Conference on

Number Theory and Spectral Theory

Friday December 3 — Saturday December 4 , 1999
Department of Mathematical Sciences
University of Aarhus

Introduction

This conference dealt with a broad spectrum of number theory, including transcendental numbers, modular forms, spectral theory and statistical analysis of the Riemann zeros.

In this leaflet we have collected brief accounts of the subjects of the talks given during the conference. Furthermore, at the end of the leaflet, the programme and the list of participants of the the Conference are included.

We wish to thank all participants — the speakers in particular — for contributing to the Conference.

Erik Balslev and Alexei Venkov
Aarhus, March 2000.

Contents

Random Matrices and the distribution of zeros of the Riemann zeta function — a survey of results of Odlyzko

Uffe Haagerup*

Numerical computations of the roots of the Riemann zeta function on the critical line $Re(s) = 1/2$ unveils a certain randomness in the separation of the zeros: Assuming the Riemann hypothesis, the average distance between neighboring zeros on the line near $1/2+iT$ is approximately $2\pi/\log T$, but sometimes the actual distance is much smaller or much larger, than this number (see e.g. [Ed1, p. 178]). In order to give a mathematical explanation to this phenomena, Montgomery [Mo] posed in 1973 the pair correlation conjecture, which states, that for $0 < a < b$ the ratio

$$\frac{\#\left\{(\gamma,\gamma') \mid 0 \leq \gamma,\, \gamma' \leq T,\ \frac{2\pi}{\log T}a < \gamma - \gamma' < \frac{2\pi}{\log T}b\right\}}{\#\left\{\gamma \mid 0 \leq \gamma \leq T\right\}}$$

converges to

$$\int_a^b \left(1 - \left(\tfrac{\sin \pi t}{\pi t}\right)^2\right) dt$$

as T goes to infinity. Here γ and γ' denote the imaginary parts of two zeros of the zeta function on the line $Re(s) = 1/2$. The above conjecture was posed on basis of number theoretical results proved in [Mo], but the conjecture happens to be very similar to the formula for pair correlation of eigenvalues of certain complex, selfadjoint random matrices (the Gaussian unitary ensemble, "GUE"). The mathematical theory for these random matrices was developed by Dyson, Mehta, Wigner and others in the 50'es and 60'es (see [Me]). With a suitable normalization, the distribution of the eigenvalues of the $N \times N$ random matrices in the GUE-ensemble converges as N goes to infinity to the probability measure on the real line with density (see [Me, p. 93]):

$$\frac{1}{\pi}\sqrt{1 - x^2/4} \cdot \mathbf{1}_{[-2,2]}(x).$$

This is Wigner's semicircle law. Using the same normalization, the pair correlation result for the GUE-ensemble can be stated as follows (see [Me, p. 94]):

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{\#\left\{(\lambda,\lambda') \mid 0 \leq \lambda,\, \lambda' \leq \varepsilon,\ \frac{a}{\pi N} < \lambda - \lambda' < \frac{b}{\pi N}\right\}}{\#\left\{\lambda \mid 0 \leq \lambda \leq \varepsilon\right\}} = \int_a^b \left(1 - \left(\tfrac{\sin \pi t}{\pi t}\right)^2\right) dt,$$

where λ and λ' denote two eigenvalues of the $N \times N$ random matrix from the GUE-ensemble.

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Similar results holds for real Gaussian random matrices (the GOE-ensemble) and for quaternionic Gaussian random matrices (the GSE-ensemble), but the function $1 - \left(\frac{\sin \pi t}{\pi t}\right)^2$ in the pair correlation formula has to be changed (see [Me, p.136 and p.167]).

In the 80'es, Odlyzko developed numerical methods to compute zeros of the Riemann zeta function with large imaginary part. Based on a sample of 10^5 zeros around the 10^{12}'th zero on the critical line, Odlyzko tested in 1987 Montgomery's pair correlation conjecture (see [Od1] and [Me, p.26]). This gave only a moderately good fit between the conjecture and the "experimental" data, and actually the pair correlation function for the GSE-ensemble appears to fit equally good for this sample. However 2 years later Odlyzko published a similar investigation based on 79 million zeros in the neighborhood of zero no. 10^{20} ([Od2] and [Me, p. 27]) and in this case there is a very convincing fit between Montgomery's conjectured formula and the "experimental" data. Indeed, it is clear that among the three choices of random matrix ensembles GOE, GUE and GSE, only the pair correlation function for the GUE-ensemble fits with the numerical data based on this large sample.

The random matrix theory gives also very precise information about the distribution of the distance of two neighboring eigenvalues (next neighbor separation) for the GOE-, GUE- and GSE-ensembles. For the the next neighbor separation of roots of the Riemann zeta function Odlyzko found again, based on his 79 million sample of roots, a very convincing fit with the next neighbor separation of eigenvalues for the GUE-ensemble (see [Me, p.30]).

The Montgomery pair correlation conjecture has also been investigated for the Dirichlet L-functions and for zeta functions based on other number fields, see [RS], [KS] and references given there.

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Number theory, dynamical systems and statistical mechanics

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Abstract

We shortly review recent work interpreting the quotient $\zeta(s-1)/\zeta(s)$ of Riemann zeta functions as a dynamical zeta function. The corresponding interaction function (Fourier transform of the energy) has been shown to be ferromagnetic, i.e. positive.

On the additive group

$$\mathbf{G}_k := (\mathbb{Z}/2\mathbb{Z})^k, \quad \text{with} \quad \mathbb{Z}/2\mathbb{Z} = (\{0,1\},+).$$

we set inductively

$$\mathbf{h}_0 := 1, \quad \mathbf{h}_{k+1}(\sigma,0) := \mathbf{h}_k(\sigma) \quad \text{and} \quad \mathbf{h}_{k+1}(\sigma,1) := \mathbf{h}_k(\sigma) + \mathbf{h}_k(1-\sigma), \tag{1}$$

where $\sigma = (\sigma_1,\ldots,\sigma_k) \in \mathbf{G}_k$ and $1-\sigma := (1-\sigma_1,\ldots,1-\sigma_k)$ is the inverted configuration. The sequences $\mathbf{h}_k(\sigma)$ of integers, written in lexicographic order, coincide with the denominators of the modified Farey sequence.

We now formally interpret $\sigma \in \mathbf{G}_k$ as a *configuration* of a spin chain with k spins and *energy function*

$$\mathbf{H}_k := \ln(\mathbf{h}_k).$$

Thus we may interpret

$$Z_k(s) := \sum_{\sigma\in\mathbf{G}_k} \exp(-s\cdot\mathbf{H}_k(\sigma))$$

as the *partition function* of that finite spin chain for *inverse temperature* s. The quotient

$$Z(s) := \frac{\zeta(s-1)}{\zeta(s)} \equiv \sum_{n=1}^{\infty} \varphi(n) n^{-s} \tag{2}$$

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is the *thermodynamic limit*

$$\lim_{k\to\infty} Z_k(s) = Z(s) \qquad (\Re(s) > 2) \tag{3}$$

of partition functions $Z_k(s) = \sum_{n=1}^{\infty} \varphi_k(n) n^{-s}$. That *number-theoretical spin chain* was introduced in [9], see also Cvitanović [5], and [8, 4] for a translation invariant version. The *Gibbs measure* for inverse temperature $\beta \in \mathbb{R}$ assigns probabilities

$$\sigma \mapsto \frac{\exp(-\beta \mathbf{H}_k(\sigma))}{Z_k(\beta)} \qquad (\sigma \in \mathbf{G}_k) \tag{4}$$

to the configurations of the spin chain. We denote the expectation of a random variable by

$$\langle f \rangle_k (\beta) := \sum_{\sigma\in\mathbf{G}_k} f(\sigma) \frac{\exp(-\beta \mathbf{H}_k(\sigma))}{Z_k(\beta)} \qquad (f : \mathbf{G}_k \to \mathbb{R}).$$

To show that the analogy with statistical mechanics is not only formal, the *Fourier coefficients* of $\mathbf{H}_k$ were estimated in [9]. One notes that the dual group $\mathbf{G}_k^*$ of $\mathbf{G}_k$ is naturally isomorphic to $\mathbf{G}_k$, since the characters on $\mathbf{G}_k$ can be written in the form

$$\chi_t : \mathbf{G}_k \to \{-1, 1\} \quad , \quad \chi_t(\sigma) := (-1)^{\sum_{l=1}^{k} \sigma_i t_i} \qquad (t \in \mathbf{G}_k^*).$$

The Fourier coefficients

$$j_k(t) := -2^{-k} \sum_{\sigma\in\mathbf{G}_k} \mathbf{H}_k(\sigma) \cdot \chi_t(\sigma) \qquad (t \in \mathbf{G}_k^*)$$

of $-\mathbf{H}_k$ are called *interaction coefficients* in the statistical mechanics terminology, and

$$\mathbf{H}_k(\sigma) = -\sum_{t\in\mathbf{G}_k^*} j_k(t) \cdot \chi_\sigma(t).$$

The negative mean $j_k(0)$ of $\mathbf{H}_k$ has special properties. In the thermodynamic limit it is asymptotic to $j_k(0) \sim -c \cdot k$ for some $c > 0$ [10], but it is the only coefficient whose value does not affect the Gibbs probability measure (4).

When we write $t \equiv (t_1, \ldots, t_k) \in \mathbf{G}_k^* \setminus \{0\}$ in the form

$$t = (0, \ldots, 0, 1, t_{l+1}, \ldots, t_{r-1}, 1, 0, \ldots, 0),$$

$s := r - l$ will be called the *size of* t, and $d := \min(l, k + 1 - r)$ its *distance* from the ends of the chain at 1 and k. Finally we say that t is *even* (*odd*) if $\sum_{i=1}^{k} t_i$ is *even* (*odd*). With these notations the following estimates were shown.

Theorem [9].

- The even interactions decay exponentially in the size:

  $$j_k(t) < 2^{-s} \qquad (t \in \mathbf{G}_k^* \setminus \{0\} \text{ even}), \tag{5}$$

  whereas in the odd case one even has

  $$j_k(t) < 2^{-(k-l)} \qquad (t \in \mathbf{G}_k^* \text{ odd}). \tag{6}$$

  So odd interactions are small in comparison to the even ones except near the right end of the chain.

- The interaction is *asymptotically translation invariant* in the sense that, up to a relative error which is exponentially small in the distance from the ends, the interactions only depend on the relative positions of the spins involved:

  $$0 \le (j_{k+1}(0,t) - j_{k+1}(t,0)) \cdot 2^s < C \cdot 2^{-d} \qquad (t \in \mathbf{G}_k^*).$$

- The interaction has a *thermodynamic limit* in the sense

  $$0 \le (j_{k+1}(0,t) - j_k(t)) \cdot 2^s < C \cdot 2^{-d} \qquad (t \in \mathbf{G}_k^* \setminus \{0\}).$$

  For $\beta > 0$ the thermodynamic limit

  $$F(\beta) := \lim_{k\to\infty} F_k(\beta) \quad \text{with} \quad F_k(\beta) := -\frac{1}{\beta \cdot k} \ln(Z_k(\beta)) \tag{7}$$

  of the *free energy* per spin exists [11].

