Concentrated Advanced Course (January 7-8, 1999)
and Workshop (January 11-13, 1999) on

Product Integrals and Pathwise Integration

Foreword

In the week January 7-13, 1999 a course and a workshop on *Product Integrals and Pathwise Integration* was held by MaPhySto at the Department of Mathematical Sciences, University of Aarhus.

The course and workshop was organized by Ole E. Barndorff-Nielsen (Aarhus, Denmark), Svend Erik Graversen (Aarhus, Denmark) and Thomas Mikosch (Groningen, The Netherlands).

In this leaflet we have gathered the program, the list of participants and the workshop abstracts. The notes for the course appeared as the first volume of the MaPhySto Lecture Notes series and may be fetched from our web-site `www.maphysto.dk`; hardcopies may also be ordered.

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1 Course Program

Thursday January 7

9.30-10.30 **R. Dudley:** *How should integrals of Stieltjes type be defined? .*

COFFEE/TEA

10.50-11.50 **R. Norvaisa:** *Stochastic integrals .*

12.00-12.30 **R. Dudley:** *Lyons' work on obstacles to pathwise Itô integration.*

12.30-13.30 LUNCH

13.30-14.15 **R. Dudley:** *Differentiability with respect to p-variation norms (I).*

14.20-15.05 **R. Dudley:** *Differentiability with respect to p-variation norms (II).*

COFFEE/TEA

15.20-16.00 **R. Norvaisa:** *Properties of p-variation (I).*

16.15-17.00 **R. Norvaisa:** *Properties of p-variation (II).*

Friday January 8

9.30-10.30 **R. Norvaisa:** *Stochastic processes and p-variation.*

COFFEE/TEA

10.45-11.45 **R. Dudley:** *Empirical processes and p-variation.*

12.00-13.00 LUNCH

13.00-14.00 **R. Norvaisa:** *Integration.*

14.10-14.40 **R. Dudley:** *Ordinary differential equations and product integrals.*

COFFEE/TEA

15.00-16.00 **R. Norvaisa:** *Product integrals.*

16.10-16.40 **R. Dudley:** *Other aspects of product integrals.*

2 Workshop Program

Monday January 11

09.30-10.00	Registration and coffee/tea

Chairman: Thomas Mikosch

10.00-10.10	**Ole E. Barndorff-Nielsen and Thomas Mikosch:** *Introduction.*
10.10-11.00	**Richard Gill:** *Product-integration and its applications in survival analysis.*
11.10-12.00	**Norbert Hofmann:** *Optimal Pathwise Approximation of Stochastic Differential Equations.*
12.00-14.00	Lunch

Chairman: Svend Erik Graversen

14.00-14.30	**Esko Valkeila:** *Some maximal inequalities for fractional Brownian motions.*
14.35-15.25	**Richard Dudley:** *On Terry Lyons's work.*
Coffee/tea	
16.00-16.50	**Rimas Norvaiša:** *p-variation and integration of sample functions of stochastic processes.*
17.00-17.50	**Søren Asmussen:** *Martingales for reflected Markov additive processes via stochastic integration.*

Tuesday January 12

Chairman: Jørgen Hoffmann-Jørgensen

09.00-09.40	**Philippe Carmona:** *Stochastic integration with respect to fractional Brownian motion.*
09.45-10.35	**Rama Cont:** *Econometrics without probability: measuring the pathwise regularity of price trajectories.*
Coffee/tea	
11.00-11.50	**Francesco Russo:** *Calculus with respect to a finite quadratic variation process.*
12.00-14.00	Lunch

Chairman: Søren Asmussen

14.00-14.50	**Pierre Vallois:** *Stochastic calculus related to general Gaussian processes and normal martingales.*
14.55-15.45	**Imme van den Berg:** *Stochastic difference equations, discrete Fokker-Planck equation and finite path-integral solutions.*
COFFEE/TEA	
16.10-17.00	**Rudolf Grübel:** *Differentiability properties of some classical stochastic models.*
17.05-17.45	**Donna M. Salopek:** *When is the stop-loss start gain strategy self-financing?.*
17.50-18.30	DISCUSSION

Wednesday January 13

Chairman: Ole E. Barndorff-Nielsen

09.00-09.40	**Bo Markussen:** *Graphical representation of interacting particle systems.*
09.45-10.35	**Jan Rosiński:** *Independence of multiple stochastic integrals.*
COFFEE/TEA	
11.00-11.50	**Zbigniew J. Jurek:** *Infinite divisibility revisited.*
12.00-14.00	LUNCH

3 Combined list of participants to the Course and to the Workshop

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4 Abstracts

On the following pages you will find the (extended) abstracts submitted to the organizers. The contributions of R. Dudley and R. Norvaiša can be seen as addendums to their lecture notes *An Introduction to p-variation and Young Integrals*, MaPhySto Lecture Notes No. 1, Aarhus, January 1999.

MARTINGALES FOR REFLECTED MARKOV ADDITIVE PROCESSES VIA STOCHASTIC INTEGRATION

SØREN ASMUSSEN

Let X_t be an additive process on a finite Markov process J_t and consider $Z_t = X_t + Y_t$ where Y_t is an adapted process of finite variation, say the local time at one or two boundaries. We construct a family of vector–valued martingales for (J_t, Z_t) as certain stochatic integrals related to the exponential Wald martingales, thereby generalizing a construction of Kella & Whitt (1992) for Lévy processes.

The applicability of the martingales is demonstrated via a number of examples, including fluid models, a storage model and Markov–modulated Brownian motion with two reflecting boundaries.

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This is based on joint work with Offer Kella.

STOCHASTIC INTEGRATION WITH RESPECT TO FRACTIONAL BROWNIAN MOTION

PHILIPPE CARMONA AND LAURE COUTIN

Abstract. For a general class of Gaussian processes X, wich contains type II fractional Brownian motion of index $H \in (0,1)$ and fractional Brownian motion of index $H \in (1/2, 1)$, we define a stochastic integral

$$\int a(s)dX(s) = \lim \int a(s)dX_n(s)$$

which is the limit of classical semi martingale integrals.

Extended Abstract

Fractional Brownian motion was originally defined and studied by Kolmogorov within a Hilbert space framework. Fractional Brownian motion of Hurst index $H \in (0,1)$ is a centered Gaussian process W^H with covariance

$$\mathbb{E}\left[W_t^H W_s^H\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \qquad (s,t \geq 0)$$

(for $H = \frac{1}{2}$ we obtain ordinary Brownian motion).
Fractional Brownian motion has stationary increments

$$\mathbb{E}\left[(W^H(t) - W^H(s))^2\right] = |t-s|^{2H} \qquad (s,t \geq 0),$$

is H-self similar

$$(\frac{1}{c^H} W^H(ct)\,; t \geq 0\,) \stackrel{d}{=} (W^H(t)\,; t \geq 0\,)\,,$$

and, for every $\beta \in (0, H)$, its sample paths are almost surely Hölder continuous with exponent β.
However, in general (i.e. $H \neq \frac{1}{2}$) fractional Brownian motion is not a semi-martingale (see, e;g;, Decreusefond and Ustunel[1], Rogers , Salopek). Therefore, integration with respect to fractional Brownian motion cannot be defined in the standard way (the semimartingale approach).

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Existing definitions of the integral. We are aware of at least three ways of defining the integral with respect to fractional Brownian motion. The first method amounts to define a *pathwise integral* by taking adavantage of the Hölder continuity of sample paths of W^H. Given a process a such that almost all sample paths $s \to a_s(\omega)$ of a have bounded p variation on $[0,t]$, for $\frac{1}{p} + H > 1$, the integral

$$\int_0^t a(s) dW^H(s)$$

almost surely exists in the Riemann-Stieltjes sense (see Young). Let us recall that the p-variation of a function f over an interval $[0,t]$ is the least upper bound of sums $\sum_i |f(x_i) - f(x_{i-1})|^p$ over all partitions $0 = x_0 < x_1 < \ldots < x_n = T$ which may be finite or infinite.
The second method is to define the integral for *deterministic* processes a by an L^2 isometry (see Norros et al.). More precisely, if $H > \frac{1}{2}$, then $\int a(s)dW^H(s)$ is defined for functions a in L^2_Γ, the space of measurable functions f such that $\langle\langle f, f\rangle\rangle_\Gamma < +\infty$, with the inner product defined as

$$\langle\langle f, g\rangle\rangle_\Gamma = H(2H-1)\int_0^\infty \int_0^\infty f(s)g(t)|s-t|^{2H-2}\, ds\, dt\,.$$

Eventually, the third method is the analysis of the Wiener space of the fractional Brownian motion (see Decreusefond et Ustunel[1], and Duncan et al).
On the one hand, the pathwise integral enables us to consider random integrands. For instance, if $H > \frac{1}{2}$,

$$\int_0^t W^H(s)\, dW^H(s)$$

almost surely exists in the Riemann-Stieltjes sense.
On the other hand, with the deterministic integral

- We require less regularity from the sample paths of the integrand a;
- we can compute the expectations

$$\mathbb{E}\left[\int_0^\infty a(s)\, dW^H(s)\right] = 0\,, \qquad \mathbb{E}\left[\left(\int_0^\infty a(s)\, dW^H(s)\right)^2\right] = \langle\langle a, a\rangle\rangle_\Gamma\,.$$

The class of integrators considered. The aim of this paper is to define a stochastic integral that tries to get the best of both worlds, not only for fractional Brownian motion, but for a general class of Gaussian process. The starting point of our approach is the construction of fractional Brownian motion given by Mandelbrot and van Ness [2]:

$$W^H(t) = c_H \int_{-\infty}^t ((t-u)_+^{H-1/2} - (-u)_+^{H-1/2})\, dB_u\,,$$

where c_H is a normalizing constant, $x_+ = \sup(x, 0)$ denotes the positive part, and $(B_u, u \in \mathbb{R})$ is a Brownian motion on the real line, that is $(B_u, u \geq 0)$ and $(B_{-u}, u \geq 0)$ are two independent standard Brownian motions.
It is natural to consider the process

$$V_h(t) = \int_0^t h(t-u)\, dB_u$$

where h is locally square integrable. We shall consider more general processes

$$W_K(t) = \int_0^t K(t,u)\, dB_u$$

where the kernel K is measurable and such that

$$\text{for all } t > 0, \quad \int_0^t K(t,u)^2\, du < +\infty\,.$$

It is to be noted that integrating on $(0,t)$ instead of $(-\infty, t)$ causes no loss in generality. Indeed, see e.g. Theorem 5.2 of Norros et al , we can construct in this way fractional Brownian motion. More precisely, for $H > \frac{1}{2}$, W_K is fractional Brownian motion of index H for the choice

$$K(t,s) = (H - \frac{1}{2}) c_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2}\, du\,.$$

Another process of interest in this class is type II fractional Brownian motion of index $H \in (0,1)$

$$V^H(t) = c'(H) \int_0^t (t-u)^{H-\frac{1}{2}}\, dB_u \quad (t \geq 0)$$

where $c'(H)$ is a normalizing constant such that $\mathbb{E}\left[(V^H(t))^2\right] = t^{2H}$. It is a H self similar centered Gaussian process whose almost all sample paths are Hölder continuous of index β, for $\beta \in (0, H)$; however it has not stationary increments.

The main idea. Let us explain now the main idea of this paper. When h is regular enough, then V_h is a semi martingale. Therefore we can define $\int a(s) dV_h(s)$ as an ordinary semimartingale integral, when a is a nice adapted process. In some cases we can restrict a to a space of good integrands and find a sequence h_n converging to h in such a way that $\int a(s) dV_{h_n}(s)$ converges in L^2 to a random variable which we note $\int a(s) dV_h(s)$. The space of good integrands will be defined by using the analysis of the Wiener space of the driving Brownian motion B, and not the Wiener space of the process V_h or W_K.
Consequently, our first result is to characterize the V_h which are semi-martingales. To this end we introduce the space $L^p_{loc}(\mathbb{R}_+)$ of functions f which locally of p-th power integrable:

$$f \in L^p_{loc}(\mathbb{R}_+) \text{ if } \quad \forall t > 0, \qquad \int_0^t |f(s)|^p \, ds < +\infty.$$

From now on h is a locally square integrable function such that h' exists almost everywhere.

Theorem 1. *The process V_h is a semimartingale if and only if $h' \in L^2_{loc}(\mathbb{R}_+)$. When this is the case, its decomposition is*

$$v_h(t) = h(0)B_t + \int_0^t V_{h'}(s) \, ds \,.$$

We shall now introduce the space of good integrands (the basic definitons needed from Malliavin Calculus can be found in Nualart's book). Let $I = [0, t]$; given $q \geq 2$, we set for $a \in \mathbf{L}_{1,2}(I)$

$$n_{q,t}(a) = \mathbb{E}\left[\int_0^t |a(s)|^q \, ds\right]^{1/q} + \mathbb{E}\left[\int_0^t \int_0^t du \, dv \, |D_u a(v)|^q\right]^{1/q} .$$

The space $GI_{q,t}$ is the set of $a \in \mathbf{L}_{1,2}$ such that $n_{q,t}(a) < +\infty$.

