B; \rho_{in}) = \frac{(\mathcal{M}(B))(\rho_{in})}{\mu(B; \rho_{in})},$$

following a single experimental trial, is described by a positive bounded *linear* mapping \mathcal{M} on $\mathcal{F}_{\Omega} \times \mathcal{T}_{\mathcal{H}}$. This assumption is usually justified by a unitary linear "dynamics" of a quantum state of the extended quantum system, which includes the quantum environment of a measuring device.

In our terminology, the latter assumption implies:

For the description of experiments upon a quantum system, only linear (extended, system and outcome) generalized observables are allowed. Hence, with respect to a system, any quantum measurement is perturbing.

Summing up, the probabilistic model for the description of non-destructive quantum measurements upon a quantum system S_q is given by the pair

$$(\mathcal{U}_{\mathcal{S}q}, \mathcal{A}_{\mathcal{S}_a}^{ext})$$

where:

(a) $\mathcal{U}_{Sq} = {\mathcal{U}_{\mathcal{K}_{\gamma}} : \forall \gamma \in \Gamma}$, with each $\mathcal{U}_{\mathcal{K}_{\gamma}}$ defined by (41), being the family of an input and output information state spaces of S_q ;

(b) \mathcal{A}_{Sq}^{ext} is the family of all non-trivial extended normalized σ -additive positive operator valued measures

$$A: \mathcal{F}_{\Omega} \otimes \mathcal{B}_{\mathcal{P}_{1}(\mathcal{K}_{\gamma_{2}})} \to \mathcal{L}^{(+)}(\mathcal{K}_{\gamma_{1}}), \qquad A(\Omega \times \mathcal{P}_{1}(\mathcal{K}_{\gamma_{2}})) = I_{\mathcal{K}_{\gamma_{1}}}, \quad \forall \gamma_{1}, \gamma_{2} \in \Gamma,$$

on any outcome space $(\Omega, \mathcal{F}_{\Omega})$, allowed under quantum measurements.

6.2 Reduced quantum probabilistic model

According to our discussion in section 5, for experiments, represented by only linear extended generalized observables, we can equivalently replace the probabilistic model by its reduced version.

From our consideration in section 6.1.2, it follows that, in the quantum case, the reduced probabilistic model is given by

$$(\mathfrak{R}_{\mathcal{S}_q}, \mathcal{A}_{\mathcal{S}_q}^{ext}).$$

with

$$\mathfrak{R}_{\mathcal{S}_q} = \{ \mathcal{R}_{\mathcal{K}_{\gamma}} : \gamma \in \Gamma \}$$

being the family of the sets of density operators on corresponding Hilbert spaces and $\mathcal{A}_{\mathcal{S}_q}^{ext}$ being the family of all non-trivial extended normalized σ -additive positive operator valued measures, defined in the point (b), section 6.1.2.

6.2.1 Quantum state instrument

Consider a non-destructive quantum measurement $\mathcal{E} \in [\Upsilon_{lin}^{(q)}]$, with an outcome space $(\Omega, \mathcal{F}_{\Omega})$, upon a quantum system \mathcal{S}_q , described initially in terms of a Hilbert space \mathcal{H} .

Suppose that, immediately after this experiment, the information on S_q is described in terms if a separable complex Hilbert space \mathcal{K} . Let

$$A: \mathcal{F}_{\Omega} \otimes \mathcal{B}_{\mathcal{P}_1(\mathcal{K})} \to \mathcal{L}^{(+)}(\mathcal{H}), \qquad A(\Omega \times \mathcal{P}_1(\mathcal{K})) = I_{\mathcal{H}}$$

be the non-trivial extended normalized σ -additive positive operator valued measure, which is put, due to the relation

$$(\Upsilon_{lin}^{(q)}(\cdot))(T) = \operatorname{tr}\{TA_{\Upsilon_{lin}^{(q)}}(\cdot)\}, \qquad \forall T \in \mathcal{T}(\mathcal{H}),$$
(61)

 $^{^{18}\}mathrm{See},$ for example, in [16] and [17-19].

into one-to-one correspondence with an extended linear generalized observable $\Upsilon_{lin}^{(q)} : \mathcal{F}_{\Omega} \otimes \mathcal{B}_{\mathcal{P}_1(\mathcal{K})} \to \mathcal{T}^*_+(\mathcal{H})$ of \mathcal{S}_q .