- The interaction is *ferromagnetic*, that is,

  $$j_k(t) \ge 0 \qquad (t \in \mathbf{G}_k^* \setminus \{0\}). \tag{8}$$

  This is in accordance with earlier speculations (see Ruelle [18]). Note, however, that the system is not of Ising type, since multi-body interactions are present.

- The *effective interaction*

  $$A_k(l,r) := \sum_{t' \in \mathbf{G}_{r-l-1}^*} j_k(0,\ldots,0,1,t_{l+1},\ldots,t_{r-1},1,0,\ldots,0)$$

  between spins at positions l and r decays quadratically with their distance $s = r - l$ in the bulk:

  $$A_k(l,r) \le \frac{1}{s^2} + \frac{2^{-(k-r)}}{s}. \tag{9}$$

  This is just the borderline decay rate for a phase transition

A proof of the ferromagnetic property can be based on abstract polymer models, see [6], and this kind of combinatorics appears naturally in the context of number-theoretical zeta functions [3].

Switching to a multiplicative representation $s_i(\sigma) := (-1)^{\sigma_i}$ of the ith spin, an important variable is the *mean magnetization* per spin

$$M_k(\beta) := \frac{1}{k}\sum_{i=1}^{k} \langle s_i \rangle_k (\beta)$$

and its thermodynamic limit $M(\beta)$. By analyzing a Perron-Frobenius operator with *PF* eigenvalue $\exp(-\beta \cdot F(\beta))$, the following statements were proved.

Theorem [2]. The only phase transition of the number-theoretical spin chain occurs for inverse temperature $\beta_{\mathrm{cr}} := 2$. For lower temperatures

$$F(\beta) = U(\beta) = 0 \quad \text{and} \quad M(\beta) = 1 \qquad (\beta > \beta_{\mathrm{cr}}),$$

whereas for high temperatures

$$U(\beta) \geq \tfrac{1}{4}(\beta - \beta_{\mathrm{cr}}) \quad (1 < \beta < \beta_{\mathrm{cr}}) \quad \text{and} \quad M(\beta) = 0 \quad (0 \leq \beta < \beta_{\mathrm{cr}}).$$

The energy function can be interpreted as the time delay of scattering geodesics in the modular domain [12]. There exist direct relations with the works [16], [17] by Mayer, and [15] by Lanford and Ruedin, and the Riemann Hypothesis can be related to a problem concerning the spectral radius of a related Markov chain [13]. See, however the work by Heiss [7] disproving a conjecture in [13].

More details can be found in the lecture notes [14]. This abstract is an up to date version of the abstract in [1].

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Integer Points in Plane Regions and Exponential Sums

M. N. Huxley*

An old problem in mathematics is to find the area inside a closed curve. A practical method is counting squares: put a grid of squares over the curve, and count the number of squares which are inside the curve. This number should be proportional to the area. A rule is needed about squares cut by the curve. Counting them all as 1/2 seems to give a slight over-estimate. Counting them when the centre of the square is inside the curve makes mathematical sense, but it is hard to judge the centre by eye. Counting them when the bottom left corner is inside the curve is essentially the same rule (shift the grid of squares so that corners are now where centres were before), and it is easier to apply.

Can you get prescribed accuracy (d significant figures, say) by taking squares small enough? Suppose that 10^n squares fit across the curve. The curve must be smooth (piecewise convex and sufficiently differentiable), or you cannot say anything.

Usually (for most placings of the square grid) you get d-figure accuracy for n a little larger than $2d/3$.

Classical result (van der Corput 1917). You can guarantee d-figure accuracy by taking n a little larger than $3d/4$.

Modern result (Iwaniec, Mozzochi 1989). You can guarantee d-figure accuracy with n a little larger than $11d/15$ $(11/15 = 0.7333...)$.

Recent improvements: (Huxley 1993). You can guarantee d-figure accuracy with n a little larger than $73d/100$ $(73/100 = 0.73)$. Now being typed: improvement to $208d/285$ $(208/285 = 0.7298...)$.

Limit of the Iwaniec-Mozzochi method would be $8d/11$ $(8/11 = 0.7273...)$.

The points counted are the lattice points, the intersection points in the grid of squares. Renormalize so that the squares are unit squares and the lattice points are integer points. The CAT maps

$$x \to ax + by, \ y \to cx + dy,$$

where a, b, c, d are integers with $ad - bc = 1$, are automorphisms of the set of lattice points, but they send the squares of the grid to parallelograms. Shapes are distorted, but areas remain the same. So the number of bottom left corners of squares inside the closed curve, and the area inside the curve both remain unchanged after a CAT map.

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If the square grid is thought of as fixed, defining the (x, y)-coordinate system, and the curve is shifted by a vector (x, y), then the area stays the same, but the number of squares counted changes as a periodic function of x and y. Accordingly it has a Fourier series of the form $\sum c(h, k)e(hx + ky)$, where $e(t)$ is written for $\exp(2\pi it)$. The constant term is the area. The other Fourier coefficients can be calculated first as integrals over the interior, then, by the divergence theorem, as integrals round the curve of stationary phase type. The points of stationary phase occur where the label (h, k) is a normal vector to the curve, $(-h, k)$ is a tangent vector, so if you look along the curve, the integer points line up along straight lines with gradient $-h/k$. The 'usually $2d/3$' result follows by Parseval's theorem (Kendall 1948).

The Fourier series does not lend itself to easy estimation. Iwaniec and Mozzochi used a divide and conquer method, based on that of Bombieri and Iwaniec for the single exponential sum. The area problem seems to be more fundamental, in that the steps can be explained geometrically.

Divide the curve into short arcs. Look along the curve, see rows of integer points lining up in straight lines with some gradient a/q. The arc has now been labelled by a 'Farey fraction' a/q.

Approximate the curve by a quadratic equation

$$qy = a(x - m) + b + \kappa + \mu q(x - m)^2,$$

where a, b, q, m are integers, and the last two terms are real number corrections.

Combine the patterns from different arcs in some way.

In more detail, the 'approximate' step continues by choosing a new basis (q, a), $(-\bar{a}, \bar{q})$, where $a\bar{a} + q\bar{q} = 1$. Choose a CAT map to take these to the unit vectors (1,0) and (0,1). The arc is now approximately horizontal. Count integer points using the row of teeth function $\rho(t) = [t] - t + \frac{1}{2}$, where $[t]$ means the largest integer n with $n \leq t$. This function has the property that the number of integers n in an interval $\alpha < n \leq \beta$ is $\beta - \alpha + \rho(\beta) - \rho(\alpha)$. It has a well-known Fourier series, which is used to translate into exponential sums.

The use of exponential sums is based on the observation that $e(x) := \exp(2\pi ix)$ depends only on the fractional part of x, $x - [x]$. When these are studied for their own sake, the most interesting sums are the single sum

$$\sum_{m=M}^{M_2} e(F(m))$$

for $M_2 < 2M$, $(\log F''(x))/(\log M)$ in an interval around zero, and the double sum

$$\sum_{h=H}^{H_2} \sum_{m=M}^{M_2} e(hf(m))$$

for $H_2 < 2H$, $M_2 < 2M$, and $(\log f'(x))/(\log M)$ in an interval around zero. There are two things to do with an exponential sum.

Poisson summation transforms $\sum_m e(F(m))$ into a sum of type $\sum_n G(n)$, where $F(x)$ and $G(y)$ are complementary functions, in the sense that their derivatives $F'(x)$ and $G'(y)$

are inverse functions. If $F(x)$ represents the area below a graph, then $G(y)$ represents the area between the same graph and the y-axis.

Rearrange the modulus squared of sums. One way to do this gives the so-called large sieve inequality that estimates the square of a double sum

$$\left| \sum_i \sum_j a_i b_j e\left(\mathbf{x}^{(i)} \cdot \mathbf{y}^{(j)}\right) \right|$$

with the inner product of vectors inside the exponential, in terms of not just the sum of the squares of the coefficients a_i, but the sum over a 'neighbourhood' of the diagonal: the sum $\sum_i \sum_k |a_i a_k|$ taken over pairs of vectors with $\mathbf{x}^{(i)}$ close to $\mathbf{x}^{(k)}$, and the corresponding sum for the $\mathbf{y}$ vectors.

Using the row of teeth function leads to a two variable exponential sum. This is transformed by Poisson summation, and the resulting mess is raised to the fourth power. After simplification the exponent takes the form $\mathbf{x}.\mathbf{y}$, where $\mathbf{x}$ is a four-dimensional vector derived from the integer summands, and $\mathbf{y}$ is a vector formed from the data in the quadratic approximation on the short arc.

The large sieve inequality is the 'combine' step of the divide and conquer method. It says that the worst case doe not happen very often. But 'often' is counted on the simplified forms after the CAT map. If the simplified forms of two different short arcs coincide up to the accuracy of approximation, then the corresponding $\mathbf{y}$ vectors count as close together in the large sieve. In terms of the approximation data we have

$$\begin{pmatrix} a' & -\bar{q}' \\ q' & \bar{a}' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & -\bar{q} \\ q & \bar{a} \end{pmatrix}, \tag{1}$$

where the matrices are in $SL(2, \mathbb{Z})$: integer entries and determinant one, and A, B, C, and D are small. Geometrically, the new lattice bases are closely related.

$$\mu' q'^3 \doteqdot \mu q^3. \tag{2}$$

Geometrically, residual areas are equal.

$$\frac{\bar{a}' b'}{q'} \doteqdot \frac{\bar{a} b}{q} \text{ modulo one,} \tag{3}$$

meaning the fractional parts are approximately equal. Geometrically, the constant terms correspond.

$$\kappa' \doteqdot \kappa. \tag{4}$$

Geometrically, the residual constants correspond.

To complete the argument one must estimate how many pairs of short arcs have the same simplified form under the CAT maps. This is itself a divide and conquer argument. There is a CAT map that takes one short arc to the other short arc. We ask: for each possible CAT map, how many short arcs can it take into coincidence with other short arcs. The equations that say that one short arc is taken to another short arc can be interpreted (this is my contribution) as saying that there is an integer point that approximately satisfies some equation. So the integer point is close to the graph of this equation, which I call a resonance curve. I am told that this is an outmoded use of 'resonance' which has a very precise meaning in terms of poles of a scattering matrix, and the other uses of the word have been replaced by 'constructive interference'. But it is too late to change the terminology in my book.

The resonance curve detects the coincidence after CAT maps of the short arcs of the curve with gradients a/q and a'/q', for the gradients a/q and a'/q' in intervals

$$\frac{e}{r} < \frac{a}{q} < \frac{f}{s}, \qquad \frac{e'}{r'} < \frac{a'}{q'} < \frac{f'}{s'},$$

on which (1) and (2) hold throughout, so that

$$\begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} f & e \\ s & r \end{pmatrix}$$

in $SL(2, \mathbb{Z})$; all three matrices have integer entries and determinant one. If (3) and (4) hold on the short arc labelled a/q, then there is an integer point (c, d) close to a curve $C = C\left(\frac{e}{r}, \frac{f}{s}; \frac{e'}{r'}, \frac{f'}{s'}\right)$. The curve has negative gradient, and usually has a cusp where two branches join. The construction of C used point and line duality in the plane, the implicit function theorem and the complementary function construction that also occurs in the Poisson summation formula.

