# GEOMETRIC CONSTRUCTION OF MODULAR FUNCTORS FROM CONFORMAL FIELD THEORY

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Abstract. We give a geometric construct of a modular functor for any simple Lie-algebra and any level by twisting the constructions in [48] and [51] by a certain fractional power of the abelian theory first considered in [32] and further studied in [2].

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## 1. Introduction

This is the second paper in a series of papers aimed at providing a geometric construction of modular functors and topological quantum field theories from conformal field theory building on the constructions in [48] and [51] and [32]. In this paper we will provide the geometric construction of a modular functor $V_\ell^{\mathfrak{g}}$ for each simple Lie algebra $\mathfrak{g}$ and a positive integer $\ell$ (the *level*). By a very general construction any modular functor with duality induces a topological quantum field theory in dimension $2+1$ by the work of Kontsevich [33] and Walker [53] and also Grove [25]. Our third paper [3] in this series provides an explicit isomorphism of the modular functor underlying the Reshetikhin-Turaev TQFT for $U_q(\mathsf{sl}(n))$ and the one constructed in this paper for the Lie algebra $\mathsf{sl}(n)$. This uses the Skein theory approach to the Reshetikhin-Turaev TQFT of Blanchet, Habegger, Masbaum and Vogel [12], [13] and [11].

Fix a simple Lie algebra $\mathfrak{g}$ and normalize the invariant inner product on it by requiring the highest root to have length squared equal to 2. Let $\ell$ be a positive integer and consider the finite (label) set $P_\ell$ of integrable highest weight representations at level $\ell$ of the affine Lie algebra of $\mathfrak{g}$. By the usual highest weight vector representations, this finite set $P_\ell$ is naturally identified with a subset of the dominant integrable weights of $\mathfrak{g}$ (see formula (1)).

The main idea is to construct a modular functor $V_\ell^{\mathfrak{g}}$ by associating to each labeled marked surface the *space of vacua* using the given labels for the Lie algebra $\mathfrak{g}$ at level $\ell$ for some complex structure on the marked surface, as defined in [48] [50] and [51]. In order to make this construction independent of the complex structure it must be understood in terms of bundles with connections over the hole of Teichmüller space of the surface, relying on parallel transport to provide the required identifications between the different spaces of vacua. The consistencies of these identifications translates to flatness requirements on these connections. However, the *sheaf of vacua* construction in [48] and [51] gives a bundles with a connections, which is only projectively flat, over Teichmüller space of the surface. By tensoring this bundle with a line bundle with a connection with the opposite curvature, we get a flat bundle over Teichmüller space and the vector space we associate to the labeled marked surface is the vector space of covariant constant sections of this resulting bundle. This line bundle is constructed as a fractional power of a certain rank 1 abelian sheaf of vacua, which we considered in the first paper in this series [2] from the same point of view as [48] and [51]. The extraction of this fractional power brings in central extensions of mapping classes as the natural morphisms on which the resulting functor is defined.

The construction and properties of this flat bundle primarily rely on the complex algebraic constructions and results of [48] and [51] on the the sheaf of vacua construction yielding a conformal field theory for each simple Lie algebra $\mathfrak{g}$ and level $\ell$. For the 1-dimensional correction theory, we draw on the work [2], which in turn relies on [32].

The paper is organized as follows.

In section 2 we give the axioms for a modular functor. We introduce the notion of a *marked* surface, which is a closed smooth oriented surface, with a finite subset of points with projectiv tangent vectors and a Lagrangian subspace of the first integer homology of the surface. These form a category on which there is the operation of disjoint union and the operation of orientation reversal. There is also the process of glueing on this category. If we have a finite set, we can label the finite set of points on a marked surface by elements from this finite *label* set and get the category of labeled marked surfaces. A modular functor based on some finite label set, is a functor from the category of labeled marked surfaces to the category of finite dimensional complex vector spaces, which takes the disjoint union operation to the tensor product operation and which takes the glueing process to a certain direct sum construction, such that some compatibilities holds, as described in details in Definition 2.11. A modular functor is said to be with duality if further the operation of orientation reversal is taken to the operation of taking the dual vector space.

In sections 3 to 6 we describe in detail how any simple Lie algebra and a level $\ell$, via the sheaf of vacua constructions in [51] yields a holomorphic vector bundles with a projectively flat connection over Teichmüller spaces of pointed surfaces equipped with symplectic basis of the first homology. In sections 7 to 10 we describe in detail how the abelian sheaf of vacua constructions in [2] yields a holomorphic line bundles with a projectively flat connection and a preferred non-vanishing section over Teichmüller spaces of pointed surfaces equipped with a symplectic basis of the first homology.

In section 11 we describe our global geometric construction of a modular functor for any simple Lie algebra and a level $\ell$. Theorem 11.1 and 11.2 summarizes the constructions from sections 3 to 10. The preferred section of the abelian theory allows us to construct a certain fractional power of this line bundle as stated in Theorem 11.3, which we tensor onto this holomorphic vector bundle, so as to obtain a holomorphic vector bundle with a *flat* connection over Teichmüller space. The modular functor is then defined (Definition 11.3) by taking covariant constant sections of this flat bundle. The section ends with the construction of the disjoint union isomorphism. The glueing isomorphism is constructed in section 12, where we also prove the needed properties of glueing.

In section 13 we establish all the axioms of a modular functor is satisfied based on the main results of the preceding sections.

## 2. The axioms for a modular functor

We shall in this section give the axioms for a modular functor. These are due to G. Segal and appeared first in [43]. We present them here in a topological form, which is due to K. Walker [53]. See also [25]. We note that similar, but different, axioms for a modular functor are given in [49] and in [6]. It is however not clear if these definitions of a modular functor is equivalent to ours.

Let us start by fixing a bit of notation. By a closed surface we mean a smooth real two dimensional manifold. For a closed oriented surface $\Sigma$ of genus $g$ we have the non-degenerate skew-symmetric intersection pairing

$$(\cdot,\cdot) : H_1(\Sigma,\mathbb{Z}) \times H_1(\Sigma,\mathbb{Z}) \to \mathbb{Z}.$$

Suppose $\Sigma$ is connected. In this case a Lagrangian subspace $L \subset H_1(\Sigma,\mathbb{Z})$ is by definition a subspace, which is maximally isotropic with respect to the intersection pairing. - A $\mathbb{Z}$-basis $(\vec{\alpha},\vec{\beta}) = (\alpha_1,\dots,\alpha_g,\beta_1,\dots\beta_g)$ for $H_1(\Sigma,\mathbb{Z})$ is called a symplectic basis if

$$(\alpha_i,\beta_j) = \delta_{ij}, \quad (\alpha_i,\alpha_j) = (\beta_i,\beta_j) = 0,$$

for all $i,j = 1,\dots,g$.

If $\Sigma$ is not connected, then $H_1(\Sigma,\mathbb{Z}) = \oplus_i H_1(\Sigma_i,\mathbb{Z})$, where $\Sigma_i$ are the connected components of $\Sigma$. By definition a Lagrangian subspace is in this paper a subspace of the form $L = \oplus_i L_i$, where $L_i \subset H_1(\Sigma_i,\mathbb{Z})$ is Lagrangian. Likewise a symplectic basis for $H_1(\Sigma,\mathbb{Z})$ is a $\mathbb{Z}$-basis of the form $((\vec{\alpha}^i,\vec{\beta}^i))$, where $(\vec{\alpha}^i,\vec{\beta}^i)$ is a symplectic basis for $H_1(\Sigma_i,\mathbb{Z})$.

For any real vector space $V$, we define $PV = (V - \{0\})/\,\mathbb{R}_+$.

**Definition 2.1.** *A* pointed surface $(\Sigma,P)$ *is an oriented closed surface* $\Sigma$ *with a finite set* $P \subset \Sigma$ *of points. A pointed surface is called* stable *if the Euler characteristic of each component of the complement of the points* $P$ *is negative. A pointed surface is called* saturated *if each component of* $\Sigma$ *contains at least one point from* $P$*.*

**Definition 2.2.** *A* morphism of pointed surfaces $f : (\Sigma_1,P_1) \to (\Sigma_2,P_2)$ *is an isotopy class of orientation preserving diffeomorphisms which maps* $P_1$ *to* $P_2$*. Here the isotopy is required not to change the induced map of the first order Jet at* $P_1$ *to the first order Jet at* $P_2$*.*

**Definition 2.3.** *A* marked surface $\mathbf{\Sigma} = (\Sigma, P, V, L)$ *is an oriented closed smooth surface* $\Sigma$ *with a finite subset* $P \subset \Sigma$ *of points with projective tangent vectors* $V \in \sqcup_{p \in P} PT_p\Sigma$ *and a Lagrangian subspace* $L \subset H_1(\Sigma, \mathbb{Z})$.

**Remark 2.1.** The notions of stable and saturated marked surfaces are defined just like for pointed surfaces.

**Definition 2.4.** *A* morphism $\mathbf{f} : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ *of marked surfaces* $\mathbf{\Sigma}_i = (\Sigma_i, P_i, V_i, L_i)$ *is an isotopy class of orientation preserving diffeomorphisms* $f : \Sigma_1 \to \Sigma_2$ *that maps* $(P_1, V_1)$ *to* $(P_2, V_2)$ *together with an integer* $s$. *Hence we write* $\mathbf{f} = (f, s)$.

**Remark 2.2.** Any marked surface has an underlying pointed surface, but a morphism of marked surfaces does not quit induce a morphism of pointed surfaces, since we only require that the isotopies preserve the induced maps on the projective tangent spaces.

Let $\sigma$ be Wall's signature cocycle for triples of Lagrangian subspaces of $H_1(\Sigma, \mathbb{R})$ (See [54]).

**Definition 2.5.** *Let* $\mathbf{f}_1 = (f_1, s_1) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ *and* $\mathbf{f}_2 = (f_2, s_2) : \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_3$ *be morphisms of marked surfaces* $\mathbf{\Sigma}_i = (\Sigma_i, P_i, V_i, L_i)$ *then the composition of* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *is*

$$\mathbf{f}_2\mathbf{f}_1 = (f_2 f_1, s_2 + s_1 - \sigma((f_2 f_1)_* L_1, f_{2*} L_2, L_3)).$$

With the objects being marked surfaces and the morphism and their composition being defined as in the above definition, we have constructed the category of marked surfaces.

The mapping class group $\Gamma(\mathbf{\Sigma})$ of a marked surface $\mathbf{\Sigma} = (\Sigma, L)$ is the group of automorphisms of $\mathbf{\Sigma}$. One can prove that $\Gamma(\mathbf{\Sigma})$ is a central extension of the mapping class group $\Gamma(\Sigma)$ of the surface $\Sigma$ defined by the 2-cocycle $c : \Gamma(\mathbf{\Sigma}) \to \mathbb{Z}$, $c(f_1, f_2) = \sigma((f_1 f_2)_* L, f_{1*} L, L)$. One can also prove that this cocycle is equivalent to the cocycle obtained by considering two-framings on mapping cylinders (see [5] and [1]).

Notice also that for any morphism $(f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$, one can factor

$$\begin{aligned}(f, s) &= ((\mathrm{Id}, s') : \mathbf{\Sigma}_2 \to \mathbf{\Sigma}_2) \circ (f, s - s') \\ &= (f, s - s') \circ ((\mathrm{Id}, s') : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_1).\end{aligned}$$

In particular $(\mathrm{Id}, s) : \mathbf{\Sigma} \to \mathbf{\Sigma}$ is $(\mathrm{Id}, 1)^s$.

**Definition 2.6.** *The operation of* disjoint union of marked surfaces *is*

$$(\Sigma_1, P_1, V_1, L_1) \sqcup (\Sigma_2, P_2, V_2, L_2) = (\Sigma_1 \sqcup \Sigma_2, P_1 \sqcup P_2, V_1 \sqcup V_2, L_1 \oplus L_2).$$

*Morphisms on disjoint unions are accordingly* $(f_1, s_1) \sqcup (f_2, s_2) = (f_1 \sqcup f_2, s_1 + s_2)$.

We see that disjoint union is an operation on the category of marked surfaces.

**Definition 2.7.** *Let $\mathbf{\Sigma}$ be a marked surface. We denote by $-\mathbf{\Sigma}$ the marked surface obtained from $\mathbf{\Sigma}$ by the* operation of reversal of the orientation. *For a morphism $\mathbf{f} = (f, s) : \mathbf{\Sigma}_1 \to \mathbf{\Sigma}_2$ we let the orientation reversed morphism be given by $-\mathbf{f} = (f, -s) : -\mathbf{\Sigma}_1 \to -\mathbf{\Sigma}_2$.*

We also see that orientation reversal is an operation on the category of marked surfaces. Let us now consider glueing of marked surfaces.

Let $(\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)$ be a marked surface, where we have selected an ordered pair of marked points with projective tangent vectors $((p_-, v_-), (p_+, v_+))$, at which we will perform the glueing.

Let $c : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$ be an orientation reversing projective linear isomorphism such that $c(v_-) = v_+$. Such a $c$ is called a *glueing map* for $\Sigma$. Let $\tilde{\Sigma}$ be the oriented surface with boundary obtained from $\Sigma$ by blowing up $p_-$ and $p_+$, i.e.

$$\tilde{\Sigma} = (\Sigma - \{p_-, p_+\}) \sqcup P(T_{p_-}\Sigma) \sqcup P(T_{p_+}\Sigma),$$

with the natural smooth structure induced from $\Sigma$. Let now $\Sigma_c$ be the closed oriented surface obtained from $\tilde{\Sigma}$ by using $c$ to glue the boundary components of $\tilde{\Sigma}$. We call $\Sigma_c$ the glueing of $\Sigma$ at the ordered pair $((p_-, v_-), (p_+, v_+))$ with respect to $c$.

Let now $\Sigma'$ be the topological space obtained from $\Sigma$ by identifying $p_-$ and $p_+$. We then have natural continuous maps $q : \Sigma_c \to \Sigma'$ and $n : \Sigma \to \Sigma'$. On the first homology group $n$ induces an injection and $q$ a surjection, so we can define a Lagrangian subspace $L_c \subset H_1(\Sigma_c, \mathbb{Z})$ by $L_c = q_*^{-1}(n_*(L))$. We note that the image of $P(T_{p_-}\Sigma)$ (with the orientation induced from $\tilde{\Sigma}$) induces naturally an element in $H_1(\Sigma_c, \mathbb{Z})$ and as such it is contained in $L_c$.

**Remark 2.3.** If we have two glueing maps $c_i : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$, $i = 1, 2$, we note that there is a diffeomorphism $f$ of $\Sigma$ inducing the identity on $(p_-, v_-) \sqcup (p_+, v_+) \sqcup (P, V)$ which is isotopic to the identity among such maps, such that $(df_{p_+})^{-1} c_2 df_{p_-} = c_1$. In particular $f$ induces a diffeomorphism $f : \Sigma_{c_1} \to \Sigma_{c_2}$ compatible with $f : \Sigma \to \Sigma$, which maps $L_{c_1}$ to $L_{c_2}$. Any two such diffeomorphims of $\Sigma$ induces isotopic diffeomorphims from $\Sigma_1$ to $\Sigma_2$.

**Definition 2.8.** *Let $\mathbf{\Sigma} = (\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)$ be a marked surface. Let*

$$c : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$$

*be a glueing map and $\Sigma_c$ the glueing of $\Sigma$ at the ordered pair $((p_-, v_-), (p_+, v_+))$ with respect to $c$. Let $L_c \subset H_1(\Sigma_c, \mathbb{Z})$ be the Lagrangian subspace constructed above from*

*$L$. Then the marked surface $\mathbf{\Sigma}_c = (\Sigma_c, P, V, L_c)$ is defined to be the* glueing *of $\mathbf{\Sigma}$ at the ordered pair $((p_-, v_-), (p_+, v_+))$ with respect to $c$.*

We observe that glueing also extends to morphisms of marked surfaces which preserves the ordered pair $((p_-, v_-), (p_+, v_+))$, by using glueing maps which are compatible with the morphism in question.

We can now give the axioms for a 2 dimensional modular functor.

**Definition 2.9.** *A* label set *$\Lambda$ is a finite set furnished with an involution $\lambda \mapsto \hat{\lambda}$ and a trivial element $1$ such that $\hat{1} = 1$.*

**Definition 2.10.** *Let $\Lambda$ be a label set. The category of* $\Lambda$-labeled marked surfaces *consists of marked surfaces with an element of $\Lambda$ assigned to each of the marked point and morphisms of labeled marked surfaces are required to preserve the labelings. An assignment of elements of $\Lambda$ to the marked points of $\mathbf{\Sigma}$ is called a labeling of $\mathbf{\Sigma}$ and we denote the labeled marked surface by $(\mathbf{\Sigma}, \lambda)$, where $\lambda$ is the labeling.*

We define a labeled pointed surface similarly.

**Remark 2.4.** The operation of disjoint union clearly extends to labeled marked surfaces. When we extend the operation of orientation reversal to labeled marked surfaces, we also apply the involution $\hat{\cdot}$ to all the labels.

**Definition 2.11.** *A* modular functor *based on the label set $\Lambda$ is a functor $V$ from the category of labeled marked surfaces to the category of finite dimensional complex vector spaces satisfying the axioms MF1 to MF5 below.*

*MF1. Disjoint union axiom*: The operation of disjoint union of labeled marked surfaces is taken to the operation of tensor product, i.e. for any pair of labeled marked surfaces there is an isomorphism

$$V((\mathbf{\Sigma}_1, \lambda_1) \sqcup (\mathbf{\Sigma}_2, \lambda_2))) \cong V(\mathbf{\Sigma}_1, \lambda_1) \otimes V(\mathbf{\Sigma}_2, \lambda_2).$$

The identification is associative.

*MF2. Glueing axiom*: Let $\mathbf{\Sigma}$ and $\mathbf{\Sigma}_c$ be marked surfaces such that $\mathbf{\Sigma}_c$ is obtained from $\mathbf{\Sigma}$ by glueing at an ordered pair of points and projective tangent vectors with respect to a glueing map $c$. Then there is an isomorphism

$$V(\mathbf{\Sigma}_c, \lambda) \cong \bigoplus_{\mu \in \Lambda} V(\mathbf{\Sigma}, \mu, \hat{\mu}, \lambda),$$

which is associative, compatible with glueing of morphisms, disjoint unions and it is independent of the choice of the glueing map in the obvious way (see remark 2.3).

*MF3. Empty surface axiom*: Let $\emptyset$ denote the empty labeled marked surface. Then

$$\dim V(\emptyset) = 1.$$

*MF4. Once punctured sphere axiom*: Let $\mathbf{\Sigma} = (S^2, \{p\}, \{v\}, 0)$ be a marked sphere with one marked point. Then

$$\dim V(\mathbf{\Sigma}, \lambda) = \begin{cases} 1, & \lambda = 1 \\ 0, & \lambda \neq 1. \end{cases}$$

*MF5. Twice punctured sphere axiom*: Let $\mathbf{\Sigma} = (S^2, \{p_1, p_2\}, \{v_1, v_2\}, \{0\})$ be a marked sphere with two marked points. Then

$$\dim V(\mathbf{\Sigma}, (\lambda, \mu)) = \begin{cases} 1, & \lambda = \hat{\mu} \\ 0, & \lambda \neq \hat{\mu}. \end{cases}$$

In addition to the above axioms one may has extra properties, namely

*MF-D. Orientation reversal axiom*: The operation of orientation reversal of labeled marked surfaces is taken to the operation of taking the dual vector space, i.e for any labeled marked surface $(\mathbf{\Sigma}, \lambda)$ there is a pairring

$$\langle \cdot, \cdot \rangle : V(\mathbf{\Sigma}, \lambda) \otimes V(-\mathbf{\Sigma}, \hat{\lambda}) \to \mathbb{C},$$

compatible with disjoint unions, glueings and orientation reversals (in the sense that the induced isomorphisms $V(\mathbf{\Sigma}, \lambda) \cong V(-\mathbf{\Sigma}, \hat{\lambda})^*$ and $V(-\mathbf{\Sigma}, \hat{\lambda}) \cong V(\mathbf{\Sigma}, \lambda)^*$ are adjoints).

and

*MF-U. Unitarity axiom* Every vector space $V(\mathbf{\Sigma}, \lambda)$ is furnished with a hermitian inner product

$$(\cdot, \cdot) : V(\mathbf{\Sigma}, \lambda) \otimes \overline{V(\mathbf{\Sigma}, \lambda)} \to \mathbb{C}$$

so that morphisms induces unitary transformation. The hermitian structure must be compatiple with disjoint union and glueing. If we have the orientation reversal property, then compatibility with the unitary structure means that we have a commutative diagrams

$$\begin{array}{ccc} V(\mathbf{\Sigma}, \lambda) & \xrightarrow[\cong]{} & V(-\mathbf{\Sigma}, \hat{\lambda})^* \\ \downarrow \cong & & \cong \downarrow \\ \overline{V(\mathbf{\Sigma}, \lambda)^*} & \xrightarrow{\cong} & \overline{V(-\mathbf{\Sigma}, \hat{\lambda})}, \end{array}$$

where the vertical identifications come from the hermitian structure and the horizontal from the duality.