Theorem 2. *Let $p \in]1, 2[$ and let q be the conjugated exponent of p ($\frac{1}{p} + \frac{1}{q} = 1$). Assume that $h' \in L^p_{loc}(\mathbb{R}_+)$ and $a \in GI_{q,t}$. Then there exists a square integrable random variable denoted by $\int_0^t a(s) \, dV_h(s)$ such that for every sequence $(h_n, n \in \mathbb{N})$ of functions verifying*

1. *for all n, h_n and h'_n are in $L^2(0, t)$,*
2. *$h_n(0) \underset{n\to\infty}{\longmapsto} h(0)$ and $h'_n \underset{n\to\infty}{\longmapsto} h'$ in $L^p(0, t)$,*

we have the convergence in L^2

$$\int_0^t a(s) \, dV_{h_n}(s) \underset{n\to\infty}{\longmapsto} \int_0^t a(s) \, dV_h(s) \,,$$

where on the left hand side we have classical semimartingales integrals.

Let us stress the fact that Malliavin Calculus is a powerful tool that may totally disappear from the results, as show the following Itô's formula

Theorem 3 (Itô's Formula)**.** *Assume that $h \in L^2_{loc}(\mathbb{R}_+)$ and that for a $p \in (0, 1)$, $h' \in L^p_{loc}(\mathbb{R}_+)$. Then, for every $F \in C^2$ such that F' and*

F'' are uniformly Lipschitz continuous, we have

$$\begin{aligned} F(V_h(t)) &= F(0) + \int_0^t F'(V_h(s))\, dV_h(s) \\ &= F(0) + \int_0^t \left(\int_s^t h'(u-s) F'(V_h(u))\, du \right) dB_s \\ &+ \int_0^t \left(\int_s^t h'(u-s) h(u-s) F''(V_h(u))\, du \right) ds \\ &+ h(0) \int_0^t F'(V_h(s))\, dB_s \\ &+ \frac{1}{2} h(0)^2 \int_0^t F''(V_h(s))\, ds\,. \end{aligned}$$

This Theorem needs the following comments

- observe first that we can check this Itô's formula by computing the expectation of both sides

$$\mathbb{E}[F(V_h(t))] = \frac{1}{2} \int_0^t F''(V_h(u)) h(u)^2\, du\,.$$

- Furthermore, we can illustrate this formula by applying it to type II fractional Brownian motion of index $H > \frac{1}{2}$:

$$(V^H(t))^2 = \int_0^t \left(\int_s^t (u-s)^{H-3/2} (H-1/2) 2 V^H(u)\, du \right) dB_s + t^{2H}\,.$$

The next logical step of our study consists of showing that the integral we defined coincides, for good regular integrands, with the pathwise integrals. To this end, it is enough to show that it is the L^2-limit of Riemann sums over a refining sequence of partitions of $(0,t)$ (for then we can find a subsequence converging almost surely).

Theorem 4. *Assume that*

- *$h \in L^2_{loc}(\mathbb{R}_+)$ and that for a $p \in (0,1)$, $h' \in L^p_{loc}(\mathbb{R}_+)$.*
- *$n \to \pi_n$ is a sequence of refining partitions of $[0,T]$ whose mesh goes to 0.*
- *$a \in GI_{q,t}$ is such that $n_{q,t}(a - a^{\pi_n}) \to 0$ where $a^{\pi_n} = \sum_{\pi_n \ni t_i} a(t_i) 1_{(t_i, t_{i+1}]}$.*

Then, we have the convergence in L^2

$$\sum_{\pi_n \ni t_i} a(t_i)(V_h(t_{i+1}) - V_h(t_i)) \to \int_0^t a(s)\, dV_h(s)\,.$$

In order to cope with true fractional Brownian motion, we have to establish the analog of Theorem 2 for processes W_K. We assume that for every $t > 0$

$$\int K(t,s)^2\,ds < +\infty\,,$$

and that $K(t,s)$ may be written, for a measurable z,

$$K(t,s) = \int_s^t z(u,s)\,du \qquad (0 \le s < t)\,.$$

Eventually we let

$$\|K\|_{p,t} = \sup_{0<u\le t}\left(\int_u^t |z(s,u)|^p\,ds\right)^{1/p}.$$

Theorem 5. *Assume that for a $p \in (0,1)$, we have $\|K\|_{p,t} < +\infty$ and that $a \in GI_{q,t}$ with $\frac{1}{p}+\frac{1}{q}=1$.*
Then there exists a square integrable random variable denoted by $\int_0^t a(s)\,dW_K(s)$ such that for every sequence $(K_n, n \in \mathbb{N})$ of functions verifying

1. *for all n, K_n has the representation*
$$K_n(t,s) = \int_s^t z_n(u,s)\,du \qquad (0 \le s < t)\,,$$
with $z_n(s,.) \in L^2(0,s)$ for every s
2. $\|K - K_n\|_{p,t} \to 0$.

we have the convergence in L^2

$$\int a(s)\,dW_{K_n}(s) \underset{n\to\infty}{\longmapsto} \int_0^t a(s)\,dW_K(s)\,,$$

where on the left hand side we have classical semimartingale integrals.

References

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ECONOMETRICS WITHOUT PROBABILITY: MEASURING THE PATHWISE REGULARITY OF PRICE TRAJECTORIES.

RAMA CONT

This talk deals with a pathwise approach to the analysis of properties of price variations. I use the notion of Holder regularity and p-variation to study stock price trajectories from an empirical and theoretical point of view and compare empirical results with theoretical results on Holder regularity of familiar stochastic processes.

Keywords: Holder regularity, singularity spectrum, wavelet transform, multifractal formalism, multiresolution analysis, Levy process.

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PICARD ITERATION AND p-VARIATION: THE WORK OF LYONS (1994)

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REVISED 13 JANUARY 1999 - MULTIDIMENSIONAL CASE

Abstract. Lyons (1994) showed that Picard iteration, a classical method in ordinary differential equations soon to be recalled, extends to certain non-linear integral and differential equations in terms of functions of bounded p-variation for $p < 2$. The product integral is a special case of Lyons's construction.

1. Classical Picard iteration.

Suppose given a non-linear ordinary differential equation

$$dy/dt \ = \ f(y) \tag{1}$$

where f is a C^1 function from $\mathbb{R}$ into $\mathbb{R}$ with a uniformly bounded derivative, so that f is globally Lipschitz. Suppose we have an initial value $y(0) = a$ and we are looking for a solution for $t \geq 0$. There is a corresponding integral equation

$$y(t) \ = \ a + \int_0^t f(y(u))du, \quad t \geq 0. \tag{2}$$

Then $y(\cdot)$ is a solution of the integral equation (2) if and only if it is a solution of (1) with $y(0) = a$. Consider the sequence of functions y_n defined by $y_0 \equiv a$ and

$$y_{n+1}(t) \ := \ a + \int_0^t f(y_n(u))du. \tag{3}$$

Then for $n = 0, 1, \cdots$, y_n is a C^1 function on $[0, \infty)$ with $y_n(0) = a$. We would like to find conditions under which y_n converges to a solution. Let $M \ := \ \|f\|_L \ := \ \sup |f'|$. Then for $n \geq 1$,

$$|(y_{n+1} - y_n)(t)| \ \leq \ \int_0^t M|(y_n - y_{n-1})(u)|du. \tag{4}$$

Let $Y_n(t) \ := \ \sup_{0 \leq u \leq t} |(y_n - y_{n-1})(u)|$. Then $Y_{n+1}(t) \leq \int_0^t MY_n(u)du$ for each $t \geq 0$ and $n = 1, 2, \cdots$. We have $Y_1(t) \ \equiv \ t|f(a)|$. Inductively, we get $Y_n(t) \leq |f(a)|M^{n-1}t^n/n!$ for $n = 1, 2, \cdots$. Since the Taylor series of e^{Mt} converges absolutely for all t, we get that $y_n(t)$ converges for all $t \geq 0$ to some $y(t)$, uniformly on any bounded interval $[0, T]$, where $y(\cdot)$ is a solution of (2). Moreover, the solution is unique: let $y(\cdot)$ and $z(\cdot)$ be two solutions of (2). Then $|(y - z)(t)| \leq \int_0^t M|(y - z)(u)|du$. Let $S(t) \ := \sup_{0 \leq u \leq t} |(y - z)(u)|$. Then $S(t) \leq MtS(t)$. Thus for $t < 1/M$ we get $S(t) = 0$. So we get $y \ \equiv \ z$ successively on $[0, 1/(2M)]$, $[1/(2M), 1/M]$, $[1/M, 3/(2M)], \cdots$, so $y(t) = z(t)$ for all $t \geq 0$. Alternatively, convergence of $\{y_n\}$ could be proved as follows: by (4), $Y_{n+1}(t) \leq MtY_n(t)$, so restricting to $0 \leq t \leq 1/(2M)$, absolute and uniform convergence follows from a geometric series, and again one can iterate to further intervals of length $1/(2M)$.

Example. Consider the analytic but not globally Lipschitz function $f(y) = y^2$. Solving (1) by separation of variables gives $dy/y^2 = dt$, $-1/y = t+c$, $y = -1/(t+c)$. For $a > 0$, setting $c := -1/a$ we get a solution $y(\cdot)$ of (1) and (2) for $0 \le t < 1/a$ which goes to ∞ as $t \uparrow 1/a$.

2. Lyons's extension and composition.

Let $(S, |\cdot|)$ be a normed vector space. The two main examples in view will be $S = \mathbb{R}^d$ with its usual Euclidean norm $|\cdot|$ and $S = M_d$, the space of $d \times d$ real matrices A, with the matrix norm $\|A\| := \sup\{|Ax'| : x \in \mathbb{R}^d, |x| = 1\}$ where x is a $1 \times d$ vector and x' its transpose, a $d \times 1$ column vector. For a function f from an interval $[0,T]$ into S and $0 < p < \infty$, the p-variation $v_p(f)$ will be defined as usual but with $|\cdot|$ denoting the given norm on S. Thus if $S = \mathbb{R}^d$, $f = (f_1, \cdots, f_d)$ with $f_j : [0,T] \mapsto \mathbb{R}$ and $v_p(f_j) \le v_p(f)$ for each j. The set of functions $f : [0,T] \mapsto S$ with $v_p(f) < \infty$ will be called $\mathcal{W}_p([0,T], S)$, or just $\mathcal{W}_p$ if $S = \mathbb{R}$. If H and $x(\cdot)$ map an interval $[0,T]$ into $\mathbb{R}^d$ then $\int_0^T H \cdot dx$ will be defined as $\sum_{j=1}^d \int_0^T H_j(t) dx_j(t)$ if the d integrals exist. If instead H maps $[0,T]$ into M_d and again $x(\cdot)$ maps $[0,T]$ into $\mathbb{R}^d$ then $\int_0^T H \cdot dx$ will be the vector whose ith component is $\sum_{j=1}^d \int_0^T H_{ij}(t) dx_j(t)$ if the d^2 integrals exist.

Lyons (1994) considered an extension of (2) with

$$z(t) \;=\; a + \int_0^t F(z(u)) \cdot dx(u), \qquad 0 \le t \le T, \tag{5}$$

where $F : \mathbb{R}^d \mapsto M_d$ is a suitable function, $a \in \mathbb{R}^d$, and $x(\cdot) \in \mathcal{W}_p([0,T], \mathbb{R}^d)$ for some p, $1 \le p < 2$. In Lyons's work, since the functions appearing are continuous, the integrals can be taken in the ordinary Riemann-Stieltjes sense. Note that the integral equations for the product integral (CAC Sections 7.1, 7.4) have the same form as (5) where F is the identity function except that here $z(\cdot)$ and $x(\cdot)$ are vector-valued rather than matrix-valued. Lyons discovered, and it will be proved below, that the equation can be solved by Picard iteration whenever F has a gradient ∇F everywhere, satisfying a global Hölder condition of order α where $p < 1 + \alpha \le 2$. The integral will exist by the Love-Young inequality (CAC, section 4.4) if $F \circ z \in \mathcal{W}_q([0,T], \mathbb{R}^d)$ with $p^{-1} + q^{-1} > 1$. Moreover, we have:

Theorem 1. *On an interval $[0,T]$, if $f \in \mathcal{W}_q([0,T], M_d)$ and $g \in \mathcal{W}_p([0,T], \mathbb{R}^d)$ with $p^{-1} + q^{-1} > 1$, then for $h(x) := \int_0^x f \cdot dg$, the indefinite integral $h(\cdot)$ is in $\mathcal{W}_p([0,T], \mathbb{R}^d)$, with $\|h\|_{(p)} \le \zeta(p^{-1} + q^{-1}) \|f\|_{[q]} \|g\|_{(p)}$.*

Proof. This follows from the proofs in Section 4.4 of CAC, which don't require that f and g have scalar values. The integrals are in the refinement-Young-Stieltjes sense, see CAC sections 2.1, 2.2 and 6.2. □

Previously, we had considered composition operators $(f,g) \mapsto (F+f) \circ (G+g) \in \mathcal{L}^p$ with f in spaces $\mathcal{W}_p$ and $G, g \in \mathcal{L}^r$, $1 \le p < r$, and suitable G (CAC, Section 3.3). Note that for a C^∞ function G we can have $F \circ G$ non-regulated for $F \in \mathcal{W}_1 \subset \mathcal{W}_p$ for all p, specifically $F := 1_{[0,\infty)}$, $G(x) := e^{-1/x}\sin(1/x)$, $x \neq 0$, $G(0) := 0$. Here we will consider the Nemitskii operator $g \mapsto F \circ (G+g) : \mathcal{W}_r([0,T],\mathbb{R}^d) \mapsto \mathcal{W}_r([0,T],\mathbb{R}^d)$ for fixed sufficiently smooth F, as follows. Let $(S, |\cdot|)$ and $(U, \|\cdot\|)$ be normed vector spaces. For a subset $C \subset S$ and $0 < \alpha \leq 1$, $H_\alpha(C,U)$ is the space of Hölder functions $f : C \mapsto U$ with seminorm

$$\|f\|_{\{(\alpha)\}} := \sup_{x \neq y} \|f(x) - f(y)\|/|x-y|^\alpha < \infty,$$

and $H_{\alpha,\infty}$ denotes the space of bounded functions in H_α with the norm

$$\|f\|_{\{\alpha\}} := \|f\|_{\sup} + \|f\|_{\{(\alpha)\}}$$

where the supremum norm is denoted by $\|h\|_{\sup} := \sup_x |h(x)|$ and the supremum is over the domain of h for any bounded function h. Let $H_\alpha := H_\alpha(\mathbb{R},\mathbb{R})$. Recall that on a bounded interval in $\mathbb{R}$, $H_\alpha \subset \mathcal{W}_{1/\alpha}$. To see that H_α is not included in $\mathcal{W}_p$ for any $p < 1/\alpha$ one can consider the lacunary Fourier series examples of L. C. Young, e.g. CAC, proof of Theorem 4.29.