The analytic construction of the resonance curve goes as follows. The original curve is $y = f(x)$. Take the gradient $v = f'(x)$ as the independent variable, and put $\frac{1}{2}f''(x) = h(v)$. The data are two arcs $e/r < v < f/s$, $e'/r' < v < f'/s'$. Parametrize by a new variable x in $0 < x < \infty$ as

$$v_1(x) = \frac{ex + f}{rx + s}, \qquad v_2(x) = \frac{e'x + f'}{r'x + s'}$$

(when x takes rational values u/t, this is the fractal self-similarity of the rational numbers). Put $h_1(x) = h(v_1(x))$, $h_2(x) = h(v_2(x))$. The function $G_1(x)$ is the solution of the differential equation

$$G_1''(x) = \frac{1}{2h_1(x)(rx + s)^3}$$

with boundary conditions $G_1(\infty) = 0$, $G_1'(\infty) = 0$, and $G_2(x)$ is constructed similarly.

The resonance-detecting function is

$$g(x) = G_1(x) - G_2(x),$$

which is small when (2) holds; $h_1(x)$ is μ, $rx + s$ corresponds to q. The derivatives $g'(x)$ and $g''(x)$ are connected with the conditions (3) and (4). The 'resonance curve' C lies in some dual space, the locus of points (y, z) with

$$\frac{dy}{dx} = -g''(x), \qquad \frac{dz}{dx} = xg''(x),$$

for certain boundary conditions at $x = \infty$. The curve C has gradient $-x$. The conditions for the coincidence of short arcs can be translated in terms of the resonance curve using the identity

$$f(m+n) = f(m) + nf'(m) + \int_m^{m+n} (m+n-x)f''(x)dx.$$

The relations (2), (3) and (4) are approximate, which leads to the integer point (c, d) lying only approximately on the resonance curve. Approximation theory wants to be linear, but the construction of the curve C is non-linear, and the estimations are delicate. There is one number-theoretical device which gains accuracy. If the real number α is close to an integer n, and t is a factor of n, then the real number α/t is very close to the integer n/t.

The resonance curve $C = C\left(\frac{e}{r}, \frac{f}{s}; \frac{e'}{r'}, \frac{f'}{s'}\right)$ depends on two intervals of gradients $e/r < v < f/s$, $e'/r' < v < f'/s'$. There should be a relation between the resonance curves for an interval and for a subinterval. The matrix products

$$\begin{pmatrix} f_0 & e_0 \\ s_0 & r_0 \end{pmatrix} = \begin{pmatrix} f & e \\ s & r \end{pmatrix} \begin{pmatrix} l & k \\ j & h \end{pmatrix}, \qquad \begin{pmatrix} f_0' & e_0' \\ s_0' & r_0' \end{pmatrix} = \begin{pmatrix} f' & e' \\ s' & r' \end{pmatrix} \begin{pmatrix} l & k \\ j & h \end{pmatrix},$$

where the matrices all have integer entries and determinant one, and h, $l \geq 1$, j, $k \geq 0$, give subintervals, with

$$\frac{e}{r} \leq \frac{e_0}{r_0} < \frac{f_0}{s_0} \leq \frac{f}{s}, \qquad \frac{e'}{r'} \leq \frac{e_0'}{r_0'} < \frac{f_0'}{s_0'} \leq \frac{f'}{s'}.$$

The resonance curve C_0 constructed from the subintervals is related to C in the nicest possible way, by an affine map

$$(z_0, y_0) = (z, y) \begin{pmatrix} l & k \\ j & h \end{pmatrix} + (\beta, \alpha).$$

The transposed vectors indicate that the resonance curve lies in some dual space. A careful calculation shows that (β, α) is very close to an integer vector. So integer points close to C_0 lift to integer points close to C (not quite so close, because the CAT map $\begin{pmatrix} l & k \\ j & h \end{pmatrix}$ distorts shapes and usually increases distances). This brings us to the question: can there be many integer points on a curve or very close to it? Besides the exponential sums used to analyse the method of counting squares, there is a direct approach using interpolation determinants. If points $P(m_1, n_1)$ and $Q(m_2, n_2)$ lie on a curve $y = f(x)$, then

$$\frac{n_2 - n_1}{m_2 - m_1} = f'(x)$$

for some x. If P, Q, and also R (m_3, n_3) all lie on the curve, then

$$\frac{1}{(m_2 - m_1)(m_3 - m_2)(m_3 - m_1)} \begin{vmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{f''(x)}{2}$$

for some x, and so on. The 3×3 determinant is an integer (twice the area of the triangle PQR, and the second derivative has the dimensions of one over length, and so is small. The product $(m_2 - m_1)(m_3 - m_2)(m_3 - m_1)$ must be large, and the points P, Q and R cannot all be close together. So there cannot be too many integer points on the curve.

When we allow points close to the curve, as well as points on the curve, there are two cases.

Major arc. The points P, Q, R lie on a straight line with rational gradient. These cases can be counted carefully.

Minor arc. The points P, Q, R do not lie on a straight line, and the 3×3 determinant is a non-zero integer.

Swinnerton-Dyer's refinement on the minor arcs is to compare 4×4 and 3×3 determinants. Formally this involves dividing the curve into regions, counting the integer points close to each region of the curve, and considering the fourth power mean of the numbers. The fourth power mean of minor arc sums in the Iwaniec-Mozzochi method is strangely analogous. If P, Q, R and S are four integer points close to the curve forming a convex polygon, then the minor determinants

$$D_1 = 2\ \text{area}(QRS), \quad D_3 = 2\ \text{area}(PQR), \quad D_2 = 2\ \text{area}(PQRS) - D_1 - D_3$$

are small positive integers. The triple D_1, D_2, D_3 is almost unique, in the sense that knowing the values of the D_1 gives very few possibilities for the integer points P, Q, R, S. Probably knowing D_3 alone should be enough.

To sketch the results, some notation: for integer points (m,n) close to curve, with $|n - f(m)| \leq \delta$, suppose that m and n run through intervals of length M and N, and let $R = \sqrt{MN}$, $\alpha = (\log N)/(\log M)$. The probabilistic expectation is $2\delta(M+1)$ integer points. We want an upper bound of the form

$$O\left(\delta M + R^{\kappa} (\log R)^{\lambda}\right).$$

Van der Corput and Vinogradov both got this with $\kappa = \frac{2}{3}$ for all α. Swinnerton-Dyer's refinement gets $\kappa = \frac{3}{5}$ for α near one, but there are other terms in the upper bound involving fractional powers of δ.

For integer points (m,n) inside a closed curve, with M, N, R, α as above, we want

$$\text{Area} + O\left(R^{K} (\log R)^{\Lambda}\right).$$

Again van der Corput obtained this with $K = \frac{2}{3}$ for α near one, and he hinted that a result in the circle problem with the smaller exponent $K = \frac{27}{41} = 0,6585...$ should hold more generally. My latest published result (using resonance curves trivially: the number of integer points is at most the length of the curve plus one) has $K = \frac{46}{73} = 0,6301...$ for $\frac{30}{43} < \alpha < \frac{43}{30}$. The result [5] now being typed says that if the estimate for integer points close to a curve holds for $\frac{1}{2} \leq \alpha \leq 2$ (and if $\kappa \geq \frac{1}{4}$), then one gets a slightly smaller K in the counting squares problem. The partial result with $\kappa = \frac{3}{5}$ gives $K = \frac{131}{208} = 0,6298...$, and a slightly better exponent 0,156098... [4] for the growth of the Riemann zeta function $\zeta(s)$ on its critical line $\operatorname{Re} s = \frac{1}{2}$.

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Transfer operators and period functions for subgroups of $PSL(2,\mathbb{Z})$

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0 Introduction

In the approach through the transfer operator belonging to the geodesic flow on the modular surface, Selberg's zeta function for $PSL(2,\mathbb{Z})$ can be expressed as the Fredholm determinant of a meromorphic family of nuclear operators $\mathcal{L}_\beta$. They generalize the transfer matrices of statistical mechanics which serve to calculate so called partition functions. Since the nontrivial zero's of Selberg's function are closely related to the eigenvalues of the Laplace-Beltrami operator on the modular surface, the transfer operator $\mathcal{L}_\beta$ connects the geodesic flow trough its classical properties to the spectrum of the quantized system. Indeed, it turned out that also the eigenfunctions of $\mathcal{L}_\beta$ corresponding to the eigenvalue $\lambda = 1$ can be directly related to the Maass wave forms for $PSL(2,\mathbb{Z})$. These eigenfunctions generalize, as shown by J. Lewis and D. Zagier, the period polynomials of Eichler, which also show up as eigenfunctions for $\mathcal{L}_\beta$ with eigenvalue $\lambda = 1$ for certain β-values.

In the present note we report on first results to extend the above results to subgroups of the full modular groups $PSL(2,\mathbb{Z})$.

In detail the paper is organized as follows: we briefly recall the definition of the Ruelle zeta function for the geodesic flow for a Fuchsian group Γ with some representation $\chi : \Gamma \to \mathrm{end}(V)$ and how it is related to the transfer operator. Next we consider subgroups $\Gamma \subset PSL(2,\mathbb{Z})$ of finite index and the symbolic dynamics of the geodesic flow on the surface $\Gamma/\mathbb{H}$. This allows us to construct the transfer operator $\mathcal{L}_\beta^\Gamma$ for Γ. We show that $\mathcal{L}_\beta^\Gamma$ is closely related to the transfer operator for $PSL(2,\mathbb{Z})$ together with its representation χ_{ind} induced from the trivial representation of Γ. In the next chapter we consider the special congruence subgroups $\Gamma_0(2), \Gamma^0(2), \Gamma_\vartheta$ and $\Gamma(2)$ and their transfer operators. We briefly recall Eichler's automorphic integrals and the period polynomicals of holomorphic cusp forms for a general Fuchsian group and their cocycle relations, which in the case $\Gamma = PSL(2,\mathbb{Z})$ have been related by Zagier to a functional equation of Lewis. We derive the analogous functional equation for the group $\Gamma_0(2)$ and discuss the solutions of this equation. It turns out that these solutions fall into two classes, so called old and new solutions. They obviously correspond to old and new forms in Atkin-Lehner's theory. Indeed in the case of old solution we can establish a relation with the old forms for these simple congruence groups whereas for

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the new solutions besides some polynomial solutions not much is known up to now besides numerical hints. But even in case of the polynomials we do not know the explicit relation to the new cusps forms for instance for the group $\Gamma_0(2)$.

1 Ruelle's zeta function and the transfer operator

On the hyperbolic plane $\mathbb{H} = \{z = x + iy : y > 0\}$ with metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, acts the modular group

$$\Gamma(1) = PSL(2, \mathbb{Z}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ \det g = 1 \right\} / (\pm 1)$$

by Möbius transformations

$$gz = \frac{az + b}{cz + d}. \tag{1}$$

In the following we are especially interested in subgroups $\Gamma \subset \Gamma(1)$ of finite index $[\Gamma(1) : \Gamma] = \mu < \infty$. The geodesics on the surface $\Gamma/\mathbb{H}$ are just the projections onto $\Gamma/\mathbb{H}$ of the geodesics on $\mathbb{H}$ which are half circles based on the real line or straight lines parallel to the imaginary axis.

