The time to ruin for a class of Markov additive risk processes

Martin Jacobsen*
Department of Applied Mathematics and Statistics
University of Copenhagen
5, Universitetsparken
DK-2100 Copenhagen Ø
Denmark

Abstract

Risk processes are considered, which locally behave as a Brownian motion with some drift and variance, both depending on an underlying Markov chain that is used also to generate the claims arrival process. Thus claims arrive according to a renewal process with waiting times of phase-type. The claims are assumed to form an iid sequence, independent of everything else, and with a distribution with a Laplace transform that is a rational function.

In the main results of the paper, the joint Laplace transform of the time to ruin and the undershoot at ruin as well as the probability of ruin is determined explicitly. Furthermore, both the Laplace transform and the ruin probability is decomposed according to the type of ruin: ruin by jump or ruin by continuity.

The methods used involve finding certain martingales by first finding partial eigenfunctions for the generator of the Markov process composed of the risk process and the underlying Markov chain. Results from complex function theory are used as an important tool.

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1. Introduction

In this paper, the work initiated by Jacobsen [8] is generalised to a large class of risk processes, see (2.3) below, leading to explicit forms for the joint Laplace transform for the time to ruin and the undershoot at ruin, and also for the ruin probability whether ruin is caused by a claim or by the risk process sliding continuously into ruin. In that earlier paper [8], the simple risk model

$$X_t = x_0 + \beta t - \sum_{n=1}^{N_t} U_n \tag{1.1}$$

was considered, assuming that claims arrive according to a renewal counting process $N = (N_t)$ with waiting times between claims that are of phase-type, while the U_n form an iid sequence of strictly positive claims, also independent of N. Assuming that the Laplace transform for the U_n be a rational function, an explicit expression for the Laplace transform of the time to ruin and the ruin probability was found, using a certain family of martingales, determined not only by the risk process X itself, but also involving in an essential manner the Markov chain J used to generate the claims arrival process N.

Essentially, finding the relevant martingales amounts to finding for any $\theta \geq 0$, partial eigenfunctions (see (2.14) below) for the generator for the piecewise deterministic Markov process (X, J). What will be shown in this paper is that the structure of the relevant eigenfunctions and martingales found in Jacobsen [8] pertains also to the much more general model to be discussed presently.

Many of the existing results in the litterature on ruin problems involve an 'extra' Laplace transform: if x_0 is the initial state for the risk process and $\mathrm{p_r}(x_0)$ is the corresponding probability of ruin, one does not determine $\mathrm{p_r}(x_0)$ directly, but finds the Laplace transform $\int_0^\infty e^{-\nu x}\mathrm{p_r}(x)\,dx$ instead. It is stressed that in this paper these 'extra' Laplace transforms are avoided.

The model itself, see (2.3), is an example of a Markov additive process X (see e.g Asmussen [1], Section II.5 for the definition and basic properties), behaving as a Brownian motion with a drift and variance determined by an underlying Markov chain J, that is used also to generate the times at which claims arrive – in particular, just as in the model (1.1), claims arrive according to a renewal process with phase-type waiting times.

The simple model (1.1) with a renewal process for the arrivals of claims has been studied recently in a number of papers: a particular case was discussed by Dickson and Hipp [6], while Avram and Usábel [3], [4] obtained general distribution results concerning the time to ruin and the undershoot using a method entirely different from that of Jacobsen [8] and the present paper. For earlier work, see also Asmussen [1], Chapter 5.

With Poisson arrivals, (1.1) is of course a Lévy process (a compound Poisson process plus linear drift). Adding an independent Brownian motion yields another well studied Lévy process that is also a special case of (2.3) below. For this model, with the variation that both positive and negative jumps are allowed, Asmussen et al [2] determined the joint Laplace transform of the time to ruin and the undershoot. For Lévy processes that are general subordinators, Winkel [11] described not only (in the terminology used here) the joint distribution of the time to ruin and the undershoot, but also considered other quantities related to the time of ruin such as the size of the claim causing ruin, time elapsed since the last previous claim etc.

In order to treat the general model (2.3), it is as already noted vital that one is able to determine partial eigenfunctions for the generator of the Markov process (X, J), which in turn yields the martingales required for the main results. The idea of using partial eigenfunctions has certainly appeared before, see e.g Paulsen and Gjessing [10] who studied a risk model not of the form (2.3). The martingales are martingales for the filtration generated by (X, J), but *not* for that generated by X alone. In their study of risk processes of the form (1.1) with N e.g a Cox process, Embrechts et al [7] similarly used an enlarged filtration to find the relevant martingales, coining the phrase *Markovization* for this useful trick.

The general model (2.3) studied in this paper is introduced in Section 2. The joint Laplace transform

$$\mathbb{E}^{(x_0,i_0)} e^{-\theta \mathrm{T_r} - \zeta \mathrm{Y_r}}$$

of the time to ruin $\mathrm{T_r}$ and the undershoot $\mathrm{Y_r}$, corresponing to an arbitrary initial state (x_0, i_0) for (X, J) is determined in Section 3 for $\theta > 0$, $\zeta \geq 0$. Taking $\zeta = 0$ and letting $\theta \downarrow 0$ would then yield the probability of ruin, but in Section 4 it is shown how to find the ruin probability directly, a result that is in some sense more difficult than finding the joint Laplace transform! A numerical examples is used to illustrate the ease with which ruin probabilities may be computed. As another illustration, the short Section 5 surveys the results for the simple model where X is the sum of a compound Poisson process with exponential claims and a Brownian motion with some drift and a variance ≥ 0. The final Section 6 briefly mentions variations and possible extensions of the model (2.3).

The main results below are Theorems 1 and 2. In both, the joint Laplace transform, resp. the ruin probability is decomposed according to the type of ruin: ruin caused by a sufficiently large claim (ruin by jump) or ruin caused by the risk process moving continuously through the level 0 (ruin by continuity). Although the results are explicit they can certainly not be given in closed form, but it is emphasised that all that is required in order to obtain numerical results is that one should find the roots with strictly negativ real parts of a certain family of polynomials, invert certain matrices and finally solve a certain system of linear equations.

2. The model

We consider a risk process $X = (X_t)_{t\geq 0}$ which is a real-valued Markov additive process defined as follows: suppose given a Markov chain $J = (J_t)_{t\geq 0}$, time-homogeneous with a finite state-space E, and a counting process $N = (N_t)_{t\geq 0}$ (in particular $N_0 \equiv 0$) such that (J, N) is a homogeneous Markov chain with state-space $E \times \mathbb{N}_0$ and transition intensities $q_{(i,n),(i',n')}$ for $i, i' \in E$ and $n, n' \in \mathbb{N}_0$ with $(i, n) \neq (i', n')$ that are > 0 only if $n' = n$ or $n' = n + 1$ and then

$$
\begin{aligned}
q_{(i,n),(j,n)} &= q_{ij} \quad (i \neq j)\,, & (2.1)\\
q_{(i,n),(j,n+1)} &= \lambda_i a_j \quad (i, j \in E) & (2.2)
\end{aligned}
$$

with all $q_{ij} \geq 0$, all $\lambda_i \geq 0$, and all $a_j \geq 0$ with $\sum_j a_j = 1$. Suppose also given a one-dimensional standard Brownian motion $B = (B_t)_{t\geq 0}$ independent of (J, N) and a sequence $(U_n)_{n\geq 1}$ of iid strictly positive claims, independent of (J, N, B). Then for $(\beta_i)_{i\in E}$ and $(\sigma_i^2)_{i\in E}$ given constants and all $\beta_i \in \mathbb{R}$, all $\sigma_i^2 \geq 0$, subject to the initial condition $X_0 \equiv x_0$, X is given by

$$
X_t = x_0 + \int_0^t \beta_{J_s}\, ds + \int_0^t \sigma_{J_{s-}}\, dB_s - \sum_{n=1}^{N_t} U_n. \tag{2.3}
$$

Thus, as long as $J_t \equiv i$ and N does not jump, X behaves as a Brownian motion with drift β_i and variance σ_i^2 (with $\sigma_i^2 = 0$ allowed corresponding to $t \mapsto X_t$ a straight line with slope β_i when $J_t \equiv i$ and there are no jumps for N). X is continuous except at the times when N jumps; at these times a claim arrives forcing a matching downward jump in X. The Markov chain J may jump simultaneously with N but may also have jumps between the jumps for N. These in-between jumps are governed by the intensities q_{ij} from (2.1), while for each i, λ_i is the intensity for a claim to be triggered when J is in state i, a_i being the probability that J remains in i and a_j for $j \neq i$ the probability that J jumps to j simultaneously with the arrival of the claim. Note that for the chain (J, N) the total intensity for a jump from (i, n) is $q_i + \lambda_i$, where we write

$$
q_i = \sum_{j\neq i} q_{ij} \quad (i \in E)\,. \tag{2.4}
$$

Remark 1. *It follows from the above that J is indeed a Markov chain with transition intensities (corresponding to true jumps for J) $\tilde{q}_{ij}$ for $i \neq j$ given by*

$$
\tilde{q}_{ij} = q_{ij} + \lambda_i a_j. \tag{2.5}
$$

For the risk process X we define the time to ruin as

$$\mathrm{T_r} = \inf\{t \geq 0 : X_t < 0\}$$

with $\mathrm{T_r} = \infty$ if $X_t \geq 0$ for all t. Assuming that $X_0 \equiv x_0 > 0$, we shall below determine in particular the Laplace transform for $\mathrm{T_r}$ and also describe the joint distribution of $(\mathrm{T_r}, \mathrm{Y_r}, \mathrm{I_r})$ with $\mathrm{Y_r} = -X_{\mathrm{T_r}}$ the size of the *undershoot* at the time of ruin and $\mathrm{I_r}$ an indicator specifying whether there is *ruin by jump* (ruin caused by a sufficiently large claim, i.e $Y_\mathrm{r} > 0$) or *ruin by continuity* (X moving continuously through the level 0, i.e $Y_\mathrm{r} = 0$).

The ruin problem will be discussed subject either to $(X_0, J_0) \equiv (x_0, i_0)$ for given, but arbitrary, $x_0 > 0$, $i_0 \in E$, in which case we write $\mathbb{P}^{x_0,i_0}, \mathbb{E}^{x_0,i_0}$ for the underlying probability and the matching expectation, or subject to $X_0 \equiv x_0 > 0$ with J_0 having distribution $\underline{a} = (a_i)$ and being independent of B and the (U_n), in which case we write $\mathbb{P}^{x_0,\underline{a}}, \mathbb{E}^{x_0,\underline{a}}$. If a formula applies to either situation, we just write $\mathbb{P}, \mathbb{E}$. It is unproblematic to set up all the probabilities $\mathbb{P}^{x_0,i_0}$ on the same space and then one may simply define $\mathbb{P}^{x_0,\underline{a}} = \sum_{i_0} a_{i_0} \mathbb{P}^{x_0,i_0}$.

Some further comments on the model: the multiplicative structure of the intensities (2.2) is used in an essential manner for the proof of the main results, Theorem 1 and 2 below. It implies in particular that if $T_n = \inf\{t \geq 0 : N_t = n\}$ is the time of arrival of the n'th claim (with $T_0 \equiv 0$), then the sequence $(T_n)_{n\geq 1}$ is a (possibly delayed) renewal sequence such that the waiting times $V_n = T_n - T_{n-1}$ are independent and for $n \geq 2$ iid and of phase-type with

$$\mathbb{P}(V_n > v) = \underline{a}^T e^{Q_V v} \mathbf{1} \tag{2.6}$$

where $\underline{a}^T$ is the row vector with elements a_i, Q_V is the *sub-intensity* matrix with elements

$$q_{V,ij} = \begin{cases} q_{ij} & (i \neq j), \\ -(q_i + \lambda_i) & (i = j), \end{cases} \tag{2.7}$$

and $\mathbf{1}$ is the column vector of 1's. Under $\mathbb{P}^{x_0,\underline{a}}$, V_1 has the same law as the V_n for $n \geq 2$.

Usually phase-type distributions are described as the distribution of the time to absorption for a Markov chain on a finite state-space E with an additional absorbing state. For us the intensity for 'absorption' from i is λ_i but of course J is not absorbed but returned instantly to E using the entrance law (a_j).

Once a claim has arrived, J will move only through states i such that either $a_i > 0$ or i can be reached by $q_{jj'}$-transitionsfrom some $i' \in E$ with $a_{i'} > 0$. The following basic assumption requires all $i \in E$ to have this property and also for ruin to be possible from any state.

Assumption (A). (i) For any $i \in E$, either $a_i > 0$ or there exists $n \geq 1$ and $i_0, \ldots, i_n \in E$ with $i_n = i$ and all $i_k \neq i_{k-1}$, such that $a_{i_0} > 0$ and all $q_{i_{k-1}, i_k} > 0$. (ii) For any $i \in E$, either $\lambda_i > 0$ or there exists $n \geq 1$ and $i_0, \ldots, i_n \in E$ with $i_0 = i$ and all $i_k \neq i_{k-1}$, such all $q_{i_{k-1}, i_k} > 0$ and $\lambda_{i_n} > 0$.

An alternative formulation of Assumption (A) is that the Markov chain on E with transition intensities $\tilde{q}_{ij}$ given by (2.5) for $i \neq j$ (and, afortiori, $\tilde{q}_{ii} = -\left(q_i + \lambda_i \left(1 - a_i\right)\right)$) is irreducible with some $\lambda_i > 0$. The assumption implies that there will be an infinity of claims: for any $n \geq 1$, $\mathbb{P}\left(V_n = \infty\right) = 0$.

The reader is reminded that under Assumption(A), the sub-intensity matrix Q_V from (2.7) is non-singular (Jacobsen [8], Lemma 1, see also Lemma 2 below) and that since $Q_V \mathbf{1} = -\underline{\lambda}$,

$$Q_V^{-1} \underline{\lambda} = -\mathbf{1}, \tag{2.8}$$

where $\underline{\lambda}$ denotes the column vector $\left(\lambda_i\right)_{i \in E}$.

The renewal structure of the claims arrival process is somewhat restrictive. It does however for a strong dependence between X and N, since e.g the behaviour of X between the jumps for N may well indicate that J is in a state i with λ_i large so there is high probability of the short term arrival of a new claim. (In general the behaviour of X allows one to distinguish between equivalence classes of states in E according to the equivalence relation $\sim$, where $i \sim j$ if either $\sigma_i^2 = \sigma_j^2 > 0$ or if $\sigma_i^2 = \sigma_j^2 = 0$, $\beta_i = \beta_j$: to determine the class to which J_t belongs, look at the path for X in a sufficiently small neighborhood to the right of t. If X follows a straight line, deduce that $\sigma_{J_t}^2 = 0$ and read off β_{J_t} as the slope of the line. If X does not follow a line, compute the quadratic variation $[X]$ of X which necessarily satisfies $[X]_{t''} - [X]_{t'} = \sigma_{J_t}^2 \left(t'' - t'\right)$ in this small neighborhood, and then read off the value of $\sigma_{J_t}^2$. Thus, if all equivalence classes contain just one state, J_t is completely determined from the behaviour of X_s for, say $s \in [t, t + \varepsilon]$ for any $\varepsilon > 0$, while if all states belong to the same class, X contains no precise information about J).

