Workshop on

Stochastics and Quantum Physics

October 21-26, 1999

Centre for Mathematical Physics and Stochastics — MaPhySto

University of Aarhus

1 Introduction

The Workshop focused on some of the areas where concepts and techniques from Stochastics (i.e. Probability and Mathematical Statistics) are, or seem likely soon to become, of real quantum physical importance.

By bringing together leading physicists and mathematicians, having an active interest in the themes of the Workshop, it was sought to foster fruitful discussions and collaboration on the role and use of Stochastics in Quantum Physics.

In this leaflet we have gathered the (extended) abstracts of the talks given. We thank all contributors for taking upon them the extra work of writing these extended abstracts. We hope that this booklet may be of some use to mathematicians as well as physicists working in the area of interplay between Quantum Physics and Stochastics.

At the end of the booklet, the schedule of the workshop and the list of participants is reproduced.

We wish to thank all participants — the speakers in particular — for contributing to the workshop.

Ole E. Barndorff-Nielsen and Klaus Mølmer.

Contents

2 (Extended) Abstracts of Talks

The abstracts/papers are ordered alphabetically after the last name of the author who presented the work. *

Luigi Accardi

The stochastic limit of quantum theory and the dilation problem†*.*

My task was to discuss the connections between the stochastic limit of quantum theory and the dilation problem.

The **dilation problem** is the following: *given a Markov semigroup to construct a Markov process whose canonically associated semi–group is the given one.*

The **stochastic limit** studies the following problem: *given a Hamiltonian system (classical or quantum), depending on a parameter λ, study the behaviour of this system in a time scale of order t/λ^2.* In the case of interest, the parameter λ is small and therefore the stochastic limit is related to long time scales (as scattering theory). On the other hand the smallness of the parameter λ reflects a weak interaction (as in perturbation theory). Thus the stochastic limit is a new asymptotic technique in the study of dynamical (Hamiltonian–for the moment) systems, combining together scattering and perturbation theory with the new ideas on quantum Markov processes, stochastic calculus, central limit theorems, which arose from quantum probability. This mixture brought a multiplicity of results both in physics [AcLuVo00] and in mathematics (cf. [AcLuVo97] for the notion of interacting Fock space and Fock module, [Ske99] for their relationships, [AcLuVo99] for the white noise approach to classical and quantum stochastic calculus).

Apparently the two topics are far apart, but there is a connection: the unique feature of the stochastic limit with respect to all the up to now known classical and quantum asymptotic methods, is that: *the dominating contribution (in a suitable topology) to the unitary evolution is still a unitary evolution.* Even more: it is a unitary Markovian cocycle and therefore, by a (now standard) technique introduced in [Ac78] it allows to construct a Markov process and a Markov semigroup. Thus, using the language of dilations we could say that *in the stochastic limit, the original Hamiltonian evolution converges to a unitary dilation of a Markov semi–group.*

A first natural question is: which Markov semigroups can be obtained with the stochastic limit technique? The answer is: all those semigroups of which a dilation can be constructed by means of classical or quantum stochastic calculus. One would like to have a

*The contribution of Richard D. Gill has appeared in the separate note *Asymptotics in Quantum Statistics*, Miscellanea No. 15, October 1999, Centre for Mathematical Physics and Stochastics, University of Aarhus.

†Unfortunately L. Accardi had to cancel his participation in the workshop. This is the manuscript for the talk he would have given.

definitive result of the type: *all the Markov semigroups admit a dilation obtained through the stochastic limit*. Up to now the obstruction to such a result was that an infinitesimal characterization of isometric flows, analogue to the Stone theorem for strongly continuous unitary groups or to the Hille–Yoshida theorem for C_0–semigroups, was absent. Recently such a characterization has been obtained in [AcKo99b] where it is proved that, if $\mathcal{B}$ is an arbitrary C^*–algebra, any completely positive flow on the space

$$\mathcal{B} \otimes \Gamma(L^2(\mathbb{R})) \quad (\Gamma^2(L^2(\mathbb{R}) \text{being the Boson Fock space on } L^2(\mathbb{R}))$$

is characterized by a single, completely positive (but non Markovian), semigroup on $M_2(\mathcal{B})$, the 2×2 matrices with coefficients in $\mathcal{B}$. This *extended semigroup* is strongly continuous if and only if the associated flow has this property. Therefore this theorem reduces the classification of strongly continuous completely positive flows to the known classification of C_0–semigroups, achieved via the Hille–Yoshida theorem. This result, combined with the stochastic golden rule, allowed to give what seems to be the first deduction of the flows associated to the Glauber–Kawasaki type dynamics, and in fact of all the dynamics used in the theory of *interacting particle systems*, from a Hamiltonian model, as well as a single unified proof of the existence of such flows in arbitrary dimensions [AcKo99a].

This result is new even when restricted to classical (commutative) flows which, as it is well known, include all the classical stochastic processes which satisfy a stochastic differential equation driven by Wiener or compound Poisson processes.

Another natural question is: from the point of view of physics, what is the relation between a dilation constructed by stochastic calculus and one obtained by the stochastic limit? The answer is simple: the same relation existing between a phenomenological model and a physical law, deduced by basic principles. In fact, given a Markov semigroup, one can a priori *invent* uncountably many unitary dilations of it: classical, Boson, Fermi, free, q–deformed, Fock, finite temperature, squeezing,... Moreover, even if we want to restrict ourselves to the Boson Fock case, still there are uncountably many choices which can be made and which are completely equivalent from a purely mathematical point of view. The usual constructions, in the physical literature, of dilations of Markov semigroups are based on the following steps:

i) one starts from the generator of a Markov semigroup (master equation)

ii) on the basis of more or less plausible physical arguments, one chooses, among the infinitely many unitary dilations of it, a definite one

iii) one describes some physical phenomena using this dilation. This procedure is not very satisfactory from the physical point of view for the following reasons:

i1) The master equation is itself an approximation, so it should be one of the final results of the construction of a model and not its starting point.

i2) The structure of the noise itself, driving the stochastic equation, used to construct the dilation, has a deep physical meaning which cannot be invented, but is one of the essential characteristics to be *deduced* from the physical model.

The stochastic limit bypasses these problems because it starts from the well established Hamiltonian models of quantum physics and to each of them it associates in a unique way a unitary (or isometric) flow and to this, by the quantum Feynman–Kac formula of [Ac78], a Markov semigroup. The flow is the limit of the Heisenberg evolution (in interaction representation) of the original Hamiltonian system and the Markov semigroup is the limit of the expectation of the flow for the reference state of the fast degrees of freedom of the system (reservoir, environment, field, gas,...). This means that the Heisenberg equation of motion converges to a (stochastic) Langevin equation whose expectation gives the master equation. It follows that all the parameters which enter in these equations have a microscopic interpretation and can be experimentally controlled.

In the class of physical systems to which the stochastic limit technique can be applied, one can distinguish 3 levels in increasing order of difficulty:

Level I corresponds to the standard open system scheme, i.e. a discrete spectrum system interacting with a continuous spectrum one. It should be noted that *all the models studied up to now in the physical literature concern this level*. The situation in the remaining two levels is too complex to be handled by plausibility arguments.

For the models in this class the **stochastic golden rule** gives to the physicists a simple recepee which allows to solve, with very few elementary calculations, the following problem: *given the Hamiltonian model, how to write the stochastic Schrödinger equation obtained by taking the stochastic limit?* Since this rule is extremely simple to apply and since all the master equations which are the (more or less implicit) starting point of the unitary dilations built up to now in the physical literature, presuppose an underlying Hamiltonian model, the stochastic limit procedure offers to the physicist the opportunity to replace the, up to now standard, scheme:

Hamiltonian system $\rightarrow$ master equation $\rightarrow$ dilation

by the stochastic limit scheme:

Hamiltonian system $\rightarrow$ dilation $\rightarrow$ master equation

which is much more satisfactory because now not only the master equation, but also the noise and the Langevin equation become uniquely determined by the original Hamiltonian system.

Level II has to do with the low density limit and is much more difficult than Level I because, while in case of Level I the stochastic golden rule gives the possibility to guess the correct stochastic equation by simple inspection of the first and second order terms of the iterated series (which, in the limit, give respectively the martingale and the drift term of the stochastic equation), in the case of Level II, the drift term of the stochastic equation receives contributions from each term of the iterated series and to single out these contributions and resum them into the 2–particle scattering operator is a subtle point.

Level III replaces the discrete–continuum interactions of the first two levels by continuum–continuum ones. Here dramatically new phenomena arise, the most important of which is

the **breaking of the commutation (or anticommutation) relations** and the subsequent replacement of the Fock space by the interacting Fock space and emergence of Fock modules. Another non trivial point is the **emergence of new statistics** based on non crossing diagrams rather than on the usual boson or fermion ones (in which all crossing diagrams are allowed). The mathematical interpretation of these diagrams in terms of *free independence* as well as their connection with the semi–circle law was discovered by Voiculescu. The stochastic limit gave rise to more sophisticated and more physically interesting notions of independence in which the role of the *Gaussian is played by different measures* whose explicit form is, in many cases, still unknown (even if all their momenta) can be written explicitly.

The absence of explicit formulae is one of the common features of nonlinear problems: the semicircle law corresponds to a linear problem (absence of interaction, in physical terms) and, as usual in linear problems, in this case all calculations can be made explicitly.

The fact that the non crossing diagrams give the dominating contribution to the quantum dynamics was first discovered, in the case of QED without dipole approximation, in the paper [AcLu92] and the fact that in the huge literature devoted to this topic (surely much larger in volume than that devoted to the last Fermat problem and involving people such as Dirac, Fermi, Heisenberg, Landau,...) such an important phenomenon was not even conjectured, is an indication of how hidden it was. In fact the original proof was rather elaborated but a much simpler and intuitive one was given later in [AcKoVo98] where the following intuitive picture was derived: before the stochastic limit (finite coupling constant λ) by effect of the nonlinearity the time rescaled fields obey a q–commutation relation with the constant q depending both on time and on λ. In the stochastic limit ($\lambda \to 0$) this quantity tends to zero (this give an intuitive explanation of why only the non crossing diagrams survive). The explanation of the Hilbert module structure and the explicit form of the (new type of) quantum stochastic equation requires more work and a good reference for this is Skeide's paper [Ske99].

From the paper [AcLu92] several new mathematical notions emerged: the notion of full Fock module (in a particular case: the general case was dealt with one year later by Pimsner), the notion of interacting Fock space, of stochastic integration on Hilbert module (mathematically developed by Lu and later by Speicher and Skeide), the white noise approach to stochastic calculus on Boltzmannian interacting Fock space (which includes the free case). In particular the notion of interacting Fock space turned out to have a multiplicity of unexpected connections with apparently totally unrelated fields of mathematics such as orthogonal polymomials, wavelets, solvable models in statistical mechanics, new forms of independence and of central limit theorems,... Among the new physical implications of this paper we mention the power decay law in the polaron model [AcKoVo99c].

In conclusion Level III of the stochastic limit provides a clear illustration, in a multiplicity of fundamental physical models, of the basic philosophy of this theory namely: the physically interesting dilations of Markovian semigroups should be deduced from the basic Hamiltonian equations. The results emerged from the realization of this program show that the wealth and beauty of the structures hidden in the basic physical laws by far exceeds the fantasy displayed in the construction of phenomenological models.

Bibliography

[AcLuVo00] Accardi L., Y.G. Lu, I. Volovich: Quantum Theory and its Stochastic Limit. to be published in the series Texts and monographs in Physics, Springer Verlag (2000); Japanese translation, Tokyo–Springer (2000)

[AcLuVo97c] Accardi L., Lu Y.G., I. Volovich The QED Hilbert module and Interacting Fock spaces. Publications of IIAS (Kyoto) (1997)

[AcLuVo99] Accardi, L., Lu, Y.G., Volovich, I.V.: A white noise approach to classical and quantum stochastic calculus, Preprint of Centro Vito Volterra N.375, Rome, July 1999, World Scientific (2000)

[Ac78] Accardi L.: On the quantum Feynmann-Kac formula. Rendiconti del seminario Matematico e Fisico, Milano 48 (1978) 135-180

[AcLu92] Accardi L., Lu Y.G.: The Wigner Semi–circle Law in Quantum Electro Dynamics. Commun. Math. Phys., **180** (1996), 605–632. Volterra preprint N.126 (1992)

[AcKo99a] L. Accardi, S.V. Kozyrev: The stochastic limit of quantum spin system. invited talk at the 3rd Tohwa International Meeting on Statistical Physics, Tohwa University, Fukuoka, Japan, November 8-11, 1999. to appear in the proceedings published by American Physical Society

[AcKo99b] L. Accardi, S.V. Kozyrev: On the structure of Markov flows. to appear in: Chaos, Solitons and Fractals (2000)

[AcKoVo98] L. Accardi, S.V. Kozyrev and I.V. Volovich: Dynamical origins of q-deformations in QED and the stochastic limit Journal of Physics A, Math. Gen. 32 (1999) 3485–3495 q-alg/9807137

[AcKoVo99c] L. Accardi, S.V. Kozyrev and I.V. Volovich: Non-Exponential Decay for Polaron Model Phys. Letters A 260 (1999) 31–38

[Ske96] Skeide M.: *Hilbert modules in quantum electro dynamics and quantum probability.* Volterra preprint N. 257 (1996). Comm. Math. Phys. (1998)

Quantum stochastic models of two-level atoms and electromagnetic cross sections.

A. Barchielli

Dipartimento di Matematica, Politecnico di Milano,

Piazza Leonardo da Vinci 32, I-20133 Milano, Italy

and *Istituto Nazionale di Fisica Nucleare, Sezione di Milano* – E-mail: barchielli@mate.polimi.it

Quantum stochastic processes

Quantum stochastic calculus (QSC) [1-4], a noncommutative analog of the classical Ito's stochastic calculus, revealed to be a powerful tool to construct mathematical models of quantum optical systems [3, 5-12] and to develop a theory of photon detection [13-16]. Just at the beginning of QSC, Hudson and Parthasarathy proposed a quantum stochastic Schrödinger equation for quantum open systems [1, 4]:

$$\mathrm{d}U(t) = \left\{ \sum_j R_j \,\mathrm{d}A_j^\dagger(t) + \sum_{i,j} (S_{ij} - \delta_{ij})\,\mathrm{d}\Lambda_{ij}(t) - \sum_{i,j} R_i^\dagger S_{ij}\,\mathrm{d}A_j(t) - \mathrm{i}K\,\mathrm{d}t \right\} U(t)\,. \quad (1)$$

The annihilation, creation and gauge (or number) processes $A_j(t)$, $A_j^\dagger(t)$, $\Lambda_{ij}(t)$ are the fundamental ingredients of QSC; they are Bose fields, acting on the symmetric Fock space $\mathcal{F} = \mathcal{F}(\mathcal{X})$ over the "one-particle space" $\mathcal{X} = \mathcal{Z} \otimes L^2(\mathbb{R}_+) \simeq L^2(\mathbb{R}_+; \mathcal{Z})$ ($\mathcal{Z}$ is a separable complex Hilbert space with a c.o.n.s. $\{e_j\}$). We denote by $e(f)$ a normalized coherent vector for the field: $A_j(t)e(f) = \int_0^t f_j(s)\mathrm{d}s\, e(f)$. Moreover, R_i, $i \geq 1$, S_{ij}, $i,j \geq 1$, K, H are bounded operators in $\mathcal{H}$ – another separable complex Hilbert space (the system space) – such that $K = H - \frac{\mathrm{i}}{2}\sum_j R_j^\dagger R_j$, $H^\dagger = H$, $\sum_i R_i^\dagger R_i$ is strongly convergent to a bounded operator, and $\sum_{i,j} S_{ij} \otimes |e_i\rangle\langle e_j| = S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{Z})$ (unitary operators in $\mathcal{H} \otimes \mathcal{Z}$). Then [4], there exists a unique unitary operator-valued adapted process $U(t)$ satisfying Eq. (1) with the initial condition $U(0) = \mathbb{1}$.

In usual applications the term containing the gauge process does not appear, i.e. $S_{ij} = \delta_{ij}$ is taken. In Refs. [17, 18], whose results I present here, we have studied the possibility of using the full Hudson-Parthasarathy equation as a phenomenological model for the simplest photoemissive source, namely a two-level atom stimulated by a laser. The points I want to discuss are:

(a) how to determine the system operators in Eq. (1) by means of physical considerations; here, a central role is played by a balance equation saying that the mean number of outgoing photons plus the mean number of photons stored in the atom is equal to the mean number of ingoing photons;

(b) how to obtain the electromagnetic cross sections and the atomic fluorescence spectrum by the theory of measurements continuous in time, via the heterodyne detection scheme [15, 16];

(c) how the new terms modify the cross sections and the spectrum of an atom stimulated by a monochromatic laser; in the usual case the dependence of the total cross section on the frequency of the stimulating laser can present only a Lorentzian shape, while in our case the full variety of Fano profiles can appear [19, 20]; for what concerns the spectrum the known triplet structure obtained by Mollow [21] is distorted by the presence of the new terms and made asymmetric.

A quantum stochastic model for a two-level atom

In order to describe a two-level atom, we take $\mathcal{H} = \mathbb{C}^2$. When no photon is injected into the system (initial state $\xi \otimes e(0)$, where $e(0)$ is the Fock vacuum and ξ is a generic state of the atom) it is natural to ask that the atom can emit at most one photon and that it exists a unique equilibrium state for the reduced dynamics of the atom. Under these conditions we prove that

$$H = \frac{1}{2}\,\omega_0\sigma_z\,, \qquad \omega_0 \in \mathbb{R}, \qquad R_j = \langle e_j|\alpha\rangle\,\sigma_-\,, \qquad \alpha \in \mathcal{Z}\,,\ \alpha \neq 0\,. \tag{2}$$

Now, let us denote by $N(t) = \sum_j \Lambda_{jj}(t)$ the observable "total number of photons entering the system up to time t" and take as initial state $\Psi(\xi, f) \in \mathcal{H}\otimes\mathcal{F}$ a generic state for the atom and a coherent vector for the field, i.e. $\Psi(\xi, f) = \xi \otimes e(f)$, $\xi \in \mathcal{H}$, $\|\xi\| = 1$, $f \in L^2(\mathbb{R}_+\,;\mathcal{Z})$. Then, the quantity

$$\langle N(t)\rangle_f = \langle U(t)\Psi(\xi, f)|N(t)\,U(t)\Psi(\xi, f)\rangle \tag{3}$$

represents the mean number of outgoing photons leaving the system in the time interval $[0, t]$, while

$$\langle N(t)\rangle_f^0 = \langle \Psi(\xi, f)|N(t)\,\Psi(\xi, f)\rangle = \int_0^t \|f(s)\|^2 \mathrm{d}s \tag{4}$$

gives the mean number of ingoing photons entering the system in the time interval $[0, t]$. Now we require a balance equation on the number of photons: the mean number of outgoing photons up to time t plus the mean number of photons stored in the atom must be equal to the mean number of ingoing photons, i.e. $\forall t$, $\forall\xi$, $\forall f$ we require the balance equation

$$\langle N(t)\rangle_f + \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}}\left\{\sigma_z\big[\rho(t) - \rho(0)\big]\right\} = \langle N(t)\rangle_f^0, \tag{5}$$

where $\rho(t)$ is the reduced density matrix for the atom. We prove that this implies

$$S = P_+ \otimes S^+ + P_- \otimes S^-\,, \qquad S^\pm \in \mathcal{U}(\mathcal{Z})\,, \tag{6}$$

where $P_\pm = \frac{1}{2}(1 \pm \sigma_z)$ are the projectors on the excited and ground states.

For physical reasons, we take also $\omega_0 > 0$ and, in order to have an atom stimulated by a monochromatic coherent wave, we take

$$f(t) = \mathrm{e}^{-\mathrm{i}\omega t}1_{[0,T]}(t)\,\lambda\,, \qquad \lambda \in \mathcal{Z}\,, \quad \omega > 0\,; \tag{7}$$

$1_{[0,T]}(t)$ is the indicator function of the set $[0,T]$, so that $f(t)$ represents a monochromatic wave for $T \to +\infty$.

Finally, we particularize our model to the case of a spherically symmetric atom stimulated by a well collimated laser. If we consider only not polarized light, the one-particle space $\mathcal{Z}$ has to contain only the degrees of freedom linked to the direction of propagation [22], so that we can take $\mathcal{Z} = L^2\big(\Upsilon,\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\big)$, $\Upsilon = \{0 \le \theta \le \pi,\ 0 \le \phi < 2\pi\}$. Then, in order to describe a laser beam propagating along the direction $\theta = 0$, we have to take

$$\lambda = \Omega \|\alpha\|^2\, \mathrm{e}^{\mathrm{i}\delta} \lambda_0\,, \qquad \Omega > 0\,, \qquad \delta \in [0, 2\pi)\,, \qquad \lambda_0(\theta,\phi) = \frac{1_{[0,\Delta\theta]}(\theta)}{\Delta\theta \sqrt{2\pi(1-\cos\Delta\theta)}}\,; \quad (8)$$

in all the physical quantities the limit $\Delta\theta \downarrow 0$ will be taken. Moreover, by denoting by $Y_{lm}(\theta,\phi)$ the spherical harmonic functions, the spherical symmetry of the atom requires

$$\alpha(\theta,\phi) = \|\alpha\|\, Y_{00}(\theta,\phi) = \|\alpha\|/\sqrt{4\pi}\,, \qquad S^{\pm} = \sum_{lm} \mathrm{e}^{2\mathrm{i}\delta_l^{\pm}}\, |Y_{lm}\rangle\langle Y_{lm}|\,, \qquad (9)$$

where the quantities δ_l^+ and δ_l^- are phase shifts.

Heterodyne detection

The best way to obtain the spectrum of our stimulated atom is by means of the *balanced heterodyne detection* scheme; the output current of the detector is represented by the operator [15, 23]

$$I(\nu, h; t) = \int_0^t F(t-s) j(\nu, h; \mathrm{d}s)\,, \qquad (10)$$

where $F(t)$ is the detector response function, say $F(t) = k_1 \sqrt{\frac{\gamma}{4\pi}}\, \exp\big(-\frac{\gamma}{2}\, t\big)$, $\gamma > 0$, $k_1 \neq 0$ has the dimensions of a current, j is essentially a field quadrature

$$j(\nu, h; \mathrm{d}s) = \overline{q}\, \mathrm{e}^{\mathrm{i}\nu s}\, \mathrm{d}A_h(s) + \mathrm{h.c.}\,, \qquad \mathrm{d}A_h(t) = \sum_j \langle h|e_j\rangle\, \mathrm{d}A_j(t)\,, \qquad (11)$$

q is a phase factor, $q \in \mathbb{C}$, $|q| = 1$, ν is the frequency of the local oscillator and $h \in \mathcal{Z}$, $\|h\| = 1$; h contains information on the localization of the detector, say

$$h(\theta', \phi') = \frac{1}{\sqrt{|\Delta\Upsilon|}}\, 1_{\Delta\Upsilon}(\theta', \phi')\,, \qquad (12)$$

where $\Delta\Upsilon$ is a small solid angle around (θ,ϕ) (in all physical quantities the limit $\Delta\Upsilon \downarrow \{(\theta,\phi)\}$ is understood). From the canonical commutation relations for the fields one has that $I(\nu_1, h_1; t_1)$ and $I(\nu_2, h_2; t_2)$ are compatible observables for any choice of the times either if $\nu_1 = \nu_2$ and $h_1 = h_2$ either if $\langle h_1|h_2\rangle = 0$. Under the same conditions also the j's commute. This means that the operators $I(\nu, h; t)$, $t \ge 0$, can be jointly diagonalized and the joint probability law obtained; in other terms, once the initial state is fixed, a (classical) stochastic process can be obtained from the continuously observed operators $I(\nu, h; t)$. All statistical properties of this process can be obtained by means of the technique of the characteristic functional [15] or by transforming the quantum

stochastic equations into classical ones [16] (see the notions of a posteriori state or conditional state, quantum filtering equations or quantum trajectories, ... [24, 25]). However, to compute the fluorescence spectrum and the cross sections, we do not need the full theory of continuous measurements, but only the second moments of $I(\nu, h; t)$. In the following for the quantum expectation of any operator B we shall use the notation $\langle B\rangle_T = \langle U(T)\Psi(\xi, f)|BU(T)\Psi(\xi, f)\rangle$.

In the long run the output mean power is given by

$$P(\nu, h) = \lim_{T\to+\infty} \frac{k_2}{T} \int_0^T \left\langle \big(I(\nu, h; t)\big)^2 \right\rangle_T \mathrm{d}t\,; \tag{13}$$

$k_2 > 0$ has the dimensions of a resistance, it is independent of ν, but it can depend on the other features of the detection apparatus. As a function of ν, $P(\nu, h)$ gives the *power spectrum* observed in the "channel h"; in the case of the choice (12) it is the spectrum observed around the direction (θ, ϕ). Next Proposition relates $P(\nu, h)$ to normal ordered quantum expectations of products of field operators and gives a sum rule which relates $P(\nu, h)$ to $\|\lambda\|^2$; let us note that $\hbar\omega\|\lambda\|^2$ is the total power of the input monochromatic state $f(t)$ (7). Moreover this Proposition identifies an elastic and an inelastic contribution to the power and reduces the computation of $P(\nu, h)$ to the solution of a master equation with Liouvillian (23). For the use of QSC in the computation of the spectrum of a two-level atom see also Ref. [26].

Proposition. *The mean power $P(\nu, h)$ can be expressed as*

$$P(\nu, h) = \frac{k}{4\pi} + \lim_{T\to+\infty} \frac{k}{2\pi T} \left\{ \left\langle \int_0^T \mathrm{d}A_h^\dagger(t) \int_0^t \mathrm{d}A_h(s)\, \mathrm{e}^{-\left(\frac{\gamma}{2}+\mathrm{i}\nu\right)(t-s)} \right\rangle_T + \text{c.c.} \right\}, \tag{14}$$

where $k = k_1^2 k_2$; Eq. (14) holds almost everywhere in ν. We have also

$$\int_{-\infty}^{+\infty} \left[P(\nu, h) - \frac{k}{4\pi} \right] \mathrm{d}\nu = \lim_{T\to+\infty} \frac{k}{T} \langle \Lambda_{hh}(T)\rangle_T\,, \tag{15}$$

where $\Lambda_{hh}(T) = \sum_{ij} \langle e_i|h\rangle \Lambda_{ij}(T) \langle h|e_j\rangle$; moreover, the following sum rule holds:

$$\sum_j \int_{-\infty}^{+\infty} \left[P(\nu, e_j) - \frac{k}{4\pi} \right] \mathrm{d}\nu = k\|\lambda\|^2\,. \tag{16}$$

The mean power can be decomposed as the sum of three positive contributions

$$P(\nu, h) = \frac{k}{4\pi} + P_{\text{el}}(\nu, h) + P_{\text{inel}}(\nu, h)\,, \tag{17}$$

where

$$P_{\text{el}}(\nu, h) = k\,|r(h)|^2\, \frac{1}{\pi}\, \frac{\gamma/2}{(\nu-\omega)^2 + \gamma^2/4}\,, \tag{18}$$

$$P_{\text{inel}}(\nu, h) = \frac{k}{2\pi} \int_0^{+\infty} \mathrm{d}t\, \exp\left[-\left(\frac{\gamma}{2} + \mathrm{i}(\nu-\omega) \right) t \right] \mathrm{Tr}\left\{ D(h)^\dagger \left(\mathrm{e}^{\mathcal{L}_\lambda t} \left[D(h)\rho_{\text{eq}}^\lambda \right] \right) \right\} + \text{c.c.}, \tag{19}$$

$$
\begin{aligned}
D(h) &= R(h) - r(h)\,, && (20)\\
r(h) &= \mathrm{Tr}\left\{R(h)\rho^{\lambda}_{\mathrm{eq}}\right\}, && (21)\\
R(h) &= \mathrm{e}^{-\mathrm{i}\beta}\langle h|\alpha\rangle\sigma_- + \langle h|S^+\lambda\rangle P_+ + \langle h|S^-\lambda\rangle P_-\,, \qquad \beta = \arg\left\{-\langle S^-\lambda|\alpha\rangle\right\}, && (22)\\
\mathcal{L}_\lambda[\rho] &= -\mathrm{i}[H_\lambda\,,\rho] + \frac{1}{2}\sum_j \left(\left[R(e_j)\rho\,,\,R(e_j)^\dagger\right] + \left[R(e_j)\,,\,\rho R(e_j)^\dagger\right]\right), && (23)\\
H_\lambda &= \frac{1}{2}\,(\omega_0-\omega)\sigma_z - \frac{1}{2}\,|\langle\alpha|S^-\lambda\rangle|\sigma_y\,; && (24)
\end{aligned}
$$

$\rho^{\lambda}_{\mathrm{eq}}$ is the equilibrium state for the master equation with Liouvillian (23).

Notice that in the decomposition (17) the term $k/(4\pi)$, independent of ν, is apparently a white noise contribution to the power; $P_{\mathrm{el}}(\nu,h)$ is the elastic contribution, as one sees from Eq. (18) which gives $P_{\mathrm{el}}(\nu,h)\propto\delta(\nu-\omega)$ for $\gamma\downarrow 0$; finally, $P_{\mathrm{inel}}(\nu,h)$ is the inelastic contribution (from Eq. (19) one can see that no delta term develops for $\gamma\downarrow 0$).

Cross sections and fluorescence spectrum

Let us consider now the case of the spherically symmetric atom, stimulated by a well collimated laser beam (8), (9). We also assume that the detector spans a small solid angle, so that h is given by Eq. (12) with $\Delta\Upsilon\downarrow\{(\theta,\phi)\}$, $|\Delta\Upsilon|\simeq\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. Moreover, we assume that the transmitted wave does not reach the detector, i.e. $\theta>0$, and so $\langle h|\lambda\rangle=0$. Then, we obtain the elastic and inelastic contributions to the power per unit of solid angle $\frac{1}{|\Delta\Upsilon|}P_{\mathrm{el}}(\nu,h)\simeq P_{\mathrm{el}}(\nu;\theta,\phi)$, $\frac{1}{|\Delta\Upsilon|}P_{\mathrm{inel}}(\nu,h)\simeq P_{\mathrm{inel}}(\nu;\theta,\phi)$. For the elastic and inelastic cross sections we shall have $\sigma_{\mathrm{el}}(\nu)\propto\int_0^\pi\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,P_{\mathrm{el}}(\nu;\theta,\phi)$, $\sigma_{\mathrm{inel}}(\nu)\propto\int_0^\pi\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\,P_{\mathrm{inel}}(\nu;\theta,\phi)$; moreover, we set $\sigma_{\mathrm{el}}=\int_{-\infty}^{+\infty}\sigma_{\mathrm{el}}(\nu)\,\mathrm{d}\nu$, $\sigma_{\mathrm{inel}}=\int_{-\infty}^{+\infty}\sigma_{\mathrm{inel}}(\nu)\,\mathrm{d}\nu$, $\sigma_{\mathrm{TOT}}=\sigma_{\mathrm{el}}+\sigma_{\mathrm{inel}}$. Let us stress that σ_{TOT} can be also obtained via the direct detection scheme [17, 18]. Finally, we can introduce the spectrum as

$$
\Sigma_{\mathrm{TOT}}(x)=\frac{\omega^2\|\alpha\|^2}{6\pi c^2}\left(\sigma_{\mathrm{el}}(\nu)+\sigma_{\mathrm{inel}}(\nu)\right), \qquad (25)
$$

where $x=(\nu-\omega)/\|\alpha\|^2$ is the reduced frequency ($\|\alpha\|^2$ is the natural line width) and we have taken the normalization $\int_{-\infty}^{+\infty}\Sigma_{\mathrm{TOT}}(x)\,\mathrm{d}x=\frac{\omega^2}{6\pi c^2}\sigma_{\mathrm{TOT}}$.

By solving the master equation and by using the proposition given before, the cross sections and the spectrum can be computed; however, the expressions are long and I refer to [18]. Here I limit myself to some comments and I give some plots obtained by choosing the parameters in such a way that the on resonance spectrum be distorted, but not too different from the Mollow one.

For what concerns the total cross section, according to the values of the various coefficients, different line shapes appear, which are known as Fano profiles (see Ref. [20] pp. 61–63); these shapes are typical of the interference among various channels. Some plots of $\frac{\omega^2}{6\pi c^2}\,\sigma_{\mathrm{TOT}}$ are given in Fig. 1 as functions of the reduced detuning $z=(\omega-\omega_0)/\|\alpha\|^2$; the same figure contains plots of elastic and inelastic cross sections. Let us recall that in the usual case σ_{TOT} has a Lorentzian shape. Whichever the line shape be, there is a strong variation of the cross section for ω around ω_0+ 'an intensity dependent shift', shift which has received various names in the literature; a very suggestive one is *lamp shift*, a name suggested by A. Kastler [27]. Let us stress that also the width of the resonance and the whole line shape are intensity dependent.

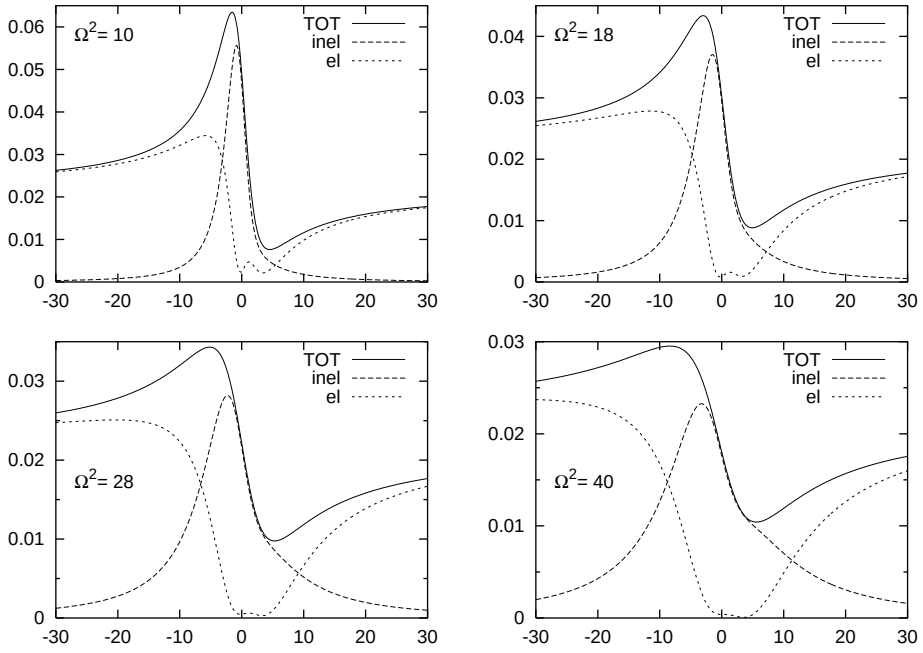

Figure 1: $\frac{\omega^2}{6\pi c^2} \times$ the cross sections as functions of the reduced detuning z for $\Omega^2 = 10, 18, 28, 40$.

For what concerns the spectrum, according to the values of the various parameters, a well resolved triplet structure can appear, but also single-maximum structures can be shown. With the choice of parameters of Fig. 1 and with an instrumental width $\gamma/\|\alpha\|^2 = 0.6$, the on resonance spectrum for $\Omega^2 = 10, 18, 28, 40$ is given in Fig. 2 (solid lines); the dashed lines give the Mollow spectrum for the same values of Ω^2 and γ (Ω is essentially the reduced Rabi frequency and it is proportional to the square root of the laser intensity). The parameters in Fig. 2 have been chosen in such a way that a triplet structure appears, not too different from the usual one, but with a well visible asymmetry in the frequency x. Experiments [28-32] confirm essentially the triplet structure; some asymmetry has been found, whose origin has been attributed to various causes.

Finally, in Fig. 3 we show some out of resonance spectra (reduced detunings $z = -4, -2, 3, 6$) for $\Omega^2 = 28$ and the other parameters as in Figs. 1 and 2 (solid lines); again, the dashed lines give the Mollow spectrum. Now, a strong difference from the usual case is shown, consistent with the strong asymmetry in z shown by the total and the elastic cross sections in Fig. 1.

References

[1] R.L. Hudson and K.R. Parthasarathy, Commun. Math. Phys. **93**, 301 (1984).

[2] C.W. Gardiner and M.J. Collet, Phys. Rev. A **31**, 3761 (1985).

[3] C.W. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).