The geodesic flow $\phi_t : S_1\Gamma/\mathbb{H} \longrightarrow S_1\Gamma/\mathbb{H}$ is the flow with constant velocity along the geodesics on $\Gamma/\mathbb{H}$. An orbit γ of ϕ_t is periodic iff $\pi\gamma$ is a closed geodesic where $\pi : S_1\Gamma/\mathbb{H} \longrightarrow \Gamma/\mathbb{H}$ denotes the projection of the unit tanget bundle $S_1\Gamma/\mathbb{H}$ onto $\Gamma/\mathbb{H}$. For unit velocity the period $T(\gamma)$ of a periodic orbit γ is identical to the length $l(\pi\gamma)$ of the closed geodesic $\pi\gamma$ which we denote simply by $l(\gamma)$.

Consider some finite dimensional representation $\chi : \Gamma \longrightarrow \mathrm{end}(V)$. Ruelle's dynamical zeta function $\zeta^\chi(\beta)$ for the geodesic flow ϕ_t with representation χ is then defined for $\mathrm{Re}\,\beta > 1$ as

$$\zeta^\chi(\beta) = \prod_{\gamma \text{ prime}} det\left(1 - \chi(\sigma_\gamma) e^{-\beta l(\gamma)}\right)^{-1} \tag{2}$$

where $\sigma_\gamma \in \Gamma$ has the property that it fixes a lift $\tilde{\gamma}$ of the geodesic γ in $\mathbb{H}$.

The generalized Selberg zeta function $Z^{\Gamma,\chi}$ of Γ with representation χ is then given for $\mathrm{Re}\,\beta > 1$ as

$$Z^{\Gamma,\chi}(\beta) = \prod_{k=0}^{\infty} \zeta^\chi(\beta + k)^{-1} \tag{3}$$

The classical approach to the analytic properties, the zero's and poles and to the functional equation of this function, is through Selberg's trace formula [14]. Another approach, which grew out from the thermodynamic formalism and its applications in the ergodic theory of dynamical dystems [12], uses the so called transfer operator of the geodesic flow ϕ_t. The main idea behind this appoach is the following: consider a Poincaré section Σ transversal to

ϕ_t and the induced Poincaré map $P : \Sigma \to \Sigma$. Obviously an orbit γ is periodic under ϕ_t iff the point $x \in \gamma \cap \Sigma$ is periodic under the Poincaré map P, that is $P^n x = x$ for some $n \geqslant 1$. The period $l(\gamma)$ can be expressed through the recurrence time function $r : \Sigma \to \mathbb{R}_+$ with $Px = \phi_{r(x)} x$ as

$$l(\gamma) = \sum_{k=0}^{n-1} r(P^k x) \tag{4}$$

if $P^n x = x$.

By simple calculation one then shows [13]

Lemma. The Ruelle function $\zeta^\chi(\beta)$ can be written as

$$\zeta^\chi(\beta) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} Z_n^\chi(P,\beta)\right)$$

with $Z_n^\chi(P,\beta)$ the partition functions

$$Z_n^\chi(P,\beta) = \sum_{x \in Fix P^n} \operatorname{trace} \chi(\sigma_x) \exp\left(-\beta \sum_{k=0}^{n-1} r(P^k x)\right)$$

where $\sigma_x \in \Gamma$ fixes the closed orbit γ through x.

The main problem now is to calculate the partition function Z_n. Long time ago there has been developed in Statistical Mechanics the so called transfer matrix method to transform this combinatorial problem into an algebraic one: one constructs a matrix, or more generally a trace class operator, whose traces determine the numbers $Z_n(\beta)$.

This program obviously demands the explicit knowledge of the Poincaré map P, its fixed points and the recurrence time function $r : \Sigma \to \mathbb{R}_+$.
Surprisingly, these quantities could be determined explicitly for the modular group and the geodesic flow on the modular surface $\Gamma(1)/\mathbb{H}$ [11].

The Poincaré section Σ can be choosen in a certain coordinate system of $S_1\Gamma(1)/\mathbb{H}$ as $\Sigma = I_2 \times \mathbb{Z}_2$, where I_2 denotes the unit square in $\mathbb{R}^2$. The Poincaré map $P : I_2 \times \mathbb{Z}_2 \to I_2 \times \mathbb{Z}_2$ then reads

$$P(x_1, x_2, \varepsilon) = \left(\frac{1}{x_1} \mod 1, \frac{1}{x_2 + \left[\frac{1}{x_1}\right]}, -\varepsilon\right) \tag{5}$$

with $\left[\frac{1}{x_1}\right]$ the largest integer smaller than $\frac{1}{x_1}$.

The recurrence time function r turns out to be the function

$$r(x_1, x_2, \varepsilon) = -\log x_1^2.$$

From these data the transfer operator $\mathcal{L}_\beta$ for the map P restricted to the expanding directions which in our case are just $(x_1, \varepsilon), \varepsilon = \pm 1$, can be constructed in a standard manner and reads

[4], [3]

$$\tilde{\mathcal{L}}_\beta^\chi f(x_1,\varepsilon) = \sum_{n=1}^\infty \left(\frac{1}{x_1+n}\right)^{2\beta} \chi(QT^{n\varepsilon}) f\left(\frac{1}{x+n}, -\varepsilon\right) \tag{6}$$

where T and Q denote the generators $Tz = z+1,\ Qz = -\frac{1}{z}$ of the modular group $\Gamma(1)$.

Chosing the function $f : I_2 \times \mathbb{Z}_2 \to V$ to be holomorphic for instance in the disc $D = \{z : |z-1| < \frac{3}{2}\}$, the operator $\tilde{\mathcal{L}}_\beta^\chi$ is a well defined operator for $\mathrm{Re}\,\beta > \frac{1}{2}$ in the Banach space $B(D\times\mathbb{Z}_2, V)$ of functions holomorphic in D with values in the linear space V. Indeed, one can show the following properties [4]

- $\tilde{\mathcal{L}}_\beta^\chi$ is a meromorphic family of nuclear operators with possible poles only at $\beta = \beta_k = \frac{1-k}{2},\ k = 0, 1, \ldots$
- $Z_n^\chi(\beta) = \mathrm{trace}(\tilde{\mathcal{L}}_\beta^\chi)^n - (-1)^n\, \mathrm{trace}(\tilde{\mathcal{L}}_\beta^\chi)^n$
- $\zeta^\chi(\beta) = \det(1 - \tilde{\mathcal{L}}_{\beta+1}^\chi) \big/ \det(1 - \tilde{\mathcal{L}}_\beta^\chi)$
- $Z^{\Gamma(1),\chi}(\beta) = \det(1 - \tilde{\mathcal{L}}_\beta^\chi)$

This shows, that the zeros and poles of Selberg's zeta function are closely related to spectral properties of the transfer operator $\tilde{\mathcal{L}}_\beta^\chi$. An analogous interpretation of zeros has been given, as is well known, for the zeros of the zeta function of Artin and Weil defined for projective algebraic varieties over finite fields by Dwork and Deligne [7]. Whether also the zeros of Riemann's zeta function allow for such an interpretation is being discussed at the moment with great intensity.

Quite a lot is known also about the eigenfunctions with eigenvalue $\lambda = 1$ of the transfer operator $\tilde{\mathcal{L}}_\beta^\chi$ for the trivial representation $\chi = 1$. In this case $\tilde{\mathcal{L}}_\beta^\chi$ has the form

$$\tilde{\mathcal{L}}_\beta^\chi = \begin{pmatrix} D & \mathcal{L}_\beta \\ \mathcal{L}_\beta & 0 \end{pmatrix}$$

with $\mathcal{L}_\beta f(z) = \sum\limits_{n=1}^\infty (\frac{1}{z+n})^{2\beta} f(\frac{1}{z+n}), Re\beta > \frac{1}{2}, f \in B(D)$.

Then Selberg's function $Z^{\Gamma(1)}(\beta)$ can be written as

$$Z^{\Gamma(1)}(\beta) = \det(1 - \mathcal{L}_\beta)\det(1 + \mathcal{L}_\beta) \tag{7}$$

Hence one has to understand the eigenfunctions of $\mathcal{L}_\beta$ with eigenvalues $\lambda_\beta = \pm 1$. If f_β is such an eigenfunction then it satisfies the functional equation [9]

$$\lambda_\beta f_\beta(z) = \lambda_\beta f_\beta(z+1) + (\tfrac{1}{z+1})^{2\beta} f_\beta(\tfrac{1}{z+1}), \qquad \lambda_\beta = \pm 1. \tag{8}$$

This equation has been derived by J. Lewis in his study of the Maass wave forms of the modular group by methods in harmonic analysis. One can show also that any solution of this

Lewis equation holomorphic in the complex plane cut along $(-\infty, 1)$ with a certain growth condition at infinity defines an eigenfunction of $\mathcal{L}_\beta$ with eigenvalue λ_β [3],[10].

Work by Zagier, Lewis, Chang and Mayer [3], [5] has shown the following results for the eigenfunctions f_β with eigenvalue $\lambda_\beta = \pm 1$:

- for $\beta = -n,\ n = 1, 2, \ldots$, corresponding to the trivial zeros of $Z(\beta)$, the functions $f_\beta(z-1)$ are the odd $(\lambda_\beta = 1)$ respectively even parts $(\lambda_\beta = -1)$ of Eichler's period polynomials $p_Q(z)$ of the holomorphic cusps forms of weight $2n+2$ of $\Gamma(1)$ respectively the odd/even parts of Zagier's period functions $p_Q(z)$ of the holomorphic Eisenstein series of weight $2n+2$ of $\Gamma(1)$.
- for $\mathrm{Re}\,\beta = \frac{1}{2}$, corresponding to the so called spectral zeros of $Z(\beta)$, the functions $f_\beta(z-1)$ are the Lewis transforms of the even $(\lambda_\beta = 1)$ respectively odd Maass cusp forms $\varphi_\beta(z)$ of $\Gamma(1)$ obeying $-\triangle\varphi(z) = \beta(1-\beta)\varphi(z)$ and $\varphi(-x, y) = \pm\varphi(x, y)$
- for $\mathrm{Re}\,\beta > 0$ such that $\zeta(2\beta) = 0$, corresponding to the nontrivial zeros of Riemann's zeta function $\zeta(z)$, the function $f_\beta(z-1)$ is the Lewis transform of the nonholomorphic Eisenstein series $E(\beta, z)$ of $\Gamma(1)$.

  The Lewis transforms relating Maass- and Eisenstein wave forms to period functions read [9]

  $$f_\beta(z-1) = \phi_\beta(z) = z\int_0^\infty y^\beta \varphi_\beta(iy)(z^2+y^2)^{-\beta-1}dy \tag{9}$$

  for even forms φ_β and [10]

  $$f_\beta(z-1) = \phi_\beta(z) = \int_0^\infty \frac{\partial}{\partial x}\varphi_\beta(iy)(z^2+y^2)^{-\beta}y^\beta dy \tag{10}$$

  for odd forms φ_β.
- for $\beta = 1$ the function $f_\beta(z-1) = \frac{1}{z}$ is the Lewis transform of the Maass form $\varphi_\beta(z) = const.$

Remark. The transfer operator $\mathcal{L}_\beta$ for $\beta = 1$ is the well known Perron-Frobenius operator of ergodic theory for the Gauß-map. It eigenfunction $f_\beta(z) = \frac{1}{z+1}$ is up to normalization the invariant density of the Gauß measure.