According to our consideration in section 4.5, the complete statistical description of a quantum measurement $\mathcal{E} \in [\Upsilon_{lin}^{(q)}]$ is given by the notion of a mean information state instrument

$$(\mathcal{M}^{st}_{\Upsilon^{(q)}_{lin}}(\cdot))(\cdot):\mathcal{F}_{\Omega}\times\mathcal{R}_{\mathcal{H}}\to\mathcal{T}(\mathcal{K})$$

defined, in the most general settings, by (56).

In the quantum case, the mapping $\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}$ is uniquely extended to all of $\mathcal{F}_{\Omega} \times \mathcal{T}(\mathcal{H})$.

Definition 21 (Quantum state instrument) We call the mapping

$$(\mathcal{M}^{st}_{\Upsilon^{(q)}_{lin}}(\cdot))(\cdot): \quad \mathcal{F}_{\Omega} \times \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$$

defined by a non-trivial quantum extended linear generalized observable $\Upsilon_{lin}^{(q)}$ through the relation

$$(\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(B))(T) = \int_{\mathcal{P}_1(\mathcal{K})} p'(\Upsilon_{lin}^{(q)}(B \times dp'))(T), \tag{62}$$

for all $B \in \mathcal{F}_{\Omega}$, $T \in \mathcal{T}(\mathcal{H})$, a quantum state instrument.

From (62) it follows that, for a quantum state instrument, the mapping $\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(\cdot)$ is a σ additive¹⁹ measure on $(\Omega, \mathcal{F}_{\Omega})$ with values that are non-negative bounded linear mappings $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$.

From (58) it follows also that, for each experimental situation $\mathcal{E} + \mathcal{S}(\rho_{in}), \rho_{in} \in \mathcal{R}_{\mathcal{H}}$, the probability distribution $\mu_{\Upsilon^{(q)}}(\cdot; \rho_{in})$ of outcomes in $(\Omega, \mathcal{F}_{\Omega})$ satisfies the relation

$$\mu_{\Upsilon^{(q)}_{lin}}(B;\rho_{in}) = \operatorname{tr}\{(\mathcal{M}^{st}_{\Upsilon^{(q)}_{lin}}(B))(\rho_{in})\}, \quad \forall B \in \mathcal{F}_{\Omega}.$$

We have the following statement.

Proposition 5 For a quantum state instrument $\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{(st)}$, defined by (62):

(i) tr{ $\{(\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(\Omega))(T)\}$ = tr{ $\{T\}$, $\forall T \in \mathcal{T}(\mathcal{H})$, that is, the mapping $\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(\Omega)$ is a dynamical map; (ii) each value $\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(B)$, $B \in \mathcal{F}_{\Omega}$, of the measure

$$\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{(st)}(\cdot):\mathcal{F}_{\Omega}\to\mathcal{L}^{(+)}(\mathcal{T}(\mathcal{H}),\mathcal{T}(\mathcal{K}))$$

is a completely positive bounded linear mapping $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$.

Proof. Due to (61) and (62),

$$(\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(B))(T) = \int_{\mathcal{P}_1(\mathcal{K})} p' \mathrm{tr}\{TA(B \times dp')\}, \quad \forall T \in \mathcal{T}(\mathcal{H}), \quad \forall B \in \mathcal{F}_{\Omega}.$$
 (63)

The relation in (i) follows trivially from (63). In the language, accepted in mathematical physics literature²⁰, this means that the mapping $\mathcal{M}_{\gamma^{(q)}}^{st}(\Omega) \in \mathcal{L}^{(+)}(\mathcal{T}(\mathcal{H}), \mathcal{T}(\mathcal{K}))$ is a dynamical map.

To prove (ii), recall that a linear mapping $W : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$ is called completely positive if, for any finite families of elements $\{f_i \in \mathcal{K} : i = 1, ..., N < \infty\}$ and $\{T_i \in \mathcal{T}(\mathcal{H}) : i = 1, ..., N < \infty\}$, the sum

$$\sum_{i,j} \langle f_i, W(T_i^*T_j)f_j \rangle_{\kappa} \ge 0.$$

¹⁹In the strong operator topology in the Banach space $\mathcal{L}(\mathcal{T}(\mathcal{H}), \mathcal{T}(\mathcal{K}))$ of bounded linear mappings $\mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{K})$. ²⁰See, for example, in [16].