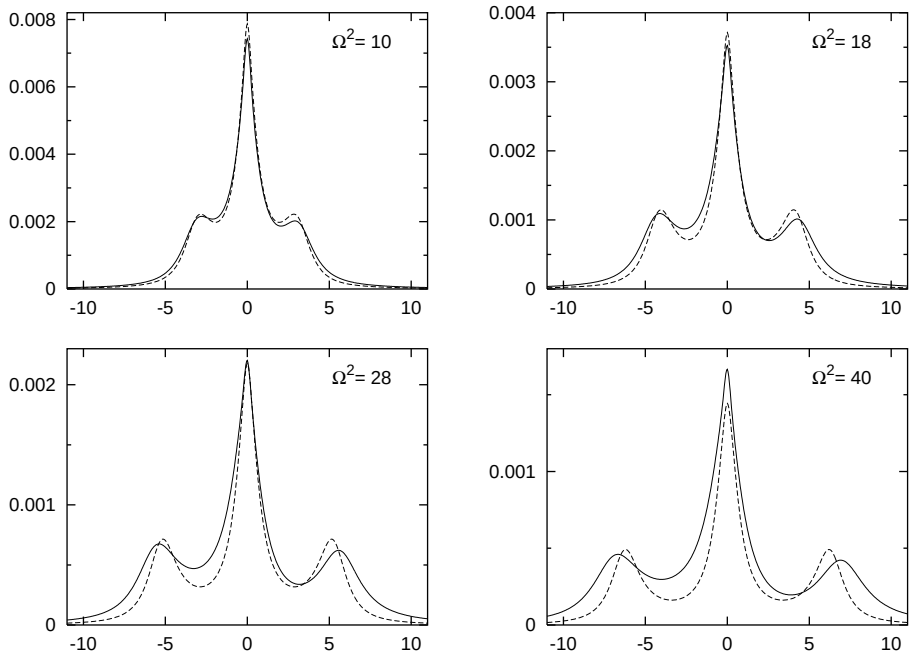

Figure 2: Total spectrum as a function of the frequency x for $z = 0$ and $\Omega^2 = 10, 18, 28, 40$; solid line: the same parameters as in Fig. 1; dashed line: the Mollow case.

[4] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Birkhäuser, Basel, 1992).

[5] C.W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).

[6] A. Barchielli, J. Phys. A: Math. Gen. **20**, 6341 (1987).

[7] T. Kennedy and D.F. Walls, Phys. Rev. A **37**, 152 (1988).

[8] P. Alsing, G.J. Milburn, and D.F. Walls, Phys. Rev. A **37**, 2970 (1988).

[9] A.S. Lane, M.D. Reid, and D.F. Walls, Phys. Rev. A **38**, 788 (1988).

[10] M.A. Marte, H. Ritsch, and D.F. Walls, Phys. Rev. A **38**, 3577 (1988).

[11] M.J. Collet and D.F. Walls, Phys. Rev. Lett. **61**, 2442 (1988).

[12] H.M. Wiseman and G.J. Milburn, Phys. Rev. A **49**, 4110 (1994).

[13] A. Barchielli and G. Lupieri, J. Math. Phys. **26**, 2222 (1985).

[14] A. Barchielli, Phys. Rev. A **34**, 1642 (1986).

[15] A. Barchielli, Quantum Opt. **2**, 423 (1990).

[16] A. Barchielli and A.M. Paganoni, Quantum Semiclass. Opt. **8**, 133 (1996).

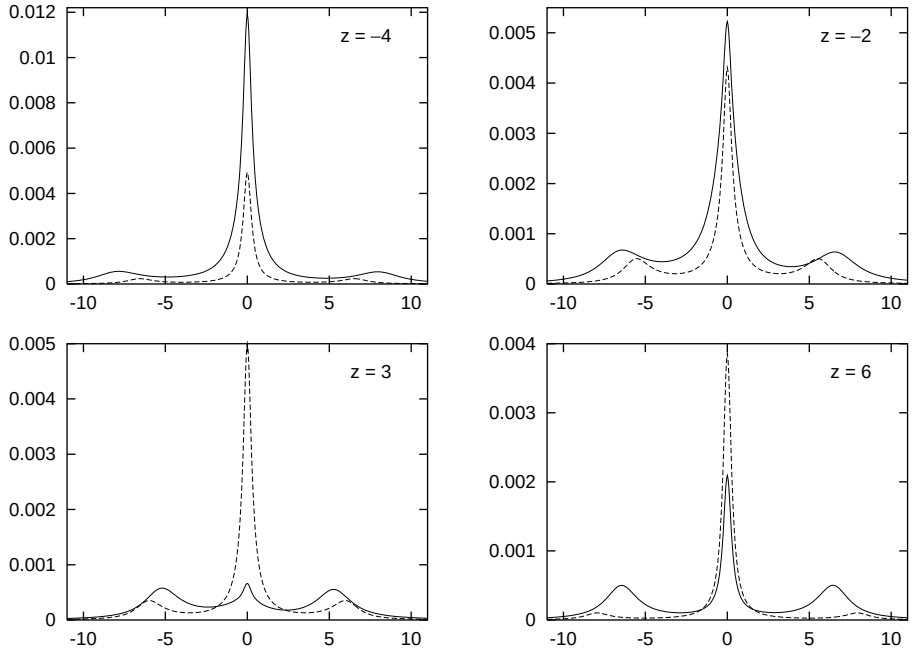

Figure 3: Total spectrum as a function of the frequency x for $\Omega^2 = 28$ and $z = -4, -2, 3, 6$; solid line: the same parameters as in Fig. 1; dashed line: the Mollow case.

[17] A. Barchielli and G. Lupieri, in R. Alicki, M. Bozejko, W.A. Majewski, *Quantum Probability*, Banach Center Publications, Vol. 43 (Polish Academy of Sciences, Institute of Mathematics, Warsawa, 1998), pp. 53–62.

[18] A. Barchielli and G. Lupieri, Quaderni del Dipartimento di Matematica, Politecnico di Milano, n. 356/P Marzo 1999 — quant-ph/9904065.

[19] U. Fano, Phys. Rev. **124**, 1866 (1961).

[20] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (Wiley, New York, 1992).

[21] B.R. Mollow, Phys. Rev. **188**, 1969 (1969).

[22] A. Barchielli, in O. Hirota, A.S. Holevo and C.M. Caves (eds.), *Quantum communication, computing, and measurement* (Plenum, New York, 1997) pp. 243–252.

[23] A. Barchielli, in H.D. Doebner, W. Scherer, F. Schroeck Jr. (eds.), *Classical and Quantum Systems — Foundations and Symmetries — Proceedings of the II International Wigner Symposium*, (World Scientific, Singapore, 1993) pp. 488–491.

[24] A. Barchielli and V.P. Belavkin, J. Phys. A: Math Gen. **24**, 1495 (1991).

[25] H. Carmichael, *An Open System Approach to Quantum Optics*, Lect. Notes Phys. **m18** (Springer, Berlin, 1993).

[26] H. Maassen, Rep. Math. Phys. **30**, 185 (1992).

[27] A. Kastler, J. Opt. Soc. Am. **53**, 902 (1963).

[28] S. Ezekiel and F.Y. Wu, in J.H. Eberly and P. Lambropoulos (eds.), *Multiphoton Processes* (Wiley, New York, 1978), pp. 145–156.

[29] F. Schuda, C.R. Stroud Jr., M. Hercher, J. Phys. B: Atom. Molec. Phys. **7**, L198 (1974).

[30] W. Harting, W. Rasmussen, R. Schieder, H. Walther, Z. Physik A **278**, 205 (1976).

[31] R.E. Grove, F.Y.Wu, S. Ezekiel, Phys. Rev. A **15**, 227 (1977).

[32] J.D. Cresser, J. Häger, G. Leuchs, M. Rateike, H. Walther, in R. Bonifacio (ed.), *Dissipative Systems in Quantum Optics*, Topics in Current Physics Vol. 27 (Springer, Berlin, 1982), pp. 21-59.

Stochastic wave functions, quantum evaporation and Lévy flights

François Bardou*

Institut de Physique et de Chimie des Matériaux de Strasbourg,
23 rue du Loess, F-67037 Strasbourg Cedex, France

1 Introduction

Stochastic wave function approaches, also called Monte-Carlo wave function approaches, describe the evolution of open quantum systems by sequences of hamiltonian evolutions of wave functions interrupted at random times by quantum jumps [DCM92, DZR92, Car93]. These approaches, complementary to the usual master equation formalism, are now widely used as numerical methods in quantum optics. Stochastic wave functions perform random walks in Hilbert spaces which are qualitatively similar to the random walks in real space associated to Brownian motion in classical physics (see Fig. 1). By stressing both random walks (in Hilbert spaces) and wave function propagation, stochastic wave functions provide insights which stimulate new theoretical studies of certain quantum systems. We present here two results inspired by stochastic wave functions.

We first study how a small momentum transfer associated to a quantum jump can modify wave function propagation. We find a new effect, called 'quantum evaporation', in which small momentum transfers increase dramatically the transmission probability of a particle impinging on a potential barrier.

Second, at the 'statistical' level, we examine the random walk properties of atoms undergoing subrecoil laser cooling, the laser cooling method that leads to the lowest temperatures (nanokelvin range). The random walks of the atoms appear to be dominated by rare events which, although rare, play a crucial role in the cooling process. Such anomalous random walks are called 'Lévy flights'. This approach provides an analytical theory of subrecoil cooling and the gained insight enables to improve the cooling strategies.

*E-mail: Francois.Bardou@ipcms.u-strasbg.fr. Temporary address : Department of Physics, University of Newcastle, Newcastle-upon-Tyne NE1 7RU, United Kingdom.

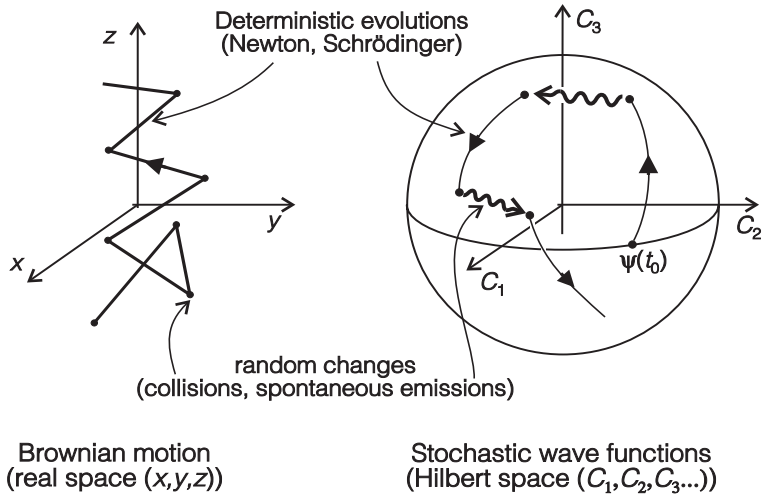

Figure 1. Analogy between stochastic wave functions and a classical random walk. For simplicity, we have represented a 3D Hilbert space.

2 Quantum evaporation

We have investigated the behaviour of a 1D quasi-monochromatic wave function undergoing a momentum transfer while impinging on a potential barrier [BoB99] (see Fig. 2). This problem can be seen, in the framework of stochastic wave functions, as the elementary part of a quantum diffusion process.

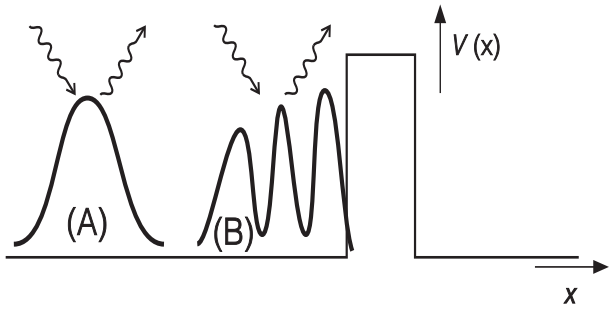

Figure 2. Quantum evaporation. A wave function undergoes a momentum transfer before (A) or while (B) bouncing on a potential barrier.

If the momentum $\hbar q$ is transferred to the wave function before (or, of course, after) its interaction with the barrier (case A in Fig. 2), the effects of the momentum transfer on the transmission probability $T(q)$ of the barrier are relatively small and are trivially related to the energy changes. On the other hand, if the momentum transfer occurs while the wave function is bouncing on the barrier (case B in Fig. 2), the transmisssion probability $T(q)$ is greatly enhanced, even if the momentum transfer $\hbar q$ is small, i.e. if the average kinetic energy of the wave function after the transfer remains much smaller than the height of the barrier. This is what we call 'quantum evaporation'.

This quantum mechanical effect is found to result from the population of high energy states with an unexpectedly large amplitude, decaying only algebraically at high energies. Thus, even in the case of a small momentum transfer, states with energies *above* the barrier are easily populated, giving rise to large transmission probabilities $T(q)$. The transmission $T(q)$ is found to vary as $T(0) + T_2 q^2 + ...$ for small q. Remarkably, $T(q)$ is therefore independent on the sign of the momentum transfer $\hbar q$.

We think that quantum evaporation could be observed, for instance, in laser cooled atomic gases or in field emission of electrons.

3 Lévy flights in subrecoil laser cooling

We have studied laser cooling of atomic gases in the subrecoil regime [1], i.e. when the final atomic kinetic energy is less than the kinetic energy transferred to an atom at rest by the absorption of a single photon. Subrecoil cooling relies on a diffusion coefficient (related to the spontaneous emission rate) which depends on the atomic momentum p and vanishes at $p = 0$ due to quantum interference effects [AAK89]. This enables to accumulate 'cooled' atoms in the vicinity of $p = 0$.

In some particularly interesting situations, the random walk of the stochastic wave functions of the atoms undergoing subrecoil cooling reduces to a random walk in momentum space [CBA91, Bar95]. Thus, it becomes relatively simple to study these otherwise complex quantum processes of laser cooling with *classical* random walk techniques.

However, an inspection of individual stochastic wave function histories (see Fig. 3) reveals that their random walks are strongly anomalous [BBE94, Bar95]. Indeed, most histories are completely dominated by very few (one or two typically) trapping events in the vicinity of $p = 0$.

This unusual statistical behaviour can be understood within the framework of the Generalized Central Limit Theorem demonstrated by Paul Lévy in the thirties [BoG90]. The probability densities of characteristic times exhibit slowly decaying power law tails (such that their variance or their average value is infinite) which dominate the statistical properties. These 'broad' distributions generate random walks which are dominated by rare events and which are called 'Lévy flights'.

We have obtained an analytical theory of subrecoil cooling which is based on the properties of broad distributions [BBE94, Bar95, BBA99]. Its predictions have now been verified by several experiments [RBB95, SHK97, SLC99]. Schemes have also be proposed to measure directly key statistical distributions [SSY99]. At last, recent mathematical developments shed interesting light on the anomalous random walks

[1] A longer introductory paper on this subject can be found in the 'mini-proceedings' of a previous MaPhySto conference : see F. Bardou, *Cooling gases with Lévy flights: using the generalized central limit theorem in physics*, in Conference on 'Lévy processes: theory and applications' Aarhus 18-22 january 1999, MaPhySto Publication (Miscellanea no. 11, ISSN 1398-5957), O. Barndorff-Nielsen, S.E. Graversen and T. Mikosch (eds.).

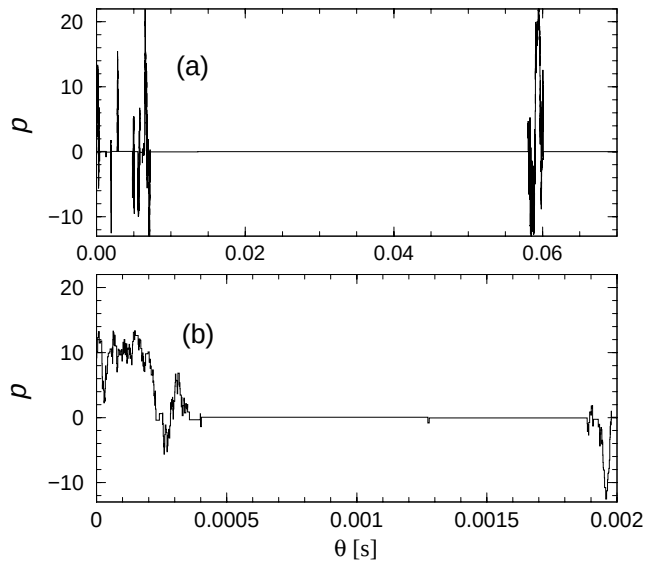

Figure 3. (a) Example of a momentum random walk resulting from a Monte-Carlo simulation of subrecoil cooling of metastable helium atoms. The unit of atomic momentum p is the momentum $\hbar k$ of the photons. The zoom (b) of the beginning of the time evolution is statistically analogous to the evolution at large scale, a fractal property typical of a Lévy flight.

associated to subrecoil cooling by relating them, in particular, to the framework of renewal processes [BaB99].

References

[AAK89] A. Aspect, E. Arimondo, R. Kaiser, N. Vansteenkiste and C. Cohen-Tannoudji, *Laser cooling below the one-photon recoil energy by velocity-selective coherent population trapping : theoretical analysis*, J. Opt. Soc. Am. B **6**, 2112-2124 (1989).

[BaB99] O.E. Barndorff-Nielsen and F.E. Benth, *Laser cooling and stochastics*, MaPhySto Publication (ISSN 1398-2699), Research Report no. 38 (1999); and to be published.

[Bar95] F. Bardou, Ph. D. Thesis, University of Paris XI Orsay, chapter V (1995).

[BBA99] F. Bardou, J.-P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji, *Non-ergodic cooling: subrecoil laser cooling and Lévy statistics*, in preparation.

[BBE94] F. Bardou, J.-P. Bouchaud, O. Emile, A. Aspect and C. Cohen-Tannoudji, *Subrecoil Laser Cooling and Lévy Flights*, Phys. Rev. Lett. **72**, 203-206 (1994).

[BoB99] D. Boosé, and F. Bardou, *A quantum evaporation effect*, submitted (1999).

[BoG90] J.P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Phys. Rep. **195**, 127-293 (1990).

[Car93] H. Carmichael, *An Open Systems Approach to Quantum Optics*, Springer-Verlag (1993).

[CBA91] C. Cohen-Tannoudji, F. Bardou, and A. Aspect, *Review of fundamental processes in laser cooling*, Proceedings of Laser Spectroscopy X (Font-Romeu, 1991), edited by M. Ducloy, E. Giacobino, and G. Camy, 3-14 (World Scientific, Singapore, 1992).

[DCM92] J. Dalibard, Y. Castin, and K. Mølmer, *Wave-Function Approach to Dissipative Processes in Quantum Optics*, Phys. Rev. Lett. **68**, 580-583 (1992).

[DZR92] R. Dum, P. Zoller, and R. Ritsch, *Monte-Carlo simulation of the atomic master equation for spontaneous emission*, Phys. Rev. A **45**, 4879-4887 (1992).

[RBB95] J. Reichel, F. Bardou, M. Ben Dahan, E. Peik, S. Rand, C. Salomon and C. Cohen-Tannoudji, *Raman Cooling of Cesium below 3 nK: New Approach Inspired by Lévy Flight Statistics*, Phys. Rev. Lett. **75**, 4575-4578 (1995).

[SHK97] B. Saubaméa, T.W. Hijmans, S. Kulin, E. Rasel, E Peik, M. Leduc, and C. Cohen-Tannoudji, *Direct measurement of the spatial correlation function of ultracold atoms*, Phys. Rev. Lett. **79**, 3146-3149 (1997).

[SLC99] B. Saubaméa, M. Leduc, and C. Cohen-Tannoudji, *Experimental Investigation of Non-Ergodic Effects in Subrecoil Laser Cooling*, Phys. Rev. Lett. **83**, 3796-3799 (1999).

[SSY99] S. Schaufler, W.P. Schleich, and V.P. Yakovlev, *A key hole look at Lévy flights in subrecoil cooling*, Phys. Rev. Lett. **83**, 3162-3165 (1999).

Viacheslav P. Belavkin

Quantum Stochastics as a Boundary Value Problem, and Classification of Quantum Noise.

ABSTRACT: Using a white (Poisson) analysis in Fock triples we formulate a class of boundary value problems for quantized fields which interact with a quantum system at the boundary. We prove that in the interaction representation the scattered fields plus the boundary satisfy a quantum stochastic equation for the Markovian unitary evolution in Fock space with respect to a quantum Poisson noise as an ultrarelativistic limit of the input fields.

We give the complete classification of quantum noises as stochastic processes with independent increments indexed by arbitrary non-Abelian Itô algebra, and prove that each such process can be decomposed into the orthogonal sum of quantum independent Brownian and Lévy motions. Every quantum stochastic unitary evolution driven by such noise corresponds to a unique self-adjoint boundary value problem for a free quantum field with the singular boundary interaction.

Howard Carmichael

Physical principles of quantum trajectories.

ABSTRACT: The earliest proposal of a stochastic evolution in quantum optics was that of Einstein, who put forward his so-called A and B theory to account for the approach to equilibrium of matter in interaction with black body radiation. Modern quantum trajectory methods are close relatives of the Einstein proposal. In this talk I trace the differences between quantum trajectories and the Einstein stochastic process, and the physical reasons that they must be introduced. The logic of a quantum trajectory as a conditioned evolution is set out and illustrated by a number of examples from quantum optics. The physical meaning of the term "quantum jump" is explored in relation to ongoing experiments in cavity QED.

The Propagation of Molecular Chaos by Quantum Systems: an extended abstract

Alexander David Gottlieb

The purpose of this abstract is to show how the classical notion of molecular chaos can be generalized to quantum many-particle systems. The concept of molecular chaos is due to Boltzmann [2], who assumed, in order to derive the fundamental equation of the kinetic theory of gases, that the molecules of a nonequilibrium gas are in a state of "molecular disorder." Kac [9, 10] called molecular chaos "the Boltzmann property" and used it to derive the homogeneous Boltzmann equation in the infinite-particle limit of certain Markovian gas models. This idea was further developed in [6, 22]. McKean [13, 14] proved the propagation of chaos for systems of interacting diffusions that yield diffusive Vlasov equations in the mean-field limit. See [23] and [15] for two definitive surveys of propagation of chaos and its applications.

Classical molecular chaos is a type of stochastic independence of particles that manifests itself in an infinite-particle limit. If S^n is the n-fold Cartesian power of a measurable space S, a probability measure P on S^n is called *symmetric* if

$$P(E_1 \times E_2 \times \cdots \times E_n) = P(E_{\pi(1)} \times E_{\pi(2)} \times \cdots \times E_{\pi(n)})$$

for all measurable sets $E_1, \ldots, E_n \subset S$ and all permutations π of $\{1, 2, \ldots, n\}$. For $k \leq n$, the *k-marginal* of P, denoted $P^{(k)}$, is the probability measure on S^k satisfying

$$P^{(k)}(E_1 \times E_2 \times \cdots \times E_k) = P(E_1 \times \cdots \times E_k \times S \times \cdots \times S)$$

for all measurable sets $E_1, \ldots, E_k \subset S$. One may define molecular chaos as follows [23]:

Definition 1 (Classical Molecular Chaos) *Let S be a separable metric space. Let P be a probability measure on S, and for each $n \in \mathbb{N}$, let P_n be a symmetric probability measure on S^n.*

*The sequence $\{P_n\}$ is P-***chaotic** *if the k-marginals $P_n^{(k)}$ converge weakly to $P^{\otimes k}$ as $n \longrightarrow \infty$, for each fixed $k \in \mathbb{N}$.*

We proceed to the quantum version of molecular chaos:

Let $\mathbb{H}$ be a Hilbert space whose vectors represent the pure states of some quantum system. The *statistical* states of that quantum system are identified with the normal positive linear functionals on $\mathcal{B}(\mathbb{H})$ that assign 1 to the identity operator. These positive linear functionals on $\mathcal{B}(\mathbb{H})$ are also called states. A state ω on $\mathcal{B}(\mathbb{H})$ is *normal* if

$$\sum_{a \in A} \omega(P_a) = 1$$

whenever $\{P_a\}_{a \in A}$ is a family of commuting projectors that sum to the identity operator (i.e., the net of finite partial sums of the projectors converges in the weak operator topology to the identity).

Normal states are precisely those states that can be represented by density operators. If D is a density operator on $\mathbb{H}$, i.e., a positive trace class operator with trace 1, then $A \mapsto \mathrm{Tr}(DA)$ defines a state on $\mathcal{B}(\mathbb{H})$. Conversely, every normal state ω on $\mathcal{B}(\mathbb{H})$ is of the form $\omega(A) = \mathrm{Tr}(DA)$ for some density operator D.

The Hilbert space of pure states of a collection of n *distinguishable* systems (the i^{th} system having a Hilbert space $\mathbb{H}_i$ of pure states) is $\mathbb{H}_1 \otimes \cdots \otimes \mathbb{H}_n$. The Hilbert space for n distinguishable particles of the same species will be denoted $\mathbb{H}^{\otimes n}$. If D_n is a density operator on $\mathbb{H}^{\otimes n}$, then its k-marginal, or partial contraction, is a density operator on $\mathbb{H}^{\otimes k}$ that gives the statistical state of the first k particles. The k-marginal is denoted $\mathrm{Tr}^{(k)} D_n$, and may be defined as follows: Let $\mathcal{O}$ be any orthonormal basis of $\mathbb{H}$. If $x \in \mathbb{H}^{\otimes k}$ with $k < n$ then for any $w, x \in \mathbb{H}^{\otimes k}$

$$\left\langle \mathrm{Tr}^{(k)} D_n(w), x \right\rangle = \sum_{y_1, \ldots, y_{n-k} \in \mathcal{O}} \left\langle K(w \otimes y_1 \otimes \cdots \otimes y_{n-k}), x \otimes y_1 \otimes \cdots \otimes y_{n-k} \right\rangle .$$

The trace-class operators form a Banach space wherein $\|T\| = \mathrm{Tr}(|T|)$.

A state on $\mathcal{B}(\mathbb{H}^{\otimes n})$ is *symmetric* if it satisfies

$$\omega_n(A_1 \otimes \cdots \otimes A_n) = \omega_n(A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)})$$

for all permutations π of $\{1, 2, \ldots, n\}$ and all $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{H})$. For each permutation π of $\{1, 2, \ldots, n\}$, define the unitary operator U_π on $\mathbb{H}^{\otimes n}$ whose action on simple tensors is

$$U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(n)}. \tag{1}$$

A density operator D_n corresponds to a symmetric state if and only D_n commutes with each U_π. A density operator D_n corresponds to the statistical state of a system of n bosons, a *Bose-Einstein state*, if and only if $D_n U_\pi = D_n$ for all permutations π.

Definition 2 (Quantum Molecular Chaos) *Let D be a density operator on $\mathbb{H}$, and for each $n \in \mathbb{N}$, let D_n be a symmetric density operator on $\mathbb{H}^{\otimes n}$.*

The sequence $\{D_n\}$ is **D-chaotic** *if, for each fixed $k \in \mathbb{N}$, the density operators $\mathrm{Tr}^{(k)} D_n$ converge in trace norm to $D^{\otimes k}$ as $n \longrightarrow \infty$.*

The sequence $\{D_n\}$ is **molecularly chaotic** *if it is D-chaotic for some density operator D on $\mathbb{H}$.*

A sequence, indexed by n, of n-particle dynamics *propagates chaos* if molecularly chaotic sequences of initial distributions remain molecularly chaotic for all time under the n-particle dynamical evolutions. For the sake of generality, we allow the transformation of states to be the dual of a completely positive and unital map, that is, the state $A \mapsto \mathrm{Tr}(DA)$ may be transformed into a state of the form $A \mapsto \mathrm{Tr}(D\phi(A))$ where ϕ is a (normal) completely positive unital endomorphism of $\mathcal{B}(\mathbb{H}^{\otimes n})$. A linear map $\phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ of C* algebras is *completely positive* if, for each $n \in \mathbb{N}$, the map from $\mathcal{A}_1 \otimes \mathcal{B}(\mathbb{C}^n)$ to $\mathcal{A}_2 \otimes \mathcal{B}(\mathbb{C}^n)$ that sends $A \otimes B$ to $\phi(A) \otimes B$ is positive. It is known that all normal completely positive unit preserving maps from $\mathcal{B}(\mathbb{H})$ to $\mathcal{B}(\mathbb{K})$ are of the form

$$\phi(A) = \sum_{a \in J} W_a^* A W_a \ , \tag{2}$$

where the family $\{W_a\}_{a \in J}$ of bounded operators is such that $\sum_{a \in J} W_a^* W_a$ converges strongly to the identity operator. The class of completely positive maps is important in quantum dynamics, for it includes the unitarily implemented automorphisms $A \mapsto UAU^*$ of the Heisenberg picture of quantum dynamics, but it also includes transformations $A \mapsto A'$ effected by the intervention of measurements, randomization, and temporary coupling to

other systems. A normal completely positive unital map ϕ induces a trace-preserving map ϕ_* on the trace class operators defined implicitly by

$$\mathrm{Tr}(\phi_*(D)A) = \mathrm{Tr}(D\phi(A))$$

for all $A \in \mathcal{B}(\mathbb{H})$. If ϕ has the form (2) then

$$\phi_*(D) = \sum_{a \in J} W_a D W_a^* \ ,$$

where the series converges in the trace norm [18].

Definition 3 (Propagation of Molecular Chaos) *For each $n \in \mathbb{N}$, let ϕ_n be a normal completely positive map from $\mathbb{H}^{\otimes n}$ to itself that fixes the identity and which commutes with permutations, i.e., such that*

$$\phi_n(U_\pi^* A U_\pi) = U_\pi^* \phi_n(A) U_\pi \tag{3}$$

for all $A \in \mathcal{B}(\mathbb{H}^{\otimes n})$ and all permutations π of $\{1, 2, \ldots, n\}$, where U_π is as defined in (1).

The sequence $\{\phi_n\}$ **propagates chaos** *if the molecular chaos of a sequence of density operators $\{D_n\}$ entails the molecular chaos of the sequence $\{\phi_{n*}(D_n)\}$.*

We will now describe a class of deterministic many-particle systems that propagates molecular chaos. Let V be a bounded Hermitian operator on $\mathbb{H} \otimes \mathbb{H}$ such that $V(x \otimes y) = V(y \otimes x)$ for all $x, y \in \mathbb{H}$, representing a two-body potential. Let $V_{1,2}^n$ denote the operator on $\otimes^n \mathbb{H}$ defined by

$$V_{1,2}^n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = V(x_1 \otimes x_2) \otimes x_3 \otimes \cdots \otimes x_n, \tag{4}$$

and for each $i, j \leq n$ with $i < j$, define V_{ij}^n similarly, so that it acts on the i^{th} and j^{th} factors of each simple tensor. This may be accomplished by setting $V_{ij}^n = U_\pi^{n*} V_{1,2}^n U_\pi^n$, where $\pi = (2j)(1i)$ is a permutation that puts i in the first place and j in the second place, and U_π^n is as defined in (1). Define the n-particle Hamiltonians H_n as the sum of the pair potentials V_{ij}^n, with common coupling constant $1/n$:

$$H_n = \frac{1}{n} \sum_{i<j} V_{ij}^n. \tag{5}$$

If D_n is a state on $\mathbb{H}^{\otimes^n}$, let $D_n(t)$ denote the state of an n-particle system that was initially in state D_n and which has undergone t units of the temporal evolution governed by the Hamiltonian (5):

$$\phi_{n*}(D_n) \equiv D_n(t) = e^{-iH_n t/\hbar} D_n e^{iH_n t/\hbar}. \tag{6}$$

Theorem 1 *Suppose D is a density operator on $\mathbb{H}$ and $\{D_n\}$ is a D-chaotic sequence of symmetric density operators on $\mathbb{H}^{\otimes^n}$. Then the sequence of density operators $\{D_n(t)\}$ defined in (5) and (6) is $D(t)$-chaotic, where $D(t)$ is the solution at time t of the following ordinary differential equation in the Banach space of trace-class operators:*

$$\begin{aligned} \frac{d}{dt}D(t) &= -\frac{i}{\hbar}\mathrm{Tr}^{(1)}[V, D(t)\otimes D(t)] \\ D(0) &= D. \end{aligned} \tag{7}$$

Theorem 1 can be applied to mean-field spin models of ferromagnetism, where the spin angular momentum of each atom of a crystal is supposed to be coupled to the average spin and to an external magnetic field. The traditional approach to the dynamics of mean-field spin models has been to construct the infinite-particle dynamics as a limit of finite-particle dynamics [4], as is customary in quantum statistical mechanics. However, although the infinite-particle dynamics of spin models with finite-range interactions (such as the Ising model) can be defined without difficulty in this manner [19], defining the infinite-particle dynamics of spin models with long-range interactions (such as the Curie-Weiss model) is a much more subtle affair [1].

Theorem 1 provides us with an alternative approach: Consider the Curie-Weiss model for spin-$\frac{1}{2}$ atoms. The Hamiltonian for the n-spin Curie-Weiss model is

$$\mathcal{H}_n = \frac{1}{n}\sum_{i,j=1}^{n}\left(-J\sigma_i^z\sigma_j^z - H\sigma_i^z\right), \tag{8}$$

where J is a positive coupling constant and H is another constant whose magnitude is proportional to and whose sign reflects the direction of the external magnetic field. If D_n is the initial density of an n-spin system, then $D_n(t)$ defined by

$$D_n(t) = e^{-i\mathcal{H}_n t/\hbar} D_n e^{i\mathcal{H}_n t/\hbar}.$$

is the density operator at time t. For any density operator D on $\mathbb{C}^2$, let $[D]$ denote the 2×2 matrix that represents D. If the sequence $\{D_n\}$ of initial n-spin states is D-chaotic with $[D_\rho] = \begin{pmatrix} a & c \\ \bar{c} & d \end{pmatrix}$, the sequence of n-spin states at time t is $D(t)$-chaotic for each $t \geq 0$, where

$$\begin{aligned} [D(t)] &= \begin{pmatrix} a & ce^{it\Theta} \\ \bar{c}e^{-it\Theta} & d \end{pmatrix} \\ \Theta &= H + \hbar J(a-d). \end{aligned}$$

Another consequence of Theorem 1 is that there exist many molecularly chaotic sequences of Bose-Einstein states: For any $\psi \in \mathbb{H}$, let D_ψ denote the orthogonal projection onto the span of ψ; this is the density operator for the pure state $A \mapsto \langle A\psi, \psi \rangle$. The sequence of Bose-Einstein states $\{D_\psi^{\otimes n}\}$ is D_ψ-chaotic, and Theorem 1 shows that the sequence of states obtained after t seconds of temporal evolution governed by Hamiltonians of the form (5) is also molecularly chaotic. These states are also Bose-Einstein states because of the symmetry of the Hamiltonians. In contrast to this, we note that it is not possible for a sequence of Fermi-Dirac states to be molecularly chaotic [8].

We now describe another class of particle systems that propagate quantum molecular chaos. Let $\{W_a\}_{a \in A}$ be a family of bounded operators on $\mathbb{H} \otimes \mathbb{H}$ such that

$$\sum_{a \in A} W_a^* W_a = I \tag{9}$$

in the sense of strong convergence, and $W_a(x \otimes y) = W_a(y \otimes x)$ for all $x, y \in \mathbb{H}$ and $a \in A$. For each $n \geq 2$ and each $1 \leq i < j \leq n$, define $W_{a,ij}^n \in \mathcal{B}(\mathbb{H}^{\otimes n})$ by

$$W_{a,ij}^n = U_{(2j)(1i)}^{n*}(W_a \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) U_{(2j)(1i)}^n,$$

where $U_{(2j)(1i)}^n$ is the permutation operator defined in (1) for the permutation $(2j)(1i)$. For each $n \geq 2$, define a completely positive unital map ϕ_n of $\mathcal{B}(\mathbb{H}^{\otimes n})$ into itself by

$$\phi_n(A) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \sum_{a \in A} W_{a,ij}^{n*} A W_{a,ij}^n.$$

The map ϕ_{n*} of trace class operators whose dual is ϕ_n is

$$\phi_{n*}(D) = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} \sum_{a \in A} W_{a,ij}^n D W_{a,ij}^{n*}. \tag{10}$$

In physical terms, the map ϕ_{n*} describes how the state of an n-component quantum system changes when two of the n components are selected at random and made to interact with one another and an external environment using the family of interaction operators $\{W_a\}_{a \in A}$.