2 The transfer operator for subgroups $\Gamma \subset \Gamma(1)$

Since $[\Gamma(1) : \Gamma] = \mu < \infty$ the surface $\Gamma/\mathbb{H}$ is a μ-fold, in general a branched covering of the modular surface $\Gamma(1)/\mathbb{H}$. This allows [4] one to lift both the Poincare surface and the Poincare map from $S_1\Gamma(1)/\mathbb{H}$ to $S_1\Gamma/\mathbb{H}$: $\Sigma_\Gamma = I_1 \times \mathbb{Z}_2 \times \Gamma \setminus \Gamma(1)$ and $P_\Gamma : \Sigma_\Gamma \to \Sigma_\Gamma$ is given by

$$P_\Gamma(x_1, x_2, \varepsilon, [g]) = \left(\frac{1}{x_1} \mod 1, \frac{1}{x_2 + [\frac{1}{x_1}]}, -\varepsilon, [gT^{n\varepsilon}Q]\right) \tag{11}$$

where $n = n(x_1) = [\frac{1}{x_1}]$.
From this, one next constructs the transfer operator $\tilde{\mathcal{L}}_\beta^\Gamma$ in the usual way [4]

$$\tilde{\mathcal{L}}_\beta^\Gamma f(z, \varepsilon, [g]) = \sum_{n=1}^\infty \left(\frac{1}{z+n}\right)^{2\beta} f\left(\frac{1}{z+n}, -\varepsilon, [gQT^{n\varepsilon}]\right). \tag{12}$$

Denoting by $\chi^\Gamma : \Gamma(1) \to \mathcal{C}^\mu$ the representation of $\Gamma(1)$ induced from the trivial representation of the subgroup Γ, the operator $\tilde{\mathcal{L}}_\beta^\Gamma$ can be written as

$$\tilde{\mathcal{L}}_\beta^\Gamma \underline{f}(z, \varepsilon) = \sum_{n=1}^\infty \left(\frac{1}{z+n}\right)^{2\beta} \chi^\Gamma(QT^{n\varepsilon}) \underline{f}\left(\frac{1}{z+n}, -\varepsilon\right) \tag{13}$$

with $\underline{f} : D \times \mathbb{Z}_2 \to \mathbb{C}^\mu$ holomorphic in $z \in D$.

Comparing this with expression (6) we see, that $\tilde{\mathcal{L}}_\beta^\Gamma$ is identical $\tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma}$ the transfer operator for $\Gamma(1)$ with representation χ^Γ. Hence if $\chi^\Gamma = \bigoplus \chi_i$ is the decomposition of χ^Γ into irreducible representations χ_i, this induces a decomposition of $\tilde{\mathcal{L}}_\beta^\Gamma$ into a direct sum of transfer operators $\bigoplus\limits_i \tilde{\mathcal{L}}^{\Gamma(1),\chi_i}$.

The transfer operator $\tilde{\mathcal{L}}_\beta^\Gamma$ has the following properties [4]

- $\tilde{\mathcal{L}}_\beta^\Gamma$ is a meromorphic family of nuclear operators with possible poles only at $\beta = \beta_k = \frac{1-k}{2},\ k = 0, 1, 2 \ldots.$
- $Z^\Gamma(\beta) = \det(1 - \tilde{\mathcal{L}}_\beta^\Gamma) = \det(1 - \tilde{\mathcal{L}}_\beta^{\Gamma(1),\chi^\Gamma}) = Z^{\Gamma(1),\chi^\Gamma}(\beta)$ which is a special case of the Artin-Venkov-Zograf formula.
- If $\chi^\Gamma(T^2) = 1$, then $\tilde{\mathcal{L}}_\beta^\Gamma$ can be written as $\tilde{\mathcal{L}}_\beta^\Gamma = \begin{pmatrix} 0 & \tilde{\mathcal{L}}_\beta^\Gamma \\ \tilde{\mathcal{L}}_\beta^\Gamma & 0 \end{pmatrix}$ where the operator $\tilde{\mathcal{L}}_\beta^\Gamma$ acts as

$$\tilde{\mathcal{L}}_\beta^\Gamma \underline{f}(z) = \sum_{n=1}^\infty \left(\frac{1}{z+n}\right)^{2\beta} \chi^\Gamma(QT^n) \underline{f}\left(\frac{1}{z+n}\right). \tag{14}$$

Therefore the Selberg function can be written as

$$Z^\Gamma(\beta) = \det(1 - \tilde{\mathcal{L}}_\beta^\Gamma)\det(1 + \tilde{\mathcal{L}}_\beta^\Gamma) \tag{15}$$

As an example consider the congruence subgroups $\Gamma = \Gamma_0(2), \Gamma^0(2)$ or Γ_ϑ. In all these cases one finds:

$$\chi^\Gamma = \chi_1 \oplus \chi_2$$

where χ_1 denotes the trivial 1-dim. representation, and χ_2 denotes the irreducible 2-dim. representation of $\Gamma(1)$.

For the principal congruence subgroup $\Gamma(2)$ one finds

$$\chi^\Gamma = \chi_1 \oplus \chi_1' \oplus \chi_2 \oplus \chi_2$$

where χ_1' is the nontrivial irreducible representation of $\Gamma(1)$. From this one can see already that for $\Gamma(2)$ the transfer operator $\tilde{\mathcal{L}}_\beta^{\Gamma(2)}$ decomposes into operators corresponding to the groups $\Gamma(1)$ and $\Gamma_0(2)$. The only new part is the one corresponding to the representation χ_1'. We will see later how this fits nicely into Atkin and Lehner's theory of old and new forms.

3 The period functions for $\Gamma \subset \Gamma(1)$

In the following we restrict our discussion to the case $\Gamma = \Gamma_0(2)$, we remark however, that in principle we can treat any subgroup $\Gamma \subset \Gamma(1)$ of finite index.

The group $\Gamma_0(2)$ has the generators $e_1 = T, e_2 = QT^{-2}Q$ and $e_3 = T^{-1}QT^{-2}Q$ with relations $e_1e_2e_3 = 1$ and $e_3^2 = 1$.

In Eichler's theory [8] of automorphic integrals and period polynomials to every holomorphic cusp form φ_k of weight k of a Fuchsian group Γ can be defined a family $p_\sigma, \sigma \in \Gamma$, of polynomials of degree $\leq k-2$ with

$$p_\sigma(z) = \frac{1}{(k-2)!}\int_{\tau_0}^{\tau_0}(az+b+(c\tau+d))^{k-2}\varphi_k(\tau)d\tau + \Theta(z)\mid_{2-k}\sigma - \Theta(z) \tag{16}$$

where Θ is an arbitrary polynomial of degree $\leq k-2$ and the slash operator $|_\sigma$ is defined as

$$f(z)\mid_k \sigma = (cz+d)^{-k}f(\sigma z) \tag{17}$$

when $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The polynomials p_σ fulfill the following cocycle relations

$$p_{\sigma_1\sigma_2} = p_{\sigma_1}\mid_{2-k}\sigma_2 + p_{\sigma_2} \tag{18}$$

which show that the polynomials for the generators of Γ determine the family p_σ completely. Since for T the polynomial can be chosen $p_T \equiv 0$ [8] it follows from the relations for $\Gamma_0(2)$ that p_{e_3} determines p_{e_2}. The relation $e_3^2 = 1$ leads to the following equation for p_{e_3}:

$$p_{e_3}(z) + p_{e_3}\left(\frac{z+1}{-2z-1}\right)(-2z-1)^{k-2} = 0. \tag{19}$$

From Eichler's theory it follows that the polynomial solutions of this functional equation are just the period polynomials p_{e_3} for the group $\Gamma_0(2)$.

Trivial solutions of this equation are obviously provided by the period polynomials $p_Q(z)$ of the modular group $\Gamma(1)$ through [6]

$$p_{e_3}(z) = p_Q(z) + z^{k-2}p_Q\left(-\frac{2z+1}{z}\right). \tag{20}$$

In the work of J. Lewis and D. Zagier for $\Gamma(1)$ there is a close relation of the corresponding cocycle relations for p_Q and the functional equation of Lewis. To extend their approach to general subgroups, especially for $\Gamma_0(2)$, we need the analogous functional equation for this group. This we will discuss next.

Consider an eigenfunction f_β of the operator $\mathcal{L}_\beta^{\Gamma_0(2)}$ with $\mathcal{L}_\beta^{\Gamma_0(2)} f_\beta(z) = \lambda_\beta f_\beta(z)$. For $\lambda_\beta = \pm 1$ we find after a simple calculation for the function $\phi_\beta(z) = f_\beta(z-1)$ the functional equation [6]:

$$\lambda_\beta[\phi_\beta(z) - \chi^{\Gamma_0(2)}(QTQ)\phi_\beta(z+1)] - z^{-2\beta}\chi^{\Gamma_0(2)}(QT)\phi_\beta(1+\tfrac{1}{z}) = \underline{0}. \tag{21}$$

Inserting the representation $\chi^{\Gamma_0(2)}$ this leads to the following equations for $\phi_\beta = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$, where we suppressed the dependence of the ϕ_i on β:

$$\begin{aligned} \lambda_\beta[\phi_1(z) - \phi_3(z+1)] - z^{-2\beta}\phi_3(1+\frac{1}{z}) &= 0 \\ \lambda_\beta[\phi_2(z) - \phi_2(z+1)] - z^{-2\beta}\phi_1(1+\frac{1}{z}) &= 0 \\ \lambda_\beta[\phi_3(z) - \phi_1(z+1)] - z^{-2\beta}\phi_2(1+\frac{1}{z}) &= 0 \end{aligned} \tag{22}$$

This shows that the function $\hat{\phi}(z) := \phi_1(z) + \phi_2(z) + \phi_3(z)$ solves Lewis equation for the group $\Gamma(1)$

$$\lambda_\beta \hat{\phi}(z) = \lambda_\beta \hat{\phi}(z+1) + \frac{1}{z^{2\beta}}\hat{\phi}(1+\tfrac{1}{z}). \tag{23}$$

It is also easy to see [6] that the solution of the above system of equations can be reduced to the solution of the following equation for the component $\phi_2(z)$:

$$\lambda_\beta[\phi_2(z) - \phi_2(z+1)] = (2z+1)^{-2\beta}\left[\lambda_\beta \phi_2\left(\frac{z}{2z+1}\right) + \phi_2\left(\frac{z+1}{2z+1}\right)\right] \tag{24}$$

Namely, if ϕ_2 solves this equation then

$$\phi(z) = \begin{pmatrix} \lambda_\beta\left[\phi_2\left(\frac{1}{z-1}\right) - \phi_2\left(1+\frac{1}{z-1}\right)\right](z-1)^{-2\beta} \\ \phi_2(z) \\ \lambda_\beta z^{-2\beta}\phi_2(\frac{1}{z}) \end{pmatrix} \tag{25}$$

solves equation (21) identically.

There is a class of solutions of equation (24) and hence of equation (21) closely related to the solutions of Lewis equation (23) for $\Gamma(1)$ [6].