A final comment on the model for X is that it shares with Lévy processes the following *additivity property:* for arbitrary x_0, x_1 and i_0, under the probability $\mathbb{P}^{x_0, i_0}$ the distribution of the process $X + x_1 - x_0$ is the same as the distribution of X itself under $\mathbb{P}^{x_1, i_0}$. This is obvious from the definition of X and a genuine restriction on the class of risk processes considered. However, as will be discussed at the end of Section 6 it is still possible to use the results of the paper to get exact results concerning ruin probabilities and Laplace transforms for processes that are not additive, e.g for suitable diffusion processes with jumps.

We shall introduce some further notation. Let $p = |E| \geq 1$ be the number of states for J (with $p = 1$ certainly allowed). Also let E_{j} denote the set of $i \in E$

such that ruin by jump is possible when $J_t = i$, i.e

$$E_{\mathrm{j}} = \{i \in E : \lambda_i > 0\}$$

with $E_{\mathrm{j}} \neq \emptyset$ by Assumption (A). Similarly, let E_{c} denote the states from which ruin by continuity is possible, i.e

$$E_{\mathrm{c}} = \left\{ i \in E : \sigma_i^2 > 0 \text{ or } \beta_i < 0 \right\}.$$

Here $E_{\mathrm{c}} = \emptyset$ may occur: all $\sigma_i^2 = 0$ and all $\beta_i \geq 0$. We write p_{j} and p_{c} for the number of elements in E_{j} and E_{c} respectively, i.e

$$p_{\mathrm{j}} = \sum_{i \in E} 1_{(\lambda_i > 0)}, \quad p_{\mathrm{c}} = \sum_{i \in E} 1_{\left(\sigma_i^2 > 0 \text{ or } \beta_i < 0\right)}. \tag{2.9}$$

Example 1. *Suppose all $\beta_i = \beta$, all $\sigma_i^2 = \sigma^2$ so that*

$$X_t = x_0 + \beta t + \sigma B_t - \sum_{n=1}^{N_t} U_n. \tag{2.10}$$

In this case J serves only to generate the renewal sequence of claims arrival times with the case $\beta > 0$ and $\sigma^2 = 0$ the model studied in Jacobsen [8].

If also $p = 1$, X is a Lévy process which is the sum of a scaled Brownian motion with drift and an independent compound Poisson process with strictly negative jumps. In particular N is then homogeneous Poisson with intensity $\lambda > 0$, which if $\sigma^2 > 0$ is independent of B. (Asmussen et al [2] studied this model when allowing for two-sided jumps. See Winkel [11] for a treatment of ruin problems for general Lévy processes).

For arbitrary p, one finds that $p_{\mathrm{c}} = p$ iff either $\sigma^2 > 0$ or $\beta < 0$, and $p_{\mathrm{c}} = 0$ otherwise ($\sigma^2 = 0$ and $\beta \geq 0$).

The simple case $p = 1$ with exponential claims is discussed in detail in Section 5 below.

Consider now the joint process (X, J). This is a homogeneous Markov process with state-space $\mathbb{R} \times E$, adapted to the filtration $(\mathcal{F}_t)$ generated by (B, J, C) where $C_t = \sum_{n=1}^{N_t} U_n$, and with infinitesimal generator of the form, for nice enough functions $f : \mathbb{R} \times E \to \mathbb{R}$,

$$\begin{aligned} Af(x,i) &= \beta_i D_x f(x,i) + \tfrac{1}{2}\sigma_i^2 D_{xx}^2 f(x,i) + \sum_{j \neq i} q_{ij} \left(f(x,j) - f(x,i) \right) \qquad (2.11) \\ &\quad + \lambda_i \sum_j a_j \int_0^\infty F_U(dy) \left(f(x-y,j) - f(x,i) \right) \end{aligned}$$

for $x \in \mathbb{R}$, $i \in E$, writing F_U for the distribution of the claims U_n.

Let next $\mathcal{D}$ denote the domain of bounded functions $f : \mathbb{R} \times E \to \mathbb{R}$ such that for all $i \in E$, $x \mapsto f(x,i)$ is twice continuously differentiable for $x \geq 0$ (but not necessarily for $x < 0$) with $D_x f(x,i)$ and $D^2_{xx} f(x,i)$ bounded for $x \geq 0$. Then for $f \in \mathcal{D}$, by Itô's formula (see Appendix A for the martingale representations required and the form of Itô's formula used), assuming that $X_0 > 0$,

$$f\left(X_{\mathrm{T_r}\wedge t}, J_{\mathrm{T_r}\wedge t}\right) = f\left(X_0, J_0\right) + \int_0^{\mathrm{T_r}\wedge t} Af\left(X_s, J_s\right)\, ds + M_t \tag{2.12}$$

where M is an $\mathcal{F}_t$-martingale with, obviously, $M_0 \equiv 0$. (See Appendix A for the precise description of M. Note that in the integral, $X_s \geq 0$ and $Af(X_s, J_s)$ is well defined except possibly at the single point $s = \mathrm{T_r}$). From this it follows directly that for any $\theta \in \mathbb{R}$,

$$\begin{aligned} e^{-\theta(\mathrm{T_r}\wedge t)} f\left(X_{\mathrm{T_r}\wedge t}, J_{\mathrm{T_r}\wedge t}\right) &= f\left(X_0, J_0\right) + \int_0^{\mathrm{T_r}\wedge t} e^{-\theta s}\left(Af(X_s, J_s) - \theta f(X_s, J_s)\right) ds \\ &\quad + \int_0^{\mathrm{T_r}\wedge t} e^{-\theta s}\, dM_s. \end{aligned}$$

If $\theta \geq 0$ the last term is again a mean 0 martingale so for $x_0 > 0$, $i_0 \in E$,

$$\begin{aligned} \mathbb{E}^{x_0,i_0} e^{-\theta(\mathrm{T_r}\wedge t)} f\left(X_{\mathrm{T_r}\wedge t}, J_{\mathrm{T_r}\wedge t}\right) &= f\left(x_0, i_0\right) \\ &\quad + \mathbb{E}^{x_0,i_0} \int_0^{\mathrm{T_r}\wedge t} e^{-\theta s}\left(Af(X_s, J_s) - \theta f(X_s, J_s)\right) ds. \end{aligned} \tag{2.13}$$

Suppose now that $f = f_\theta \in \mathcal{D}$ is a partial eigenfunction for A in the sense that

$$Af\left(x,i\right) = \theta f\left(x,i\right) \quad \left(x \geq 0, i \in E\right). \tag{2.14}$$

Then the integral in the last term of (2.13) vanishes and (2.13) reduces to

$$\mathbb{E}^{x_0,i_0}\left[e^{-\theta \mathrm{T_r}} f\left(X_{\mathrm{T_r}}, J_{\mathrm{T_r}}\right); \mathrm{T_r} \leq t\right] + \mathbb{E}^{x_0,i_0}\left[e^{-\theta t} f\left(X_t, J_t\right); \mathrm{T_r} > t\right] = f\left(x_0, i_0\right). \tag{2.15}$$

Assuming now not only that $\theta \geq 0$ but that $\theta > 0$, taking limits as $t \to \infty$, dominated convergence yields

$$\mathbb{E}^{x_0,i_0}\left[e^{-\theta \mathrm{T_r}} f\left(X_{\mathrm{T_r}}, J_{\mathrm{T_r}}\right); \mathrm{T_r} < \infty\right] = \mathbb{E}^{x_0,i_0} e^{-\theta \mathrm{T_r}} f\left(X_{\mathrm{T_r}}, J_{\mathrm{T_r}}\right) = f\left(x_0, i_0\right), \tag{2.16}$$

one key identity to be exploited in the sequel.

3. The joint Laplace transform

Consider the risk process given by (2.3) with fixed initial state $x_0 > 0$ and recall the definition of the sets of states E_{j} and E_{c} from which ruin by jump, resp. ruin by continuity is possible. Recall also that $E_{\mathrm{j}} \neq \emptyset$ while $E_{\mathrm{c}} = \emptyset$ is possible, and that the number of elements in the two sets are denoted p_{j} and p_{c} respectively. For the statement of Theorem 1 below we shall distinguish $1 + p_{\mathrm{c}}$ different types of ruin corresponding to the events A_{j} and $A_{\mathrm{c},i}$ for $i \in E_{\mathrm{c}}$, where

$$A_{\mathrm{j}} = \left(X_{\mathrm{T_r}} < 0, \mathrm{T_r} < \infty\right) \tag{3.1}$$

is the event that ruin occurs by jump, while

$$A_{\mathrm{c},i} = \left(X_{\mathrm{T_r}} = 0, J_{\mathrm{T_r}} = i, \mathrm{T_r} < \infty\right) \tag{3.2}$$

is the event that ruin occurs by continuity with the Markov chain J in state i.

We shall also need the following notation: for $z \in \mathbb{C}$, $\theta \geq 0$, $Q(z, \theta) = \left(q_{ij}(z, \theta)\right)_{i,j \in E}$ denotes the matrix given by

$$q_{ij}(z, \theta) = \begin{cases} \phi_i(z) - q_i - \lambda_i - \theta & \text{if } i = j, \\ q_{ij} & \text{if } i \neq j, \end{cases} \tag{3.3}$$

where

$$\phi_i(z) = \beta_i z + \tfrac{1}{2}\sigma_i^2 z^2$$

is a polynomium of degree ≤ 2 associated with the scaled Brownian motion with drift that X follows when J is in state i.

The matrix $Q_V = Q(0, 0)$ is the sub-intensity matrix used for the description of the phase-type distribution of the V_n for $n \geq 2$, cf. (2.7) above. Thus the Laplace transform of the waiting times between claims is

$$L_V(\nu) = \mathbb{E}e^{-\nu V_n} = -\underline{a}^T \left(Q_V - \nu I\right)^{-1} \underline{\lambda} \quad (\nu \geq 0), \tag{3.4}$$

for $n \geq 2$. (It is a consequence of Assumption (A) that $Q_V - \nu I$ is non-singular, see Lemma 1 in Jacobsen [8] or Lemma 2 below). Of course, with the notation used here, $Q_V - \nu I = Q(0, \nu)$).

The final assumption we require is that the distribution of the claims has a Laplace transform L_U which is a rational function,

$$L_U(\nu) = \mathbb{E}e^{-\nu U_n} = \frac{P_U(\nu)}{R_U(\nu)} \quad (\nu \geq 0), \tag{3.5}$$

where P_U, R_U are polynomials, standardised so that they have no common complex roots and the leading coefficient for R_U is 1. We write $m \geq 1$ for the degree of R_U and note that necessarily P_U is of degree $\leq m-1$. Below we shall need $P_U(z)$ and $R_U(z)$ for all $z \in \mathbb{C}$ but remind the reader that the resulting extension of L_U to $\bar{L}_U(z) = \frac{P_U(z)}{R_U(z)}$ for $z \in \mathbb{C}$ is meaningless as an expectation: the identity

$$\mathbb{E}e^{-zU_n} = \int_0^\infty e^{-zu} F_U(du) = \frac{P_U(z)}{R_U(z)} \tag{3.6}$$

is guaranteed only for z with $\mathrm{Re}(z) > -\varepsilon$ for some small enough $\varepsilon > 0$. The fact that (3.6) is always true if $\mathrm{Re}(z) \geq 0$ is important: it implies that the m roots z for $R_U(z)$ (counted with multiplicity) must satisfy that $\mathrm{Re}(z) < 0$, an observation used frequently below. It may be noted that $z \mapsto \bar{L}_U(z)$ is analytic in $\mathbb{C}$ except for finitely many poles, located where R_U has its roots.

Notation. For any $z \in \mathbb{C}$, $\theta \geq 0$, denote by $Q^*(z,\theta) = \left(q^*_{ij}(z,\theta)\right)_{i,j\in E}$ the matrix with

$$q^*_{ij}(z,\theta) = (-1)^{i+j} m_{ji}, \tag{3.7}$$

m_{ji} denoting the minor (subdeterminant) of $Q(z,\theta)$ obtained by deleting the j'th row and i'th column. In particular, if $Q(z,\theta)$ is non-singular,

$$Q^*(z,\theta) = (\det Q(z,\theta))\, Q^{-1}(z,\theta).$$

Note that if $p=1$, $\det Q(z,\theta) = \phi_1(z) - \lambda_1 - \theta$ and $Q^*(z,\theta) = 1$ for all z, θ.

Suppose Assumption (A) holds. Then, since for $\theta \geq 0$, $z \mapsto \det Q(z,\theta)$ is a polynomium that is not $\equiv 0$ (cf. Lemmas 2 and 3 below), $Q(z,\theta)$ is non-singular for all but finitely many z. It is a consquence of Lemma 2 below that $Q(z,\theta)$ is non-singular whenever all $\mathrm{Re}(\phi_i(z)) \leq 0$.

For the statement of the main theorem we shall consider two versions of the Cramèr-Lundberg equation. We shall refer to

$$R_U(\gamma) = -P_U(\gamma)\, \underline{a}^T Q^{-1}(\gamma,\theta)\, \underline{\lambda} \tag{3.8}$$

as the *Cramèr-Lundberg* equation, and for $\theta \geq 0$ given call $\gamma \in \mathbb{C}$ a solution to this equation if $Q(\gamma,\theta)$ is non-singular and (3.8) holds. By the the *modified Cramèr-Lundberg* equation we shall understand the equation

$$R_U(\gamma) \det Q(\gamma,\theta) = -P_U(\gamma)\, \underline{a}^T Q^*(\gamma,\theta)\, \underline{\lambda}. \tag{3.9}$$

Note that here both the left and right hand side are polynomials in γ.

Theorem 1 below deals with solutions to (3.8) or (3.9) with $\mathrm{Re}(\gamma) < 0$. It is clear that any solution to (3.8) is also a solution to (3.9), but it is entirely possible that (3.9) may have more solutions than (3.8). In particular this happens if $\mathrm{Re}\,(\gamma) < 0$ and

$$\det Q\,(\gamma,\theta) = 0, \quad \underline{a}^T Q^*\,(\gamma,\theta)\,\underline{\lambda} = 0, \tag{3.10}$$

see Remark 4 below for further discussion.

Recall that the size of the undershoot at the time of ruin is denoted $\mathrm{Y_r} = -X_{\mathrm{T_r}}$.

Theorem 1. *Consider the risk process X given by (2.3) and assume that the Laplace transform for the distribution of the claims is given by (3.5) with the degree of R_U equal to m. Assume that Assumption (A) holds.*

(i) For any $\theta > 0$, the modified Cramèr-Lundberg equation (3.9) has precisely $m + p_{\mathrm{c}}$ solutions (counted with multiplicity) $(\gamma_\ell)_{1\le\ell\le m+p_{\mathrm{c}}} = (\gamma_\ell(\theta))$ with $\mathrm{Re}\,(\gamma_\ell) < 0$.

(ii) For any $\theta > 0$, γ with $\mathrm{Re}\,(\gamma) < 0$ is a solution to the Cramèr-Lundberg equation (3.8) if and only if γ is a solution to the modified equation (3.9) with $Q\,(\gamma,\theta)$ non-singular.