Theorem 2 *Suppose D is a density operator on $\mathbb{H}$ and $\{D_n\}$ is a D-chaotic sequence of density operators on $\mathbb{H}^{\otimes^n}$. For each $n \ge 2$, let $m(n) \in \mathbb{N}$ be such that $\lim_{n\to\infty} \frac{m(n)}{n} = t$. Let $\phi_{n*}^{m(n)}$ denote ϕ_{n*} composed with itself $m(n)$ times.*

Then $\{\phi_{n*}^{m(n)}(D_n)\}$ is a $D(t)$-chaotic sequence of density operators, where $D(t)$ is the solution at time t of

$$\begin{aligned} \frac{d}{dt}D(t) &= 2\mathrm{Tr}^{(1)}\left(\sum_{a \in A} W_a(D(t) \otimes D(t))W_a^*\right) - 2D(t) \\ D(0) &= D. \end{aligned} \tag{11}$$

Equation (11) is analogous to the Boltzmann equation. Using the propagation of chaos and the properties of entropy, we can prove an H-theorem for (11). The *entropy of E relative to D* is

$$S(E|D) = -\mathrm{Tr}(E \log E - E \log D).$$

Corollary 1 *Let D_∞ be a density operator on $\mathbb{H}$ such that*

$$D_\infty \otimes D_\infty = \sum_{a \in A} W_a(D_\infty \otimes D_\infty)W_a^*. \tag{12}$$

Let $D(t)$ be a solution of (11). Then $S(D(t)|D_\infty)$ is nondecreasing as t increases.

Proof of Corollary:

Let $s, t \geq 0$ and let $D(s)$ and $D(s+t)$ be the solutions to equation (11) at times $s, s+t$. Then $D(s+t)$ equals the solution at time t of

$$\begin{aligned} \frac{d}{dt}X(t) &= 2\mathrm{Tr}^{(1)}\left(\sum_{a\in A} W_a(X(t)\otimes X(t))W_a^*\right) - 2X(t) \\ X(0) &= D(s). \end{aligned}$$

Choose a sequence $\{m(n)\}$ such that $\lim \frac{m(n)}{n} = t$. By Theorem 2,

$$D(s+t) = \lim_{n\to\infty} \mathrm{Tr}^{(1)}\phi_n^m(D(s)^{\otimes n}). \tag{13}$$

Condition (12) on D_∞ implies that $\phi_n(D_\infty^{\otimes n}) = D_\infty^{\otimes n}$ for all n. Completely positive unital maps increase the relative entropy of densities [12], so

$$S(D(s)^{\otimes n}|D_\infty^{\otimes n}) \leq S(\phi_n^m(D(s)^{\otimes n})|D_\infty^{\otimes n}).$$

It follows that

$$\begin{aligned} S(D(s)|D_\infty) &\leq \liminf_{n\to\infty} \frac{1}{n} S(\phi_n^m(D(s)^{\otimes n})|D_\infty^{\otimes n}) \\ &\leq \liminf_{n\to\infty} S(\mathrm{Tr}^{(1)}\phi_n^m(D(s)^{\otimes n})|D_\infty) \\ &\leq S(D(s+t)|D_\infty). \end{aligned}$$

The final inequality follows from (13) and the upper semicontinuity of relative entropy. The second-to-last inequality is the subadditivity property of relative entropy. See [17] for these properties of relative entropy. □

References

[1] F. Bagarello and G. Morchio. Dynamics of mean-field spin models from basic results in abstract differential equations. *Journal of Statistical Physics* 66: 849 - 866, 1992.

[2] L. Boltzmann. *Lectures on Gas Theory.* Dover Publications, New York, 1995.

[3] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the $\frac{1}{n}$ limit of interacting classical particles. *Communications in Mathematical Physics* 56: 101-113, 1977.

[4] G. G. Emch and H. J. F. Knops. Pure thermodynamical phases as extremal KMS states. *Journal of Mathematical Physics* 11 (no. 10): 3008 - 3018, 1970.

[5] A. D. Gottlieb. Propagation of molecular chaos by quantum systems and the dynamics of the Curie-Weiss model. *Center for Pure and Applied Mathematics, University of California at Berkeley.* Report no. 764.

[6] F. A. Grünbaum. Propagation of chaos for the Boltzmann equation. *Archive for Rational Mechanics and Analysis* 42: 323-345, 1971.

[7] E. Hewitt and L. Savage. Symmetric measures on Cartesian products. *Transactions of the American Mathematical Society* 80: 470-501, 1955.

[8] R.L. Hudson and G.R. Moody. Locally normal symmetric states and an analogue of de Finetti's theorem. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 33: 343-351, 1976.

[9] M. Kac. Foundations of kinetic theory. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol III.* University of California Press, Berkeley, California, 1956.

[10] M. Kac. *Probability and Related Topics in Physical Sciences.* American Mathematical Society, Providence, Rhode Island, 1976.

[11] O. E. Lanford III and D. W. Robinson. Mean entropy of states in quantum statistical mechanics. *Journal of Mathematical Physics* 9 (7): 1120-1125, 1968.

[12] G. Lindblad. Completely positive maps and entropy inequalities. *Communications in Mathematical Physics* 40: 147-151, 1975.

[13] H. P. McKean, Jr. A class of Markov processes associated with nonlinear parabolic equations. *Proceedings of the National Academy of Science* 56: 1907-1911, 1966.

[14] H. P. McKean, Jr. Propagation of chaos for a class of nonlinear parabolic equations. *Lecture Series in Differential Equations* 7: 41-57. Catholic University, Washington, D.C., 1967.

[15] S. Méléard. Asymptotic behavior of some interacting particle systems; McKean-Vlasov and Boltzmann models. *Lecture Notes in Mathematics*, 1627. Springer, Berlin, 1995.

[16] H. Narnhofer and G. Sewell. Vlasov hydrodynamics of a quantum mechanical model. *Communications in Mathematical Physics* 79: 9-24, 1981.

[17] M. Ohya and D. Petz. *Quantum Entropy and its Use*. Springer, Berlin, 1993.

[18] K. R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus* Birkhäuser, Basel, 1992.

[19] D. Ruelle. *Statistical Mechanics: Rigorous Results*. Addison Wesley, Redwood City, California, 1969.

[20] J. Messer and H. Spohn. Statistical mechanics of the isothermal Lane-Emden equation. *Journal of Statistical Physics* 29 (3): 561-578, 1982.

[21] E. Størmer. Symmetric states of infinite tensor products of C*-algebras. *Journal of Functional Analysis* 3: 48-68, 1969.

[22] A. Sznitman. Équations de type de Boltzmann, spatialement homogènes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 66: 559-592, 1984.

[23] A. Sznitman. Topics in propagation of chaos. *Lecture Notes in Mathematics*, 1464. Springer, Berlin, 1991.

ALEXANDER DAVID GOTTLIEB
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720

Inge S. Helland

Experiments, symmetries and quantum mechanics.

Looking at the development of quantitative methodology in this century, one of the most striking observation is that we throughout the entire period have had two different cultures, mathematical statistics and quantum theory, both working with prediction under uncertainty, but where there up to now has been practically no scientific contact between the two disciplines. One reason may be that the domains of application for the theories of the two disciplines are different, and there may possibly also be some real differences that can never be removed. Nevertheless, a systematic search for common ground may well be worth while, and may even lead to new insight on both sides.

A very important issue will then be the language used in the various theories. We all know how efficient the use of probability theory and of the theory of decisions made from classes of probability measure is on the one side, and of the efficiency of concepts derived from functional analysis and operator theory on the other side. However, to be able to find a common ground, I think that we must try if possible to simplify concepts, even to such an extent that we in principle should be able to explain the conceptual foundation to lay persons. I feel that in the end, nobody could claim to have real understanding of a subject without having addressed such an aim. This does of course not mean that mathematics is unimportant. On the contrary, strong mathematics will always be very important in developing theories. But the basic foundation should be simple - at least ideally.

Consider first the concept of experiment from mathematical statistics. The formal structure found in textbooks is $(\mathcal{X}, \mathcal{F}, \{P_\theta; \theta \in \Theta\})$, where $\mathcal{X}$ is the space of possible outcomes of the experiments, $\mathcal{F}$ is a σ-algebra (Boolean algebra) of subsets of $\mathcal{X}$, and we then have a class of probability measures on $(\mathcal{X}, \mathcal{F})$, indexed by θ. By use of some time and patience, I think that the basic idea here can be conveyed in relatively simple terms: The possible outcomes of an experiment may always be assumed to belong to some given space; the purpose of the experiment is to get some information on an unknown parameter which can be taken as a label of the probability measures that together constitute the model, and so on. On the other hand: The formal concept above obviously lacks many of the features that people link to the concept of 'experiment': First of all the preparations: choice of treatments, blocking, randomization and so on; this is rather trivial and well appreciated. But even as a way to describe the outcome of real experiments, the formal concept is defective: In this formal world nothing unexpected can happen, but in the real world this happens quite often. At best we can look upon the formal structure as some simplified frame that is useful when handling certain questions in statistical decision theory.

Next consider the concept of 'state' of a system. In quantum mechanics this is of course a ray in a Hilbert space or a density matrix. But for a layman this must seem like a very strange concept, and in this case it indeed seems very difficult to explain the concept in simple terms.

One possible approach might be to make a comparison to a completely different area where the concept of state is also used: Look at a medical patient. The state of this patient can then in principle be defined as the collection of all results of all tests/experiments that can be performed with him, some of which may be mutually incompatible.

A complication is that such experiments may have random measurement errors, so in the state definition we should in some way talk about ideal experiments. From a statistical point of view, an alternative might perhaps be several large sets of independent, identically distributed experiments on the same patient, but this will represent a further complication in several ways, so we will avoid that. An alternative that is important in the quantum framework, would have been to include the measurement aparatus in the modelling framework - leading to socalled generalized measurements. This was pointed out to me afterwards by Richard Gill, but was not included in the talk. However, the issue seems to be possible to address by extending the models discussed below.

The main purpose of this talk, then, is to discuss simply to which extent one can pass from the rather straightforward state concept of the medical patient to a similar state concept in quantum mechanics. More precisely: Let $\mathcal{A}$ be a set of potential experiments $\mathcal{E}_a = (\mathcal{X}_a, \mathcal{F}_a, \{P^a_{\theta_a}; \theta_a \in \Theta_a\})$ for $a \in \mathcal{A}$. Define a proposition as $\mathcal{P} = (a, E_a)$, where $E_a \in \mathcal{F}_a$. (Perform an experiment, then observe an event in it.) Finally, we can always define ϕ such that each $\theta_a = \theta_a(\phi)$. The state should somehow be determined by ϕ in such a way that all the probability measures $P^a_{\theta>_a}$ can be found from the state.

The basic question is to what an extent one can pass from this setting to the Hilbert space setting. We look at two approaches, one based upon quantum lattice theory and one based upon group representation theory. In the last case we will also have to assume in addition that there exists a group G on $\Phi = \{\phi\}$.

In the quantum lattice approach (Helland, 1999) we first have to order partially the propositions that we have just defined. For propositions from the same experiment the ordering is obvious; in general we say that $\mathcal{P}_1 \leq \mathcal{P}_2$ if $P^{a_1}_{\theta_{a_1}(\phi)} \leq P^{a_2}_{\theta_{a_2}(\phi)}$ for all ϕ. We have to identify events $\mathcal{P}_1$ and $\mathcal{P}_2$ if both $\mathcal{P}_1 \leq \mathcal{P}_2$ and $\mathcal{P}_2 \leq \mathcal{P}_1$.

Under some additional assumptions this partially ordered set of propositions will form an orthomodular, orthocomplemented lattice. Specifically, these assumptions turn out to be essentially:

1) If we define the orthogonal complement of a proposition by $(a, E)^\perp = (a, E^c)$, then we demand that pairwise orthogonality ($\mathcal{P}_i \leq \mathcal{P}_j^\perp$ for all i and j) should imply orthogonality in the sense:

$$\sum_{i=1}^{k} P^{a_i}_{\theta_i(\phi)}(E_i) \leq 1, \ \forall \phi.$$

A similar condition is fundamental in the axiom set of Mackey (1963).

2) If the supremum of a proposition set $\{\mathcal{P}_i\}$ (corresponding parameters θ_i) exists, then

it will be a proposition (with parameter θ) such that

$$\{f : f(\phi) = \tilde{f}(\theta(\phi))\} \subseteq \bigvee_i \{f : f(\phi) = \tilde{f}(\theta_i(\phi))\}.$$

The main point now is that there exist deep theorems (Beltrametti and Cassinelli, 1981, and references there) to the following effect: Under the conditions above and some additional technical assumptions (atomicity, covering property, separability) a Hilbert space model of the quantum theory type for the propositions can be constructed. In the discrete case these additional assumptions are automatically satisfied, so if we combine with Gleason's Theorem we get the conclusion:

In the case where all experiments are discrete, and the assumptions above hold, there is a complex, separable Hilbert space $\mathcal{H}_0$ such that (assuming that the dimension of $\mathcal{H}_0$ is ≥ 3) each proposition $\mathcal{P} = (a, E)$ can be associated uniquely with a projection operator $\Pi_{a,E}$ in $\mathcal{H}_0$ in the sense that

$$P^a_{\theta_a(\phi)}(E) = \mathrm{tr}(\rho \Pi_{a,E}),$$

where $\rho = \rho(\phi)$ is a density operator.

Results of this kind definitively give some information about the interpretation of the quantum mechanical state concept and about the possiblity of finding a more classical interpretation of quantum mechanics. However this approach also has weaknesses:

1) The technical conditions needed above are disturbing.

2) The approach gives no explicit construction of the projection operators and of the density matrices.

The alternative, symmetry based approach seems to be an improvement to the quantum logic approach with regard to both these aspects. It also seems to tie up with some recent development in statistical methodology, related to model reduction under symmetry. The details of the method are rather complex however, and work is still being done on improving some of these technical points. The main idea is very simple, however:

Group representation theory gives for free a vector space and operators on this vector space related to any given group. This is well known and used as a tool in several quantum mechanical calculations. However, in our setting we aim at being more fundamental and possibly base the construction of state vectors on this vector space. An interesting point is the following: The connection from the original group to a matrix group in the representation is a homomorphism. Homomorphisms also appear in statistical estimation theory when parametric functions are to be estimated.

Bohr & Ulfbeck (1995) have formulated a symmetry based quantum theory, to some extent on qualitative considerations. There are quantitative details to fill out, and, in the spirit of the present talk, the possibility of a translation from that theory to ordinary statistical theory should be investigated.

Here are some details of our approach:

1. Consider a model for a closed physical system of the type formulated above, where ϕ is a hyperparameter, but where only certain parametric functions can be estimated.

2. A representation of the underlying group G on the Hilbert space $\mathrm{L}^2(\Phi, \nu)$, with ν being Haar measure, is given by

$$U(g)f(\phi) = f(g^{-1}\phi).$$

3. If ψ is an invariantly estimable function of ϕ (i.e., $\psi(g\phi)$ is always a function of $\psi(\phi)$), then

$$V = \{f : f(\phi) = \tilde{f}(\psi(\phi))$$

is an invariant space of the operators $\{U(g)\}$ above. This creates a 1-1 orderpreserving correspondance between the invariantly estimable functions and certain invariant subspaces. One can also construct a correspondence between parameter values and vectors in the space, given by $f_{\phi_1}(\phi) = f_0(g^{-1}\phi)$ when $\phi_1 = g\phi_0$. Here, f_0 and g_0 are fixed.

4. Corresponding roughly to model reduction under scarce data, one can regard a fixed irreducible invariant space $\mathcal{H}$ as the state space.

5. Fix $a \in \mathcal{A}$ and a function q. Using the Fourier transform of group representations one can show that there is a unique selfadjoint operator A_a^q on $L^2(\Phi, \nu)$ such that

$$f_{\phi_1}^{\dagger}(A_a^q f_{\phi_1}) = q(\theta_a(\phi_1))$$

for all ϕ_1.

6. Apart from technicalities, the only essential thing that seems to be missing from this scheme to create quantum formalism, is to postulate that the transition probabilities are symmetric: $P(u \to v) = P(v \to u)$.

We hope to complete the paper on this last approach before too long. We are also working on large scale statistical methods that appear to have at least some (admittedly, at present rather weak) relationship to this framework. The hope in the really long term, however, is still that there is a feasible road to the unity of science - a science where formal constructions are welcome as tools for doing calculations, but where the conceptual foundation somehow may be understood in simple terms.

References.

Beltrametti, E.G. & G. Cassinelli (1981) *The Logic of Quantum Mechanics.* Addison-Wesley, London

Bohr. A. & O. Ulfbeck (1995) Primary manifestation of symmetry. Origin of quantal indeterminacy. *Rev. Mod. Phys.* **67**, 1-35.

Helland, I.S. (1999) Quantum mechanics from symmetry and statistical modelling. *Int. J. Theor. Phys.* **38**, 1851-1881

Mackey, G.W. (1963) *The Mathematical Foundations of Quantum Mechanics.* Benjamin, New York.

Coding Theorems for Quantum Channels

A. S. Holevo[1]
Steklov Mathematical Institute, Moscow

I. The capacity and the entropy bound

The more than thirty years old issue of the information capacity of quantum communication channels was dramatically clarified during the last period, when a number of direct quantum coding theorem was discovered. To considerable extent this progress is due to an interplay between the quantum communication theory and quantum information ideas related to more recent development in quantum computing. It is remarkable, however, that many probabilistic tools underlying the treatment of quantum case have their roots, and in some cases direct prototypes, in classical Shannon's theory. In this paper we address the problem of *classical* capacity of quantum channel (see [*Holevo 1998*] for a detailed survey).

Let $\mathcal{H}$ be a Hilbert space providing a quantum-mechanical description for the physical carrier of information. A simple model of quantum communication channel consists of the input alphabet $A = \{1, ..., a\}$ and a mapping $i \to S_i$ from the input alphabet to the set of quantum states in $\mathcal{H}$. A quantum state is a *density operator*, i. e. positive operator S in $\mathcal{H}$ with unit trace, $\mathrm{Tr}S = 1$. Sending a letter i results in producing the signal state S_i of the information carrier.

Like in the classical case, the input is described by an *a priori probability distribution* $\pi = \{\pi_i\}$ on A. At the receiving end of the channel a quantum measurement is performed, which mathematically is described by a *resolution of identity* in $\mathcal{H}$, that is by a family $X = \{X_j\}$ of positive operators in $\mathcal{H}$ satisfying $\sum_j X_j = I$, where I is the unit operator in $\mathcal{H}$ [*Holevo 1973*]. The index j runs through some finite output alphabet. The probability of the output j conditioned upon the input i by definition is equal to $P(j|i) = \mathrm{Tr}S_iX_j$. The classical case is embedded into this picture by assuming that all operators in question commute, hence are diagonal in some basis labelled by index ω; in fact by taking $S_i = \mathrm{diag}[S(\omega|i)], X_j = \mathrm{diag}[X(j|\omega)]$, we have a classical channel with transition probabilities $S(\omega|i)$ and the classical decision rule $X(j|\omega)$, so that $P(j|i) = \sum_\omega X(j|\omega)S(\omega|i)$. We call such channel *quasiclassical*.

The Shannon information is given by the usual formula

$$I_1(\pi, X) = \sum_j \sum_i \pi_i P(j|i) \log \left(\frac{P(j|i)}{\sum_k \pi_k P(j|k)} \right). \qquad (1)$$

Denoting by $H(S) = -\mathrm{Tr}S\log S$ the von Neumann entropy of a state S, we *assume* that $H(S_i) < \infty$. If $\pi = \{\pi_i\}$ is an apriori distribution on A, we denote

$$\bar{S}_\pi = \sum_{i \in A} \pi_i S_i, \qquad \bar{H}_\pi(S_{(\cdot)}) = \sum_{i \in A} \pi_i H(S_i)$$

and

$$\Delta H(\pi) = H(\bar{S}_\pi) - \bar{H}_\pi(S_{(\cdot)}).$$

[1]E-mail address: holevo@mi.ras.ru

The quantity $\Delta H(\pi)$ is well defined and is continuous in π. The famous *quantum entropy bound* says that

$$\sup_X I_1(\pi, X) \leq \Delta H(\pi), \tag{2}$$

with the equality achieved if and only if all the operators $\pi_i S_i$ commute. The inequality was explicitly conjectured in [*Gordon 1964*], and discussed in [*Forney 1963*], [*Levitin 1969*] and elsewhere in the context of conventional quantum measurement theory. The first published proof appeared in [*Holevo 1973*]. It is worthwhile to mention also that this bound is closely related to the fundamental property of decrease of quantum relative entropy under completely positive maps developed in the series of papers [*Lindblad 1973-1975*] and in [*Uhlmann 1977*] (see [*Yuen and Ozawa 1993*] for history and some generalizations of the entropy bound).

In the same way we can consider the product channel in the tensor product Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes ... \otimes \mathcal{H}$ with the input alphabet A^n consisting of words $w = (i_1, ..., i_n)$ of length n, and with the density operator $S_w = S_{i_1} \otimes ... \otimes S_{i_n}$ corresponding to the word w. If π is an apriori distribution on A^n and X is a resolution of identity in $\mathcal{H}^{\otimes n}$, we define the information quantity $I_n(\pi, X)$ by the formula similar to (1). Denoting $C_n = \sup_{\pi,X} I_n(\pi, X)$, we have the property of superadditivity $C_n + C_m \leq C_{n+m}$, hence the following limit exists

$$C = \lim_{n\to\infty} C_n/n,$$

which is called the *(classical) capacity* of the initial channel [*Holevo 1979*]. This definition is justified by the fact easily deduced from the classical Shannon's coding theorem, that C is the least upper bound of the rate (bits/symbol) of information which can be transmitted with asymptotically vanishing error. More precisely, we call by *code* (W, X) *of size* M a sequence $(w^1, X_1), ..., (w^M, X_M)$, where w^k are words of length n, and $\{X_k\}$ is a family of positive operators in $\mathcal{H}^{\otimes n}$, satisfying $\sum_{k=1}^M X_k \leq I$. Defining $X_0 = I - \sum_{k=1}^M X_k$, we have a resolution of identity in $\mathcal{H}^{\otimes n}$. The average error probability for such a code is

$$\bar{\lambda}(W, X) = \frac{1}{M} \sum_{k=1}^{M} [1 - \mathrm{Tr} S_{w^k} X_k]. \tag{3}$$

Let us denote $p(M, n)$ the minimum of this error probability with respect to all codes of the size M with words of length n. Then $p(2^{n(C-\delta)}, n) \to 0$ and $p(n, 2^{n(C+\delta)}) \not\to 0$, where $\delta > 0$, as $n \to \infty$.

Applied to $I_n(\pi, X)$ and combined with the additivity and continuity properties of $\Delta H(\pi)$ the entropy bound (2) implies $C \leq \max_\pi \Delta H(\pi) \equiv \bar{C}$. Thus

$$C_1 \leq C \leq \bar{C}.$$

For a classical channel $C_n = nC_1$, and all the three quantities coincide. A striking feature of quantum case is possibility of the inequality $C_1 < C$ implying strict superadditivity of the information quantities C_n [*Holevo 1979*]. In a sense, there is a kind of "quantum memory" in channels, which are the analog of classical memoryless channels. This fact is just another manifestation of the "quantum nonseparability", and in a sense is dual to the existence of Einstein - Podolsky - Rosen correlations: the latter are due to entangled (non-factorizable) states and hold for disentangled measurements while the superadditivity is due to entangled measurements and holds for disentangled states.

The inequality $C \neq C_1$ raised the problem of the actual value of the capacity C. A possible conjecture was $C = \bar{C}$, but the proof for it came only in 1996, first for the pure state (noiseless) channels in the paper of Hausladen, Jozsa, Schumacher, Westmoreland and Wootters [*Hausladen et al. 1996*], and then for the general case in [*Holevo 1996*], [*Schumacher and Westmoreland 1997*]. Since the entropy bound (2) and the classical weak converse provide the converse of the quantum coding theorem, the main problem was the proof of the direct coding theorem, i. e. of the inequality $C \geq \bar{C}$.

II. The pure state channel

Following Dirac's formalism, we shall denote vectors of $\mathcal{H}$ as $|\psi>$, and hermitean conjugate vector of the dual space – as $<\psi|$. Then $<\phi|\psi>$ is the inner product of $|\phi>, |\psi>$ and $|\psi><\phi|$ is the outer product, i. e. operator A of rank 1, acting on vector $|\chi>$ as $A|\chi>=|\psi><\phi|\chi>$. If $|\psi>$ is a unit vector, then $|\psi><\psi|$ is the orthogonal projection onto $|\psi>$. This is a special density operator, representing *pure* state of the system. Pure states are precisely extreme points of the convex set of all states; an arbitrary state can be represented as a mixture of pure states, i. e. by imposing classical randomness on pure states. In this sense pure states are "noiseless", i. e. they contain no classical source of randomness.

Let us consider a *pure state channel* with $S_i = |\psi_i><\psi_i|$. Since the entropy of a pure state is zero, $\Delta H(\pi) = H(\bar{S}_\pi)$ for such a channel. If a decision rule $X = \{X_j\}$ is applied at the output then $P(j|i) =<\psi_i|X_j\psi_i>$. A system $\{|\phi_j>\}$ of (unnormalized) vectors is called *overcomplete* if $\sum_j |\phi_j><\phi_j| = I$. Every overcomplete system gives rise to the decision rule $X_j = |\phi_j><\phi_j|$ for which $P(j|i) = |<\psi_i|\phi_j>|^2$.

The first step in getting a lower bound for the capacity C has geometric nature and amounts to obtaining a tractable upper bound for the average error (3) minimized over all decision rules. Sending a word $w = (i_1, \ldots, i_n)$ produces the tensor product vector $\psi_w = \psi_{i_1} \otimes \ldots \otimes \psi_{i_n} \in \mathcal{H}^{\otimes n}$. Let (W, X) be a code of size M. Let us restrict for a while to the subspace of $\mathcal{H}^{\otimes n}$ generated by the vectors $\psi_{w^1}, \ldots, \psi_{w^M}$, and consider the Gram matrix $\Gamma(W) = [<\psi_{w^i}|\psi_{w^j}>]$ and the Gram operator $G(W) = \sum_{k=1}^M |\psi_{w^k}><\psi_{w^k}|$. This operator has the matrix $\Gamma(W)$ with respect to the overcomplete system

$$|\hat{\psi}_{w^k}>= G(W)^{-1/2}|\psi_{w^k}>; \quad k = 1, \ldots, M\ . \tag{4}$$

The resolution of identity of the form

$$X_k = |\hat{\psi}_k><\hat{\psi}_k| \tag{5}$$

approximates the quantum maximum likelihood decision rule (which in general cannot be found explicitly); the necessary normalizing factor $G(W)^{-1/2}$ is a major source of analytical difficulties in the noncommutative case. Note that the vectors $\psi_{w^1}, \ldots, \psi_{w^M}$ need not be linearly independent; in the case of linearly independent coherent state vectors (5) is related to the "suboptimal receiver" described in [*Helstrom 1976*], Sec. VI.3(e). It was shown in [*Holevo 1978*] that by using this decision rule one obtains the upper bound

$$\min_X \bar{\lambda}(X, W) \leq \frac{2}{M} \operatorname{Sp}\left(E - \Gamma(W)^{1/2}\right) = \frac{1}{M} \operatorname{Sp}\left(E - \Gamma(W)^{1/2}\right)^2, \tag{6}$$

where E is the unit $M \times M$-matrix and Sp is the trace of $M \times M$-matrix. As explained in[*Holevo 1978*], this bound is "tight" in the sense that there is a similar lower bound.

However it is difficult to use because of the presence of square root of the Gram matrix. A simpler but coarser bound is obtained by using the operator inequality $(E-\Gamma(W)^{1/2})^2 \leq (E-\Gamma(W))^2$:

$$\min_X \bar{\lambda}(W,X) \leq \frac{1}{M}\,\mathrm{Sp}\,(E-\Gamma(W))^2 = \frac{1}{M}\,\mathrm{Tr}\sum\sum_{r\neq s} S_{w^r}S_{w^s}. \tag{7}$$

As shown in [*Holevo 1978*], this bound is asymptotically equivalent (up to the factor 1/4) to the tight bound (6) in the limit of "almost orthogonal" states $\Gamma(W)\to E$. On the other hand, different words are "decoupled" in (7) which makes it suitable for application of the random coding.

Just as in the classical case, we assume that the words $w^1,...,w^M$ are chosen at random, independently and with the probability distribution

$$\mathsf{P}\{w=(i_1,\ldots,i_n)\} = \pi_1\ldots\pi_n. \tag{8}$$

Then for each word w the expectation

$$\mathsf{E}\,S_w = \bar{S}_\pi^{\otimes n}, \tag{9}$$

and by taking the expectation of the coarse bound (7) we obtain, due to the independence of w^r, w^s

$$p(M,n) \leq \mathsf{E}\min_X \bar{\lambda}(W,X) \leq (M-1)\mathrm{Tr}(\bar{S}_\pi^{\otimes n})^2 = (M-1)2^{-n\log \mathrm{Tr}\bar{S}_\pi^2}.$$

By denoting

$$\tilde{C} = -\log\min_\pi \mathrm{Tr}\bar{S}_\pi^2 = -\log\min_\pi \sum_{i,j}\pi_i\pi_j|<\psi_i|\psi_j>|^2, \tag{10}$$

we conclude that $C\geq\tilde{C}$. There are cases (e. g. pure state binary channel) where $\tilde{C}>C_1$, so this is sufficient to establish $C>C_1$, and hence the strict superadditivity of C_n [*Holevo 1979*], but not sufficient to prove the coding theorem, since $\tilde{C}<\bar{C}$ unless the channel is quasiclassical. A detailed comparison of the quantities $C_1,\bar{C}$ for different quantum channels was made by Ban, Hirota, Kato, Osaki and Suzuki [*Kato et al. 1996*]. The quantity $\tilde{C}$ was discussed in [*Holevo 1979*], [*Stratonovich and Vantsjan 1978*], but its real information theoretic meaning as the *cutoff rate* is elucidated only in connection with the quantum reliability function (see (15) below).

The proof of the inequality $C\geq\bar{C}$ given in [*Hausladen et al. 1996*] achieves the goal by using the approximate maximum likelihood improved with projection onto the "typical subspace" of the density operator $\bar{S}_\pi^{\otimes n}$ and the correspondingly modified coarse bound for the error probability. The coarseness of the bound is thus compensated by eliminating "non-typical" (and hence far from being orthogonal) components of the signal state vectors. More precisely, let us fix small positive δ, and let λ_j be the eigenvalues, $|e_j>$ the eigenvectors of $\bar{S}_\pi$. Then the eigenvalues and eigenvectors of $\bar{S}_\pi^{\otimes n}$ are $\lambda_J=\lambda_{j_1}\cdot...\cdot\lambda_{j_n}$, $|e_J>=|e_{j_1}>\otimes...\otimes|e_{j_n}>$ where $J=(j_1,...,j_n)$. The spectral projector onto the *typical subspace* is defined as

$$P=\sum_{J\in B}|e_J><e_J|,$$

where $B = \{J : 2^{-n[H(\bar{S}_\pi)+\delta]} < \lambda_J < 2^{-n[H(\bar{S}_\pi)-\delta]}\}$. This concept plays a central role in "quantum data compression" [*Jozsa and Schumacher 1994*]. In a more mathematical context a similar notion appeared in [*Ohya and Petz 1993*], Theorem 1.18. Its application to the present problem relies upon the following two basic properties: first, by definition,

$$\|\bar{S}_\pi^{\otimes n} P\| < 2^{-n[H(\bar{S}_\pi)-\delta]}. \tag{11}$$

Second, for fixed small positive ϵ and large enough n

$$\mathrm{Tr}\bar{S}_\pi^{\otimes n}(I-P) \leq \epsilon, \tag{12}$$

because a sequence $J \in B$ is typical for the probability distribution given by eigenvalues λ_J in the sense of classical information theory [*Gallager 1968*], [*Cover and Thomas 1991*].

By replacing the signal state vectors $|\psi_{w^k}>$ with unnormalized vectors $|\tilde{\psi}_{w^k}>= P|\psi_{w^k}>$, defining the corresponding approximate maximum likelihood decision rule,and denoting $\tilde{\Gamma}(W)$ the corresponding Gram matrix, the modified upper bound

$$\min_X \bar{\lambda}(W,X) \leq \frac{1}{M}\{\mathrm{Sp}\left(E-\tilde{\Gamma}(W)\right) + \mathrm{Sp}\left(E-\tilde{\Gamma}(W)\right)^2\}$$

$$\leq \frac{1}{M}\sum_r\{2\mathrm{Tr}S_{w^r}(I-P) + \sum_{s\neq r}\mathrm{Tr}S_{w^r}PS_{w^s}P\}$$

is obtained in [*Hausladen et al. 1996*]. Applying the random coding and using (9) and the properties (11), (12) of the typical subspace, one gets for large n

$$p(M,n) \leq \{2\epsilon + (M-1)2^{-n[H(\bar{S}_\pi)-\delta]}\},$$

resulting in the inequality $C \geq \bar{C}$.

It is known, however, that in classical information theory the coding theorem can be proved without resorting to typical sequences, by mere use of clever estimates for the error probability [*Gallager 1968*]. Moreover, in this way one obtains the exponential rate of convergence for the error probability, the so called reliability function

$$E(R) = \lim_{n\to\infty}\sup\frac{1}{n}\log\frac{1}{p(\mathrm{e}^{nR},n)}\ , \quad 0<R<C\ .$$

This puts us onto the idea of trying to apply the random coding procedure directly to the tight bound (6) in the quantum case. This is realized in [*Burnashev and Holevo 1997*].

Theorem 1. *For all M, n and $0 \leq s \leq 1$*

$$\mathsf{E}\min_X \bar{\lambda}(W,X) \leq 2(M-1)^s\left[\mathrm{Tr}\,\bar{S}_\pi^{1+s}\right]^n. \tag{14}$$

It is natural to introduce the function $\mu(\pi,s)$ similar to analogous function in classical information theory [*Gallager 1968*], Sec. 5.6

$$\mu(\pi,s) = -\log\mathrm{Tr}\,\bar{S}_\pi^{1+s} = -\log\sum_j \lambda_j^{1+s}.$$

Then

$$E(R) \geq \max_\pi\max_{0\leq s\leq 1}\left(\mu(\pi,s)-sR\right) \equiv E_r(R).$$

On the other hand, it appears possible to apply in the quantum case the "expurgation" technique from [*Gallager 1968*], Sec. 5.7, resulting in the bound

$$E(R) \geq \max_{\pi} \max_{s \geq 1} (\tilde{\mu}(\pi, s) - sR) \equiv E_{ex}(R),$$

where

$$\tilde{\mu}(\pi, s) = -s \ln \sum_{i,k} \pi_i \pi_k | < \psi_i | \psi_k > |^{\frac{2}{s}}.$$

The behavior of the lower bounds $E_r(R), E_{ex}(R)$ can be studied by the methods of classical information theory, see [*Burnashev and Holevo 1997*]. In particular, it follows easily that $C \geq \max_\pi \mu'(\pi, 0) = \bar{C}$. Thus the rate $C - \delta$ can be attained with the approximate maximum likelihood decision rule (5), (4) without projecting onto the typical subspace.

We also remark that

$$\tilde{\mu}(\pi, 1) = \mu(\pi, 1) = -\log \mathrm{Tr} \bar{S}_\pi^2, \tag{15}$$

and that the common linear portion of the functions $E_r(R), E_{ex}(R)$ is just $\mu(\pi, 1) - R$.