The rest of the paper is concerned with the detailed geometric construction of modular functors using conformal field theory. However, we shall assume the reader is familiar with [51] and [2] and freely use the notations of these two papers in this paper.

## 3. Families of curves with formal neighbourhoods and Teichmüller space

Let us first review some basic Teichmüller theory. Let $\Sigma$ be a closed oriented smooth surface and let $P$ be finite set of points on $\Sigma$.

**Definition 3.1.** *A* marked curve $\mathbf{C}$ *is a smooth projective curve $C$ over $\mathbb{C}$ with a finite set of marked points $Q$ and non-zero tangent vectors $W \in T_Q C = \bigsqcup_{q\in Q} T_q C$.*

**Definition 3.2.** *A* morphism *between marked curves is a biholomorphism of the underlying curves which induces a bijection between the two sets of marked points and tangent vectors at the marked points.*

The notions of stable and saturated is defined just like for pointed surfaces.

**Definition 3.3.** *A* complex structure *on $(\Sigma, P)$ is a marked curve $\mathbf{C} = (C, Q, W)$ together with an orientation preserving diffeomorphism $\phi : \Sigma \to C$ mapping the points $P$ onto the points $Q$. Two such complex structures $\phi_j : (\Sigma, P) \to \mathbf{C}_j = (C_j, Q_j, W_j)$ are* equivalent *if there exists a morphism of marked curves*

$$\Phi : \mathbf{C}_1 \to \mathbf{C}_2$$

*such that $\phi_2^{-1}\Phi\phi_1 : (\Sigma, P) \to (\Sigma, P)$ is isotopic to the identity through maps inducing the identity on the first order neighbourhood of $P$.*

We shall often in our notation suppress the diffeomorphism, when we denote a complex structure on a surface.

**Definition 3.4.** *The* Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ *of the pointed surface $(\Sigma, P)$ is by definition the set of equivalence classes of complex structures on $(\Sigma, P)$.*

We note there is a natural projection map from $\mathcal{T}_{(\Sigma,P)}$ to $T_P\Sigma = \sqcup_{p\in P} T_p\Sigma$, which we call $\pi_P$.

**Theorem 3.1** (Bers)**.** *There is a natural structure of a complex analytic manifold on Teichmüller space $\mathcal{T}_{(\Sigma,P)}$. Associated to any morphism of pointed surfaces $f : (\Sigma_1, P_1) \to (\Sigma_2, P_2)$ there is a biholomorphism $f^* : \mathcal{T}_{(\Sigma_1,P_1)} \to \mathcal{T}_{(\Sigma_2,P_2)}$ which is induced by mapping a complex structure $\mathbf{C} = (C, Q, W)$, $\phi : \Sigma_1 \to C$ to $\phi \circ f^{-1} : \Sigma_2 \to C$. Moreover, compositions of morphisms go to compositions of induced biholomorphisms.*

There is an action of $\mathbb{R}_+^{\mathrm{P}}$ on $\mathcal{T}_{(\Sigma,P)}$ given by scaling the tangent vectors. This action is free and the quotient $\mathcal{T}^{(\mathrm{r})}{}_{(\Sigma,P)} = \mathcal{T}_{(\Sigma,P)} \,/\, \mathbb{R}_+^{\mathrm{P}}$ is a smooth manifold, which we call the *reduced* Teichmüller space of the pointed surface $(\Sigma, P)$. Moreover the projection map $\pi_P$ descend to a smooth projection map from $\mathcal{T}^{(\mathrm{r})}{}_{(\Sigma,P)}$ to $\sqcup_{p\in P} P(T_p\Sigma)$, which we denote $\pi_P^{(r)}$. We denote the fiber of this map over $V \in \sqcup_{p\in P} P(T_p\Sigma)$ by $\mathcal{T}_{(\Sigma,P,V)}$. Teichmüller space of a marked surface $\mathbf{\Sigma} = (\Sigma, P, V, L)$ is by definition $\mathcal{T}_{\mathbf{\Sigma}} = \mathcal{T}_{(\Sigma,P,V)}$, which we call the Teichmüller space of the marked surface. Morphisms of marked surfaces induce diffeomorphism of the corresponding Teichmüller spaces of marked surfaces, which of course also behaves well under composition. We observe that the self-morphism $(\mathrm{Id}, s)$ of a marked surface acts trivially on the associated Teichmüller space for all integers $s$. General Teichmüller theory implies that

**Theorem 3.2.** *The Teichmüller space $\mathcal{T}_{\mathbf{\Sigma}}$ of any marked surface $\mathbf{\Sigma}$ is contractible.*

Now let us recall the definition of a formal neighbourhood of a point on a curve.

**Definition 3.5.** *Let $C$ be a smooth projective curve and $q$ a point on $C$. Let $\mathcal{O}_{C,q}$ be the stalk of $\mathcal{O}_C$ at $q$ and let $m_q$ the maximal ideal in $\mathcal{O}_{C,q}$. We note that $m_q^n$, $n = 0, 1, 2, \ldots$, gives a filtration of $\mathcal{O}_{C,q}$. A formal $n$'th-order neighbourhood at $q$ is a filtration preserving isomorphism*

$$\mathcal{O}_{C,q}/m_q^{n+1} \cong \mathbb{C}[[\xi]]/(\xi^{n+1}).$$

*Let $\hat{\mathcal{O}}_{C,q} = \lim_{n\to\infty} \mathcal{O}_{C,q}/m_q^n$ be the completion of $\mathcal{O}_{C,q}$ with respect to the filtration. A formal neighbourhood at $q$ is a filtration preserving isomorphism*

$$\hat{\mathcal{O}}_{C,q} \cong \mathbb{C}[[\xi]].$$

We note that we have a canonical isomorphism

$$\begin{array}{rcl} \mathcal{O}_{C,q}/\mathfrak{m}_q^2 & \simeq & \mathbb{C} \oplus T_q^*C, \\ f & \mapsto & (f(q), df_q). \end{array}$$

Hence a formal 1'st order neighbourhood induces and is determined by an isomorphism of $T_q^*C$ with $\mathbb{C}$. Hence a formal 1'st order neighbourhood determines and is determined by a non-zero vector in $T_q^*C$, specified by the property that it maps to $1 \in \mathbb{C}$ or equivalently a vector in $T_qC$ pairing to unity with this vector. Recall the definition of an $N$-pointed curve with formal neighbourhoods $\mathfrak{X}$ from [51], Definition 1.1.3.

**Definition 3.6.** *For an $N$-pointed curve with formal neighbourhoods $\mathfrak{X}$, we denote by $c(\mathfrak{X})$ the underlying marked curve. For a labeled curve with formal neighbourhoods*

$(\mathfrak{X}, \vec{\lambda})$, *we denote by* $c(\mathfrak{X}, \vec{\lambda}) = (c(\mathfrak{X}), \lambda)$ *the underlying labeled marked curve. Here* $\lambda$ *denotes the labeling of the marked points of* $c(\mathfrak{X})$ *induced by* $\vec{\lambda}$.

Let $\Sigma$ be a closed oriented smooth surface and let $P$ be finite set of $N$ marked points on $\Sigma$, i.e. $(\Sigma, P)$ is a pointed surface.

Let now $\mathcal{B}$ be a connected smooth complex manifold. Let $Y$ be the smooth manifold $Y = \Sigma \times \mathcal{B}$ and $\pi : Y \to \mathcal{B}$ the projection map. Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; \vec{s}; \vec{\eta})$ be a family of $N$-pointed curves with formal neighbourhoods as in Definition 1.2.1 in [51] and assume we have a smooth fiber preserving diffeomorphism $\Phi_{\mathfrak{F}}$ from $Y$ to $\mathcal{C}$ taking the marked points to the sections $\vec{s}$ and inducing the identity on $\mathcal{B}$. This data induces a unique holomorphic map $\Psi_{\mathfrak{F}}$ from $\mathcal{B}$ to the Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ of the surface $(\Sigma, P)$.

**Definition 3.7.** *The pair* $(\mathfrak{F}, \Phi_{\mathfrak{F}})$ *is called a family of pointed curves with formal neighbourhoods* on $(\Sigma, P)$. *If* $P' \subset P$ *is a strict subset, we say that* $(\mathfrak{F}, \Phi_{\mathfrak{F}})$ *is called a family of pointed curves with formal neighbourhoods* over $(\Sigma, P')$.

Often we will suppress $\Phi_{\mathfrak{F}}$ in our notation and just write $\mathfrak{F}$ is a family of pointed curves with formal neighbourhoods on $(\Sigma, P)$.

**Definition 3.8.** *If a family* $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; \vec{s}; \vec{\eta})$ *of pointed curves with formal neighbourhoods on* $(\Sigma, P)$, *as above, has the properties, that the base* $\mathcal{B}$ *is biholomorphic to an open ball and that the induced map* $\Psi_{\mathfrak{F}}$ *is a biholomorphism onto an open subset of Teichmüller space* $\mathcal{T}_{(\Sigma,P)}$ *then the family is said to be good.*

**Proposition 3.1.** *For a stable and saturated pointed curve* $(\Sigma, P)$ *the Teichmüller space* $\mathcal{T}_{(\Sigma,P)}$ *can be covered by images of such good families.*

This follows from Theorem 1.2.9 in [51].

Suppose now that we have two stable and saturated families $\mathfrak{F}_i$, $i = 1, 2$ with the property that they have the same image $\Psi_{\mathfrak{F}_1}(\mathcal{B}_1) = \Psi_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ and that $\mathfrak{F}_2$ is a good family.

**Proposition 3.2.** *For such a pair of families there exists a unique fiber preserving biholomorphism* $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ *covering* $\Psi_{\mathfrak{F}_2}^{-1}\Psi_{\mathfrak{F}_1}$ *such that* $\Phi_{\mathfrak{F}_2}^{-1}\Phi\Phi_{\mathfrak{F}_1} : (Y, P) \to (Y, P)$ *is isotopic to* $\Psi_{\mathfrak{F}_2}^{-1}\Psi_{\mathfrak{F}_1} \times \mathrm{Id}$ *through such fiber preserving maps inducing the identity on the first order neighbourhood of* $P$.

This follows from uniqueness of the $\Phi$ in Definition 3.3.

We note that there is some permutation $S$ of $\{1, \ldots, N\}$ such that $(S\Phi^*(\vec{\eta}_2))^{(1)} = (\vec{\eta}_1)^{(1)}$, i.e. $S\Phi^*(\vec{\eta}_2)$ induce the same first order formal neighbourhoods as $\vec{\eta}_1$ does.

Suppose now $f$ is an orientation preserving diffeomorphism from $(\Sigma_1, P_1)$ to $(\Sigma_2, P_2)$. Let $\mathfrak{F}_1$ be a family of pointed curves with formal neighbourhoods of $(\Sigma_1, P_1)$. By composing $\Phi_{\mathfrak{F}_1}$ with $f^{-1}\times \mathrm{Id}$ we get a family of pointed curves with formal neighbourhoods of $(\Sigma_2, P_2)$. We note that $\Psi_{\mathfrak{F}_2} = f^* \circ \Psi_{\mathfrak{F}_1}$, where $f^*$ is the induced map between the Teichmüller space. This operation on families clearly behaves well under compositions of diffeomorphisms.

## 4. Definition of the space of vacua associated to a labeled marked curve.

Let us based on the definition of the space of vacua associated to a labeled pointed curve with formal neighbourhoods as given in [51], define the space of vacua associated to a label marked curve. So we fix a simple Lie algebra $\mathfrak{g}$ over the complex numbers. We normalize the Cartan-Killing form as in (2.1.2) in [51], i.e. the length squared of the longest root $\theta$ is 2. We fix a positive integer $\ell$, which is called the *level*. Let $P_+$ be the set of dominant integral weights of $\mathfrak{g}$ and

$$P_\ell = \{\lambda \in P_+ \mid 0 \leq (\theta, \lambda) \leq \ell\}. \tag{1}$$

The level $\ell$ integrable highest weight representations of the affine Lie-algebra associated to $\mathfrak{g}$ is indexed by their highest weight vector in the finite set $P_\ell$. For $\lambda \in P_\ell$, we denote the corresponding representation $\mathcal{H}_\lambda$. We also define an involution $\dagger : P_\ell \to P_\ell$ by

$$\lambda^\dagger = -w(\lambda),$$

where $w$ is the longest element of the Weyl group of $\mathfrak{g}$. We notice that $0 \in P_\ell$ and $0^\dagger = 0$. We refer the reader to chapter 2 & 3 of [51] for properties of these representations, the involution $\dagger$ and the definition of the space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ associated to a pointed labeled curve with formal neighbourhoods $\mathfrak{X}$.

**Remark 4.1.** We notice that the definition (3.1.10) in [51] of the space of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ associated to an $N$-pointed curve with formal neighbourhoods $\mathfrak{X} = (C, \vec{Q}, \vec{\eta})$ depends on the ordering of the marked points $\vec{Q} = (Q_1, \dots, Q_N)$. However, if the points are permuted by a permutation, the same permutation acting on the tensor product of the $\mathcal{H}_{\vec{\lambda}}$'s induce isomorphism between the corresponding space of (co-)vacua. Clearly, compositions of permutations go to compositions of isomorphisms.

Let $\mathfrak{X}' = (C, \vec{Q}, \vec{\eta}')$ be another $N$-pointed curve with formal neighbourhoods such that $c(\mathfrak{X}) = c(\mathfrak{X}')$. Let $\vec{h}$ be the formal change of coordinates from $\vec{\eta}$ to $\vec{\eta}'$. Then Corollary 3.2.6 in [51] states that

$$G[\vec{h}] : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \to \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}')$$

is an isomorphism.

**Definition 4.1.** *Let* $\mathbf{C} = (C, Q, W)$ *be a stable and saturated marked curve. Let* $\lambda$ *be a labeling of* $\mathbf{C}$ *using the set* $P_\ell$. *The space of vacua associated to the labeled marked curve* $(\mathbf{C}, \lambda)$ *is by definition*

$$\mathcal{V}_\lambda^\dagger(\mathbf{C}) = \coprod_{c(\mathfrak{X},\vec{\lambda})=(\mathbf{C},\lambda)} \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X})/\sim,$$

*where the disjoint union is over all labeled curves with formal neighbourhoods with* $(\mathbf{C}, \lambda)$ *as the underlying labeled marked curve,* $\vec{\lambda}$ *is compatible with the labeling* $\lambda$ *and* $\sim$ *is the equivalence relation generated by the preferred isomorphisms described in remark 4.1 above.*

That the relation $\sim$ is an equivalence relation follows from 2. in Theorem 3.2.4 in [51]. Further it is clear that

**Proposition 4.1.** *The natural quotient map from* $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X})$ *to* $\mathcal{V}_\lambda^\dagger(\mathbf{C})$ *is an isomorphism for all labeled curves with formal neighbourhoods* $(\mathfrak{X}, \vec{\lambda})$ *with* $c(\mathfrak{X}, \vec{\lambda}) = (\mathbf{C}, \lambda)$.

Suppose $(\mathbf{C}_i, \lambda_i)$ are labeled marked curves and $\Phi : (\mathbf{C}_1, \lambda_1) \to (\mathbf{C}_2, \lambda_2)$ is a morphism of labeled marked curves. Let $(\mathfrak{X}_2, \vec{\lambda}_2)$ be a labeled curve with formal neighbourhoods such that $c(\mathfrak{X}_2, \vec{\lambda}_2) = (\mathbf{C}, \lambda_2)$. Let $\Phi^*\mathfrak{X}_2 = \mathfrak{X}_1$. Then $\Phi$ is a morphism of labeled marked curves with formal neighbourhoods. We obviously have that

**Proposition 4.2.** *The identity map on* $\mathcal{H}_{\vec{\lambda}}^\dagger$ *induces a linear isomorphism from* $\mathcal{V}_{\vec{\lambda}_1}^\dagger(\mathfrak{X}_1)$ *to* $\mathcal{V}_{\vec{\lambda}_2}^\dagger(\mathfrak{X}_2)$, *which induces a well defined linear isomorphism* $\mathcal{V}^\dagger(\Phi)$ *from* $\mathcal{V}_{\lambda_1}^\dagger(\mathbf{C}_1)$ *to* $\mathcal{V}_{\lambda_2}^\dagger(\mathbf{C}_2)$. *Compositions of morphisms of labeled marked curves go to compositions of the induced linear isomorphisms.*

Let $\mathfrak{X} = (C; \vec{Q}; \vec{\eta})$ be a curve with formal neighbourhoods and let $Q_{N+1}$ be a further point on the curve $C$ and $\eta$ a formal neighbourhood of $C$ at $Q_{N+1}$. Put $\vec{\vec{Q}} = (Q_1, \ldots, Q_N, Q_{N+1})$ and $\vec{\vec{\eta}} = (\eta_1, \ldots, \eta_N, \eta_{N+1})$. Let

$$\widetilde{\mathfrak{X}} = (C; \vec{\vec{Q}}; \vec{\vec{\eta}}).$$

Let $P_{\widetilde{\mathfrak{X}},\mathfrak{X}}$ be the canonical isomorphism from $\mathcal{V}_{\vec{\lambda},0}^\dagger(\widetilde{\mathfrak{X}})$ to $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X})$ as given in Theorem 3.3.1 in [51]. This isomorphism is called the *Propagation of vacua isomorphism* and it is simply induced from the following inclusion map $\mathcal{H}_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$ given by $|v\rangle \mapsto |v\rangle \otimes |0\rangle$. Suppose now $\vec{\xi}$ is another formal neighbourhood at $\vec{Q}$ and that $\xi_{N+1}$ is a formal neighbourhood at $Q_{N+1}$. Let then $\vec{\vec{\xi}} = (\xi, \xi_{N+1})$, $\mathfrak{X}' = (C; \vec{Q}; \vec{\xi})$ and $\widetilde{\mathfrak{X}} = (C; \vec{\vec{Q}}; \vec{\vec{\xi}})$. Let $\vec{h}$ be the formal coordinate change $\vec{\xi} = \vec{h}(\vec{\eta})$ and $\widetilde{\vec{h}}$ the formal coordinate change $\vec{\tilde{\xi}} = \vec{\tilde{h}}(\vec{\tilde{\eta}})$. Assume now $c(\mathfrak{X}, \vec{\lambda}) = c(\mathfrak{X}', \vec{\lambda})$ and $c(\widetilde{\mathfrak{X}}, \vec{\lambda}, 0) = c(\widetilde{\mathfrak{X}}', \vec{\lambda}, 0)$. Then $h_j \in \mathcal{D}^{p_j}$, $p_j \geq 1$ and $\widetilde{h}_j \in \mathcal{D}^{\widetilde{p}_j}$, $\widetilde{p}_j \geq 1$. By Theorem 3.2.5 in [51] we then get the following diagram:

$$\begin{array}{ccc} \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}) & \xrightarrow{P_{\widetilde{\mathfrak{X}},\mathfrak{X}}} & \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \\ {\scriptstyle G[\vec{h}]}\Big\downarrow & & \Big\downarrow{\scriptstyle G[\vec{\tilde{h}}]} \\ \mathcal{V}^{\dagger}_{\vec{\lambda},0}(\widetilde{\mathfrak{X}}') & \xrightarrow{P_{\widetilde{\mathfrak{X}}',\mathfrak{X}'}} & \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}') \end{array} \tag{2}$$

**Proposition 4.3.** *The diagram (2) is commutative.*

*Proof.* Using the notation of section 3.2 in [51], we recall that

$$G[\tilde{h}_{N+1}] = \exp(-T[\underline{\lambda}_{N+1}])$$

where $\tilde{h}_{N+1} = \exp(\underline{\lambda}_{N+1})$ and $\underline{\lambda}_{N+1} \in d^{p_{N+1}}$. An explicit calculation shows that

$$L_k|0\rangle = 0$$

if $k > 0$ and

$$L_0|0\rangle = \Delta_0|0\rangle.$$

But by (2.2.8) in [51] $\Delta_0 = 0$. From this we immediately get that

$$G[\tilde{h}_{N+1}]|0\rangle = |0\rangle,$$

which by the very construction of the propagation of vacua isomorphism makes the above diagram commute.