(iii) For $\theta > 0$ given such that if $(\tilde{\gamma}_k)_{1\le k\le m}$ are any m of the solutions to (3.8) with $\mathrm{Re}\,(\tilde{\gamma}_k) < 0$, and these solutions are distinct with all the matrices $Q\,(\tilde{\gamma}_k,\theta)$ non-singular, it holds for all $x_0 > 0$, $i_0 \in E$ and all $\zeta \ge 0$ that

$$\begin{aligned}\sum_{i\in E_{\mathrm{c}}}\sum_{k=1}^{m} r_k \left(Q^{-1}\,(\tilde{\gamma}_k,\theta)\,\underline{\lambda}\right)_i \mathbb{E}^{x_0,i_0}\left[e^{-\theta \mathrm{T_r}}; A_{\mathrm{c},i}\right] \\ -\frac{\sum_{k=1}^m r_k}{L_U\,(\zeta)}\mathbb{E}^{x_0,i_0}\left[e^{-\theta\mathrm{T_r}-\zeta\mathrm{Y_r}}; A_{\mathrm{j}}\right] \\ = \sum_{k=1}^{m} r_k \left(Q^{-1}\,(\tilde{\gamma}_k,\theta)\,\underline{\lambda}\right)_{i_0} e^{\tilde{\gamma}_k x_0}\end{aligned} \tag{3.11}$$

where $r_k = r_k\,(\theta,\zeta)$ is given by

$$r_k = -\frac{P_U\,(\tilde{\gamma}_k)}{(\tilde{\gamma}_k-\zeta)\prod_{k'\neq k}(\tilde{\gamma}_k-\tilde{\gamma}_{k'})}. \tag{3.12}$$

(iv) If all the solutions $(\gamma_\ell)_{1\le\ell\le m+p_{\mathrm{c}}}$ to (3.8) with $\mathrm{Re}\,(\gamma_\ell) < 0$ are distinct with all the matrices $Q\,(\gamma_\ell,\theta)$ non-singular, using (3.11) $p_{\mathrm{c}}+1$ times with, say, $(\tilde{\gamma}_k)_{1\le k\le m} = \left(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_{m-1},\tilde{\gamma}_{m+s}\right)$ for $s = 0,\ldots,p_{\mathrm{c}}$, a system of linear equations is obtained that, provided the matrix of coefficients to the $p_{\mathrm{c}}+1$ unknowns

$$\mathbb{E}^{x_0,i_0}\left[e^{-\theta\mathrm{T_r}-\zeta\mathrm{Y_r}}; A_{\mathrm{j}}\right] \quad \text{and} \quad \mathbb{E}^{x_0,i_0}\left[e^{-\theta\mathrm{T_r}}; A_{\mathrm{c},i}\right] \quad (i\in E_{\mathrm{c}}) \tag{3.13}$$

is non-singular, can be solved uniquely.

Remark 2. *It is quite possible that the system of equations in (iv) always have a unique solution (except, possibly, for a few values of* (θ, ζ)*), but we have no proof of this.*

In (iv) there are of course a multitude of ways in which to choose $p_c + 1$ *equations. The solutions do of course not depend on this choice, as may be verified directly and as is verified numerically in Example 2 below. When choosing the equations it is vital that* all *the* γ_ℓ *for* $1 \leq \ell \leq m + p_c$ *are used.*

Remark 3. *The technique used for proving Theorem 1 was first used by Jacobsen [8]. The main result there, Theorem 6 (dealing with the Laplace transform of* $\mathrm{T_r}$ *only), corresponds to the model with all* $\beta_i = \beta > 0$ *and all* $\sigma_i^2 = 0$*, the traditional risk model with a fixed premium rate and where claims arrive according to a renewal process with interarrival times of phase-type. We thus have* $E_c = \emptyset$ *and see that since all* $\phi_i(z) = \beta z$ *so that*

$$Q(z, \theta) = Q_V - (\theta - \beta z) I,$$

(3.9) becomes (cf. (3.4))

$$L_V(\theta - \beta\gamma) \bar{L}_U(\gamma) = 1,$$

one of the forms presented in Jacobsen [8].

Remark 4. *Theorem 1 is really intended for the situation assumed in (iv) where (3.9) has* $m + p_c$ *distinct solutions* γ *with* $\mathrm{Re}(\gamma) < 0$ *and all the matrices* $Q(\gamma, \theta)$ *are non-singular. The main purpose of this remark is to discuss what happens when the assumptions from (iv) fail.*

Consider the roots of (3.9) as θ *varies. It may well occur that except for finitely many values of* θ*, the* $m + p_c$ *roots are distinct while moving* θ *across an exceptional value may e.g cause two real roots to collapse into one and then split intp two complex conjugate roots. In this case Theorem 1 may still be used to find the partial Laplace transforms (3.13) since they are all continuous functions of* θ*.*

It may however also occur that the assumptions made in (iv) fail systematically, because for all $\theta > 0$*, (3.10) holds for some* $\gamma = \gamma(\theta)$ *solving (3.9), in particular* $Q(\gamma, \theta)$ *is singular, and in this case Theorem 1 is useless. Assumption (A) provides one safeguard against this unfortunate situation: without Assumption (A) there might be some state* $i_0 \in E$ *such that* $a_{i_0} = 0$ *and* $q_{ii_0} = 0$ *for all* $i \neq i_0$*, in particular* i_0 *is never visited after the first claim. But then* $Q(z, \theta)$ *has the form (identifying* i_0 *with the first row/column and denoting by* $\bar{Q}(z, \theta)$ *that part of* $Q(z, \theta)$ *obtained when deleting the row and column corresponding to* i_0*)*

$$Q(z, \theta) = \begin{pmatrix} \psi_{i_0}(z, \theta) & Q_{i_0}(z, \theta) \\ 0 & \bar{Q}(z, \theta) \end{pmatrix}$$

with $\psi_{i_0}(z,\theta) = \phi_{i_0}(z) - q_{i_0} - \lambda_{i_0} - \theta$, a polynomium that will have one root < 0, no matter what is the value of $\theta > 0$, provided $\sigma^2_{i_0} > 0$ or $\beta_{i_0} < 0$. Clearly $\det Q(z,\theta)$ *contains the factor $\psi_{i_0}(z,\theta)$ and it is also easy to see that the minor m_{ji} (see (3.7)) contains this factor for $i \geq 2$, $j \geq 2$ while obviously $m_{ji} = 0$ if $i \geq 2$, $j = 1$. Thus $Q^*(z,\theta)$ has the form*

$$Q^*(z,\theta) = \begin{pmatrix} \det \bar{Q}(z,\theta) & Q^*_{i_0}(z,\theta) \\ 0 & \psi_{i_0}(z,\theta)\,\bar{Q}^*(z,\theta) \end{pmatrix}$$

and since $a_{i_0} = 0$ by the assumption on i_0, it follows that $a^T Q^*(z,\theta) = 0$ if $\psi_{i_0}(z,\theta) = 0$. We have thus shown that with this i_0 present, if for all $\theta > 0$, $z = \gamma(\theta) < 0$ solves $\psi_{i_0}(z,\theta) = 0$, then (3.10) holds and Theorem 1 is meaningless.

There is one more situation where the assumptions in (iv) may fail systematically. Consider first the simple model with $p = 1$ (see Example 1 above). The same model may be obtained with an arbitrary $p \geq 2$ using the following silly parametrisation: simply put all $\sigma^2_i = \sigma^2$, $\beta_i = \beta$, $\lambda_i = \lambda$ and also put $q_{ij} = 0$ for all $i \neq j$. Then $\phi_i(z) = \phi(z) = \beta z + \frac{1}{2}\sigma^2 z^2$ for all i and

$$Q(z,\theta) = (\phi(z) - \lambda - \theta)\, I$$

so that

$$\det Q(z,\theta) = (\phi(z) - \lambda - \theta)^p, \quad Q^*(z,\theta) = (\phi(z) - \lambda - \theta)^{p-1}\, I$$

and if $z = \gamma(\theta) < 0$ solves $\phi(z) - \lambda - \theta = 0$ we again have that (3.10) holds and the theorem is void. This is an example of the following more general situation where two or more states in E may be collapsed into one without affecting the model. Without discussing the details we claim that such a merger of states may be performed for $i' \in E' \subset E$ provided the $\beta_{i'}, \sigma^2_{i'}, \lambda_{i'}, a_{i'}$ do not depend on $i' \in E'$, the $q_{i'j}$ for $i' \in E'$, $j \in E \backslash E'$ do not depend on $i' \in E'$ and the sums $\sum_{j \in E' \backslash i'} q_{i'j}$ do not depend on $i' \in E'$.

The overall message is that in order for Theorem 1 to be of interest, not only must Assumption (A) hold, but it is also essential that the model be parametrised in a suitable minimal fashion.

Remark 5. *It is worth emphasising that for Theorem 1, the solutions to the Cramèr-Lundberg equations with strictly* negative *real parts are required. The work by Avram and Usabél [3], [4] and Asmussen et al [2] involving special cases of the model (2.3) uses the solutions with* positive *real parts.*

Before giving the proof of Theorem 1, we present two lemmas which are proved in Appendix B. Below the complex unit is denoted $\mathrm{i} = \sqrt{-1}$.

Lemma 2. *Assume that Assumption (A) holds. For $\theta \geq 0$ and $y \in \mathbb{R}$, the matrix $Q(\mathrm{i}y, \theta)$ is non-singular and for any $\theta \geq 0$, $y \in \mathbb{R}$ and $j \in E$,*

$$\left| \left(Q^{-1} (\mathrm{i}y, \theta) \underline{\lambda} \right)_j \right| \leq \frac{q_j + \lambda_j}{q_j + \lambda_j + \theta}, \tag{3.14}$$

in particular

$$\left| \left(Q^{-1} (\mathrm{i}y, \theta) \underline{\lambda} \right)_j \right| < 1 \tag{3.15}$$

if $\theta > 0$.

Lemma 3. *Assume that Assumption (A) holds. For any $\theta \geq 0$, the polynomium $z \mapsto \det Q(z, \theta)$ is of degree*

$$d = 2 \sum_{j \in E} 1_{\left(\sigma_j^2 > 0\right)} + \sum_{j \in E} 1_{\left(\sigma_j^2 = 0, \beta_j \neq 0\right)} \tag{3.16}$$

and has exactly p_{c} roots z with $\mathrm{Re}(z) < 0$.

Proof. (Theorem 1). (ii) and (iv) are obvious. We focus first on (iii) and after that prove (i). Let $\theta > 0$ and $\zeta \geq 0$ be given. Consider $f : \mathbb{R} \times E \to \mathbb{R}$ of the form

$$f(x, i) = \begin{cases} \sum_{k=1}^{m} c_{ik} e^{\gamma_k x} & (x \geq 0), \\ K e^{\zeta x} & (x < 0). \end{cases} \tag{3.17}$$

If all $\mathrm{Re}(\gamma_k) \leq 0$, then $f \in \mathcal{D}$. Suppose also that $\gamma_1, \ldots, \gamma_m$ are distinct solutions to (3.8) with $\mathrm{Re}(\gamma_k) < 0$. We shall show that

$$A f(x, i) = \theta f(x, i) \tag{3.18}$$

for $x \geq 0$, $i \in E$, where the k'th column $c_{|k}$ of the matrix $(c_{ik})_{i \in E, 1 \leq k \leq m}$ and the constant K are given by the expressions

$$c_{|k} = r_k Q^{-1} (\gamma_k, \theta) \underline{\lambda} \tag{3.19}$$

and

$$K = -\frac{\sum_{k=1}^{m} r_k}{L_U(\zeta)} \tag{3.20}$$

respectively where r_k is as in (3.12). (Note that the γ_k depend on θ but not on ζ while $r_k, c_{|k}$ and K depend on both θ and ζ).

From (3.18) it follows by (2.16) that for initial states $x_0 > 0$, $i_0 \in E$,

$$\mathbb{E}^{x_0, i_0} \left[e^{-\theta \mathrm{T_r}} \sum_{k=1}^{m} c_{J_{\mathrm{T_r}}, k}; \mathrm{Y_r} = 0 \right] + \mathbb{E}^{x_0, i_0} \left[K e^{-\theta \mathrm{T_r} - \zeta \mathrm{Y_r}}; \mathrm{Y_r} > 0 \right] = \sum_{k=1}^{m} c_{i_0 k} e^{\gamma_k x_0}$$

which is precisely (3.11) for $\tilde{\gamma}_k(\theta) = \gamma_k$ for $1 \le k \le m$.

Thus (iii) follows by verifying that (3.18) holds when $r_k, c_{|k}$ and K are given by (3.12), (3.19) and (3.20) respectively, and this we now show.

With f as in (3.17), since by (2.11),

$$\begin{aligned} Af(x,i) &= \sum_{k=1}^{m} \Bigg[c_{ik}\phi_i(\gamma_k) e^{\gamma_k x} + \sum_{j\neq i} q_{ij}(c_{jk} - c_{ik}) e^{\gamma_k x} \\ &\quad + \lambda_i \sum_j a_j \left\{ \int_{]0,x]} F_U(dy) \sum_k c_{jk} e^{\gamma_k (x-y)} + \int_{]x,\infty[} F_U(dy)\ K e^{\zeta(x-y)} - c_{ik} e^{\gamma_k x} \right\} \Bigg], \end{aligned}$$

it is seen that (3.18) is equivalent to

$$\begin{aligned} 0 &= \sum_k e^{\gamma_k x} \left(Q(\gamma_k, \theta) c_{|k} \right)_i \\ &\quad + \lambda_i \sum_j a_j \left[\int_{]0,x]} F_U(dy) \sum_k c_{jk} e^{\gamma_k (x-y)} + \int_{]x,\infty[} F_U(dy)\ K e^{\zeta(x-y)} \right] \end{aligned} \tag{3.21}$$

which we need for all $x \ge 0$, $i \in E$. But using (3.19) allows us to eliminate the dependence on i since a common factor λ_i appears and we are left with

$$\sum_k r_k e^{\gamma_k x} + \sum_j a_j \left[\int_{]0,x]} F_U(dy) \sum_k c_{jk} e^{\gamma_k (x-y)} + \int_{]x,\infty[} F_U(dy)\ K e^{\zeta(x-y)} \right] = 0 \tag{3.22}$$

which must hold for all $x \ge 0$. Taking $x = 0$ shows that

$$\sum_k r_k + K L_U(\zeta) = 0$$

in agreement with (3.20).

Because all $\mathrm{Re}(\gamma_k) < 0$, (3.22) holds iff it holds for the Laplace transform: multiplying by $e^{-\nu x}$ and integrating x from 0 to ∞ gives that (3.22) is equivalent to

$$\sum_k r_k \left(\frac{1}{\nu - \gamma_k} - \frac{1}{\nu - \zeta} \left(1 - \frac{L_U(\nu)}{L_U(\zeta)} \right) \right) + \sum_k s_k \frac{L_U(\nu)}{\nu - \gamma_k} = 0 \tag{3.23}$$

for $\nu \ge 0$ with

$$s_k = \sum_j a_j c_{jk} = r_k a^T Q^{-1}(\gamma_k, \theta)\, \underline{\lambda}. \tag{3.24}$$

(To deduce (3.23) from (3.22) one uses (3.20) and the elementary formulas

$$\int_0^\infty dx\, e^{-\nu x} \int_{]0,x]} F_U(dy)\ e^{\gamma_k (x-y)} = \frac{L_U(\nu)}{\nu - \gamma_k}$$

and

$$\int_0^\infty dx\, e^{-\nu x} \int_{]x,\infty[} F_U(dy)\, e^{\zeta(x-y)} = \frac{1}{\nu-\zeta}\left(L_U(\zeta) - L_U(\nu)\right).$$

(3.23) gives

$$\begin{aligned} L_U(\nu) &= \frac{-\sum_k r_k\left(\frac{1}{\nu-\gamma_k} - \frac{1}{\nu-\zeta}\right)}{\sum_k\left(s_k\frac{1}{\nu-\gamma_k} + \frac{r_k}{(\nu-\zeta)L_U(\zeta)}\right)} \\ &= \frac{-\sum_k r_k(\gamma_k-\zeta)\,\pi_{\setminus k}(\nu)}{\sum_k\left(s_k(\nu-\zeta)\,\pi_{\setminus k}(\nu) + \frac{r_k}{L_U(\zeta)}(\nu-\gamma_k)\,\pi_{\setminus k}(\nu)\right)} \end{aligned} \tag{3.25}$$

using the notation $\pi_{\setminus k}(\nu) = \prod_{k':k'\neq k}(\nu-\gamma_{k'})$.