III. General signal states with finite entropy

The general case is substantially more complicated already on the level of quantum Bayes problem; in particular, so far no upper bound for the average error probability is known, generalizing appropriately the geometrically simple bound (6). The proof given in [*Holevo 1996*], [*Schumacher and Westmoreland 1997*] is based rather on a non-commutative generalization of the idea of "jointly typical" sequences in classical theory [*Cover and Thomas 1991*]. This is realized by substituting in the average error probability (3) the decision rule

$$X_{w^k} = (\sum_{l=1}^{M} P P_{w^l} P)^{-\frac{1}{2}} P P_{w^k} P (\sum_{l=1}^{M} P P_{w^l} P)^{-\frac{1}{2}}, \tag{16}$$

where P_{w^k} is a proper generalization of the typical projection for the density operators S_{w^k} . The essential properties of P_{w^l} are

$$P_{w^k} \leq S_{w^k} 2^{n[\bar{H}_\pi(S_{(\cdot)}) + \delta]}, \tag{17}$$

$$\mathsf{E} \mathrm{Tr} S_{w^k} (I - P_{w^k}) \leq \epsilon, \tag{18}$$

After substituting (16) into (3) and performing a number of rather laborious steps intended to get rid of the normalization factors in (16) and thus to obtain an expression in which the different words are "decoupled", one arrives at the estimate

$$\min_X \bar{\lambda}(W, X) \leq \frac{1}{M} \sum_{k=1}^{M} \{ 3 \mathrm{Tr} S_{w^k} (I - P) + \sum_{l \neq k} \mathrm{Tr} P S_{w^k} P P_{w^l} + \mathrm{Tr} S_{w^k} (I - P_{w^k}) \}. \tag{19}$$

Taking the expectation and using (9), (12), (18), one obtains

$$\mathsf{E} \min_X \bar{\lambda}(W, X) \leq 4\epsilon + (M-1) \| \bar{S}_\pi^{\otimes n} P \| \mathrm{Tr} \, \mathsf{E} P_w,$$

for n large enough, hence by (11), (17)

$$p(M, n) \leq 4\epsilon + (M-1) 2^{-n[\Delta H(\pi) - 2\delta]}$$

implying $C \geq \bar{C}$. Combined with the entropy bound, this gives

Theorem 2. *The capacity of the channel with $H(S_i) < \infty$ is given by*

$$C = \max_{\pi}[H(\sum_{i\in A} \pi_i S_i) - \sum_{i\in A} \pi_i H(S_i)]. \tag{20}$$

For quasiclassical channel where the signal states are given by commuting density operators S_i one can use the classical bound of Theorem 5.6.1 [*Gallager 1968*] with transition probabilities $S(\omega|i)$, where $S(\omega|i)$ are the eigenvalues of S_i. In terms of the density operators it takes the form

$$\mathsf{E} \min_X \bar{\lambda}(W, X) \leq \min_{0\leq s\leq 1} (M-1)^s \left(\mathrm{Tr} \left[\sum_{i\in A} \pi_i S_i^{\frac{1}{1+s}} \right]^{1+s} \right)^n. \tag{21}$$

The righthand side of (21) is meaningful for arbitrary density operators, which gives a hope that this estimate could be generalized to the noncommutative case (note that for pure states S_i Theorem 1 gives twice the expression (21)). This would not only give a different proof of Theorem 2, but also a lower bound for the quantum reliability function in the case of general signal states, eventually with infinite entropy.

IV. Quantum channels with constrained inputs

In classical information theory direct coding theorems for channels with additive constraints are proved by using random coding with probability distribution (8) modified with a factor concentrated on words, for which the constraint holds close to the equality [*Gallager 1968*], Sec. 7.3. The same tool can be applied to quantum channels [*Holevo 1997*]. For definiteness in this section we take for the input alphabet A an arbitrary Borel subset in a finite-dimensional Euclidean space $\mathcal{E}$. We assume that the channel is given by *weakly continuous* mapping $x \to S_x$ from the input alphabet A to the set of density operators in $\mathcal{H}$. We assume that a *continuous* function f on $\mathcal{E}$ is fixed and consider the set $\mathcal{P}_1$ of probability measures π on A satisfying

$$\int_A f(x)\pi(dx) \leq E. \tag{22}$$

For arbitrary $\pi \in \mathcal{P}_1$ consider the quantity

$$\Delta H(\pi) = H(\bar{S}_\pi) - \int_A H(S_x)\pi(dx), \tag{23}$$

where $\bar{S}_\pi = \int_A S_x \pi(dx)$. Assuming the condition

$$\sup_{\pi\in\mathcal{P}_1} H(\bar{S}_\pi) < \infty, \tag{24}$$

we denote

$$\bar{C} = \sup_{\pi\in\mathcal{P}_1} \Delta H(\pi). \tag{25}$$

Let $p(M, n)$ denote the infimum of the average error probability over all codes of size M with words $w = (x_1, \ldots, x_n)$ satisfying the additive constraint

$$f(x_1) + \ldots + f(x_n) \leq nE. \tag{26}$$

Theorem 3. *Under the condition (24) the capacity of the channel with the input constraint (26) is given by (25), i. e.* $p(e^{n(\bar{C}-\delta)}, n) \to 0$ *, and* $p(e^{n(\bar{C}+\delta)}, n) \not\to 0$ *for* $\delta > 0$ *as* $n \to \infty$.

The proof uses the inequality (19) with the random coding modified as described in [*Gallager 1968*], Sec. 7.3. The same method combined with the estimate (14) for pure state channels gives lower bound for the reliability function modified with the factor const $\cdot \mathrm{e}^{r[f(x)-E]}$, with $r \geq 0$.

Theorem 3, when applied to quantum memoryless Gaussian channels with the energy constraint [*Holevo 1997*], allows us to prove for the first time their asymptotic equivalence, in the sense of the information capacity, to the corresponding quasiclassical "photon channels", extensively studied from the beginning of quantum communications [*Gordon 1964*], [*Lebedev and Levitin 1966*], [*Caves and Drummond 1994*]. In [*Holevo 1998*] it is shown that the equivalence extends also to waveform channels, in particular , that the infinite-band photon channel capacity [*Lebedev and Levitin 1966*]

$$C = \sqrt{\frac{\pi}{3}} \left[\sqrt{\frac{N+E}{\hbar}} - \sqrt{\frac{N}{\hbar}} \right]$$

is equal to the properly defined capacity of the quantum Gaussian channel

$$Y(t) = x(t) + Z(t); \qquad t \in [0,T], \quad T \to \infty,$$

where $x(t)$ is the classical signal subject to the energy constraint $\frac{1}{4\pi}\int_0^T x(t)^2 dt \leq ET$, and $Z(t)$ is the equilibrium quantum Gaussian noise having the commutator

$$[Z(t), Z(s)] = 2i\hbar \int_0^\infty \omega \sin\omega(s-t) d\omega = 2i\hbar\pi\delta'(t-s),$$

zero mean, and the correlation function $\langle Z(t)Z(s)\rangle = B_N(t-s) + K(t-s)$, with

$$B_N(t) = 2\hbar \int_0^\infty \omega \frac{\cos\omega t}{\mathrm{e}^{\beta\hbar\omega}-1} d\omega = 2\hbar\,\mathrm{Re} \sum_{k=1}^\infty \frac{1}{(k\beta\hbar + it)^2},$$

N and β are related by $N = \frac{1}{4\pi}B_N(0) = \pi/12\hbar\beta^2$, and

$$K(t) = \hbar \int_0^\infty \omega \mathrm{e}^{-i\omega t} d\omega = -\hbar[t^{-2} - i\pi\delta'(t)]$$

is the zero temperature correlation ($B_0(t) \equiv 0$) .

The classical capacity of the squeezed-state channel was computed by Holevo, Sohma and Hirota [*Holevo et al. 1999*].

V. Some further problems

The present paper was entirely devoted to the "classical-quantum" channels, in terminology of [*Holevo 1977*]), and even in this case there are open problems, some of which were mentioned above. Such channels can alternatively be described by (completely) positive maps from noncommutative algebra of operators in $\mathcal{H}$ to commutative algebra of functions on the input alphabet. More general "quantum-quantum" channels are described by completely positive maps between noncommutative algebras. The definition of

capacity and the quantum entropy bound can be generalized to this case [*Holevo 1977*], [*Ohya and Petz 1993*]. However for such channels the new difficult problem of optimization with respect to coding maps arises. In particular, it is not yet known, whether the entropy bound optimized in this way is an additive function on the product channel. An interesting preliminary investigation of this situation is contained in the paper by Bennett, Fuchs and Smolin [*Bennett et al. 1996*].

All these problems address transmission of classical information through quantum channels. There is yet "more quantum" domain of problems concerning reliable transmission of entire quantum states under a given fidelity criterion. The very definition of the relevant "quantum information" is far from obvious. Important steps in this direction were made by Barnum, Nielsen and Schumacher [*Barnum et al. 1997*] , who in particular suggested a tentative converse of the relevant coding theorem. However the proof of the corresponding direct theorem remains an open question.

References

[*Barnum et al. 1997*] H. Barnum, M. A. Nielsen, B. Schumacher,LANL Report no. quant-ph/9702049, Feb. 1997.

[*Bennett et al. 1996*] C. H. Bennett, C. A. Fuchs, J. A. Smolin, "Entanglement-enhanced classical communication on a noisy quantum channel," LANL Report no. quant-ph/9611006, Nov. 1996. Appeared in *Quantum Communication, Computing and Measurement*, Proc. QCM96, ed. by O. Hirota, A. S. Holevo and C. M. Caves, New York: Plenum 1997, pp. 79-88.

[*Burnashev and Holevo 1997*] M. V. Burnashev, A. S. Holevo, "On reliability function of quantum communication channel," LANL Report no. quant-ph/9703013, March 1997.

[*Caves and Drummond 1994*] C. M. Caves, P. B. Drummond, "Quantum limits of bosonic communication rates," *Rev. Mod. Phys.*, vol. 66, no. 2, pp. 481-538 1994.

[*Cover and Thomas 1991*] T. M. Cover, J. A. Thomas, *Elements of Information Theory*. New York: Wiley 1991.

[*Forney 1963*] G. D. Forney, Jr., S. M. Thesis, MIT 1963 (unpublished).

[*Gallager 1968*] R. G. Gallager, *Information Theory and Reliable Communications*. New York: J. Wiley 1968.

[*Gordon 1964*] J. P. Gordon, "Noise at optical frequencies; information theory," in *Quantum Electronics and Coherent Light*, Proc. Int. School Phys. "Enrico Fermi", Course XXXI, ed. P. A. Miles, New York: Academic Press 1964, pp.156-181.

[*Hausladen et al. 1996*] P. Hausladen, R. Jozsa, B. Schumacher, M. Westmoreland, W. Wootters, "Classical information capacity of a quantum channel," *Phys. Rev. A*, vol. 54, no. 3, pp. 1869-1876 1996.

[*Helstrom 1976*] C. W. Helstrom, *Quantum Detection and Estimation Theory*. New York: Academic Press 1976.

[*Holevo 1973]* A. S. Holevo, "Some estimates of the information transmitted by quantum communication channel," *Probl. Peredachi Inform.*, vol. 9, no. 3, pp. 3-11 1973. (English transl.: *Problems of Inform. Transm.*, vol. 9, no. 3, pp. 177-183 1973).

[*Holevo 1977]* A. S. Holevo, "Problems in the mathematical theory of quantum communication channels," *Rep. Math. Phys.*, vol. 12, no. 2, pp. 273-278 1977.

[*Holevo 1978]* A. S. Holevo, "On asymptotically optimal hypotheses testing in quantum statistics," *Theor. Probab. and Appl.*, vol. 23, no. 2, pp. 429-432 1978.

[*Holevo 1979]* A. S. Holevo, "On the capacity of quantum communication channel," *Probl. Peredachi Inform.*, vol. 15, no. 4, pp. 3-11 1979. (English transl.: *Problems of Inform. Transm.*, vol. 15, no. 4, pp. 247-253 1979).

[*Holevo 1982]* A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*. Amsterdam: North Holland 1982.

[*Holevo 1996]* A. S. Holevo, "The capacity of quantum communication channel with general signal states", LANL Report no. quant-ph/9611023, Nov. 1996. Appeared in *IEEE Trans. Inform. Theory*, 1998.

[*Holevo 1997]* A. S. Holevo, "On quantum communication channels with constrained inputs," LANL Report no. quant-ph/9705054, May 1997.

[*Holevo 1998]* A. S. Holevo, "Quantum coding theorems," Uspekhi Mat. Nauk, vol. 53, N6, 1998, 193-230 (rus); Extended English version: LANL e-print quant-ph/9809023.

[*Holevo et al. 1999]* A. S. Holevo, M. Sohma and O. Hirota, "The capacity of quantum Gaussian channels," Phys. Rev. A, vol. 59, N3, 1999, 1820-1828.

[*Jozsa and Schumacher 1994]* R. Jozsa, B. Schumacher, "A new proof of the quantum noiseless coding theorem," *J. Modern Optics*, vol. 41, no. 12, pp. 2343-2349 1994.

[*Kato et al. 1996]* K. Kato, M. Osaki, T. Suzuki, M. Ban, O. Hirota, "Upper bound of the accessible information and lower bound of the Bayes cost in quantum signal detection processes," *Phys. Rev. A*, vol. 54, no. 4, pp. 2718-2727 1996. in *Quantum Communication, Computing and Measurement*, Proc. QCM96, ed. by O. Hirota, A. S. Holevo and C. M. Caves, New York: Plenum 1997, pp.63-71.

[*Lebedev and Levitin 1966]* D. S. Lebedev, L. B. Levitin, "The maximal amount of information transmissible by an electromagnetic field," *Information and Control*, vol. 9, pp. 1-22 1966.

[*Levitin 1969]* L. B. Levitin, "On quantum measure of information," Proc. IV All-Union Conference on information transmission and coding theory, pp. 111-115, Tashkent 1969.

[*Lindblad 1973-1975]* G. Lindblad, "Entropy, information and quantum measurements," *Commun. Math. Phys.*, vol. 33, pp. 305-322 1973; "Expectations and entropy inequalities for finite quantum systems," *ibid.*, vol. 39, pp. 111-119 1974; "Completely positive maps and entropy inequalities," *ibid.*, vol. 40, pp. 147-151 1975.

[*Ohya and Petz 1993]* M. Ohya, D. Petz, *Quantum Entropy and Its Use*. Berlin: Springer-Verlag, 1993.

[*Schumacher and Westmoreland 1997]* B. Schumacher, M. D. Westmoreland, "Sending classical information via noisy quantum channel," Phys. Rev. A. 1997.

[*Stratonovich and Vantsjan 1978]* R. L. Stratonovich, A. G. Vantsjan, "On asymptotically errorless decoding in pure quantum channels," *Probl. Control Inform. Theory*, vol. 7, no.3, pp. 161-174 1978.

[*Uhlmann 1977]* A. Uhlmann, "Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory," *Commun. Math. Phys.*, vol. 54, pp. 21-32 1977.

[*Yuen and Ozawa 1993]* H. P. Yuen and M. Ozawa, "Ultimate information carrying limit of quantum systems," *Phys. Rev. Lett.*, vol. 70, No. 4, pp. 363-366 1993.

Peter Høyer

The Subgroup Problem via *State Distinguishability.*

Abstract: All known quantum algorithms which run super-polynomially faster than the most efficient probabilistic classical algorithm solve special cases of what is called the Abelian Subgroup Problem. This general formulation includes Shor's algorithms for factoring and finding discrete logarithms.

The Abelian subgroup problem has a simple and beautiful quantum solution, and we give a very brief review of it. The heart of the idea behind the Abelian solution is Fourier analysis on Abelian groups and thus it is natural to consider how Fourier analysis on non-commutative groups might help in solving the general hidden subgroup problem. This is so far the most studied approach for possible generalizations to non-commutative groups. It is the difficulties of Fourier analysis on non-commutative groups that makes this a challenging approach.

The hidden subgroup problem might be most easily understood when casted as a state-distinguishability problem: We are given a source that emits a fixed state out of a set of possible non-orthogonal mixed states. Our problem is to determine which one of those possible states the source emits. We can request several copies of the state and we are allowed to perform measurements on the joint system. For the subgroup problem, the set of possible states contains one state for each subgroup, and the state emitted by the source is the state corresponding to the hidden subgroup. Determining which state the source emits also determines the hidden subgroup.

From the outcomes of our measurements, we can consider probabilistic analysis like Bayesian analysis (to conclude that, say, certain states are more likely than others), or we can limit ourselves to analytical tools as used in search theory (to rule out those states for which the outcomes would not have been possible).

We discuss and exemplify all of the ideas above. No prior knowledge about the non-commutative problem will be assumed.

Uffe Haagerup

Spectra of random matrices and random operators on Hilbert spaces.

Abstract: The eigenvalue distribution of selfadjoint random matrices were first studied by E. Wigner in 1955, where he proved the so-called semicircle law for the eigenvalue distribution of large symmetric random matrices A with independent entries $a(i,j)$ for $i \leq j$. In 1967 a similar analysis were carried out by Marchenko and Pastur for the asymptotic eigenvalue distribution of Wishart matrices, i.e. symmetric random matrices of the form $S = B^T B$, (where B is an $m \times n$ random matrix with independent entries $b(i,j)$ in the limit where m and n both goes to infinity while their ratio m/n stays almost constant). In both cases further developments have been made by mathematical physicists and by probabilists, particularly the asymptotic behaviour of the largest and smallest eigenvalue have been studied by Tracy and Widom et. al. in the "Wigner case" and by Geman, Silverstein, Bai, Yin et. al. in the "Wishart case".

In the talk, I will give a survey of these result, and present some recent results by Steen Thorbjørnsen and the speaker, where the asymptotic behaviour of the upper and lower bound of the spectrum of certain random operators on an infinite dimensional Hilbert space is studied. The motivation for the latter research does not come from physics or probability theory, but from problems in pure mathematics, more precisely from K-theory of C^*-algebras.

Göran Lindblad

Gaussian maps and processes in quantum systems.

Abstract: Quasifree maps on the CCR algebra can be used to construct models of relaxation processes, information transfer, measurement processes, and many related types of irreversible quantum dynamics. This set of models are simple and convenient, allowing rather explicit calculations in many cases; they are particularly useful in quantum optics problems. In this talk I will give one or two examples. One if them involves a family of quasifree quantum cloning maps. They have a partial order which allow us to discuss the minimal noise added in such copying of quantum information. Multiple clones are easily handled. The formalism is very close to that for quantum amplifiers in quantum optics.

STOCHASTIC CALCULATIONS IN QUANTUM PHYSICS VIA NUMERICAL INTEGRATION IN METRIC SPACES [1]

Yu. Yu. Lobanov [2]

Lab. of Computing Techniques and Automation,
Joint Institute for Nuclear Research,
Dubna, Moscow Region, 141980 Russia

A method of numerical evaluation of functional integrals with respect to probabilistic measures of the Gaussian type is described. This method does not require any preliminary discretization of time and allows to use the more preferable deterministic algorithms in computations instead of traditional probabilistic ones. This approach is proven to have important advantages over the conventional lattice Monte Carlo method including the higher efficiency of computations. The method is suitable for numerical study of the complex quantum systems, singularities like phase transitions and critical phenomena, nonperturbative problems in Euclidean quantum field theory, etc. Examples of application of the method are presented.

Numerical functional integration is one of the most important methods for calculation of characteristics of the complex systems in quantum and statistical physics [1,2]. It is very useful when other methods like perturbation expansion, semiclassical approximation, etc. cannot be applied, e.g. in the study of the topological structure of vacuum in gauge theories [3], study of the Markovian open quantum systems [4], etc. Functional integrals provide a useful tool for studying the variety of systems which are otherwise not amenable to definitive analysis through perturbative, variational or stationary-phase approximations, etc. (see [5]). However, the existing approaches to functional integrals in physics (path integrals) are not always quite correct in a mathematical sense and the usual method of their computation is Monte Carlo (MC), which assumes the lattice discretization of space and time and gives the results only as probabilistic averages while requiring too much computer resources to obtain the good statistics. When the nuclear many-body problem is being formulated on a lattice, the computation of characteristics of heavy nucleus would require the lattice size which is four to five orders of magnitude more than any lattice gauge calculation [5]. The results of computation of the binding energy even for the light nuclei by means of variational or Green function Monte Carlo method, as well as by the coupled cluster and Faddeev equation methods differ from one another and from the experimental values more than the estimated errors of calculation (see [6],[7]). It is clear that the creation of new methods for solution of such complicated problems is of high importance.

Based on the rigorous definition of an integral in complete separable metric space we elaborated the new method of computation of functional integrals [8]. This method does not require preliminary discretization of space and time which is important for studying the singularities like phase transitions. Our approach allows to obtain the mathematically well-grounded physical results with a guaranteed (not probabilistic) error estimate. We have found this method to be suitable for studying the complex quantum systems [9]. Numerical computations show [10] that our method gives significant economy of computer time and memory versus conventional

[1] The work was partially supported by RFBR, Grant 98-01-00190

[2] E-mail: lobanov@main1.jinr.ru

MC method used by other authors in the problems which we have considered. This approach is also proven to have advantages in the case of high dimensions [11].

According to the Feynman-Kac formula, matrix element $Z_{if}(\beta)$ of the system evolution operator $\exp\{-\beta H\}$, where the Hamiltonian $H = -\frac{1}{2}\Delta + V$ and V is the potential of interaction is written as follows:

$$Z_{if}(\beta) = \int\limits_{C_{\mathrm{x}_i,\mathrm{x}_f,\beta}} \exp\Big\{-\int\limits_0^\beta V[x(t)]\,dt\Big\}\,d_W x. \tag{1}$$

The integration in (1) is performed over the manifold of continuous functions $x(t) \in C[0,\beta]$ satisfying $x(0) = \mathrm{x}_i$, $x(\beta) = \mathrm{x}_f$ with respect to the conditional Wiener measure $d_W x$. It should be noted that as distinct from the conventional path-integral approach, integral (1) does not contain the kinetic term (the first derivative squared) since it is included now to the measure of integration, and this circumstance simplifies the numerical simulations. Taking the trace (calculating the statistical sum)

$$Z(\beta) = \mathrm{Tr}\,\exp\{-\beta\,H\} = \int\limits_{-\infty}^{\infty} Z(\mathrm{x},\mathrm{x},\beta)\,d\mathrm{x}$$

we can compute various physical quantities [10]. Making the change of the functional variables we represent $Z(\mathrm{x},\mathrm{x},\beta)$ in a form of the integral with respect to normalized conditional Wiener measure in the space $C \equiv \{C[0,1],\ x(0) = x(1) = 0\}$. In order to calculate this integral we use one of our approximation formulas. For example, under certain conditions on a real functional F we have proven that the following relation for the m-fold conditional Wiener integral is valid:

$$\int\limits_{C^m} P[x_1,\ldots,x_m]\,F[x_1,\ldots,x_m]\,d_W^{(m)}x =$$

$$(2m)^{-1}\prod_{i=1}^{m}\Big(\frac{\sqrt{2p_i}}{\sin\sqrt{2p_i}}\Big)^{\frac{1}{2}}\exp\Big\{\frac{q_i^2}{(2p_i)^{3/2}}\Big[\tan\sqrt{\frac{1}{2}p_i} - \sqrt{\frac{1}{2}p_i}\Big]\Big\} \tag{2}$$

$$\times\sum_{i=1}^{m}\int\limits_{-1}^{1} F[a_1(\cdot),\ldots,a_{i-1}(\cdot),\sqrt{m}\,\Psi_i(v,\cdot)+a_i(\cdot),a_{i+1}(\cdot),\ldots,a_m(\cdot)]\,dv + \mathcal{R}_m(F).$$

Here $P[x_1,\ldots,x_m] = \exp\{\sum_{i=1}^m \int\limits_0^1 [p_i\,x_i^2(t) + q_i\,x_i(t)]\,dt\}$ is the weight functional, $a_i(t)$, $\Psi_i(v_i,t)$ are some expressions and $\mathcal{R}_m(F)$ is a remainder term which can be estimated in accordance with our theorem on convergence of approximations [11]. Approximation formula (2) is exact (i.e. in this case $\mathcal{R}_m(F) = 0$) when $F[x]$ is a polynomial functional of the third total degree on the space $C^m = C \times \ldots \times C$. For the functional integrals without weight, formula (2) can be applied by setting $p_i = q_i = 0$. We compute the Riemann integral in (2) using Gaussian or Tchebyshev quadratures with the relative accuacy 0.01.

One of the important areas of application of the functional integration method is calculation of characteristics of the interacting many-particle systems [12]. We have studied the nuclear model proposed in [13] with the following potential of interaction:

$$V(\mathrm{x}) = \sum_{k=1}^{2}\frac{V_k}{\sigma_k\sqrt{\pi}}\exp\{-\frac{\mathrm{x}^2}{\sigma_k^2}\},$$

$V_1 = 12,\ V_2 = -12,\ \sigma_1 = 0.2,\ \sigma_2 = 0.8,\ \hbar = m = 1$, in units of length $l_0 = 1.89Fm$ and energy $E_0 = \hbar^2/(m\,l_o^2) = 11.6\ MeV$.

For the system of two nucleons (deuteron) our result of computation of the binding energy is $E_d = 2.4\,MeV$ which can be compared with the experimental data $E_{ex} = 2.2\,MeV$ and with the prediction of the semi-empirical formula [14] $E_{se} = 3.5\,MeV$. The result can be considered as satisfactory for such a simple model and it provides the basis for study of the more realistic types of interaction.

For the system of four nucleons (α-particle) we computed the binding energy with result $E_F = 27.6\,MeV$ which is close to the experimental value $E_{ex} = 28.3\,MeV$ [6]. The prediction of the semi-empirical formula is $E_{se} = 18.8\,MeV$. We compare our results with those obtained in [13] by means of the lattice Monte Carlo simulations in the framework of the same model. Since the results of [13] are given in a graphical form, we reproduce them in Fig.1. It shows the binding energy of 4 particles in the dimensionless units E/E_0 as a function of the lattice spacing ε, obtained in [13] by simulation of 10^4 events. E_T and E_N denote the trial energy and the normalization energy respectively, and E_M are the values obtained by the Metropolis algorithm [13]. The problem of extrapolation of results to the continuum limit ($\varepsilon \to 0$) has been discussed in [13] and [15] and found to be not simple enough. In contrast, in our approach we do not have such problems since we do not introduce the lattice discretization and consider the quantities directly in continuum limit. Our functional-integral result E_F/E_0 is shown in Fig.1 at the point $\varepsilon = 0$.

Numerical experiment based on approximate path integration is widely used for investigation of the topological structure of vacuum in gauge theories. However, the values of the topological susceptibility (the average topological charge) obtained in various works by the lattice MC simulations differ much from one another and from the phenomenological estimates. The origin of such a difference may lie both in the non-uniqueness of definition of topological charge on a lattice and in the possibility of existence of specific systematic error introduced by the finiteness of the lattice spacing. We have studied the topological effects on example of the quantum pendulum, i.e. system described by the Hamiltonian

$$H = \frac{1}{2}\,\lambda\,\hat{p}^2 + V, \qquad V(\mathrm{x}) = \frac{\omega^2}{\lambda}\,(1 - \cos\,\mathrm{x}),$$

ω is a frequency of the small oscillations, λ is a coupling constant. The results of computation of the topological susceptibility $\frac{1}{\beta_0} < Q^2 >$ where Q is the topological charge are shown in Fig.2 as a function of the parameter $\rho = \frac{\omega}{\lambda}$, $\beta_0 = \omega\,\beta$, for $\beta_0 = 7$, $\lambda = 1$, in a logarithmic scale. The crosses in Fig.2 represent the value $\frac{1}{\varepsilon N} < Q^2 >$, obtained in [16] by the MC method on a lattice with $N = 100$, $\varepsilon = \omega a = 1$ (N is a number of nodes, a is a lattice spacing). Full line denotes the theoretical prediction of the dilute instanton gas approximation (DGA), which is valid for the large ρ, dotted line is the high temperature expansion (HTE) in the continuum limit ($\varepsilon \to 0$, $N \to \infty$, εN is fixed). In order to compare $< Q^2 >$ with the theoretical estimates more carefully, let us consider the quantity

$$D = \frac{1}{\beta_0} < Q^2 > \rho^{-1/2}\, e^{S\rho}.$$

In the continuum limit (in this case $S = 8$) and for sufficiently large ρ we have $D = \frac{8}{\sqrt{\pi}} \simeq 4.51$. The results of our computations are shown in Fig.3; they agree well with the theoretical predictions. The crosses denote the results of [16] obtained with $\varepsilon = 1$, $N = 100$, $S = 7.87$ (action for an instanton on such a lattice). ¿From the results of paper [16] one can not draw the conclusion about the validity of the dilute instanton gas approximation. For the sake of clarity, in [16] the computatins with $\varepsilon = 0.6$ (at the point $\rho = 0.7$) have been performed, and the result was $D = 3.3$, that is greater than with $\varepsilon = 1$, but still less than the theoretical value. Further decreese of ε in [16] has not been performed due to the difficulties connected with the growth

of the instanton size and with the slowdown of the convergence of iterations. In contrast, in our approach these problems do not arise since we do not introduce the lattice discretization (i.e. $\varepsilon = 0$).

Fitting the parameters by means of the least square method in [16] the authors obtained $D = 2.98$; $p = 0.46$. According to our results shown in Fig.3, we have $D = 4.25$; $p = 0.493$, which agrees well with the theoretical predictions (in continuum limit the theoretical value is $p = 0.5$).

The values of the θ-vacua energy are shown in Fig.4 for $\rho = 1$ and for $\rho = 1.2$. It is seen that our resuls are in a good agreement with the dilute instanton gas approximation which is also shown in Fig.4 by the full and by the dotted lines.

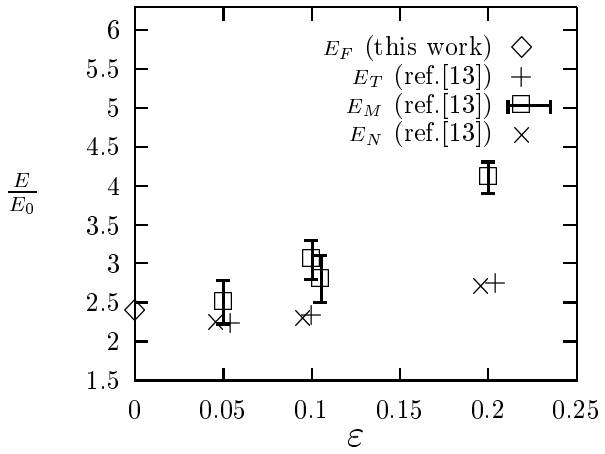

Fig.1 Binding energy of 4-particle bound state

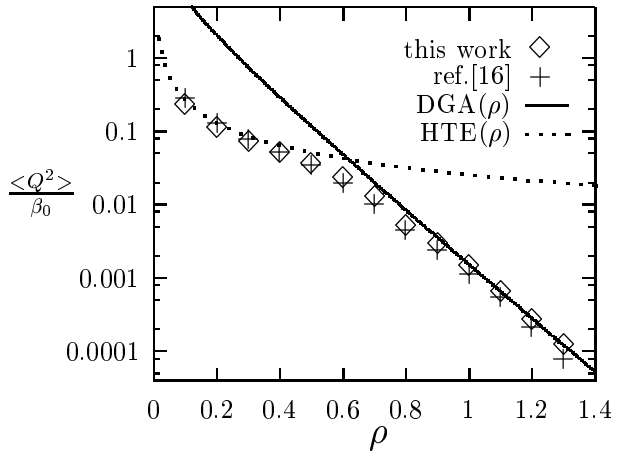

Fig.2 The topological susceptibility

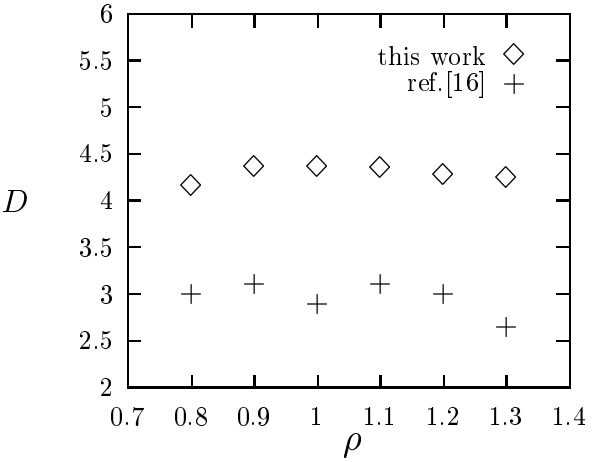

Fig.3 Comparison with the theory

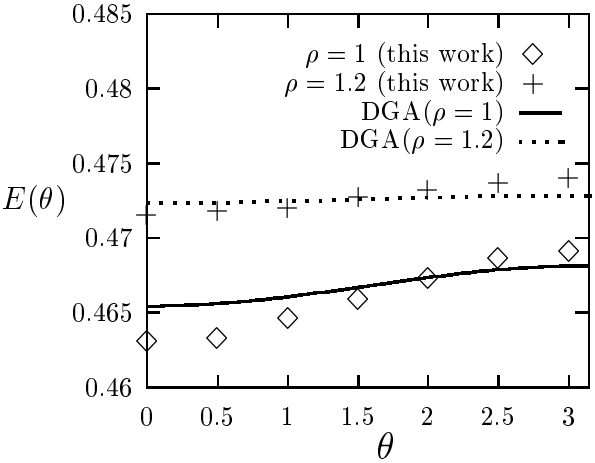

Fig.4 The θ-vacua energy

We have considered the problem of calculating the probability of nuclear fusion in reactions with heavy ions. This problem stems from the necessity of choosing the optimum conditions in the experiments on synthesis of super-heavy elements. The relevant model for studying the processes in the system of two nuclei has been proposed in [17]. It has been shown in [18],[19] that the problem of quantum tunneling with dissipation in this model can be studied in the framework of the methods developed for open quantum systems. Path integrals appear to be a convenient tool in this approach. Based on the Markovian master equation [20] the following expression for the propagator of the system via Lagrangian path integral has been derived [4]:

$$J(q,q',t;q_0,q_0',0) = \int\limits_{(q_0,0)}^{(q,t)} D[q] \int\limits_{(q_0',0)}^{(q',t)} D[q'] \ \exp\Big\{\frac{i}{\hbar}(S_{cl}[q] - S_{cl}[q'])\Big\}\, F[q,q']. \qquad (3)$$

Here S_{cl} is the classical action, $F[q,q']$ is the influence functional [4].

We have obtained the expression for the right-hand side in the form of conditional Wiener integral and computed it using our numerical method [21]. We use various types of the potential V related to the considered model, i.e. of rectangular and of polynomial form. In the particular case of harmonic oscillator potential there is an explicit expression for the propagator [4]. It allows us to compare the numerical results with the exact values and to study the accuracy of the method. This comparison for $V(x) = \frac{1}{2}x^2$, $q = q_0 = 0.7$, $q' = q_0' = 1.0$ and various t is given in Table 1.

Table 1: Propagator of the open quantum system

t	computed	exact
0.2	0.8305	0.8363
0.4	0.4412	0.4391
0.6	0.3084	0.3070
0.8	0.2404	0.2410
1.0	0.1967	0.2015

It is seen that the agreement of results is good. Further work on computations in this model is in progress.

References

1. Yu.Yu.Lobanov and E.P.Zhidkov (Eds), Programming and Mathematical Techniques in Physics (World Scientific, Singapore, 1994).

2. J.Glimm and A.Jaffe, Quantum Physics. A Functional Integral Point of View (Springer, New York, 1987).

3. H.Abe, *Int. J. Mod. Phys. A* **12** (1997) 3775.

4. W.T.Strunz, *J. Phys. A: Math. Gen.* **30** (1997) 4053.

5. J.W.Negele, *J. Stat. Phys.* **43** (1986) 991.

6. R.B.Wiringa, in: Recent Progress in Many-Body Theories, (Plenum Press, New York, 1992) p 39.

7. J.Carlson, *Phys. Rev. C* **36** (1987) 2026.

8. Yu.Yu.Lobanov, in: Path Integrals from meV to MeV: Tutzing'92 (World Scientific, Singapore, 1993) p 26.

9. Yu.Yu.Lobanov, *Math. and Comput. in Simul.*, **39** (1995) 239.

10. Yu.Yu.Lobanov, *Comp. Phys. Comm.*, **99** (1996) 59.

11. Yu.Yu.Lobanov, R.R.Shahbagian and E.P.Zhidkov, *J. Comput. Appl. Math.* **70** (1996) 145.

12. Yu.Yu.Lobanov, *J. Phys. A: Math. Gen.* **29** (1996) 6653.

13. J.W.Negele, in: Time-Dependent Hartree-Fock and Beyond (Springer, Berlin, 1982) p 198.

14. A.Bohr and B.Mottelson, Nuclear Structure (W.A.Benjamin, Inc., New York, 1969).

15. C.Alexandrou and J.W.Negele *Phys. Rev. C* **37** (1988) 1513.

16. B.Bunk and U.Wolff, *Nucl. Phys. B*, **215** (1983) 495.

17. E.A.Cherepanov, V.V.Volkov, N.V.Antonenko, V.B.Permjakov and A.V.Nasirov, *Nucl. Phys. A*, **459** (1996) 145.

18. G.G.Adamian, N.V.Antonenko and W.Scheid, Preprint JINR E4-97-366, Dubna, 1997.

19. R.V.Jolos, A.K.Nasirov and A.I.Muminov, Preprint JINR E4-97-302, Dubna, 1997.

20. G.Lindblad, *Commun. Math. Phys.*, **48** (1976) 119.

21. Yu.Yu.Lobanov and V.D.Rushai, in: Path Integrals from peV to TeV: 50 Years after Feynman's Paper, World Scientific, 1999, p 581.

Elena R. Loubenets

Continuous in Time Nondemolition Observation of an Open System.