□

This proposition states that propagation of vacua is compatible with formal coordinate change, and hence the propagation of vacua isomorphisms decent to a corresponding isomorphisms between spaces of vacua associated to labeled marked curves as defined above in Definition 4.1.

Let $(\mathbf{C}, \lambda)$ be a labeled marked curve which might not be stable or which might not be saturated.

Consider all stable saturated labeled marked curves $(\mathbf{C}', \lambda')$ obtained from $(\mathbf{C}, \lambda)$ by adding points labeled with the trivial label $0 \in P_\ell$.

**Definition 4.2.** *The space of vacua associated to the labeled marked curve* $(\mathbf{C}, \lambda)$ *is by definition*

$$\mathcal{V}^{\dagger}_{\lambda}(\mathbf{C}) = \coprod_{(\mathbf{C}',\lambda')} \mathcal{V}^{\dagger}_{\lambda'}(\mathbf{C}')/\sim,$$

*where the disjoint union is over all labeled marked curves* $(\mathbf{C}', \lambda')$ *discussed above and* $\sim$ *is the equivalence relation generated by the preferred isomorphisms in the propagation of vacua given in Theorem 3.3.1 in* [51] *and change of formal coordinates given in Theorem 3.2.5 in* [51].

## 5. Definition of the bundle of vacua over Teichmüller space

Recall the definition of the sheaf of vacua $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ associated to a labeled family $(\mathfrak{F}, \vec{\lambda})$ of stable and saturated pointed curves with formal coordinates as given in Section 4.1 in [51].

We will need the following basic obvious property of the sheaf of vacua construction on top of many of the properties already stated and proved in [51].

**Lemma 5.1.** *Let* $\mathfrak{F}_i$ *be two families of pointed stable curves over the same base* $\mathcal{B}$*. Let* $\Phi : (\mathfrak{F}_1, \lambda_1) \to (\mathfrak{F}_2, \lambda_2)$ *be an isomorphism of labeled families, which induces the identity map on the base. Then the identity map on* $\mathcal{H}^\dagger_{\vec{\lambda}}(\mathcal{B})$ *induces a canonical isomorphism*

$$\mathcal{V}^\dagger(\Phi) : \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathfrak{F}_1) \to \mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathfrak{F}_2). \tag{3}$$

Suppose that we have two families of $N$-pointed curves with formal neighbourhoods $\mathfrak{F}_i$, $i = 1, 2$ on a pointed surface $(\Sigma, P)$, with the property that they have the same image $\Psi_{\mathfrak{F}_1}(\mathcal{B}_1) = \Psi_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ and that $\mathfrak{F}_2$ is a good family. For such a pair of families there exists by Proposition 3.2 a unique fiber preserving biholomorphism $\Phi_{12} : \mathcal{C}_1 \to \mathcal{C}_2$ covering $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1}$ such that $\Phi^{-1}_{\mathfrak{F}_2}\Phi_{12}\Phi_{\mathfrak{F}_1} : (Y, P) \to (Y, P)$ is isotopic to $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1} \times \mathrm{Id}$ through such fiber preserving maps inducing the identity on the first order neighbourhood of $P$.

We note that $(\Phi^*_{12}(\vec{\eta}_2))^{(1)} = (S\vec{\eta}_1)^{(1)}$, where $S$ is some permutation of $\{1, \ldots, N\}$, i.e. $\Phi^*_{12}(\vec{\eta}_2)$ induce the same first order formal neighbourhoods as $S\vec{\eta}_1$ does. Let $\vec{h}$ be the formal change of coordinates from $S\vec{\eta}_1$ to $\Phi^*_{12}(\vec{\eta}_2)$. Let $\widetilde{\mathfrak{F}_1} = \Phi^*_{12}(\mathfrak{F}_2)$. Then $\Phi_{12}$ induces an isomorphism of families from $\widetilde{\mathfrak{F}_1}$ to $(\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1})^*(\mathfrak{F}_2)$. Choose labelings $\vec{\lambda}_1$ of $\mathfrak{F}_1$ and $\vec{\lambda}_2$ of $\mathfrak{F}_2$, which are compatible under the above isomorphism.

By Theorem 4.1.6 in [51] and Corollary 4.2.4 in [51] we have that $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_i)$ are holomorphic vector bundles over $\mathcal{B}_i$.

**Proposition 5.1.** *The action of* $S$ *on* $\mathcal{H}^\dagger_{\vec{\lambda}_1}(\mathcal{B}_1)$ *induces an isomorphism from* $\mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathfrak{F}_1)$ *to* $\mathcal{V}^\dagger_{\vec{\lambda}_2}(S\mathfrak{F}_1)$*, where* $S$ *acts on the family* $\mathfrak{F}_1$ *by permuting the numbering of the formal coordinates and the sections. Furthermore* $G[\vec{h}] : \mathcal{H}^\dagger_{\vec{\lambda}_2}(\mathcal{B}_1) \to \mathcal{H}^\dagger_{\vec{\lambda}_2}(\mathcal{B}_1)$ *induces an isomorphism from* $\mathcal{V}^\dagger_{\vec{\lambda}_2}(S\mathfrak{F}_1)$ *to* $\mathcal{V}^\dagger_{\vec{\lambda}_2}(\widetilde{\mathfrak{F}_1})$*. The pull back isomorphisms provided by Lemma 4.1.3 in* [51] *and the families isomorphism (3) provided by Lemma 5.1 induces* *an isomorphism from* $\mathcal{V}^\dagger_{\vec{\lambda}_2}(\widetilde{\mathfrak{F}_1})$ *to* $(\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1})^*\mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathfrak{F}_2)$*. The composite of these three isomorphism gives the transformation isomorphism*

$$G_{12} : \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathfrak{F}_1) \to (\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1})^*\mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathfrak{F}_2). \tag{4}$$

*The isomorphisms satisfies the cocycle condition*

$$G_{12}(\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1})^* G_{23} = G_{13}.$$

*Proof.* The cocycle condition follows, since $G_{12}$ is induced from the permutation $S$ and the isomorphism $G[\vec{h}]$ on the $\mathcal{H}$-level, combined with formula (2) of Theorem 3.2.4 in [51].

□

**Definition 5.1.** *Let* $\Sigma$ *be a closed oriented smooth surface and let* $P$ *be a finite set of* $N$ *marked points on* $\Sigma$*. Let* $\lambda$ *be a labeling of* $(\Sigma, P)$ *and assume* $(\Sigma, P)$ *is stable and saturated. We now define a holomorphic vector bundle* $\mathcal{V}^\dagger_{\vec{\lambda}} = \mathcal{V}^\dagger_{\vec{\lambda}}(\Sigma, P)$ *over Teichmüller space* $\mathcal{T}_{(\Sigma,P)}$ *using the cover* $\{\Psi_{\mathfrak{F}}(\mathcal{B})\}$*, where* $\mathfrak{F}$ *runs over the good families of complex structures on* $(\Sigma, P)$*. Over* $\Psi_{\mathfrak{F}}(\mathcal{B})$ *we specify the bundle as* $(\Psi^{-1}_{\mathfrak{F}})^*\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$*. On overlaps of the image of two good families, we use the glueing isomorphism (4) to glue the corresponding bundles together.*

Proposition 5.1 trivially implies that.

**Proposition 5.2.** *For any stable and saturated family* $\tilde{\mathfrak{F}}$ *of pointed curves with formal coordinates on* $(\Sigma, P)$ *we have a preferred isomorphism*

$$\Upsilon_{\tilde{\mathfrak{F}}} : \mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}}) \to \Psi^*_{\tilde{\mathfrak{F}}}\mathcal{V}^\dagger_{\vec{\lambda}}(\Sigma, P)$$

*induced by the transformation isomorphism between* $\mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}})$ *and* $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$*, for good families* $\mathfrak{F}$ *of complex structures on* $(\Sigma, P)$ *such that* $\Psi_{\mathfrak{F}}(\mathcal{B})$ *intersect* $\Psi_{\tilde{\mathfrak{F}}}(\mathcal{B}')$ *nonempty.*

Let $(\Sigma, P, \lambda)$ be a general labeled pointed surface, i.e. $(\Sigma, P)$ might not be stable nor saturated. Let $(\Sigma, P_i, \lambda_i)$, $i = 1, 2$ be any labeled marked surfaces obtained from $(\Sigma, P, \lambda)$ by labeling further points by $0 \in P_\ell$. Assume that $(\Sigma, P_i)$ are stable and saturated pointed surfaces. Let $\bar{P} = P_1 \cup P_2$ and $\bar{\lambda}$ be the induced labeling of $\bar{P}$. Note that $(\Sigma, \bar{P})$ is also stable and saturated.

We get holomorphic projection maps $\pi_i : \mathcal{T}_{(\Sigma,\bar{P})} \to \mathcal{T}_{(\Sigma,P_i)}$. As a direct consequence of Proposition 4.3 we get the following.

**Proposition 5.3.** *Iterations of the propagation of vacua isomorphism induces natural isomorphisms of bundles*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\Sigma, \bar{P}) \cong \pi_1^*\mathcal{V}^\dagger_{\lambda_1}(\Sigma, P_1) \cong \pi_2^*\mathcal{V}^\dagger_{\lambda_2}(\Sigma, P_2),$$

*which satisfies associativity.*

Suppose now $f:(\Sigma_1,P_1)\to(\Sigma_2,P_2)$ is a morphism of stable and saturated pointed surfaces. Then of course $f$ induces a morhpism $f^*$ from $\mathcal{T}_{(\Sigma_1,P_1)}$ to $\mathcal{T}_{(\Sigma_2,P_2)}$. Let $\lambda_1$ be a labeling of $(\Sigma_1,P_1)$ and let $\lambda_2$ be the induced labeling on $(\Sigma_2,P_2)$ such that $f:(\Sigma_1,P_1,\lambda_1)\to(\Sigma_2,P_2,\lambda_2)$ is a morphism of labeled pointed surfaces. Let now $\mathfrak{F}_1$ be a good family of stable pointed curves with formal neighbourhoods of $(\Sigma_1,P_1)$. Then by composing with $f^{-1}\times\mathrm{Id}$ we get a good family $\mathfrak{F}_2$ of stable pointed curves with formal neighbourhoods on $(\Sigma_2,P_2)$ over the same base $\mathcal{B}_1$. The identity morphism on $\mathcal{H}^\dagger_{\vec\lambda}(\mathcal{B}_1)$ then induces a morphism $\mathcal{V}^\dagger(f):\mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{F}_1)\to\mathcal{V}^\dagger_{\vec\lambda}(\mathfrak{F}_2)$ which covers the identity on the base. This is precisely the morphism induced from the morphism of families $\Phi_f=f\times\mathrm{Id}:\mathfrak{F}_1\to\mathfrak{F}_2$ by Lemma 5.1. This intern then induces a morphism $\mathcal{V}^\dagger(f):(\Psi^{-1}_{\mathfrak{F}_1})^*(\mathcal{V}^\dagger_{\vec\lambda_1}(\mathfrak{F}_1))\to(\Psi^{-1}_{\mathfrak{F}_2})^*(\mathcal{V}^\dagger_{\vec\lambda_2}(\mathfrak{F}_2))$ which covers $f^*:\Psi_{\mathfrak{F}_1}(B_1)\to\Psi_{\mathfrak{F}_2}(B_1)$.

**Proposition 5.4.** *The above construction provides a well defined lift of*

$$f^*:\mathcal{T}_{(\Sigma_2,P_2)}\to\mathcal{T}_{(\Sigma_1,P_1)}$$

*to a morphism* $\mathcal{V}^\dagger(f):\mathcal{V}^\dagger_{\lambda_1}(\Sigma_1,P_1)\to\mathcal{V}^\dagger_{\lambda_2}(\Sigma_2,P_2)$ *which behaves well under compositions.*

*Proof.* Let $f'$ be a diffeomorphism whose first order neighbourhood from $P_1$ to $P_2$ is the same as $f$'s and such that $f'$ is isotopic to $f$ among such. Let $\mathfrak{F}_{2'}$ be obtained from $\mathfrak{F}_1$ by composing with $(f')^{-1}\times\mathrm{Id}$. Then

$$\Phi=\Phi_{\mathfrak{F}_1}\circ(((f')^{-1}\circ f)\times\mathrm{Id})\circ\Phi^{-1}_{\mathfrak{F}_1}:\mathcal{C}_1\to\mathcal{C}_1$$

is the unique biholomorphism from Proposition 3.2. Hence we get a commutative diagram

$$\begin{array}{ccc}(\Psi_{\mathfrak{F}_1})^*(\mathcal{V}^\dagger_{\vec\lambda_1}(\mathfrak{F}_1)) & \xrightarrow{\mathcal{V}^\dagger(f)} & (\Psi_{\mathfrak{F}_2})^*(\mathcal{V}^\dagger_{\vec\lambda_2}(\mathfrak{F}_2)) \\ {=}\big\downarrow & & \big\downarrow G_{22'} \\ (\Psi_{\mathfrak{F}_1})^*(\mathcal{V}^\dagger_{\vec\lambda_1}(\mathfrak{F}_1)) & \xrightarrow{\mathcal{V}^\dagger(f')} & (\Psi_{\mathfrak{F}_{2'}})^*(\mathcal{V}^\dagger_{\vec\lambda_2}(\mathfrak{F}_{2'}))\end{array}$$

which shows that $\mathcal{V}^\dagger(f):\mathcal{V}^\dagger_{\vec\lambda_1}(\Sigma_1,P_1)\to\mathcal{V}^\dagger_{\vec\lambda_2}(\Sigma_2,P_2)$ is well defined. It is obvious that $\mathcal{V}^\dagger(fg)=\mathcal{V}^\dagger(f)\mathcal{V}^\dagger(g)$.

□

Assume that $(\Sigma_i,P_i,\lambda_i)$ are labeled pointed surfaces, which need not be neither stable nor saturated. Let $f:(\Sigma_1,P_1,\lambda_1)\to(\Sigma_2,P_2,\lambda_2)$ be an orientation preserving diffeomorphism of labeled pointed surfaces. Let $(\Sigma_i,P'_i,\lambda'_i)$ be labeled pointed surfaces obtained from $(\Sigma_i,P_i,\lambda_i)$ by labeling further points by $0\in P_\ell$ such that $(\Sigma_i,P'_i,\lambda'_i)$ are stable and saturated labeled pointed surfaces such that $f : (\Sigma_1, P_1', \lambda_1') \to (\Sigma_2, P_2', \lambda_2')$ is a morphism of labeled pointed surfaces. We obviously have the following result.

**Proposition 5.5.** *The lift of* $f^* : \mathcal{T}_{(\Sigma_1,P_1')} \to \mathcal{T}_{(\Sigma_2,P_2')}$ *to a morphism*

$$\mathcal{V}^\dagger(f) : \mathcal{V}^\dagger_{\lambda_1'}(\Sigma_1, P_1') \to \mathcal{V}^\dagger_{\lambda_2'}(\Sigma_2, P_2')$$

*as given by Theorem 5.4 is compatible with the isomorphisms given in Proposition 5.3.*

## 6. The connection in the bundle of vacua over Teichmüller space.

Let $\Sigma$ be a closed oriented smooth surface and let $P$ be a set of $N$ marked points on $\Sigma$. Assume that $(\Sigma, P)$ is stable and saturated pointed surface.

Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}; \vec{s}; \vec{\eta})$ be a family of $N$-pointed curves with formal neighbourhoods on $(\Sigma, P)$. Recall the discussion of symmetric bidifferentials from section 1.4 in [51]. We now introduce the notion of a normalized symmetric bidifferential.

**Definition 6.1.** *A symmetric bidifferential* $\omega \in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ *with* $\mathrm{Res}^2(\omega) = 1$ *is called a* normalized *symmetric bidifferential for the family* $\mathfrak{F}$*.*

We have the following lemma as a consequence of the construction in Section 1.4 in [51].

**Lemma 6.1.** *For any family of stable pointed curves with formal neighbourhoods* $\mathfrak{F}$ *on* $(\Sigma, P)$ *and any symplectic basis* $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ *of* $H_1(\Sigma, \mathbb{Z})$*, there is a unique normalized symmetric bidifferential* $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ *determined by formula (1.4.31) in* [51]*, but with the roles of* $\alpha$ *and* $\beta$ *reversed.*

Please do note that $\alpha$ and $\beta$ play the reverse roles in [51], but the same as in [2].

We note that for such a family we get a well defined holomorphic connection on $\mathcal{H}_{\vec{\lambda}}(\mathcal{B})$ by formula (5.1.5) in [51], which in turn induces a holomorphic connection in $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$.

**Theorem 6.1.** *Let* $\mathfrak{F}$ *be a family of stable pointed curves with formal neighbourhoods on* $(\Sigma, P)$ *and choose a symplectic basis* $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ *of* $H_1(\Sigma, \mathbb{Z})$*. Let* $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ *be the normalized symmetric biddifferential determined by this data. Then there is the connection* $\nabla^{(\omega)}$ *in the bundle* $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})$ *whose* $(1,0)$*-part is given by formula (5.1.9) in* [51] *and whose* $(0,1)$*-part is just the* $\overline{\partial}$*-operator in this holomorphic bundle. The* $(2,0)$*-part of the curvature of this connection is given by the formula at the end of the proof of Theorem 5.1.5 in* [51] *and the* $(1,1)$ *and (0,2)-part vanishes. From this we see that this connection is projectively flat.*

*Proof.* It follows from the definition of the connection in formula (5.1.9) in [51] and corollary 5.1.2 in [51] that the $(1,1)$ and $(0,2)$-part of the curvature vanishes.

□

Suppose now that we have two good families $\mathfrak{F}_i$, $i = 1, 2$ with the property that they have the same image $\Psi_{\mathfrak{F}_1}(\mathcal{B}_1) = \Psi_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_{(\Sigma,P)}$.

**Lemma 6.2.** *Let $\nabla_i^{(\omega)}$ be the connection in $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}_i)$ described in Theorem 6.1. Then we have that*

$$G_{12}^*(\nabla_2^{(\omega)}) = \nabla_1^{(\omega)}.$$

*Proof.* Since the connection is descended from the $\mathcal{H}$-level and $G_{12}$ is also descended from this level, we just need to check the transformation rule on this level. Up on the $\mathcal{H}$-level it follows straight from Theorem 3.2.4 (3) in [51].