At this stage the reader is reminded that if $\mathcal{P}$ is a polynomium of degree $\leq m-1$, then

$$\mathcal{P}(z) = \sum_k \frac{\mathcal{P}(\gamma_k)}{\pi_{\setminus k}(\gamma_k)}\pi_{\setminus k}(z) \quad (z\in\mathbb{C}), \tag{3.26}$$

see e.g Lemma 4 in Jacobsen [8].

Inserting now (3.12) into the numerator of (3.25) and using (3.26) shows that the numerator equals $P_U(\nu)$. To identify the denominator with $R_U(\nu)$, first note that

$$S(\nu) = \frac{1}{\nu-\zeta}\left(R_U(\nu) - \frac{P_U(\nu)}{L_U(\zeta)}\right)$$

for $\nu \geq 0$ defines a polynomium of degree $\leq m-1$, simply because $R_U(\nu) - \frac{P_U(\nu)}{L_U(\zeta)}$ is a polynomium of degree m which has $\nu=\zeta$ as a root; hence, by (3.26),

$$S(\nu) = \sum_k \frac{1}{(\gamma_k-\zeta)\,\pi_{\setminus k}(\gamma_k)}\left(R_U(\gamma_k) - \frac{P_U(\gamma_k)}{L_U(\zeta)}\right)\pi_{\setminus k}(\nu). \tag{3.27}$$

Next, rewrite the denominator in (3.25) as

$$\tilde{R}(\nu) = (\nu-\zeta)\sum_k\left(s_k + \frac{r_k}{L_U(\zeta)}\right)\pi_{\setminus k}(\nu) + \frac{1}{L_U(\zeta)} \times \text{numerator}$$

and recall that the numerator equals $P_U(\nu)$. But

$$s_k + \frac{r_k}{L_U(\zeta)} = -\frac{P_U(\gamma_k)}{(\gamma_k-\zeta)\,\pi_{\setminus k}(\gamma_k)}\left(a^T Q^{-1}(\gamma_k,\theta)\,\underline{\lambda} + \frac{1}{L_U(\zeta)}\right)$$

using (3.24) and (3.12), and because of (3.8) this

$$= \frac{1}{(\gamma_k-\zeta)\,\pi_{\setminus k}(\gamma_k)}\left(R_U(\gamma_k) - \frac{P_U(\gamma_k)}{L_U(\zeta)}\right)$$

and comparing with (3.27) we finally arrive at

$$\tilde{R}(\nu) = (\nu - \zeta) S(\nu) + \frac{P_U(\nu)}{L_U(\zeta)} = R_U(\nu)$$

as wanted. This completes the proof of (iii).

We proceed with the proof of (i). As noted below (3.9), both the left and right hand sides are polynomials. Because of Lemma 3, the left hand side $\mathcal{P}_l$ is of degree $m+d$, with d given by (3.16). By inspection, every minor m_{ij} of $Q(z,\theta)$ is seen to be a polynomium of degree $\leq d$ and consequently, (cf (3.7)), the right hand side $\mathcal{P}_r$ of (3.9) is of degree $\leq (m-1)+d < m+d$.

Let now, for $\rho > 0$, Γ_ρ denote the interior of the subset of $\mathbb{C}$ determined by the outer boundary

$$\partial\Gamma_\rho = \{z : |z| = \rho, \mathrm{Re}(z) < 0\} \cup \{z : z = \mathrm{i}y, -\rho \leq y \leq \rho\}. \tag{3.28}$$

By Rouché's theorem from complex function theory, $\mathcal{P}_l$ and the difference $\mathcal{P}_l - \mathcal{P}_r$ will have the same number of zeros (counted with multiplicity) in Γ_ρ provided

$$\left|P_U(z)\, \underline{a}^T Q^*(z,\theta)\, \underline{\lambda}\right| = |\mathcal{P}_r(z)| < |\mathcal{P}_l(z)| = |R_U(z) \det Q(z,\theta)| \quad (z \in \partial\Gamma_\rho). \tag{3.29}$$

Because $\mathcal{P}_r$ is of a degree strictly less than that of $\mathcal{P}_l$, (3.29) is obvious for the $z \in \partial\Gamma_\rho$ with $\mathrm{Re}(z) < 0$ if only ρ is large enough. And for $z = \mathrm{i}y \in \partial\Gamma_\rho$, since by Lemma 2 $Q(\mathrm{i}y,\theta)$ is non-singular, (3.29) is equivalent to

$$\left|P_U(\mathrm{i}y)\, \underline{a}^T Q^{-1}(\mathrm{i}y,\theta)\, \underline{\lambda}\right| < |R_U(\mathrm{i}y)|$$

and this follows from (3.15) (which implies that

$$\left|\underline{a}^T Q^{-1}(\mathrm{i}y,\theta)\, \underline{\lambda}\right| \leq \sum_j a_j \left|\left(Q^{-1}(\mathrm{i}y,\theta)\, \underline{\lambda}\right)_j\right| < 1)$$

and the fact that

$$\frac{P_U(\mathrm{i}y)}{R_U(\mathrm{i}y)} = \bar{L}_U(\mathrm{i}y) = L_U(\mathrm{i}y) = \int_0^\infty e^{\mathrm{i}yu}\, F_U(du)$$

with the last term ≤ 1 in absolute value.

Thus (3.29) holds for ρ large enough and letting $\rho \to \infty$ we see that the number of solutions γ to (3.9) with $\mathrm{Re}(\gamma) < 0$, is the same as the number of roots γ with $\mathrm{Re}(\gamma) < 0$ (counted with multiplicity) of the polynomium $\mathcal{P}_l = R_U \det Q(\cdot,\theta)$. That this number is $m + p_c$ as claimed, is a direct consequence of Lemma 3 and the fact that R_U is a polynomium of degree m with all the roots γ satisfying $\mathrm{Re}(\gamma) < 0$, cf. 10. This completes the proof of (i). ■

4. The probability of ruin

The probability of ultimate ruin is

$$\begin{aligned}
\mathrm{p_r} &= \mathbb{P}^{x_0,i_0}\left(\mathrm{T_r}<\infty\right)\\
&= \mathbb{P}^{x_0,i_0}\left(A_{\mathrm{j}}\right)+\sum_{i\in E_{\mathrm{c}}}\mathbb{P}^{x_0,i_0}\left(A_{\mathrm{c},i}\right) \qquad (4.1)
\end{aligned}$$

and here each term may be determined from the expressions for the quantities in (3.13) by taking $\zeta=0$ and letting $\theta\downarrow 0$ since for any event A,

$$\mathbb{P}^{x_0,i_0}\left(\mathrm{T_r}<\infty,A\right)=\lim_{\theta\downarrow 0,\theta>0}\mathbb{E}^{x_0,i_0}\left[e^{-\theta\mathrm{T_r}};A\right].$$

Thus Theorem 1 makes it possible to determine each of the terms in (4.1), but for direct calculation it is of course preferable to avoid taking limits and we shall now discuss how this may be done. Compared with Theorem 1 and its proof, the discussion is more intricate and involves a number of subtleties of an analytic nature.

It is natural that one should try and use the Cramèr-Lundberg equations (3.8) or (3.9) with $\theta=0$. However, allowing $\theta=0$ invalidates the sharp inequality in (3.15) (for $y=0$ the inequality is simply wrong when $\theta=0$, cf. (2.8)) which was used in an essential manner in order to apply Rouché's theorem in the proof of Theorem 1, see p.17, so it is necessary to be careful – in particular, as we shall see, the analogue Theorem 2 below of Theorem 1 used for computing the terms in (4.1) has two versions according as $\mathrm{p_r}<1$ or $\mathrm{p_r}=1$.

Write $\xi=\mathbb{E}U_n$ for the expected claim size and $\mu=\mathbb{E}V_n$ for the expected times between claims for $n\leq 2$, so that (e.g use (2.6))

$$\mu=-\underline{a}^TQ_V^{-1}\mathbf{1}. \qquad (4.2)$$

Also let $\underline{\alpha}=(\alpha_i)_{i\in E}$ denote the unique invariant probability for the irreducible Markov chain J, i.e $\alpha_i>0$, $\sum\alpha_i=1$ and

$$\sum_{i\in E}\alpha_i\tilde{q}_{ij}=0 \quad (j\in E) \qquad (4.3)$$

with $\tilde{q}_{ij}$ given by (2.5) for $i\neq j$ and $\tilde{q}_{ii}=-\left(q_i+\lambda_i\left(1-a_i\right)\right)$. Then in fact the row vector $\boldsymbol{\alpha}^T$ is given by

$$\underline{\alpha}^T=-\frac{1}{\mu}\underline{a}^TQ_V^{-1} \qquad (4.4)$$

as is argued as follows: use (2.7) to rewrite (4.3) as

$$\sum_i\alpha_iq_{V,ij}=-a_j\sum_i\alpha_i\lambda_i.$$

Thus, with $C = \sum_i \alpha_i \lambda_i$, $\underline{\alpha}^T Q_V = -C\underline{a}^T$ and (4.4) follows when using that $\underline{\alpha}^T \mathbf{1} = 1$ together with (4.2).

Proposition 1. *For $x_0 > 0$, $i_0 \in E$, the ruin probability $\mathrm{p_r} = \mathbb{P}^{x_0,i_0}(\mathrm{T_r} < \infty)$ equals 1 if and only if*

$$\sum_{i \in E} \alpha_i \beta_i \leq \frac{\xi}{\mu}. \tag{4.5}$$

Proof. By (2.3),

$$\frac{1}{t}(X_t - x_0) = \frac{1}{t}\int_0^t \beta_{J_s}\,ds + \frac{1}{t}\int_0^t \sigma_{J_{s-}}\,dB_s - \frac{1}{t}\sum_{n=1}^{N_t} U_n. \tag{4.6}$$

Since J is ergodic,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta_{J_s}\,ds = \sum_{i\in E} \alpha_i \beta_i$$

$\mathbb{P}^{x_0,i_0}$-a.s. By standard properties of Brownian motion, the second term on the right of (4.6) converges to 0 a.s., while because of the renewal structure of the claims arrival process (T_n) and the independence between this and (U_n),

$$\lim_{t\to\infty} \frac{1}{t}\sum_{n=1}^{N_t} U_n = \frac{\xi}{\mu}$$

a.s. Thus $X_t \to -\infty$ $\mathbb{P}^{x_0,i_0}$-a.s. if $\sum_{i\in E} \alpha_i\beta_i < \frac{\xi}{\mu}$ and $X_t \to \infty$ $\mathbb{P}^{x_0,i_0}$-a.s. if $\sum_{i\in E}\alpha_i\beta_i > \frac{\xi}{\mu}$ and this certainly shows that $\mathrm{p_r} = 1$ if (4.5) holds with sharp inequality and also, since the drift of X to ∞ may set in with probability > 0 before any claim has arrived, that $\mathrm{p_r} < 1$ if (4.5) does not hold. If there is equality in (4.5) one may e.g argue directly using (2.3) that $\mathrm{p_r} = \mathrm{p_r}(\varepsilon)$ is a continuous function of $\varepsilon \geq 0$, where $\mathrm{p_r}(\varepsilon)$ is the ruin probability for the process obtained by replacing β_i by $\beta_i - \varepsilon$ but retaining all the other parameters and then of course $\mathrm{p_r}(0) = \lim_{\varepsilon\downarrow 0,\varepsilon>0} \mathrm{p_r}(\varepsilon) = 1$ since for the $(\beta_i - \varepsilon)$-process, (4.5) holds with sharp inequality. ■

The Cramèr-Lundberg equations (3.8) and (3.9) for $\theta = 0$ take the forms

$$R_U(\gamma) = -P_U(\gamma)\,\underline{a}^T Q^{-1}(\gamma, 0)\,\underline{\lambda} \tag{4.7}$$

and

$$R_U(\gamma)\det Q(\gamma,0) = -P_U(\gamma)\,\underline{a}^T Q^*(\gamma,0)\,\underline{\lambda} \tag{4.8}$$

respectively with the elements of $Q^*(\gamma, 0)$ given as in (3.7) when taking $\theta = 0$.

The first thing to note is that $\gamma = 0$ is a solution to both (4.7) and (4.8): $Q(0,0) = Q_V$ is non-singular and by (2.8), $\underline{a}^T Q_V^{-1} \underline{\lambda} = -1$ while also $R_U(0) = P_U(0)$. But $\tilde{\gamma}_k = 0$ cannot be used to define the r_k from (3.12) when $\zeta = 0$, so the 0-solution is to be avoided when describing the terms in (4.1). What we shall show is that if $\mathrm{p_r} < 1$, then (4.8) has precisely $m + p_\mathrm{c}$ solutions γ_ℓ with $\mathrm{Re}(\gamma_\ell) < 0$ (exactly as in the case $\theta > 0$), while if $\mathrm{p_r} = 1$ there are only $m + p_\mathrm{c} - 1$ solutions (one of the solutions from the case $\theta > 0$ has moved to 0). If $\mathrm{p_r} < 1$ the $p_\mathrm{c} + 1$ terms in (2.6) may be found in complete analogy with Theorem 1, (iii) and (iv), while if $\mathrm{p_r} = 1$, since the terms of (2.6) now add to 1, only p_c of these have to be found and for this it suffices to exploit the $m + p_\mathrm{c} - 1$ solutions γ to (4.8) with $\mathrm{Re}(\gamma) < 0$.

Recall that the events for ruin, A_j and $A_{\mathrm{c},i}$ as defined by (3.1) and (3.2) respectively, are subsets of the set $(\mathrm{T_r} < \infty)$.

Theorem 2. *Consider the risk process X given by (2.3) and assume that the Laplace transform for the distribution of the claims is given by (3.5) with the degree of R_U equal to m. Assume that Assumption (A) holds.*

(i) If $\mathrm{p_r} < 1$ the modified Cramèr-Lundberg equation (4.8) for the ruin probabilities has precisely $m+p_\mathrm{c}$ solutions (counted with multiplicity) $(\gamma_\ell)_{1 \le \ell \le m+p_\mathrm{c}} = (\gamma_\ell(\theta))$ with $\mathrm{Re}(\gamma_\ell) < 0$. If $\mathrm{p_r} = 1$ (4.8) has precisely $m + p_\mathrm{c} - 1$ solutions (counted with multiplicity) $(\gamma_\ell)_{1 \le \ell \le m+p_\mathrm{c}} = (\gamma_\ell(\theta))$ with $\mathrm{Re}(\gamma_\ell) < 0$.

(ii) γ with $\mathrm{Re}(\gamma) < 0$ is a solution to the Cramèr-Lundberg equation (4.7) if and only if γ is a solution to the modified equation (4.8) with $Q(\gamma, 0)$ non-singular.