1. Introduction

The problem of description of continuous in time measurement of an open system was considered in the well known papers of A.Barchielli, V.P.Belavkin, E.B. Davies, L.Diosi, A.S.Holevo, M.Ozawa, A.Peres [2-18, 22,23].
The special case of continuous in time indirect measurement of an open system called a nondemolition case was introduced by V.P.Belavkin in [9-15]. The well known quantum filtering equation [9-15, 5] describes the posterior quantum stochastic evolution of an open system subjected to continuous in time nondemolition observation in the special nondemolition measurement model of the so called quantum stochastic mechanics.

In this talk we present the approach [1,20] to the description of continuous in time indirect nondemolition observation of an open system which is based on the methods of quantum theory and the Schrödinger equation but not on the methods of quantum stochastic calculus.

2. The description of continuous in time direct observation of a quantum system

Continuous in time direct measurement of a quantum observable is possible if and only if in the Heisenberg picture this observable is a nondemolition one, that is if it satisfies the conditions

$$\hat{A}_H(t) = \hat{A}_H^+(t), \qquad [\hat{A}_H(t),\ \hat{A}_H(t_1)] = 0, \quad \text{for} \forall t, t_1. \tag{1}$$

The process of continuous in time direct measurement of a nondemolition observable $\hat{A}_H(s)$ until the moment t is described by a projection-valued measure $\hat{P}^{(0,t]}_{\hat{A}_H}(\cdot)$ on the product space $\Omega^{(0,t]}$ of all sets of possible outcomes of continuous observation of $\hat{A}_H(s)$ at all moments of time $0 < s \le t$ until t. For different moments of time the introduced measures commute and are compatible.

3. The description of continuous in time indirect measurement of an open system

Let in the interaction picture induced by the free dynamics of reservoir the unitary time evolution of the extended system on the Hilbert space $\boldsymbol{H}_S \otimes \boldsymbol{H}_R$ be described by the unitary operator $\hat{U}(t,s)$. We consider the case when the indirect observation of an open system is performed by means of continuous in time direct observation of a nondemolition observable $\hat{Q}_H(t)$ of the reservoir modelling the measuring device

$$\hat{Q}_H(t) = \hat{U}^+(t,0)(\hat{I} \otimes \hat{Q}(t))\hat{U}(t,0). \tag{2}$$

The POV measure of the indirect measurement is given by

$$\hat{M}^{(0,t]}(E^{(0,t]}) = <\varphi_R, \hat{P}^{(0,t]}_{\hat{Q}_H}(E^{(0,t]})\,\varphi_R>_{H_R}, \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]} \tag{3}$$

on the product space $\Omega^{(0,t]}$ of all sets of possible outcomes of continuous in time indirect observation until the moment *t.*
The description of the process of continuous in time indirect measurement of an open system is similar to the description of a classic stochastic process with a scalar probability measure

(4) $$\mu^{(0,t]}(E^{(0,t]}) = \left\langle \psi_0, \hat{M}^{(0,t]}(E^{(0,t]})\psi_0 \right\rangle_{H_S}, \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]}$$

on $\Omega^{(0,t]}$ induced by initial states ψ_0 and φ_R of an open system and a reservoir, respectively. The consideration of the case when an open system and a reservoir are described initially by density operators is obvious.
The family of POV measures (3) (the family of scalar probability measures (4) as well) is compatible.

3. Nondemolition measurement

Consider a special case of continuous in time indirect measurement called a nondemolition case. The main principles of nondemolition observation were introduced by V.P.Belavkin in [9-15] and imply that in the Heisenberg picture the continuously observed nondemolition observable $\hat{Q}_H(t)$ commutes with all observables of an open system under $s \le t$.

(5) $$[\hat{Z}_H(t), \hat{Q}_H(s)] = 0,$$

In this representation we identify a nondemolition measurement as one defined by the conditions

(6) $$\begin{aligned} &\hat{Q}(t) = \hat{Q}^+(t), \quad [\hat{Q}(t), \hat{Q}(t_1)] = 0, \quad \text{for} \forall t, t_1. \\ &[\hat{U}(t,s), (\hat{I} \otimes \hat{Q}(r))] = 0, \quad \forall\, r < s \le t, \end{aligned}$$

which are sufficient for a continuous in time indirect measurement to be a nondemolition one according to V.P.Belavkin's definition. The families of nondemolition observables $\{\hat{Q}(s), 0 < s \le t\}$ and $\{\hat{Q}_H(s),\ 0 < s \le t\}$ are usually called [9-15] as input and output operator-valued processes, respectively.
In the case of continuous in time nondemolition measurement we derive:

(7) $$\begin{aligned} &\hat{P}^{(0,t]}_{\hat{Q}_H}(E^{(0,t]}) = \hat{U}^+(t_1, 0)(\hat{I} \otimes \hat{P}^{(0,t]}_{\hat{Q}}(E^{(0,t]}))\hat{U}(t_1, 0), \\ &\textit{for} \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]}, \forall t_1 \ge t, \end{aligned}$$

where $\hat{P}^{(0,t]}_{Q}(\cdot)$ is a projection -valued measure describing the process of continuous in time direct measurement of the input nondemolition observable $\hat{Q}(t)$ until the moment *t.* Consequently, the POV measure $\hat{M}^{(0,t]}(\cdot)$ given by (3) in the nondemolition case has the form:

(8) $$\begin{aligned} &\hat{M}^{(0,t]}(E^{(0,t]}) = \left\langle \varphi_R, \hat{U}^+(t,0)(\hat{I} \otimes \hat{P}^{(0,t]}_{\hat{Q}}(E^{(0,t]}))\hat{U}(t,0)\varphi_R \right\rangle_{H_R}, \\ &\forall E^{(0,t]} \subseteq \Omega^{(0,t]}. \end{aligned}$$

For simplicity, suppose the input observable $\hat{Q}(t)$ is continuous with respect to t in the strong operator topology and for any moment t the family of commuting operators $\{\hat{Q}(s)\}_{s\in(0,t]}$ on $\boldsymbol{H}_R$ is full. Denote by $Q(s) \in R$ a possible outcome of a direct measurement of $\hat{Q}_H(s)$ at the moment s. A time-ordered sequence $Q^t = \{Q(s)\}_{s\in(0,t]}$ of possible outcomes of continuous measurement of $\hat{Q}_H(s)$ at all moments of time until *t* is an element of $\Omega^{(0,t]}$. Due to the properties of a projection-valued measure corresponding to a self-adjoint operator, a space $\Omega(s)$ of possible outcomes of measurement at the moment s can be identified with $\boldsymbol{R}$. Consequently, the product space $\Omega^{(0,t]}$ can be identified with a space of

trajectories. Due to our assumption of continuity of the operator $\hat{Q}(t)$, $\Omega^{(0,t]}$ is a space of trajectories "continuous" with respect to t in the sense of the scalar probability measure induced by the input process

(9) $$\nu_{\hat{Q}}^{(0,t]}(E^{(0,t]}) = <\varphi_R, \hat{P}_{\hat{Q}}^{(0,t]}(E^{(0,t]})\,\varphi_R>.$$

At any moment of t the following resolution of identity on the Hilbert space $\boldsymbol{H}_R$ of a reservoir is valid

(10) $$\hat{I} = \int\limits_{Q^t \in \Omega^{(0,t]}} \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t).$$

Introduce a linear operator $\hat{V}[Q^t, t; \varphi_R]$ on the Hilbert space $\mathbf{H}_S$ of an open system by the following relation

(11) $$(\hat{V}[Q^t, t; \varphi_R] \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))(\psi \otimes \varphi_R) = (\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))\,\hat{U}(t,0)(\psi \otimes \varphi_R),$$
$$for \quad \forall \psi \in H_S,$$

understood in the infinitesimal sense. The POV measure, defined by (8), is

(12) $$\hat{M}^{(0,t]}(E^{(0,t]}) = \int\limits_{Q^t \in E^{((0,t]}} \hat{V}^+[Q^t, t; \varphi_R]\,\hat{V}[Q^t, t; \varphi_R]\,\nu_{\hat{Q}}^{(0,t]}(dQ^t), \quad for \quad \forall E^{(0,t]} \subseteq \Omega^{(0,t]}.$$

The scalar probability measure $\mu^{(0,t]}(\cdot)$ of the output process is given through a scalar probability measure of the input process by

(13) $$\mu^{(0,t]}(E^{(0,t]}) = \int\limits_{Q^t \in E^{(0,t]}} \left\langle \psi_0, \hat{V}^+[Q^t, t; \varphi_R]\,\hat{V}[Q^t, t; \varphi_R]\,\psi_0 \right\rangle_{H_S} \nu_{\hat{Q}}^{(0,t]}(dQ^t).$$

From the definition (11) the following representations follow

(14a) $$\hat{U}(t,0)(\psi \otimes \varphi_R) = \int\limits_{Q^t \in \Omega^{(0,t]}} (\hat{V}[Q^t, t; \varphi_R] \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t))\,(\psi \otimes \varphi_R), \qquad for\ \forall \psi \in H_S$$

(14b) $$<\varphi_R, \hat{U}(t,0)\varphi_R>_{H_R} = \int\limits_{Q^t \in \Omega^{(0,t]}} \hat{V}[Q^t, t; \varphi_R]\,\nu_{\hat{Q}}^{(0,t]}(dQ^t).$$

The operator $\hat{V}[Q^t, t; \varphi_R]$ describes the irreversible in time stochastic evolution of an open system under the condition that the output process $\hat{Q}_H(s)$ was continuously observed until the moment t and found to have the trajectory Q^t. The state

(15) $$\chi[Q^t, t; \varphi_R] = \hat{V}[Q^t, t; \varphi_R]\,\psi_0,$$
$$for \quad \forall \psi_0 \in H_S$$

can be interpreted as the posterior state (unnormalized) of an open system subjected to continuous in time nondemolition observation

In the general case of nondemolition measurement, considering the Cauchy problem for the operator $\hat{U}(t,0)$, we derive the following integral equation for the posterior state:

(16) $$\chi[Q^t, t; \varphi_R] = \psi_0 + \left(-\frac{i}{\hbar}\right)\int_0^t \hat{H}_S\,\chi[Q^\tau, \tau; \varphi_R]\,d\tau +$$
$$+\left(-\frac{i}{\hbar}\right)\int_0^t \left(\int\limits_{Q_1^\tau \in \Omega^{(0,t]}} \hat{W}[Q^\tau, Q_1^\tau, \tau; \varphi_R]\,\chi[Q_1^\tau, \tau; \varphi_R]\,\nu_{\hat{Q}}^{(0,\tau]}(dQ_1^\tau) \right) d\tau.$$

where

$$(\hat{W}[Q^t, Q_1^t, t; \varphi_R]\, \nu_{\hat{Q}}^{(0,\,t]}(dQ_1^t)) \otimes \hat{P}_{\hat{Q}}^{(0,t]}(dQ^t)(\psi \otimes \varphi_R) =$$

$$= (\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,\,t]}(dQ^t))\hat{H}_{\text{int}}(t)(\hat{I} \otimes \hat{P}_{\hat{Q}}^{(0,\,t]}(dQ_1^t))(\psi \otimes \varphi_R), \tag{17}$$

for $\forall \psi \in H_S$,

and $\hat{H}_S, \hat{H}_{\text{int}}(t)$ are the Hamiltonians of an open system and of the interaction (in the interaction picture), respectively. In the general case of nondemolition measurement

$$\hat{W}[Q^t, Q_1^t, t; \varphi_R] = \hat{W}_1[Q^t, Q_1^t, t; \varphi_R]\, \delta_\nu(Q_1^{(0,t)} - Q^{(0,t)}), \tag{18}$$

so that the third term in (16) can be essentially simplified.

The notion of the posterior state of an open system under continuous in time observation was first introduced by V.P.Belavkin in [9-15] in the special nondemolition measurement model of quantum stochastic calculus. The notion similar to the notion of the quantum stochastic evolution operator $\hat{V}[Q^t, t; \varphi_R]$ was introduced in [9-15] through the concept of the generating map of an instrument [2] and defined in the integral form, so that the derivation of the posterior state equation based on the use of quantum stochastic calculus is rather complicated. The equation for the posterior state derived in [9-15] is valid only for the case of the nondemolition measurement model of quantum stochastic calculus.

Our approach to the description of continuous in time indirect measurement, our definition (11) of the quantum stochastic evolution operator $\hat{V}[Q^t, t, \varphi_R]$ allow us to derive the integral equation (16) describing the posterior dynamics of an open system under continuous in time observation in the general nondemolition case.

5. The special nondemolition measurement model of quantum stochastic calculus, the extended variant

In the case of the special nondemolition measurement model of quantum stochastic calculus the unitary time evolution $\hat{U}(t,0) \equiv \hat{U}(t)$ of the extended system in the interaction picture is described by the quantum stochastic differential equation

$$d\hat{U}(t) = \left((-\hat{K} \otimes \hat{I})\, dt - (\hat{L}^+ \hat{R}) \otimes d\hat{A}(t) + \hat{L} \otimes d\hat{A}^+(t) + (\hat{R} - \hat{I}) \otimes d\hat{\Lambda}(t)\right)\hat{U}(t), \tag{19}$$

introduced by Hudson R.L. and Parthasarathy K.R. [19].

In (19) $\hat{L}$; $\hat{R}$; $\hat{K} + \hat{K}^+ = \hat{L}^+\hat{L}$ are operators on the Hilbert space $\boldsymbol{H}_S$ of an open system, $\hat{R}$ is unitary. The exact relation between the operator $\hat{K}$ in (19) and the Hamiltonian $\hat{H}_S$ of an open system in the case when $\hat{R} \neq -\hat{I}$ is given in [20]:

$$\hat{K} = \frac{i}{\hbar} H_S + \hat{L}^+ \hat{R}(\hat{R} + \hat{I})^{-1} \hat{L}\,. \tag{20}$$

The reservoir, modelling the measuring device, is described as a Bose field and is represented by a symmetric Fock space $\Gamma(Z))$ over a single particle Hilbert space $\boldsymbol{Z}$ which in applications in quantum stochastic mechanics is taken to be $L^2(R)$ **.** The annigilation, creation and gauge (or number) operators $\hat{A}(t), \hat{A}^+(t), \hat{\Lambda}(t) = \hat{\Lambda}^+(t)$ describing the free dynamics of the Bose field form the annigilation, creation and gauge operator-valued processes, respectively. Denote by $e(f), f \in L^2(R)$ -- a normalized coherent vector for the Bose field:

$$\hat{A}(t)e(f) = (\int_0^t \tilde{f}(\tau)\, d\tau)e(f), \tag{21}$$

where $\tilde{f}(t)$ is a Fourier transform of a function $f \in L^2(R)$ .Considering the Hamiltonian [20] of the Schrödinger equation corresponding (when $\hat{R} \neq -\hat{I}$) to the quantum stochastic

differential equation of Hudson- Parthasarathy, we derive the following equation for the posterior state of an open system in the case of diffusion observation and the initial state of the reservoir $\varphi_R = e(f)$:

(22a)

$$\chi[Q^t,t;e(f)] = \psi_0 - \int_0^t (\frac{i}{\hbar}\hat{H}_S + \hat{G}(\tau))\chi[Q^\tau,\tau;e(f)]\,d\tau +$$

$$+ \int_0^t q(\tau)\{\hat{L} + \tilde{f}(\tau)(\hat{R}-\hat{I})\}\chi[Q^\tau,\tau;e(f)]\,d\tau, \qquad \forall \psi_0 \in H_S,$$

where we introduce

(22b)

$$\hat{G}(t) = \frac{1}{2}\hat{L}^2 + \hat{L}^+\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L} + \frac{1}{2}\tilde{f}(t)(\hat{R}-\hat{I})\hat{L} +$$

$$+\frac{1}{2}\tilde{f}(t)\hat{L}(\hat{R}+\hat{I}) + \tilde{f}(t)\hat{L}^+\hat{R} + \frac{1}{2}\tilde{f}^2(t)(\hat{R}^2-\hat{I}),$$

and $q(t) = \dfrac{dQ(t)}{dt}$ - the generalized derivative of the function $Q(t)$. In the considered case of diffusion observation, $q(t)$ is a piecewise continuous function. Using the methods of the classic stochastic calculus, we can rewrite the integral equation (22) in the stochastic form:

$$\chi[Q^t,t;e(f)] = \psi_0 -$$

(23)
$$-\int_0^t \{\frac{i}{\hbar}\hat{H}_S + \hat{L}^+\hat{R}(\hat{R}+\hat{I})^{-1}\hat{L} + \tilde{f}(\tau)(\hat{L}^+\hat{R}+\hat{L}) + \tilde{f}^2(\tau)(\hat{R}-\hat{I})\}\chi[Q^\tau,\tau;e(f)]\,d\tau +$$

$$+\int_0^t \{\hat{L} + \tilde{f}(\tau)(\hat{R}-\hat{I})\}\chi[Q^\tau,\tau;e(f)]\,dQ(\tau), \qquad \forall \psi_0 \in H_S.$$

The stochastic integral in (23) is understood in Ito's sense, $dQ(t)$ is a stochastic differential of the classic diffusion process corresponding to the continuous in time direct observation, when $\varphi_R = e(f)$, of the input nondemolition observable $\hat{Q}(t) = \hat{A}^+(t) + \hat{A}(t)$.

The scalar probability measure of the input diffusion process is defined by (9).

We would like to mention that the stochastic integral equation (23) can be also derived by substituting $\hat{U}(t,0)$, defined by (19), directly into the definition (11).

The stochastic integral equation (23) for the posterior state of an open system under continuous in time observation of diffusion type coincides with the quantum filtering equation for the case of diffusion observation [15] only if the initial state of a reservoir is vacuum and $\hat{R} = \hat{I}$.

3. Concluding remarks

We present an approach to the description of continuous in time indirect measurement which allows us:

- to derive the new results in the general case of nondemolition measurement;
- to derive the new integral equation for the posterior state of an open system in the special nondemolition measurement model which is the extended variant (including the gaude term) usually considered in quantum stochastic mechanics.

REFERENCES

[1] Åberg , C., Loubenets, E.R. (1994*). Quantum stochastic calculus and nondemolition measurements. Notes on a recent approach to quantum measurements.* Report No **94-28**, 48pp., Institute of Theoretical Physics, Göteborg , Sweden.

[2] Barchielli, A., Lanz, L., Prosperi, G.M. (1983). *Statistic of continuous trajectories in quantum mechanics: operation valued stochastic processes*. Found.Phys, **13**, 79-812.

[3] Barchielli, A. (1985).*Quantum stochastic calculus, operation-valued stochastic processes and continual measurements in quantum mechanics*. J.Math.Phys. **26**, 2222-2230**.**

[4] Barchielli, A. (1986). *Measurement theory and stochastic differential equations in quantum mechanics.* Phys. Rev. A, **34**, 1642-1648.

[5] Barchielli, A.. Belavkin, V.P. (1991).*Measurements continuous in time and a posteriori states in quantum mechanics*.J.Phys.A; Math.Gen. **24**,1495-1514.

[6] Barchielli, A. (1993).*On the quantum theory of measurements continuous in time.* Reports on Math.Phys. **33**, 1/2, 21-34.

[7] Barchielli, A., Holevo, A.S. (1995). *Constructing quantum measurement processes via classical stochastic calculus*. Stochastic Processes Appl. **58**, 293-317

[8] Barchielli, A. (1998). *On stochastic differential equations and semigroups of probability operators in quantum probability*. Stoch. Proc. and their Applications **73**, 69-86.

[9] Belavkin, V.P. (1988). *Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes*. In: Blaquiere, A. (Ed.), Lecture Notes in Control and Information Sciences **121**, 245-265, Springer, Berlin.

[10] Belavkin, V.P. (1989). *Stochastic calculus of input-output processes and nondemolition filtering*. VINITI **36**, 29-67, Moscow.

[11] Belavkin, V.P. (1989). *A continuous counting observation and posterior quantum dynamics.* J. Phys.A **22**, L1109-L1114.

[12] Belavkin, V.P. (1989). *A new wave equation for a continuous nondemolition measurement.* Phys. Lett. A **140**, 355-358 .

[13] Belavkin, V.P. (1990). *Stochastic equations of quantum filtering.* Proceedings of the Fifth Int.Conference "Probab.Theory and Math.Statistic", Vilnus.

[14] Belavkin, V.P.(1992). *Quantum continual measurements and a posteriori collapse on CCR.* Commun. Math.Phys.**146**, 611-635.

[15] Belavkin, V.P. and Staszewski, P. (1992). *Nondemolition observation of a free quantum particle* Phys. Rev. A, **45**, 1347-1356.

[16] Davies, E. B. (1976). *Quantum Theory of Open Systems.* Academic Press, London.

[17] Diosi, L.(1988). *Continuous quantum measurement and Ito formalism*. Phys.Lett.A **129**, 419-423.

[18] Holevo, A.S. (1987), *Quantum estimation.* Advances in Statistical Signal Processing **1**, 157-202 , JAI Press Inc.

[19] Hudson, R.L., Parthasarathy, K.R. (1984). *Quantum Ito's formula and stochastic evolutions.* Comm. in Math. Phys., **93**, 301-323.

[20] Loubenets, E.R. (1999).*Quantum stochastic evolutions and the Schrödinger equation.* Russian Journal of Mathematical Physics. (To appear).

[21] Loubenets, E.R. (1999) . *Quantum stochastic evolution of an open system under continuous in time nondemolition measurement.* Research Report No**. 45**, Centre for Mathematical Physics and Stochastics, University of Aarhus, Denmark.

[22] Ozawa, M.(1884). *Quantum measuring processes of continuous observables.* J.Math.Phys.**25**, 79-87.

[23] Peres, A. (1993). *Quantum Theory: Concepts and Methods*. Kluwer, Dordrecht.

An ergodic theorem for repeated and continuous measurement

Hans Maassen, Nijmegen (Netherlands)
work in collaboration with B. Kümmerer, (Stuttgart, Germany);
paper in preparation.

Abstract: We prove an ergodic theorem for repeated measurement, indicating its significance for quantum trajectories in discrete time. We roughly sketch the extension to continuous time, and some connections to the algebraic theory of quantum Markov processes.

1. *Measurement in an operational approach.*

A measurement, whether quantummechanical or not, is an operation performed on a physical system which results in the extraction of information from that system, while possibly changing its state.
So before the measurement there is the physical system, described by a state ρ, (a probability measure in the classical case, a density matrix in the quantum case), and afterwards there is a piece of information, say an outcome $i \in \{1, 2, \ldots, k\}$, and there is the system itself, in some new (or *posterior*) state θ_i:

$$\rho \longrightarrow (i, \theta_i) \ .$$

Now, a probabilistic theory rather than predicting an outcome i, gives a probability distribution $(\pi_1, \pi_2, \ldots, \pi_k)$ on the possible outcomes. In fact the measurement operation is described by an affine map

$$M_* : \rho \mapsto (\pi_1\theta_1, \pi_2\theta_2, \ldots, \pi_k\theta_k) \ ,$$

taking a state ρ on the algebra $\mathcal{A}$ of observables of the system to a state on the tensor product of $\mathcal{C} := \mathbb{C}^k$ with $\mathcal{A}$. In the literature on measurement theory this is called an *operation valued measure* or *instrument* [Dav, Hel, Hol1, BGL]. We shall call the i'th component $\pi_i\theta_i$ of the right hand side: $(T_i)_*(\rho)$. The maps M_* and $(T_i)_*$ are the (pre)duals of completely positive maps

$$M : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \quad \text{and} \quad T_i : \mathcal{A} \to \mathcal{A} \ .$$

T_i describes the effect on the system's observables of the occurrence of an outcome i. The effect of the measurement on the system, when we ignore the outcome, is given by the map

$$T : \mathcal{A} \to \mathcal{A} : x \mapsto M(\mathbf{1} \otimes x) = \sum_{i=1}^{k} T_i(x) \ .$$

On the other hand, if we ignore the system after the measurement, we obtain the map

$$Q : \mathcal{C} \to \mathcal{A} : f \mapsto M(f \otimes \mathbf{1}) = \sum_{i=1}^{k} f(i) T_i(\mathbf{1}) \ ,$$

which is known as a *positive operator valued measure* or *generalised observable*. We note that M, T and Q are all completely positive and identity preserving linear operators on C*-algebras. Such maps are called *operations*.

Example 1: Classical measurement with error.

We measure the length X of a bar by means of a measuring stick. The length X is a random variable having distribution ρ with support $[0, K]$, say. Its algebra of observables $\mathcal{A}$ is $L^\infty([0, K], \rho)$. After the measurement the bar has the same length X as before, but a second random variable Y has arisen whose values depend in a stochastic way on X. Let us assume that Y is the result of measuring the length X with some random error, rounded off to an integer number of millimeters. Then Y takes values from 0 to k, where k is number of millimeters in the upper bound K.
This example is described by

$$\left((T_i)_*(\rho)\right)(d\xi) = \pi_i(\xi)\rho(d\xi) \ ,$$

where $\pi_i(\xi)$ is the probability that the length $\xi \in [0, K]$ will be 'measured' as i millimeters. In the dual 'Heisenberg' picture the measurement is given by

$$M : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} : \quad M(f \otimes g)(\xi) = \sum_{i=1}^{k} f(i)\pi_i(\xi)g(\xi) \ .$$

In this example the measurement has no effect on the system, as is expressed by the relation

$$(Tg)(\xi) = \sum_{i=1}^{k} (T_i g)(\xi) = \left(\sum_{i=1}^{k} \pi_i(\xi) \right) g(\xi) = g(\xi) \ .$$

The generalised random variable Q is given by

$$Q(f)(\xi) = \sum_{i=1}^{k} f(i)\pi_i(\xi) \ .$$

Example 2: von Neumann measurement.

Let $\mathcal{A} := M_n$, the algebra of all complex $n \times n$-matrices. We think of $\mathcal{A}$ as the obserable algebra of some finite quantum system. Let $p_1, p_2, \ldots p_k$ be mutually orthogonal projections in $\mathcal{A}$ adding up to $\mathbf{1}$.
If some physical quantity is described by a self-adjoint matrix in $\mathcal{A}$ whose eigenspaces are the ranges of the p_i, then according to von Neumann's projection postulate a measurement of this quantity is described by

$$(T_i)_*(\rho) = p_i \rho p_i, \quad \text{so} \quad M(f \otimes x) = \sum_{i=1}^{k} f(i) p_i x p_i \ .$$

Example 3: von Neumann measurement followed by unitary evolution.
Modify the above example by taking

$$M(f \otimes x) := \sum_{i=1}^{k} p_i u^* x u p_i \ .$$

Each von Neumann measurements is now followed by a fixed unitary time evolution. This will have the effect of making repetitions of this operation more interesting.

Example 4: Kraus measurement.
Couple the finite quantum system with observable algebra $\mathcal{A}$ to an finite 'apparatus' with observable algebra $\mathcal{B}$ in the initial state β. Let the two systems evolve for a while, say according to a unitary matrix $u \in \mathcal{B} \otimes \mathcal{A}$, and then perform a von Neumann measurement on $\mathcal{B}$ described by the mutually orthogonal projections $p_1, \ldots, p_k \in \mathcal{B}$. Then obtain

$$(T_i)_*(\rho) : x \mapsto (\beta \otimes \rho)(u^*(p_i \otimes x)u) \ ,$$

or, in the 'Heisenberg picture',

$$T_i(x) = (\beta \otimes \mathrm{id})(u^*(p_i \otimes x)u) \ .$$

Let us call this indirect von Neumann measurement *perfect* if β is a pure state and the p_i are one-dimensional projections. (This corresponds to maximal information concerning the apparatus, and maximally efficient measurement.) If this is the case, let us write $\beta(y) = \langle v, yv \rangle_{\mathcal{B}}$ and $p_i = |e_i\rangle\langle e_i|$. Then T_i is of the form

$$T_i(x) = a_i^* x a_i \ ,$$

where the *Kraus matrices* $a_1, \ldots, a_k$ [Kra] are given by

$$a_i = \sum_{j=1}^{k} \langle e_i, u e_j \rangle_{\mathcal{B}} \langle e_j, v \rangle$$

Here we have used the notation

$$\langle e_i, (y \otimes x) e_j \rangle_{\mathcal{B}} := \langle e_i, y e_j \rangle x \quad (x \in \mathcal{A}, y \in \mathcal{B}).$$

2. *Repeated measurement.*
By repeating the measurement operation of the previous section indefinitely, we obtain for every initial state ρ of the finite quantum system a stochastic process in discrete time, taking values in the outcome space $\mathcal{X} := \{1, 2, \ldots, k\}$. We shall prove an ergodic theorem for this type of process.
Let $\Omega := \mathcal{X}^{\mathbb{N}}$, and let for $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathcal{X}$ the cylinder sets $\Lambda_{i_1,\ldots,i_m} \subset \Omega$ be given by

$$\Lambda_{i_1,\ldots,i_m} := \{\omega \in \Omega \,|\, \omega_1 = i_1, \ldots, \omega_m = i_m \} \ .$$

Denote by Σ the σ-algebra generated by all these Σ_m.
Let $\mathcal{A}$ be a finite-dimensional von Neumann algebra, and let T_i $(i = 1, \ldots, k)$ be completely positive operators $\mathcal{A} \to \mathcal{A}$ such that their sum maps $\mathbf{1}_{\mathcal{A}}$ to itself.

Proposition 1. *There exists a unique $\mathcal{A}$-valued probability measure Q_∞ on (Ω, Σ) such that*

$$Q_\infty(\Lambda_{i_1,\ldots,i_m}) = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbf{1}) \ .$$

In particular, if ρ is a state on $\mathcal{A}$, then $\mathbb{P}_\rho := \rho \circ Q_\infty$ is an ordinary $[0,1]$-valued probability measure on (Ω, Σ).

Proof. By the reconstruction theorem of Kolmogorov and Daniel it suffices to prove consistency: for all $i_1, \ldots, i_m \in \mathcal{X}$,

$$\sum_{i=1}^{k} Q_\infty\left(\Lambda_{i_1,\ldots,i_m,i}\right) = Q_\infty\left(\Lambda_{i_1,\ldots,i_m}\right) \ .$$

Indeed, since $T(\mathbf{1}) = \mathbf{1}$, the l.h.s. is equal to

$$\begin{aligned}\sum_{i=1}^{k} T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m} \circ T_i(\mathbf{1}) &= T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m} \circ T(\mathbf{1}) \\ &= T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbf{1}) \ ,\end{aligned}$$

which is equal to the r.h.s. □

We shall consider the left shift σ on Ω given by

$$(\sigma\omega)_j := \omega_{j+1} \ .$$

A probability measure μ on (Ω, Σ) is called *stationary* if for all $B \in \Sigma$ we have

$$\mu(\sigma^{-1}(B)) = \mu(B) \ .$$

Proposition 2. *If $\rho \circ T = \rho$, then $\mathbb{P}_\rho$ is stationary.*

Proof. Since any probability measure on (Ω, Σ) is determined by its values on the cylinder sets $\Lambda_{i_1,\ldots,i_m}$, it suffices to prove the equality

$$\mathbb{P}_\rho\left(\sigma^{-1}\left(\Lambda_{i_1,\ldots,i_m}\right)\right) = \mathbb{P}_\rho\left(\Lambda_{i_1,\ldots,i_m}\right) \ .$$

Now,

$$\sigma^{-1}\left(\Lambda_{i_1,\ldots,i_m}\right) = \bigcup_{i=1}^{k} \Lambda_{i,i_1,\ldots,i_m} \ .$$

Therefore, if $\rho \circ T = \rho$, the l.h.s. of the equality to be proved is equal to

$$\begin{aligned}\sum_{i=1}^{k} \mathbb{P}_\rho\left(\Lambda_{i,i_1,\ldots,i_m}\right) &= \sum_{i=1}^{k} \rho \circ T_i \circ T_{i_1} \circ \cdots \circ T_{i_m}(\mathbf{1}) \\ &= \rho \circ T \circ T_{i_1} \circ \cdots \circ T_{i_m}(\mathbf{1}) \\ &= \rho \circ T_{i_1} \circ \cdots \circ T_{i_m}(\mathbf{1}) \ ,\end{aligned}$$

which is equal to the r.h.s. □

In preparation of our ergodic theorem we prove the following lemma.

Lemma 3. *For all $B \in \Sigma$, $m \in \mathbb{N}$, and $i_1, i_2, \ldots, i_m \in \mathcal{X}$ we have*

$$Q_\infty\left(\Lambda_{i_1,\ldots,i_m} \cap \sigma^{-m}(B)\right) = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m} \circ Q_\infty(B) \ ,$$

in particular,

$$Q_\infty(\sigma^{-1}(B) = T \circ Q_\infty(B) \ .$$

Proof. It suffices to prove the first equality for $B = \Lambda_{j_1,\ldots,j_l}$ (some $l \in \mathbb{N}$, $j_1, \ldots, j_l \in \mathcal{X}$). But then

$$\Lambda_{i_1,\ldots,i_m} \cap \sigma^{-m}(B) = \Lambda_{i_1,\ldots,i_m,j_1,\ldots,j_l} \ ,$$

so that both sides of the first equality are equal to

$$T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m} \circ T_{j_1} \circ T_{j_2} \circ \cdots \circ T_{j_l}(\mathbf{1}) \ .$$

The second equality follows from the first since its l.h.s. side is equal to

$$\sum_{i=1}^{k} Q_\infty(\Lambda_i \cap \sigma^{-1}(B)) \ .$$

□

We shall call an $\mathcal{A}$-valued probability measure R on (Ω, Σ) *ergodic* if for all $E \in \Sigma$ we have

$$\sigma^{-1}(E) = E \quad \Rightarrow \quad R(E) \in \{0, \mathbf{1}\} \ .$$

Theorem 4. *If T_* has a unique invariant state, then Q_∞ is ergodic.*

An important consequence of the above ergodicity theorem is that path averages are equal to quantummechanical expectations:

Corollary 5. (Ergodic theorem for repeated measurement.) *If T_* has a unique invariant state $\rho \in \mathcal{A}_*$, then for any initial state $\theta \in \mathcal{A}_*$ and any sequence $i_1, \ldots, i_m \in \{1, 2, \ldots, k\}$, we have almost surely with respect to $\mathbb{P}_\theta$*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \#\{j < n \,|\, \omega_{j+1} = i_1, \omega_{j+2} = i_2, \ldots, \omega_{j+m} = i_m \} = \rho \circ T_{i_1} \circ \cdots \circ T_{i_m}(\mathbf{1}) \ .$$

Proof of the Corollary. By Proposition 2, $\mathbb{P}_\rho$ is stationary. By Birkhoff's individual ergodic theorem, the path average on the l.h.s., $F(\omega)$ say, exists for almost all $\omega \in \Omega$. Since $F = F \circ \sigma$, the events $E_{[a,b]} := \{\omega \in \Omega | a \leq F(\omega) \leq b\}$ are σ-invariant, hence by Theorem 4 they all have Q_∞-measure either 0 or $\mathbf{1}$. This implies that for some $c \in \mathbb{R}$ we have $Q_\infty(E_{\{c\}}) = \mathbf{1}$, hence $\mathbb{P}_\theta(E_{\{c\}}) = 1$ for all states $\theta \in \mathcal{A}_*$. In particular c must be the expectation $\mathbb{E}_\rho(F)$ of F under $\mathbb{P}_\rho$. Using the stationarity of ρ we may calculate:

$$c = \mathbb{E}_\rho(F) = \mathbb{E}_\rho\left(1_{\Lambda_{i_1,\ldots,i_m}}\right) = \mathbb{P}_\rho\left(\Lambda_{i_1,\ldots,i_m}\right) = \rho \circ T_{i_1} \circ \cdots \circ T_{i_m}(\mathbf{1}) \ .$$

□

Proof of Theorem 4. As $\mathcal{A}$ is finite-dimensional, uniqueness of the T_*-invariant state ρ implies that all T-invariant elements of $\mathcal{A}$ are multiples of $\mathbf{1}$. Now let $E \in \Sigma$ be such that $\sigma^{-1}(E) = E$. Then by Lemma 3,

$$Q_\infty(E) = Q_\infty(\sigma^{-1}(E)) = T \circ Q_\infty(E) \ ,$$

so $Q_\infty(E) = \lambda \cdot \mathbf{1}$. It remains to show that $\lambda = 0$ or 1. For this purpose, define an $\mathcal{A}$-valued measure Q_E on (Ω, Σ) by

$$Q_E(B) := Q_\infty(B \cap E), \quad (B \in \Sigma).$$

By Lemma 3 we have for all $m \in \mathbb{N}$, $i_1, i_2, \ldots, i_m \in \mathcal{X}$,

$$\begin{aligned} Q_E\left(\Lambda_{i_1,\ldots,i_m}\right) &= Q_\infty\left(\Lambda_{i_1,\ldots,i_m} \cap E\right) \\ &= Q_\infty\left(\Lambda_{i_1,\ldots,i_m} \cap \sigma^{-m}(E)\right) \\ &= T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(Q_\infty(E)) \\ &= \lambda \cdot T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbf{1}) \\ &= \lambda Q_\infty\left(\Lambda_{i_1,\ldots,i_m}\right) \ . \end{aligned}$$

And since a measure on (Ω, Σ) is determined by its values on the cylinder sets, we conclude that for all $B \in \Sigma$:

$$Q_E(B) = \lambda Q_\infty(B) \ .$$

Applying this relation to E itself, we find that

$$\lambda \cdot \mathbf{1} = Q_\infty(E) = Q_\infty(E \cap E) = Q_E(E) = \lambda Q_\infty(E) = \lambda^2 \cdot \mathbf{1} \ .$$

Therefore $\lambda = 0$ or 1. □

3. *Application to the examples.*

Example 1: Classical measurement with error.