□

**Theorem 6.2.** *Let $(\Sigma, P, \lambda)$ be a closed oriented stable and saturated marked surface and let $(\vec{\alpha}, \vec{\beta}) = (\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ be a symplectic basis of $H_1(\Sigma, \mathbb{Z})$. There is a unique connection $\nabla^{(\vec{\alpha},\vec{\beta})} = \nabla^{(\vec{\alpha},\vec{\beta})}(\Sigma, P)$ in the bundle $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\Sigma, P)$ over $\mathcal{T}_{(\Sigma,P)}$ with the property that for any good family $\mathfrak{F}$ of stable pointed curves with formal neighbourhoods on $(\Sigma, P)$ we have that*

$$\Psi_{\mathfrak{F}}^*(\nabla^{(\vec{\alpha},\vec{\beta})}) = \nabla^{(\omega)}.$$

*In particular the connection is holomorphic and projectively flat with $(2,0)$-curvature as described in Theorem 6.1. If we act on the symplectic basis $(\vec{\alpha}, \vec{\beta})$ by an element $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{Z})$ as in Section 1.4 of [51] (roles of $\alpha$ and $\beta$'s reversed) so as to obtain $\Lambda(\vec{\alpha}, \vec{\beta})$, then*

$$\nabla^{\Lambda(\vec{\alpha},\vec{\beta})} - \nabla^{(\vec{\alpha},\vec{\beta})} = -\frac{c_v}{2}\Pi^*(d \log \det(C\tau + D)), \tag{5}$$

*where $\Pi$ is given by (1.4.4) in [51]. If $f : (\Sigma_1, P_1, \lambda_1) \to (\Sigma_2, P_2, \lambda_1)$ is an orientation preserving diffeomorphism of labeled pointed surfaces which maps the symplectic basis $(\vec{\alpha}^{(1)}, \vec{\beta}^{(1)})$ of $H_1(\Sigma_1, \mathbb{Z})$ to the symplectic basis $(\vec{\alpha}^{(2)}, \vec{\beta}^{(2)})$ of $H_1(\Sigma_2, \mathbb{Z})$ then we have that*

$$\mathcal{V}^{\dagger}(f)^*(\nabla^{(\vec{\alpha}^{(2)},\vec{\beta}^{(2)})}) = \nabla^{(\vec{\alpha}^{(1)},\vec{\beta}^{(1)})}.$$

*Proof.* The existence of the connection is a consequence of Lemma 6.2. The transformation law (5) is proved in section 5.2 in [51].

□

**Proposition 6.1.** *For any stable and saturated family $\tilde{\mathfrak{F}}$ of pointed curves with formal coordinates on $(\Sigma, P)$ the preferred isomorphism*

$$\Upsilon_{\tilde{\mathfrak{F}}} : \mathcal{V}^{\dagger}_{\tilde{\lambda}}(\tilde{\mathfrak{F}}) \to \Psi^*_{\tilde{\mathfrak{F}}}\mathcal{V}^{\dagger}_{\lambda}(\Sigma, P)$$

*given in Proposition 5.2 preserves connections.*

This follows directly from Lemma 6.2.

□

**Proposition 6.2.** *The $\mathbb{R}^{\mathrm{P}}_{+}$-action on $\mathcal{T}_{(\Sigma,P)}$ lifts by the use of the connection $\nabla^{(\vec{\alpha},\vec{\beta})}$ to an action $\mathcal{V}^{\dagger}_{\lambda}$ of $\mathbb{R}^{\mathrm{P}}_{+}$ on $\mathcal{V}^{\dagger}_{\lambda}(\Sigma, P)$ which preserves the connection $\nabla^{(\vec{\alpha},\vec{\beta})}$.*

*Proof.* Since the $\mathbb{R}^{\mathrm{P}}_{+}$ action on the formal coordinates is just obtained by scaling the coordinates by positive scalars, we get a well defined homomorphism from $\mathbb{R}^{\mathrm{P}}_{+}$ to the group of formal coordinates changes for any family of $N$-pointed curves with formal coordinates, hence by composing with the group homomorphism $G$ we get an action of $\mathbb{R}^{\mathrm{P}}_{+}$ on $\mathcal{V}^{\dagger}_{\lambda}(\Sigma, P)$. Note that we have here used Theorem 3.2.4. (2) for $p = 0$ in [51], but only for these special real coordinates changes. This action preserved the connection by Theorem 3.2.4. (3) in [51].

□

Let now $\mathbf{\Sigma}_i$, $i = 1, 2$ be marked surface and let $f_j : (\Sigma_1, P_1) \to (\Sigma_2, P_2)$, $j = 1, 2$ be diffeomorphisms of pointed surfaces, which induce the same morphism of marked surfaces from $\mathbf{\Sigma}_1$ to $\mathbf{\Sigma}_2$. Then there exists a unique $v \in \mathbb{R}^{\mathrm{P}}_{+}$ such that $v \cdot d_{P_1} f_1 = d_{P_2} f_2$.

**Lemma 6.3.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{V}^{\dagger}_{\lambda_1}(\Sigma_1, P_1) & \xrightarrow{\mathcal{V}^{\dagger}(f_1)} & \mathcal{V}^{\dagger}_{\lambda_2}(\Sigma_2, P_2) \\ = \downarrow & & \downarrow \mathcal{V}^{\dagger}_{\lambda_2}(v) \\ \mathcal{V}^{\dagger}_{\lambda_1}(\Sigma_1, P_1) & \xrightarrow{\mathcal{V}^{\dagger}(f_2)} & \mathcal{V}^{\dagger}_{\lambda_2}(\Sigma_2, P_2), \end{array}$$

*Proof.* By the construction of $\mathcal{V}^{\dagger}_{\lambda_i}(\Sigma_i, P_i)$ we just need to check the commutativity on the $\mathcal{H}$-level, where this just amounts to $G$ being a homomorphism, which again is the content of Theorem 3.2.4. (2) in [51].

□

Let $(\Sigma, P, \lambda)$ be a general labeled pointed surface which might not be stable or saturated. Let now $(\tilde{P}, \tilde{\lambda})$ be obtained from $(P, \lambda)$ by labeling further points not in $P$ by $0 \in P_\ell$ such that $(\tilde{\Sigma}, \tilde{P}, \tilde{\lambda})$ is a stable and saturated labeled pointed surface. Let $\tilde{\pi} : \mathcal{T}_{(\Sigma,\tilde{P})} \to \mathcal{T}_{(\Sigma,P)}$ be the natural projection map.

**Proposition 6.3.** *The connection* $\nabla^{(\vec{\alpha},\vec{\beta})} = \nabla^{(\vec{\alpha},\vec{\beta})}(\Sigma, P)$ *is flat with trivial holonomy when restricted to any of the fibers of* $\tilde{\pi} : \mathcal{T}_{(\Sigma,\tilde{P})} \to \mathcal{T}_{(\Sigma,P)}$.

*Proof.* The propagation of vacua isomorphism applied along the fibers of $\tilde{\pi}$ is compatible with the connection. In the notation of chapter 5 of [51] this is seen by the following calculation. Let $\tilde{N} = |\tilde{P}|$. Let $\mathfrak{F}$ be a family of $\tilde{N}$-pointed curves with formal neighbourhoods on $(\Sigma, \tilde{P})$ such that $\Psi_{\mathfrak{F}}(\mathcal{B})$ is contained in a fiber of $\tilde{\pi}$. Further we can assume that when we consider $\mathfrak{F}$ over $(\Sigma, P)$, then it is simply a product family of a fixed $N$-pointed curve with formal neighbourhoods crossed with the base $\mathcal{B}$, i.e. we only vary the coordinates and the points in $\tilde{P} - P$. Then for any tangent field $X$ on $\mathcal{B}$, we choose a corresponding $\vec{\ell} = (\ell_1, \dots, \ell_N, \ell_{N+1}, \dots, \ell_{\tilde{N}}) \in \mathcal{L}(\mathfrak{F})$, such that $\ell_j = 0$, $j = 1, \dots, N$ and $\ell_j \in \mathbb{C}[[\xi]]$, $j = N+1, \dots, \tilde{N}$. Then for a section of $\mathcal{H}_{\tilde{\lambda}}(\mathcal{B})$ of the form $F \otimes |\Phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle$ we compute that

$$\begin{aligned}
\nabla^{(\omega)}_X(F|\Phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle) &= X(F)|\Phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle \\
&\quad - F \sum_{j=N+1}^{\tilde{N}} \rho_j(T[\ell_j])(|\Phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle) \\
&\quad - a_\omega(\vec{\lambda}) F|\Phi\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle \\
&= X(F)|\Phi\rangle) \otimes |0\rangle \otimes \dots \otimes |0\rangle.
\end{aligned}$$

Here we have used that $\rho_j(T[\ell_j])(|0\rangle) = 0$ because $\ell_j \in \mathbb{C}[[\xi]]$ and $a_\omega(\vec{\lambda}) = 0$ for the same reason. Hence we see that $\nabla^{(\omega)}$ is the trivial connection in $\mathcal{H}_{\tilde{\lambda}}(\mathcal{B})$. Hence the connection is flat with trivial holonomy on the subbundle $\mathcal{V}^\dagger_{\tilde{\lambda}}(\mathfrak{F}) \subset \mathcal{H}^\dagger_{\tilde{\lambda}}(\mathcal{B})$.

□

Let $(\Sigma, P, \lambda)$ be a general labeled pointed surface, i.e. $(\Sigma, P)$ might not be stable nor saturated. Let $(\Sigma, P', \lambda')$ and $(\Sigma, P'', \lambda'')$ be labeled marked surfaces obtained from $(\Sigma, P, \lambda)$ by labeling further points not in $P$ by $0 \in P_\ell$. Assume that $(\Sigma, P')$ and $(\Sigma, P'')$ are stable and saturated pointed surfaces. Let $\bar{P} = P' \cup P''$ and $\bar{\lambda}$ be the induced labeling of $\bar{P}$. Note that $(\Sigma, \bar{P})$ is also stable and saturated.

**Proposition 6.4.** *Let* $(\vec{\alpha}, \vec{\beta})$ *be a symplectic basis of* $H_1(\Sigma, \mathbb{Z})$. *The isomorphisms given in Theorem 5.3 satisfies*

$$\nabla^{(\vec{\alpha},\vec{\beta})}(\Sigma, \bar{P}) = (\pi')^* \nabla^{(\vec{\alpha},\vec{\beta})}(\Sigma, P') = (\pi'')^* \nabla^{(\vec{\alpha},\vec{\beta})}(\Sigma, P'').$$

*Proof.* We only have to consider the case of adding one point to a stable and saturated curve, i.e. say $\bar{P} = P'$ and $P'$ is obtained from $P''$ by adding one more point. Let $\bar{N}$ be the number of points in $\bar{P}$ and $\mathfrak{F}$ be a family of $\bar{N}$-pointed curves with formal neighbourhoods on $(\Sigma, \bar{P})$. For any tangent field $X$ on $\mathcal{B}$, we choose a corresponding

$\vec{\ell} = (\ell_1, \ldots, \ell_{\bar{N}}, \ell_{\bar{N}+1}) \in \mathcal{L}(\mathfrak{F})$, such that $\ell_{\bar{N}+1} \in \mathbb{C}[[\xi]]$. Then the same computation as above shows that

$$\nabla^{(\omega)}(F|\Phi\rangle \otimes |0\rangle) = \nabla^{(\omega)}(F|\Phi\rangle) \otimes |0\rangle.$$

The Proposition follows directly from this.

□

Let now $(\Sigma_i, P_i', \lambda_i')$ and $(\Sigma_i, P_i'', \lambda_i'')$ be stable and saturated labeled pointed surfaces, obtained from the labeled pointed surfaces $(\Sigma_i, P_i, \lambda_i)$, by labeling further points with the zero-label. Assume that $f' : (\Sigma_1, P_1', \lambda_1') \to (\Sigma_2, P_2', \lambda_2')$ and $f'' : (\Sigma_1, P_1'', \lambda_1'') \to (\Sigma_2, P_2'', \lambda_2'')$ are diffeomorphisms which induce isotopic maps from $(\Sigma_1, P_1)$ to $(\Sigma_2, P_2)$, where the isotopy is through maps which induces the same map from $PT_{P_1}\Sigma_1$ to $PT_{P_2}\Sigma_2$.

**Proposition 6.5.** *With respect to the propagation of vacua isomorphisms, we get that*

$$(\pi')^* \mathcal{V}^\dagger(f') = (\pi'')^* \mathcal{V}^\dagger(f'').$$

This follows directly from the way the morphisms for $f'$ and $f''$ are defined.

## 7. Definition of the space of abelian vacua associated to a curve.

Let $\mathbf{C}$ be a smooth projective curve over $\mathbb{C}$. For a stable and saturated pointed curve with formal neighbourhoods $\mathfrak{X}$ we denote by $\tilde{c}(\mathfrak{X})$ the underlying curve. The definition of the space of abelian vacua $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{X})$ is given in Definition 3.1 in [2]. Suppose we now have two pointed curves with formal neighbourhoods $\mathfrak{X}_i$ such that $\tilde{c}(\mathfrak{X}_1) = \tilde{c}(\mathfrak{X}_2)$. Choose for each component of $\mathbf{C}$ a point with a formal coordinate, which is not a point with formal coordinates of $\mathfrak{X}_i$, i=1,2. Let $\mathfrak{X}_0$ be the resulting stable and saturated pointed curve with formal neighbourhoods (if $\mathfrak{X}_0$ is not stable, then add further points with formal coordinates). Then iterations of the propagation of vacua isomorphism determined by the inclusion of $\mathcal{F}_N$ into $\mathcal{F}_{N+1}$ given by $|v\rangle \mapsto |v\rangle \otimes |0\rangle$, induces isomorphisms from $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{X}_0)$ to $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{X}_i)$, $i = 1, 2$. It is elementary to check that the resulting isomorphism from $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{X}_1)$ to $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{X}_2)$ is independent of $\mathfrak{X}_0$.

Furthermore, we get from the commutativity of the following diagram (see section 6 in [2] for the definition of $G[h]$)

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{=} & \mathcal{F}_1 \\ {\scriptstyle \mathrm{Id}\otimes|0\rangle}\downarrow & & \downarrow{\scriptstyle Id\otimes|0\rangle} \\ \mathcal{F}_2 & \xrightarrow{\mathrm{Id}\otimes G[h]} & \mathcal{F}_2, \end{array}$$

which follows from the fact that $G[h]|0\rangle = |0\rangle$, that these isomorphisms are also compatible with the change of formal coordinates isomorphism induced by $G[h]$.

**Definition 7.1.** *The space of abelian vacua associated to the curve* $\mathbf{C}$ *is by definition*

$$\mathcal{V}^\dagger_{\text{ab}}(\mathbf{C}) = \coprod_{\tilde{\mathfrak{c}}(\mathfrak{X})=\mathbf{C}} \mathcal{V}^\dagger_{\text{ab}}(\mathfrak{X})/\sim,$$

*where the disjoint union is over all curves with formal neighbourhoods with* $\mathbf{C}$ *as the underlying curve and* $\sim$ *is the equivalence relation generated by the isomorphisms discussed above.*

It is obvious that

**Proposition 7.1.** *The natural quotient map from* $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{X})$ *to* $\mathcal{V}^\dagger_{\text{ab}}(\mathbf{C})$ *is an isomorphism for all curves with formal neighbourhoods* $\mathfrak{X}$ *with* $\tilde{\mathfrak{c}}(\mathfrak{X}) = \mathbf{C}$*.*

Suppose $\mathbf{C}_i$ are curves and $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$ is a morphism of labeled marked curves. Let $\mathfrak{X}_2$ be a curve with formal neighbourhoods such that $\tilde{\mathfrak{c}}(\mathfrak{X}_2) = \mathbf{C}_2$. Let $\Phi^*\mathfrak{X}_2 = \mathfrak{X}_1$. Then $\Phi$ is a morphism of curves with formal neighbourhoods. We clearly have that

**Proposition 7.2.** *The identity map on* $\mathcal{F}_N$ *induces a linear isomorphism from* $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{X}_1)$ *to* $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{X}_2)$*, which induces a well defined linear isomorphism* $\mathcal{V}^\dagger_{\text{ab}}(\Phi)$ *from* $\mathcal{V}^\dagger_{\text{ab}}(\mathbf{C}_1)$ *to* $\mathcal{V}^\dagger_{\text{ab}}(\mathbf{C}_2)$*. Compositions of morphisms of labeled marked curves go to compositions of the induced linear isomorphisms.*

## 8. Definition of the line bundle of abelian vacua over Teichmüller space

Recall the definition of the sheaf of abelian vacua $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{F})$ associated to a family $\mathfrak{F}$ of stable and saturated pointed curves with formal neighbourhoods as given in Section 4 in [2].

Let $\mathfrak{F}_i$, $i = 1, 2$ be two families of stable and saturated pointed curves with formal neighbourhoods. Assume we have a morphism of families (not necessarily preserving sections nor formal coorinates)

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\Phi} & \mathcal{C}_2 \\ \downarrow & & \downarrow \\ \mathcal{B}_1 & \xrightarrow{\Psi} & \mathcal{B}_2, \end{array}$$

which is a fiberwise biholomorphism.

Let now $\mathfrak{F}_0 = (\mathcal{C}_1 \to \mathcal{B}_1; \vec{s}_0, \vec{\eta}_0)$ be obtained from $\mathfrak{F}_1$, by replacing $(\vec{s}_1, \vec{\eta}_1)$ by $(\vec{s}_0, \vec{\eta}_0)$ such that $\vec{s}_0(\mathcal{B}_1)$ is disjoint from $\vec{s}_1(\mathcal{B}_1)$ and from $\vec{\tilde{s}}_2(\mathcal{B}_1)$, where $\Phi\vec{\tilde{s}}_2 = \vec{s}_2\Psi$. The propagation of vacua isomorphism induces an isomorphism between $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{F}_0)$ and $\mathcal{V}^\dagger_{\text{ab}}(\mathfrak{F}_1)$.

Furthermore the propagation of vacua induces an isomorphism between $\mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_0)$ and $\mathcal{V}^\dagger_{\rm ab}(\tilde{\mathfrak{F}}_2)$, where $\tilde{\mathfrak{F}}_2 = (\mathcal{C}_1 \to \mathcal{B}_1; \vec{\tilde{s}}_2, \vec{\tilde{\eta}}_2)$ and $\Phi\vec{\tilde{\eta}}_2 = \Psi^*\vec{\eta}_2$. The identity on $\mathcal{F}_N(\mathcal{B}_1)$ induces an isomorphism between $\mathcal{V}^\dagger_{\rm ab}(\tilde{\mathcal{F}}_2)$ and $\mathcal{V}^\dagger_{\rm ab}(\Psi^*(\mathcal{F}_2))$. Composing these with the pull back isomorphism just as in the non-abelian case, we arrive at the following proposition.

**Proposition 8.1.** *We get an induced bundle morphism*

$$\begin{array}{ccc} \mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_1) & \xrightarrow{\mathcal{V}^\dagger_{\rm ab}(\Phi)} & \mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_2) \\ \downarrow & & \downarrow \\ \mathcal{B}_1 & \xrightarrow{\Psi} & \mathcal{B}_2, \end{array} \tag{6}$$

*determined as above. Moreover, composition of such family morphisms goes to composition of the induced bundle morphisms.*

Suppose now that we have two families $\mathfrak{F}_i$, $i = 1, 2$ over $\Sigma$ with the property that they have the same image $\tilde{\Psi}_{\mathfrak{F}_1}(\mathcal{B}_1) = \tilde{\Psi}_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_\Sigma$ and that $\mathfrak{F}_2$ is a good family with respect to $\Sigma$. For such a pair of families there exists by Proposition 3.2 a unique fiber preserving biholomorphism $\Phi_{12} : \mathcal{C}_1 \to \mathcal{C}_2$ covering $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1}$ such that $\Phi^{-1}_{\mathfrak{F}_2}\Phi_{12}\Phi_{\mathfrak{F}_1} : Y \to Y$ is isotopic to $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1} \times \mathrm{Id}$ through such fiber preserving maps.