Due to (63), for any $B \in \mathcal{F}_{\Omega}$, we have

$$\sum_{i,j} \langle f_i, (\mathcal{M}_{\Upsilon_{lin}^{(q)}}^{st}(B))(T_i^*T_j)) f_j \rangle_{\mathcal{K}} = \int_{\mathcal{P}_1(\mathcal{K})} \sum_{i,j} \langle f_i, p'f_j \rangle_{\mathcal{K}} \operatorname{tr}\{T_i^*T_j A(B \times dp')\}$$

$$= \int_{\mathcal{O}_{\mathcal{K}}} \operatorname{tr}\{G(\psi')A'(B \times d\psi')\},$$
(64)

where A' is the normalized σ -additive positive operator valued measure

$$A': \mathcal{F}_{\Omega} \times \mathcal{F}_{\mathcal{O}_{\mathcal{K}}} \to \mathcal{L}^{(+)}(\mathcal{H}), \qquad A'(\Omega \times \mathcal{O}_{\mathcal{K}}) = I_{\mathcal{H}},$$
$$\mathcal{F}_{\mathcal{O}_{\mathcal{K}}} = \{R^{-1}(F): F \in \mathcal{F}_{\mathcal{P}_{1}(\mathcal{K})}\},$$

defined by the equation

$$A'(B \times R^{-1}(F)) = A(B \times F), \quad \forall F \in \mathcal{B}_{\mathcal{P}_1(\mathcal{K})}, \quad \forall B \in \mathcal{F}_{\Omega},$$

and, for any $\psi' \in \mathcal{O}_{\mathcal{K}}$,

$$G(\psi') = \sum_{ji} T_i^* T_j \langle f_i', \psi' \rangle_{\kappa} \langle \psi', f_j \rangle_{\kappa} \in \mathcal{T}^{(+)}(\mathcal{H}).$$

Since $G(\psi') \ge 0, \forall \psi' \in \mathcal{O}_{\mathcal{K}}$, in (64) we finally have:

$$\sum_{i,j} \langle f_i, (\mathcal{M}_{\Upsilon^{(q)}_{lin}}^{st}(B))(T_i^*T_j)f_j \rangle_{\kappa} \ge 0, \qquad \forall B \in \mathcal{F}_{\otimes}.$$

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We underline that, in contrast to our general framework, in the frame of the operational approach to the description of quantum measurements, the relation between an instrument and conditional posterior quantum states, as well as the complete positivity of a quantum instrument, are always introduced axiomatically rather than actually proved as in this paper.

The basics of the quantum stochastic approach to the description of quantum measurements, formulated in [17-19], correspond to the general framework, formulated in this paper.

6.3 Quantum statistical model

From our presentation in sections 5 and 6.1 it follows that, for a quantum system S_q , described initially in terms of \mathcal{H} , the quantum statistical model is given by

$$(\mathcal{U}_{\mathcal{H}},\mathfrak{M}^{outcome}_{\mathcal{H}})$$

where

$$\mathcal{U}_{\mathcal{H}} = (\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})}, [\mathcal{V}_{(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})}])$$

is an initial information state space of S_q and $\mathfrak{M}_{\mathcal{H}}^{outcome}$ is the family of all non-trivial POV measures:

$$M: \mathcal{F}_{\Omega} \to \mathcal{L}^{(+)}(\mathcal{H}), \quad M(\Omega) = I_{\mathcal{H}},$$
(65)

with any outcome space $(\Omega, \mathcal{F}_{\Omega})$.

Correspondingly, the reduced statistical model for the description of quantum measurements upon a system S_q is given by the pair

$$(\mathcal{R}_{\mathcal{H}}, \mathfrak{M}_q^{outcome}),$$

where $\mathcal{R}_{\mathcal{H}}$ is the set of all density operators on \mathcal{H} .

The above reduced statistical model coincides with the well-known quantum statistical model, considered in [6,8,9,11,12,14-16,21].

6.3.1 "No-go" theorem

The problem of the relation between Kolmogorov's model and the statistical model of quantum theory is the point of intensive discussions for already more than 70 years.

The so-called "no-go" theorems²¹ state that the properties of quantum observables can not be explained in terms of some "underlying" Kolmogorov's probability space.

However, different mathematical formulations of "no-go" theorems still leave a loophole for doubts. The numerous papers on Bell-like inequalities are a good confirmation that these doubts still exist.