In this example T is the identity map on $\mathcal{A}$. So the assumption of the Theorem is that $\dim(\mathcal{A})$=1. Since $\mathcal{A} = L^\infty([0, K], \rho)$, this means that $\rho = \delta_\xi$ for some length $\xi \in [0, K]$. In that case the measurement process $\omega_1, \omega_2, \ldots$ is a sequence of independent random variables all with distribution $\pi(\xi)$. Such a sequence is indeed ergodic by the law of large numbers. Note however, that if different values ξ_1 and ξ_2 can occur with positive probability, then the path average would still exist, but could take different values according to chance.

Example 2: Repeated von Neumann measurement.

This is not an interesting case. The first measurement determines the outcome, and all later measurements confirm it. Uniqueness of ρ amounts to $k = 1$, i.e., we are measuring a sure observable without error.

Example 3: Alternating von Neumann measurement and Schrödinger evolution.
In this case the condition of uniqueness of the invariant state becomes

$$\{u, p_1, p_2, \ldots, p_k\}' = \mathbb{C}\mathbf{1} \ .$$

The unique invariant state is the *trace* state on $\mathcal{A} = M_n$: $\rho(x) = \frac{1}{n}\mathrm{tr}(x)$. If we take for p_i one-dimensional projections, say $p_i = |e_i\rangle\langle e_i|$, then the stochastic sequence of outcomes is a Markov chain with transition probabilities

$$|\langle e_i, ue_j\rangle|^2 \ .$$

This is a bistochastic transition matrix, indeed having equipartition as an equilibrium distribution. The condition that $\{u, p_1, p_2, \ldots, p_k\}' = \mathbb{C} \cdot \mathbf{1}$ makes the transition matrix irreducible and the equilibrium distribution unique.

Example 4: Davies processes or quantum trajectories in discrete time.
This is our most interesting example. Let us take repeated Kraus measurements, i.e.

$$T_i(x) = a_i^* x a_i, \quad (i = 1, \ldots, k) \ ,$$

for some $a_1, a_2, \ldots, a_k \in \mathcal{A} = M_n$ with $\sum_i a_i^* a_i = \mathbf{1}$. Then, if $\rho \circ T = \rho$ we have a stationary measurement sequence satisfying

$$\mathbb{P}_\rho[\omega_1 = i_1, \omega_2 = i_2, \ldots, \omega_m = i_m] = \rho(a_{i_1}^* a_{i_2}^* \cdots a_{i_m}^* a_{i_m} \cdots a_{i_2} a_{i_1}).$$

In general, this is not a Markov chain. However, it is intimately connected with the following Hilbert space valued Markov chain.
On $(\Omega, \Sigma, \mathbb{P}_\rho)$, consider the stochastic process $\Psi_0, \Psi_1, \Psi_2, \ldots$ with values in $\mathcal{H} := \mathbb{C}^n$ given by

$$\Psi_m(\omega) := \frac{a_{\omega_m} a_{\omega_{m-1}} \cdots a_{\omega_2} a_{\omega_1} \psi_0}{\|a_{\omega_m} a_{\omega_{m-1}} \cdots a_{\omega_2} a_{\omega_1} \psi_0\|} \ .$$

The process Ψ is called the *quantum trajectory* associated to this repeated Kraus measurement.

Proposition 6. *In the situation of Example 4 (perfect case), the stochastic process $\Psi_0, \Psi_1, \Psi_2, \ldots$ is a classical Markov chain on the unit sphere of $\mathcal{H}$ with initial condition $\Psi_0 = \psi_0$ and transition probabilities*

$$P(\psi, \theta) = \sum_{i=1}^{k} \|a_i\psi\|^2 \delta_\theta \left(\frac{a_i \psi}{\|a_i \psi\|} \right) \ ,$$

where

$$\delta_{\theta_1}(\theta_2) := \begin{cases} 1 & \text{if } \theta_1 = \theta_2, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is a straightforward verification.

This Hilbert space valued version of the repeated Kraus measurement is very well suited for numerical simulation, and has been fruitfully employed in areas such as quantum optics [CSVR, WiM]. Our ergodic theorem implies that, if ρ is the unique T_*-invariant state, then the jump process of this quantum trajectory is ergodic, i.e. a single path reveals all the statistical properties of the whole jump process.

4. *Continuous measurement.*

In this Section we roughly sketch how the ergodic theorem of Section 2 can be extended to continuous measurement.
Making minimal assumptions, still allowing essentially the same proof, we arrive at the following structure.
For Σ we take a σ-algebra of subsets of some sample space Ω, an for all $0 \leq a \leq b$ we assume that we have a sub-σ-algebra $\Sigma_{[a,b]}$ of Σ such that, for $0 \leq a \leq b \leq c$,

$$\Sigma_{[a,b]} \cap \Sigma_{[b,c]} = \{\emptyset, \Omega\} \quad \text{and} \quad \Sigma_{[a,b]} \vee \Sigma_{[b,c]} = \Sigma_{[a,c]} \ ,$$

expressing the localisation in time of the measurement outcomes. We assume that for all $t \geq 0$ a (left) time shift $\sigma_t : \Omega \to \Omega$ is given, i.e. for all $t \geq 0$ and all a, b with $0 \leq a \leq b$ we must have:

$$\{\sigma_t^{-1}(A) \,|\, A \in \Sigma_{[a,b]}\} = \Sigma_{[a+t,b+t]} \ .$$

Let $\mathcal{A}$ be our finite-dimensional von Neumann algebra, and for all $t \geq 0$ let a $\mathrm{CP}(\mathcal{A})$-valued measure M_t on $\Sigma_{[0,t]}$ be given such that for all $s, t \geq 0$,

(a) $T_t := M_t(\Omega)$ maps $\mathbf{1}_{\mathcal{A}}$ to itself;

(b) if $A \in \Sigma_{[0,t]}$ and $B \in \Sigma_{[0,s]}$, then $M_{t+s}(A \cap \sigma_t^{-1}(B)) = M_t(A) \circ M_s(B)$.

Then one proves along the same lines as in Section 2 that the family of $\mathcal{A}$-valued probability measures

$$Q_t : \Sigma_{[0,t]} \to \mathcal{A} : B \mapsto M_t(B)(\mathbf{1}_{\mathcal{A}})$$

is consistent and extends to a single $\mathcal{A}$-valued probability measure Q_∞ on Σ. Moreover, this measure is ergodic provided that the semigroup $((T_t)_*)_{t \geq 0}$ admits only a single invariant state on $\mathcal{A}$.

The above abstract scheme contains all the examples of continuous measurement termed 'Markovian', such as the jump processes of Srinivas and Davies [SrD], the diffusions of Gisin [Gis], and any infinitely divisible instrument in the sense of Holevo [Hol2, BaH].

5. *Some algebraic connections.*

In Section 1 we have seen that a *measurement* on a system with observable algebra $\mathcal{A}$ can be viewed as an operation

$$M : \mathcal{C} \otimes \mathcal{A} \to \mathcal{A} \ .$$

with $\mathcal{C}$ abelian. In the spirit of Example 4 (Kraus measurement) we may extend this idea somewhat by allowing the information extracted from the system to be quantum information: we replace the abelian algebra $\mathcal{C} \subset \mathcal{B}$ by $\mathcal{B}$ itself, thus

postponing the choice of the abelian subalgebra to a later stage. So let us define a *generalised measurement operation* as an operation

$$M : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \ .$$

Repeating this generalised measurement indefinitely leads to the scheme

$$\begin{array}{cccc} & & & \mathcal{A} & \\ & & \overbrace{\nearrow \quad \nearrow}^{} & & M \\ & \mathcal{B} \otimes & \mathcal{B} \otimes & \mathcal{A} & \\ & & \overbrace{\nearrow \quad \nearrow \quad \nearrow}^{} & & \mathrm{id} \otimes M \\ \mathcal{B} \otimes & \mathcal{B} \otimes & \mathcal{B} \otimes & \mathcal{A} & \\ & & \overbrace{\nearrow \quad \nearrow \quad \nearrow \quad \nearrow}^{} & & \mathrm{id} \otimes \mathrm{id} \otimes M \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

In this way any state ρ on $\mathcal{A}$ leads to a state on the infinite tensor product $\cdots \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A}$. This is closely related to Accardi's early version of a Quantum Markov Process [Acc]. If ρ is invariant, i.e., if $\rho(M(\mathbf{1} \otimes x)) = \rho(x)$ for all $x \in \mathcal{A}$, then the above state naturally defines a shift-invariant state on $\bigotimes^{\mathbb{Z}} \mathcal{B}$, which was exploited by Fannes, Nachtegaele and Werner to describe states on spin chains [FNW].
In the algebraic notation the m-fold measurement of Section 2 is described by the operation $M^{(m)} : \bigotimes_{i=1}^{m} \mathcal{B} \otimes \mathcal{A} \to \mathcal{A}$ given by

$$M^{(m)} := M \circ (\mathrm{id} \otimes M) \circ \cdots \circ (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes M) \ .$$

For comparison we note that $M^{(m)}(p_1 \otimes \cdots \otimes p_m \otimes \mathbf{1}_{\mathcal{A}}) = Q_\infty(\Lambda_{i_1,\ldots,i_m})$.
By attaching an infinite product of copies of $(\mathcal{B}, \beta)$ to the right of the diagram above, which we interpret as a chain of measurement devices queuing up to be coupled to the system $\mathcal{A}$, we obtain a *dilation* in the sense of Kümmerer of the semigroup $(T^n)_{n \geq 0}$ to a group of automorphisms. This is indicated in the following diagram, which commutes for all $n \geq 0$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T^n} & \mathcal{A} \\ {\scriptstyle \mathbf{1} \otimes \mathrm{id}} \downarrow & & \uparrow {\scriptstyle \left(\bigotimes_{\mathbb{Z}} \beta\right) \otimes \mathrm{id}} \\ \left(\bigotimes_{\mathbb{Z}} \mathcal{B}\right) \otimes \mathcal{A} & \xrightarrow{\widehat{T}^n} & \left(\bigotimes_{\mathbb{Z}} \mathcal{B}\right) \otimes \mathcal{A} \end{array} \ .$$

Here, $\widehat{T}$ is given by

$$\widehat{T}(y \otimes x) := u^*(Sy \otimes x)u \ ,$$

where S denotes the right shift on the infinite tensor power of $\mathcal{B}$, and $u \in \mathcal{B} \otimes \mathcal{A}$ is the unitary of Example 4, acting only on the 0-th component of this infinite tensor power.

The connection between the dilation and the repeated measurement is expressed by the following relation:

$$M^{(m)}(y_1\otimes\cdots\otimes y_m\otimes x) = \left(\left(\bigotimes_{\mathbb{Z}}\beta\right)\otimes\mathrm{id}\right)\left(\widehat{T}^m(\cdots\otimes\mathbf{1}\otimes y_1\otimes\cdots\otimes y_m\otimes \overset{x}{\overset{\otimes}{\mathbf{1}}}\otimes\mathbf{1}\otimes\cdots)\right).$$

Davies processes in discrete time are obtained by restriction to some abelian subalgebra $\bigotimes_{\mathbb{Z}}\mathcal{C}$ of $\bigotimes_{\mathbb{Z}}\mathcal{B}$.

References.

[Acc] L. Accardi: On the quantum Feynman-Kac formula. *Rend. Sem. Mat. Fis. Milano* **48** (1978), 135–179.

[BaH] A. Barchielli, A.S. Holevo: Constructing quantum measurement processes via classical stochastic calculus. *Stoch. Proc. Appl.* **58** (1995), 293–317.

[BGL] P. Busch, M. Grabovski, P.J. Lahti: Operational quantum physics. Lect. Notes Phys. **m31**, Springer 1997.

[CSVR] H.J. Carmichael, S. Singh, R. Vyas, P.R. Rice: *Phys. Rev.* **A 39** (1989), 1200.

[Dav] E.B. Davies: *Quantum theory of open systems.* Academic Press, London, New York, San Francisco 1976.

[FNW] M. Fannes, B. Nachtegaele, R.F. Werner: Finitely correlated states on quantum spin chains. *Commun. Math. Phys.* **144** (1992), 443–490.

[Gis] N. Gisin: Stochastic quantum dynamics and relativity. *Helv. Phys. Acta* **62** (1989), 363–371.

[Hel] C.W. Helström: *Quantum detection and estimation theory.* Academic Press, New York 1976.

[Hol1] A.S. Holevo: *Probabilistic and statistical aspects of quantum theory.* Nauka, Moscow 1980 (in Russian); North Holland, Amsterdam 1982 (English translation).

[Hol2] A.S. Holevo: On the Lévy-Khinchin formula in the non-commutative probability theory. *Theor. Probab. and Appl.* **37** (1993), 211–216.

[Kra] K. Kraus: General state changes in quantum theory. *Ann. Phys.* **64** (1971), 331–335.

[Küm] B. Kümmerer: Markov Dilations on W*-algebras. *J. Funct. Anal.* **62** (1985), 139–177.

[SrD] M.D. Srinivas, E.B. Davies: Photon counting probabilities in quantum optics. *Optica acta* **28** (1981), 981–996.

[WiM] H.M. Wiseman, G.J. Milburn: Interpretation of quantum jump and diffusion processes illustrated on the Bloch sphere. *Phys. Rev.* **A 47** (1993), 1652–1666.

Günther Mahler

Statistical measures for the characterization of quantum trajectories.

Abstract: It has been a widespread believe that the notion of a trajectory has to be abandoned in quantum mechanics. This certainly holds for the classical phase-space; nevertheless, the unitary motion of the state vector in Hilbert space may well be interpreted as generating an abstract trajectory. For open quantum systems, though, such a concept remained obscure until the advent of stochastic unravelling techniques of the underlying master-equation. While there is substantial progress in the description of non-classical states, the notion of "non-classical" trajectories has received much less attention. For stationary models we consider the statistics of 4 different features:
(i) coherence measures (distributions),
(ii) uncertainty of "relevant" observables (variances),
(iii) information aspects (entropies), and
(iv) jump distances (pertinent distributions).
Typical results are illustrated by means of numerical simulations.

Serge Massar

How much classical communication is required to simulate quantum communication and quantum entanglement? .

Abstract: I will analyze how much classical communication is required to simulate quantum communication and quantum entanglement. I first show that it is impossible to classically simulate quantum communication and entanglement by exchanging only a finite number of classical bits. However if the two parties share an infinite number of random bits (ie. an infinite number of local hidden variables) then simulation of quantum communication and entanglement is possible with only a finite number of bits of communication, as shown recently by Brassard, Cleve and Tapp. Furthermore even if the two parties do not share an infinite number of random bits, simulation of quantum communication and entanglement is possible with an amount of communication which is finite in the mean.

Ian Percival

Generalized Bell inequalities.

1 Introduction

The original Bell inequality is one example of a wide class of inequalities that relate the conditional probabilities of classical events in spacetime. The inequalities can be expressed in a beautiful geometrical form, which is a many-dimensional generalization of a three-dimensional problem in diamond-cutting.

A prominent diamond-cutter from Amsterdam has a rich and eccentric customer, who has given precise specifications for a diamond, in terms of the coordinates of each vertex. But the cutting of diamonds is not defined in terms of the vertexes, it is defined by the planes of the cutter, which follow the plane surfaces or facets of the diamond. So the diamond-cutter needs the equations of the facets, not the coordinates of the vertexes. The problem of finding all the generalized Bell inequalities is like the problem of the diamond-cutter, extended to arbitrarily high dimensions. The shape of a diamond of arbitrarily high dimension is called a polytope, and the space occupied by the diamond is the convex hull of its vertexes.

General Bell inequalities are of interest to physicists and philosophers of science, but their presentation is particularly simple in the language of the engineer's input-output systems, particularly stochastic systems, which require the use of probability theory.

A summary of the theory of transfer functions for stochastic systems in spacetime is given in Percival (1999). It expands on sections 2-4 of this abstract, with illustrative figures.

2 Cause and effect

Traditionally causes come before effects, but according to special relativity, things are not that simple, and the Bell inequalities show that for interaction between classical and quantum systems, the relation between cause and effect in spacetime is even more subtle.

3 Deterministic systems

Our systems are like the engineer's systems, with discrete classical inputs i and outputs j, but unlike the engineer's systems, which occupy a given region of space, ours occupy a region of spacetime. The input and output ports occupy restricted regions of spacetime, which for our purposes may be considered as points. The relations between an input and an output depends on whether they are separated by a spacelike interval or a timelike interval, and for the latter on the sign of the time. For classical deterministic and stochastic systems the input can only affect the output if the output port is in the forward light cone of the input port.

The inputs and outputs may be connected by classical systems, as for the gates of classical computers, or by quantum systems, as for experiments to test Bell's inequality. For such experiments the causal relations between input and output in spacetime are not so simple.

Suppose that for a system with one input and one output port the number of possible inputs is $N(i)$ and the number of possible outputs is $N(j)$. Then there are $N(i) \times N(j)$ possible transitions between the two. A particular deterministic system is defined by all the transitions $(i \to j)$, one from each input. This is the transition picture. These transitions can be considered together as a functional relation between the input and the output, given by a transfer function F, where

$$j = F(i). \tag{1}$$

This is the transfer picture. The number of possible transfer functions is

$$N(F) = N(j)^{N(i)}. \tag{2}$$

There is a trivial relation between the transition and transfer pictures for deterministic systems, but not for stochastic systems.

Now suppose we have two subsystems A and B of a system A+B that do not interact with one another. The constraint of no interactions can be expressed in terms of transfer functions. We denote the inputs, outputs and transfer functions for the subsystems by suffixes A and B. For deterministic systems, this independence can be expressed in terms of transfer functions. The subsystems A and B have separate transfer functions F_A, F_B such that

$$j_A = F_A(i_A), \qquad j_B = F_B(i_B). \tag{3}$$

The transfer function for the whole system is F, where

$$(j_A, j_B) = F(i_A, i_B) \qquad \text{where} \qquad F = (F_A, F_B) \tag{4}$$

This is the independence constraint on the transfer function F for the whole system. The number of transfer functions for these two independent systems is much less than the number of transfer functions for an arbitrary system with two inputs and two outputs of the same type.

4 Stochastic systems and Bell experiments

In the transition picture the relation between the input and the output is uniquely defined in terms of the transition or conditional probabilities

$$P(j|i) = P(i \to j) \qquad (0 \le P(i \to j) \le 1, \quad \sum_j P(i \to j) = 1). \tag{5}$$

In the transfer picture the behaviour of the stochastic system is defined in terms of the probabilities $P(F)$ that the stochastic system behaves like a deterministic system with transfer function F. The relation between the two pictures is given by

$$P(i \to j) = \sum_F P(F)\delta(j, F(i)). \tag{6}$$

In a standard Bell experiment, two photons with total spin zero are produced by down-conversion, and travel in opposite directions to distant locations labelled A and B, where

their polarizations are measured at A and B by using a polarizing beam splitter and detectors. The angle θ_A of the beam-splitter determines the direction in which the polarization is measured, with two possible output values j_A. The same applies at B with an angle θ_B and two possible values for the output j_B. For different experiments there are a different possible values of the angles, which are labelled by the input values i_A, i_B. There are a finite number of possible values for these integer variables, typically two or three, depending on the type of experiment.

The settings of the angles and the recordings of the polarizations are the classical events. The whole experiment may be considered as a single system, with input $i = (i_A, i_B)$ and output $j = (j_A, j_B)$, or as a pair of subsystems A and B. To test locality in this experiment, it is essential that *both* the input and output ports (events) at A should be separated by spatial spacetime intervals from *both* the input and output events at B.

In that case, according to special relativity, no signal can be sent from an input or output of A to an output of B, nor from an input or output of B to an output of A. For stochastic systems this can be expressed in terms of probabilities of transfer functions:

$$P(F) = 0 \qquad \text{unless} \qquad F = (F_A, F_B). \tag{7}$$

This does *not* mean that the outputs for systems A and B are uncorrelated. The correlation can come about because

$$P(F_A, F_B) \neq P(F_A)P(F_B), \tag{8}$$

which is not excluded, because the transfer functions can be correlated through interactions in their common past.

Now look at equation (6). For a given set of transition probabilities, and no constraint on the transfer function probabilities, it is always possible to find $P(F)$ to satisfy this equation. Usually there are many solutions. But as we remove transfer functions one by one, setting the corresponding $P(F)$ to zero, (6) becomes more and more difficult to satisfy, until a point is reached for which there are $P(i \to j)$ that satisfy the conditions of (5), but cannot be expressed in the form (6). For the constraint (7), these are the transition probabilities that violate the Bell inequalities. A more detailed account of transfer functions for deterministic and stochastic systems, with a derivation of the simplest Bell inequality in terms of transition probabilities can be found on the lanl server: quant-ph/9906005.

General Bell inequalities

Now we are ready to generalize. Suppose there are $N(s)$ subsystems, with all the inputs and outputs of each subsystem spatially separated from all the inputs and outputs of all the other subsystems. Suppose each subsystem has $N'(i)$ inputs and $N'(j)$ outputs, so that the whole system has

$$N(i) = N(s)N'(i) \qquad \text{and} \qquad N(j) = N(s)N'(j) \tag{9}$$

inputs and outputs.

To analyse these systems, we work in the space of the transition probabilities $P(i \to j)$ of

the whole system. There are $\nu(i)\nu(j)$ such transition probabilities, of which $\nu(i)(\nu(j)-1)$ are independent, due to the summation over the output probabilities for a given input adding to unity.

A deterministic system is a special example of stochastic system in which all the transition probabilities are either zero or one. The values are determined by the transfer function F. Now look again at equation (6). The delta function can be considered as the coordinates of the transfer function F in the space of transition probabilities. The whole equation expresses the coordinates of the transition probabilities as a weighted mean over the coordinates of the transfer functions. Clearly if there are constraints on the transfer functions then the weights for the forbidden transfer functions are zero and for the allowed transfer functions can be nonzero.

The geometric interpretation of this weighted mean is that a point which represents an allowed set of conditional probabilities must lie in the convex hull of the points which represent the allowed transfer functions. This convex hull is the allowed polytope, a many-dimensional generalization of the three-dimensional diamond of the first section. The allowed sets of conditional probabilities lie inside this polytope. Each facet of the polytope defines an independent linear inequality that has to be satisfied by the point representing the conditional probabilities. These are the generalized Bell inequalities.

By special relativity, if all the subsystems are linked classically, then subsystems must be independent in the sense of having separate transfer functions, and the transition probabilities must satisfy these generalized Bell inequalities.

The remarkable thing about systems which are linked quantum-mechanically is that the transition probabilities need not satisfy the generalized Bell inequalities.

Applications

Generalized Bell inequalities can be applied to experiments with quantum systems with higher spin, as has been done by Garg and Mermin. They can also be applied directly to the raw data of imperfect experiments, which is important, because all real experiments are imperfect. For example in the case of the original Bell experiment for which $\nu(s) = 2$, a perfect experiment would have $\nu'(j) = 2$ corresponding to the two orthogonal states of polarization. But detectors are far from perfect, and often fail to count the photons. In addition, the photons may be absorbed before they reach the detectors. So there are three possible outputs for each subsystem, corresponding to the two polarizations of one photon, and zero photons. So instead of $\nu(j) = 4$ for the perfect experiment, we have $\nu(j) = 9$, and generalized Bell inequalities are required.

The generalized inequalities are also needed for planning future experiments of a more general kind.

There is an extensive literature on the subject of general Bell inequalities, of which the most accessible for physicists is the recent paper of Peres. Those more familiar with stochastic terminology and methods may prefer the book of Deza and Laurent.

References for physicists

J.S.Bell 1987 Speakable and unspeakable in quantum mechanics. Cambridge University Press.

A.Peres 1995 Quantum theory: concepts and methods. Kluwer, Dordrecht.

E.P.Wigner 1970 On hidden variables and quantum mechanical probabilities. Am. J. Phys. **38** 1005-1009.

J.F.Clauser and A. Shimony 1978 Bell's theorem: experimental tests and implications. Rep. Prog. Phys. **41** 1881-1192.

A.Garg and N.D.Mermin 1984 Farkas' lemma and the nature of reality. Found. Phys. **14** 1-39.

A.Garg and N.D.Mermin 1987 Detector inefficiencies in the Einstein-Podolsky-Rosen experiment. Phys. Rev. D **35** 3831-3835.

A.Peres 1999 All the Bell inequalities. Found. Phys. **29** 589-614.

I.C.Percival 1999 Quantum measurement breaks Lorentz symmetry. LANL quant-ph/9906005

References for others

I.Pitowsky 1989 Quantum probability - quantum logic. Springer Lect. Notes on Physics **321**, Springer, Berlin

M.M.Deza and M.Laurent 1997 Geometry of cuts and metrics. Springer, Berlin.

Relativistic Quantum Measurements*

Asher Peres

Department of Physics, Technion—Israel Institute of Technology, 32 000 Haifa, Israel

The measuring process [1–3] is the interface of the classical and quantum worlds. It is a classical intervention in a quantum system. That intervention has two consequences. One is the acquisition of information by means of an apparatus that produces a record. This step is the measurement itself. Its outcome is in general unpredictable, except statistically. The other consequence is a change of the environment in which the quantum system will evolve after completion of the intervention. For example the intervening apparatus may generate a new Hamiltonian that depends on the recorded result. In particular, classical signals may be emitted for controlling the execution of further interventions. If these signals are limited to the velocity of light, we obtain a relativistic version of quantum measurement theory.

Typical detectors are bubbles in a bubble chamber, or small segments of wire in a wire chamber. The time required for the irreversible act of amplification (the formation of a microscopic bubble, or the initial stage of the electric discharge) is extremely brief, typically of the order of an atomic radius divided by the velocity of light. Once irreversibility has set in, the rest of the amplification process is essentially classical. It is noteworthy that the time and space needed for initiating the irreversible processes are incomparably smaller than the macroscopic resolution of the detecting equipment.

Interventions are mathematically represented by completely positive maps of the density matrix. A detailed dynamical description of the measuring process involves unitary interactions with a measuring apparatus and with an unknown environment that causes decoherence, and then the optional deletion of a subsystem. The Hilbert space for the resulting quantum system may have a different number of dimensions than the initial one. Thus, a quantum system whose description starts in a given Hilbert space may evolve into a set of Hilbert spaces with different dimensions.

An intervention is described by a set of parameters that include the spacetime coordinates of the location at which the intervention occurs, referred to an arbitrary coordinate system. We also have to specify the speed and orientation of the apparatus in that coordinate system and various other input parameters that control the apparatus, such as the strength of a magnetic field, or that of an RF pulse used in the experiment, and so on. The input parameters are determined by classical information received from past interventions, or they may be chosen arbitrarily by the observer who prepares that intervention. A crucial physical assumption is that there exists an objective time ordering of the various interventions in an experiment. There are no closed causal loops. This time ordering defines the notions earlier and later. The input parameters of an intervention are deterministic (or possibly stochastic) functions of the parameters of earlier interventions, but not of the stochastic outcomes resulting from later ones.

*The complete articles are in `quant-ph/9906023` and `quant-ph/9906034` and were submitted to Physical Review A.

The probabilities of the various outcomes of an intervention can be predicted by using a suitable theory. Quantum mechanics is fundamentally statistical. In the laboratory, any experiment has to be repeated many times in order to infer a law; in a theoretical discussion, we may imagine an infinite number of replicas of our gedankenexperiment, so as to have a genuine statistical ensemble. The various experiments that we consider all start in the same way, with the same initial state ρ_0, and the first intervention is the same. However, later stages of the experiment may involve different types of interventions, possibly with different spacetime locations, depending on the outcomes of the preceding events. Yet, assuming that each intervention has only a finite number of outcomes, there is for the entire experiment only a finite number of possible records. (A "record" is the complete list of outcomes that occurred during the experiment.) Each one of these records has a definite probability in the statistical ensemble. In the laboratory, experimenters can observe its relative frequency among all the records that were obtained. The role of theoretical physics is to predict the probability of each record, given the inputs of the various interventions (both the inputs that are actually controlled by the local experimenter and those determined by the outputs of earlier interventions). Each record is objective: everyone agrees on what happened (e.g., which detectors clicked). Therefore, everyone agrees on what the various relative frequencies are, and the theoretical probabilities are also the same for everyone.

The measuring process involves several participants: first, there are the physical system under study and a measuring apparatus whose states belong to macroscopically distinguishable subspaces. These interact unitarily and form a composite system $\mathcal{C}$. There is also an "environment" which includes unspecified degrees of freedom of the apparatus and the rest of the world. These unknown degrees of freedom interact with the relevant ones, but they are not under the control of the experimenter and cannot be described explicitly. In order to keep the discussion as general as possible, I do not introduce any "ancilla," contrary to current fashion. This omission is not an oversight, it is intentional and deserves an explanation. According to von Neumann's classic treatise [1], the various outcomes of a measurement correspond to a complete set of orthogonal projection operators in the Hilbert space of the quantum system under study. However, von Neumann's approach is too restricted, because a measuring process may have more distinct outcomes than the number of dimensions of that Hilbert space. The appropriate formalism is that of a *positive operator valued measure* (POVM) [4, 5]. Namely, the various outcomes of the measurement correspond to positive operators E_μ, which sum up to the unit operator but need not commute. This raises the problem of the actual implementation of a given POVM. A possible answer given by Helstrom [6] was to introduce an auxiliary quantum system that he called *ancilla*. Helstrom showed that any POVM can be realized as an ordinary von Neumann measurement of a composite system that consists of the original system and an ancilla having a sufficient number of dimensions. In real life this is usually not how measurements are actually performed. In the present work, the description of the measuring process involve no ancilla, and yet any POVM can be implemented by a unitary interaction of the quantum system with a suitable apparatus.

The apparatus itself is an utterly complicated system and some radical assumptions are needed in order to proceed with explicit calculations. Let us assume that the composite system $\mathcal{C}$ can be fully described by the theory. Its complete description involves both

"macroscopic" variables and "microscopic" ones. The difference between them is that the microscopic degrees of freedom can be considered as adequately isolated from the environment for the duration of the experiment, so that their evolution is in principle perfectly controlled, while the macroscopic ones cannot be isolated from the unknown environment and the dynamical evolution cannot be completely predicted. Statistical hypotheses are required in order to make plausible predictions, as explained below. Any other degrees of freedom of the apparatus, for which no explicit description is provided, are considered as part of the environment.

An essential property of the composite system $\mathcal{C}$, which is necessary to produce a meaningful measurement, is that its states form a finite number of orthogonal subspaces which are distinguishable by the observer. Each macroscopically distinguishable subspace corresponds to one of the outcomes of the intervention and defines a POVM element E_μ which is given explicitly by the unitary interaction of the system and the apparatus. Once this interaction is specified, the quantum evolution is well defined and it is in principle reversible. It would remain so if the macroscopic degrees of freedom of the apparatus could be perfectly isolated from their environment, and in particular from the "irrelevant" degrees of freedom of the apparatus itself. This demand is of course self-contradictory, since we have to read the result of the measurement if we wish to make any use of it.

The unitary interaction of the unknown environment with $\mathcal{C}$ generates an evolution whose description is necessarily incomplete. However, reasonable assumptions about the randomness of the environment lead to the conclusion that the density matrix of $\mathcal{C}$ becomes nearly exactly block-diagonal, the blocks corresponding to macroscopically distinguishable subspaces of $\mathcal{C}$. The latter are stable on the time scale of the experiment. This is the property called *decoherence*. From this moment on, the macroscopic degrees of freedom of $\mathcal{C}$ have entered into the classical domain [7, 8] and can be used to trigger amplification mechanisms (detector "clicks") for the convenience of the experimenter.

The final step of the intervention is to discard part of the composite system $\mathcal{C}$. In the case of a von Neumann measurement, the subsystem that is discarded and thereafter ignored is the measuring apparatus itself. In general, it may be a different subsystem: the discarded part may depend on the outcome μ and in particular its dimensions may depend on μ. We thus obtain the "quantum jump"

$$\rho \to \rho'_\mu = \sum_m A_{\mu m}\, \rho\, A^\dagger_{\mu m}, \tag{1}$$

which is the most general completely positive linear map [9]. Here, $(A_{\mu m})_{\sigma s} \equiv U_{s\mu\sigma m}$, where the indices s and σ refer to the original system under study and to the final one, respectively, and U is the unitary matrix for the interaction of the quantum system and the apparatus. Clearly, the "quantum jump" $\rho \to \rho'_\mu$ is not a dynamical process that occurs in the quantum system by itself. It results from the introduction of an apparatus, followed by its deletion or that of another subsystem. In the quantum folklore, an important role is played by the "irreversible act of amplification." The latter is irrelevant to the present issue. The amplification is solely needed to help the experimenter. A jump in the quantum state occurs even when there is no detector click or other macroscopic amplification because we impose abrupt changes in our way of delimiting the object that we consider as the quantum system under study. The precise location of the intervention,

which is important for the relativistic discussion below, is the point from which classical information may be sent and affect the input of other interventions.

We may also wish to consider the evolution of the quantum system in its unknown environment, without measurements. We write the Hamiltonian as $H = H_0 + H_{\rm env} + H_{\rm int}$, with obvious notations. The last two terms generate a stochastic, rapidly fluctuating motion. The exact evolution, taking everything into account, is a Brownian motion (a kind of random walk) superimposed on the ideal motion generated by H_0. By smoothing out the fluctuations, one obtains Lindblad's equation [10].

Quantum jumps as in Eq. (1) are quasi-instantaneous processes. In particular, they affect the wave function instantaneously throughout the entire configuration space. Is this quasi-instantaneous change of the quantum state, caused by a local intervention, consistent with relativity theory? The answer is not obvious. The wave function itself is not a material object forbidden to travel faster than light, but we may still ask how the dynamical evolution of an extended quantum system that undergoes several measurements in distant spacetime regions is described in different Lorentz frames. Difficulties were pointed out long ago by Bloch [11], Aharonov and Albert [12], and many others [13]. Some authors [14] considered detectors in relative motion, and therefore at rest in different Lorentz frames. However a detector in uniform motion is just as good as one that has undergone an ordinary spatial rotation. The point is not how individual detectors happen to move, but how the effects due to these detectors are described in different ways in one Lorentz frame or another.

Indeed, the rules for computing quantum probabilities involve explicitly the spacetime coordinates of the interventions. Lorentz invariance says that if the classical spacetime coordinates are subjected to a particular linear transformation, the probabilities remain the same. This invariance is not trivial because the rule for computing the probability of occurrence of a given record involves a sequence of mathematical operations corresponding to the time ordered set of all the relevant interventions. If we only consider the Euclidean group, all we have to know is how to transform the classical parameters, and the wave function, and the various operators, under translations and rotations of the coordinates. However, when we consider genuine Lorentz transformations, we have not only to Lorentz-transform the above symbols, but we are faced with a new problem: the natural way of calculating the result of a sequence of interventions, namely by considering them in chronological order, is different for different inertial frames. The issue is not only a matter of covariance of the symbols at each intervention and between consecutive interventions. There are genuinely different prescriptions for choosing the sequence of mathematical operations in our calculation. Einstein's principle of relativity asserts that there are no privileged inertial frames. Therefore these different orderings ought to give the same set of probabilities, and this demand is not trivial.

Note also that while interventions are localized in spacetime, quantum systems are pervasive. In each experiment, irrespective of its history, there is only one quantum system. The latter typically consists of several particles or other subsystems, some of which may be created or annihilated by the various interventions. For instance, consider the evolution of the quantum state in a Lorentz frame where intervention A is the first one to occur and has outcome μ, and B is the second intervention, with outcome ν. There will now be in Eq. (1) products of two Kraus matrices, $A_{\mu m}$ and $B_{\nu n}$, with a unitary evolution

between them. Then consider the same physical situation as described in another frame, where B occurs before A. Not only we have to Lorentz transform these matrices, but their order will be different. Since there is no privileged inertial frame, both descriptions given above are equally valid. Formally, the final states ρ_f and ρ'_f have to be Lorentz transforms of each other. This requirement imposes severe restrictions on the Kraus matrices that appear in Eq. (1).