Proposition. Let $\phi(z)$ be a solution of Lewis equation (23) for $\Gamma(1)$ for some β and $\lambda_\beta = \pm 1$. Then $\phi(z)$ and $\phi(2z)$ solve equation (24) for the same β and λ_β.

The solutions ϕ for equation (21) have the form

$$\phi_1(z) = \begin{pmatrix} \phi(z) \\ \phi(z) \\ \phi(z) \end{pmatrix} \qquad \text{respectively} \qquad \phi_2(z) = \begin{pmatrix} 2^{-2\beta}(\phi\left(\frac{z-1}{2}\right) - z^{-2\beta}\phi\left(\frac{z-1}{z}\right) \\ \phi(2z) \\ 2^{-2\beta}\phi\left(\frac{z}{2}\right) \end{pmatrix}.$$

To understand these old solutions we recall briefly the theory of old and new forms of Atkin and Lehner [1] for $\Gamma_0(2)$ respectively $\Gamma^0(2)$ [2].

If $g(z)$ is a modular form for $\Gamma(1)$ then $g(z), g(2z)$ are modular forms for $\Gamma_0(2), g(z)$ and $g\left(\frac{z}{2}\right)$ are modular forms for $\Gamma^0(2)$. If $h(z)$ is a modular form for $\Gamma_0(2)$ respectively for $\Gamma^0(2)$ then $h\left(\frac{z}{2}\right)$ is a modular form for $\Gamma^0(2)$ respectively $h(2z)$ is a modular form for $\Gamma_0(2)$.

Obviously the solutions of the above Proposition are "old" solutions in this sense.

Let us remark only without giving details that those solutions $\phi(z)$ of Lewis equation (23) for $PSL(2,\mathbb{Z})$ which fulfill certain asymptotic relations at infinity and hence determine also eigenfunctions of the transfer operator determine through the formulas in the Proposition also eigenfunctions of the transfer operator $\mathcal{L}_\beta^{\Gamma_0(2)}$. They hence are called old eigenfunctions being determined by the eigenfunctions of $\mathcal{L}_\beta^{\Gamma_0(1)}$.

Besides the old eigenfunctions of the transfer operator respectively old solutions of Lewis equation there should exist also new solutions. Since we do not know how the theory of Zagier and Lewis for $\Gamma(1)$ can be extended to $\Gamma_0(2)$ we looked in a first step after polynomial solutions of equation (24). Indeed for $\beta = -n$, we found such solutions numerically [6]: for $\lambda_\beta = 1$ they have the form

$$\phi_{odd}(z) = \begin{pmatrix} -\phi_o(z) - e\phi_o\left(\frac{z}{2}\right)2^n \\ \phi_o(z) \\ e\phi_o\left(\frac{z}{2}\right)2^n \end{pmatrix} \tag{26}$$

where $\phi_o(z)$ is an odd polynomial solving equation (24) of degree $< 2n$.
For $\lambda_\beta = -1$ on the other hand we found solutions of the form

$$\phi_{even}(z) = \begin{pmatrix} -\phi_e(z) - e\phi_e\left(\frac{z}{2}\right)2^n + cp_{2n}^+(z) \\ \phi_e(z) \\ e\phi_e\left(\frac{z}{2}\right)2^n \end{pmatrix} \tag{27}$$

with $\phi_e(z)$ an even polynomial solution of equation (24), $0 \neq c \in \mathbb{Q}$ and $e = \pm 1$. The polynomial $p_{2n}^+(z)$ is the even part of the period function of the holomorphic Eisenstein series $E_{2n+2}(z)$ [4].

One sees that for the odd solutions the sum of the components of ϕ_{odd} vanishes and hence the solutions cannot be old solutions. Furthermore the solutions ϕ_{odd}, ϕ_{even} exist only for those weights $2n+2$ for which there exist new holomorphic cusp forms for the group $\Gamma_0(2)$ and the number of solutions found for small values of n numerically coincide exactly with this number of new forms. This obviously supports the expectation that these solutions

ϕ_{odd}, ϕ_{even} are in 1-1 correspondence with the new cusp forms of $\Gamma_0(2)$. The form of these solutions show that the third component of ϕ_{even} and ϕ_{odd} describes the group $\Gamma^0(2)$, since with $\varphi(z)$ an automorphic form for $\Gamma_0(2)$ the function $\varphi\left(\frac{z}{2}\right)$ is an automorphic form for $\Gamma^0(2)$. One furthermore could conjecture that the first component in $\underline{\phi}$ is related to the group Γ_ϑ, since as we have seen, the three groups $\Gamma_0(2), \Gamma^0(2)$ and Γ_ϑ lead to the same transfer operator and hence to the same equation (21) for $\underline{\phi}$.

What is known about the relation between the second component ϕ_2 of $\underline{\phi}$ and the automorphic forms or functions for the group $\Gamma_0(2)$? Surprisingly, for the old forms this relation seems to be given by the transformation determined by J. Lewis for Maass forms respectively the Eichler formula for holomorphic cusps forms for $\Gamma(1)$: in both cases a simple calculation shows that [6] if

$$p_Q(z) = \int_0^{i\infty} \varphi_{2n+z}(z')(z-z')^{2n} dz' = T_n \varphi_{2n+z}(z) \tag{28}$$

is Eichler's transformation for cusp forms φ_{2n+2}, then

$$T_n \tilde{\varphi}_{2n+2}(z) = c_n p_Q(2z) \qquad \text{for} \qquad \tilde{\varphi}(z) = \varphi(2z) \tag{29}$$

respectively

$$T_n \tilde{\varphi}_{2n+2}(z) = d_n p_Q\left(\tfrac{z}{2}\right) \qquad \text{for} \qquad \tilde{\varphi}(z) = \varphi\left(\tfrac{z}{2}\right) \tag{30}$$

where c_n and d_n are some constants. Similarly one finds for the transformation of Lewis

$$\phi_\beta(z) = (L_\beta \varphi_\beta)(z) = z \int_0^{\infty} \varphi_\beta(iy)(z^2 + y^2)^{-\beta-1} \mu^\beta dy \tag{31}$$

for Maass-wave forms or Eisenstein forms:

$$(L_\beta \tilde{\varphi}_\beta)(z) = c_n' \phi_\beta(2z) \qquad \text{for} \qquad \tilde{\varphi}_\beta(z) = \varphi_\beta(2z) \tag{32}$$

respectively

$$(L_\beta \tilde{\varphi}_\beta)(z) = d_n' \phi_\beta(\tfrac{z}{2}) \qquad \text{for} \qquad \tilde{\varphi}_\beta(z) = \varphi_\beta\left(\tfrac{z}{2}\right). \tag{33}$$

with c_n', d_n' some constants.

Concerning the relation between new forms and new solutions, respectively new eigenfunctions of the transfer operator for $\Gamma_0(2)$, unfortunately nothing is known at the moment. It is also not known how the solutions ϕ_2 of Lewis equation for $\Gamma_0(2)$ are related to the solutions of the cocycle equation (19) for the period polynomials of the group $\Gamma_0(2)$. Numerical investigations of the polynomial solutions show however that the number of such solutions is identical to the number of solutions of equation (24), at least for small values of n. This obviously supports our expectation that these two equations are closely related to each other.

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Spectral decomposition of Iwahori-Hecke algebras

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Abstract

The Iwahori-Hecke algebra is a subalgebra of the convolution algebra of a p-adic reductive group. Its representations have been studied in detail, mainly using algebraic geometry. In this talk I will discuss the Iwahori-Hecke from an analytic viewpoint, and discuss a method to obtain its spectral decomposition explicitly.

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Capacities and Jacobi Forms

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1 Summary

The purpose of this contribution is to present recent results obtained with Thérèse Falliero on Capacities of an union of several intervals and related topics. Let K be a compact set in the complex plane, for simplicity we suppose itis infinite. According to Fekete, the transfinite diameter of K may be defined as follows: For each positive integer n, we consider

$$d_n = \max_{z_i \in K} \prod_{1 \le i < j \le n} |z_i - z_j|^{\frac{2}{n(n-1)}}.$$

$d_2 = \max_{z,z' \in K} |z - z'|$ is the "geometrical diameter" of K.

If $z_1, \cdots, z_n \in K$, $\prod_{1 \le i < j \le n} |z_i - z_j|^{\frac{2}{n(n-1)}}$ is the geometrical mean of the different distances $|z_i - z_j|, i \neq j$ and d_n is the greatest possible value of these geometrical means. The sequence $(d_n)_n$ is decreasing and its limit $C(K)$ is called the transfinite diameter or capacity of K. It is known that if $K = [a \ , \ b]$, then $C(K)$ is equal to $\frac{1}{4}(b - a)$. The formula giving the capacity of an union of two intervals is given by Akhieser and here we give the result for any finite union of closed disjoint intervals. There are many motivations for studying capacities. For instance, it is known that for closed intervals, the length 4 is critical for the problem of conjugate algebraic integers. This means that, any longer interval contains infinitely many sets of conjugate algebraic integers, whereas any shorter interval contains only a finite number of such sets. We believe that it is important to study a such problem for an union of closed intervals. We mention that in the case of 3 intervals, an account of the essential features (including the main theorems) has already appeared in [FS1]. The details and the used references are in [FS2] and in [FS3].

2 Case of 3 intervals

Let K be a compact subset of the real axis of the form

$$K = [e_1, e_2] \cup [e_3, e_4] \cup \ldots \cup [e_{2k-1}, e_{2k}]$$

with $e_i < e_j$ if $i < j$, we denote $p = g+1$. Let G be the Green's function of $\widehat{K} = \mathbb{P}^1 \backslash K$ with pole at infinity. The capacity $C(K)$ of K is also given by the constant term in the following asymptotic expansion:

$$G(z) = \log|z| - \log C(K) + o(1), \qquad z \to \infty.$$

We consider the hyper-elliptic curve X of genus $g \geq 1$ of the form:

$$w^2 = f(z) = \prod_{1}^{2g+2} (z - e_i)$$

We associate to M a canonical basis of cycles ${}^t\{a, b\} = \{a_1, ..., a_g, b_1, ..., b_g\}$ and $\{\zeta\} = {}^t\{dv_1, ..., dv_g\}$ a basis of holomorphic differentials, normalized and canonically dual to the basis ${}^t\{a, b\}$. If τ is the matrix of periods $\tau = (\tau_{i,j}),\ \tau_{i,j} = \displaystyle\int_{b_j} dv_i$, then τ is a Riemann matrix. The Riemann theta function is defined by its Fourier series:

$$\theta(z) = \theta(z, \tau) = \sum_{N \in \mathbb{Z}^g} \exp\left(\pi i \left({}^tN\tau N + 2{}^tNz\right)\right), \qquad z \in \mathbb{C}^g$$

A *θ-function with characteristics* is defined as follows: We introduce a $2 \times g$ matrix

$$\left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right] = \left[\begin{array}{ccc} \epsilon_1, & \ldots & , \epsilon_g \\ \epsilon'_1, & \ldots & , \epsilon'_g \end{array}\right]$$