For function spaces $\mathcal{F}, \mathcal{G}, \mathcal{H}$, $\mathcal{F} \circ \mathcal{G} \subset \mathcal{H}$ will mean that $f \circ g \in \mathcal{H}$ for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$. The following are then easy to verify:

For any three normed spaces S, U, V, $H_\alpha(U,V) \circ H_\beta(S,U) \subset H_{\alpha\beta}(S,V)$ for $0 < \alpha, \beta \leq 1$;

$H_\alpha \circ H_\beta$ is not $\subset H_\gamma$ if $\gamma > \alpha\beta$ for $0 < \alpha, \beta \leq 1$: consider $g(x) := x^\beta$, $f(y) := y^\alpha$, $0 \leq x, y \leq 1$.

For normed spaces S, U and an interval $[0,T]$, $H_\alpha(S,U) \circ \mathcal{W}_p([0,T],S) \subset \mathcal{W}_{p/\alpha}([0,T],U)$, with $\|f \circ g\|_{(p/\alpha)} \leq \|f\|_{\{(\alpha)\}} \|g\|_{(p)}^\alpha$ for $0 < \alpha \leq 1$, $1 \leq p < \infty$, $f \in H_\alpha(S,U)$, $g \in \mathcal{W}_p([0,T],S)$.

It is known, moreover, that for $S = U = \mathbb{R}$, given $0 < \alpha \leq 1$ and $1 \leq p < \infty$, $f \in H_\alpha$ is not only sufficient, but necessary so that $\{f\} \circ \mathcal{W}_p \subset \mathcal{W}_{p/\alpha}$: see Ciemnoczolowski and Orlicz (1986), Theorem 1, in which we suppose that "$\leq \infty$ for x" means "$< \infty$ for all $x(\cdot)$."

For $0 < \alpha \leq 1$ the space $H_{1+\alpha,d}$ will be defined as the set of C^1 functions $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ such that $\nabla\psi \in H_{\alpha,\infty}(\mathbb{R}^d,\mathbb{R}^d)$. Also, $H_{1+\alpha,d}^{(d)}$ will be the set of functions $F : \mathbb{R}^d \mapsto M_d$ for which each entry $F_{ij} \in H_{1+\alpha,d}$. The next theorem gives a kind of Lipschitz property of the Nemitskii operator $g \mapsto \psi \circ (G+g)$: $\mathcal{W}_r([0,T],\mathbb{R}^d) \mapsto \mathcal{W}_r([0,T],\mathbb{R})$ under sufficient assumptions on ψ, G. The theorem is not in Lyons's paper and seems to provide a shorter proof than his, not requiring his new extension of the Love-Young inequality (cf. CAC, section 7.5) but only the original inequality. We have:

Theorem 2. *If $0 < \alpha \leq 1$, $1 \leq r < \infty$, $g \in \mathcal{W}_r([0,T],\mathbb{R}^d)$, and $G \in \mathcal{W}_{\alpha r}([0,T],\mathbb{R}^d)$, with $\psi \in H_{1+\alpha,d}$, we have $\psi \circ (G+g)$ and $\psi \circ G \in \mathcal{W}_r([0,T],\mathbb{R})$, $\|\psi \circ (G+g) - \psi \circ G\|_{\sup} \leq \|\nabla\psi\|_{\sup}\|g\|_{\sup}$, and*

$$\|\psi \circ (G+g) - \psi \circ G\|_{(r)} \leq \|\nabla\psi\|_{\sup}\|g\|_{(r)} + \|g\|_{\sup}\|\nabla\psi\|_{\{(\alpha)\}}\|G\|_{(\alpha r)}^\alpha.$$

Proof. We have $G \in \mathcal{W}_r([0,T],\mathbb{R}^d)$ and since $\psi \in H_1$ we have $\psi \circ (G+g)$ and $\psi \circ G$ in $\mathcal{W}_r([0,T],\mathbb{R})$ as noted above. The second conclusion is also clear. For $a \leq t < u \leq b$ we have

$$|\psi((G+g)(u)) - \psi(G(u)) - \psi((G+g)(t)) + \psi(G(t))| \;\leq\; S + T$$

where

$$S \;:=\; |\psi((G+g)(u)) - \psi(G(u)+g(t))| \;\leq\; \|\nabla\psi\|_{\sup}|g(u)-g(t)|,$$
$$T \;:=\; |\psi(G(u)+g(t)) - \psi((G+g)(t)) - \psi(G(u)) + \psi(G(t))|.$$

If $g(t) = 0$ then $T = 0$. Otherwise $g(t)/|g(t)|$ is a unit vector and

$$T \;=\; \left|\int_0^{|g(t)|}\left[\nabla\psi\left(G(u)+s\frac{g(t)}{|g(t)|}\right) - \nabla\psi\left(G(t)+s\frac{g(t)}{|g(t)|}\right)\right]\cdot\frac{g(t)}{|g(t)|}ds\right|$$
$$\leq\; |g(t)|\|\nabla\psi\|_{\{(\alpha)\}}|G(u)-G(t)|^{\alpha}.$$

For any partition $0 = x_0 < x_1 < \cdots < x_n = T$ of $[0,T]$, we apply the above bounds for S and T to $t = x_{j-1}$ and $u = x_j$, $j = 1,\cdots,n$, sum the rth powers over j and apply Minkowski's inequality, proving Theorem 2. □

The next theorem shows how Picard iteration works under Lyons's conditions. Denote the set of continuous functions in $\mathcal{W}_p([0,T],\mathbb{R}^d)$ by $C\mathcal{W}_p([0,T],\mathbb{R}^d)$.

Theorem 3. *(Lyons) Let $0 < \alpha \leq 1$, $1 \leq p < 1+\alpha$ and $x(\cdot) \in C\mathcal{W}_p([0,T],\mathbb{R}^d)$ for some $T > 0$. Let $F \in H^{(d)}_{1+\alpha,d}$. For $a \in \mathbb{R}^d$ let $z_0(t) \equiv a$, and for $n = 1,2,\cdots$, $z_n(t) \;:=\; a + \int_0^t F(z_{n-1}(u))\cdot dx(u)$. Then z_n converges in $\mathcal{W}_p([0,T],\mathbb{R}^d)$ to some z which is the unique solution of (5) for $0 \leq t \leq T$.*

Proof. Theorem 2 will be applied with $r \;:=\; p/\alpha$ and where $\psi = F_{ij}$ for each i,j. Then $r^{-1}+p^{-1} = p^{-1}(1+\alpha) > 1$. We have $\mathcal{W}_p([0,T],\mathbb{R}^d) \subset \mathcal{W}_r([0,T],\mathbb{R}^d)$ since $p < r$. Clearly $z_0 \in \mathcal{W}_p([0,T],\mathbb{R}^d)$. It will be shown inductively that $z_n \in \mathcal{W}_p([0,T],\mathbb{R}^d)$ for each n. Suppose $z_{n-1} \in \mathcal{W}_p([0,T],\mathbb{R}^d)$. Then $F\circ z_{n-1} \in \mathcal{W}_p([0,T],M_d) \subset \mathcal{W}_r([0,T],M_d)$ since F is Lipschitz. Thus by Theorem 1, $z_n \in \mathcal{W}_p([0,T],\mathbb{R}^d)$ as stated.

Let $\|h\|_{[0,t],(p)}$ be the p-variation seminorm of h on $[0,t]$. For each $n = 0,1,2,\cdots$ and $t > 0$, let $A_n(t) \;:=\; \|z_{n+1}-z_n\|_{[0,t],(p)}$. Note that $(z_n - z_{n-1})(0) = 0$ for all n. Thus $\|z_n - z_{n-1}\|_{\sup} \leq \|z_n - z_{n-1}\|_{(p)}$ for $1 \leq p \leq \infty$. Then Theorem 2 with $g \;:=\; z_n - z_{n-1}$ and $G \;:=\; z_{n-1}$ gives by the Love-Young inequality in the form of Theorem 1

$$A_n(t) \;\leq\; B(2C + D\|G\|^{\alpha}_{(p)})N_t A_{n-1}(t) \tag{6}$$

where $B \;:=\; \zeta((1+\alpha)/p)$, $C \;:=\; \sum_{i,j=1}^d \|\nabla F_{ij}\|_{\sup}$, $D \;:=\; \sum_{i,j=1}^d \|\nabla F_{ij}\|_{\{(\alpha)\}}$, and $N_t \;:=\; \|x\|_{[0,t],(p)}$. We have

$$A_0(t) \;=\; \|z_1(\cdot)\|_{[0,t],(p)} \;\leq\; |F(a)|N_t \;\leq\; 1 \tag{7}$$

for $0 \leq t \leq \delta_1$ small enough. Take $0 < \delta \leq \delta_1$ small enough so that

$$B(2C+2D)N_t \;<\; 1/2 \tag{8}$$

for $0 \leq t \leq \delta$. It will be shown by induction on n that for $0 < t \leq \delta$ and $n = 0,1,2,\cdots$, both

$$A_{n+1}(t) \;\leq\; A_n(t)/2 \;\leq\; 2^{-n-1}, \tag{9}$$

and

$$\|z_n(\cdot)\|_{[0,t],(p)} \;\leq\; 2 - 2^{1-n}. \tag{10}$$

For $n = 0$, (10) holds, as does the right inequality in (9) by (7), which also gives (10) for $n = 1$. For each $n = 0, 1, 2, \cdots$, (10) and (8) for n, and (6) for $n+1$, imply the first inequality in (9). Thus, inductively, the second inequality also holds. Since $\|z_{n+1}\| \leq \|z_n\| + A_n$, (10) and (9) for a given n imply (10) and thus (9) for $n+1$, completing the induction step and the proof of (9) and (10) for all n. From (9) it follows that $\{z_n\}$ is a Cauchy sequence for $\|\cdot\|_{[0,t],(p)}$, so it converges to some $z \in \mathcal{W}_p[0,\delta]$.

We can repeat the whole process on an interval $[\delta, \delta+\delta']$ for some $\delta' > 0$. Since $\|x\|_{[0,T],(p)} < \infty$, and B, C, D remain constant, after finitely many steps we have z defined on the whole interval $[0,T]$. By continuity, $z(\cdot)$ satisfies (2).

For uniqueness, suppose $y(\cdot)$ is another solution of (5) on $[0,T]$. Let $B(t) := \|y - z\|_{[0,t],(p)}$. Then by the proof of (9), $B(t) \leq B(t)/2$ for $t \leq \delta$, so $B(t) = 0$ on $[0,\delta]$ and by iteration, $y(t) = z(t)$ on $[0,T]$. Theorem 3 is proved. □

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APPLICATIONS OF PRODUCT-INTEGRATION IN SURVIVAL ANALYSIS

RICHARD GILL

In the talk I will describe a number of applications of product-integration in survival analysis. In particular I will focus on

- the Kaplan-Meier estimator (use of the Duhamel equation to study its properties)
- the Aalen-Johansen estimator of the transition probability matrix of a markov chain based on observations with censoring and delayed-entry
- the Dabrowska multivariate product-limit estimator.

In each case I will use different probabilistic methods, namely: function-indexed empirical processes and the van der Laan identity; martingale methods; and compact-differentiability methods respectively.