In order to investigate this problem, consider a continuous Lorentz transformation from one frame to the other. As long as the order of occurrence of A and B is not affected by this continuous transformation of the spacetime coordinates, the latter is implemented in the quantum formalism by unitary transformations of the various operators. These unitary transformations obviously do not affect the observable probabilities. Therefore it is sufficient to consider just two Lorentz frames where A and B are almost simultaneous: either A occurs just before B, or just after B. There is of course no real difference in the actual physical situations and the Lorentz "transformation" between these two arbitrarily close frames is performed by the unit operator. The only difference resides in our method for calculating the final quantum state: first A then B, or first B then A. Consistency of the two results is obviously achieved if $[A_{\mu m}, B_{\nu n}] = 0$. This condition is always satisfied if the operators $A_{\mu m}$ and $B_{\nu n}$ are direct products of operators pertaining to the two subsystems: $A_{\mu m} = a_{\mu m} \otimes \mathbf{1}$ and $B_{\nu n} = \mathbf{1} \otimes b_{\nu n}$, where $\mathbf{1}$ denotes the unit matrix of a subsystem. This relationship is obviously fulfilled if the dynamical variables of the quantum subsystems commute. This is indeed a necessary condition for legitimately calling them subsystems.

Quantum nonlocality has led some authors to suggest the feasibility of superluminal communication by means of quantum measurements performed on correlated systems far away from each other [15, 16]. Such a possibility is definitely ruled out by the present relativistic formalism. We have already assumed that there exists a partial ordering of events. Superluminal communication would mean that the deliberate choice [17] of the test performed by an observer could influence in a deterministic way, at least statistically, the outputs of tests located at a space-like distance from that observer (or apparatus) and having a later time-coordinate. If this were true for any pair of space-like separated events, this would lead to the possibility of propagating information backwards in time between events with time-like separation. However, the assumption of Lorentz invariance, and the existence of random inputs, and the restriction of causal relationships between time-like related events to the future direction only, preclude causal relationships at spatial distances.

In summary, relativistic causality cannot be violated by quantum measurements. The fundamental physical assumption that is needed is that Lorentz transformations of the spacetime coordinates are implemented in quantum theory by *unitary* transformations of the various operators. This is the same as saying that the Lorentz group is a valid symmetry of the physical system.

References

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

[2] J. A. Wheeler and W. H. Zurek (eds.), *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).

[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).

[4] J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).

[5] E. B. Davies and J. T. Lewis, Comm. Math. Phys. **17**, 239 (1970).

[6] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[7] N. Bohr, in *New Theories in Physics* (Internat. Inst. of Intellectual Cooperation, Paris, 1939).

[8] O. Hay and A. Peres, Phys. Rev. A **58**, 116 (1998).

[9] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).

[10] G. Lindblad, Comm. Math. Phys. **48**, 119 (1976).

[11] I. Bloch, Phys. Rev. **156**, 1377 (1967).

[12] Y. Aharonov and D. Z. Albert, Phys. Rev. D **24**, 359 (1981); **29**, 228 (1984).

[13] A. Peres, in *Fundamental Problems in Quantum Theory* ed. by D. M. Greenberger and A. Zeilinger, Ann. New York Acad. Sci. **755**, 445 (1995); and references therein.

[14] A. Suarez and V. Scarani, Phys. Letters A **232**, 9 (1997).

[15] N. Herbert, Found. Phys. **12**, 1171 (1982).

[16] A. Garuccio, in *Quantum Interferometry*, ed. by F. De Martini, G. Denardo, and Y. Shih (VCH, Weinheim, 1996).

[17] A. Peres, Found. Phys. **16**, 573 (1986).

Relative entropy in quantum information theory

Dénes Petz*
Mathematics Institute
Technical University of Budapest
H-1521 Budapest XI. Sztoczek u. 2, Hungary

1 Relative entropy and its properties

The relative entropy is an information measure representing the uncertainty of a state with respect to another state. In the quantum (or noncommutative) case, states correspond to positive normalized functionals on an operator algebra and they are often given by density operators. When densities are available, the entropy of ω with respect to φ is defined by

$$S(\omega,\varphi)=\begin{cases}\mathrm{Tr}\,\mathrm{D}_\omega(\log\mathrm{D}_\omega-\log\mathrm{D}_\varphi) & \text{if } \operatorname{supp}D_\varphi \geq \operatorname{supp}D_\omega\\ +\infty & \text{otherwise .}\end{cases}$$

This formula was given by *Umegaki* as a generalization of the Kullback-Leibler information for discrimination [18].

The relative entropy may be defined for linear functionals of an arbitrary C*-algebra. The general definition due to *Araki* goes through von Neumann algebras and is based on the concept of relative modular operator, see [1]. An equivalent definition works for nuclear algebras:

$$S(\omega,\varphi)=\sup\{S(\omega\circ\alpha,\varphi\circ\alpha):\alpha\},$$

where sup is taken over all unital Schwarz maps $\alpha:\mathcal{A}_0\to\mathcal{A}$ such that $\mathcal{A}_0$ is of finite dimension. (For the states $\omega\circ\alpha$ and $\varphi\circ\alpha$ densities are available and the previous definition applies.)

Theorem 1.1 *The relative entropy of positive functionals of a C*-algebra posesses the following properties.*

(1a) $(\omega,\varphi)\mapsto S(\omega,\varphi)$ *is convex and lower semicontinuous.*

(1b) $\|\varphi-\omega\|^2\leq 2S(\omega,\varphi)$ *if* $\varphi(I)=\omega(I)=1$.

(1c) $S(\omega,\varphi_1)\geq S(\omega,\varphi_2)$ *if* $\varphi_1\leq\varphi_2$.

(1d) The relation $S(\omega\circ\alpha,\varphi\circ\alpha)\leq S(\omega,\varphi)$ *holds for a unital Schwarz map* $\alpha:\mathcal{A}_0\to\mathcal{A}$.

(1e) $S(\omega,\varphi)+\sum_{i=1}^n S(\omega_i,\omega)=\sum_{i=1}^n S(\omega_i,\varphi)$ *for* $\omega=\sum_i\omega_i$.

*Partially supported by Hungarian National Science Foundation Grant OTKA F023447

These properties have been obtained in different papers at different levels of generality. (1a) is related to [1] and [10], (1b) is essentially from [6], (1d) is a result from [17] and (1e) is from [2] and [4].

Now we list the properties of the relative entropy functional which were used in an axiomatic characterization in [14]:

(2a) Conditional expectation property:
Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and there exists a projection of norm one E of $\mathcal{B}$ onto $\mathcal{A}$ such that $\varphi \circ E = \varphi$. Then for every state ω of $\mathcal{B}$ $S(\omega, \varphi) = S(\omega|\mathcal{A}, \varphi|\mathcal{A}) + S(\omega, \omega \circ E)$ holds.

(2b) Monotonicity property:
For every unital Schwarz mapping α we have $S(\omega, \varphi) \geq S(\omega \circ \alpha, \varphi \circ \alpha)$.

(2c) Direct sum property:
Assume that $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$. Let $\varphi_{12}(a \oplus b) = \lambda\varphi_1(a) + (1-\lambda)\varphi_2(b)$ and $\omega_{12}(a \oplus b) = \lambda\omega_1(a) + (1-\lambda)\omega_2(b)$ for every $a \in \mathcal{B}_1, b \in \mathcal{B}_2$ and some $0 < \lambda < 1$. Then $S(\omega_{12}, \varphi_{12}) = \lambda S(\omega_1, \varphi_1) + (1-\lambda)S(\omega_2, \varphi_2)$.

(2d) Lower semicontinuity property:
The function $(\omega, \varphi) \mapsto S(\omega, \varphi)$ is lower semicontinuous on the state space of a C*-algebra.

Theorem 1.2 *If an extended positive valued functional $R(\omega, \varphi)$ is defined for states of nuclear C*-algebras such that it has properties (2a)–(2d), then there exists a constant $c \in \mathbf{R} \cup \{+\infty\}$ such that*

$$R(\omega, \varphi) = c\, S(\omega, \varphi)\,.$$

The von Neumann entropy of a state of a finite quantum system is the relative entropy with respect to the tracial state, at least up to a sign and an additive constant:

$$S(\omega) = -\mathrm{Tr}\,(\mathrm{D}_\omega \log \mathrm{D}_\omega)$$

Hence several properties of the von Neumann entropy follow from those of the relative entropy.

2 Model for quantum communication

Let $\alpha : \mathcal{A} \to \mathcal{B}$ be positive unitial mapping and φ be a state of $\mathcal{B}$. Such α may be interpreted as a channel in the sense that a state φ of $\mathcal{B}$ is transformed into another state $\varphi \circ \alpha$ of $\mathcal{A}$. φ is the initial or input state and $\varphi \circ \alpha$ is the corresponding output state. It is always convenient to assume that α is not only a positive mapping in the usual sense but completely positive.

To show a concrete example, consider the Stokes parametrization of 2×2 density matrices.

$$D_x = \tfrac{1}{2}(I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3),$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices and $(x_1, x_2, x_3) \in \mathbf{R}^3$ with $x_1^2 + x_2^2 + x_3^2 \leq 1$. For a positive semi-definite 3×3 matrix A the application $\Gamma^* : D_x \mapsto D_{Ax}$ gives a channeling transformation when $\|A\| \leq 1$. This channel was introduced in [5] under the name of *symmetric binary quantum channel*. (In order to have the dual of a completely positive mapping we need to impose some conditions on the eigenvalues of A.)

The quantum mutual entropy is defined after [11] as

$$I(\varphi;\alpha) = \sup\{\textstyle\sum_j \lambda_j S(\varphi_j \circ \alpha, \varphi \circ \alpha) : \sum_j \lambda_j \varphi_j = \varphi\}, \tag{2.6}$$

where the least upper bound is over all (orthogonal) extremal decompositions. The purely quantum capacity $C(\alpha)$ of α is the least upper bound of the mutual information quantities $I(\varphi;\alpha)$, where φ is varying over a set of states, possibly over all states.

In order to estimate the quantum mutual information, we introduce the concept of *divergence center*. Let $\{\omega_i : i \in I\}$ be a family of states and $R > 0$. We say that the state ω is a divergence center for $\{\omega_i : \ i \in I\}$ with radius $\leq R$ if

$$S(\omega_i, \omega) \leq R \qquad \text{for every } i \in I.$$

In our discussion about the geometry of relative entropy (or divergence as it is called in information theory) the ideas of [3] can be recognized very well.

Let $\{\omega_i : i \in I\}$ be a family of state. We say that the state ω is an *exact divergence center with radius* R if

$$R = \inf_{\varphi} \sup_{i} \{S(\omega_i, \varphi)\}$$

and ω is a minimizer for the right hand side. (When R is finite, then there exists a minimizer, because $\varphi \mapsto \sup\{S(\omega_i, \varphi) : i \in I\}$ is lower semicontinuous with compact level sets, cf. Proposition 5.27 in [12].)

The following result is from [13].

Theorem 2.1 *Let Λ^* be a channeling transformation sending density matrices on the Hilbert space $\mathcal{H}$ to those of the other Hilbert space $\mathcal{K}$. Assume that $\mathcal{K}$ is finite dimensional. Then the capacity of Λ^* is the divergence radius of the range of Λ^*.*

We want to compute the capacity of the above Γ^*. Since a unitary conjugation does not change capacity obviously, we may assume that A is diagonal with eigenvalues $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. The range of Γ^* is visualized as an ellipsoid with (Euclidean) diameter $2\lambda_1$. It is not difficult to see that the tracial state τ is the exact divergence center of the segment connected the states $(I \pm \lambda_1\sigma_1)/2$ and hence τ must be the divergence center of the whole range. The divergence radius is

$$\begin{aligned} &S\left(\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \tau\right) \\ &= \log 2 - \mathrm{S}\left(\frac{1}{2}\begin{pmatrix}1+\lambda & 0\\ 0 & 1-\lambda\end{pmatrix}\right) = \log 2 - \eta((1+\lambda)/2) - \eta((1-\lambda)/2). \end{aligned}$$

This gives the capacity according to the previous theorem.

3 An infinite system setting of the source

Let X^n denote the set of all messages of length n. If $x^n \in X^n$ is a message then a quantum state $\varphi(x^n)$ of the n-fold quantum system is corresponded with it. The Hilbert space of the n-fold system is the n-fold tensor product $\mathcal{H}^{\otimes n}$ and $\varphi(x^n)$ has a statistical operator $D(x^n)$. If messages of length n are to be transmitted then our quantum source should be put in the mixed state $\varphi_n = \sum_{x^n} p(x_n)\varphi(x^n)$ with statistical operator $D_n = \sum_{x^n} p(x_n)D(x^n)$, where

$p(x_n)$ is the probability of the message x_n. Since we want to let $n \to \infty$, it is reasonable to view all the n-fold systems as subsystem of an infinite one. In this way we can conveniently use a formalism standard in statistical physics.

Let an infinitely extended system be considered over the lattice $\mathbf{Z}$ of integers. The observables confined to a lattice site $k \in \mathbf{Z}$ form the selfadjoint part of a finite dimensional matrix algebra $\mathcal{A}_k$, that is the set of all operators acting on the finite dimensional space $\mathcal{H}$. It is assumed that the local observables in any finite subset $\Lambda \subset \mathbf{Z}$ are those of the finite quantum system

$$\mathcal{A}_\Lambda = \bigotimes_{\substack{i=1 \\ k\in\Lambda}}^{n} \mathcal{A}_k.$$

The quasilocal algebra $\mathcal{A}$ is the norm completion of the normed algebra $\mathcal{A}_\infty = \cup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with finite intervals $\Lambda \subset \mathbf{Z}^\nu$.

A state φ of the infinite system is a positive normalized functional $\mathcal{A} \to \mathbf{C}$. It does not make sense to associate a statistical operator to a state of the infinite system in general. However, φ restricted to a finite dimensional local algebra $\mathcal{A}_\Lambda$ admits a density matrix D_Λ. We regard the algebra $\mathcal{A}_{[1,N]}$ as the set of all operators acting on the N-fold tensor product space $\mathcal{H}^{\otimes N}$. Moreover, we assume that the density D_N from the first part of this section is identical with $D_{[1,N]}$. Under this assumptions we call the state φ the state of the (infinite) channel. Roughly speaking, all the states used in the transmission of messages of length n are marginals of this φ. Coding, transmission and decoding could be well formulated using the states $\varphi_N \equiv \varphi_{[1,N]}$. However, it is more convenient to formulate our setting in the form of an infinite system, in particularly because we do not want to assume that the channel state φ is a product type. This corresponds to the possibility that our quantum source has a memory effect. It is the main point in our stationary source coding theorem that the mean entropy of the channel state appears in the role of source coding rate.

First we present our positive source coding theorem for a completely ergodic source. The result says that that the source coding rate may approach the mean entropy while we can keep the fidelity arbitarily good.

Theorem 3.1 *Let $\mathcal{H}$ be a finite dimensional Hilbert space, $d := \dim \mathcal{H}$, and φ be a completely ergodic state on $B(\mathcal{H})^{\otimes\infty}$ with mean entropy s. Then for every $\varepsilon, \delta > 0$ there exists $n_{\varepsilon,\delta} \in \mathbb{N}$ such that for $n \geq n_{\varepsilon,\delta}$ there is a subspace $\mathcal{K}_n(\varepsilon,\delta)$ of $\mathcal{H}^{\otimes n}$ such that*

(i) $\log \dim \mathcal{K}_n(\varepsilon,\delta) < n(s+\delta)$ *and*

(ii) for every decomposition $D_n = \sum_{i=1}^m p_i D^{(i)}$ one can find an encoding $D^{(i)} \mapsto \tilde{D}^{(i)}$ with density matrices $\tilde{D}^{(i)}$ supported in $\mathcal{K}_n(\varepsilon,\delta)$ such that the fidelity $F := \sum_{i=1}^m p_i \mathrm{Tr}\, D_i \tilde{D}^{(i)}$ exceeds $1-\varepsilon$.

The negative part of the coding theorem tells that the source coding rate cannot exceeds the mean entropy when the fidelity is good.

Theorem 3.2 *Let $\mathcal{H}$ be a finite dimensional Hilbert space, $d := \dim \mathcal{H}$, and φ be a completely ergodic state on $B(\mathcal{H})^{\otimes\infty}$ with mean entropy s. Then for every $\varepsilon, \delta > 0$ there exists $n_{\varepsilon,\delta} \in \mathbb{N}$ such that for $n \geq n_{\varepsilon,\delta}$*

(i) for all subspaces $\mathcal{K}_n(\varepsilon,\delta)$ of $\mathcal{H}^{\otimes n}$ with the property log *dim $\mathcal{K}_n(\varepsilon,\delta) < n(s-\delta)$ and*

(ii) for every decomposition $D_n = \sum_{i=1}^m p_i D^{(i)}$ and for every encoding $D^{(i)} \mapsto \tilde{D}^{(i)}$ with density matrices $\tilde{D}^{(i)}$ supported in $\mathcal{K}_n(\varepsilon,\delta)$, the fidelity $F := \sum_{i=1}^m p_i \mathrm{Tr}\, D^{(i)} \tilde{D}^{(i)}$ is smaller than ε.

The detailed proofs are given in the paper [15]. This result extends Schumacher's source coding theorem obtained originally for memoryless channels, see [16] and [9]. It is worthwile to note that [8] contains more material on quantum coding.

References

[1] Araki, H. (1977): Relative entropy of states of von Neumann algebras II, Publ. RIMS Kyoto Univ. **13**, 173–192

[2] Araki, H. (1987): Recent progress on entropy and relative entropy, in *VIIIth Int. Cong. on Math. Phys.* (World Sci. Publishing, Singapore) 354–365

[3] Csiszár, I. (1975): I-divergence geometry of probability distributions and minimization problems, Ann. Prob. **3**, 146–158

[4] Donald, M.J. (1990): Relative Hamiltonian which are not bounded from above, J. Functional Anal. **91**, pp. 143–173

[5] A. Fujiwara, H. Nagaoka (1994): Capacity of memoryless quantum communication channels, Math. Eng. Tech. Rep. **94-22**, University of Tokyo

[6] Hiai, F., Ohya, M. Tsukada, M. (1981): Sufficiency, KMS condition and relative entropy in von Neumann algebras, Pacific J. Math. **96**, 99–109

[7] Hiai, F., Petz, D. (1991): The proper formula for relative entropy and its asymptotics in quantum probability, Comm. Math. Phys. **143**, 99–114

[8] A.S. Holevo (1998): Quantum coding theorems, Russian Math. Surveys, **53**, 1295–1331

[9] R. Jozsa, B. Schumacher, A new proof of the quantum noiseless coding theorem, J. Modern Optics, 1994.

[10] Kosaki, H. (1986): Relative entropy for states: a variational expression, J. Operator Theory **16**, 335–348

[11] Ohya, M (1983): On compound state and mutual information in quantum information theory, IEEE Trans. Information Theory, **29**, 770–777

[12] Ohya, M., Petz, D. (1993): *Quantum Entropy and Its Use* (Springer-Verlag, Berlin, Heidelberg, New York)

[13] M. Ohya, D. Petz, N. Watanabe (1997): On capacities of quantum channels, Prob. Math. Stat. **17**, 179–196

[14] Petz, D. (1992): Characterization of the relative entropy of states of matrix algebras, Acta Math. Hungar. **59**, 449–455

[15] Petz, D., Mosonyi, M. (1999): Stationary quantum source coding, to be published

[16] B. Schumacher (1995): Quantum coding, Phys. Rev. A **52**, 2738–2747

[17] Uhlmann, A. (1977): Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory, Commun. Math. Phys. **54**, 21–32

[18] Umegaki, H. (1962): Conditional expectations in an operator algebra IV (entropy and information), Kodai Math. Sem. Rep. **14**, 59–85

Adaptive Quantum Measurements (Summary)

H.M. Wiseman
School of Science, Griffith University, Nathan 4111, Brisbane, Queensland, AUSTRALIA

In this presentation I will: **I** Review the general theory of quantum measurements; **II** Introduce "incomplete" and "adaptive" measurements; **III** Describe different formulations of conditioned state evolution; **IV** Discuss unraveling the master equation; **V** Present the theory of "dyne" measurements; **VI** Derive the theory of complete dyne measurements; **VII** Apply this to adaptive phase measurements; and **VIII** Conclude.

I. QUANTUM MEASUREMENT THEORY

A. Orthodox Quantum Measurement Theory

An observable Λ is represented by an operator

$$\Lambda = \sum_\lambda \lambda \Pi_\lambda, \tag{1.1}$$

where

$$\lambda \in \Re\ ,\ \ \Pi_\lambda \Pi_{\lambda'} = \delta_{\lambda,\lambda'} \Pi_\lambda\ ,\ \ \sum_\lambda \Pi_\lambda = 1. \tag{1.2}$$

The probability for obtaining the result λ is

$$P_{\lambda(t)} = \mathrm{Tr}[\tilde{\rho}_\lambda(t+T)], \tag{1.3}$$

where

$$\tilde{\rho}_\lambda(t+T) = \Pi_\lambda \rho(t) \Pi_\lambda. \tag{1.4}$$

Here $\rho(t)$ is the system state matrix at time t (the start of the measurement).

The system state matrix at the end of the measurement (time $t+T$) is

$$\rho_\lambda(t+T) = \tilde{\rho}_\lambda(t+T)/P_\lambda. \tag{1.5}$$

If the initial state is pure then we can write

$$|\tilde{\psi}_\lambda(t+T)\rangle = \Pi_\lambda |\psi(t)\rangle. \tag{1.6}$$

B. More General Quantum Measurements

Still restricting the discussion to *efficient* measurements (in which final state is pure if initial state is pure), we can generalize orthodox measurements:

$$\Lambda \to \text{trash}, \tag{1.7}$$

$$\Pi_\lambda \to \Omega_\lambda. \tag{1.8}$$

The $\{\Omega_\lambda(T)\}$ are arbitrary, subject only to

$$\sum_\lambda \Omega_\lambda^\dagger \Omega_\lambda = 1. \tag{1.9}$$

I will call them *measurement operators*.

The unnormalized conditioned state is

$$|\tilde{\psi}_\lambda(t+T)\rangle = \Omega_\lambda |\psi(t)\rangle. \tag{1.10}$$

The probability for result λ is

$$P_\lambda(t) = \langle \tilde{\psi}_\lambda(t+T) | \tilde{\psi}_\lambda(t+T)\rangle. \tag{1.11}$$

C. Most General Quantum Measurements

We now allow for *inefficient* measurements (in which the final state may be mixed even if initial state is pure).

The unnormalized conditioned state is

$$\tilde{\rho}_\lambda(t+T) = \sum_k \Omega_{\lambda k} \rho(t) \Omega_{\lambda k}^\dagger. \tag{1.12}$$

We can also write this as

$$\tilde{\rho}_\lambda(t+T) = \mathcal{O}_\lambda \rho(t), \tag{1.13}$$

where the *operation* [1] $\mathcal{O}_\lambda = \sum_k \mathcal{J}[\Omega_{\lambda k}]$, where

$$\mathcal{J}[A]B \equiv ABA^\dagger. \tag{1.14}$$

The probability for the result λ is

$$P_\lambda(t) = \mathrm{Tr}[\tilde{\rho}_\lambda(t+T)] = \mathrm{Tr}[\rho(t) F_\lambda], \tag{1.15}$$

where the *effect* [1] $F_\lambda = \sum_k \Omega_{\lambda k}^\dagger \Omega_{\lambda k}$.

The only restriction on $\{\mathcal{O}_\lambda\}$ is conservation of probability which requires

$$\sum_\lambda F_\lambda = 1. \tag{1.16}$$

II. INCOMPLETE AND ADAPTIVE MEASUREMENTS

A. Classifying Measurements

There are many different, overlapping classes based on the properties of $\{\mathcal{O}_\lambda\}$. For example

- Back-action evading (and efficient) [2]:

$$\forall\ \lambda\ \ \Omega_\lambda = \Omega_\lambda^\dagger. \tag{2.1}$$

- Sharp: $\forall\ \lambda\ \ \mathrm{rank}\,[F_\lambda] = 1$ [3]

$$\text{i.e. } \forall\ \lambda\ \ \mathcal{O}_\lambda = \sum_k \mathcal{J}\left[|\theta_{\lambda k}\rangle\langle\phi_\lambda|\right]. \tag{2.2}$$

(θ and ϕ denote unnormalized states.)

Here I introduce a (perhaps new) class.

- **Complete**:

$$\forall\ \lambda\ \ \mathcal{O}_\lambda = \sum_{jk} \mathcal{J}\left[|\theta_{\lambda k}\rangle\langle\phi_{\lambda j}|\right]. \tag{2.3}$$

Obviously sharp $\Longrightarrow$ complete, but not *vice versa* except for efficient measurements.

B. Why 'Complete'?

Because the final state is determined solely by the measurement result

$$\rho_\lambda(t+T) = \frac{\mathcal{O}_\lambda\rho(t)}{P_\lambda(t)} \tag{2.4}$$

$$= \frac{\sum_{jk}|\theta_{\lambda k}\rangle\langle\phi_{\lambda j}|\rho(t)|\phi_{\lambda j}\rangle\langle\theta_{\lambda k}|}{\sum_{jk}\langle\theta_{\lambda k}|\theta_{\lambda k}\rangle\langle\phi_{\lambda j}|\rho(t)|\phi_{\lambda j}\rangle} \tag{2.5}$$

$$= \frac{\sum_{jk}|\theta_{\lambda k}\rangle\langle\theta_{\lambda k}|}{\sum_{jk}\langle\theta_{\lambda k}|\theta_{\lambda k}\rangle} \tag{2.6}$$

so no more information about $\rho(t)$ can be obtained from measuring the system again.

For efficient measurements, complete $\equiv$ sharp:

$$\forall\ \lambda\ \ \Omega_\lambda = |\theta_\lambda\rangle\langle\phi_\lambda|, \tag{2.7}$$

and an *incomplete* measurement is defined by

$$\exists\ \lambda : \Omega_\lambda \neq |\theta_\lambda\rangle\langle\phi_\lambda|. \tag{2.8}$$

An example of an incomplete measurement on a harmonic oscillator $[a, a^\dagger] = 1$ is

$$\Omega_1 = a(aa^\dagger)^{-1/2}\sin(\epsilon aa^\dagger)\ ,\ \ \Omega_2 = \cos(\epsilon a^\dagger a) \tag{2.9}$$

C. Adaptive Measurements

If a measurement is incomplete, $\rho_\lambda(t+T)$ depends on $\rho(t)$ and on λ. Subsequent measurements can reveal more information about the preparation of $\rho(t)$.

To optimize the information gained, subsequent measurements should (in general) depend on λ. That is, the optimum scheme is *adaptive*.

I will consider the example of adaptive phase measurements. To understand this we first need to understand

- Linear, nonlinear and semilinear formulations of conditioned state evolution
- Lindblad master equations
- Dyne measurements

III. FORMULATIONS FOR THE CONDITIONED STATE EVOLUTION

See for example Ref. [4].

A. Nonlinear

Generate

$$\rho_\lambda(t+T) = \mathcal{O}_\lambda\rho(t)/P_\lambda(t) \tag{3.1}$$

according to the probability distribution

$$P_\lambda(t) = \mathrm{Tr}[\mathcal{O}_\lambda\rho(t)] \tag{3.2}$$

and weight all equally.

B. Linear

Generate

$$\tilde{\rho}_\lambda(t+T) = \mathcal{O}_\lambda\rho(t) \tag{3.3}$$

at random, and weight by

$$\mathrm{Tr}[\tilde{\rho}_\lambda(t+T)] = P_\lambda(t). \tag{3.4}$$

C. Semilinear

Generate

$$\bar{\rho}_\lambda(t+T) = \mathcal{O}_\lambda\rho(t)/Q_\lambda(t) \tag{3.5}$$

according to an arbitrary probability distribution $Q_\lambda(t)$, and weight by

$$\mathrm{Tr}[\bar{\rho}_\lambda(t+T)] = P_\lambda(t)/Q_\lambda(t). \tag{3.6}$$

IV. UNRAVELING THE MASTER EQUATION

A. Unconditioned state evolution

If we ignore measurement results then we get the *unconditioned* state matrix

$$\rho(t+T) = \sum_\lambda \rho_\lambda(t+T)P_\lambda(t) \tag{4.1}$$
$$= \sum_\lambda \tilde{\rho}_\lambda(t+T) \tag{4.2}$$
$$= \sum_\lambda \bar{\rho}_\lambda(t+T)Q_\lambda(t) \tag{4.3}$$
$$= \mathcal{O}_{\rm unc}\rho(t) \tag{4.4}$$

where the unconditioned evolution operation is

$$\mathcal{O}_{\rm unc} = \sum_\lambda \mathcal{O}_\lambda. \tag{4.5}$$

For $T = dt$, $\mathcal{O}_{\rm unc} = 1 + O(dt)$. The most general form in this case is the Lindblad master equation [5]

$$\mathcal{O}_{\rm unc} = 1 + \mathcal{L}dt, \tag{4.6}$$

$$\mathcal{L}\rho = -i[H,\rho] + \sum_\mu \mathcal{D}[c_\mu]\rho, \tag{4.7}$$

where

$$\mathcal{D}[c]\rho \equiv c\rho c^\dagger - \tfrac{1}{2}c^\dagger c\rho - \tfrac{1}{2}\rho c^\dagger c\,; \quad H = H^\dagger \tag{4.8}$$

B. Unraveling the master equation

There are many different measurements $\{\mathcal{O}_\lambda\}$ giving the same unconditioned evolution $\mathcal{O}_{\rm unc}$.

For $\mathcal{O}_{\rm unc} = 1 + \mathcal{L}dt$ these are called different *unravelings* of the master equation (ME) [6].

Each unraveling describes an ensemble of *quantum trajectories* for the system state which *on average* reproduces the ME.

The different unravelings correspond to different detection schemes. For example, efficient direct detection of photons escaping a single-mode cavity gives quantum jumps (for result 1):

$$\Omega_1 = \sqrt{\gamma\,dt}\,a\,; \quad \Omega_0 = 1 - \tfrac{1}{2}\gamma\,dt\,a^\dagger a, \tag{4.9}$$

where γ is the cavity linewidth. The ME is

$$\dot{\rho} = \mathcal{L}\rho = \gamma\mathcal{D}[a]\rho. \tag{4.10}$$

V. "DYNE" DETECTION

Instead of directly detecting the light from system, one can interfere it with coherent light (the "local oscillator") before detection.

For a strong local oscillator the photon flux is too great for individual photons to be resolved. Instead, one measures a *photocurrent* $I_t \in \Re$.

If we discretize time in intervals $[t, t+dt)$, then we can normalize the photocurrent $I(t)$ (with dimension time$^{-1/2}$) so that

$$\left\langle I_t^2 \right\rangle = 1/dt. \tag{5.1}$$

A. Dyne Detection Measurement Operators

For efficient detection, we must use measurement operators indexed by the continuous variable I_t [7]

$$\Omega(I_t) = \sqrt{Q\left(I_t\right)}\left[1 + dt\left(I_t e^{-i\Phi}\sqrt{\gamma}\,a - \tfrac{1}{2}\gamma\,a^\dagger a\right)\right], \tag{5.2}$$

where Φ is the local oscillator phase and

$$Q\left(I_t\right) = (dt/2\pi)^{1/2}\exp\left(-\tfrac{1}{2}dtI_t^2\right). \tag{5.3}$$

The Q-distribution for I_t is that appropriate for Gaussian white noise dW/dt:

$$\langle I_t\rangle_Q \equiv \int_{-\infty}^{\infty} dI_t\,Q(I_t)\,I_t = 0 \tag{5.4}$$
$$\left\langle (I_t)^2\right\rangle_Q \equiv \int_{-\infty}^{\infty} dI_t\,Q(I_t)\,(I_t)^2 = 1/dt. \tag{5.5}$$

¿From these moments, we can show that

$$\int_{-\infty}^{\infty} dI_t\,\Omega^\dagger(I_t)\Omega(I_t) = 1, \tag{5.6}$$
$$\int_{-\infty}^{\infty} dI_t\,\mathcal{J}\left[\Omega(I_t)\right] = 1 + \mathcal{L}dt. \tag{5.7}$$

B. Linear Quantum Trajectories

Measurement operators for dyne detection are ideal for semilinear evolution formulation.

$$|\bar{\psi}_I(t+dt)\rangle = \frac{\Omega(I_t)|\psi(t)\rangle}{\sqrt{Q(I_t)}} \tag{5.8}$$
$$= \left[1 + dt\left(I_t e^{-i\Phi}\sqrt{\gamma}\,a - \tfrac{1}{2}\gamma\,a^\dagger a\right)\right]|\psi(t)\rangle$$

Thus $|\bar{\psi}_{\mathbf{I}}(t)\rangle$ obeys a *linear* stochastic Schrödinger equation [7]

$$d|\bar{\psi}_{\mathbf{I}}(t)\rangle = dt\left(I_t e^{-i\Phi}\sqrt{\gamma}\,a - \tfrac{1}{2}\gamma\,a^\dagger a\right)|\bar{\psi}_{\mathbf{I}}(t)\rangle. \tag{5.9}$$

Here the $\mathbf{I}$ subscript shows the state is conditioned on the complete record $\mathbf{I}_{[0,t)} = \{I_u : u \in [0,t)\}$.

The results I_t are generated according to the *ostensible* [4] distribution $Q(I_t)$, so ostensibly

$$d|\bar{\psi}_{\mathbf{I}}(t)\rangle = \left(\sqrt{\gamma}\, dW(t) e^{-i\Phi} a - \tfrac{1}{2}\gamma\, dt\, a^\dagger a\right) |\bar{\psi}_{\mathbf{I}}(t)\rangle. \quad (5.10)$$

The *actual* probability for the record $\mathbf{I}_{[0,t)}$ is

$$P_{\rm act}\left(\mathbf{I}_{[0,t)}\right) = P_{\rm ost}\left(\mathbf{I}_{[0,t)}\right) \langle\bar{\psi}_{\mathbf{I}}(t)|\bar{\psi}_{\mathbf{I}}(t)\rangle. \quad (5.11)$$

C. Solving the Linear Quantum Trajectory

Let $\gamma = 1$ so $\dot{\rho} = \mathcal{L}\rho = \mathcal{D}[a]\rho$ and

$$d|\bar{\psi}_{\mathbf{I}}(t)\rangle = \left(dW(t) e^{-i\Phi} a - \tfrac{1}{2} dt\, a^\dagger a\right) |\bar{\psi}_{\mathbf{I}}(t)\rangle. \quad (5.12)$$

Now using the Itô calculus,

$$|\bar{\psi}_{\mathbf{I}}(t+dt)\rangle = \exp\left(-\tfrac{1}{2} a^\dagger a dt\right) \exp\left(dW(t) e^{-i\Phi(t)} a\right) \exp\left(-\tfrac{1}{2} e^{-2i\Phi(t)} a^2 dt\right) |\bar{\psi}_{\mathbf{I}}(t+dt)\rangle.$$

This suggests the following solution [4] which can be verified by substitution

$$|\bar{\psi}_{\mathbf{I}}(t)\rangle = \exp\left(-\tfrac{1}{2} a^\dagger a t\right) \exp\left(\tfrac{1}{2} S_t^* a^2 + R_t^* a\right) |\psi(0)\rangle, \quad (5.13)$$

where

$$R_t = \int_0^t e^{i\Phi(s)} e^{-s/2} I(s) ds \quad (5.14)$$

$$S_t = -\int_0^t e^{2i\Phi(s)} e^{-s} ds. \quad (5.15)$$

That is to say, the solution at time t depends not on the full function $\mathbf{I}_{[0,t)}$, but only on these two complex functionals. [1]

VI. COMPLETE DYNE MEASUREMENTS

A dyne measurement is only complete for $t \to \infty$:

$$|\bar{\psi}_{\mathbf{I}}(\infty)\rangle = |0\rangle\langle 0| \exp\left(\tfrac{1}{2} B^* a^2 + A^* a\right) |\psi(0)\rangle \quad (6.1)$$

$$\equiv |0\rangle\langle\phi_{A,B}|\psi(0)\rangle \quad (6.2)$$

where $A = R_\infty, B = S_\infty$.