Here $\epsilon, \epsilon' \in \mathbb{R}^g$ and a new function is defined for each $z \in \mathbb{C}^g$ by :

$$\begin{aligned} x\theta\left[\begin{array}{c} \epsilon \\ \epsilon' \end{array}\right](z, \tau) &= \sum_{N \in \mathbb{Z}^g} \exp\left(\pi i({}^t(N + \epsilon/2)\tau(N + \epsilon/2) + 2{}^t(N + \epsilon/2)(z + \epsilon'/2))\right) \\ &= \exp(2\pi i[\tfrac{1}{8}{}^t\epsilon\tau\epsilon + \tfrac{1}{2}{}^t\epsilon z + \tfrac{1}{4}{}^t\epsilon\epsilon'])\theta(z + I\tfrac{\epsilon'}{2} + \tau\tfrac{\epsilon}{2}, \tau) \end{aligned}$$

The vector of Riemann constants is given here by

$$K_{e_1} = \frac{\epsilon'}{2} + \tau\frac{\epsilon}{2}, \epsilon, \epsilon' \in \mathbb{Z}^g.$$

A fundamental monic polynomial $h(z) = h_0 + h_1 z + ... + h_{g-1}z^{g-1} + z^g$ of degree $g = p - 1$ is uniquely determined by the conditions on the periods:

$$\int_{e_{2j}}^{e_{2j+1}} h(z)|f(z)|^{-1/2}\, dz = 0, \qquad j = 1, ..., g.$$

$h(z)$ is a polynomial of real coefficients and of real zeros $z_k,\ k = 1, ..., g$. These zeros are the critical points of the Green's function and play a central role in our problem. They will be called *the equilibrium points*. Each zero z_k is in the gap $[e_{2k}, e_{2k+1}],\ k = 1, ..., p-1$. More precisely, the positions of the zeros z_k are intimately related to some questions on Jacobi Forms. We will see a specific example in section 4. Returning to the Green's function, we note the following useful result:

Proposition 1 *The capacity and the Green's function are connected to the polynomial* $h(z)$ *and to the Riemann theta function by the following formulas:*

$$
\begin{aligned}
C(E) &= \exp\left(\int_{-\infty}^{e_1}\left(\frac{h(\zeta)}{f(\zeta)^{1/2}} - \frac{1}{\zeta-(e_1+1)}\right)d\zeta\right)\\
G(z) &= \log\{z-(e_1+1)\} + \int_{e_1}^{z}\left(\frac{h(\zeta)}{f(\zeta)^{1/2}} - \frac{1}{\zeta-(e_1+1)}\right)d\zeta\\
G(z) &= \Re\Omega(z)
\end{aligned}
$$

where

$$
\Omega(z) = -\log\frac{\theta(\int_{e_1}^{z}\zeta - \int_{e_1}^{\infty_1}\zeta + K_{e_1},\tau)}{\theta(\int_{e_1}^{z}\zeta - \int_{e_1}^{\infty_2}\zeta + K_{e_1},\tau)} \qquad (1)
$$

In the case of 3 intervals where the curve M has genus $g = 2$, we use the Rosenhain relations to arrange the formula (1) of the previous proposition. In fact, if $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ stands for $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau)$, we obtain:

Theorem 1 *The capacity of* $K = [e_1,e_2]\cup[e_3,e_4]\cup[e_5,e_6]$ *of the real axis is given by:*

$$
C(E) = \frac{1}{2}\frac{\sqrt{(e_3-e_1)(e_4-e_2)(e_5-e_2)(e_6-e_2)}}{e_2-e_1}\times
$$

$$
\left|\frac{\theta\begin{bmatrix}1&1\\1&1\end{bmatrix}\theta\begin{bmatrix}0&0\\1&1\end{bmatrix}\theta\begin{bmatrix}1&0\\0&1\end{bmatrix}\theta\begin{bmatrix}0&0\\0&1\end{bmatrix}}{\theta\begin{bmatrix}1&1\\1&0\end{bmatrix}\left(2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}du, 2\int_{-\infty}^{\frac{e_3-e_1}{e_3-e_2}}dv\right)\theta\begin{bmatrix}0&0\\1&0\end{bmatrix}\theta\begin{bmatrix}1&0\\0&0\end{bmatrix}\theta\begin{bmatrix}0&0\\0&0\end{bmatrix}}\right|
$$

3 Difference Operators with periodic coefficients

There is another approach to determine the capacity of general a compact set K. The n'th Tchebyshev polynomial of K is the unique monic n'th polynomial $T_n(z)$ with a maximum modulus on K as small as possible. If we define $M_n = \max|T_n(z)|,\ z\in K$, then:

$$
\lim_{n\to\infty} M_n^{1/n} = C(K)
$$

It is natural to ask when one Tchebyshev polynomial suffices to give the value of $C(K)$.The answer is in almost all the cases if K is an union of closed intervals of the real axis. We will consider here only a typical circumstance. A second order symmetric *difference operator* C*, periodic of period* N, with complex coefficients is an operator acting on sequences by:

$$
(Cf)_k = b_{k-1}f_{k-1} + a_kf_k + b_kf_{k-1}
$$

with $a_{k+N} = a_k$, $b_{k+N} = b_k$. We will assume that $\prod_{0\le i\le N-1} b_i$ is different from zero. Our basic result is the following:

Theorem 2 *The spectrum $\sigma(C)$ of the operator C is compact and its capacity is*

$$\sigma(C) \;=\; \sqrt[N]{\prod_{i=0}^{N-1} |b_i|}.$$

It is also possible to give a family of Tchebyshev polynomials of $\sigma(C)$ in terms of $a_k, b_k;\ 0 \leq k \leq N-1$, and a similar formula for $C(K)$ is available in the almost periodic case. The inverse problem can also be solved: to a compact set $K = [e_1, e_2] \cup [e_3, e_4] \cup \ldots \cup [e_{2k-1}, e_{2k}]$ with $e_i < e_j$ if $i < j$, we can associate with the *equilibrium points* $z_1, \ldots\ z_{p-1}$ a second order difference operator C with almost periodic coefficients such that the spectrum $\sigma(C)$ is exactly K. The operator C is periodic if and only if the following conditions hold:

$$\int_{e_{2k}}^{e_{2k+1}} \sqrt{Q(x)}dx = 0, \qquad 1 \leq k \leq p-1,$$

$$\int_{e_{2k-1}}^{e_{2k}} \sqrt{Q(x)}dx = \pm r_k \frac{i\pi}{r}, \qquad 1 \leq k \leq p,$$

where :

$$Q(z) = \frac{(z-z_1)^2 \cdots (z-z_{p-1})^2}{(z-e_1)(z-e_2)\cdots(z-e_{2p-1})(z-e_{2p})}.$$

4 Motion of the equilibrium point

We would like to point out that some results of Eichler and Zagier on Jacobi forms can be used to give a complete parametrization of the equilibrium point at least in the case of two intervals. For this purpose, let us take $-1 < \alpha < \beta < 1$ and map conformally the complement of the compact set $K = [-1, \alpha] \cup [\beta, 1]$ upon the annulus $A_{1,r} = \{r < |v| < 1\}$ where $r = e^{-\pi K/K'}$ and:

$$\begin{aligned}
k^2 &= \frac{2(\beta-\alpha)}{(1+\beta)(1-\alpha)} \qquad \in]0,1[, \qquad k'^2 = 1 - k^2 \\
K &= \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}} \\
M &= \int_0^{\sqrt{\frac{1-\alpha}{2}}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{K'\log s}{\pi} \\
\tau &= iK'/K, \qquad q = e^{i\pi\tau}, \qquad r = e^{-i\pi/\tau}, \qquad u = \frac{K'\log v}{\pi}
\end{aligned}$$

Explicitly, the Green's function of $A_{1,r}$ is given by the classical formula:

$$\begin{aligned} G(v,s) &= -\Re \log sv^{-lns/lnr} \frac{1-v/s}{1-vs} \frac{\prod_{n=1}^{\infty}(1-r^{2n}v/s)(1-r^{2n}s/v)}{\prod_{n=1}^{\infty}(1-r^{2n}vs)(1-r^{2n}/vs)} \\ &= -\Re \log \frac{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\frac{u-M}{2K}, \tau)}{\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\frac{u+M}{2K}, \tau)} \end{aligned}$$

It has been pointed out by A.Maria and others that the critical point or the unique equilibrium point of G(v,s) (for fixed pole s) is on the same diameter as the pole, and two values a, b exist, $r < a < b < 1$, $ab = r^2$ such that $a < |v(s)| < b$ for each s in $A_{1,r}$. Our exposition reveals a link between the geometrical mean $\sqrt{r} = ab$ and some properties of the Bergman metric and kernel of the annulus. By using a result of Eichler and Zagier, we give a complete parametrization of the unique equilibrium point, the unique zero of the Bergman kernel of the annulus and of the values a and b. All these statements are transformed into equations of the form $\wp(z,\tau) = \varphi(\tau)$ where $\wp$ is the Weierstrass $\wp$-function and φ is a meromorphic function in τ [FS3]. The similar problems in the case of several intervals are still under investigation.

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Exceptional Hecke operators for the groups $\Gamma_0(N)$ with primitive character

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In the paper [1] we studied embedded eigenvalues of automorphic Laplacians $A(\bar\Gamma_0(N),\chi)$, for Hecke groups with primitive character χ and developed the corresponding Hecke theory for Maass cusp forms. We proved that $A(\bar\Gamma_0(N),\chi)$ with the Hecke operators $T(p)$, $p\nmid N$, p prime, determined common eigenfunctions uniquely (multiplicity one theorem). The Hecke operators $U(q)$, $q|N$, q prime, were proved to be unitary with eigenvalues ±1, and the continuous spectrum equaled the unit circle. From this follows the analogue of Selberg's small eigenvalue conjecture for the exceptional Hecke operators.

I will show here briefly how to prove these results. For more details and references see [1].

Let f be a continuous $(\bar\Gamma_0(N),\chi)$ automorphic function. Then the Hecke operators are defined by

$$T(n)f(z)=\frac{1}{\sqrt{n}}\sum_{ad=n}\chi(a)\sum_{b \bmod d} f\left(\frac{az+b}{d}\right) \tag{1}$$

and $T(n)f(z)$ is again a continuous $(\bar\Gamma_0(N),\chi)$ automorphic function. From (1) follows the basic relation

$$T(m)T(n)=\sum_{d|(m,n)}\chi(d)T(mn/d^2). \tag{2}$$

The most fundamental are the Hecke operators $T(p)$ which correspond to primes. Here we first have to distinguish two cases, 1) $p\nmid N$, 2) $p|N$. For convenience we introduce the notation $U(q)=T(q)$ for $q|N$, while $T(p)$ is reserved for $p\nmid N$. From (1) follows

$$T(p)f(z)=\frac{1}{\sqrt{p}}\chi(p)f(pz)+\frac{1}{\sqrt{p}}\sum_{b \bmod d} f\left(\frac{z+b}{p}\right), p\nmid N$$

$$U(q)f(z)=\frac{1}{\sqrt{q}}\sum_{b \bmod q} f\left(\frac{z+b}{q}\right),\ q|N,$$

$\chi(q)=0$.