In our framework, it is clear that, in the quantum probabilistic model, formulated in section 6, there are no beables. In contrast to a beable, any quantum generalized observable represents an experiment which, in general, perturbs a quantum system state.

However, on the level of statistical models, where the concept of a posterior state does not appear, this point is not obvious.

Based on the concept of *reducible* models, which we introduced in section 5.1, we proceed to formulate and to prove the theorem on irreducibility of the quantum statistical model to Kolmogorov's statistical model.

We note that, among different versions of "no-go" theorems, the mathematical setting of our theorem is most general and includes the mathematical settings of all other "no-go" theorems²² as particular cases.

Theorem 3 The quantum statistical model is **not reducible** to Kolmogorov's statistical model.

Proof. We are based on the definitions, in sections 5.1, 5.2, 6.3, of the notions of reducible models, Kolmogorov's statistical model and the quantum statistical model, respectively.

Suppose that the quantum statistical model is reducible to Kolmogorov's statistical model.

Then there must exist an information state space $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}])$ such that:

(a) the quantum information state space $(\mathcal{P}_{1}(\mathcal{H}), \mathcal{B}_{\mathcal{P}_{1}(\mathcal{H})}, [\mathcal{V}_{(\mathcal{P}_{1}(\mathcal{H}), \mathcal{B}_{\mathcal{P}_{1}(\mathcal{H})})}])$ is induced by an information state space $(\Theta_{in}, \mathcal{F}_{\Theta_{in}}, [\mathcal{V}_{(\Theta_{in}, \mathcal{F}_{\Theta_{in}})}]);$

(b) any quantum outcome generalized observable $M_{lin}^{(q)}$ on $(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})$ and any beable $S^{(be)}$ on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$, with the outcome space $(\mathcal{P}_1(\mathcal{H}), \mathcal{B}_{\mathcal{P}_1(\mathcal{H})})$, generate, due to (22), a beable

$$\mathbf{M}^{(be)}(B) = (\mathbf{M}_{lin}^{(q)}(B) \circ \mathbf{S}^{(be)}), \quad \forall B \in \mathcal{F}_{\Omega}$$

on $(\Theta_{in}, \mathcal{F}_{\Theta_{in}})$. Since, due to (60),

$$(\mathbf{M}_{lin}^{(q)}(B))(p) = \operatorname{tr}\{pM(B)\}, \quad \forall p \in \mathcal{P}_1(\mathcal{H}), \ \forall B \in \mathcal{F}_{\Omega},$$

where M is a POV measure, we derive:

$$(\mathbf{M}^{(be)}(B))(\theta_{in}) = \int_{\mathcal{P}_1(\mathcal{H})} \operatorname{tr}\{pM(B)\}(\mathbf{S}^{(be)}(dp))(\theta_{in}),\tag{66}$$

for all $B \in \mathcal{F}_{\Omega}, \ \theta_{in} \in \Theta_{in}$.

Due to the definition of Kolmogorov's model, section 5.2, and definition 14 of a beable, there must exist:

a probability state space $(\Theta, \mathcal{F}_{\Theta}, [\mathcal{V}_{(\Theta, \mathcal{F}_{\Theta})}]);$ an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Theta_{in}}$ -measurable mapping $\phi_{in} : \Theta \to \Theta_{in};$ an $\mathcal{F}_{\Theta}/\mathcal{F}_{\Omega}$ -measurable mapping $\varphi : \Theta \to \Omega;$ an $\mathcal{F}_{\Theta}/\mathcal{B}_{\mathcal{P}_1(\mathcal{H})}$ -measurable mapping $\Phi : \Theta \to \mathcal{P}_1(\mathcal{H});$ such that

$$(\mathbf{E}_{\phi_{in}^{-1}}^{(be)}(B))(\theta) \equiv (\mathbf{E}^{(be)}(B))(\phi_{in}(\theta)) = \chi_{\varphi^{-1}(B)}(\theta), \quad \forall B \in \mathcal{F}_{\Omega}, \quad \forall \theta \in \Theta,$$

$$(\mathbf{S}_{\phi_{in}^{-1}}^{(be)}(F))(\theta) \equiv (\mathbf{S}^{(be)}(F))(\phi_{in}(\theta)) = \chi_{\Phi^{-1}(F)}(\theta), \quad \forall F \in \mathcal{B}_{\mathcal{P}_{1}(\mathcal{H})}, \quad \forall \theta \in \Theta.$$

$$(67)$$

²¹See, for example, duscussion and references in [14,16].