A and B are *sufficient statistics* for $\mathbf{I}_{[0,t)}$, and

$$P_{\rm act}(A,B) = {\rm Tr}[\rho(0) F(A,B)], \quad (6.3)$$

where the effect is

$$F(A,B) = P_{\rm ost}(A,B) |\phi_{A,B}\rangle\langle\phi_{A,B}|, \quad (6.4)$$

where $P_{\rm ost}(A,B)$ is that implied by

$$P_{\rm ost}(I_t) = Q(I_t) \ \ \forall\, t. \quad (6.5)$$

i.e. $P_{\rm ost}(A,B)$ is that which would result if

$$I(t) = dW(t)/dt. \quad (6.6)$$

A. Example: Homodyne

*Homo*dyne means the local oscillator has the *same* frequency as the system, so Φ is constant and

$$A_{\rm ost} = e^{i\Phi} \int_0^\infty e^{-t/2} dW(t) \to e^{i\Phi} X_\Phi. \quad (6.7)$$

$$B = -e^{2i\Phi} \int_0^\infty e^{-t} dt \to -e^{2i\Phi} \quad (6.8)$$

These imply

$$P_{\rm ost}(A,B)\, d^2A\, d^2B \to \frac{e^{-X_\Phi^2/2}}{\sqrt{2\pi}}\, dX_\Phi. \quad (6.9)$$

and

$$(a e^{-i\Phi} + a^\dagger e^{i\Phi}) |\phi_{A,B}\rangle = X_\Phi |\phi_{A,B}\rangle, \quad (6.10)$$

which gives

$$F(A,B) \to F_{\rm hom}(X_\Phi) = |X_\Phi\rangle\langle X_\Phi|. \quad (6.11)$$

Thus a completed homodyne measurement is a measurement of one quadrature of the field.

B. Example: Heterodyne

*Hetero*dyne means the local oscillator has a different frequency so $\Phi = t\Delta$ with $\Delta \gg 1$ and

$$A_{\rm ost} = \int_0^\infty e^{i\Delta t - t/2} dW(t) \to \alpha \quad (6.12)$$

$$B = -\int_0^\infty e^{2it\Delta - t} dt \to 0. \quad (6.13)$$

These imply

$$P_{\rm ost}(A,B)\, d^2A\, d^2B \to \frac{e^{-|\alpha|^2}}{\pi}\, d^2\alpha \quad (6.14)$$

[1] $\Phi(s)$ may depend on $\mathbf{I}_{[0,s)}$.

and

$$|\phi_{A,B}\rangle = e^{|\alpha|^2/2}|\alpha\rangle \tag{6.15}$$

which gives

$$F(A,B) \to F_{\rm het}(\alpha) = \pi^{-1}|\alpha\rangle\langle\alpha|. \tag{6.16}$$

Thus a completed heterodyne measurement is a measurement of the complex amplitude $\alpha = re^{i\phi}$.

This is the standard way to measure an optical phase:

$$F_{\rm het}(\phi) = \int_0^\infty r\,dr\,\pi^{-1}|re^{i\phi}\rangle\langle re^{i\phi}|. \tag{6.17}$$

VII. ADAPTIVE PHASE MEASUREMENTS

A better way to measure phase is *adaptively*. We can adjust $\Phi(t)$ *in real time* to make a homodyne measurement of the *estimated* phase quadrature:

$$\Phi(t) = \hat{\phi}(t) + \pi/2. \tag{7.1}$$

Here $\hat{\phi}(t)$ is a phase estimate based on R_t, S_t.

The simplest example uses [8]

$$\hat{\phi}(t) = \arg R_t. \tag{7.2}$$

Now ostensibly

$$R_t = \int_0^t e^{i\Phi(s)}e^{-s/2}dW(s), \tag{7.3}$$

so

$$dR_t = e^{i\Phi(t)}e^{-t/2}dW(t) = i\frac{R_t}{|R_t|}e^{-t/2}dW(t). \tag{7.4}$$

This has the solution

$$A = R_\infty = \exp\left(i\int_0^\infty \frac{dW(t)}{\sqrt{e^t-1}}\right), \tag{7.5}$$

which is random number of unit norm.

A. Special Case: at most one photon in field

If there is at most one photon in the field then $F(A,B)$ can be replaced by

$$F(A) = P_{\rm ost}(A)\left(|0\rangle + A|1\rangle\right)\left(\langle 0| + A^*\langle 1|\right). \tag{7.6}$$

The effect for a measurement of $\phi = \arg A$ is

$$F(\phi) = \int_0^\infty |A|\,d|A|\,F(A). \tag{7.7}$$

Since the adaptive technique implies $|A| = 1$,

$$F_{\rm adapt}(\phi) = \frac{1}{2\pi}|\phi\rangle\langle\phi|\ ,\ \ |\phi\rangle = |0\rangle + e^{i\phi}|1\rangle, \tag{7.8}$$

which is the optimum effect for a phase measurement on this system.

By contrast the heterodyne effect gives

$$F_{\rm het}(\phi) = \frac{\sqrt{\pi}}{2}F_{\rm adapt}(\phi) + \left(1-\frac{\sqrt{\pi}}{2}\right)\frac{1}{2\pi}. \tag{7.9}$$

i.e. as if it were an optimum measurement 88% of time, but gave a random result 12% of time.

VIII. CONCLUSIONS

1. Generalized measurement theory is needed to describe real measurements e.g. in quantum optics.

2. Measurements may be *incomplete* which implies that future measurements will provide more information about the state preparation.

3. To optimize the information obtained, we can use *adaptive* control of future measurements.

4. Dyne measurements on optical systems are well described by *linear* quantum trajectories.

5. Analytical solutions are possible for homodyne, heterodyne, and a simple adaptive scheme.

6. For a system with at most one photon, the adaptive phase measurement is superior to the heterodyne phase measurement.

7. Continuing research:

   - adaptive phase measurements
   - other applications of linear quantum trajectories
   - numerical simulations using nonlinear quantum trajectories
   - generalization of quantum trajectory theory (e.g. to non-Markovian systems)

[1] E.B. Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976).
[2] V.B. Braginsky and F.Y. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
[3] H. Martens and W.M. de Muynck, Found. Phys. **20**, 255 (1990); *ibid.*, 357 (1990)
[4] H.M. Wiseman, Quantum Semiclass. Opt **8**, 205 (1996).
[5] G. Lindblad, Commun. math. Phys. **48**, 199 (1976).
[6] H.J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993).
[7] P. Goetsch and R. Graham, Phys. Rev. A **50**, 5242 (1994).
[8] H.M. Wiseman, Phys. Rev. Lett. **75**, 4587 (1995).

Workshop on Stochastics and Quantum Physics Aarhus 21/10/99 -- 26/10/99

Stochastic Analysis as a tool for the foundations of Quantum Mechanics

J.C. Zambrini,

GFM, Av. Prof. Gama Pinto 2, 1649-003 LISBOA, PORTUGAL

A short story of quantum mechanics and probability.

To make this story short indeed, I'll consider only the probabilistic status of pure states, for systems of Hamiltonian $H = -\frac{\hbar^2}{2}\Delta + V$. For mixture of pure states, there is nothing special about the probability involved.

There are basically two kinds of motivations for the basic introduction of probability theory in Quantum Mechanics (QM):

a) Technical Motivations

Origin: M. Kac's work (1949) inspired by a lecture of R. Feynman on his path integral representation of solutions of the wave equation in $L^2(\mathbb{R}^n)$:

$$\begin{cases} i\hbar\frac{\partial\psi}{\partial t} = H\psi \\ \psi(x,0) = \varphi(x) \end{cases} \quad \text{i.e. } \psi_\varphi(x,t) = \int \varphi(\omega(0))\, e^{\frac{i}{\hbar}S[\omega,t]} D\omega$$

$$\Omega^{t,x} = \{\omega \in C([0,t],\mathbb{R}^n) \text{ s.t } \omega(t) = x\}$$

where $D\omega$ denotes the flat "measure" $\prod_{0\le\tau\le t} d\omega(\tau)$ and

$$S[\omega,t] = \int_0^t \left(\frac{1}{2}\,|\dot\omega(\tau)|^2 - V(\omega(t))\right) d\tau \equiv S_0[\omega,t] - \underbrace{\int_0^t V(\omega(t))\, d\tau}_{perturbation} < \infty$$ is the classical action

functional of the system.

Unfortunately, $\lim_{disc}\left\{ e^{\frac{i}{\hbar}S_0[\omega;t]} \prod_{\tau\in\{t_1,\ldots,\,t_j=t\}} d\omega(\tau) \right\}$, where the limit is in L^2 sense when the discretization goes to zero, is not σ - additive $\Rightarrow$ there is no such complex measure on $\Omega^{t,x}$!

This is not only a technical detail, as some radical physicists would love to think . It means that there is no such thing as Feynman's fundamental integral over a space of continuous paths in Quantum Theory and that any consequence drawn from it is under legitimate suspicion.

On the other hand, after $0 < t \to -it$, Schrödinger equation $\to \begin{cases} -\hbar\frac{\partial\eta_*}{\partial t} = H\eta_* \\ \eta_*(x,0) = \chi(x) \end{cases}$ i.e. the heat equation.

In this case, it is well known that $\lim_{disc}\left\{ e^{-\frac{1}{\hbar}S_0[\omega;t]} \prod d\omega(\tau) \right\} = d\mu_w^{\hbar}$, the Wiener measure with coeff $\sqrt{\hbar}$ and, for a large class of potentials V the following path integral representation makes sense:

$$\eta_\chi^*(x,t) = E^{t,x}\left[\chi(w(0))\, e^{-\frac{1}{\hbar}\int_0^t V(w(\tau))d\tau} \right]$$

where $E^{t,x}$ denotes the conditional expectation given that $w(\tau) = x$.

NB: The Wiener process $w(\tau)$ is used exclusively as a technical tool here. Its familiar irreversible properties have nothing to do with the (reversible) one of free quantum dynamics. The way probability enters here has nothing to do with the way it enters in Quantum Mechanics according to Born: $\int_B \psi\overline{\psi}(x,t)dx = \text{Prob}\{\text{system} \in B \text{ at time } t\}$. All crucial features of quantum mechanical probability (interference etc…) are missing.

There are many variations on Kac's theme since 1949. For example, one can introduce complex Markov processes with diffusions coefficient $(1+i)\sqrt{\hbar}$ the way Doss did (1980).

Then Kac's representation holds for Schrödinger equation but for a narrow class of potentials V and of solutions ψ_t. Of course, Born interpretation of ψ is lost. Unfortunately, the structure of Quantum theory lies more in this interpretation than in an underlying hypothetical process.

b) Foundations of Q.M.

b_1) ***Schrödinger's (1932) Idea:*** To construct a model of classical physics (in order to avoid any metaphysical contamination) where Probability enters like in Q.M.

Consider the Cauchy problem for the free diffusion equation: $-\frac{\partial \eta_*}{\partial t} = -\frac{1}{2}\frac{\partial^2 \eta_*}{\partial x^2} = H_0 \eta_*$, with $\eta_{*_{-T/2}} = \rho_{-T/2}$ given**.** We know all about it since Einstein's time. Now consider the new assumption: the observer knows another probability density in the future $\rho_{-T/2}(dy)$. For $+\frac{\partial \eta}{\partial t} = -\frac{\partial^2 \eta}{\partial x^2}$ this is an O.K data. For $\{\rho_{-T/2}$ and $\rho_{T/2}\}$, the most probable evolution is $\rho_t(x) = \eta\eta^*(x,t)dx$, compatible with these data. This (Euclidean) Born interpretation lies at the foundation of (EQM) Euclidean Quantum Mechanics. On $I = \left[-\frac{T}{2}, \frac{T}{2}\right]$, the interpolating real valued diffusion Z_t is well defined (this is a Bernstein diffusion). Its qualitative properties are very "non classical".

For example: If $\eta_* = (\rho^{1/2} e^{-S})(x,t)$ is a positive free solution, $\eta^*(x-\ell,t) + \eta^*(x+\ell,t)$, (for $\ell = cste$) is another one. The probability of the superposed Bernstein diffusion is

$$p_S(x,t) = p(x-\ell,t) + p(x+\ell,t) + \underbrace{2p^{1/2}(x-\ell,t)p^{1/2}(x+\ell,t)\cosh[S(x-\ell,t) - S(x+\ell,t)]}_{\text{Euclidean Interference}}$$

Observe that if the phase S was purely imaginary, p_s would coincide with the quantum interference formula. So, in spite of what we are told all the time, there are classical experiments, indeed, with apparently non-classical probabilistic outcomes.

Also notice that the abovementioned decomposition of the positive solution η_* involves a term, ρ, even under time-reversal and a phase S odd under this transformation. Clearly, this suggests that our Euclidean counterpart of the complex conjugate is $\eta_* = (\rho^{1/2} e^{-S})(x,t) \to \eta = (\rho^{1/2} e^{+S})(x,t)$ justifying Schrödinger's interpretation of the pair of adjoint heat equations as counterpart of the quantum wave equation and its complex conjugate. Of course, well defined path integral respresentations of η_* and η are now available.

The physically relevant Bernstein diffusions are Markovian. But, in contrast with most Markovian processes familiar in physics, they respect the time-symmetry of Markov properties, precisely expressed by $E[f(z_t) \mid P_s \cup F_u] = E[f(z_t) \mid z_s, z_u]$, $\forall f$ bounded

measurable, where P_s, F_u denote, respectively, the past (increasing) and future (decreasing) filtrations.

b_2) ***Nelson Stochastic Mechanics (1966):*** Construction of a real valued Markov process compatible with Born interpretation: $|\psi_t|^2\, dx = \rho(x,t)dx, \quad \forall t \in \mathrm{IR}$. But a careful analysis shows that Nelson's theory is, in fact, a probabilistic interpretation of Bohm theory. Any perturbation of Wiener measure involves not the physical (local) V but Bohm (nonlocal) potential: $\vartheta(x,t) = \hbar^2 \frac{\Delta\rho^{1/2}}{\rho^{1/2}}(x,t) - V(x)$.

This implies, among other disagreements, that all the (well defined) multi times correlations of Nelson's processes are without relations with quantum mechanical predictions.

b_3) ***Change of the (Kolmogorovian) foundations of probability theory:*** Some of the participants in this meeting defend this radical point of view.

I intend to summarize recent progress of EQM. Our ultimate goal is to embed regular Quantum Mechanics in a nonprobabilistic framework but much closer to probability theory than Feynman path integral approach and, a fortiori, than QM in Hilbert space. The only hope to convince physicists that the detour via stochastic analysis makes sense is to use it for the introduction of new conceptual (and then technical) results in regular QM. So we want to use stochastic analysis to widen the foundations of regular QM through a better analogy than the ones already known, but without claiming that QM is a true (Kolmogorovian) probabilistic theory. Actually, we believe that this claim is hopelessly wrong. A few successful examples of the above mentioned strategy are known today. I'll sketch their main ideas and results. In particular, it has provided a fresh insight on quantum symmetries, always richer than the one given by the Hilbert space approach.

This is in this sense that we are entitled to believe that stochastic analysis is indeed a natural tool for quantum mechanics, the only physical theory where probability lies at the foundations and, we are told, in an irreducible way.

References

a) For Physicists:

J.C.Z. "From Quantum Physics to Probability Theory and Back" in "Chaos, the interplay between Stochastic and Deterministic behaviour", Springer Lecture Notes in Physics, Nº 457, Edit. P. Garbaczewski et al., Spinger, Berlin 1995.

b) For Mathematical Physicists:

- M. Thieullen, J.C.Z., "Probability and Quantum Symmetries I", Ann. Inst. Henri Poincaré (Physique Théorique), Vol. 67, nº 3, (1997) p297.
- S. Albeverio, J. Rezende, J.C.Z., "Probability and Quantum Symmetries II", in preparation.

c) For Mathematicians:

- A.B. Cruzeiro, J.C.Z., "Malliavin Calculus and Euclidean Quantum Mechanics I. Functional Calculus", Journal of Funct. Analysis, Vol. 96, Nº 1, Feb. 15 1991, p62.
- A.B. Cruzeiro, Wu Liming, J.C.Z., "Bernstein processes associated with a Markov Process", to appear in Proceedings of the Third International Workshop on Stochastic Analysis and Mathematical Physics", R. Rebolledo (Ed.), Birkhäuser Progress in Probability Series, to appear.
- M. Thieullen, J.C.Z., "Probability Theory and Related Fields", 107 (1997), 401.

White noise calculus for fractional Brownian motion and application to stochastic partial differential equations

Bernt Øksendal[1),2)]

[1)] Department of Mathematics , University of Oslo
Box 1053 Blindern , N-0316, Oslo, Norway ,
Email: oksendal@math.uio.no
[2)] Norwegian School of Economics and Business administration ,
Helleveien 30 , N-5035, Bergen-Sandviken, Norway

Extended abstract

1 The 1-parameter case

Recall that if $0 < H < 1$ then the *fractional Brownian motion with Hurst parameter* H is the Gaussian process $B_H(t)$; $t \in \mathbf{R}$ with mean $E(B_H(t)) = 0$ and covariance

$$E\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

for all $s, t \in \mathbf{R}$. Here E denotes the expectation with respect to the probability law for $B_H = B_H(t, \omega)$. For simplicity we assume $B_H(0) = 0$.

If $H = \frac{1}{2}$, then $B_H(t)$ coincides with the standard Brownian motion $B(t)$. If $H > \frac{1}{2}$ then $B_H(t)$ has a *long range dependence*, in the sense that if we put

$$r(n) = \text{cov}\ \left(B_H(1)\,, (B_H(n+1) - B_H(n))\right)$$

then

$$\sum_{n=1}^{\infty} r(n) = \infty\,.$$

AMS 1991 subject classifications. Primary 60H05, 60H10; Secondary 90A09, 90A12.
Key words and phrases: Fractional Brownian motions; Fractional white noises; Chaos expansion; Wick calculus; Fractal Black and Scholes market; Arbitrage; Price formula; Replicating portfolio.

For any $H \in (0,1)$ the process $B_H(t)$ is self-similar in the sense that $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$ for any $\alpha > 0$. See Mandelbrot and Van Ness [MV] for more information about fractional Brownian motion.

Because of these properties $B_H(t)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$ has been suggested as a useful tool in many applications, including physics and finance [M].

A major difficulty with these processes is that they are not semimartingales and not Markov processes, so many of the powerful tools from stochastic analysis cannot be used for $B_H(t)$. However, we shall see that it is possible to develop a white noise calculus based on $B_H(t)$, $\frac{1}{2} < H < 1$.

We will use this to study Itô type stochastic differential equations driven by fractional white noise $W_H(t) = \frac{dB_H(t)}{dt}$. Moreover, we will introduce a fractional Malliavin calculus and prove a generalized fractional Clark-Haussmann-Ocone formula. This presentation will follow the paper [HØ] closely.

2 The multiparameter case

As in [H1], [H2] we define *d-parameter fractional Brownian motion* $B_H(x)$; $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ with *Hurst parameter* $H = (H_1, \dots, H_d) \in (0,1)^d$ as a Gaussian process on $\mathbf{R}^d$ with mean

$$E[B_H(x)] = 0 \qquad \text{for all } x \in \mathbf{R}^d \tag{2.1}$$

and covariance

$$E[B_H(x)B_H(y)] = (\tfrac{1}{2})^d \prod_{i=1}^{d} (|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}) \tag{2.2}$$

We also assume that

$$B_H(0) = 0 \quad \text{a.s.} \tag{2.3}$$

From now on we will assume that

$$\tfrac{1}{2} < H_i < 1 \qquad \text{for } i = 1, \dots, d\,. \tag{2.4}$$

In part 2 we will extend the fractional white noise theory to the multiparameter case and use this theory to study some stochastic partial differential equations, driven by multiparameter fractional white noise $W_H(x) = \frac{\partial^d B_H(x)}{\partial x_1 \dots \partial x_d}$; $x \in \mathbf{R}^d$.

This presentation will be based on the paper [HØZ].

For example, we will prove the following result from [HØZ]:

Theorem **a)** *For any dimension d there is a unique strong solution $U(x) : \bar{D} \to (\mathcal{S})^*_H$ satisfying the fractional Poisson equation*

$$\Delta U(x) = -W_H(x)\,; \qquad x \in D \subset \mathbf{R}^d \tag{2.5}$$

$$U(x) = 0 \qquad \textit{for} \quad x \in \partial D\,. \tag{2.6}$$

The solution is given by

$$U(x) = \int_D G(x,y) W_H(y) dy = \int_D G(x,y) dB_H(y) \ . \tag{2.7}$$

The solution belongs to $C(\bar{D}) \cap C^2(D)$.

b) *If*

$$d - \sum_{i=1}^{d} H_i < 2 \, , \tag{2.8}$$

then

$$U(x) \in L^2(\mu_\varphi) \qquad \textit{for all} \quad x \in \bar{D} \, ,$$

so the solution exists as an ordinary stochastic field.

Remark Note that if $H_i = \frac{1}{2}$ for all i then the condition (2.8) coincides with the known condition $d < 4$. But if $H_i > \frac{1}{2}$ for all i we may get $L^2(\mu_\varphi)$ solutions also in higher dimensions. Thus it is easier to obtain $L^2(\mu_\varphi)$ solutions the higher the values of H_i are. This seems reasonable from the point of view that the paths of $B_H(x)$ get more regular as H increases.

References

[AØU] K. Aase, B. Øksendal, J. Ubøe: White noise generalizations of the Clark-Ocone theorem with application to mathematical finance. Preprint, MaPhySto, Aarhus, 1998.

[B] F.E. Benth: Integrals in the Hida distribution space $(\mathcal{S})^*$. In T. Lindstrøm, B. Øksendal and A.S. Üstünel (editors): Stochastic Analysis and Related Topics. Gordon and Breach 1993, pp. 89–99.

[BG] F.E. Benth and H. Gjessing: A non-linear parabolic equation with noise. A reduction method. Potential Analysis (to appear).

[D] A. Dasgupta: Fractional Brownian motion: Its properties and applications to stochastic integration. Ph. D Thesis, Dept of Statistics, University of North Carolina at Chapel Hill, 1997.

[DH] W. Dai and C.C. Heyde: Itô formula with respect to fractional Brownian motion and its application. *J. Appl. Math. Stoch. Anal.* **9** (1996), 439–448.

[DHP] T.E. Duncan, Y. Hu and B. Pasik-Duncan: Stochastic calculus for fractional Brownian motion. I. Theory. To appear in SIAM Journal of Control and Optimization.

[DK] A. Dasgupta and G. Kallianpur: Arbitrage opportunities for a class of Gladyshev processes. To appear in Appl. Math. & Optimization.

[DU] L. Decreusefond and A.S. Üstünel: Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10** (1999), 177–214.

[GN] G. Gripenberg and I. Norros: On the prediction of fractional Brownian motion. *J. Appl. Prob.* **33** (1996), 400–410.

[H1] Y. Hu: Heat equation with fractional white noise potentials. Preprint 1998.

[H2] Y. Hu: A class of stochastic partial differential equations driven by fractional white noise. Preprint 1999.

[H3] Y. Hu: Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities. *Journal of Theoretical Probability*, **10** (1997), 835–848.

[HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: White Noise Analysis. Kluwer 1993.

[HØ] Y. Hu and B. Øksendal: Fractional white noise calculus and applications to finance. Preprint, Dept. of Mathematics, University of Oslo 1999.

[HØ1] Y. Hu and B. Øksendal: Wick approximation of quasilinear stochastic differential equations. In H. Körezlioglu, B. Øksendal and A.S. Üstünel (editors): Stochastic Analysis and Related Topics, Vol. 5. Birkhäuser 1996, pp. 203–231.

[HØS] Y. Hu, B. Øksendal and A. Sulem: Optimal portfolio in a fractional Black & Scholes market. Preprint, Dept. of Mathematics, University of Oslo, No. 13, August 1999.

[HØUZ] H. Holden, B. Øksendal, J. Ubøe and T. Zhang: Stochastic Partial Differential Equations. Birkhäuser 1996.

[HØZ] Y. Hu, B. Øksendal and T. Zhang: Stochastic partial differential equations driven by multiparameter fractional white noise. Preprint 1999.

[K] H.-H. Kuo: White Noise Distribution Theory. CRC Press 1996.

[L] A. Le Breton: Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. *Stat. Prob. Letters* **38** (1998), 263–274.

[LC] T. Lundgren and D. Chiang: Solution of a class of singular integral equations. *Quart. J. Appl. Math.* **24**, 303–313.

[Li] S.J. Lin: Stochastic analysis of fractional Brownian motion. *Stochastics and Stochastic Reports* **55** (1995), 121–140.

[LØU] T. Lindstrøm, B. Øksendal and J. Ubøe: Wick multiplication and Itô-Skorohod stochastic differential equations. In S. Albeverio, J.E. Fenstad, H. Holden and T. Lindstrøm (editors): Ideas and Methods in Mathematical Analysis, Stochastics, and Applications, Cambridge Univ. Press, 1992, pp. 183–206.

[M] B.B. Mandelbrot: Fractals and Scaling in Finance: Discontinuity, Concentration, Risk. Springer-Verlag, 1997.

[MV] B.B. Mandelbrot and J.W. Van Ness: Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** (1968), 422–437.

[NVV] I. Norros, E. Valkeila and J. Virtamo: An elementary approach to a Girsanov formula and other analytic results on fractional Brownian motions. *Bernoulli* **5** (1999), 571–587.

[Ø] B. Øksendal: Stochastic Differential Equations. Springer, fifth edition, 1998.

[PT] J. Potthoff and M. Timpel: On a dual pair of smooth and generalized random variables. *Potential Analysis* **4** (1995), 637–654.

[R] L.C.G. Rogers: Arbitrage with fractional Brownian motion. *Math. Finance* **7** (1997), 95–105.

[S] A. Shiryaev: On arbitrage and replication for fractal models: In A. Shiryaev and A. Sulem (editors): Workshop on Mathematical Finance, INRIA, Paris 1998.

[So] T. Sottinen: Fractional Brownian motion, random walks and binary market models. Preprint, University of Helsinki, June 1999.

[Sz] G. Szego: Orthogonal Polynomials. *Amer. Math. Soc. Colloq. Pub.* **23**, Providence, R.I. 1967.

[T] S. Thangavelu: Lectures of Hermite and Laguerre Expansions. Princeton University Press, 1993.

[V] E. Valkeila: On some properties of geometric fractional Brownian motion, Preprint, University of Helsinki, May 1999.

3 Workshop Program (revised)

Thursday October 21 (in Auditorium G2, building 532)

09.00-9.50 Registration and coffee/tea

Chairman: Ole E. Barndorff-Nielsen

10.10-11.00 **Bernt Øksendal:**
Short course on Wick Products, Malliavin Calculus and their applications in Physics.

11.10-12.00 **Bernt Øksendal:**
Short course on Wick Products, Malliavin Calculus and their applications in Physics.

12.30-14.00 Lunch

Chairman: Klaus Mølmer

14.00-14.50 **Denes Petz:**
Relative entropy in quantum information theory.

Coffee/tea

15.10-16.00 **Ian Percival:**
Generalized Bell inequalities.

16.10-17.00 **Uffe Haagerup:**
Spectra of random matrices and random operators on Hilbert spaces.

17.00-18.00 Welcome Reception

Friday October 22 (in Auditorium G2, building 532)

Chairman: Richard Gill

9.00-9.50 **François Bardou:**
Stochastic wave-functions, Lévy flights and quantum evaporation.

Coffee/tea

Bernt Øksendal:
10.10-11.00 *Short course on Wick Products, Malliavin Calculus and their applications in Physics.*

Bernt Øksendal:
11.10-12.00 *Short course on Wick Products, Malliavin Calculus and their applications in Physics.*

12.30-14.00 Lunch

Chairman: Viacheslav Belavkin

Günther Mahler:
14.00-14.50 *Statistical measures for the characterization of quantum trajectories.*

Coffee/tea

Jean Claude Zambrini:
15.10-16.00 *Stochastic analysis as a tool for the foundations of quantum mechanics.*

16.10-17.00 **Eugene Polzik:**
Short introduction to Quantum Optics & Lab-Tour.

Saturday October 23 (in Auditorium G2, building 532)

Chairman: Göran Lindblad

9.00-9.50	**Alexander Holevo:** *Coding Theorems for Quantum Channels.*
Coffee/tea	
10.10-11.00	**Viacheslav P. Belavkin:** *Quantum Stochastics as a Boundary Value Problem.*
11.10-12.00	**Viacheslav P. Belavkin:** *Classification of Quantum Noise.*
12.00-13.20	Lunch

Chairman: Günther Mahler

13.20-14.10	**Alberto Barchielli:** *Quantum stochastic models of two-level atoms.*
Coffee/tea	
14.30-15.20	**Hans Maassen:** *Ergodicity of continuous measurement in quantum optics.*
15.30-16.20	**Serge Massar:** *How much classical communication is required to simulate quantum communication and quantum entanglement?.*
16.30	Departure for Excursion/Conference Dinner

Monday October 25 (in Auditorium G1, building 532)

Chairman: François Bardou

9.00-9.50 **Asher Peres:**
Relativistic Quantum Measurement.

Coffee/tea

10.10-11.00 **Howard Carmichael:**
Physical principles of quantum trajectories.

11.10-12.00 **Howard Wiseman:**
Monitoring Open Quantum Systems: Theory and Applications to Adaptive Quantum Measurements.

12.00-12.20 **Elena R. Loubenets:**
The Quantum Stochastic Evolution of an Open System under Continuous in Time Non-demolition Measurement.

12.30-14.00 Lunch

Chairman: Alexander Holevo

14.00-14.50 **Richard Gill:**
Quantum asymptotic statistics and "non-locality without entanglement".

Coffee/tea

15.10-16.00 **Alexander Gottlieb:**
Quantum molecular chaos and the dynamics of mean-field spin models.

16.10-17.00 **Yuri Yu. Lobanov:**
Stochastic calculations in quantum physics via numerical integration in metric spaces.

Tuesday October 26 (in Auditorium G1, building 532)

Chairman: Elena Loubenets

9.00-9.50 **Inge Helland:**
Experiments, symmetries and quantum mechanics.

Coffee/tea

10.10-11.00 **Peter Høyer:**
The Subgroup Problem via *State Distinguishability.*

11.10-12.00 **Göran Lindblad:**
Gaussian maps and processes in quantum systems.

12.30-14.00 Lunch

4 List of participants

Luigi Accardi
Università degli Studi di Roma "Tor Vergata"
Centro Vito Volterra
Via di Tor Vergata, snc
I-00133 Roma, Italy
Email: accardi@volterra.mat.uniroma2.it
WWW: http://volterra.mat.uniroma2.it

Alberto Barchielli
Universita di Milano
Dipartimento di Matematica
Via Saldini 50
I-20133 Milano, Italy
Email: albbar@pop2.mate.polimi.it

Francois Bardou
Department of Physics
University of Newcastle
Newcastle Upon Tyne, NE1 7RU
United Kingdom
Email: bardou@morgane.u-strasbg.fr

Ole E. Barndorff-Nielsen
Department of Mathematical Sciences
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: oebn@imf.au.dk

Viacheslav P. Belavkin
School of Mathematical Sciences
University of Nottingham
Nottingham NG7 2RD
United Kingdom
Email: vpb@maths.nott.ac.uk

Fred Espen Benth
Department of Mathematics
University of Oslo
P.B. 1053 Blindern
N-0316 Oslo
Norway
Email: fredb@math.uio.no

Howard Carmichael
University of Ulm
Department of Quantum Physics
D-89069 Ulm
Germany
Email: Howard.Carmichael@physik.uni-ulm.de

Richard D. Gill
University of Utrecht
Mathematical Institute
Budapestlaan 6
NL-3585 CD Utrecht, The Netherlands
Email: Richard.Gill@math.ruu.nl
WWW: http://www.math.uu.nl/people/gill/

Alexander Gottlieb
Lawrence Berkeley National Laboratory
1 Cyclotron Road, Mail Stop 50A-1148
94720 Berkeley, CA
USA
Email: gottlieb@math.berkeley.edu
WWW: http://www.lbl.gov/~gottlieb/

Svend Erik Graversen
Department of Mathematical Sciences
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: matseg@imf.au.dk

Inge S. Helland
Department of Mathematics
University of Oslo
P.O.Box 1053 Blindern
N-0316 Oslo, Norway
Email: ingeh@math.uio.no
WWW: http://www.math.uio.no/~ingeh/

Alexander S. Holevo
Steklov Mathematical Institute
Russian Academy of Sciences
Moscow
Russia
Email: holevo@genesis.mi.ras.ru

Peter Høyer
BRICS
Department of Computer Science
University of Aarhus
DK-8000 Aarhus C, Denmark
Email: hoyer@brics.dk

Uffe Haagerup
Matematisk Institut
Odense Universitet
Campusvej 55
DK-5230 Odense M, Denmark
Email: haagerup@imada.sdu.dk

Jens Ledet Jensen
Department of Mathematical Sciences
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: jlj@imf.au.dk

Peter E. Jupp
School of Mathematical and Computational Sciences
University of St. Andrews
North Haugh
St Andrews KY16 9SS, Scotland
Email: pej@st-andrews.ac.uk

Martin Leitz-Martini
University of Tübingen
Mathematical Institute
Auf der Morgenstelle 10
D-72076 Tübingen, Germany
Email: martin.leitz@uni-tuebingen.de

Mykola Leonenko
Department of Mathematics
Kiev University
Volodimirska str. 64
252601 Kyiv
Ukraine
Email: leonenko@aki.pp.kiev.ua

Göran Lindblad
Theoretical Physics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
Email: gli@theophys.kth.se

Yuri Yu. Lobanov
Lab. of Computing Techniques and Automation
Joint Institute for Nuclear Research
Dubna, Moscow Region
141980 Russia
Email: lobanov@main1.jinr.ru

Elena R. Loubenets
Applied Mathematics Department
Moscow State Institute of
Electronics and Mathematics
Moscow, Russia
Email: erl@erl.msk.ru

Alessandra Luati
Dipartimento di Scienze Statistiche
Paolo Fortunati
via Belle Arti 40
I-40126 Bologna, Italy
Email: luati@stat.unibo.it

Hans Maassen
University of Nijmegen
Toernooiveld 1
6525 ED Nijmegen
The Netherlands
Email: maassen@sci.kun.nl

Günther Mahler
Institute for Theoretical Physics I
Pfaffenwaldring 57
D-70550 Stuttgart
Germany
Email: mahler@theo.physik.uni-stuttgart.de

David Márquez-Carreras
Facultat de Matemátiques
Universitat de Barcelona
Gran Via 585
E-08007 Barcelona, Spain
Email: marquez@mat.ub.es

Serge Massar
Service de Physique Theorique
Universite Libre de Bruxelles
CP 225, Bvd du Triomphe
B1050 Bruxelles, Belgium
Email: smassar@ulb.ac.be

Milan Mosonyi
Panorama u.90
2083 Solymar
Hungary
Email: mmm@ludens.elte.hu

Klaus Mølmer
Institute of Physics and Astronomy
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: moelmer@ifa.au.dk

Ian Percival
Department of Physics
Queen Mary and Westfield College
Mile End Road
London E1 4NS, United Kingdom
Email: I.C.Percival@qmw.ac.uk

Asher Peres
Department of Physics
Technion
32000 Haifa
Israel
Email: peres@photon.technion.ac.il

Denes Petz
Department of Analysis
Mathematics Institute
Technical University of Budapest
H-1521 Budapest XI, Hungary
Email: petz@math.bme.hu

Goran Peskir
Department of Mathematical Sciences
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: goran@imf.au.dk

Eugene Polzik
Institute of Physics and Astronomy
University of Aarhus
DK-8000 Aarhus C
Denmark
Email: polzik@dfi.aau.dk

Daniel Terno
Department of Physics
Technion
32000 Haifa
Israel
Email: terno@physics.technion.ac.il

Steen Thorbjørnsen
Dept. of Mathematics
Odense University
Campusvej 55
DK-5230 Odense M, Denmark
Email: steenth@imada.sdu.dk

Flemming Topsøe
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø, Denmark
Email: topsoe@math.ku.dk

Howard Wiseman
School of Science
Griffith University
Nathan 4111, Brisbane
Queensland, Australia
Email: H.Wiseman@mailbox.gu.edu.au

Jean Claude Zambrini
Grupo de Fisica-Matematica
Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, Portugal
Email: zambrini@alf1.cii.fc.ul.pt

Eberhard Zeidler
Max Planck Institute for
Mathematics in the Sciences
Inselstrasse 22
D-04103 Leipzig, Germany
Email: ezeidler@mis.mpg.de

Bernt Øksendal
Department of Mathematics
University of Oslo
B.P. 1053 Blindern
N-0316 Oslo, Norway
Email: oksendal@math.uio.no