By Theorem 3.2, Corollary 4.1 & 4.3 and Theorem 4.1 in [2] we have that $\mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_i)$ are holomorphic line bundles over $\mathcal{B}_i$. By Proposition 8.1 we get induced a glueing isomorphism

$$\mathcal{V}^\dagger_{\rm ab}(\Phi_{12}) : \mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_1) \to \mathcal{V}^\dagger_{\rm ab}(\mathfrak{F}_2). \tag{7}$$

**Definition 8.1.** *Let $\Sigma$ be a closed oriented smooth surface. We now define a line bundle $\mathcal{V}^\dagger_{\rm ab} = \mathcal{V}^\dagger_{\rm ab}(\Sigma, P)$ over Teichmüller space $\mathcal{T}_\Sigma$ using the cover $\{\tilde{\Psi}_{\mathfrak{F}}(\mathcal{B})\}$, where $\mathfrak{F}$ runs over the stable and saturated good families of pointed curves with formal neighbouhoods over $\Sigma$. Over $\tilde{\Psi}_{\mathfrak{F}}(\mathcal{B})$ we specify the line bundle as $(\tilde{\Psi}^{-1}_{\mathfrak{F}})^*\mathcal{V}^\dagger_{\rm ab}(\mathfrak{F})$. On overlaps of the image of two good families, we use the glueing isomorphism $\mathcal{V}^\dagger_{\rm ab}(\Phi_{12})$ to glue the corresponding bundles together.*

We obviously have the following

**Proposition 8.2.** *For any stable and saturated family $\tilde{\mathfrak{F}}$ of pointed curves with formal coordinates over $\Sigma$ we have a preferred isomorphism*

$$\tilde{\Upsilon}_{\tilde{\mathfrak{F}}} : \mathcal{V}^\dagger_{\rm ab}(\tilde{\mathfrak{F}}) \to \tilde{\Psi}^*_{\tilde{\mathfrak{F}}}\mathcal{V}^\dagger_{\rm ab}(\Sigma)$$

*induced by the transformation isomorphism between* $\mathcal{V}^{\dagger}_{\mathrm{ab}}(\tilde{\mathfrak{F}})$ *and* $\mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F})$*, for good families* $\mathfrak{F}$ *of pointed curves with formal coordinates over* $\Sigma$ *such that* $\tilde{\Psi}_{\mathfrak{F}}(\mathcal{B})$ *intersect* $\tilde{\Psi}_{\tilde{\mathfrak{F}}}(\mathcal{B}')$ *nonempty.*

Suppose now $f : \Sigma_1 \to \Sigma_2$ is a morphism of surfaces. Then of course $f$ induces a biholomorphism $f^*$ from $\mathcal{T}_{\Sigma_1}$ to $\mathcal{T}_{\Sigma_2}$. Let now $\mathfrak{F}_1$ be a good family of stable pointed curves with formal neighbourhoods over $\Sigma_1$. Then by composing with $f^{-1} \times \mathrm{Id}$ we get a good family $\mathfrak{F}_2$ of stable pointed curves with formal neighbourhoods over $\Sigma_2$ over the same base $\mathcal{B}_1$. The identity morphism on $\mathcal{F}_N(\mathcal{B}_1)$ then induces a morphism $\mathcal{V}^{\dagger}_{\mathrm{ab}}(f) : \mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F}_1) \to \mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F}_2)$ which covers the identity on the base. This is precisely the morphism induced from the morphism of families $\Phi_f = f \times \mathrm{Id} : \mathfrak{F}_1 \to \mathfrak{F}_2$ by Proposition 8.1. This in turn induces a morphism $\mathcal{V}^{\dagger}_{\mathrm{ab}}(f) : (\tilde{\Psi}^{-1}_{\mathfrak{F}_1})^*(\mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F}_1)) \to (\tilde{\Psi}^{-1}_{\mathfrak{F}_2})^*(\mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F}_2))$ which covers $f^* : \tilde{\Psi}_{\mathfrak{F}_1}(B_1) \to \tilde{\Psi}_{\mathfrak{F}_2}(B_1)$.

**Proposition 8.3.** *The above construction provides a well defined lift of* $f^* : \mathcal{T}_{\Sigma_2} \to \mathcal{T}_{\Sigma_1}$ *to a morphism* $\mathcal{V}^{\dagger}_{\mathrm{ab}}(f) : \mathcal{V}^{\dagger}_{\mathrm{ab}}(\Sigma_1) \to \mathcal{V}^{\dagger}_{\mathrm{ab}}(\Sigma_2)$ *which behaves well under compositions.*

The proof is exactly the same as the proof of Proposition 5.4.

## 9. The connection in the line bundle of abelian vacua over Teichmüller space.

**Proposition 9.1.** *Let* $\mathfrak{F}$ *be a family of stable and saturated pointed curves with formal neighbourhoods on* $(\Sigma, P)$ *and choose a symplectic basis* $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ *of* $H_1(\Sigma, \mathbb{Z})$*. Let* $\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}}(2\Delta))$ *be the normalized symmetric biddifferential determined by this data. Then there is the connection* $\tilde{\nabla}^{(\omega)}$ *in the bundle* $\mathcal{V}^{\dagger}_{\mathrm{ab}}(\mathfrak{F})$ *whose* $(1,0)$*-part is given by formula (4.23) in* [2] *and whose* $(0,1)$*-part is just the* $\overline{\partial}$*-operator in this holomorphic line-bundle. The curvature of this connection is given by the formula in Theorem 4.2, formula (4.25) in* [2].

*Proof.* By the definition of $\tilde{\nabla}^{(\omega)}$ and the definition of $b_\omega$ in formula (4.21) in [2], we see that the $(1,1)$ and $(0,2)$-part of the curvature is zero.

□

Suppose now that we have two good families $\mathfrak{F}_i$, $i = 1, 2$ with the property that they have the same image $\tilde{\Psi}_{\mathfrak{F}_1}(\mathcal{B}_1) = \tilde{\Psi}_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_\Sigma$. For such a pair of families there exists by Proposition 3.2 a unique fiber preserving biholomorphism $\Phi_{12} : \mathcal{C}_1 \to \mathcal{C}_2$ covering $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1}$ such that $\Phi^{-1}_{\mathfrak{F}_2}\Phi_{12}\Phi_{\mathfrak{F}_1} : (Y, P) \to (Y, P)$ is isotopic to $\Psi^{-1}_{\mathfrak{F}_2}\Psi_{\mathfrak{F}_1} \times \mathrm{Id}$.

**Lemma 9.1.** *Let $\tilde{\nabla}_i^{(\omega)}$ be the connection in $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_i)$ described in Proposition 9.1. Then we have that*

$$\mathcal{V}^\dagger_{\mathrm{ab}}(\Phi_{12})^*(\tilde{\nabla}_2^{(\omega)}) = \tilde{\nabla}_1^{(\omega)}.$$

This follows from Theorem 6.1 in [2], by the same argument as in the non-abelian case.

**Theorem 9.1.** *Let $\Sigma$ be a closed oriented surface and let $(\vec{\alpha},\vec{\beta}) = (\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g)$ be a symplectic basis of $H_1(\Sigma,\mathbb{Z})$. There is a unique connection $\tilde{\nabla}^{(\vec{\alpha},\vec{\beta})} = \tilde{\nabla}^{(\vec{\alpha},\vec{\beta})}(\Sigma,P)$ in the bundle $\mathcal{V}^\dagger_{\mathrm{ab}}(\Sigma,P)$ over $\mathcal{T}_\Sigma$ with the property that for any good family $\mathfrak{F}$ of stable pointed curves with formal neighbourhoods over $\Sigma$ we have that*

$$\tilde{\Psi}^*_{\tilde{\mathfrak{F}}}(\tilde{\nabla}^{(\vec{\alpha},\vec{\beta})}) = \tilde{\nabla}^{(\omega)}.$$

*In particular the connection is compatible with the holomorphic line-bundle structure on $\mathcal{V}^\dagger_{\mathrm{ab}}(\Sigma,P)$. The curvature is of type $(2,0)$ as stated in Proposition 9.1.*

*If we act on the symplectic basis $(\vec{\alpha},\vec{\beta})$ by an element $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z})$ so as to obtain $\Lambda(\vec{\alpha},\vec{\beta})$, as described in Section 1.4 of* [51] *(with the roles of $\alpha$ and $\beta$ reversed), then*

$$\tilde{\nabla}^{\Lambda(\vec{\alpha},\vec{\beta})} - \tilde{\nabla}^{(\vec{\alpha},\vec{\beta})} = \frac{1}{2}\Pi^*(d\log\det(C\tau + D)), \tag{8}$$

*where $\Pi$ is given by (1.4.4) in* [51]. *If $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism of surfaces which maps the symplectic basis $(\vec{\alpha}^{(1)},\vec{\beta}^{(1)})$ of $H_1(\Sigma_1,\mathbb{Z})$ to the symplectic basis $(\vec{\alpha}^{(2)},\vec{\beta}^{(2)})$ of $H_1(\Sigma_2,\mathbb{Z})$ then we have that*

$$\mathcal{V}^\dagger_{\mathrm{ab}}(f)^*(\tilde{\nabla}^{(\vec{\alpha}^{(2)},\vec{\beta}^{(2)})}) = \tilde{\nabla}^{(\vec{\alpha}^{(1)},\vec{\beta}^{(1)})}.$$

*Proof.* The existence is a consequence of Lemma 9.1. The transformation law 8 follows from Theorem 4.2 in [2].

□

**Proposition 9.2.** *For any stable and saturated family $\tilde{\mathfrak{F}}$ of pointed curves with formal coordinates over $\Sigma$ the preferred isomorphism*

$$\tilde{\Upsilon}_{\tilde{\mathfrak{F}}} : \mathcal{V}^\dagger_{\mathrm{ab}}(\tilde{\mathfrak{F}}) \to \tilde{\Psi}^*_{\tilde{\mathfrak{F}}}\mathcal{V}^\dagger_{\mathrm{ab}}(\Sigma)$$

*given by Propostition 8.2 preserves connections and is compatible with the lift $\mathcal{V}^\dagger_{\mathrm{ab}}(f)$.*

This follows directly from Lemma 9.1 and Theorem 9.1.

## 10. The preferred non-vanishing section of the bundle of abelian vacua.

Let $\Sigma$ be a closed oriented surface. Assume first that $\Sigma$ is connected. As described just above Theorem 6.2 in [2], the choice of a symplectic basis gives a preferred section in the line bundle of abelain vacua associated to any family of stable and saturated pointed curves with formal neighbourhoods over $\Sigma$. We have that

**Theorem 10.1.** *Let $\mathfrak{F}$ be a family of stable and saturated pointed curves with formal neighbourhoods over $\Sigma$ and choose a symplectic basis $(\vec{\alpha},\vec{\beta})$ of $H_1(\Sigma,\mathbb{Z})$. Then there is a preferred non-vanishing holomorphic section $s_{\mathfrak{F}}^{(\vec{\alpha},\vec{\beta})}$ in the bundle $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F})$ given by (6.2-4) and the following formula in* [2]*. If we act on the symplectic basis $(\vec{\alpha},\vec{\beta})$ by an element $\Lambda = \begin{pmatrix} (U^t)^{-1} & B \\ 0 & U \end{pmatrix} \in Sp(g,\mathbb{Z})$ in order to obtain $\Lambda(\vec{\alpha},\vec{\beta})$ as described in Section 1.4 of* [51] *(roles of $\alpha$ and $\beta$ reversed), then*

$$s^{\Lambda(\vec{\alpha},\vec{\beta})} = \det U s^{(\vec{\alpha},\vec{\beta})}. \tag{9}$$

The transformation law is given in Theorem 6.3 in [2].

From the transformation rule, we see that the section only really depends on the Lagrangian subspace $L = \mathrm{Span}\{\beta_i\}$ and we therefore denote it $s_{\mathfrak{F}}(L)$.

Suppose now that $\Sigma$ is not connected and that $\Sigma = \coprod_i \Sigma_i$ is the decomposition of $\Sigma$ into its connected components $\Sigma_i$. Let $\mathfrak{F}$ be a family of stable pointed curves with formal neighbourhoods over $\Sigma$. Let $\mathfrak{F}_i$ be the restriction of $\mathfrak{F}$ to $\Sigma_i$. Let $N_i$ be the number of sections of $\mathfrak{F}_i$ and $N = \sum_i N_i$ the number of sections of $\mathfrak{F}$. We obviously have the following lemma.

**Lemma 10.1.** *The isomorphism $\mathcal{F}_N \cong \otimes_i \mathcal{F}_{N_i}$ induces an isomorphism of holomorphic line bundles*

$$\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}) \cong \otimes_i \mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_i),$$

*which is compatible with the connections.*

Suppose now that $(\vec{\alpha},\vec{\beta}) = ((\vec{\alpha}_i,\vec{\beta}_i))$ is a symplectic basis of $H_1(\Sigma,\mathbb{Z})$. We then define the preferred section to be

$$s_{\mathfrak{F}}^{(\vec{\alpha},\vec{\beta})} = \otimes_i s_{\mathfrak{F}_i}^{(\vec{\alpha}_i,\vec{\beta}_i)}.$$

For the rest of this section $\Sigma$ is just any closed oriented surface.

Suppose now that we have two good families $\mathfrak{F}_i$, $i = 1,2$ with the property that they have the same image $\tilde{\Psi}_{\mathfrak{F}_1}(\mathcal{B}_1) = \tilde{\Psi}_{\mathfrak{F}_2}(\mathcal{B}_2)$ in Teichmüller space $\mathcal{T}_\Sigma$. For such a pair of families there exists by Theorem 3.2 a unique fiber preserving biholomorphism

$\Phi_{12} : \mathcal{C}_1 \to \mathcal{C}_2$ covering $\Psi_{\mathfrak{F}_2}^{-1}\Psi_{\mathfrak{F}_1}$ such that $\Phi_{\mathfrak{F}_2}^{-1}\Phi_{12}\Phi_{\mathfrak{F}_1} : Y \to Y$ is isotopic to $\Psi_{\mathfrak{F}_2}^{-1}\Psi_{\mathfrak{F}_1} \times \mathrm{Id}$ through fiber preserving diffeomorphisms.

**Lemma 10.2.** *Let $s_{\mathfrak{F}_i}^{(\vec{\alpha},\vec{\beta})}$ be the preferred sections of $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F}_i)$ described in Theorem 9.1. Then we have that*

$$\mathcal{V}_{\mathrm{ab}}^{\dagger}(\Phi_{12})(s_{\mathfrak{F}_1}^{(\vec{\alpha},\vec{\beta})}) = s_{\mathfrak{F}_2}^{(\vec{\alpha},\vec{\beta})}.$$

*Proof.* Since $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\Phi_{12})$ is induced by the propagation of vacua isomorphism and the coordinate change isomorphism, this lemma follows from Theorem 6.2 and 6.4 in [51].

□

**Theorem 10.2.** *Let $\Sigma$ be a closed oriented surface and let $(\vec{\alpha},\vec{\beta}) = (\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g)$ be a symplectic basis of $H_1(\Sigma,\mathbb{Z})$. Then there is a unique non-vanishing holomorphic section $s^{(\vec{\alpha},\vec{\beta})} = s_{\Sigma}^{(\vec{\alpha},\vec{\beta})}$ in the bundle $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\Sigma)$ over $\mathcal{T}_{\Sigma}$ with the property that for any good family $\mathfrak{F}$ of stable pointed curves with formal neighbourhoods over $\Sigma$ we have that*

$$\tilde{\Psi}_{\mathfrak{F}}^{*}(s^{(\vec{\alpha},\vec{\beta})}) = s_{\mathfrak{F}}^{(\vec{\alpha},\vec{\beta})}.$$

*The sections transforms according to the transformation rule (9. If $f : \Sigma_1 \to \Sigma_2$ is an orientation preserving diffeomorphism of surfaces which maps the symplectic basis $(\vec{\alpha}^{(1)},\vec{\beta}^{(1)})$ of $H_1(\Sigma_1,\mathbb{Z})$ to the symplectic basis $(\vec{\alpha}^{(2)},\vec{\beta}^{(2)})$ of $H_1(\Sigma_2,\mathbb{Z})$ then we have that*

$$\mathcal{V}_{\mathrm{ab}}^{\dagger}(f)^{*}(s_{\Sigma_2}^{(\vec{\alpha}^{(2)},\vec{\beta}^{(2)})}) = s_{\Sigma_1}^{(\vec{\alpha}^{(1)},\vec{\beta}^{(1)})}.$$

*Proof.* The first part of this theorem follows from the above. Since the choice of basis of meromorphic 1-forms with the relevant properties is natural w.r.t. morphisms of curves, we get that the preferred section transforms just like the bases does.

□

Likewise, we see that the section only depends on the Lagrangian subspace and we denote it therefore $s(L) = s_{\Sigma}(L)$.

**Proposition 10.1.** *For any stable and saturated family $\tilde{\mathfrak{F}}$ of pointed curves with formal coordinates over $\Sigma$ the preferred isomorphism*

$$\tilde{\Upsilon}_{\tilde{\mathfrak{F}}} : \mathcal{V}_{\mathrm{ab}}^{\dagger}(\tilde{\mathfrak{F}}) \to \tilde{\Psi}_{\tilde{\mathfrak{F}}}^{*}\mathcal{V}_{\mathrm{ab}}^{\dagger}(\Sigma)$$

*given by Proposition 8.2 preserves the preferred sections.*

This Follows from Lemma 10.2.

## 11. The geometric construction of the modular functor.

For the convenience of the reader, let us summarize the results of the sheaf of vacua constructions over Teichmüller spaces of pointed surfaces obtained in non-abelian case in sections 4 to 6 and in the abelian case in sections 7 to 10.

**Theorem 11.1.** *Let $(\Sigma, P, \lambda)$ be a stable and saturated labeled pointed surface.*

- *The sheaf of vacua construction (see Definition 5.1) yields a vector bundle $\mathcal{V}_\lambda^\dagger = \mathcal{V}_\lambda^\dagger(\Sigma, P)$ over the Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ of $(\Sigma, P)$ whose fiber at a complex structure $\mathbf{C}$ on $(\Sigma, P)$ is identified (via the isomorphism given in Proposition 5.2) with the space of vacua $\mathcal{V}_\lambda^\dagger(\mathbf{C})$ as defined in Definition 4.1.*
- *For each symplectic basis of $H_1(\Sigma, \mathbb{Z})$, we get induced a connection in $\mathcal{V}_\lambda^\dagger$ over $\mathcal{T}_{(\Sigma,P)}$. Any two of these connections differ by a global scalar-value 1-form on $\mathcal{T}_{(\Sigma,P)}$. See Theorem 6.2.*
- *Each of these connections are projectively flat and their curvature is described in details in Theorem 6.1 and 6.2.*
- *There is a natural lift of morphisms of pointed surfaces to these bundles covering induced biholomorphisms between Teichmüller spaces, which preserves compositions. See Proposition 5.4.*
- *A morphism of pointed surfaces transforms these connections according to the way it transforms symplectic basises of the first homology. See Theorem 6.2.*

**Remark 11.1.** If we choose a Lagrangian subspace $L$ of $H_1(\Sigma, \mathbb{Z})$ and constrain the symplectic basis $(\alpha_i, \beta_i)$ of $H_1(\Sigma, \mathbb{Z})$ such that $L = \mathrm{Span}\{\beta_i\}$ then we see from the transformation laws in Theorem 6.2, that we get a connection in $\mathcal{V}_\lambda^\dagger$ which depends only on $L$.

Since the connections in the vector bundle $\mathcal{V}_\lambda^\dagger$ are only projectively flat, we need a 1-dimensional theory with connections, whose curvature after taking tensor products, can cancel this curvature and result in a bundle with a flat connection. There are obstructions to doing this mapping class group equivariantly, so we expect to see central extension of the mapping class groups occurring. As we shall see below, this is exactly what happens, when one extracts the necessary root of the abelian theory treated in [2], so as to get the right scaling of the curvature. Again, the following theorem summarizes the results and constructions, now in the abelian case treated in section 7 to 10.