 $^{^{22}}$ (See in [16], section 1.4.1.

Taking ϕ_{in} -preimages of the left and the right hand sides in (66) and considering (67), we finally derive that the assumptions (a), (b) result in the following relation:

$$\chi_{\varphi^{-1}(B)}(\theta) = \operatorname{tr}\{p_{\theta}M(B)\}, \quad p_{\theta} = \Phi(\theta) \in \mathcal{P}_{1}(\mathcal{H}), \quad \forall B \in \mathcal{F}_{\Omega},$$
(68)

valid for all $\theta \in \Theta$.

Take, for example, a discrete projection valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$:

$$P(B) = \sum_{\lambda_i \in B} |\psi_i\rangle \langle \psi_i|, \quad \lambda_i \in \mathbb{R}, \quad \langle \psi_i, \psi_j \rangle_{\mathcal{H}} = \delta_{ij}, \quad i, j = 1, 2, ..., N \le \infty; \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

In the most general settings, a value $p_{\theta} = \Phi(\theta) \in \mathcal{P}_1(\mathcal{H})$ admits the representation:

$$p_{\theta} = |\psi_{\theta}\rangle\langle\psi_{\theta}|, \qquad \psi_{\theta} = \sum_{i} \beta_{i}^{(\theta)}\psi_{i}, \qquad \sum_{i} |\beta_{i}^{(\theta)}|^{2} = 1,$$
(69)

where, for given p_{θ} , a vector $\psi_{\theta} \in \mathcal{O}_{\mathcal{H}}$ is defined up to phase equivalence. Substituting (69) into (68), we derive:

$$\chi_{\varphi^{-1}(B)}(\theta) = \sum_{\lambda_i \in B} |\beta_i^{(\theta)}|^2, \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

But this relation cannot be valid for all $\theta \in \Theta$ and all $B \in \mathcal{B}(\mathbb{R})$.

Thus, assuming (a), (b), we have come to the contradiction and, hence, the quantum statistical (and, hence, probabilistic) model is not reducible to Kolmogorov's statistical model. \blacksquare

Conclusion 4 Kolmogorov's model cannot induce the properties of quantum generalized observables.

7 Concluding remarks

In the present paper we formulate a new general framework for the probabilistic description of an experiment upon a system in terms of initial information representing this system.

We introduce the notions of an information state, an information state space and a generalized observable and prove the corresponding statements.

We prove that, to any experiment upon a system, there corresponds a unique generalized observable on a system initial information state space. An initial information state space provides the knowledge on the description of only such experiments which are represented on this space by non-trivial generalized observables.

- We specify the special types of generalized observables:
- (i) observables, which describe experiments with deterministic set-up;
- (ii) beables, which describe non-perturbing experiments with deterministic set-up.

A beable describes a non-perturbing experiment on the "errorless" measurement of some property of a system, objectively existing before this experiment. Under an experiment, described by a beable, the randomness is caused only by the uncertainty encoded in the initial probability distribution in a Kolmogorov probability space. In Kolmogorov's model only beables are considered.

In general, however, a generalized observable represents an experiment upon a system in a non-separable manner and cannot be associated with any of system properties, objectively existing before this experiment.

In the most general settings, we introduce the concept of a complete information description of a non-destructive experiment upon a system. This type of description is given by the notion of an information state instrument, which we define in this paper.

We point out that the phenomenon of "reduction" of an information state is inherent, in general, to any non-destructive experiment and upon a system of any type. In case of nondestructive experiments upon quantum systems, the von Neumann "state collapse" and its further generalizations represent particular cases of this general phenomenon. For a system, described by an information state space with a Banach space based structure, we introduce, in the most general settings, the notion of a mean information state instrument.

We specify the concepts of the probabilistic model and the statistical model for the description of experiments upon a quantum system and prove, in the most general mathematical setting, the theorem on the irreducibility of the quantum statistical model to Kolmogorov's statistical model.

In the quantum case, only such a generalized observable is allowed which is represented by a σ -additive measure with values that are non-negative continuous linear functionals on the Banach space of trace class operators. To each this generalized observable there is put into the one-to-one correspondence a normalized σ -additive measure with values that are non-negative bounded linear operators on a Hilbert space. A projection valued measure represents a quantum measurement with a deterministic set-up.

We prove that a quantum mean information state instrument represents a dynamical map and is completely positive.

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