(iii) In both cases, $\mathrm{p_r} < 1$ and $\mathrm{p_r} = 1$, if $(\tilde{\gamma}_k)_{1 \le k \le m}$ are any m of the solutions to (4.7) with $\mathrm{Re}(\tilde{\gamma}_k) < 0$, and these solutions are distinct with all the matrices $Q(\tilde{\gamma}_k, 0)$ non-singular, it holds for all $x_0 > 0$, $i_0 \in E$ that

$$\sum_{i \in E_\mathrm{c}} \sum_{k=1}^{m} r_k \left(Q^{-1}(\tilde{\gamma}_k, 0) \underline{\lambda} \right)_i \mathbb{P}^{x_0, i_0}(A_{\mathrm{c},i}) - \left(\sum_{k=1}^{m} r_k \right) \mathbb{P}^{x_0, i_0}(A_\mathrm{j}) = \sum_{k=1}^{m} r_k \left(Q^{-1}(\tilde{\gamma}_k, 0) \underline{\lambda} \right)_{i_0} e^{\tilde{\gamma}_k x_0} \qquad (4.9)$$

where r_k is given by

$$r_k = -\frac{P_U(\tilde{\gamma}_k)}{\tilde{\gamma}_k \prod_{k' \ne k} (\tilde{\gamma}_k - \tilde{\gamma}_{k'})}. \qquad (4.10)$$

(iv) If $\mathrm{p_r} < 1$ *and all the solutions* $(\gamma_\ell)_{1\le\ell\le m+p_\mathrm{c}}$ *to (3.8) with* $\mathrm{Re}\,(\gamma_\ell) < 0$ *are distinct with all the matrices* $Q\,(\gamma_\ell, 0)$ *non-singular, using (4.22)* $p_\mathrm{c}+1$ *times with, say,* $(\tilde{\gamma}_k)_{1\le k\le m} = \left(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{m-1}, \tilde{\gamma}_{m+s}\right)$ *for* $s = 0, \ldots, p_\mathrm{c}$*, a system of linear equations is obtained that, provided the matrix of coefficients to the* $p_\mathrm{c} + 1$ *unknowns*

$$\mathbb{P}^{x_0,i_0}\,(A_\mathrm{j}) \qquad \textit{and} \qquad \mathbb{P}^{x_0,i_0}\,(A_{\mathrm{c},i}) \quad (i \in E_\mathrm{c})\,. \tag{4.11}$$

is non-singular, can be solved uniquely.

If $\mathrm{p_r} = 1$ *and all the solutions* $(\gamma_\ell)_{1\le\ell\le m+p_\mathrm{c}-1}$ *to (3.8) with* $\mathrm{Re}\,(\gamma_\ell) < 0$ *are distinct with all the matrices* $Q\,(\gamma_\ell, 0)$ *non-singular, using (4.22)* p_c *times with, say,* $(\tilde{\gamma}_k)_{1\le k\le m} = \left(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{m-1}, \tilde{\gamma}_{m+s}\right)$ *for* $s = 0, \ldots, p_\mathrm{c}-1$*, a system of linear equations is obtained that, provided the relevant matrix of coefficients is non-singular, can be solved uniquely for* p_c *of the* $p_\mathrm{c}+1$ *unknowns in (4.11) with the remaining unknown equal to* 1 *minus the sum of the* p_c *unknowns just found.*

Proof. We do not give all the details but focus on those parts of the proof that differ from the arguments used for the proof of Theorem 1. Also, when treating the case $\mathrm{p_r} = 1$ we shall assume that there is strict inequality in (4.5).

As for the claims of the theorem, (ii) and (iv) are obvious, and we then proceed to discuss (iii). So let $\gamma_1, \ldots, \gamma_m$ be m distinct roots of (4.7) and consider the function f given by (3.17) with (cf. (3.19) and (3.20)),

$$c_{|k} = r_k Q^{-1}\,(\gamma_k, 0)\,\underline{\lambda}, \quad K = -\sum_{k=1}^{m} r_k$$

where r_k is given by (4.10). Arguing as in the proof of Theorem 1 one finds that, cf. (3.18),

$$Af\,(x, i) = 0$$

for $x \ge 0$, $i \in E$ and consequently by (2.15), for any $t \ge 0$,

$$\mathbb{E}^{x_0,i_0}\left[f\,(X_{\mathrm{T_r}}, J_{\mathrm{T_r}})\,; \mathrm{T_r} \le t\right] + \mathbb{E}^{x_0,i_0}\left[f\,(X_t, J_t)\,; \mathrm{T_r} > t\right] = f\,(x_0, i_0)\,. \tag{4.12}$$

We now claim that

$$\lim_{t\to\infty} \mathbb{E}^{x_0,i_0}\left[f\,(X_t, J_t)\,; \mathrm{T_r} > t\right] = 0. \tag{4.13}$$

If $\mathrm{p_r} = 1$, since f is bounded this is obvious. And if $\mathrm{p_r} < 1$ it was argued in the proof of Proposition 1 that $\lim_{t\to\infty} X_t = \infty$ a.s. which, since all $\mathrm{Re}\,(\gamma_k) < 0$ implies that $\lim_{t\to\infty} f\,(X_t, J_t) = 0$ a.s. so (4.13) follows by dominated convergence.

Thus (4.13) holds and letting $t \to \infty$ in (4.12) now yields the analogue

$$\mathbb{E}^{x_0,i_0}\left[f\left(X_{\mathrm{T_r}}, J_{\mathrm{T_r}}\right); \mathrm{T_r} < \infty\right] = f\left(x_0, i_0\right)$$

of (2.16) from which (4.22) follows directly and (iii) is proved.

The main difference with the arguments yielding Theorem 1 is in the proof of (i). It is still true that the left hand side $\mathcal{P}_l$ and the right hand side $\mathcal{P}_r$ of (4.8) are both polynomials with $\mathcal{P}_l$ of degree $m+d$ (d given by (3.16)), and by Lemma 3, $\mathcal{P}_l(z)$ has exactly $m+p_{\mathrm{c}}$ roots with $\mathrm{Re}(z) < 0$. Also $\mathcal{P}_r$ is of degree $< m+d$ so to complete the proof of (i) it remains to apply Rouché's theorem, but for this it is however necessary to adjust and refine the argument from the proof of Theorem 1: because $\gamma = 0$ is always a solution to (4.8), the sharp inequality

$$\left|P_U\left(\mathrm{i}y\right)\underline{a}^T Q^*\left(\mathrm{i}y,\theta\right)\underline{\lambda}\right| < \left|R_U\left(\mathrm{i}y\right)\det Q\left(\mathrm{i}y,\theta\right)\right|$$

established there for $\theta > 0$ fails for $\theta = 0$ and $y = 0$!

Instead of using the open set Γ_ρ determined by the boundary (3.28), let $\rho > 0$ be given and for $0 < \varepsilon < \rho$ so small that $Q\left(z,0\right)$ is non-singular for $|z| \leq \varepsilon$ (possible since $Q\left(0,0\right) = Q_V$ is non-singular) and so small that

$$\bar{L}_U\left(z\right) = \int_0^\infty e^{-zu}\, F_U\left(du\right) \tag{4.14}$$

(possible by the comment following (3.6)), define $\Gamma_{\rho,\varepsilon}$ as the interior of the subset of $\mathbb{C}$ determined by the inside of the boundary

$$\begin{aligned}\partial\Gamma_{\rho,\varepsilon} &= \left\{z : |z| = \rho,\ \mathrm{Re}\left(z\right) < 0\right\} \cup \left\{z : z = \mathrm{i}y,\ y \in \mathbb{R},\ \varepsilon \leq |y| \leq \rho\right\} \\ &\quad \cup \left\{z : |z| = \varepsilon,\ \mathrm{Re}\left(z\right) < 0\right\}\end{aligned}$$

if $\mathrm{p_r} < 1$ and

$$\begin{aligned}\partial\Gamma_{\rho,\varepsilon} &= \left\{z : |z| = \rho,\ \mathrm{Re}\left(z\right) < 0\right\} \cup \left\{z : z = \mathrm{i}y,\ y \in \mathbb{R},\ \varepsilon \leq |y| \leq \rho\right\} \\ &\quad \cup \left\{z : |z| = \varepsilon,\ \mathrm{Re}\left(z\right) > 0\right\}\end{aligned}$$

if $\mathrm{p_r} = 1$. Thus $0 \notin \Gamma_{\rho,\varepsilon}$ if $\mathrm{p_r} < 1$ and $0 \in \Gamma_{\rho,\varepsilon}$ if $\mathrm{p_r} = 1$.

Rouché's theorem will imply the claim in (i) if we show that

$$\left|-P_U\left(z\right)\underline{a}^T Q^*\left(z,0\right)\underline{\lambda}\right| < \left|R_U\left(z\right)\det Q\left(z,0\right)\right| \tag{4.15}$$

for $z \in \partial\Gamma_{\rho,\varepsilon}$ when ρ is sufficiently large and ε is sufficiently small.

Here there is no problem if $|z| = \rho$ with ρ large since the polynomium on the left is of lower degree than that on the right. If $z = \mathrm{i}y$ with $|y| \geq \varepsilon$, since $Q\left(\mathrm{i}y,0\right)$ is non-singular and $\left|\underline{a}^T Q^{-1}\left(\mathrm{i}y\right)\boldsymbol{\lambda}\right| \leq 1$ by Lemma 2 it is enough to argue that

$$\left|\bar{L}_U\left(\mathrm{i}y\right)\right| < 1.$$

But if for some $y_0 \in \mathbb{R}\backslash 0$,

$$\left|\bar{L}_U\left(\mathrm{i}y_0\right)\right| = \left|\int_0^\infty e^{-\mathrm{i}y_0 u}\, F_U\left(du\right)\right| = 1$$

it follows readily that the probability with distribution function F_U is concentrated on a subset of $\left(c + \frac{2\pi}{y_0}\mathbb{Z}\right) \cap \mathbb{R}_+$ for some $c \in \mathbb{R}$ in which case also $\left|\bar{L}_U\left(m\mathrm{i}y_0\right)\right| = 1$ for all $m \in \mathbb{Z}$. Since however

$$\lim_{|m|\to\infty} \left|\bar{L}_U\left(m\mathrm{i}y_0\right)\right| = \lim_{|m|\to\infty} \left|\frac{P_U\left(m\mathrm{i}y_0\right)}{R_U\left(m\mathrm{i}y_0\right)}\right| = 0$$

this yields a contradiction.

It remains to consider $z \in \partial\Gamma_{\rho,\varepsilon}$ with $|z| = \varepsilon$. For ε small enough, $Q\left(z, 0\right)$ is non-singular and $P_U\left(z\right) \neq 0$, hence (4.15) for this case is equivalent to

$$\left|\underline{a}^T Q^{-1}\left(z, 0\right)\underline{\lambda}\right| < \frac{1}{\left|\bar{L}_U\left(z\right)\right|}$$

with $\bar{L}_U\left(z\right)$ given by the integral (4.14). Both functions

$$z \mapsto g_l\left(z\right) := -\underline{a}^T Q^{-1}\left(z, 0\right)\underline{\lambda}$$

and

$$z \mapsto g_r\left(z\right) := \frac{1}{\left|\bar{L}_U\left(z\right)\right|}$$

are analytic in a neighborhood of 0 with $g_l(0) = g_r(0) = 1$. But if a function g is analytic in a neighborhood of 0 with $g(0), g'(0)$ and $g''(0)$ $\mathbb{R}$-valued one finds for $z = x + \mathrm{i}y$ close to 0 that

$$|g(z)|^2 = g^2(0) + 2xg(0)g'(0) + y^2\left(g'^2(0) - g(0)g''(0)\right) + o(x) + o(y^2)$$

and hence the desired inequality $|g_l(z)| < |g_r(z)|$ for $z \in \partial\Gamma_{\rho,\varepsilon}$ with $|z| = \varepsilon$ will follow if we show that for $x \neq 0$, $y \neq 0$ close enough to 0,

$$xg_l'(0) < xg_r'(0), \quad y^2\left(g_l'^2(0) - g_l''(0)\right) < y^2\left(g_r'^2(0) - g_r''(0)\right),$$

i.e we must show that

$$g_l'(0) > g_r'(0) \tag{4.16}$$

in the case $\mathrm{p_r} < 1$,

$$g_l'(0) < g_r'(0) \tag{4.17}$$

in the case $\mathrm{p_r} = 1$, and that

$$g_l'^2(0) - g_l''(0) < g_r'^2(0) - g_r''(0) \tag{4.18}$$

in both cases.

For z close enough to 0 one may differentiate with respect to z under the integral sign in (4.14) and then find that

$$g_r'(0) = \xi, \quad g_r''(0) = -\left(\mathbb{E}U_1^2 - 2\xi^2\right)$$

so that

$$g_r'^2(0) - g_r''(0) = \mathrm{Var}\,(U_1) > 0.$$

Next, differentiating after z in the matrix identity $I = Q^{-1}(z,0)\,Q(z,0)$ and referring back to (3.3) yields

$$\begin{aligned} \frac{d}{dz}Q^{-1}(z,0)\,|_{z=0} &= -Q_V^{-1}\overline{\beta}Q_V^{-1}, \\ \frac{d^2}{dz^2}Q^{-1}(z,0)\,|_{z=0} &= 2Q_V^{-1}\overline{\beta}Q_V^{-1}\overline{\beta}Q_V^{-1} - Q_V^{-1}SQ_V^{-1} \end{aligned}$$

where $\overline{\beta}$ is the diagonal matrix with entries β_i and S the diagonal matrix with entries σ_i^2. Thus, using (2.8),

$$\begin{aligned} g_l'(0) &= -\underline{a}^T Q_V^{-1}\underline{\beta}, \\ g_l''(0) &= 2\underline{a}^T Q_V^{-1}\overline{\beta}Q_V^{-1}\underline{\beta} - \underline{a}^T Q_V^{-1}\underline{\sigma}^2 \end{aligned}$$

where $\underline{\beta}$ and $\underline{\sigma}^2$ denote the column vectors (β_i) and (σ_i^2) respectively.

We now first show (4.16) and (4.17): by (4.4), $g_l'(0) = \mu\sum_i \alpha_i\beta_i$ which is $> g_r'(0) = \xi$ exactly when $\mathrm{p_r} < 1$ and is $< \xi$ exactly when $\mathrm{p_r} = 1$ and there is strict inequality in (4.5).