All the operators $T(p)$, $U(q)$ are bounded in the Hilbert space of automorphic functions $\mathcal{H}(\bar\Gamma_0(N))$, also they map the subspace of cusp forms $\mathcal{H}_0\subset\mathcal{H}$ into itself. The operators

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$T(p)$ are $\chi(p)$-hermitian in $\mathcal{H}$:

$$\begin{cases} \langle T(p)f, q\rangle = \chi(p)\,\langle f, T(p)g\rangle \text{ or} \\ T(p)^* = \chi(p)T(p). \end{cases} \tag{3}$$

We introduce next two involutions, $Kf(z) = \overline{f(z)}$ is the complex conjugation, $H_N f(z) = f(-1/Nz)$. We have the following properties

$$K : \mathcal{H} \to \mathcal{H},\ H_N : \mathcal{H} \to \mathcal{H},\ KH_N = H_N K$$

$$KA(\bar{\Gamma}_0(N); \chi) = A(\bar{\Gamma}_0(N), \chi)K$$

$$KT(p) = T(p)K,\ U(q)K = KU(q).$$

Less trivial facts are

$$T^*(p) = H_N T(p) H_N,\ U^*(q) = H_N U(q) H_N \tag{4}$$

where T^* and U^* are the adjoint operators of T and U in $\mathcal{H}$ respectively. From (3), (4) follows

$$H_N T(p) = \chi(p) T(p) H_N,\ p \nmid N. \tag{5}$$

All Hecke operators have only point spectrum in the space of cusp-forms $\mathcal{H}_0(\bar{\Gamma}_0(N), \chi)$. We want to find the common basis of eigenfunctions for $A(\Gamma_0(N), \chi)$ and all Hecke operators $T(p)$, $U(q)$ in this space. Actually it is possible, because we consider primitive characters χ, which make all cusp forms new. The existence of the above-mentioned common basis follows from the important theorem (see [1]):

Theorem 1. Each Hecke exceptional operator $U(q)$, $q|N$, is a unitary operator in the Hilbert space $\mathcal{H}(\Gamma_0(N), \chi))$, $U(q)U^*(q) = U_q^* U_q = I$, where I is the identity operator in $\mathcal{H}(\bar{\Gamma}_0(N))$.

The idea of the proof is the following. We consider the more difficult case when $q|N$, but $q^2 \nmid N$ (that is also the general case, see [1]). We have to prove the equality

$$U(q) H_N U(q) H_N = I.$$

We have

$$\begin{aligned} U(q)H_N U(q) H_N f(z) &= \frac{1}{q} \sum_{b \bmod q} \sum_{b' \bmod q} f\left(\frac{z + b'}{(-N/q)bz + 1 - (N/q)bb'} \right) \\ &= f(z) + \frac{1}{q} \sum_{\substack{b \bmod q \\ b \neq 0}} \sum_{b' \bmod q} f\left(\frac{z + b'}{-(N/q)bz + 1 - (N/q)bb'} \right). \end{aligned}$$

We want to prove that the double sum on the right hand side is equal to zero. Then we get that for each pair b, b' mod q, $b \neq 0$, there exists a unique matrix depending on α' mod q

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ N_\gamma & \delta \end{pmatrix} &\in \Gamma_0(N), \text{ such that } \begin{pmatrix} 1, & b' \\ -(N/q)b, & 1 - (N/q)bb' \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ N_\gamma & \delta \end{pmatrix} \begin{pmatrix} 1, & a' \\ -N/q, & 1 - a'N/q \end{pmatrix} \end{aligned}$$

and $\delta \equiv b$ mod q, $\delta \equiv 1$ mod N/q. that means $\chi(\delta) = \chi_q(\delta) \cdot \chi_{N/q}(\delta) = \chi_q(b)$, where $\chi_q(\delta)$ is a part of the character symbol $\left(\frac{N}{\delta}\right)$, which corresponds to the period q (see [1]). Then we get that the double sum considered is equal to zero, because of the multiple

$$\sum_{b \text{ mod } q} \chi_q(b) = 0$$

and that proves the first part of the theorem. The proof of the identity $U^*(q)U(q) = I$ is similar.

From Theorem 1 follows that all operators $U^*(q)$ also commute with all Hecke operators and $A(\bar{\Gamma}_0(N), \chi)$ and that is the reason why there exists the common basis of eigenfunctions for all these operators in $\mathcal{H}_0(\bar{\Gamma}_0(N); \chi)$. In fact, it is possible to prove a much stronger result about the existence of the common basis of eigenfunctions, the so-called "multiplicity one" theorem. Unitarity of $U(q)$ does not follow from this theorem, however. This theorem is about the following. We take first the common basis of all eigenfunctions $v_j(z)$ for all $T(n)$, $(n, N) = 1$, and $A(\bar{\Gamma}_0(N), \chi)$ in the space $\mathcal{H}_0(\bar{\Gamma}_0(N), \chi)$ of cusp forms. We introduce

$$T'(n) = iT(n) \text{ if } \chi(n) = -1 \text{ and } T'(n) = T(n) \text{ if } \chi(n) = 1. \tag{6}$$

All $T'(n)$ are selfadjoint operators in $\mathcal{H}$. For $v_j(z)$ we have the Fourier series decomposition (see [1])

$$v_j(z) = \sum_{n \neq 0} \delta_j(n) \sqrt{y} K_{s_j - 1/2}(2\pi |n| y) e^{2\pi i n x} \tag{7}$$

$z = x + iy$, $\delta_j(n) \in \mathbb{C}$, $K_s(y)$ is the modified Bessel function. We have $A(\bar{\Gamma}_0(N), \chi)v_j = \lambda_j v_j$, $\lambda_j = s_j(1 - s_j)$ and

$$T(n)v_j(z) = \Delta_j(n)v_j(z),\ (n, N) = 1.$$

We have also if $\delta_j(1) = 0$, then $\delta_j(n) = 0$ for all n, $(N, n) = 1$. If $\delta_j(1) \neq 0$ we obtain for all n, $(n, N) = 1$

$$\Delta_j(n) = \frac{\delta_j(n)}{\delta_j(1)}.$$

We can always assume $\delta_j(n) = \delta_j(-n)$ or $\delta_j(-n) = -\delta_j(n)$ and that means the Fourier coefficients $\delta_j(n)$ with negative numbers n are determined by $\delta_j(n)$ with positive numbers n.

We assume that $\delta_j(1) = 0$. From that follows $\delta_j(n) = 0$, $(n, N) = 1$. The series (7) can be written as a sum of terms

$$v_j(z) = \sum_{q|N} w_{jq}(z). \tag{8}$$

Each w_{jq} is associated with a subgroup of $\Gamma_0(N)$ with character χ and level q, where the numbers q are mutually prime. Then since the whole sum (8) belongs to $(\Gamma_0(N), \chi)$ it follows that each $w_{jq} \in (\Gamma_0(N), \chi)$. From the structure of w_{jq} as a Fourier series similar to (7) follows that each w_{jq} belongs to some overgroup of $\bar{\Gamma}_0(N)$ with trivial extension of χ. Since the character χ is primitive it is only possible if each $\omega_{jq} = 0$. From that follows the multiplicity one theorem.

Theorem 2. 1) There exists a unique common basis of eigenfunctions for all operators $A(\Gamma_0(N);\chi)$, $T(n)$, $T(n)^*$, $n \geq 1$ in the space of cusp forms $\mathcal{H}_0(\bar{\Gamma}_0(N),\chi)$.

2) Each eigenfunction $v_j(z)$ of this basis with normalization $\delta_j(1) = 1$ is uniquely determined by the eigenvalues λ_j, $\Delta_j(n)$, $(n, N) = 1$.

3) We have also

$$U(q)v_j(z) = \delta_j(q)v_j(z),\ U^*(q)v_j(z) = \overline{\delta_j(q)}v_j(z)$$

and

4)

$$\delta_j(n)\delta_j(m) = \sum_{d|(m,n)} \chi(d)\delta_j(mn/d^2).$$

On the basis of Theorems 1, 2 we can prove

Theorem 3. For any $q|N$ we have $\delta_j(q) = \pm 1$.

For the proof we consider the involution $H_N K$. We have

$$T(p)H_N K v_j = \chi(p)KH_N T(p)v_j = \chi(p)\bar{\Delta}_j(p)H_N K v_j = \Delta_j(p)H_N K v_j.$$

From Theorem 2 follows that $H_N K v_j = \nu_j v_j$ with $\nu_j \in \mathbb{C}$.. Since $(H_N K)^2 = 1$ we have $\nu_j = \pm 1$. We obtain $H_N K v_j = \pm v_j$. From Theorem 1 follows

$$H_N U_{(q)} H_N U(q) = I$$

is equivalent to $H_N K U(q) H_N K U(q) = K \cdot K = I$, and we get the claim

$$\Delta_j^2(q) = \delta_j^2(q) = 1.$$

The Selberg small eigenvalue conjecture for $A(\bar{\Gamma}_0(N);\chi)$ says that all eigenvalues are embedded in the continuous spectrum $[1/4, \infty)$. It is not difficult to see that for $q|N$ the continuous spectrum of $U(q)$ is the whole unit circle. Since the only eigenvalues are ± 1, the analogue of Selberg's small eigenvalue conjecture holds true for the exceptional Hecke operators.

For regular Hecke operator $T(p)$ $(p, N) = 1$ the continuous spectrum is the interval $[-1, 1]$ if $\chi(p) = 1$, and is the interval of imaginary axis $[-i, i]$ if $\chi(p) = -1$. But it is an open question to prove that all eigenvalues of $T(p)$ are embedded in the corresponding continuous spectrum, the part of general Ramanujan's conjecture.

References

[1] E. Balslev, A. Venkov, *Spectral theory of Laplacians for Hecke groups with primitive character*, Research Report No. **41**, 1999, Centre for Mathematical Physics and Stochastics, University of Aarhus, Denmark.

Hypergeometric functions, special points on Shimura varieties and transcendence

Gisbert Wüstholz*

Abstract

In the talk we shall explain how algebraic values of hypergeometric functions are related to special points on rational curves in Shimura varieties and a conjecture of Andre and Oort. The basic ingredient is the analytic subgroup theorem which we also shall discuss briefly.

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Programme

FRIDAY

9.30-10.30 Ahmed Sebbar (Bordeaux): *Capacity, Jacobi forms and geodesics of the Bergman metric*

Coffee

10.45-11.45 Martin Huxley (Cardiff): *Integer points in plane regions and exponential sums*

12.00-13.30 Lunch (free for registered participants)

13.30-14.30 Alexei Venkov (Aarhus): *Exceptional Hecke operators for the groups* $\Gamma_0(N)$ *with primitive character*

Coffee

15.00-16.00 Gisbert Wüstholz (Zürich): *Hypergeometric functions, special points on Shimura varieties and transcendence*

SATURDAY

9.30-10.30 Uffe Haagerup (Odense): *Random matrices and the distribution of zeros of the Riemann zeta function — a survey of results of Odlyzko*

Coffee

10.45-11.45 Eric Opdam (Amsterdam): *Spectral decomposition of Iwahori-Hecke algebras*

12.00-14.00 Lunch (free for registered participants)

14.00-15.00 Dieter Mayer (Clausthal): *Transfer operators and period functions for subgroups of the modular group*

Coffee

15.30-16.30 Andreas Knauf (Erlangen): *Number theory and statistical mechanics*

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