**Theorem 11.2.** *Let $\Sigma$ be a closed oriented surface.*

- *The sheaf of abelian vacua construction (see Definition 8.1) yields a line bundle $\mathcal{V}_{\mathrm{ab}}^\dagger = \mathcal{V}_{\mathrm{ab}}^\dagger(\Sigma)$ over the Teichmüller space $\mathcal{T}_\Sigma$ of $\Sigma$, whose fiber at a complex* *structure* $\mathbf{C}$ *on* $\Sigma$ *is identified (via the isomorphism given in Proposition 8.2) with the space of abelain vacua* $\mathcal{V}^\dagger_{\rm ab}(\mathbf{C})$ *as defined in Definition 7.1.*

- *For each symplectic basis of* $H_1(\Sigma, \mathbb{Z})$*, we get induced a holomorphic connection in* $\mathcal{V}^\dagger_{\rm ab}$ *over* $\mathcal{T}_\Sigma$ *(see Theorem 9.1). The difference between the connections associated to two different basis's is the global scalar-value 1-form on* $\mathcal{T}_\Sigma$ *given in (8).*
- *The curvature of each of these connections are described in Proposition 9.1.*
- *For each symplectic basis of* $H_1(\Sigma, \mathbb{Z})$*, we also get a preferred non-vanishing section of* $\mathcal{V}^\dagger_{\rm ab}$ *as specified in Theorem 10.2. The transformation formula (9) states how the preferred sections transforms under change of the symplectic basis of* $H_1(\Sigma, \mathbb{Z})$*.*
- *There is a natural lift of morphisms of surfaces to these bundles covering induced biholomorphisms between Teichmüller space, which preserves compositions. See Proposition 8.3.*
- *A morphism of surfaces transforms these connections and the preferred sections according to the way it transforms symplectic basises of the first homology. See Theorem 9.1 and 10.2.*

**Remark 11.2.** If we choose a Lagrangian subspace $L$ of $H_1(\Sigma, \mathbb{Z})$ and constrain the symplectic basis $(\alpha_i, \beta_i)$ of $H_1(\Sigma, \mathbb{Z})$ such that $L = \mathrm{Span}\{\beta_i\}$ then we see from the transformation laws in Theorem 10.2, that we get a preferred non-vanishing section $s = s(L)$ and a connection in $\mathcal{V}^\dagger_{\rm ab}$ which only depends on $L$.

From the discussion of the curvatures of the connections in $\mathcal{V}^\dagger_\lambda$ and $\mathcal{V}^\dagger_{\rm ab}$, i.e. by comparing the curvature formula in the proof of 5.15 in [51] and then (4.25) in [2], it is clear that the root of $\mathcal{V}^\dagger_{\rm ab}$ we are seeking is $-\frac{1}{2}c_v$. The following theorem provided us with such a root.

**Theorem 11.3.** *For any marked surface* $\mathbf{\Sigma} = (\Sigma, L)$ *there exists a line bundle, which we denoted* $(\mathcal{V}^\dagger_{\rm ab})^{-\frac{1}{2}c_v}(L) = (\mathcal{V}^\dagger_{\rm ab})^{-\frac{1}{2}c_v}(\mathbf{\Sigma})$*, over* $\mathcal{T}_\Sigma$ *that satisfies the following:*

- $(\mathcal{V}^\dagger_{\rm ab})^{-\frac{1}{2}c_v}$ *is a functor from the category of marked surfaces to the category of line bundles over Teichmüller spaces of closed oriented surfaces.*
- *If we choose a symplectic basis of* $H_1(\Sigma, \mathbb{Z})$ *for a marked surface* $\mathbf{\Sigma}$ *then we get induced a connection in* $(\mathcal{V}^\dagger_{\rm ab})^{-\frac{1}{2}c_v}(L)$*, whose curvature is* $-\frac{1}{2}c_v$ *times the curvature of the corresponding connection in* $\mathcal{V}^\dagger_{\rm ab}$*. The difference between the connections associated to two different basises is* $-\frac{1}{2}c_v$ *times the global scalar-value 1-form on* $\mathcal{T}_\Sigma$ *given in (8).*

*Proof.* Let $(\mathcal{V}^\dagger_{\rm ab})^*$ be the complement of the zero section of $\mathcal{V}^\dagger_{\rm ab}$. Let $\tilde{\mathcal{V}}^\dagger_{\rm ab}$ be the fiberwise universal cover of $(\mathcal{V}^\dagger_{\rm ab})^*$ based at the section $s(L)$. This is a completely functorial construction on pairs of line bundles and non-vanishing sections. There is a unique lift of the $\mathbb{C}^*$-action on $(\mathcal{V}^\dagger_{\mathrm{ab}})^*$ to a $\mathbb{C}$-action on $\tilde{\mathcal{V}}^\dagger_{\mathrm{ab}}$ with respect to the covering map exp from $\mathbb{C}$ to $\mathbb{C}^*$. For any $\alpha \in \mathbb{C}^*$ we can now functorially define a line bundle $(\mathcal{V}^\dagger_{\mathrm{ab}})^\alpha(L)$ as follows:

$$(\mathcal{V}^\dagger_{\mathrm{ab}})^\alpha(L) = \tilde{\mathcal{V}}^\dagger_{\mathrm{ab}} \times_{\rho_\alpha} \mathbb{C},$$

where $\rho_\alpha(z) : \mathbb{C} \to \mathbb{C}$ is the linear map given by multiplication by $\exp(\alpha z)$ for all $z \in \mathbb{C}$. We emphasis the dependence of this bundle on the section $s(L)$ and hence on $L$ in the notation for this bundle.

It is clear from the construction of $(\mathcal{V}^\dagger_{\mathrm{ab}})^\alpha(L)$, that a connection in $\mathcal{V}^\dagger_{\mathrm{ab}}$ will induce a connection in $(\mathcal{V}^\dagger_{\mathrm{ab}})^\alpha(L)$, whose curvature two-form is $\alpha$ times the curvature two-form of that connection in $\mathcal{V}^\dagger_{\mathrm{ab}}$. For the construction of the functor on the morphisms of marked surfaces, we refer to [53] and [1].

□

By pulling $(\mathcal{V}^\dagger_{\mathrm{ab}})^{-\frac{1}{2}c_v}(L)$ with its connection back to $\mathcal{T}_{(\Sigma,P)}$ from $\mathcal{T}_\Sigma$, we get a line bundle with with a connection on $\mathcal{T}_{(\Sigma,P)}$, which we also denote $(\mathcal{V}^\dagger_{\mathrm{ab}})^{-\frac{1}{2}c_v}(L)$.

Let now $(\mathbf{\Sigma}, \lambda) = (\Sigma, P, V, L, \lambda)$ be a stable and saturated labeled marked surface. From the above Theorems 11.1 and 11.3, we see that there is a well defined flat connection in the vector bundle $\mathcal{V}^\dagger_\lambda \otimes (\mathcal{V}^\dagger_{\mathrm{ab}})^{-\frac{1}{2}c_v}(L)$ over $\mathcal{T}_{(\Sigma,P)}$ gotten by taking the tensor product connection of the two connections induced by any symplectic basis $(\alpha_i, \beta_i)$ of $H_1(\Sigma, \mathbb{Z})$. Now $\mathcal{T}_{(\Sigma,P)}$ forms a $\mathbb{R}^{\mathrm{P}}_+$-principal bundle over the reduced Teichmüller space $\mathcal{T}^{(\mathrm{r})}{}_{(\Sigma,P)}$. Hence we can use the flat connection to push forward this bundle to obtain a bundle with a flat connection over the reduced Teichmüller space.

**Definition 11.1.** *For the stable and saturated labeled marked surface $(\mathbf{\Sigma}, \lambda)$ we define the vector bundle $\mathcal{V}^\dagger_\lambda(\mathbf{\Sigma})$ with its flat connection $\nabla(\mathbf{\Sigma}, \lambda)$ as the push forward of the bundle $\mathcal{V}^\dagger_\lambda \otimes (\mathcal{V}^\dagger_{\mathrm{ab}})^{-\frac{1}{2}c_v}(L)$ to the reduced Teichmüller space $\mathcal{T}^{(\mathrm{r})}{}_{(\Sigma,P)}$ followed by restriction to the fiber $\mathcal{T}_{\mathbf{\Sigma}}$.*

**Remark 11.3.** By Theorem 11.1 and 11.2 and Lemma 6.3 we see that morphism of stable and saturated marked surfaces induces isomorphisms of flat vector bundles covering corresponding diffeomorphisms of Teichmüller spaces of the corresponding marked surfaces.

However, for a labeled marked surface $(\mathbf{\Sigma}, \lambda) = (\Sigma, P, V, L, \lambda)$, which is not stable or not saturated we need to say a little more. Namely, let $(\mathbf{\Sigma}', \lambda')$ be obtained from $(\mathbf{\Sigma}, \lambda)$ by further labeling points not in $P$ by the zero label $0 \in P_\ell$ and choose projectiv tangent vectors at these new labeled points, such that $\mathbf{\Sigma}'$ is both stable and saturated. Let $\pi'$ be the projection map from $\mathcal{T}_{\mathbf{\Sigma}'}$ to $\mathcal{T}_{\mathbf{\Sigma}}$.

**Proposition 11.1.** *The connection $\nabla(\mathbf{\Sigma}', \lambda')$ has trivial holonomy along the fibers of the projection map $\pi'$. The connection $\nabla(\mathbf{\Sigma}', \lambda')$ induces a flat connection in the bundle over $\mathcal{T}_{\mathbf{\Sigma}}$ obtained by push forward $\mathcal{V}^\dagger_{\lambda'}(\mathbf{\Sigma}')$ along $\pi'$ using $\nabla(\mathbf{\Sigma}', \lambda')$. If $(\mathbf{\Sigma}'', \lambda'')$ is another stable and saturated labeled marked surface obtained from $(\mathbf{\Sigma}, \lambda)$ in the same way by adding zero-labeled point to $P$, then iterations of the propagation of vacua isomorphisms given in Proposition 5.3 induces a connection preserving isomorphism between the corresponding pair of bundles over $\mathcal{T}_{\mathbf{\Sigma}}$.*

This proposition follows directly from Proposition 6.4 and the definition of the flat vector bundle $\mathcal{V}^\dagger_{\lambda'}(\mathbf{\Sigma}', P')$.

**Definition 11.2.** *We define the vector bundle with its flat connection $(\mathcal{V}^\dagger_{\lambda}(\mathbf{\Sigma}), \nabla(\mathbf{\Sigma}, \lambda))$ over $\mathcal{T}_{\mathbf{\Sigma}}$ to be $\pi'_*(\mathcal{V}^\dagger_{\lambda'}(\mathbf{\Sigma}'), \nabla(\mathbf{\Sigma}', \lambda'))$ for any stable and saturated labeled marked surface $(\mathbf{\Sigma}', \lambda')$ obtained from $(\mathbf{\Sigma}, \lambda)$ by adding zero-labeled marked points to $P$.*

Suppose now $f : (\mathbf{\Sigma}_1, \lambda_1) \to (\mathbf{\Sigma}_2, \lambda_2)$ is a morphism of labeled marked surfaces and that $(\mathbf{\Sigma}'_i, \lambda'_i)$ is obtained as above from $(\mathbf{\Sigma}_i, \lambda_i)$ by adding zero-labeled points and further that $f' : (\mathbf{\Sigma}'_1, \lambda'_1) \to (\mathbf{\Sigma}'_2, \lambda'_2)$ is any morphism of labeled marked surfaces, which induces $f$ when restricted to $(\mathbf{\Sigma}_1, \lambda_1)$. We then have the following result as a direct consequence of Proposition 6.5.

**Proposition 11.2.** *The induced morphism of flat vector bundles*

$$\mathcal{V}^\dagger(f') : \mathcal{V}^\dagger_{\lambda'_1}(\mathbf{\Sigma}'_1) \to \mathcal{V}^\dagger_{\lambda'_2}(\mathbf{\Sigma}'_2)$$

*induces a morphism of flat vector bundles from $\mathcal{V}^\dagger_{\lambda_1}(\mathbf{\Sigma}_1)$ to $\mathcal{V}^\dagger_{\lambda_2}(\mathbf{\Sigma}_2)$ which only depends on $f$ and which behaves well under compositions of morphism of labeled marked surface.*

Let us now collect the thus fare obtained in the following theorem.

**Theorem 11.4.** *The construction given above gives a functor from the category of labeled marked surfaces to the category of vector bundles with flat connections over Teichmüller spaces of marked surfaces.*

The modular functor we seek is now simply just obtained by composing with the functor which takes covariant constant sections of vector bundles with connections.

**Definition 11.3** (The functor $V^{\mathfrak{g}}_\ell$). *Let $\ell$ be a positive integer. Let $P_\ell$ be the finite set defined in (1) with the involution † as described right after (1). Let $(\mathbf{\Sigma}, \lambda) = (\Sigma, P, V, L, \lambda)$ be a labeled marked surface using the label set $P_\ell$. The functor $V^{\mathfrak{g}}_\ell$ is by definition the composite of the functor, which assigns to $(\mathbf{\Sigma}, \lambda)$ the flat vector bundle $\mathcal{V}^\dagger_{\lambda}(\mathbf{\Sigma})$ over $\mathcal{T}_{\mathbf{\Sigma}}$, and the functor, which takes covariant constant sections.*

**Remark 11.4.** For a labeled marked surface $(\mathbf{\Sigma}, \lambda)$ and a complex structure $\mathbf{C}$ on it, we see that Proposition 5.2 and 8.2 gives an isomorphism

$$V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \lambda) \cong \mathcal{V}_\lambda^\dagger(\mathbf{C}) \otimes (\mathcal{V}_{\mathrm{ab}}^\dagger)^{-\frac{1}{2}c_v}(L)(\mathbf{C}),$$

since $\mathcal{T}_{\mathbf{\Sigma}}$ is contractible. Moreover, if $\mathbf{f} : (\mathbf{\Sigma}_1, \lambda_1) \to (\mathbf{\Sigma}_2, \lambda_2)$ is a morphism of labeled marked curves, which is realized by a morphism of labeled marked curves $\Phi : \mathbf{C}_1 \to \mathbf{C}_2$, such that $\Phi^*(L_2) = L_1$, then we have the following commutative diagram

$$\begin{array}{ccc} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_1, \lambda_1) & \xrightarrow{\cong} & \mathcal{V}_\lambda^\dagger(\mathbf{C}_1) \otimes (\mathcal{V}_{\mathrm{ab}}^\dagger)^{-\frac{1}{2}c_v}(L_1)(\mathbf{C}_1) \\ {\scriptstyle V_\ell^{\mathfrak{g}}(\mathbf{f})}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{V}^\dagger(\Phi)\otimes\mathcal{V}_{\mathrm{ab}}^\dagger(\Phi)^{-\frac{1}{2}c_v}} \\ V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_2, \lambda_2) & \xrightarrow{\cong} & \mathcal{V}_\lambda^\dagger(\mathbf{C}_2) \otimes (\mathcal{V}_{\mathrm{ab}}^\dagger)^{-\frac{1}{2}c_v}(L_2)(\mathbf{C}_2). \end{array}$$

**Remark 11.5.** Let $(\mathbf{\Sigma}, \lambda)$ be a labeled marked surface and suppose that $(\mathbf{\Sigma}', \lambda')$ is obtained from $(\mathbf{\Sigma}, \lambda)$ by labeling further points by $0 \in P_\ell$, then by Proposition 11.1 we get induced an isomorophism

$$V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \lambda) \cong V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}', \lambda').$$

Let $(\mathbf{\Sigma}, \lambda)$ be a labeled marked surface. Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}, \vec{s}, \vec{\eta})$ be a family of stable and saturated curves with formal neighbourhoods over $\mathbf{\Sigma}$.

**Definition 11.4.** *We define $V_\ell^{\mathfrak{g}}(\mathfrak{F}, \lambda)$ to be the covariant constant sections of the flat bundles $\mathcal{V}_\lambda^\dagger(\mathfrak{F}) \otimes (\mathcal{V}_{\mathrm{ab}}^\dagger)^{-\frac{1}{2}c_v}(L)(\mathfrak{F})$ over $\mathcal{B}$.*

Form this definition it is clear that we get an isomorphism

$$I_{\mathfrak{F}} : V_\ell^{\mathfrak{g}}(\mathfrak{F}, \lambda) \to V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \lambda).$$

In order for the functor $V_\ell^{\mathfrak{g}}$ to be modular, we need to further construct the disjoint union isomorphism and the glueing isomorphism and to check that the axioms of a modular functor is satisfied. First we construct the disjoint union isomorphism. The glueing isomorphism will be constructed in the following section.

Let $(\mathbf{\Sigma}_i, \lambda_i) = (\Sigma_i, P_i, V_i, L_i, \lambda_i)$, $i = 1, 2$, be two stable and saturated labeled marked surfaces and let $(\mathbf{\Sigma}, \lambda) = (\mathbf{\Sigma}_1, \lambda_1) \sqcup (\mathbf{\Sigma}_2, \lambda_2)$. Let $\lambda = \lambda_1 \sqcup \lambda_2$. We have that $\mathcal{T}_{\mathbf{\Sigma}} = \mathcal{T}_{\mathbf{\Sigma}_1} \times \mathcal{T}_{\mathbf{\Sigma}_2}$. Let $\pi_i : \mathcal{T}_{\mathbf{\Sigma}} \to \mathcal{T}_{\mathbf{\Sigma}_i}$ be the projection onto the $i$'th factor. We clearly have that

**Proposition 11.3.** *The natural isomorphism $\mathcal{H}_{\vec{\lambda}} \cong \mathcal{H}_{\vec{\lambda}_1} \otimes \mathcal{H}_{\vec{\lambda}_2}$ (for any ordering $\vec{\lambda}, \vec{\lambda}_1$ and $\vec{\lambda}_2$ of $\lambda, \lambda_1$ and $\lambda_2$ respectively) induces an isomorphism of vector bundles with connections*

$$\mathcal{V}_\lambda^\dagger(\Sigma, P) \cong \pi_1^*\mathcal{V}_{\lambda_1}^\dagger(\Sigma_1, P_1) \otimes \pi_2^*\mathcal{V}_{\lambda_2}^\dagger(\Sigma_2, P_2), \tag{10}$$

*where we use the Lagrangian subspaces to fix the connections in all 3 bundles. The isomorphism is compatible with isomorphism induced by disjoint union of morphism of corresponding labeled marked surfaces.*

**Remark 11.6.** These disjoint union isomorphisms are clearly compatible with the propagation of vacua isomorphisms given in Proposition 11.1.

Further it is easy to see that

**Proposition 11.4.** *The isomorphism given in Lemma 10.1 induces an isomorphism of line bundles with connections*

$$\mathcal{V}^{\dagger}_{\mathrm{ab}}(\Sigma) \cong \pi_1^*\mathcal{V}^{\dagger}_{\mathrm{ab}}(\Sigma_1) \otimes \pi_2^*\mathcal{V}^{\dagger}_{\mathrm{ab}}(\Sigma_2), \tag{11}$$

*where we use the Lagrangian subspaces to fix the connections in all 3 bundles. The isomorphism is compatible with isomorphism induced by disjoint union of morphism of corresponding labeled marked surfaces. Moreover the preferred sections of these bundles specified by the given Lagragian subspaces are compatible with this isomorphism.*

From this proposition it then follows that we get the corresponding isomorphism of $-\frac{1}{2}c_v$-power of these bundles. Combining this with (10) we now get induced a prefered isomorphism of flat vector bundles

$$\mathcal{V}^{\dagger}_{\lambda}(\mathbf{\Sigma}) \cong \pi_1^*\mathcal{V}^{\dagger}_{\lambda_1}(\mathbf{\Sigma}_1) \otimes \pi_2^*\mathcal{V}^{\dagger}_{\lambda_2}(\mathbf{\Sigma}_2),$$

which intern induces the required isomorphism of the corresponding vector spaces of covariant constant sections:

$$V^{\mathfrak{g}}_{\ell}(\mathbf{\Sigma}, \lambda) \cong V^{\mathfrak{g}}_{\ell}(\mathbf{\Sigma}_1, \lambda_1) \otimes V^{\mathfrak{g}}_{\ell}(\mathbf{\Sigma}_2, \lambda_2)$$

which is natural with respect to disjoint union of morphisms.

## 12. Sheaf of vacua and gluing.

Let $\mathbf{\Sigma} = (\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)$ be a marked surface. Let

$$c : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$$

be a glueing map and $\Sigma_c$ the glueing of $\Sigma$ at the ordered pair $((p_-, v_-), (p_+, v_+))$ with respect to $c$ as described in section 2. We shall first assume that $(\Sigma_c, P)$ is stable and saturated.