The proof of (4.18) is more difficult. Since the right hand side is > 0 it is enough to show that

$$g_l'^2(0) - g_l''(0) \le 0. \tag{4.19}$$

Suppose first that all $\beta_i = \beta$, all $\sigma_i^2 = \sigma^2$. Then for $|z|$ small enough $g_l(z) = L_V\left(-\beta z - \frac{1}{2}\sigma^2 z^2\right) = \mathbb{E}^{x_0,\underline{a}} \exp\left(\beta z + \frac{1}{2}\sigma^2 z^2\right) V_1$, cf. (3.4), which implies that $g_l'(0) = \beta\mu$, $g_l''(0) = \sigma^2\mu + \beta^2\mathbb{E}^{x_0,\underline{a}}V_1^2$ and therefore that

$$g_l'^2(0) - g_l''(0) = -\beta^2\,\mathrm{Var}^{x_0,\underline{a}}\,V_1 - \sigma^2\mu \le 0$$

so that (4.19) holds for this particular case. Inspired by this it is natural to consider the function $\tilde{g}$ given by

$$z \mapsto \mathbb{E}^{x_0,\underline{a}} \exp \int_0^{V_1} \left(\beta_{J_s} z + \tfrac{1}{2}\sigma_{J_s}^2 z^2\right)\, ds,$$

well defined for z close enough to 0. Differentiation with respect to z may be performed under the expectation sign and yields

$$\begin{aligned}
\tilde{g}'(0) &= \mathbb{E}^{x_0,\underline{a}} \int_0^{V_1} \beta_{J_s}\, ds \\
&= \mathbb{E}^{x_0,\underline{a}} \int_0^{\infty} \beta_{J_s} 1_{(V_1>s)}\, ds \\
&= \int_0^{\infty} \underline{a}^T e^{sQ_V} \underline{\beta}\, ds \\
&= -\underline{a}^T Q_V^{-1} \underline{\beta} \\
&= g_l'(0)
\end{aligned}$$

where here and below we use that prior to the first claim, the transition matrix for the Markov chain J over a time interval of length s is e^{sQ_V}. Differentiating once more gives

$$\tilde{g}''(0) = -\underline{a}^T Q_V^{-1} \underline{\sigma}^2 + \mathbb{E}^{x_0,\underline{a}} \left(\int_0^{V_1} \beta_{J_s}\, ds \right)^2$$

and by straightforward computation, writing $\bar{\beta}$ for the diagonal matrix diag$\,(\beta_i)$

$$\begin{aligned}
\mathbb{E}^{x_0,\underline{a}} \left(\int_0^{V_1} \beta_{J_s}\, ds \right)^2 &= 2\mathbb{E}^{x_0,\underline{a}} \left(\int_0^{\infty} \beta_{J_s} 1_{(V_1>s)}\, ds \right) \int_s^{\infty} \beta_{J_t} 1_{(V_1>t)}\, dt \\
&= 2 \int_0^{\infty} ds \int_s^{\infty} dt\, \underline{a}^T e^{sQ_V} \bar{\beta} e^{(t-s)Q_V} \underline{\beta} \\
&= 2\underline{a}^T Q_V^{-1} \bar{\beta} Q_V^{-1} \underline{\beta}
\end{aligned}$$

so that $\tilde{g}''(0) = g_l''(0)$. Therefore

$$g_l'^2(0) - g_l''(0) = \underline{a}^T Q_V^{-1} \underline{\sigma}^2 - \mathrm{Var}^{x_0,\underline{a}} \int_0^{V_1} \beta_{J_s}\, ds$$

and since by (4.4) $\underline{a}^T Q_V^{-1} \underline{\sigma}^2 = -\mu \sum_i \alpha_i \sigma_i^2 \leq 0$ the inequality (4.19) and therefore also (4.18) has been shown. ■

Note. It is certainly possible, but not shown here, that in fact $\tilde{g} \equiv g_l$.

Example 2. *We shall illustrate Theorem 2 by an example: take* $E = \{0, 1\}$, $\beta_0 = 2$, $\sigma_0 = 0$, $\beta_1 = 0$, $\sigma_1 = 1$ *or* 10, $\lambda_0 = 1$, $\lambda_1 = 0$, $q_{01} = q_{10} = 1$ *and* $a_0 = 1$, $a_1 = 0$. *When* J *is in state* 0, X *follows a straight line with slope* 2 *and when* J *is* *in state* 1, X *behaves as* $\sigma_1 B$. *A claim can only arrive when* J *is in state* 0, *and* J *then stays in* 0 *for a while. Clearly ruin by continuity is possible only when* J *is in state* 1, *ruin by jump is possible only when* J *is in state* 0: $p = 2$ *and* $p_{\mathrm{c}} = p_{\mathrm{j}} = 1$.

For claims distribution we take a mixture of two exponentials,

$$L_U(\nu) = (1-\delta)\frac{1}{1+\nu} + \delta\frac{\eta}{\eta+\nu} = \frac{\eta + (1-\delta+\delta\eta)\nu}{(1+\nu)(\eta+\nu)} \tag{4.20}$$

with the idea that both δ *and* η *should be small so that on rare occasions a huge claim may appear: below we take* $\delta = 0.01$ *but allow* η *to vary.*

We have $\xi = 1 - \delta + \frac{\delta}{\eta}$. *Simple calculations yields* $\mu = q_{01} + q_{10} = 2$ *and the stationary distribution* $\alpha_0 = \alpha_1 = \frac{1}{2}$ *for* J *(essentially because* $q_{01} = q_{10}$ *and* $a_0 = 1$*). Thus the necessary and sufficient condition (4.5) for* $\mathrm{p_r} = 1$ *becomes* $\xi \geq \alpha_0\beta_0\mu = 2$, *which for* $\delta = 0.01$ *translates into* $\eta \leq \frac{0.01}{1.01}$.

The matrix $Q(z,0)$ *has the form, cf. (3.3),*

$$Q(z,0) = \begin{pmatrix} 2z-2 & 1 \\ 1 & \frac{1}{2}\sigma_1^2 z^2 - 1 \end{pmatrix},$$

so the modified Cramèr-Lundberg equation (4.8) becomes

$$R_U(\gamma)\left((2\gamma-2)\left(\tfrac{1}{2}\sigma_1^2\gamma^2 - 1\right) - 1\right) = -P_U(\gamma)\left(\tfrac{1}{2}\sigma_1^2\gamma^2 - 1\right) \tag{4.21}$$

with, see (4.20),

$$R_U(\gamma) = (1+\gamma)(\eta+\gamma), \quad P_U(\gamma) = \eta + (0.99 + 0.01\eta)\gamma.$$

Solving (4.21) amounts to finding the roots of a polynomium of degree 5 *with* 0 *always a root. In all cases below the* 5 *roots are real, with of course precisely* 3 *roots* < 0 *iff* $\mathrm{p_r} < 1$ *and precisely* 2 *roots* < 0 *iff* $\mathrm{p_r} = 1$. *For* $\eta = \frac{0.01}{1.01}$, *the critical value,* 0 *becomes a root of multiplicity* 2.

The table below collects the ruin probabilities by continuity and by jump for $\eta = 0.0001, 0.001, 0.008, 0.012, 0.1, 0.99$ *and* $\sigma_1 = 1$ *and* 10, *for initial values* $x_0 = 0.1, 1, 10, 100$ *and* $i_0 = 0$ *and* 1. *(The numbers* 0.008 *and* 0.012 *were picked to represent two values fairly close to and on either side of the critical value* $\frac{0.01}{1.01}$*). The table in particular reflects the impact on having a high volatility for the probability of ruin by continuity when* x_0 *is small, and how the presence of rare huge claims affects the ruin probability by jump. The probability of ruin by continuity is presented on top in each cell, that of ruin by jump on the bottom. The table is split into two subtables, the first with* $\sigma_1 = 1$, *the second with* $\sigma_1 = 10$.

In the case where $\mathrm{p_r} = 1$, *having found the three roots* < 0 *to (4.21), one must solve two linear equations with two unknowns and there are three choices for the set of two equations. It has been checked numerically that the solutions for the ruin probabilities do not depend on the choice of equations.*

x_0	η	$\sigma_1 = 1$ 0.0001	0.001	0.008	0.012	0.1	0.99
0.1	$i_0 = 0$	.263 .737	.263 .737	.264 .736	.265 .672	.265 .410	0.265 0.378
	1	.892 .108	.892 .108	.893 .107	.893 .095	.893 .047	0.893 0.041
1	0	.143 .857	.144 .856	.146 .854	.146 .752	.146 .328	0.146 0.277
	1	.342 .658	.342 .658	.344 .656	.344 .573	.344 .232	0.344 0.191
10	0	.002 .998	.003 .997	.006 .994	.007 .807	.005 .056	0.002 0.006
	1	.002 .998	.003 .997	.006 .994	.007 .807	.005 .057	0.003 0.007
100	0	$5 \cdot 10^{-5}$ 1	$5 \cdot 10^{-4}$ 1	.004 .996	.004 .671	$1 \cdot 10^{-6}$ $2 \cdot 10^{-5}$	– –
	1	$5 \cdot 10^{-5}$ 1	$5 \cdot 10^{-4}$ 1	.004 .996	.004 .671	$1 \cdot 10^{-6}$ $2 \cdot 10^{-5}$	– –

x_0	η	$\sigma_1 = 10$ 0.0001	0.001	0.008	0.012	0.1	0.99
0.1	$i_0 = 0$	.692 .308	.692 .308	.698 .302	.699 .297	.700 .282	0.702 0.277
	1	.997 .003	.997 .003	.997 .003	.997 .002	.997 .001	0.997 $8 \cdot 10^{-4}$
1	0	.793 .207	.794 .206	.804 .196	.806 .187	.807 .159	0.811 0.151
	1	.969 .031	.969 .031	.974 .026	.975 .021	.975 .007	0.975 0.006
10	0	.734 .266	.741 .259	.789 .211	.802 .165	.803 .024	0.800 0.010
	1	.761 .239	.767 .233	.811 .189	.823 .146	.820 .020	0.815 0.010
100	0	.072 .928	.100 .900	.310 .690	.345 .468	.170 .006	0.142 0.002
	1	.074 .926	.103 .897	.312 .688	.346 .467	.173 .006	0.144 0.002

TABLE 1. The ruin probabilities p_c (first line in each cell) and p_j (second line) for Example 2. First half of the table: $\sigma_1 = 1$, second half: $\sigma_1 = 10$.

We shall conclude this section with some comments on how one may determine the Laplace transform for the undershoot alone, i.e find the quantities

$$\mathbb{E}^{x_0,i_0}\left[e^{-\zeta \mathrm{Y}_\mathrm{r}}; A_\mathrm{j}\right]$$

jointly with the ruin probabilities

$$\mathbb{P}^{x_0,i_0}\left(A_{\mathrm{c},i}\right) = \mathbb{E}^{x_0,i_0}\left[e^{-\zeta \mathrm{Y}_\mathrm{r}}; A_{\mathrm{c},i}\right].$$

For $\zeta > 0$ these $p_\mathrm{c}+1$ unknowns always require a system of $p_\mathrm{c}+1$ linear equations for their solution – in case $\mathrm{p}_\mathrm{r} = 1$ we no longer know that they add to 1. The idea is now to solve the Cramér-Lundberg equations (3.8) or (3.9) (that do not depend on ζ) for $\theta = 0$, i.e use (4.7) or (4.8). If $\mathrm{p}_\mathrm{r} < 1$ we know that we get $m + p_\mathrm{c} + 1$ solutions γ_k with $\mathrm{Re}\left(\gamma_k\right) < 0$, and one then proceeds as in (iv) of Theorem 1, using (3.11) with $\theta = 0$ and the given ζ. If $\mathrm{p}_\mathrm{r} = 1$ one uses the $m + p_\mathrm{c}$ roots to (4.8) *and* the root 0 plugged into (3.11), which is possible because for $\zeta > 0$, r_k defined by (3.12) makes sense, also for $\tilde{\gamma}_k = 0$!

Example 3. *Suppose that the claims are exponential with rate $\eta > 0$ so that*

$$L_U\left(\nu\right) = \frac{\eta}{\eta + \nu}.$$

Then $m = 1$ and (3.11) with $\theta = 0$ simplifies to

$$\begin{aligned}\sum_{i \in E_\mathrm{c}} \left(Q^{-1}\left(\tilde{\gamma}, 0\right)\underline{\lambda}\right)_i \mathbb{P}^{x_0,i_0}\left(A_{\mathrm{c},i}\right) - \frac{1}{L_U\left(\zeta\right)}\mathbb{E}^{x_0,i_0}\left[e^{-\zeta \mathrm{Y}_\mathrm{r}}; A_\mathrm{j}\right] \\ = \left(Q^{-1}\left(\tilde{\gamma}, 0\right)\underline{\lambda}\right)_{i_0} e^{\tilde{\gamma} x_0}\end{aligned}$$

for any of the relevant $p_\mathrm{c} + 2$ roots $\tilde{\gamma}$ to (4.8). Comparing with what (4.22) gives for $m = 1$, it follows immediately that

$$\mathbb{E}^{x_0,i_0}\left[e^{-\zeta \mathrm{Y}_\mathrm{r}} \left| A_\mathrm{j}\right.\right] = L_U\left(\zeta\right),$$

i.e conditionally on ruin happening in finite time and by a jump, if the claims are exponential at rate η, so is the undershoot. *This is of course as it should be: given that ruin occurs at the jump time T_n for N from the level $x_- = X_{T_n-}$, the conditional distribution of the undershoot is simply that of $U_n - x_-$, i.e exponential at rate η by the basic lack of memory property of the exponential distribution.*

5. The simplest case

This short section is devoted to a more detailed study of the simplest model: X is Brownian motion with drift and we have Poisson arrivals of exponential claims. So X is given by (2.10) and is a Lévy process. In terms of the general model (2.3), we have $p = 1$, $m = 1$ and the important quantity $\underline{a}^T Q^{-1}(\gamma, \theta)\,\underline{\lambda}$ from (3.8) and (4.7) simplifies to

$$\frac{\lambda}{\beta\gamma + \frac{1}{2}\sigma^2\gamma^2 - \lambda - \theta}$$

with $\lambda > 0$ the rate at which claims arrive. With claims that are exponential at some rate $\eta > 0$ we have $L_U(\nu) = \frac{\eta}{\eta+\nu}$ so that $P_U(\nu) = \eta$, $Q_U(\nu) = \eta + \nu$. Further, since the claims are exponential, by Example 3 the undershoot is also exponential at rate η. Finally, by (4.5)

$$\mathrm{p_r} = 1 \text{ if and only if } \beta \leq \frac{\lambda}{\eta}. \tag{5.1}$$

If $\beta \geq 0$, $\sigma^2 = 0$ we have the most classical of all risk models and Theorem 1 yields well known expressions for the joint Laplace transform of $\mathrm{T_r}$ and $\mathrm{Y_r}$: here $p_\mathrm{c} = 0$ and we find from (3.8) and (3.11) that

$$\mathbb{E}^{x_0}\left[e^{-\theta \mathrm{T_r} - \zeta \mathrm{Y_r}}; A_\mathrm{j}\right] = \frac{\eta + \gamma_0}{\eta + \zeta} e^{\gamma_0 x_0}$$

for $\theta > 0$, $\zeta \geq 0$, where γ_0 is the only solution < 0 to the equation

$$\eta + \gamma = -\frac{\eta\lambda}{\beta\gamma - \lambda - \theta}, \tag{5.2}$$

i.e

$$\gamma_0 = \begin{cases} \frac{1}{2\beta}\left(\lambda + \theta - \eta\beta - \sqrt{(\lambda + \theta - \eta\beta)^2 + 4\beta\eta\theta}\right) & (\beta > 0,) \\ -\frac{\eta\theta}{\lambda+\theta} & (\beta = 0). \end{cases}$$

The ruin probability becomes

$$\mathrm{p_r} = \begin{cases} 1 & (\eta\beta \leq \lambda), \\ \frac{\lambda}{\eta\beta} e^{(\lambda/\beta - \eta)x_0} & (\eta\beta > \lambda). \end{cases}$$