Please find appended my contribution to the Encyclopaedia of Statistical Science, which serves as a brief introduction and contains useful references.

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Product-Integration (ESS)

All statisticians are familiar with the sum and product symbols $\sum$ and $\prod$, and with the integral symbol $\int$. Also they are aware that there is a certain analogy between summation and integration; in fact the integral symbol is nothing else than a stretched-out capital S—the S of summation. Strange therefore that not many people are aware of the existence of the product-integral $\prod\!\!\!\!\!\!\prod$, invented by the Italian mathematician Vito Volterra in 1887, which bears exactly the same relation to the ordinary product as the integral does to summation.

The mathematical theory of product-integration is not terribly difficult and not terribly deep, which is perhaps one of the reasons it was out of fashion again by the time survival analysis* came into being in the fifties. However it *is* terribly useful and it is a pity that E.L. Kaplan and P. Meier [10], the inventors (1956) of the product-limit or Kaplan-Meier* estimator (the nonparametric maximum likelihood estimator of an unknown distribution function based on a sample of censored survival times), did not make the connection, as neither did the authors of the classic 1967 and 1974 papers on this estimator by B. Efron [5] and N. Breslow and J. Crowley [3]. Only with the 1978 paper of O.O. Aalen and S. Johansen [1] was the connection between the Kaplan-Meier estimator and product-integration made explicit. It took several more years before the connection was put to use to derive new large sample properties of the Kaplan-Meier estimator (e.g., the asymptotic normality of the Kaplan-Meier mean) with the 1983 paper of R.D. Gill [8].

Modern treatments of the theory of product-integration with a view toward statistical applications can be found in [2], [6], and [9].

The Kaplan-Meier estimator is the product-integral of the Nelson-Aalen estimator* (see counting processes*) of the cumulative or integrated hazard function*; these two estimators bear the same relation to one another as the actual survival function* (one minus the distribution function) and the actual cumulative hazard function. There are many other applications of product-integration in statistics, for instance in the study of multi-state processes (connected to the theory of Markov processes) as initiated by Aalen and Johansen [1] and in the theory of partial likelihood (cf. Cox regression model*); see Andersen, Borgan, Gill and Keiding [2]. Product-integration also turns up in extreme-value theory where again the hazard rate plays an important role, and in stochastic analysis and martingale* theory (stochastic integration), where it turns up under the name Doléans-Dades exponential martingale.

Properties of integrals are often easily guessed by thinking of them as sums of many, many (usually very small) terms. Similarly, product-integration generalises the taking of products. This makes properties of product-integrals easy to guess and to understand.

Let us define product-integration at a just slightly higher level of generality than Volterra's original definition (corresponding to the transition from Lebesgue to Lebesgue-Stieltjes integration). Suppose $X(t)$ is a $p \times p$ matrix-valued function of time $t \in [0, \infty)$. Suppose also X (or if you prefer, each component of X) is right continuous with left hand limits. Let $\mathbf{1}$ denote the identity matrix. The product-integral of X over the interval $(0, t]$ is now defined as

$$\prod_{0}^{t}\bigl(\mathbf{1} + \mathrm{d}X(s)\bigr) \;=\; \lim_{\max|t_i - t_{i-1}| \to 0} \prod \Bigl(\mathbf{1} + \bigl(X(t_i) - X(t_{i-1})\bigr)\Bigr)$$

where the limit is taken over a sequence of ever finer partitions $0 = t_0 < t_1 < \ldots < t_k = t$ of the time interval $[0, t]$. For the limit to exist, X has to be of bounded variation; equivalently, each component of X is the difference of two increasing functions.

A very obvious property of product-integration is its multiplicativity. Defining the product-integral over an arbitrary time interval in the natural way, we have for $0 < s < t$

$$\prod_{0}^{t}(\mathbf{1} + \mathrm{d}X) \;=\; \prod_{0}^{s}(\mathbf{1} + \mathrm{d}X)\prod_{s}^{t}(\mathbf{1} + \mathrm{d}X).$$

We can guess (for proofs, see [6] or preferably [9]) many other useful properties of product-integrals by looking at various simple identities for finite products. For instance, in deriving asymptotic statistical theory it is often important to study the difference between two product-integrals. Now if $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ are two sequences of numbers, we have the identity:

$$\prod(1+a_i)-\prod(1+b_i) \;=\; \sum_j\prod_{i<j}(1+a_i)(a_j-b_j)\prod_{i>j}(1+b_i).$$

This can be easily proved by replacing the middle term on the right hand side of the equation, (a_j-b_j), by $(1+a_j)-(1+b_j)$. Expanding about this difference, the right hand side becomes

$$\sum_j\Bigl(\prod_{i\le j}(1+a_i)\prod_{i>j}(1+b_i)\;-\;\prod_{i\le j-1}(1+a_i)\prod_{i>j-1}(1+b_i)\Bigr).$$

This is a telescoping sum; writing out the terms one by one the whole expression collapses to the two outside products, giving the left hand side of the identity. The same manipulations work for matrices. In general it is therefore no surprise, replacing sums by integrals and products by product-integrals, that

$$\prod_0^t(\mathbf{1}+\mathrm{d}X)-\prod_0^t(\mathbf{1}+\mathrm{d}Y) \;=\; \int_{s=0}^t\prod_0^{s-}(\mathbf{1}+\mathrm{d}X)(\mathrm{d}X(s)-\mathrm{d}Y(s))\prod_{s+}^t(\mathbf{1}+\mathrm{d}Y).$$

This valuable identity is called the Duhamel equation (the name refers to a classical identity for the derivative with respect to a parameter of the solution of a differential equation).

As an example, consider the scalar case ($p=1$), let A be a cumulative hazard function and $\widehat{A}$ the Nelson-Aalen estimator based on a sample of censored survival times. In more detail, we are considering the statistical problem of estimating the survival curve $S(t)=\Pr\{T\ge t\}$ given a sample of independently censored* i.i.d. survival times $T_1,\ldots,T_n$. The cumulative hazard rate $A(t)$ is defined by

$$A(t) \;=\; \int_0^t\frac{\mathrm{d}S(t)}{S(t-)};$$

A is just the integrated hazard rate in the absoutely continuous case, the cumulative sum of discrete hazards in the discrete case. Let $t_1<t_2<\ldots$ denote the distinct times when deaths are observed; let r_j denote the number of individuals at risk just before time t_j and let d_j denote the number of observed deaths at time t_j. We estimate the cumulative hazard function A corresponding to S with the Nelson-Aalen estimator

$$\widehat{A}(t)=\sum_{t_j\le t}\frac{d_j}{r_j}.$$

This is a discrete cumulative hazard function, corresponding to the discrete estimated hazard $\widehat{\alpha}(t_j)=d_j/r_j$, $\widehat{\alpha}(t)=0$ for t not an observed death time. The product-integral of $\widehat{A}$ is then

$$\widehat{S}(t) \;=\; \prod_0^t(1-\mathrm{d}\widehat{A}) \;=\; \prod_{t_j\le t}\left(1-\frac{d_j}{r_j}\right),$$

which is nothing else than the Kaplan-Meier estimator of the true survival function S. The Duhamel equation now becomes the identity

$$\widehat{S}(t)-S(t) \;=\; \int_{s=0}^t\widehat{S}(s-)\Bigl(\mathrm{d}\widehat{A}(s)-\mathrm{d}A(s)\Bigr)\frac{S(t)}{S(s)}$$

which can be exploited to get both small sample and asymptotic results, see Gill [7,8,9], Gill and Johansen [6], Andersen, Borgan, Gill and Keiding [2]. The same identity pays off in studying Dabrowska's [4] multivariate product-limit estimator (see [9], [11]), and in studying Aalen and Johansen's [1] estimator of the transition matrix of an inhomogeneous Markov chain (see [2]). It can be rewritten ([9]) as a so-called *van der Laan* [11] *identity* expressing $\widehat{S} - S$ as a function-indexed empirical process, evaluated at a random argument, so that the classical large sample results for Kaplan-Meier (consistency, asymptotic normality) can be got by a two-line proof: without further calculations simply invoke the modern forms of the Glivenko-Cantelli* theorem and the Donsker* theorem; i.e., the functional versions of the classical law of large numbers and the central limit theorem respectively.

Taking Y identically equal to zero in the Duhamel equation yields the formula

$$\prod_0^t (\mathbf{1} + \mathrm{d}X) - \mathbf{1} \;=\; \int_{s=0}^t \prod_0^{s-} (\mathbf{1} + \mathrm{d}X)\mathrm{d}X(s).$$

This is the integral version of Kolmogorov's forward differential equation from the theory of Markov processes, and it is the type of equation:

$$Y(t) \;=\; \mathbf{1} + \int_0^t Y(s-)\mathrm{d}X(s)$$

(in unknown Y, given X), which motivated Volterra to invent product-integration. $Y(t) = \prod_0^t(\mathbf{1} + \mathrm{d}X)$ is the unique solution of this equation.

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RICHARD GILL

DIFFERENTIABILITY PROPERTIES OF SOME CLASSICAL STOCHASTIC MODELS

RUDOLF GRÜBEL

In statistics it is well known that differentiability properties of an estimator can be used to obtain asymptotic normality or to investigate the validity of bootstrap confidence regions. Here we concentrate on the use of differentiability in connection with perturbation aspects and numerical treatment of stochastic models. Our first example deals with the expansion of G/G/1 queues about M/M/1 type models, in the second we discuss the applicability of Richardson extrapolation for the computation of perpetuities. In both cases a decomposition of the functional in question into simpler components has to be achieved. In the first example the general theory of commutative Banach algebras turns out to be very useful, the analysis in the second example is based on the investigation of fixed point equations.

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Optimal Pathwise Approximation of Stochastic Differential Equations

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1 Global Error in L_2-norm

We study the pathwise (strong) approximation of scalar stochastic differential equations with additive noise

$$dX(t) = a(t, X(t))\, dt + \sigma(t)\, dW(t), \qquad t \in T, \tag{1}$$

driven by a one-dimensional Brownian motion W, on the unit interval $T = [0, 1]$.

Different notions of errors for pathwise approximation are studied in the literature on Stochastic Numerics. See Kloeden and Platen (1995), Milstein (1995), and Talay (1995) for results and references. Mainly errors of pathwise approximations are defined discretely at a finite number of points in T. In most of the cases these points coincide with the discretization of the given method. We follow a new approach by measuring the pathwise distance between the strong solution X and its approximation $\overline{X}$ globally on T in the L_2-norm $\|\cdot\|_2$ and defining the overall error of $\overline{X}$ by

$$e(\overline{X}) = \left(E(\|X - \overline{X}\|_2^2)\right)^{1/2}.$$

We aim at determining (asymptotically) optimal approximation methods. To that end we consider arbitrary methods $\overline{X}_n$, that use observations of the Brownian motion W at n points. Moreover, a finite number of function values (or derivative values) of a and σ may be used. We establish sharp lower and upper bounds for the minimal error

$\inf_{\overline{X}_n} e(\overline{X}_n)$ that can be obtained by methods of the above type. Note that upper bounds may be shown by the error analysis of a specific method, while lower bounds must hold for every method $\overline{X}_n$. The optimal order is achieved by an Euler scheme with adaptive step-size control. We state this more precisely in the sequel.

2 Euler Scheme with Adaptive Step-Size Control

For an arbitrary discretization

$$0 = \tau_0 < \cdots < \tau_n = 1 \tag{2}$$

of the unit interval the Euler scheme $\widehat{X}$, applied to equation (1), with initial value $X(0)$ is defined by

$$\widehat{X}(\tau_0) = X(0)$$

and

$$\widehat{X}(\tau_{k+1}) = \widehat{X}(\tau_k) + a(\tau_k, \widehat{X}(\tau_k)) \cdot (\tau_{k+1} - \tau_k) + \sigma(\tau_k) \cdot (W(\tau_{k+1}) - W(\tau_k))$$

where $k = 0, \dots, n-1$. The global approximation $\widehat{X}$ for X on T is defined by piecewise linear interpolation of the data $(\tau_k, \widehat{X}(\tau_k))$ with $k = 0, \dots, n$. Let

$$h > 0$$

be a basic step-size which we choose previously. We introduce the following adaptive step-size control for the Euler method $\widehat{X} = \widehat{X}^h$:

Put $\tau_0 = 0$ and

$$\tau_{k+1} = \tau_k + \min\bigl(h^{2/3}, h/\sigma(\tau_k)\bigr), \tag{3}$$

as long as the right-hand side does not exceed one. Otherwise put $\tau_{k+1} = 1$.

We study the asymptotic behaviour of the error $e(\widehat{X}^h)$ with h tending to zero, where $X(t)$ is the unique strong solution of an equation (1). The error analysis based on the following assumptions on the drift $a : T \times \mathbb{R} \to \mathbb{R}$ and the diffusion coefficient $\sigma : T \to \mathbb{R}$

(A) There exist constants $K_1, K_2, K_3 > 0$ such that

$$\left|a^{(0,1)}(t,x)\right| \leq K_1, \qquad \left|a^{(0,2)}(t,x)\right| \leq K_2,$$

and

$$|a(t,x) - a(s,x)| \leq K_3 \cdot (1 + |x|) \cdot |t-s|$$

for all $s, t \in T$ and $x \in \mathbb{R}$.