Let $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}, s_-, s_+, \vec{s}, \vec{\eta}_-, \eta_+, \eta)$ be a family of pointed curves on $\mathbf{\Sigma}$ over some simply-connected base $\mathcal{B}$. Let $D$ be the unit disk in the complex plane. Assume we have holomorphic functions $x_\pm : U_\pm \subset \mathcal{C} \to D$ such that for each $b \in B$ we have that $x_\pm \mid_{\pi^{-1}(b)}: U_\pm \cap \pi^{-1}(b) \to D$ are local coordinates for $\pi^{-1}(b)$ centered at

$p_\pm$ and further that $x_\pm = \eta_\pm$ as formal neighbourhoods. Further we assume that $d_{p_\pm}(x_\pm\, |_{\pi^{-1}(b)})(v_\pm) = 1$ and that

$$c = P(d_{p_+}(x_+\, |_{\pi^{-1}(b)}))^{-1} \circ P(\bar{\cdot}) \circ P(d_{p_-}(x_-\, |_{\pi^{-1}(b)})) : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$$

where $P(\bar{\cdot}) : P(T_0 D) \to P(T_0 D)$ is induced by the the real linear map $z \mapsto \overline{z}$. Assume that $P \subset \Sigma - (U_- \cup U_+)$.

Let us now construct a stable and saturated family of pointed curves with formal coordinates $\mathfrak{F}_c = (\pi_c : \mathcal{C}_c \to \mathcal{B}_c, \vec{s}, \vec{\eta})$, by applying the glueing construction pointwise over $\mathcal{B}$ to $\mathfrak{F}$:

Let

$$\mathcal{C}^1 = \{(z, w, \tau) \in D^{\times 3} \mid zw = \tau\}$$
$$\mathcal{C}^1_c = C^1 \times \mathcal{B}$$

and

$$\mathcal{C}^2_c = \{(y, \tau) \in \mathcal{C} \times D \mid y \in U_\pm \Rightarrow |x_\pm(y)| > |\tau|\}$$

Let then

$$\mathcal{C}_c = \mathcal{C}^1_c \cup_\phi \mathcal{C}^2_c,$$

where

$$\phi : ((U_- - p_-) \times D \cup (U_+ - p_+) \times D) \cap \mathcal{C}^2_c \to \mathcal{C}^1_c$$

is given by

$$\phi(y, \tau) = \begin{cases} (x_-(y), \tau/x_-(y), \tau, \pi(y)), & y \in U_- - p_- \\ (\tau/x_+(y), x_+(y), \tau, \pi(y)), & y \in U_+ - p_+ \end{cases}.$$

One easily checks that $\mathcal{C}_c$ is a smooth complex manifold of dimension $\dim(\mathcal{B}) + 2$ and that we have an obvious holomorphic projection map $\pi_c : \mathcal{C}_c \to \mathcal{B}_c$. Let $\mathfrak{F}_c = (\pi_c : \mathcal{C}_c \to \mathcal{B}_c, \vec{s}, \vec{\eta})$.

Set $D^* = D - \{0\}$ and $\tilde{D} = \{\zeta \in \mathbb{C} |\, \mathrm{Im}(\zeta) > 0\}$. On $\tilde{D}$ we now consider the real coordinates $(r, \theta)$ given by $r(\zeta) = \exp(-2\pi\, \mathrm{Im}(\zeta))$ and $\theta(\zeta) = \mathrm{Re}(\zeta)$. Let $(r_\pm, \theta_\pm)$ be $x_\pm$ composed with polar coordinates.

Let $\tilde{\mathcal{B}}_c = \mathcal{B} \times \tilde{D}$ and $p_c : \tilde{\mathcal{B}}_c \to \mathcal{B}^*_c = \mathcal{B} \times D^*$ be given by $p_c(b, \zeta) = (b, \exp(2\pi i \zeta))$. Then $\tilde{\mathcal{B}}_c$ is the universal cover of $\mathcal{B}_c$. Let $\tilde{\mathcal{C}}_c = p_c^* \mathcal{C}_c$, $\mathcal{C}'_c = \mathcal{C}_c\, |_{\mathcal{B}^*_c}$, $\tilde{\mathfrak{F}}_c = p_c^* \mathfrak{F}$, $\tilde{\pi}_c : \tilde{\mathcal{C}}_c \to \tilde{\mathcal{B}}_c$, $\tilde{\pi}_{\tilde{D}} : \tilde{\mathcal{B}}_c \to \tilde{D}$ and $\tilde{\pi}_{\mathcal{B}} : \tilde{\mathcal{B}}_c \to \mathcal{B}$.

Let

$$V_\pm = \Phi_{\mathfrak{F}}^{-1}(U_\pm)$$

and

$$\tilde{x}_\pm : V_\pm \to \mathcal{B} \times D$$

be given by

$$\tilde{x}_\pm = (\pi_{\mathcal{B}}, x_\pm \circ \Phi_{\mathfrak{F}}).$$

Let us now define a fiber preserving diffeomorphism

$$f : \tilde{\mathcal{C}}_c \to \Sigma_c \times \tilde{B}_c$$

by

$$f(y,r,\theta) = \begin{cases} (\tilde{x}_-^{-1}(\pi(y), \chi_r(r_-(y)), \theta_-(y) + \frac{1}{2}\frac{1-r_-(y)}{1-r^{1/2}}\theta), r, \theta) & \text{if } y \in U_-, 1 \geq r_-(y) \geq r^{1/2} \\ (\tilde{x}_+^{-1}(\pi(y), -\chi_r(r_-(y)), -\theta_-(y) - \frac{1}{2}\frac{r_-(y)-r}{r^{1/2}-r}\theta), r, \theta) & \text{if } y \in U_-, r^{1/2} \geq r_-(y) \geq r, \end{cases}$$

and extend $f$ to all of $\tilde{\mathcal{C}}_c$ by the map $\Phi_{\mathfrak{F}}^{-1} \times \mathrm{Id}$ on $(\mathcal{C} - (U_+ \cup U_-)) \times \tilde{D}$.

Here $\chi_r$ is a smooth family of diffeomorphisms

$$\chi_r : [1, r] \to [1, -1], \; r \in (0,1),$$

with the properties that $\chi_r$ is the identity near 1, $\chi_r$ maps $\rho \mapsto -r/\rho$ near $r$ and $\chi_r(r^{1/2}) = 0$ for each $r \in (0,1)$. We will furthermore assume that for all $\rho \in (0,1)$ we have that

$$\lim_{r \to 0} \chi_r(\rho) = \rho \text{ and } \lim_{r \to 0} \chi_r(r/\rho) = -\rho,$$

for all $\rho \in (0,1)$.

The extra conditions on $\chi_r$ implies that the limit $\lim_{r\to 0} q \circ f(\cdot, r, 0) : \Sigma' \to \Sigma'$ exists and is equal to $\mathrm{Id} : \Sigma' \to \Sigma'$. We observe that the monodromy $(f|_{\{b\}\times\pi^{-1}(\zeta+1)}) \circ (f|_{\{b\}\times\pi^{-1}(\zeta)})^{-1}$ is a Dehn twist in $P(T_{p_-}\Sigma)$.

Using $f^{-1} : \Sigma_c \times \tilde{\mathcal{B}}_c \to \tilde{\mathcal{C}}_c$ we see that $\tilde{\mathfrak{F}}_c$ is a family of stable and saturated curves with formal coordinates on $\Sigma_c$.

The sheaf of vacua construction applied to the family $\mathfrak{F}_c$ gives a holomorphic vector bundle $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}_c)$ over $\mathcal{B}_c$ as stated in Theorem 4.4.2 in [51]. We get an isomorphism of vector bundles

$$\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{F}_c)|_{\mathcal{B}\times\{0\}} \cong \bigoplus_{\mu \in P_\ell} \mathcal{V}_{\mu,\mu^\dagger,\lambda}^{\dagger}(\mathfrak{F}) \tag{12}$$

induced by the isomorphism given in Theorem 3.3.5 in [51]. This is the content of Theorem 4.4.9 in [51].

The abelian sheaf of vacua construction applied to the family $\mathfrak{F}_c$ gives a holomorphic line bundle $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F}_c)$ over $\mathcal{B}_c$. We get an isomorphism of vector bundles

$$\mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F}_c)|_{\mathcal{B}\times\{0\}} \cong \mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F}) \tag{13}$$

induced by the isomorphism given in Theorem 3.5 in [2]. The preferred section $s_{\mathfrak{F}_c}(L_c)$ is continuous over $\mathcal{B}_c$. Over $\pi_D^{-1}(0)$ it is mapped via the above isomorphism to the preferred section $s_{\mathfrak{F}}(L)$ of $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F})$. This follows by Theorem 6.5 and 6.7 in [2].

As discussed before the Lagragian subspace $L$ determines connections in the bundles $\mathcal{V}_{\mu,\mu^\dagger,\lambda}^{\dagger}(\mathfrak{F})$ and $\mathcal{V}_{\mathrm{ab}}^{\dagger}(\mathfrak{F})$.

**Proposition 12.1.** *The Lagragian subspace $L_c \subset H_1(\Sigma_c, \mathbb{Z})$ determines a unique normalized symmetric bidifferential $\omega_c \in H^0(\mathcal{C}_c \times_{\mathcal{B}_c} \mathcal{C}_c, \omega_{\mathcal{C}_c \times_{\mathcal{B}_c} \mathcal{C}_c}(2\Delta))$ specified by formula (1.4.31) in* [51] *for any symplectic basis $(\vec{\alpha}, \vec{\beta})$ of $H_1(\Sigma_c, \mathbb{Z})$ such that $L_c = \mathrm{Span}\{\beta_i\}$ (recalling that the roles of $\alpha$ and $\beta$ are reversed in* [51]*).*

*Proof.* Given any symplectic basis $(\vec{\alpha}, \vec{\beta})$ of $H_1(\Sigma_c, \mathbb{Z})$ such that $L_c = \mathrm{Span}\{\beta_i\}$, formula (1.4.31) in [51] defines a normalized symmetric bidifferential $\tilde{\omega}_c \in H^0(\tilde{\mathcal{C}}_c \times_{\tilde{\mathcal{B}}_c} \tilde{\mathcal{C}}_c, \omega_{\tilde{\mathcal{C}}_c \times_{\tilde{\mathcal{B}}_c} \tilde{\mathcal{C}}_c}(2\Delta))$. The monodromy of the fibration $\pi : \mathcal{C}_c|_{\tilde{\pi}_{\mathcal{B}}^{-1}(b)} \to D$ is the Dehn twist in the curve $P(T_{p_-}\Sigma)$, hence it preserves $L_c$. But then by applying the transformation law given in (5.2.7) and (5.2.8) [51], we see that $\tilde{\omega}_c$ is invariant under this monodromy and therefore $\tilde{\omega}_c$ is the pull back of a unique $\omega'_c \in H^0(\mathcal{C}'_c \times_{\mathcal{B}_c} \mathcal{C}'_c, \omega_{\mathcal{C}'_c \times_{\mathcal{B}_c} \mathcal{C}'_c}(2\Delta))$. By the general theory for bidifferentials on such families, see e.g. [19] chapter III, we have that there is a unique $\omega_c \in H^0(\mathcal{C}_c \times_{\mathcal{B}_c} \mathcal{C}_c, \omega_{\mathcal{C}_c \times_{\mathcal{B}_c} \mathcal{C}_c}(2\Delta))$ such that $\omega_c |_{\mathcal{C}'_c \times_{\mathcal{B}^*_c} \mathcal{C}'_c} = \omega'_c$.

□

By Propostion 5.1.4 and Theorem 5.1.5 in [51] we get that $\omega_c$ determines a projectively flat connection in $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_c)|_{\mathcal{B}^*_c}$ and by Theorem 4.2 in [2] a connection in $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)|_{\mathcal{B}^*_c}$.

Let us now recall the conclusion of the glueing constructions on the sheaf of vacua both in the non-abelian and abelian case applied to the family $\mathfrak{F}_c$:

The explicit formulea (4.4.2) - (4.4.3), Theorem 5.3.4 and the formula preceding that Theorem in [51] gives an isomorphism between sections of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_c)|_{\mathcal{B}\times\{0\}}$ and sections of $p^*(\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F})|_{\mathcal{B}^*_c}) \cong \mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}}_c)$, which are covariant constant along the fibers of $\tilde{\pi}_{\tilde{D}}$.

Furthermore $\omega$ determines a connection in $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)|_{\mathcal{B}^*_c}$ which extends to a connection on all of $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)$, hence we get by parallel transport along the fibers of $\tilde{\pi}_{\tilde{D}}$ an isomorphism between sections of $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)\ |_{\mathcal{B}\times\{0\}}$ and sections of $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)$, which are covariant constant along the fibers of $\tilde{\pi}_{\tilde{D}}$. Formula (5.10) in [2] gives an explicit formula for this isomorphism.

By applying the fractional power construction to the line bundle $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)$ with the preferred section $s_{\mathfrak{F}_c}(L_c)$, we get a line bundle over $\mathcal{B}_c$, which we denote $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)^{-\frac{1}{2}c_v}(L_c)$. By the very construction of this bundle we see that $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)^{-\frac{1}{2}c_v}(L_c)\ |_{\mathcal{B}\times\{0\}}$ is identified with $(\mathcal{V}^\dagger_{\mathrm{ab}})(\mathfrak{F})^{-\frac{1}{2}c_v}(L)$. We get a connection in this bundle from its construction and an isomorphism from sections of $(\mathcal{V}^\dagger_{\mathrm{ab}})(\mathfrak{F})^{-\frac{1}{2}c_v}(L)$ to sections of $\mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)^{-\frac{1}{2}c_v}(L_c)$ over $\mathcal{B}_c$, which again are covariant constant along the fibers of $\tilde{\pi}_{\tilde{D}}$.

**Theorem 12.1.** *The tensor product of these two glueing constructions gives an isomorphism $I_c(\mathfrak{F}, x_\pm)$ from covariant constant sections of $\bigoplus_{\mu\in P_\ell} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\lambda}(\mathfrak{F}) \otimes \mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F})^{-\frac{1}{2}c_v}(L)$*

*over $\mathcal{B}$ to covariant constant sections of $\mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}}_c) \otimes \mathcal{V}^\dagger_{\mathrm{ab}}(\tilde{\mathfrak{F}}_c)^{-\frac{1}{2}c_v}(L_c)$ over $\tilde{\mathcal{B}}_c$:*

$$I_c(\mathfrak{F}, x_\pm) : \bigoplus_{\mu \in P_\ell} V_\ell^{\mathfrak{g}}(\mathfrak{F}, \mu, \mu^\dagger, \lambda) \to V_\ell^{\mathfrak{g}}(\tilde{\mathfrak{F}}_c, \lambda).$$

*Proof.* From an element of $\bigoplus_{\mu\in P_\ell} V_\ell^{\mathfrak{g}}(\mathfrak{F}, \mu, \mu^\dagger, \lambda)$, we get a section of

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{F}_c) \otimes \mathcal{V}^\dagger_{\mathrm{ab}}(\mathfrak{F}_c)^{-\frac{1}{2}c_v}(L_c) \mid_{\mathcal{B}\times\{0\}} .$$

By Theorem 5.3 and Remark 5.1 in [2] we see that the covariant derivative of the section of $\mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}}_c) \otimes \mathcal{V}^\dagger_{\mathrm{ab}}(\tilde{\mathfrak{F}}_c)^{-\frac{1}{2}c_v}(L_c)$ obtained by glueing vanishes, since the contributions from the non-abelian factor exactly cancels the constribution form the $-\frac{1}{2}c_v$-power of the abelian factor. Hence glueing results in a covariant constant section of $\mathcal{V}^\dagger_{\vec{\lambda}}(\tilde{\mathfrak{F}}_c) \otimes \mathcal{V}^\dagger_{\mathrm{ab}}(\tilde{\mathfrak{F}}_c)^{-\frac{1}{2}c_v}(L_c)$ over $\tilde{\mathcal{B}}_c$. Since the two glueing constructions give isomorphisms, it is clear that $I_c$ is an isomorphism.

□

Let $\mathbf{C}^{(i)}$, $i = 1, 2$, be two complex structures on $\Sigma$ and let $x_\pm^{(i)} : U_\pm^{(i)} \to D$ be coordinates around $p_\pm$ with $d_{p_\pm} x_\pm^{(i)}(v_\pm) = 1$ such that $c = P((d_{p_+} x_+^{(i)})^{-1} \circ P(\bar{\cdot}) \circ P(d_{p_-} x_-^{(i)}) : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$. Let $\eta_j^{(i)}$ be formal coordinates around $p_j \in C^{(i)}$.

**Theorem 12.2.** *For such two pairs $(\mathbf{C}^{(i)}, x_\pm^{(i)})$, $i = 1, 2$, of complex structures and holomorphism coordinates on $(\Sigma, \{p_+, p_-\} \cup P)$ we have that*

$$I_c(\mathbf{C}^{(1)}, x_\pm^{(1)}) = I_c(\mathbf{C}^{(2)}, x_\pm^{(2)}).$$

This follows straight from Theorem 12.1, since we clearly have the following

**Lemma 12.1.** *There exists a family of pointed curves with formal neighbourhoods $\mathfrak{F} = (\pi : \mathcal{C} \to \mathcal{B}, \vec{s}, \vec{\eta})$ on $(\Sigma, P)$, holomophic functions $x_\pm : U_\pm \subset \mathcal{C} \to D$ and $b_i \in \mathcal{B}$ $i = 1, 2$ such that the following holds*

- *The base $\mathcal{B}$ is simply-connected.*
- *Restriction to the fiber*

$$(\pi^{-1}(b_i), \vec{\eta} \mid_{\pi^{-1}(b_i)}, x_\pm \mid_{\pi^{-1}(b_i)} : U_\pm \cap \pi^{-1}(b_i) \to D)$$

  *over $b_i$, $i = 1, 2$, is the same complex structure on $(\Sigma, P)$ as*

$$(\mathbf{C}^{(i)}, \vec{\eta}^{(i)}, x_\pm^{(i)} : U_\pm^{(i)} \to D)$$

  *with the same formal coordinates and the same coordinates around $p_\pm$.*
- *For each $b \in \mathcal{B}$ we have that $x_\pm \mid_{U_\pm \cap \pi^{-1}(b)} : U_\pm \cap \pi^{-1}(b) \to D$ are holomorphic coordinates around $p_\pm \in \pi^{-1}(b)$.*

**Definition 12.1.** *We define the glueing isomorphism*

$$I_c = I_c(\mathbf{\Sigma}, \lambda) : \bigoplus_{\mu \in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \mu, \mu^\dagger, \lambda) \to V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_c, \lambda)$$

*to be equal to* $I_c(\mathbf{C}, x_\pm)$ *for any pair* $(\mathbf{C}, x_\pm)$ *of a complex structure and holomorphic coordinates on* $(\Sigma, \{p_+, p_-\} \cup P)$.

Recall that it is assumed that $(\Sigma_c, P)$ is stable and saturated. Let now $(\mathbf{\Sigma}', \lambda')$ be a labeled marked surface obtained from $(\mathbf{\Sigma}, \lambda)$ by labeling further points by $0 \in P_\ell$.

**Proposition 12.2.** *We get the following commutative diagram of isomorphisms:*

$$\begin{array}{ccc} \oplus_{\mu \in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \mu, \mu^\dagger, \lambda) & \xrightarrow{I_c(\mathbf{\Sigma},\lambda)} & V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_c, \lambda) \\ \cong \downarrow & & \cong \downarrow \\ \oplus_{\mu \in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}', \mu, \mu^\dagger, \lambda') & \xrightarrow{I_c(\mathbf{\Sigma}',\lambda')} & V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}'_c, \lambda'), \end{array} \tag{14}$$

*where the vertical isomorphisms are the ones gotten from Theorem 11.1.*

*Proof.* This follows since the glueing constructions both in the non-abelian case and the in the abelian case commutes with the propagation of vacua isomorphism. This is clear from (4.4.2) in [51] and (5.10) in [2].

□

**Definition 12.2.** *In the cases where* $(\mathbf{\Sigma}, \lambda)$ *is not stable or not saturated, we define the glueing isomorphism, to be the unique isomorphism*

$$I_c = I_c(\mathbf{\Sigma}, \lambda) : \oplus_{\mu \in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}, \mu, \mu^\dagger, \lambda) \to V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_c, \lambda)$$

*which makes the diagram (14) commutative for any stable and saturated labeled marked surface* $(\mathbf{\Sigma}'_c, \lambda')$ *obtained from* $(\mathbf{\Sigma}, \lambda)$ *by labeling further points by* $0 \in P_\ell$.