If $\beta < 0$, $\sigma^2 = 0$ we have $p_\mathrm{c} = 1$ and both roots

$$\gamma_\pm = \frac{1}{2\beta}\left(\lambda + \theta - \eta\beta \pm \sqrt{(\lambda + \theta - \eta\beta)^2 + 4\beta\eta\theta}\right)$$

to (5.2) are real and < 0. In this case (3.11) gives for $\gamma = \gamma_+$ and γ_-,

$$\frac{\eta+\gamma}{\eta}\mathbb{E}^{x_0}\left[e^{-\theta \mathrm{T_r}}; A_\mathrm{c}\right] + \frac{\eta+\zeta}{\eta}\mathbb{E}^{x_0}\left[e^{-\theta \mathrm{T_r} - \zeta \mathrm{Y_r}}; A_\mathrm{j}\right] = \frac{\eta+\gamma}{\eta}e^{\gamma x_0}$$

and the two resulting equations with two unknowns are then easily solved explicitly. Of course $\mathrm{p_r} = 1$ and (4.7) which reads

$$\eta + \gamma = -\frac{\eta\lambda}{\beta\gamma - \lambda}$$

has one strictly negative solution

$$\gamma_0 = \frac{\lambda - \eta\beta}{\beta}.$$

Using this in (4.22) gives that

$$\mathbb{P}^{x_0}\left(A_\mathrm{c}\right) = 1 - \mathbb{P}^{x_0}\left(A_\mathrm{j}\right) = \tfrac{-\eta\beta}{\lambda-\eta\beta} + \tfrac{\lambda}{\lambda-\eta\beta}e^{(\lambda/\beta-\eta)x_0}.$$

It remains to discuss the case $\sigma^2 > 0$. Then $p_\mathrm{c} = 1$ always and solving (3.9) amounts to finding the roots γ with $\mathrm{Re}\,(\gamma) < 0$ of a polynomium of degree three. This can of course be done explicitly, but here we shall just determine the ruin probabilities since (4.7), which reads

$$\eta + \gamma = -\frac{\eta\lambda}{\beta\gamma + \frac{1}{2}\sigma^2\gamma^2 - \lambda}$$

after elimination of the root $\gamma = 0$ is reduced to finding the roots of a polynomium of degree two. These roots are

$$\gamma_\pm = \tfrac{1}{\sigma^2}\left(-\left(\beta + \tfrac{1}{2}\sigma^2\eta\right) \pm \sqrt{\left(\beta + \tfrac{1}{2}\sigma^2\eta\right)^2 + 2\sigma^2\left(\lambda - \eta\beta\right)}\right)$$

and are both real. From (5.1) we have $\mathrm{p_r} = 1$ iff $\lambda \geq \eta\beta$ in agreement with the fact that if $\lambda \geq \eta\beta$ only $\gamma_- < 0$, while if $\lambda < \eta\beta$ both γ_- and γ_+ are < 0.

If $\lambda \geq \eta\beta$, (4.22) gives

$$\mathbb{P}^{x_0}\left(A_\mathrm{c}\right) = 1 - \mathbb{P}^{x_0}\left(A_\mathrm{j}\right) = -\tfrac{\eta}{\gamma_-} + \left(1 + \tfrac{\eta}{\gamma_-}\right)e^{\gamma_- x_0},$$

while if $\lambda < \eta\beta$ one must take $\gamma = \gamma_-$ and γ_+ in

$$\frac{\eta+\gamma}{\eta}\mathbb{P}^{x_0}\left(A_\mathrm{c}\right) + \mathbb{P}^{x_0}\left(A_\mathrm{j}\right) = \frac{\eta+\gamma}{\eta}e^{\gamma x_0}.$$

in order to obtain two equations for finding the two ruin probabilities.

6. Extensions

As set out in (2.3), the basic model studied in this paper uses *one* Markov chain to generate changes in the parameters of the underlying Brownian motion with drift *and* generate the claims arrival process. It is quite easy instead to think of two Markov chains, one determining the Brownian motion and the other determining the arrival of claims as a renewal sequence with inter arrival times of phase-type.

To achieve this, focus first on the description of the phase-type distribution. This is the distribution of the time to absorption for a Markov chain J^*on a finite state space $E^* \cup \{\nabla\}$ where ∇ is the absorbing state, having some given initial distribution $\underline{a}^* = (a^*_{i^*})_{i^* \in E^*}$ concentrated on E^* and with transition intensity matrix of the form

$$\begin{pmatrix} Q^* & \underline{\lambda}^* \\ \mathbf{0}^T & 0 \end{pmatrix}$$

with Q^* a sub-intensity matrix and $\underline{\lambda}^* = -Q^*\mathbf{1}$. The Laplace transform for the distribution of the interarrival times V (where the waiting time till the first claim may be different) is then given by

$$\mathbb{E}e^{-\nu V} = -\underline{a}^{*T} \left(Q^* - \nu I\right)^{-1} \underline{\lambda}^* \quad (\nu \geq 0)$$

in complete analogy with (3.4). Next consider a second Markov chain $\bar{J}$ on some finite state space $\bar{E}$ with transition intensity matrix $\bar{Q}$ (with row sums 0: $\bar{q}_{\bar{i}\bar{i}} = -\bar{q}_{\bar{i}} = -\sum_{j \neq i} \bar{q}_{\bar{i}\bar{j}}$) and then define a process X as in (2.3) using for (J, N) the Markov chain on $\left(\bar{E} \times E^*\right) \times \mathbb{N}_0$ with the only possible non-zero transition intensities $q_{(\bar{i}i^*,n),(\bar{j}j^*,n')}$ for $(\bar{i}i^*, n) \neq (\bar{j}j^*, n')$ given by

$$\begin{aligned} q_{(\bar{i}i^*,n),(\bar{j}i^*,n)} &= \bar{q}_{\bar{i}\bar{j}} \quad \left(\bar{i} \neq \bar{j}\right), \\ q_{(\bar{i}i^*,n),(\bar{i}j^*,n)} &= q^*_{i^*j^*} \quad \left(i^* \neq j^*\right), \\ q_{(\bar{i}i^*,n),(\bar{j}j^*,n+1)} &= \lambda^*_{i^*}\bar{a}_{\bar{j}}a^*_{j^*} \quad \left(\bar{i}i^*, \bar{j}j^* \in \bar{E} \times E^*\right), \end{aligned}$$

cf. (2.1) and (2.2). Of course $\left(\bar{a}_{\bar{j}}\right)$ and $\left(a^*_{j^*}\right)$ are entrance laws on $\bar{E}$ and E^* respectively and with this structure, the two chains $\bar{J}$ and J^* as far as possible move independently of each other and are only correlated through the simultaneous jump of both chains with entrance law at the times when a claim arrives. (This type of simultaneous jump is forced in order to get a model of the type from (2.3) with the intensity for a claim of the form $\lambda_i a_j$ where here $\lambda_{\bar{i}i^*} = \lambda^*_{i^*}$, $a_{\bar{j}j^*} = \bar{a}_{\bar{j}}a^*_{j^*}$; as emphasized earlier this multiplicative structure is used in an essential manner for the proofs of Theorems 1 and 2).

The class of processes X just described is large enough to allow good approximations of additive processes composed of an arbitrary diffusion part and a claims process $C_t = \sum_{n=1}^{N_t} U_n$ as follows: consider a diffusion $\bar{X}$ solving a stochastic differential equation of the form (with W a standard Brownian motion)

$$d\bar{X}_t = b\left(\bar{X}_t\right)\,dt + \sigma\left(\bar{X}_t\right)\,dW_t$$

and then define

$$\begin{aligned} X_t &= x_0 + \sum_{n=0}^{N_t} \left(\bar{X}^{(n)}_{T_{n+1}\wedge t} - \bar{X}^{(n)}_{T_n}\right) - C_t \\ &= x_0 + \int_0^t b\left(\bar{X}^{(N_s)}_s\right)\,ds + \int_0^t \sigma\left(\bar{X}^{(N_{s-})}_{s-}\right)\,dB_s - C_t \end{aligned}$$

where the $\bar{X}^{(n)}$ may be thought of as independent copies of $\bar{X}$ with $\bar{X}^{(0)}$ starting from some given state $\bar{x}_0$ and the $\bar{X}^{(n)}$ for $n \geq 1$ iid with initial distribution given by some entrance law $\bar{a}\left(d\bar{x}\right)$. Generating the claim arrival times T_n by a Markov chain J^* returned to E^* using an entrance law $\left(a^*_{j^*}\right)$, and correlating $\bar{X}$ and J^* only through simultaneous jumps with entrance law $\bar{a} \otimes a^*$ at the times T_n, the process $\left(X, \bar{X}^{(\cdot)}, J^*\right)$ is Markov with generator

$$\begin{aligned} Af\left(x, \bar{x}, i^*\right) &= b\left(\bar{x}\right) D_{\bar{x}} f(x, \bar{x}, i^*) + \tfrac{1}{2}\sigma^2\left(\bar{x}\right) D^2_{\bar{x}\bar{x}} f(x, \bar{x}, i^*) \\ &\quad + \sum_{j^* \neq i^*} q_{i^* j^*}\left(f\left(x, \bar{x}, j^*\right) - f\left(x, \bar{x}, i^*\right)\right) \\ &\quad + \lambda_{i^*} \sum_{j^*} a^*_{j^*} \int \bar{a}\left(d\bar{y}\right) \int_0^\infty F_U\left(dy\right)\left(f\left(x - y, \bar{y}, j^*\right) - f\left(x, \bar{x}, i^*\right)\right) \end{aligned}$$

and here, the diffusion part $\bar{X}^{(\cdot)}$ may be approximated arbitrarily well by a finite state Markov chain $\bar{J}$ and consequently, arbitrarily good approximations for Laplace transforms of the time to ruin and the undershoot as well as the ruin probabilities, may in principle be given.

We shall conclude with a brief discussion of models that are not additive, but where the results of the paper may still be used.

Let X be given by the basic structure (2.3), and let $\phi : \mathbb{R} \to I$ be, say, strictly increasing and sufficiently smooth, with $I =]l, r[$ denoting some open interval, $-\infty \leq l < r \leq \infty$. Now consider the process $\left(\tilde{X}, J\right)$ where $\tilde{X}_t = \phi\left(X_t\right)$ for all t. Then $\left(\tilde{X}, J\right)$ is certainly time-homogeneous Markov and fixing a state $\tilde{a}_0 \in I$ and defining the passage time ('time to ruin')

$$\tau_{\tilde{a}_0} = \inf\left\{t \geq 0 : \tilde{X}_t < \tilde{a}_0\right\}$$

it is clear that subject to starting $\left(\tilde{X}, J\right)$ from $(\tilde{x}_0, i_0)$ where $\tilde{x}_0 \in I$, $\tilde{x}_0 > \tilde{a}_0$, Theorem 1 may be used to determine the joint distribution of $\tau_{\tilde{a}_0}$ and $\tilde{X}_{\tau_{\tilde{a}_0}}$: just start (X, J) from $\left(\phi^{-1}\tilde{x}_0, i_0\right)$ and look at the passage time for X to the lower level $\phi^{-1}\tilde{a}_0$. By the additivity of X it then follows that $\left(\tau_{\tilde{a}_0}, \tilde{X}_{\tau_{\tilde{a}_0}}\right)$ when $\left(\tilde{X}, J\right)$ starts from $(\tilde{x}_0, i_0)$ has the same distribution as $\left(\mathrm{T_r}, \phi\left(\phi^{-1}\tilde{a}_0 - \mathrm{Y_r}\right)\right)$ when (X, J) starts from $\left(\phi^{-1}\tilde{x}_0 - \phi^{-1}\tilde{a}_0, i_0\right)$. The generator $\tilde{A}$ for $\left(\tilde{X}, J\right)$ is determined from that of (X, J), see (2.11), by the simple recipe

$$\tilde{A}f(\tilde{x}, i) = (Af(\phi(\cdot), \cdot))\left(\phi^{-1}\tilde{x}, i\right).$$

A further generalisation of the class of models for which at least Theorem 2 applies, is obtained by subjecting $\left(\tilde{X}, J\right)$ to a random time change through an additive functional: let $\psi(\tilde{x}, i) > 0$ be a function on $I \times E$ and define

$$\mathcal{A}_t = \int_0^t \psi\left(\tilde{X}_s, J_s\right) ds.$$

Assuming that, no matter where $\left(\tilde{X}, J\right)$ starts,

$$\lim_{t\to\infty} \mathcal{A}_t = \infty \tag{6.1}$$

almost surely, define the time-changed process $(Z_u, L_u)_{u\geq 0}$ by

$$(Z_{\mathcal{A}_t}, L_{\mathcal{A}_t}) = \left(\tilde{X}_t, J_t\right) \quad (t \geq 0).$$

Then (Z, L) is a new time-homogeneous Markov process, starting from the same state as $\left(\tilde{X}, J\right)$ and, because of (6.1), with the same 'ruin probability' as $\tilde{X}$, i.e for the two processes Z and $\tilde{X}$ started from the same state above $\tilde{a}_0$, the two probabilities of ever getting below $\tilde{a}_0$ are identical. Also, the 'undershoot' for Z is the same as that for $\tilde{X}$, but the 'time to ruin' has of course changed due to the time substitution.

The state space for (Z, L) is of course $I \times E$. It may be useful to note in conclusion that the generator for (Z, L) is of the form

$$A_{Z,L}f(\tilde{x}, i) = \frac{1}{\psi(\tilde{x}, i)}\tilde{A}f(\tilde{x}, i),$$

i.e

$$A_{Z,L}f(\tilde{x}, i) \;=\; \tilde{b}_i(\tilde{x}) D_{\tilde{x}}f(\tilde{x}, i) + \tfrac{1}{2}\tilde{\sigma}_i^2(\tilde{x}) D_{\tilde{x}\tilde{x}}^2 f(\tilde{x}, i) \tag{6.2}$$

$$+\frac{1}{\psi(\tilde{x},i)}\sum_{j\neq i} q_{ij}\left(f(\tilde{x},j)-f(\tilde{x},i)\right)$$
$$+\frac{\lambda_i}{\psi(\tilde{x},i)}\sum_j a_j \int_0^\infty F_U(dy)\left[f\left(\phi\left(\phi^{-1}\tilde{x}-y\right),j\right)-f(\tilde{x},i)\right],$$

where

$$\begin{aligned}\tilde{b}_i(\tilde{x}) &= \frac{1}{\psi(\tilde{x},i)}\left(\beta_i\phi'\left(\phi^{-1}\tilde{x}\right)+\tfrac{1}{2}\sigma_i^2\phi''\left(\phi^{-1}\tilde{x}\right)\right),\\ \tilde{\sigma}_i^2(\tilde{x}) &= \frac{1}{\psi(\tilde{x},i)}\sigma_i^2\left(\phi'\left(\phi^{-1}\tilde{x}\right)\right)^2.\end{aligned}$$

Here, for a given i (but not for all i simultaneously) it is possible to obtain a general form of the diffusion part (6.2) of the generator. And if $\sigma_i^2=0$, between the jumps for J, Z behaves as a piecewise deterministic Markov process, typically following a determinstic smooth curve different from the straight line followed by the original process X.

A. Itô's formula, martingale representations

We shall here discuss the martingale representations and the version of Itô's formula used to derive (2.12) which in turn yields the key identity (2.16).