(B) The function σ is continuously differentiable and satisfies

$$\sigma(t) > 0$$

for all $t \in T$.

Furthermore we assume

(C) $X(0)$ is independent of W and

$$E|X(0)|^2 \leq K_4$$

for some constant $K_4 > 0$.

The properties (A) and (C) ensure the existence of a pathwise unique strong solution of equation (1). We use $n(h, \sigma)$ to denote the total number of steps and write $e(\overline{X}, a, \sigma, X(0))$ instead of $e(\overline{X})$. With $\|\cdot\|_p$ we denote the L_p-norm of real-valued functions on T.

Theorem 1. *Assume that (A)–(C) hold for equation* (1)*. Then*

$$\lim_{h \to 0} n(h,\sigma)^{1/2} \cdot e(\widehat{X}^h, a, \sigma, X(0)) = 1/\sqrt{6} \cdot \|\sigma\|_1$$

for the Euler approximation with discretization (3)*. The Euler approximation $\widehat{X}_n$ with constant step-size*

$$\tau_{k+1} - \tau_k = 1/n \tag{4}$$

yields

$$\lim_{n \to \infty} n^{1/2} \cdot e(\widehat{X}_n, a, \sigma, X(0)) = 1/\sqrt{6} \cdot \|\sigma\|_2.$$

Hence it is not efficient to discretize equidistantly: taking (3) instead of (4) reduces the error roughly by the factor $\|\sigma\|_1/\|\sigma\|_2$ for the same number of steps. Even a much stronger optimality property holds for the method $\widehat{X}^h$. This method is asymptotically optimal for all equations (1) among all methods that use values of W at a finite number of points. See Theorem 2 and Remark 2.

Remark 1. The term $h^{2/3}$ in (3) only matters if $h > \sigma^3(\tau_k)$. For small values of $\sigma(\tau_k)$ still a reasonably small step-size is needed to get a good approximation. The particular choice $h^{2/3}$ based on error estimates on intervals $[\tau_k, \tau_{k+1}]$.

In the case of arbitrary scalar equations

$$dX(t) = a(t, X(t))\,dt + \sigma(t, X(t))\,dW(t), \qquad t \in T, \tag{5}$$

we propose the natural generalization of (3), given by

$$\tau_{k+1} = \tau_k + \min\bigl(h^{2/3}, h/\sigma(\tau_k, \widehat{X}^h(\tau_k))\bigr), \tag{6}$$

as an adaptive step-size control. Simulation studies indicate that the discretization (6) is still superior to an equidistant discretization. The asymptotic analysis of the step-size control (6) for equations (5) is an open problem.

3 Lower Error Bounds

We present lower bounds that hold for every n-point method. For that purpose we drop all restrictions on the available information about a and σ. We fix a and σ, and we consider the corresponding equation (1). By an n-point method we mean an arbitrary method $\overline{X}_n$, that is based on a realization of the initial value $X(0)$ and on n observations of a trajectory of W. Such a method is defined by measurable mappings

$$\psi_k : \mathbb{R}^k \to T$$

for $k = 1, \ldots, n$ and

$$\phi_n : \mathbb{R}^{n+1} \to L_2(T).$$

The mapping ψ_k determines the observation point in step k in terms of the previous evaluations. A pathwise approximation is computed according to

$$\overline{X}_n = \phi_n(X(0), Y_1, \ldots, Y_n),$$

where $Y_1 = W(\psi_1(X(0)))$, and

$$Y_k = W(\psi_k(X(0), Y_1, \dots, Y_{k-1}))$$

is the observation in step $k \geq 2$. The quantity

$$e(n, a, \sigma, X(0)) = \inf_{\overline{X}_n} e(\overline{X}_n, a, \sigma, X(0))$$

is the minimal error that can be obtained by n-point methods for equation (1).

Theorem 2. *Assume that (A)–(C) hold for equation* (1). *Then*

$$\lim_{n\to\infty} n^{1/2} \cdot e(n, a, \sigma, X(0)) = 1/\sqrt{6} \cdot \|\sigma\|_1.$$

Remark 2. Due to Theorem 1 and 2 the Euler approximation with adaptive step-size control (3) is asymptotically optimal for every equation (1).

Apart from Clark and Cameron (1980), there is a lack of papers dealing with lower bounds. Clark and Cameron derive lower and upper bounds for n-point methods that are based on the equidistant discretization (4). Theorem 2 provides for the first time a lower bound for arbitrary methods which use discrete observations of a Brownian path.

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INFINITE DIVISIBILITY REVISITED

ZBIGNIEW J. JUREK

Abstract. For the class ID of all infinite divisible measures on a Banach space, we introduce a family of convolution semi-groups that sum up to the class ID. Each semi-group in question is given as a class of distributions of some random integrals with respect to some Levy processes. At the same time, it is defined as a class of limit laws.

The aim of this talk is to show usefulness of random integrals in classical limit distributions theory. Our integrals will be pathwise integrals with respect to some Lévy processes , i.e.,

$$(1) \qquad \int_{(a,b]} f(t)\mathrm{d}Y(t) \stackrel{def}{=} f(t)Y(t)|_{t=a}^{t=b} - \int_{(a,b]} Y(t-)\mathrm{d}f(t),$$

where f is a real-valued function with bounded variation and Y is a Lévy process (stationary and independent increments) with cadlag paths. Moreover,

$$\int_{(a,\infty)} f(t)\mathrm{d}Y(t) \stackrel{def}{=} \lim_{b\uparrow\infty} \int_{(a,b]} f(t)\mathrm{d}Y(t), \quad a.s.$$

By ID we denote the class of all infinitely divisible probability measures. It coincides with laws of $Y(1)$, where Y is an arbitrary Lévy process . We will define a filtration of the ID into increasing convolution semi-groups $\mathcal{U}_\beta,\ \beta \in \mathbb{R}$ Namely:

DEF1. We say that $\mu \in \mathcal{U}_\beta$ if there exists a sequence of independent Lévy processes $\xi_j(t), j = 1, 2, \ldots$, such that

$$\mathcal{L}(\tfrac{1}{n}(\xi_1 + \xi_2 + \ldots + \xi_n)(n^{-\beta})) \Longrightarrow \mu, \quad \text{as} \quad n \to \infty,$$

i.e., one evaluates the distributions of the arithmetic average at times $(n^{-\beta})$.

Theorem 1 (a) For $-2 < \beta \leq -1$,

$$(2) \qquad \text{a symmetric } \mu \in \mathcal{U}_\beta \text{ iff } \mu = \gamma_\beta * \mathcal{L}(\int_{(0,1)} t\mathrm{d}Y(t^\beta)),$$

where γ_β is symmetric stable measure with exponent $(-\beta)$ and Y is a unique Lévy process such that $\mathbb{E}[||Y(1)||^{-\beta}] < \infty$.

(b) for $-1 < \beta < 0$, $\mu \in \mathcal{U}_\beta$ iff (2) holds with γ_β as strictly stable measure with exponent $(-\beta)$.

(c) $\mu \in \mathcal{U}_0$ iff $\mu = \mathcal{L}(\int_{(0,\infty)})\mathrm{e}^{-s}\mathrm{d}Y(s) = \mathcal{L}(-\int_{(0,1)})t\mathrm{d}Y(\ln t)$
where Y is a unique Lévy process with $\mathbb{E}\log(1 + ||Y(1)||) < \infty$.
(d) For $\beta > 0$, $\mu \in \mathcal{U}_\beta$ iff $\mu = \mathcal{L}(\int_{(0,1})t\mathrm{d}Y(t)$
where Y is an arbitrary Lévy process . [To the Lévy processes in the above random integrals we refer as the BDLP, i.e., background driving Lévy processes for the corresponding distributions.]

Remark. 1) If $\mathcal{U}_\beta$ has non-degenerate distribution then $\beta \geq -2$.
2) $\mathcal{U}_{-2}$ consist of Gaussian distributions.
3) If $\beta_1 \leq \beta_2$ then $\mathcal{U}_{\beta_1} \subseteq \mathcal{U}_{\beta_2}$.
4) Each $\mathcal{U}_\beta$ is a closed convolution subsemigroup.
5) $\mathcal{U}_0 = L$, where L stands for Lévy class L distributions (also called self-decomposable distributions).

Corollary. A filtration of the class ID is:

$$\overline{\bigcup_n \mathcal{U}_{\beta_n}} = ID$$

for any sequence $\beta_n \uparrow +\infty$. [The bar means closure in weak topology].

Examples. (1) Compound Poisson distributions are in ID but they are not in class $\mathcal{U}_0 = L$ of selfdecomposable distributions.
(2) Stable laws are in L.
(3) Compound geometric laws (waiting time for the first success but not the moment of the first success) are not in L.
(4) Gaussian distributions, Student t-distribution, Fisher F-distributions, generalized hyperbolic distributions are in L; [3].

For $r > 0$ let us define the shrinking operation U_r as follows: $U_r x = 0$, if $||x|| < r$ and $U_r x = \max(||x|| - r, 0)\frac{x}{||x||}$ otherwise. In case of $x \in \mathbb{R}$, note that $x - U_r x = c_r(x)$ is the censoring at the level $r > 0$. Also, for $x > 0$, $U_r x$ is the pay-off function in some financial models, but here we are interested in the following.

DEF2. A measure μ is said to be s-selfdecomposable (s here indicates s-operations U_r), if μ is a limit of the following sequence

$$U_{r_n}X_1 + U_{r_n}X_1 + \ldots + U_{r_n}X_n + x_n,$$

where $X_1, X_2, \ldots$ are independent and the triangular array $U_{r_n}X_j$, $j = 1, 2, \ldots :$ $n \geq 1$, is the uniformly infinitesimal.

Theorem 2. μ is s-selfdecomposable iff $\mu \in \mathcal{U}_1$.

In the lecture we will discuss continuity of the random integral mappings given via the integrals in Theorem 1.

Remark. From random integrals easily follow characterization in terms of Fourier transformations. This allows to avoid the method of extreme points (Choquet's Theorem) as used by D.Kendall, S. Johansen or K. Urbanik. It seems that one might argue that random integral, as a method of describing (limiting) distributions, are more "natural or probabilistic" than the Fourier transform.

Since the "random integral representations" (as in Theorem 1) were successfully used for some other classes of limit laws, it led to the following:

Hypothesis. *Each class of limit class derived from sequences of independent random variables is an image of some subset of ID via same random integral mapping.*

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GRAPHICAL REPRESENTATION FOR INTERACTING PARTICLE SYSTEMS

BO MARKUSSEN

An Interacting Particle System is a Markov jump-process on the space $X = W^S$ of configurations on an infinite site-space S, where each particle takes a state from a phase-space W. The so-called graphical representation gives an explicit construction as a function of infinite many Poisson processes. For this construction to be well-defined we need to impose a condition on the interactions between the particles such that for every time-point T the site-space splits into finite regions which do not interact before time T. The proof of the well-definedness uses the Ito formula for counting process integrals.

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p-variation and integration of sample functions of stochastic processes

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Let h be a real–valued right–continuous function on a closed interval $[a,b]$. Consider the refinement–Riemann–Stieltjes integral equation

$$F(y) = 1 + (RRS)\int_a^y F_-^{(a)}\,dh, \qquad a \le y \le b, \tag{1}$$

for a function F. Here and below for a regulated function f on $[a,b]$, we write

$$\begin{cases} f_+^{(b)}(x) := f_+(x) := f(x+) := \lim_{z\downarrow x} f(z) & \text{if} \quad a \le x < b, \quad \text{and} \quad f_+^{(b)}(b) := f(b), \\ f_-^{(a)}(x) := f_-(x) := f(x-) := \lim_{z\uparrow x} f(z) & \text{if} \quad a < x \le b, \quad \text{and} \quad f_-^{(a)}(a) := f(a). \end{cases}$$

Let $\mathcal{W}_p = \mathcal{W}_p[a,b]$ be the set of all functions on $[a,b]$ with bounded p-variation. If $h \in \mathcal{W}_p$ for some $0 < p < 2$ then equation (1) has the unique solution in $\mathcal{W}_p$ given by the indefinite product integral

$$\mathcal{P}^h(y) = \mathcal{P}_a^h(y) := \prod_a^y (1 + dh) := \lim_{\kappa\uparrow} \prod_{i=1}^n [1 + h(x_i) - h(x_{i-1})],$$

where the limit exists under refinements of partitions $\kappa = \{x_i\colon i = 0, \ldots, n\}$ of $[a,y]$.

Several aspects leading to linear integral equation (1) were discussed in Section 7.4 of [2]. The indefinite product integral is defined with respect to any function h having bounded p-variation for some $0 < p < 2$, while h in (1) is assumed to be right–continuous. Thus one may ask what integral equation satisfies the indefinite product integral $\mathcal{P}^h$ when h is not right–continuous? In this case we have that

$$\begin{cases} \Delta^+\mathcal{P}^h(y) = \mathcal{P}^h(y)\Delta^+h(y) & \text{if } a \le y < b \\ \Delta^-\mathcal{P}^h(y) = \mathcal{P}^h(y-)\Delta^-h(y) & \text{if } a < y \le b \end{cases}$$

(see Lemmas 5.1 and 5.2 in [1]). The different values of $\mathcal{P}^h$ on the right side may look unusual, so that above relations should be considered as an evolution on the extended time scale $\{a, a+, \ldots, y-, y, y+, \ldots, b-, b\}$.