By the naturality of the glueing construction we have that

**Proposition 12.3.** *The glueing isomorphism are compatible with the isomorphisms induced by glueing morphisms of marked surfaces. That is suppose*

$$\mathbf{f} : (\Sigma^1, \{p_-^1, p_+^1\} \sqcup P^1, \{v_-^1, v_+^1\} \sqcup V^1, L^1) \to (\Sigma^2, \{p_-^2, p_+^2\} \sqcup P^2, \{v_-^2, v_+^2\} \sqcup V^2, L^2)$$

*is a morphism of marked surfaces and that there are glueing maps*

$$c_i : P(T_{p_-^i} \Sigma^i) \to P(T_{p_+^i} \Sigma^i),$$

*such that* $(d_{p^1_+} f)^{-1} c_2 d_{p^1_-} f = c_1$*, then we get the following commutative diagram*

$$\begin{array}{ccc}
\oplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}^1,\mu,\mu^\dagger,\lambda^1) & \xrightarrow{I_{c_1}(\mathbf{\Sigma}^1,\lambda^1)} & V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}^1_{c_1},\lambda^1) \\
{\scriptstyle V^{\mathfrak{g}}_\ell(\mathbf{f})}\downarrow & & {\scriptstyle V^{\mathfrak{g}}_\ell(\mathbf{f})}\downarrow \\
\oplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}^2,\mu,\mu^\dagger,\lambda^2) & \xrightarrow{I_{c_2}(\mathbf{\Sigma}^2,\lambda^2)} & V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}^2_{c_2},\lambda^2),
\end{array}$$

*for all labelings* $\lambda^i$ *of* $P^i$ *compatible with* $\mathbf{f}$*.*

We summarize the results on the glueing construction.

**Theorem 12.3.** *There is an isomorphism* $I_c$ *from* $\oplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma},\mu,\mu^\dagger,\lambda)$ *to* $V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}_c,\lambda)$ *as specified in Definition 12.1 and 12.2, which is independent of the glueing map* $c$ *in the following sense:*

*If* $c_i : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$*,* $i = 1, 2$*, are glueing maps and* $f : \mathbf{\Sigma}_{c_1} \to \mathbf{\Sigma}_{c_2}$ *is a diffeomorphism as described in Remark 2.3, then we have that*

$$I_{c_2} = V^{\mathfrak{g}}_\ell(f) I_{c_1}$$

*as isomorphisms from* $\oplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma},\mu,\mu^\dagger,\lambda)$ *to* $V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}_{c_2},\lambda)$*. Moreover the isomorphisms* $I_c$ *are compatible with glueing of morphisms of labeled marked surfaces.*

Let $(\mathbf{\Sigma}',\lambda')$ be another labeled marked surface, which is stable and saturated. Let $\mathbf{\Sigma}''$ be the disjoint union of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$. Let $\mathbf{\Sigma}''_c$ be the glueing of $\mathbf{\Sigma}''$ using the glueing map $c$. Then we clearly have that $\mathbf{\Sigma}''_c = \mathbf{\Sigma}_c \sqcup \mathbf{\Sigma}'$. It is trivial to check that

**Proposition 12.4.** *The glueing isomorphism is compatible with the disjoint union isomorphism, namely the following diagram is commutative*

$$\begin{array}{ccc}
\bigoplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}'',\mu,\mu^\dagger,\lambda,\lambda') & \longrightarrow & \bigoplus_{\mu\in P_\ell} V^{\mathfrak{g}}_\ell(\mathbf{\Sigma},\mu,\mu^\dagger,\lambda,) \otimes V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}'\lambda') \\
{\scriptstyle I_c(\mathbf{\Sigma}'',\lambda,\lambda')}\downarrow & & {\scriptstyle I_c(\mathbf{\Sigma},\lambda)\otimes \mathrm{Id}}\downarrow \\
V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}''_c,\lambda,\lambda') & \longrightarrow & V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}_c,\lambda) \otimes V^{\mathfrak{g}}_\ell(\mathbf{\Sigma}',\lambda').
\end{array}$$

Let now $\mathbf{\Sigma} = (\Sigma, \{p^{(1)}_-, p^{(1)}_+, p^{(2)}_-, p^{(2)}_+\} \sqcup P, \{v^{(1)}_-, v^{(1)}_+, v^{(2)}_-, v^{(2)}_+\} \sqcup V, L)$ be a marked surface. Let

$$c^{(i)} : P(T_{p^{(i)}_-}\Sigma) \to P(T_{p^{(i)}_+}\Sigma)$$

be glueing maps and $\Sigma_{c^{(i)}}$ the glueing of $\Sigma$ at the ordered pair $((p^{(i)}_-, v^{(i)}_-), (p^{(i)}_+, v^{(i)}_+))$ with respect to $c^{(i)}$. Let $\Sigma_{c^{(12)}}$ be the glueing with respect to $c^{(12)} = c^{(1)} \sqcup c^{(2)}$.

**Theorem 12.4.** *The glueing isomorphisms commute, meaning the following diagram is commutative*

$$\begin{array}{ccc}\bigoplus_{\mu_1,\mu_2\in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma},\mu_1,\mu_1^\dagger,\mu_2,\mu_2^\dagger,\lambda) & \xrightarrow{\bigoplus_{\mu_2\in P_\ell} I_{c^{(1)}}(\mathbf{\Sigma},\mu_2,\mu_2^\dagger,\lambda)} & \bigoplus_{\mu_2\in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_{c^{(1)}},\mu_2,\mu_2^\dagger,\lambda) \\ {\scriptstyle \bigoplus_{\mu_1\in P_\ell} I_{c^{(2)}}(\mathbf{\Sigma},\mu_1,\mu_1^\dagger,\lambda)}\downarrow & & \downarrow{\scriptstyle I_{c^{(2)}}(\mathbf{\Sigma}_{c^{(1)}},\lambda)} \\ \bigoplus_{\mu_1\in P_\ell} V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_{c^{(2)}},\mu_1,\mu_1^\dagger,\lambda) & \xrightarrow{I_{c^{(1)}}(\mathbf{\Sigma}_{c^{(2)}},\lambda)} & V_\ell^{\mathfrak{g}}(\mathbf{\Sigma}_{c^{(12)}},\lambda).\end{array}$$

*Proof.* Choose a complex structure on $\Sigma$ and let $\mathbf{C}$ denote the resulting marked curve. The complex structure $\mathbf{C}$ gives a point in the Teichmüller space $\mathcal{T}_{(\Sigma,\{p_-^{(1)},p_+^{(1)},p_-^{(2)},p_+^{(2)}\}\sqcup P)}$. Choose centered coordinates $x_\pm^{(i)} : U_\pm \to D$ around $p_\pm^{(i)}$ with $d_{p_\pm^{(i)}} x_\pm^{(i)}(v_\pm^{(i)}) = 1$ and such that $c^{(i)} = P(d_{p_+^{(i)}} x_+^{(i)})^{-1} \circ P(\bar{\cdot}) \circ P(d_{p_-^{(i)}} x_-^{(i)}) : P(T_{p_-^{(i)}}\Sigma) \to P(T_{p_+^{(i)}}\Sigma)$. The following construction of a smooth 3-dimensional complex manifold $\mathcal{C}$ with a holomorphic map $\pi : \mathcal{C} \to D \times D$ is the main ingredient in this proof :

Let

$$\mathcal{C}_1 = \{(z_1^{(i)}, w_2^{(i)}, \tau^{(1)}, \tau^{(2)}) \in (\mathbb{C}^2 \sqcup \mathbb{C}^2) \times D \times D \mid z^{(i)} w^{(i)} = \tau^{(i)}, |z^{(i)}| < 1, |w^{(i)}| < 1, |\tau^{(i)}| < 1\}$$

and

$$\mathcal{C}_2 = \{(y,\tau) \in \Sigma \times D \times D \mid y \in U_\pm^{(i)} \Rightarrow |x_\pm^{(i)}(y)| > |\tau^{(i)}|\}$$

Let then

$$\mathcal{C} = \mathcal{C}_1 \cup_\phi \mathcal{C}_2,$$

where

$$\phi : ((U_-^{(i)} - p_-^{(i)}) \times D \times D \cup (U_+^{(i)} - p_+^{(i)}) \times D \times D) \cap \mathcal{C}_2 \to \mathcal{C}_1$$

is given by

$$\phi(y,\tau) = \begin{cases} (x_-^{(i)}(y), \tau^{(i)}/x_-^{(i)}(y), \tau^{(1)}, \tau^{(2)}), & y \in U_-^{(i)} - p_-^{(i)} \\ (\tau^{(i)}/x_+^{(i)}(y), x_+^{(i)}(y), \tau^{(1)}, \tau^{(2)}), & y \in U_+^{(i)} - p_+^{(i)} \end{cases}.$$

One easily checks that $\mathcal{C}$ is a smooth complex manifold of dimension 3 and that we have an obvious holomorphic projection map $\pi : \mathcal{C} \to D \times D$. Choose formal neighbourhoods $\vec{\eta}$ for the points $P$. We thus get a family of stable and saturated curves with formal neighbourhoods $\mathfrak{F}_{c^{(12)}}$ over $D \times D$ obtained by applying the glueing construction at the two pairs $(p_-^{(1)}, p_+^{(1)})$ and $(p_-^{(2)}, p_+^{(2)})$. Let $\mathfrak{F}_{c^{(1)}}$ be the normalization of $\mathfrak{F}_{c^{(12)}} |_{D\times\{0\}}$ at $[p_-^{(2)}] = [p_+^{(2)}]$ and $\mathfrak{F}_{c^{(2)}}$ be the normalization of $\mathfrak{F}_{c^{(12)}} |_{\{0\}\times D}$ at $[p_-^{(1)}] = [p_+^{(1)}]$. Let $\mathfrak{X} = (\mathbf{C}, \{p_-^{(1)}, p_+^{(1)}, p_-^{(2)}, p_+^{(2)}\} \sqcup P, \{x_-^{(1)}, x_+^{(1)}, x_-^{(2)}, x_+^{(2)}\} \sqcup \vec{\eta})$.

The glueing constuction in the non-abelian case applied to $\mathfrak{F}_{c^{(1)}}$ (respectively to $\mathfrak{F}_{c^{(2)}}$) and then to $\mathfrak{F}_{c^{(12)}}$ results in two two-variable versions of (4.4.2) in [51] for any element of $\mathcal{V}^\dagger_{\mu_1,\mu_1^\dagger,\mu_2,\mu_2^\dagger,\nu}(\mathfrak{X})$. These two series formally satisfies differential equations of Fuchsian type (in each variable) just as in Theorem 5.2.4 in [51]. By the general theory of differential equations of Fuchsian type, these series are therefore convergent and give covariant constant sections over $\tilde{D} \times \tilde{D}$. By straight forward inspection these two power series in two variables are identical, hence they result in the same covariant constant section of $\mathcal{V}_\nu^\dagger(\tilde{\mathfrak{F}}_{c^{(12)}})$. From this we in particular see that the connection in $\mathcal{V}_\nu^\dagger(\tilde{\mathfrak{F}}_{c^{(12)}})$ is flat. So is the connection in $\mathcal{V}_{\rm ab}^\dagger(\mathfrak{F}_{c^{(12)}})$ over $D \times D$ then and the glueing construction is also independent of the order of the glueing in the abelian case. The theorem now follows.

□

## 13. Verification of the axioms

It is now straight forward to check the axioms of a modular functor given the results obtained in the previous sections.

**Theorem 13.1.** *The functor $V_\ell^{\mathfrak{g}}$ from the category of labeled marked surfaces to the category finite dimensional vector spaces is a modular functor.*

*Proof.* In order to check axiom *MF1*, we only need to check that the disjoint union isomorphisms satisfies associativity, but this follows from associativity of the isomorphisms between the corresponding $\mathcal{H}^\dagger$'s and $\mathcal{F}$'s.

We have that the glueing isomorphism $I_c$ from Theorem 12.3 is compatible with

- The disjoint union isomorphisms: Proposition 12.4.
- The glueing isomorphisms them self, i.e. the glueing isomorphisms should commute: Theorem 12.4

Hence axiom *MF2* is checked.

Axiom *MF3* is trivial, since we define $V_\ell^{\mathfrak{g}}(\emptyset) = \mathbb{C}$. Axiom *MF4* and *MF5* follows from Corollary 3.5.2 (1) and (2) in [51].

□

## References


[1] J. E. Andersen, The witten invariant of finite oder mapping tori I, To appear in *Journal für Reine und angevantes Matematik*.

[2] J. E. Andersen & K. Ueno, Abelian Conformal Field theories and Determinant Bundles, Preprint, math.QA/0304135.

[3] J. E. Andersen & K. Ueno, Construction of the Reshetikhin-Turaev TQFT from conformal field theory, Preprint in preparation.

[4] M. Artin, Lectures on deformation of singularities *Tata Institute Lecture Notes*, 54, 1976.

[5] M.F. Atiyah, On Framings of 3-manifolds, *Topology* 29 (1990) 1-7

[6] B. Bakalov; A. Jr. Kirillov, Lectures on tensor categories and modular functors, *University Lecture Series*, 21. American Mathematical Society, Providence, RI, 2001.

[7] A. Beauville and Y. Laszlo, Conformal blocks and generalized theta functions, *Comm. Math. Phys.*, 164, 385–419, 1994.

[8] A. A. Beilinson and V. V. Shechtman, Determinant bundles and Virasoro algebra, *Comm. Math. Phys.* 118, 651–701, 1988.

[9] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys. B*, 241, 333–380, 1984.

[10] D. Bernard and G. Felder, Fock representations and brst cohomology in $SL(2)$ current algebra, *Comm. Math. Phys.* 127, 145–168, 1990.

[11] C. Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants, *Topology* 39 no. 1, 193–223, 2000.

[12] C. Blanchet; N. Habegger; G. Masbaum; P. Vogel, Three-manifold invariants derived from the Kauffman bracket, *Topology* 31 no. 4, 685–699, 1992.

[13] C. Blanchet; N. Habegger; G. Masbaum; P. Vogel, Topological quantum field theories derived from the Kauffman bracket, *Topology* 34 no. 4, 883–927, 1995.

[14] E. A. Coddington and N. Levinson, Theory of ordinary differential equations *McGraw-Hill* 1955.

[15] C. De Concini, V. Kac and Kazhdan, Boson-fermion correspondence over $\mathbb{Z}$, in Infinite-dimensional Lie algebras and groups. Luminy-Marseille, 1988, *Adv. Ser. Math. Phys.*, 7, 603 – 622, 1989.

[16] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, *Pub. Math. IHES*, 36, 75–109, 1969.

[17] R. Dijkgraaf and E. Verlinde, Modular invariance and the fusion algebra, *Nucl. Phys. B(Proc. Suppl.)*, 5B, 87–97, 1988.

[18] T. Eguchi and H. Ooguri, Conformal and current algebras on a general Riemann surface, *Nucl. Phys. B*, 282, 308–328, 1987.

[19] J. D. Fay, Theta functions on Riemann surfaces, *Lecture Notes in Math.*, 352, Springer-Verlag, 1973.

[20] G. Felder, Brst approach to minimal models, *Nucl. Phys. B*, 317, 215–236, 1989.

[21] D. Friedan and S. Shenker, The analytic geometry of two-dimensional conformal field theory, *Nucl. Phys. B*, 281, 509–545, 1987.

[22] O. Gabber, The integrability of the characteristic variety, *Amer. J. Math.*, 103, 445–468, 1981.

[23] H. Garland, The arithmetic theory of loop algebra, *J. Algebra*, 53, 480–551, 1978.

[24] L. Gerritzen, F. Herrich and M. van der Put, Stable $n$-pointed trees of projective lines, *Nederl. Akad. Wetensch. Indag. Math.*, 50, 131–163, 1988.

[25] J. Grove, Constructing TQFTs from modular functors, *J. Knot Theory Ramifications* 10 no. 8, 1085–1131, 2001.

[26] R. Hartshorn, Algebraic Geometry, *Springer-Verlag*, 1977.

[27] N. J. Hitchin, Flat connections and geometric quantization, *Comm. Math. Phys.*, 131, 347–380, 1990.

[28] V. Kac, Infinite dimensional Lie algebras, third edition, *Cambridge Univ. Press*, 1990.

[29] T. Katsura, Y. Shimizu and K. Ueno, New bosonization and conformal field theory over $\mathbb{Z}$, *Comm. Math. Phys.* 121, 603–622, 1988.

[30] ———, Formal groups and conformal field theory over $\mathbb{Z}$, *Advanced Studies in Pure Mathematics*, 19, 347–366, 1988.

[31] ———, Complex cobordism ring and conformal field theory over $\mathbb{Z}$, *Math. Ann.* 291, 551–571, 1991.

[32] N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, Geometric realization of conformal field theory on Riemann surfaces, *Comm. Math. Phys.*, 116, 247–308, 1988.

[33] M. Kontsevich, Rational conformal field theory and invariants of 3-manifolds, *Preprint of Centre de Physique Theorique Marseille,* CPT-88/p2189, 1988.

[34] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions", *Nucl. Phys. B* 247, 83–103, 1984.

[35] F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves I, *Math. Scand.*, 89, 19–55, 1976.

[36] K. Kodaira, Complex manifolds and Deformation of Complex Structures, *Springer-Verlag*, 1985.

[37] T. Kohno, Three-manifold invariants derived from conformal field theory and projective representations of modular groups, *Intern. J. Modern Phys.*, 6, 1795–1805, 1992.

[38] S. Kumar, M. S. Narasimhan and A. Ramanathan, Infinite Grassmannian and moduli space of $G$-bundles, *preprint* 1993.

[39] G. Kuroki, Fock space representations of affine Lie algebras and integral representations in the Wess-Zumino-Witten models, *Comm. Math. Phys.* 142, 511–542, 1991.

[40] Y. Laszlo and C. Sorger, The line bundles on the stack of parabolic $G$-bundles over curves and their sections, *preprint*, 1995.

[41] G. Moore and N. Seiberg, Polynomial equations for rational conformal field theories, *Phys. Lett. B*, 212, 451–460, 1988.

[42] ———, Classical and quantum conformal field theory, *Comm. Math. Phys.* 123, 177–254, 1989.

[43] G. Segal, The definition of conformal field theory, *preprint* 1992.

[44] T. Suzuku, Finite-dimensionality of the space of conformal blocks, *preprint* 1994.

[45] T. Terada, Quelques propiétés géométriques du domaine de $F_1$ et le groupe de tresses colorées, *Publ. Res. Inst. Math. Sci.* 17, 95–111, 1981.

[46] S. Tsuboi, Deformations of locally stable holomorphic maps and locally trivial displacement of analytic subvarieties with ordinary singularities, *Science Report of Kagoshima Univ.* 35, 9–90, 1986.

[47] A. Tsuchiya and Y. Kanie, Vertex Operators in Conformal Field Theory on $\mathbb{P}^1$ and Monodromy Representations of Braid Group, *Advanced Studies in Pure Mathematics* 16, 297–326, 1988.

[48] A. Tsuchiya, K. Ueno and Y. Yamada", Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries, *Advanced Studies in Pure Mathematics* 19, 459–566, 1989.

[49] V. Turaev, Quantum invariants of knots and 3-manifolds, W. de Gruyter, Berlin, 1994..

[50] K. Ueno, On conformal field theory, *London Math. Soc. Lecture Note* 208, 283–345, 1995.

[51] K. Ueno, Introduction to conformal field theory with gauge symmetries, *Geometry and physics (Aarhus, 1995), Lecture Notes in Pure and Appl. Math.* 184, 603–745, Dekker, New York, 1997.

[52] E. Verlinde, Fusion rules and modular transformations in 2d conformal field theory, *Nucl. Phys. B*, 300 [FS22], 360–376, 1988.

[53] K. Walker, On Witten's 3-manifold invariants, *Preliminary version # 2, Preprint* 1991.

[54] C.T.C. Wall, Non-additivity of the signature, *Invent. Math.* 7 (1969) 269-274

[55] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* 121, 351-399, 1989.

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