Consider (J,C), the Markov chain with state space $G=E\times\mathbb{R}_0$ determined by the chain J and the accumulated claims process

$$C_t=\sum_{n=1}^{N_t}U_n.$$

Let μ be the random counting measure on $\mathbb{R}_0\times G$ that counts the jumps for (J,C) occurring over time, identifying the jumps by the state reached by J at the time of a jump together with the size of the jump for C, i.e for $t\geq 0, j\in E$,

$$\mu\left([0,t]\times(\{j\}\times\{0\})\right)=\sum_{0<s\leq t}1_{(J_{s-}\neq J_s=j)},$$

(a jump for J to state j, no jump for C) and for $t\geq 0, j\in E, h>0$,

$$\mu\left([0,t]\times(\{j\}\times]h,\infty])\right)=\sum_{n=1}^{N_t}1_{(J_{T_n}=j,U_n>h)},$$

(a jump for C of size $>h$ and, possibly, a simultaneous jump to j for J).

The compensating measure for μ (with respect to the filtration generated by (J,C), and, since B is independent of (J,C), also with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by (J,C,B)) is given by

$$\begin{aligned}\Lambda\left([0,t]\times(\{j\}\times\{0\})\right) &= \int_0^t q_{J_{s-},j} 1_{(J_{s-}\neq j)}\, ds,\\ \Lambda\left([0,t]\times(\{j\}\times\,]h,\infty])\right) &= \left(\int_0^t \lambda_{J_{s-}}\, ds\right) a_j \bar{F}_U(h),\end{aligned}$$

writing $\bar{F}_U(h)=\mathbb{P}\left(U_n>h\right)$. From the theory of marked point processes (Davis [5] or Jacobsen [9]) one now knows that if $\left(S_t^{j,u}\right)_{t\geq 0}$ is for $j\in E$, $u\geq 0$ varying a uniformly bounded family of $\mathcal{F}_t$-predictable processes, jointly measurable in (t,j,u,ω), then the stochastic integral $M^\circ(S)=\left(M_t^\circ(S)\right)_{t\geq 0}$ defined by

$$\begin{aligned}M_t^\circ(S) &= \int_{]0,t]}\int_G S_s^{j,u}\left(\mu\left(ds,d\left(j,u\right)\right)-\Lambda\left(ds,d\left(j,u\right)\right)\right)\\ &= \sum_{0<s\leq t}\sum_{j\in E} S^{j,0} 1_{(J_{s-}\neq J_s=j)}+\sum_{n=1}^{N_t}\sum_{j\in E} S_{T_n}^{j,U_n} 1_{(J_{T_n}=j)}\\ &\quad -\int_0^t ds\sum_{j\in E} q_{J_{s-},j} 1_{(J_{s-}\neq j)} S_s^{j,0}\\ &\quad -\int_0^t ds\sum_{j\in E}\int_{]0,\infty[} F_U\left(du\right)\,\lambda_{J_{s-}} a_j S_s^{j,u} \end{aligned}\tag{A.1}$$

is an $\mathcal{F}_t$-martingale. Combining this with the stochastic integral representation of Brownian martingales and again referring to the independence between (J,C) and B, it follows that all stochastic integrals of the form

$$M_t^\circ=\int_0^t Z_s\, dB_s+M_t^\circ(S)\tag{A.2}$$

are $\mathcal{F}_t$-martingales when also Z is bounded and $\mathcal{F}_t$-predictable.

Let now $f\in\mathcal{D}$, take

$$S_t^{j,u}=f\left(X_{t-}-u,j\right)-f\left(X_{t-},J_{t-}\right),\quad Z_t=\sigma_{J_{t-}} D_x f\left(X_{t-},J_{t-}\right)\tag{A.3}$$

(with Z defined formally only for $t\leq \mathrm{T_r}$ – take for instance $Z_t=0$ for $t>\mathrm{T_r}$) and consider the Itô formula (2.12),

$$f\left(X_{\mathrm{T_r}\wedge t},J_{\mathrm{T_r}\wedge t}\right)=f\left(X_0,J_0\right)+\int_0^{\mathrm{T_r}\wedge t} Af\left(X_s,J_s\right)\, ds+M_t\tag{A.4}$$

with $M_t = M^\circ_{\mathrm{T_r}\wedge t}$ when M° is given by (A.2) with S and Z as in (A.3). We need to verify this identity and observe first that the definition of S ensures precisely that the jumps of the left and right hand side, which can occur only when the counting measure μ jumps, agree for all jump times $\leq \mathrm{T_r}$. And in between jumps, if there are no jumps for μ on $[t_0, t_1[$ say, where $t_1 \leq \mathrm{T_r}$, one obtains (using Itô's formula for functions of Brownian motion) for $t \in]t_0, t_1[$ the stochastic differential

$$\begin{aligned} df\left(X_t, J_t\right) &= df\left(X_{t_0} + \beta_{J_{t_0}}(t - t_0) + \sigma_{J_{t_0}}(B_t - B_{t_0}), J_{t_0}\right) \\ &= \left(\beta_{J_{t_0}} D_x f\left(X_t, J_{t_0}\right) + \tfrac{1}{2}\sigma^2_{J_{t_0}} D^2_{xx} f\left(X_t, J_{t_0}\right)\right) dt \\ &\quad + \sigma_{J_{t_0}} D_x f\left(X_t, J_{t_0}\right) dB_t \end{aligned} \tag{A.5}$$

and for the right hand side of (A.4) the differential

$$\begin{aligned} Af\left(X_t, J_t\right) dt + dM_t &= Af\left(X_t, J_{t_0}\right) dt + Z_t\, dB_t + dM^\circ_t(S) \\ &= Af\left(X_t, J_{t_0}\right) dt + \sigma_{J_{t_0}} D_x f\left(X_t, J_{t_0}\right) dB_t \\ &\quad - \sum_{j\in E} q_{J_{t_0-},j} 1_{\left(J_{t_0-}\neq j\right)} \left(f\left(X_{t-}, j\right) - f\left(X_{t-}, J_{t_0-}\right)\right) dt \\ &\quad - \sum_{j\in E} \int_{]0,\infty[} F_U(du)\, \lambda_{J_{t_0-}} a_j \left(f\left(X_{t-} - u, j\right) - f\left(X_{t-}, J_{t_0-}\right)\right) dt \end{aligned}$$

using (A.1) and the definitions of Z and S. Referring back to the definition (2.11) of the infinitesimal generator, it is now clear that this differential agrees with (A.5), hence the increments for the processes on the left and right of (A.4) are the same between jump times and we conclude that (A.4) does indeed hold. Finally it must be noted that since $M^\circ(S)$ is an $\mathcal{F}_t$-martingale, so is the stopped process $M = \left(M^\circ_{\mathrm{T_r}\wedge t}\right)_{t\geq 0}$.

B. Proofs of Lemmas 2 and 3

Proof. (Lemma 2). For any given complex numbers z_i for $i \in E$, define the matrix $\hat{Q} = (\hat{q}_{ij})$ by

$$\hat{q}_{ij} = \begin{cases} z_i - q_i - \lambda_i & \text{if } i = j, \\ q_{ij} & \text{if } i \neq j \end{cases} \tag{B.1}$$

so that $Q(z,\theta) = \hat{Q}$ when $z_i = \phi_i(z) - \theta$ for all i, see (3.3). Since $\mathrm{Re}\left(\phi_i(\mathrm{i}y)\right) \leq 0$, in order to prove the lemma it suffices to show that whenever $\mathrm{Re}\left(z_i\right) \leq 0$ for all i, then $\hat{Q}$ is non-singular and

$$|u_j| \leq \frac{q_j + \lambda_j}{q_j + \lambda_j + c} \tag{B.2}$$

for all $j \in E$ where $c = \min |\mathrm{Re}\,(z_i)|$

$$u_j = \left(\hat{Q}^{-1}\underline{\lambda}\right)_j, \tag{B.3}$$

and this we now proceed to do, assuming that all $\mathrm{Re}\,(z_i) \leq 0$ from now on.

To argue that $\hat{Q}$ is non-singular, suppose that $v = (v_j)$ is a column vector such that $\hat{Q}v = 0$. Then

$$\sum_{j \neq i} q_{ij} v_j + \hat{q}_{ii} v_i = 0$$

for all i, and since $\mathrm{Re}\,(\hat{q}_{ii}) < 0$ because $\mathrm{Re}\,(z_i) \leq 0$ and $q_i + \lambda_i > 0$ by Assumption (A), we have $\hat{q}_{ii} \neq 0$ and obtain

$$v_i = \frac{\sum_{j \neq i} q_{ij} v_j}{-\hat{q}_{ii}}$$

which implies that

$$|v_i| \leq \frac{q_i \max |v_j|}{q_i + \lambda_i + c}.$$

If now $i = i_0$ is chosen so that $|v_{i_0}| = \max |v_j|$ and we assume that $|v_{i_0}| > 0$, we see that $q_{i_0} > 0$ and $\lambda_{i_0} = c = 0$, and also that $|v_{j_0}| = |v_{i_0}|$ for all $j_0 \neq i_0$ such that $q_{i_0 j_0} > 0$. But then also $q_{j_0} > 0$ and $\lambda_{j_0} = 0$ and continuing it follows that for all states j reachable from i_0 through q_{ij}-transitions alone, we must have $q_j > 0$, $\lambda_j = 0$. Since however by the irreducibility inherent in Assumption (A) there is some j_1 with $\lambda_{j_1} > 0$ reachable by q_{ij}-transitions from i_0 we have reached a contradiction and deduce that $\max |v_j| = 0$ as wanted.

To show (B.2), first rewrite (B.2) as

$$u_i = \frac{1}{-\hat{q}_{ii}} \left(\lambda_i + \sum_{j \neq i} q_{ij} u_j \right)$$

which implies that for all i,

$$|u_i| \leq \frac{\lambda_i + q_i \max |u_j|}{\lambda_i + q_i + c}. \tag{B.4}$$

Consider now the states i_0 with $|u_{i_0}| = \max |u_j|$ (which is > 0 since $\underline{\lambda} \neq 0$ and $\underline{u} = -\hat{Q}^{-1}\underline{\lambda}$). We argue first that for some such i_0, $\lambda_{i_0} + c > 0$: if for all these i_0, were $\lambda_{i_0} + c = 0$ we would get

$$u_{i_0} = \frac{1}{q_{i_0}} \sum_{j \neq i} q_{i_0 j} u_j$$

forcing $|u_j| = |u_{i_0}|$ for all $j \neq i_0$ with $q_{i_0 j} > 0$, hence by the assumption made also $\lambda_j + c = 0$. The irreducibility from Assumption (A) would then yield a contradiction exactly as in the first part of the proof. Thus, for some i_0 with $|u_{i_0}| = \max |u_j|$ we have $\lambda_{i_0} + c > 0$ and then from the trivial consequence

$$|u_{i_0}| \leq \frac{\lambda_{i_0} + q_{i_0} |u_{i_0}|}{\lambda_{i_0} + q_{i_0} + c}$$

of (B.4) it follows that $|u_{i_0}| \leq \frac{\lambda_{i_0}}{\lambda_{i_0}+c} \leq 1$. Using this in (B.4) gives for all i that

$$|u_i| \leq \frac{\lambda_i + q_i}{\lambda_i + q_i + c}$$

proving (B.2). ■

Note. Taking all $z_i = 0$, the proof shows in particular that the sub-intensity matrix Q_V is non-singular as was also shown by Jacobsen [8], Lemma 1.

Proof. (Lemma 3). Define $E' = \{i \in E : \sigma_i^2 > 0 \text{ or } \beta_i \neq 0\}$. The determinant

$$\det Q(z,\theta) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i \in E} q_{i,\pi(i)}(z,\theta) \tag{B.5}$$

with π ranging over the set of all permutations of the states in E. Because z appears only in the diagonal elements $q_{ii}(z,\theta)$, and then only if $i \in E'$, it is clear that a term in the sum (B.5) is always a polynomium of degree $\leq d$, with d given by (3.16), and that this polynomium is of degree precisely d iff $\pi(i) = i$ for all $i \in E'$. Adding all the terms of degree d gives a coefficient to z^d equal to

$$\left(\det Q_{E \setminus E'}(\theta)\right) \prod_{i \in E' : \sigma_i^2 > 0} \left(\tfrac{1}{2}\sigma_i^2\right) \prod_{i \in E' : \sigma_i^2 = 0} \beta_i, \tag{B.6}$$

where $Q_{E \setminus E'}(\theta)$ is the square matrix obtained from $Q(z,\theta)$ by deleting all rows and columns corresponding to states in E', in particular it does not depend on z. But the matrix $Q_{E \setminus E'}(\theta)$ is a subintensity matrix, hence by Lemma 1 in Jacobsen [8] it is non-singular and we have shown that $\det Q(z,\theta)$ is a polynomium of degree d exactly.

To show that the number of roots z for $\det Q(\cdot,\theta)$ with $\mathrm{Re}(z) < 0$ equals p_c given by (2.9) we argue as follows: with E' as above, consider first the matrix $\bar{Q}(z,\theta)$ obtained from $Q(z,\theta)$ by replacing all the off-diagonal elements q_{ij} with $i \in E'$, $j \in E$ by 0. Then

$$\det \bar{Q}(z,\theta) = \left(\det Q_{E \setminus E'}(\theta)\right) \prod_{i \in E'} \left(\phi_i(z) - q_i - \lambda_i - \theta\right). \tag{B.7}$$

For $i \in E'$ with $\sigma_i^2 > 0$, the factor $\phi_i(z) - q_i - \lambda_i - \theta$ is a polynomium of degree 2 and since this polynomium takes the value $-q_i - \lambda_i - \theta < 0$ for $z = 0$ (remember that by Assumption (A) all $\lambda_i + q_i > 0$) with the limit ∞ as $z \to \pm\infty$, it has two real roots, one < 0 and one > 0. For $i \in E'$ with $\sigma_i^2 = 0$, the factor $\phi_i(z) - q_i - \lambda_i - \theta$ is a polynomium of degree 1 and it is immediate that the root of this polynomium is < 0 if $\beta_i < 0$ and > 0 if $\beta_i > 0$. Thus all roots for the polynomium (B.7) are real and precisely p_c of these are < 0.

For the general case with $Q(z, \theta)$ given by (3.3), consider the map $s \mapsto \det Q_s(\cdot, \theta)$, defined for $0 \leq s \leq 1$ and where the elements of the matrix $Q_s(z, \theta)$ are given by

$$q_{s,ij}(z, \theta) = \begin{cases} q_{ij}(z, \theta) & \text{if } i \in E \backslash E', j \in E \\ \phi_i(z) - q_i - \lambda_i - \theta & \text{if } i = j \in E', \\ s q_{ij} & \text{if } i \in E', j \in E, i \neq j, \end{cases}$$

so that $Q_1(z, \theta) = Q(z, \theta)$ while $\det Q_0(z, \theta) = \det \bar{Q}(z, \theta)$. When s varies, the leading coefficient of the polynomium $\det Q_s(\cdot, \theta)$ is always the same and given by (B.6), hence it follows that the roots for $\det Q_s(\cdot, \theta)$ (when ordered lexicographically for example) are continuous functions of s. By Lemma 2 however, for every s, $\det Q_s(\cdot, \theta)$ has no roots on the line $\mathrm{i}\mathbb{R}$ so when s varies from 0 to 1 none of the roots can cross from the strictly negative ($\mathrm{Re}(z) < 0$) to the strictly positive ($\mathrm{Re}(z) > 0$) half of the complex plane. But as shown in the beginning of the proof, $\det Q_0(\cdot, \theta)$ has precisely p_c roots z with $\mathrm{Re}(z) < 0$, hence also $\det Q_1(\cdot, \theta)$ has precisely p_c roots z with $\mathrm{Re}(z) < 0$. ■

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