Because the (RRS) integral need not exist when jumps of an integrand and integrator appear at the same point on the same side, we have to replace it by more general integrals discussed in [2]. For example, consider the refinement–Young–Stieltjes integral equation

$$F(y) = 1 + (RYS)\int_0^y F_-^{(0)}\,dh, \qquad 0 \le y \le 2, \tag{2}$$

with respect to function h defined by

$$h(x) := \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2. \end{cases} \tag{3}$$

Then

$$\mathcal{P}^h(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 1 + 1/2 & \text{if } x = 1 \\ 2 + 1/4 & \text{if } 1 < x \le 2, \end{cases}$$

and

$$1 + (RYS)\int_0^y (\mathcal{P}^h)_-^{(0)}\, dh = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \le 2 \end{cases} \ne \mathcal{P}^h(y) \quad \text{for } y \ge 1.$$

Thus $\mathcal{P}^h$ does not satisfy (2).

Let f and g be regulated functions on $[a,b]$. Define the *Left Young integral*, or the (LY) integral, by

$$(LY)\int_a^b g\, df := (RRS)\int_a^b g_-^{(a)}\, df_+^{(b)} + [g\Delta^+ f](a) + \sum_{(a,b)} \Delta^- g \Delta^+ f$$

provided the (RRS) integral exists and the sum converges absolutely. We say that g is (LY) integrable with respect to f on $[a,b]$. The (LY) integral was defined in [1, Relation (3.44)] for Banach algebra valued functions to provide a suitable extension of Duhamel's formula for product integrals. To compare the (LY) integral with the (RYS) integral we recall the (CY) integral defined by L. C. Young [5] as follows:

$$(CY)\int_a^b g\, df := (RRS)\int_a^b g_-^{(a)}\, df_+^{(b)} + [g\Delta^+ f](a) + \Big(\sum_{(a,b)} \Delta^- g \Delta^{\pm} f\Big) + [\Delta^- g \Delta^- f](b)$$

provided the (RRS) integral exists and the sum converges absolutely. By Proposition 3.17 of [1], or by Theorem 6.20 of [2], if the (RYS) integral exists then so does the (CY) integral, and the two are equal.

It is instructive to check that the indefinite product integral with respect to the function h defined by (3) satisfies the Left Young integral equation

$$1 + (LY)\int_0^y \mathcal{P}^h\, dh = 1 + (RRS)\int_0^y (\mathcal{P}^h)_-^{(0)}\, dh_+^{(y)} + [\mathcal{P}^h \Delta^+ h](0) + \sum_{(0,y)} \Delta^- \mathcal{P}^h \Delta^+ h = \mathcal{P}^h(y)$$

for $0 \le y \le 2$.

Define the *Right Young integral*, or the (RY) integral, by

$$(RY)\int_a^b g\, df := (RRS)\int_a^b g_+^{(b)}\, df_-^{(a)} - \sum_{(a,b)} \Delta^+ g \Delta^- f + [g\Delta^- f](b)$$

provided the (RRS) integral exists and the sum converges unconditionally. We say that g is (RY) integrable with respect to f on $[a,b]$.

Next follows from Theorem on Stieltjes integrability of L. C. Young [5, p. 264].

Theorem 1. *Let $g \in \mathcal{W}_p[a,b]$ and $f \in \mathcal{W}_q[a,b]$ for some $p, q > 0$ such that $1/p + 1/q > 1$. Then the integrals $(LY)\int_a^b g\,df$ and $(RY)\int_a^b g\,df$ are defined.*

The following statement is proved in [1, Theorems 5.21 and 5.22] for Banach algebra valued functions.

Theorem 2. *Let $f \in \mathcal{W}_p[a,b]$ with $0 < p < 2$. Then the indefinite product integral*

$$\mathcal{P}_a^f(x) := \prod_a^x (1 + df), \qquad a \le x \le b,$$

exists and is the unique solution in $\mathcal{W}_r[a,b]$, for any $r \ge p$ such that $1/p + 1/r > 1$, of the (LY) integral equation

$$F(x) = 1 + (LY)\int_a^x F\,df, \qquad a \le x \le b.$$

Similarly, the indefinite product integral

$$\mathcal{P}_b^f(y) := \prod_y^b (1 + df), \qquad a \le y \le b,$$

exists and is the unique solution in $\mathcal{W}_r[a,b]$, for any $r \ge p$ such that $1/p + 1/r > 1$, of the (RY) integral equation

$$G(y) = 1 + (RY)\int_y^b G\,df, \qquad a \le y \le b.$$

A different proof of this statement based on the chain rule formula (Theorem 6 below), is due to Mikosch and Norvaiša [3]. In the same manner they also solved the non-homogeneous linear (LY) and (RY) integral equations.

For illustration we formulate several properties of the (LY) and (RY) integrals. Their proofs are given in [4].

Theorem 3. *Let g, f be regulated functions on $[a,b]$, and let $a \le c \le b$. For $A = LY$ or RY, $(A)\int_a^b g\,df$ exists if and only if $(A)\int_a^c g\,df$ and $(A)\int_c^b g\,df$ both exist, and then*

$$(A)\int_a^b g\,df = (A)\int_a^c g\,df + (A)\int_c^b g\,df.$$

Next is the *integration by parts* formula for the LY and RY integrals.

Theorem 4. *Let g and f be regulated functions on $[a,b]$. If either of the two integrals $(LY)\int_a^b g\,df$ and $(RY)\int_a^b f\,dg$ exists then both exist, and*

$$(LY)\int_a^b g\,df + (RY)\int_a^b f\,dg = f(b)g(b) - f(a)g(a).$$

Define the indefinite (LY) and (RY) integrals by

$$\Psi(x) := (LY)\int_a^x g\,dh \qquad \text{and} \qquad \Phi(y) := (RY)\int_y^b g\,dh,$$

respectively for $x, y \in [a,b]$. Next is the *substitution rule* for the LY and RY integrals.

Theorem 5. *Let $h \in \mathcal{W}_p[a,b]$ and $f, g \in \mathcal{W}_q[a,b]$ for some $p, q > 0$ with $p^{-1}+q^{-1} > 1$. Then g and fg are (LY) integrable with respect to h, f is (LY) integrable with respect to the indefinite (LY) integral Ψ, and*

$$(LY)\int_a^b f\,d\Psi = (LY)\int_a^b fg\,dh\,.$$

Similarly, g and fg are (RY) integrable with respect to h, f is (RY) integrable with respect to the indefinite RY integral Φ, and

$$(RY)\int_a^b f\,d\Phi = (RY)\int_a^b fg\,dh\,.$$

Itô's formula gives a stochastic integral representation of a composition of a smooth function and a Brownian motion. We prove the Left Young integral representation for a composition $\phi \circ f$ of a smooth function ϕ and a function f having bounded p-variation with $p \le 2$. Recall that almost every sample function of a Brownian motion has bounded p-variation for each $p > 2$. Besides its applications to sample functions of stochastic processes the chain rule formula extends Theorem 1 of L. C. Young to the case when $1/p + 1/q = 1$. Notice that this extension concerns the existence of integrals with integrands of a special form.

Let d be a positive integer and let ϕ be a real-valued function on a d-dimensional cube $[s,t]^d := [s,t] \times \cdots \times [s,t]$. We write $\phi \in \mathcal{H}_{1,0}([s,t]^d)$ if ϕ satisfies the condition: (1) ϕ is differentiable on $[s,t]^d$ with continuous partial derivatives $\phi_l'(u) := \frac{\partial \phi}{\partial u_l}(u)$ for $l = 1, \ldots, d$. For each $\phi \in \mathcal{H}_{1,0}([s,t]^d)$, let $K_0 := 2\max_{1\le l\le d}\sup\{\phi_l'(u)\colon u \in [s,t]^d\}$. Also, for $\alpha \in (0,1]$, we write $\phi \in \mathcal{H}_{1,\alpha}([s,t]^d)$ if, in addition to condition (1), ϕ satisfies the condition: (2) there is a finite constant K_α such that the inequality

$$\max_{1\le l\le d} |\phi_l'(u) - \phi_l'(v)| \le K_\alpha \sum_{k=1}^{d} |u_k - v_k|^\alpha$$

holds for all $u = (u_1, \ldots, u_d), v = (v_1, \ldots, v_d) \in [s,t]^d$.

The following statement provides the (LY) integral representation of a composition under the boundedness of p-variation condition.

Theorem 6. *For $\alpha \in [0,1]$, let $\boldsymbol{f} = (f_1, \ldots, f_d)\colon [a,b] \mapsto (s,t)^d$ be a vector function with coordinate functions $f_l \in \mathcal{W}^*_{1+\alpha}[a,b]$ for $l = 1, \ldots, d$, let $\phi \in \mathcal{H}_{1,\alpha}([s,t]^d)$ and let h be a regulated function on $[a,b]$. Then the equality*

$$(LY)\int_a^b h\,d(\phi\circ\boldsymbol{f}) = \sum_{l=1}^{d}(LY)\int_a^b h(\phi_l'\circ\boldsymbol{f})\,df_l$$

$$+\sum_{(a,b]} h_-[\Delta^-(\phi\circ\boldsymbol{f}) - \sum_{l=1}^{d}(\phi_l'\circ\boldsymbol{f})_-\Delta^- f_l] + \sum_{[a,b)} h[\Delta^+(\phi\circ\boldsymbol{f}) - \sum_{l=1}^{d}(\phi_l'\circ\boldsymbol{f})\Delta^+ f_l],$$

holds meaning that all $d+1$ integrals exist provided any d integrals exist, and the two sums converge unconditionally.

We plan to discuss applications of these results to sample functions of stochastic processes.

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INDEPENDENCE OF MULTIPLE STOCHASTIC INTEGRALS

JAN ROSIŃSKI

The problem of independence of multiple stochastic integrals with respect to infinitely divisible random measures is investigated. It is remarked that multiple Ito-Wiener integrals are independent if and only if their squares are uncorrelated. This criterion breaks down for multiple Poisson integrals.

The necessary and sufficient condition for the independence of multiple integrals of arbitrary (not necessary equal) orders with respect to a symmetric stable random measure is established. To this aim the product formula for multiple stochastic integrals is studied through series representations. The knowledge of the tail asymptotic behavior for multiple stable integrals and their products plays the crucial role in our method, its role is similar to Ito's isometry for Ito-Wiener integrals.

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This is based on joint work with Gennady Samorodnitsky.

FORWARD AND BACKWARD CALCULUS

WITH RESPECT TO FINITE QUADRATIC VARIATION PROCESSES

by Francesco RUSSO and Pierre VALLOIS

1 Stochastic integration via regularization

This project which is carried on since 1991 in several papers, see [RV1; RV2; RV3; RV4; RV5; RV6; ER; ERV], but also in [W1;W2;W3] and [RVW]. It covers the topic of the following two talks in Aarhus.

- F. Russo: Calculus with respect to a finite quadratic variation process.
- P. Vallois: Stochastic calculus related to general Gaussian processes and normal martingales.

The aim of the project was to develop a stochastic calculus which has essentially four features.

1. It belongs to the context of "pathwise" stochastic calculus. To this extent, one of the basic reference is the article of H. Föllmer ([F]), continued for instance in [Be] but also by Dudley and Norvaisa, see contributions in this volume. Föllmer's method is based on integrator discretization, ours works out regularization tools.
2. One aspect of our approach is the particular simplicity. In fact, many rules are directly derived using first year calculus arguments and finite Taylor expansions, uniform continuity and so on.
3. Our approach establishes a bridge between anticipating and non-anticipating integrals, see for instance [NP], [N]. Concerning anticipating calculus, there are two main techniques at least as far as the integrator is a Brownian motion: Skorohod integration which is based on functional

analysis, see for instanse [NP], [N], and enlargement of filtration (see e.g. [J]) tools which are based on classical stochastic calculus method. Our integrals help to relate those two concepts.

4. Our theory should allow to go beyond semimartingales. Our aim was to understand what could be done when integrators are just Gaussian processes, convolution of martingales, as fractional Brownian motions and Dirichlet type processes (sum a local martingale and a zero-quadratic variation processes).

All the processes will be supposed to be continuous (for simplicity). Extensions to the jump case are done in [ERV].

Let $(X_t)_{t\geq 0}$, $(Y_t)_{t\geq 0}$ be (continuous) processes. We set

$$\begin{aligned}
\int_0^t Y d^- X &= ucp - \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds \\
\int_0^t Y d^+ X &= ucp - \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon}\, ds \\
\int_0^t Y d^o X &= ucp - \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{(s-\varepsilon)\vee 0}}{2\varepsilon}\, ds \\
[X,Y]_t &= ucp - \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds
\end{aligned}$$

if previous quantities do exist; ucp denotes the uniform convergence in probability on each compact.

$\int_0^t Y d^- X$ (resp. $\int_0^t Y d^+ X, \int_0^t Y d^o X$) is called the **forward** (resp. **backward, symmetric** integral) of Y with respect to X provided that those integrals do exist.

$[X,Y]$ is called the **covariation** of X and Y. $[X,X]$ is also called quadratic variation of X and X.

If $[X,X]$ exists X will be said to be a **finite quadratic variation process**. In this case, it is an increasing processes, being a limit of increasing processes; therefore it is a classical integrator in the sense of measure theory of $I\!R+$.

Let $X^1,\ldots,X^m$ be some processes.

Definition $\{X^1, \ldots, X^m\}$ are said to have all **their mutual brackets** if $[X^i, X^j]$ exists $\forall\, 1 \le i, j \le m$.
In this case :

$$[X^i + X^j, X^i + X^j] = [X^i, X^i] + 2[X^i, X^j] + [X^j, X^j]$$

and

$$[X^i, X^j] \text{ are (locally) with bounded variation.}$$

Properties

We first state some elementary properties which directly follow from the definition and ordinary calculus.

1. $[X, Y]_t = \int_0^t Y\, d^+ X - \int_0^t Y\, d^- X.$

2. $$\int_0^t Y\, d^o X = \frac{\int_0^t Y\, d^+ X + \int_0^t Y\, d^- X}{2}$$

3. For $T > 0$ we set $\hat{X}_t = X_{T-t}$. Then

$$\int_0^t Y\, d^+ X = -\int_{T-t}^T \hat{Y}\, d^- \hat{X}$$

4. Let $f, g \in C^1(\mathbb{R})$, $\{X, Y\}$ having all their mutual brackets. Then $\{f(X), g(Y)\}$ have the same property and

$$[f(X), g(Y)]_t = \int_0^t f'(X_s)\, g'(Y_s)\, d[X, Y]_s.$$

5. **Itô formula**. Let $f \in C^2$ and X a finite quadratic variation process. Then :

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X) d^{\mp} X \pm \frac{1}{2} \int_0^t f''(X_s) d[X, X]_s \\ \int_0^t f''(X)\, d[X, X] &= [f'(X), X]_t. \end{aligned}$$

6. If $[X, X]$ exists, $[Y, Y] = 0$ then $\{X, Y\}$ have all their mutual brackets and $[X, Y] = 0$.

7. **Integration by parts**.

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t X d^- Y + \int_0^t Y d^+ X \\ &= X_0 Y_0 + \int_0^t X d^- Y + \int_0^t Y d^- X + [X, Y]_t \end{aligned}$$

2 Examples

Even if the theory developed until now is very simple, it would be empty, if we cannot provide a rich enough class of examples.

1. If X has bounded variation

$$\int_0^t Y d^- X = \int_0^t Y d^+ X = \int_0^t Y dX$$

The fact that the first and second members equal the third is a consequence of the definition of forward and backward integrals, of Fubini theorem and of the dominated convergence theorem.

2. Let X, Y two semimartingales with respect to some usual filtration $(\mathcal{F}_t)$, H^i some adapted processes, $i = 1, 2$. Then $[X, Y]$ is the usual bracket of M^1 and M^2, those processes being the local martingale part of X and Y. Then $\int_0^t H^1 d^- X$ coincides with the classical Itô integral and

$$\left[\int_0^{\cdot} H^1 dM^1, \int_0^{\cdot} H^2 dM^2 \right]_t = \int_0^t H^1 H^2 d[M^1, M^2].$$

We remark that that property will play in the sequel the role of a definition of a "good" finite quadratic variation process.

3. When X is a Brownian motion B, under suitable assumptions on a process H, it is possible to relate the Skorohod integral $\int_0^{\cdot} H \partial B$ with

the forward and backward integrals $\int_0^\cdot H d^{\mp} B$. If $H \in \mathcal{D}_{2,1}(L^2([0,T]))$, $\mathcal{D}_{2,1}$ being Wiener-Sobolev spaces of Malliavin-Watanabe type,

$$DH = (D_r H_s : (r,s) \in [0,T]^2) \in L^2(\Omega \times [0,T]^2).$$

then

$$\int_0^t H d^{\mp} B = \int_0^t H \delta B + Tr^{\mp} DH(t)$$

where

$$Tr^- DH(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t dr \int_{(r-\varepsilon)\vee 0}^r ds\, D_r H_s$$

$$\left[\int_0^\cdot H^1 \delta B ; \int_0^\cdot H^2 \delta B \right]_t = \int_0^t H_s^1 H_s^2 ds.$$

For this see [RV1, SU, Z].

4. Since the forward integral does not see any filtration, it also coincides with any enlargement of filtration integral. Previous point relates precisely the integral constructed by enlargement of filtrations and Skorohod integral.

5. **Dirichlet-Fukushima process.**

   A Dirichlet process (in the pathwise sense) is a process X of the form $X = M + A$ where M is a $(\mathcal{F}_t)$- local martingale, and A is a zero quadratic variation $(\mathcal{F}_t)$-adapted process. This concept has been inspired by [Fu] who considered functions of a good stationary process coming out from a Dirichlet form for which he could obtain the above decomposition. We observe that

   - $[X, X] = [M, M]$
   - If $f \in C^1$, $Y = f(X)$ is again a Dirichlet process with $Y = \tilde{M} + \tilde{A}$ and $\tilde{M}_t = f(X_0) + \int_0^t f'(X)\, dM$ where $\tilde{A}_t = f(X_t) - \tilde{M}_t$. Using the bilinearity of the covariation, property d) stated in section 1 and example 2 we easily obtain $[\tilde{A}, \tilde{A}] \equiv 0$.

6. **Extended Lyons-Zheng processes**

   This example is treated in details in [RVW]. For simplicity we consider here only processes indexed by $[0, 1]$.

- For $X = (X(t),\, t \in [0,1])$, we set $\hat{X}(t) = X(1-t)$.
- Let $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,1]}$, $\mathcal{H} = (\mathcal{H}_t)_{t\in[0,1]}$ be two filtrations.
- If $Y(t) - Y(0)$ is $\mathcal{F}$-adapted and $\hat{Y}(t) - \hat{Y}(0)$ is $\mathcal{H}$-adapted then we say that Y is **weakly** $(\mathcal{F}, \mathcal{H})$**-adapted**

Definition

A continuous weakly $(\mathcal{F}, \mathcal{H})$-adapted process X is called a $(\mathcal{F}, \mathcal{H})$- **LZ process** if there are $M^i = (M^i(t),\, t \in [0,1])$, $i = 1, 2$, $V = (V(t),\, t \in [0,1])$, such that

$$X = \frac{1}{2}(M^1 + M^2) + V$$

and the following conditions are satisfied:

a) M^1 is a local $\mathcal{F}$-martingale with

$$M^1(0) = 0.$$

b) $\hat{M}^2$ is a local $\mathcal{H}$-martingale with

$$M^2(1) = 0.$$

c) V is a bounded variation process.

d) $M^1 - M^2$ is a zero quadratic variation process.

Remark 2.1 Let X be an $(\mathcal{F}, \mathcal{H})$-LZ process.

1) $[X, X] = \frac{1}{2}([M^1, M^1] + [M^2, M^2])$.

2) A time reversible semimartingale is a LZ process with respect to the natural filtrations.

3) If Y is a $(\mathcal{F}, \mathcal{H})$-weakly adapted process then we define the LZ-symmetric integral by

$$\int_0^t Y \circ dX = \frac{1}{2}\int_0^t Y\, d^- M^1 - \frac{1}{2}\int_{1-t}^1 \hat{Y}\, d^- \hat{M}^2 + \int_0^t Y\, dV.$$

If $[Y, M^1 - M^2] = 0$ then

$$\int_0^t Y \circ dX = \int_0^t Y d^\circ X$$

7. **Delayed processes**

   Let $\tau > 0, (S_t)$ a $(\mathcal{F}_t)$-semimartingale and (X_t) a $(\mathcal{F}_{(t-\tau)\vee 0})$-adapted process. Then

   $[X,S] \equiv 0, \int_0^t X d^- S = \int_0^t X d^+ S = \int_0^t X dS$

8. **Convolution of martingales**.

   This example has been explicitely discussed in [ER]. Let M be a martingale and g be a continuous real function such that $X_t = \displaystyle\int_0^t g(t-s)\, dM_s$.

   If $[g(u-\cdot), g]$ exists, for any $u \geq 0$, then $[X,X]$ exists and can be explicitely calculated.

9. **Substitution formulae.**

   Let $(X(t,x), t \geq 0, x \in I\!R^d)$, $(Y(t,x),\ t \geq 0,\ x \in I\!R^d)$ semimartingales depending on a parameter x, $(H(t,x), t \geq 0, x \in I\!R^d)$ predictable processes depending on x .

   We make assumptions of Garsia-Rudemich-Rumsey type. Then

$$\int_0^t H(s,Z)\, d^- X(s,Z) = \int_0^t H(s,x)\, dX(s,x)\Big|_{x=Z}$$

$$[X(\cdot,Z), Y(\cdot,Z)] = [X(\cdot,x), Y(\cdot,x)]\Big|_{x=Z}$$

   Those formulas are useful for proving existence results for SDE's driven by semimartingales with anticipating initial conditions.

10. **Gaussian case.**

    Let (X_t) be a Gaussian process such that

$$m(t) = E(X_t),$$

$$K(s,t) = \mathrm{Cov}(X_s, X_t) = E(X_s X_t) - m(s)\, m(t)$$

    Provided $[m,m] = 0$, it is enough to suppose that X is a mean-zero process.

Proposition 2.2 We suppose :

$$K \in C^1(\Delta+)$$

where

$$\Delta+ = \{(s,t) : 0 \le s \le t\}$$

We denote $D_2K(s,s+)$ the restriction of D_2K on the diagonal.

Then $[Y,Y]_t$ exists, it is deterministic and it equals

$$a(t) = K(t,t) - K(0,0) - 2\int_0^t ds\, D_2K(s,s+).$$

Particular cases:

- $X = B$: Brownian motion. $K(s,t) = s \wedge t$

$$\int_0^t ds\, D_2K(s,s+) = 0 \Rightarrow a(t) = t.$$

- Fractional Brownian motion.

$$K(s,t) = K_\alpha(s,t) = \frac{1}{2}(|s|^\alpha + |t|^\alpha - |s-t|^\alpha)$$

$$a(t) = 0 \qquad 1 < \alpha < 2.$$

In such a case it is possible to show that the $\alpha-$ variation of such a process equals t. We recall that a stochastic integral and calculus for fractional Brownian motion has been for instance developed by [DU] and [Z1;Z2].

Concerning forward integration, we have the following. Supposing that K is of class C^1 on Δ_+, we set

- $A_1(f,f)(t) = 2E[\int\int 1_{\{u \le v \le t\}} f(X_u)f(X_v)\{\tau_1(u,v)X_u^2 + \tau_2(u,v)X_v^2 + \tau_{1,2}(u,v)X_uX_v + \tau(u,v)\}dudv,$

  $\tau_1, \tau_2, \tau_{1,2}$ and τ being four functions defined explicitely through K.
- $A_2(f,f)(t) = E[\int_O^t f^2(X_s)\left\{\frac{\partial K}{\partial u}(s,s) - \frac{\partial K}{\partial v}(s,s)\right\} ds.$

- $A = A_1 + A_2$.

Theorem 2.3

(a) $A(f,f)\,(t) \geq 0 \quad \forall\, t \geq 0$, for every continuous and bounded real function f .

(b) Suppose that $A(f,f) < \infty$ then $\displaystyle\int_0^t f(X_s)\,\frac{X(s+\varepsilon)-X(s)}{\varepsilon}\,ds$, converges, as $\varepsilon \to 0_+$, in L^2 (to $\displaystyle\int_O^t f(X_s)\,d^- X_s$).

(c) Moreover

$$E\left[\left(\int_0^t f(X_s) d^- X_s\right)^2\right] = A(f,f)\,(t)$$

$$E\left[\int_0^t f(X_s) d[X,X]\,(s)\right] = A_2(f,f)\,(t)$$

$$E\left[\int_0^t f(X_s) d^- X_s\right] = \int_0^t E[f(X_s)\,X_s]\,\frac{\frac{\partial K}{\partial u}\,(s,s_r)}{K(s,s)}\,ds.$$

11. **Normal martingales as integrators**

The basis for developing forward integration with respect to a normal martingale has been developed in [RV5].

Suppose M is a normal martingale of it is locally square integrable martingale and $\langle M,M\rangle(t), t\ \forall\, t \geq 0$, $\langle M,M\rangle$ being the dual projection of the classical bracket $[M,M]$ (ref. [DMM]).

Let $(X_t; t \in [0,1])$ be a square integrable process, which means

(2.1) $$E\left[\int_0^1 X_s^2 ds\right] < \infty.$$

We suppose

(2.2) $$X(t) = \sum_{n\geq 0} I_n(f_n(t,\cdot)), \qquad \forall\, t \in [0,1]$$

where $I_n(f_n(t,\cdot))$ is the n multiple iterated stochastic integral of $f_n(t,\cdot)$ with respect to M. $f_n(t,\cdot)$ is a square integrale and symmetric function defined on $I\!R_+^n$.

If M has the chaos decomposition property, every square integrable random variable has such a decomposition. In particular (2.1) implies that (2.2) holds. We set $\delta(X)$ the Skorohod integral of X with respect to M :

$$\delta(X) = \sum_{n\geq 0} I_{n+1}(\tilde{f}_n),$$

provided the series converges in L^2, $\tilde{f}_n$ denoting the symmetrization of f_n considered as a function of $n+1$ variables.

In the Brownian case, $\int_0^t X\, d^- M$ is the sum of $\delta(X)$ plus a trace term. The proof is based on the following identity :

$$(2.3) \qquad I_n(f)\, I_1(g) = I_{n+1}(f \widetilde{\otimes} g) + n\, I_{n-1}\left(\int_0^\infty f(t,\cdot)\, g(t)\, dt\right).$$

This property can be generalized to M, namely

$$I_n(f)\, I_1(g) = I_{n+1}(f \widetilde{\otimes} g) + n \int_0^\infty I^1_{n-1}(f)\,(t)\, g(t)\, d^-[M,M]\,(t)$$

where $(I^1_{n-1}(f)\,(t); t \geq 0)$ is the unique element verifying

$$\begin{aligned} E\left[\int_0^\infty I^1_{n-1}(f)\,(t)^2 d[M,M](t)\right] &= \left[\int_0^\infty I^1_{n-1}(f)\,(t)\, dt\right] \\ &= n!\, \|f\|^2_{L^2(I\!R^n_+)}. \end{aligned}$$

Then

$$(2.4) \qquad \int_0^1 X^- dM = \delta(X) + Tr(X)$$

$$Tr(X) = \int_0^1 d[M,M]\,(s) \left\{\sum_n n\, \tilde{I}^1_{n-1}(f_n(s_-, s, \cdot))\right\}$$

$$\tilde{I}^1_{n-1}(f_n(s_-, s, \cdot) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{s-\varepsilon}^s I^1_{n-1}(f_n(t,\cdot))\,(s)\, dt.$$

if the two series converges in $L^2(\Omega)$.

3 Generalized Itô processes.

For a one-dimensional process X, the existence of the quadratic variation is equivalent to the validity of Itô formula in the sens of property 4, section 1. This is illustrated by the following lemma.

Lemma 3.1 Let X be a process. X is a finite quadratic variation process if and only if

$$\int_0^{\cdot} g(X)\, d^{\mp}X \textit{ exists} \qquad \forall\, g \in C^1(\mathbb{R}). \tag{3.1}$$

Proof

Let $G \in C^2$ such that $G' = g$. Using Itô formula,

$$\int_0^t g(X) d^{\mp}X = G(X_t) - G(X_0) \mp \frac{1}{2}\int_0^t g'(X)\, d[X,X].$$

Conversely if (3.1) holds,

$$[g(X), X]_t = \int_0^t g(X)\, d^+X - \int_0^t g(X)\, d^-X \text{ exists.}$$

Taking $g(x) = x$, X is proven to be a finite quadratic variation process. □

This result motivates the n-dimensional case.

Let $\{X^1, \ldots, X^n\}$ having all their mutual brackets. $(X^1, \ldots, X^n)$ is called **(generalized) Itô process** if

$$\int_0^{\cdot} g(X) d^{\mp}X^i \text{ exists, } \forall\, g \in C^1(\mathbb{R}^n). \tag{3.2}$$

This means $g(X) \in I_X$ where I_X is the class of processes (Z_t) such that

$$\int_0^{\cdot} Z d^{\mp}X^i \text{ exists,} \quad 1 \le i \le n. \tag{3.3}$$

Proposition 3.2 Let $X = (X^1, \ldots, X^n)$ an Itô process, $\varphi \in C^2$, $Y = \varphi(X)$. Let (Z_t) another process. If $Zg(X) \in I_X$, $\forall\, g \in C^1(I\!R^n)$, then $Z \in I_Y$ and

(3.4)
$$\int_0^t Z d^{\mp} Y = \int_0^t Z_s \Big(\sum_{i=1}^n \partial_i \varphi(X_s) d^{\mp} X_s^i \Big) \pm \frac{1}{2} \int_0^t Z_s \Big(\sum_{i,j=1}^n \partial_{ij}^2 \varphi(X_s)\, d[X^i, X^j]_s \Big)$$

Remark 3.3

i) If $Z = f(X)$, $f \in C^1$ then (3.4) can be applied.

ii) (3.4) formally can be expressed as :

$$d^{\mp} Y_s = \sum_{i=1}^n \partial_i \varphi(X_s)\, d^{\mp} X_s^i \pm \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 \varphi(X_s)\, d[X^i, X^j]_s \tag{3.5}$$

Definition A process $X = (X^1, \ldots, X^n)$ will be called **vector Itô** process if

$$\Big[\int_0^{\cdot} f(X) d^- X^i, \int_0^{\cdot} g(X) d^- X^j \Big]_t = \int_0^t f(X_s) g(X_s) d[X^i, X^j]_s.$$

Remark 3.4 If $n = 1$, any finite quadratic variation process is a vector Itô process.

4 Stochastic differential equations

Let $(\xi_t)_{t \geq 0}$ a finite quadratic variation process. We are interested by

$$d^- X_t = \sigma(X_t)\, d^- \xi_t, \quad X_0 = \alpha \tag{4.1}$$

$\sigma \in C_b^2$, α any random variable.

Remark 4.1 If (ξ_t) is a semimartingale (even for the case σ Lipschitz) there is a solution $(X_t)_{t\geq 0}$ of

$$X_t = \alpha + \int_0^t \sigma(X_s)\, d^-\xi_s. \tag{4.2}$$

Proof We apply substitution method, see section 2. □

However a serious uniqueness result was missing. The problem is probably not well-posed in the (too large) class of all processes for which the equation makes sense. So we have the following alternative.

a) either we restrict the class of processes X,

b) or we modify a little bit the sens of solution; of course it will be necessary to include the classical cases.

We have chosen this second solution.

Definition A process (X_t) is solution of (4.1) if

a) (X, ξ) is a vector Itô process.

b) For every $Z_t = \varphi(X_t, \xi_t)$, φ regular,

$$\int_0^t Z_s d^- X_s = \int_0^t Z_s \sigma(X_s)\, d^-\xi_s$$

Remark 4.2 If (ξ_t) is a semimartingale, then the solution which is obtained by substitution fulfills a) and b).

Proposition 4.3 Let (ξ_t) be a finite quadratic variation process, $\sigma \in C_b^2$. There is a unique solution X to problem (4.1).

Proof We give an idea of the uniqueness proof, since the existence involves very similar arguments.

We apply here the method developed for classical one-dimensional diffusions by [Do] and [Su]. We consider the deterministic flow $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ solution of

$$\left\{ \begin{array}{l} \dfrac{\partial F}{\partial r}(r,x) = \sigma(F(r,x)) \\ F(0,x) = x. \end{array} \right\}.$$

F is C^2 and there is $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of class C^2 such that

$$F(r, H(r,x)) \equiv x, \;\; H(r, F(r,x)) \equiv x. \tag{4.3}$$

We set

$$Y_t = H(\xi_t, X_t).$$

By Itô formula we have:

$$Y_t = Y_0 + \int_0^t \frac{\partial H}{\partial r}(\xi_s, X_s)\, d^- \xi_s + \int_0^t \frac{\partial H}{\partial x}(\xi_s, X_s)\, d^- X_s + BV \tag{4.4}$$

where BV is a bonded variation process.

Through (4.3), we have

$$\frac{\partial H}{\partial r}(r,x) = -\sigma(x)\, \frac{\partial H}{\partial x}(r,x)$$

(ξ, X) is a vector Itô process. Using Proposition 3.2 the two stochastic integrals appearing in (4.4) cancel. Then we solve pathwise (4.4) .

Extensions are possible in the following cases:

$$\begin{array}{rcl} d^- X_t &=& \sigma(X_t)\, d^- \xi_t + \beta(t, X_t)\, d^- S_t \\ X_0 &=& \alpha \end{array}$$

when

a) (S_t) semimartingale, α non-anticipating and β is Lipschitz.

b) (S_t) having bounded variation, α any random variable.

5 Extended Itô formulae

A particular class of generalized vector Itô processes is constituted by processes $X = (X_t)_{t \geq 0}$ such that

$$(5.1) \quad Z^{\pm}(g) = \int_0^{\cdot} g(X) d^{\pm} X$$

exist for any $g \in C^{\circ}(I\!R)$.

Proposition 5.1

$$(5.1)) \qquad G(X_t) = G(X_0) \pm \int_0^t G'(X) d^{\mp} X \pm [G'(X), X]_t$$

for any $G \in C^1(I\!R)$.

Proof (5.1) holds for $G \in C^2$. Then

$$(5.2) \qquad g \to \int_0^t g(X_s) \frac{X_{s+\varepsilon} - X_s}{\varepsilon} \, ds, \qquad g \to \int_0^t g(X_s) \frac{X_s - X_{(s-\varepsilon)\vee 0}}{\varepsilon} \, ds$$

are continuous from $C(I\!R)$ to the space $\mathcal{C}(I\!R)$ of continuous processes equipped with the ucp topology. The conclusion follows easily by Banach - Steinaus theorem and (5.1).

Example: X: time reversible semimartingale.

$$\int_0^t g(X) d^+ X = -\int_{T-t}^T g(\hat{X}) d\hat{X}.$$

We remark that $[G'(X), X]$ **does not need to be of bounded variation.** If X is a Brownian motion B and $f \in C^o(I\!R)$. Then $[G'(X), X]$ is of bounded variation if and only if $G(X)$ is a semimartingale. This is only possible if G is difference of convex functions. *Therefore* $\{G'(X), X\}$ *have not all the mutual brackets.*

Proposition 5.2 The following properties are equivalent.

1. $[Z^-(g), Z^-(g)]$ exists for any $g \in C^{\circ}(I\!R)$.

2. $[[g(X), X], [g(X), X]] = 0, \quad g \in C^\circ(I\!R)$.

3. $[Z^+(g), Z^+(g)]$ exists for any $g \in C^\circ(I\!R)$.

4. $[Z^-(g), Z^-(g)] = \int_0^t g^2(X_s) d[X,X]_s$.

5. $[Z^+(g), Z^+(g)] = \int_0^t g^2(X_s) d[X,X]_s$.

Remark 5.3 Results in this direction have been obtained by [RV4], for the case of time-reversible semimartingales; [BY] obtained a similar result for semimartingales where the bracket term is expressed by a generalized integral involving local time. [FPS] for the case of Brownian motion, [BJ] for the case of elliptic diffusions and [MN] for non-degenerate martingales, explore the case of f being in $W^{1,2}_{loc}$.

Remark 5.4 If $X = M + A$, M being a time reversible semimartingale, then

$$[g(X), X], \quad g \in C^\circ(I\!R)$$

exists and it has zero quadratic variation. Then

$$\int_0^\cdot g(X) d^\mp A$$

exists and it has zero bracket.

Proposition 5.5 (Generalized Itô-Tanaka formula)

Let $X = (X_t)_{t \geq 0}$ be a finite quadratic variation process.

[g(X),X] exists for any increasing continuous function g if and only if $\int_0^\cdot g(X) d^\mp X$ exists for any increasing continuous function g.

Then

$$G(X_t) = G(X_0) + \int_0^t G'(X_s) d^\mp X_s + \frac{1}{2}[G'(X), X]_t$$

for any $G \in C^1(I\!R)$.

Remark 5.6 If (ξ_t) in section 4 has previous properties, it is possible to relax the assumption under which the equation

$$d^\circ X_t = \sigma(X_t) d^\circ \xi_t$$

has a solution.

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TOLERANCE TO ARBITRAGE AND THE STOP-LOSS START-GAIN STRATEGY

DONNA M. SALOPEK

We contruct an arbitrage opportunity in a frictionless stock market when price processes have continuous sample paths of bounded p-variation with $p \in [1, 2)$. In addition, we review when the stop-loss start gain strategy is self financing for the various price processes.

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SOME MAXIMAL INEQUALITIES FOR FRACTIONAL BROWNIAN MOTIONS

ESKO VALKEILA

ABSTRACT. We give an analogue of the Burkholder-Davis-Gundy inequalities for fractional Brownian motions. The proof is based on kernel transformations, which transform a fractional Brownian motion to a Gaussian martingale and back. This allows us to use the martingale version of the Burkholder-Davis-Gundy inequality for the proof.

1. INTRODUCTION

1.1. **Fractional Brownian motions.** We work in probability space (Ω, F, P). A real valued process $Z = Z^H$ is a fractional Brownian motion with Hurst index H, if it is a continuous Gaussian process with stationary increments and has the following properties:

- **i:** $Z_0 = 0$.
- **ii:** $EZ_t = 0$ for all $t \geq 0$.
- **iii:** $EZ_tZ_s = \frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right)$ for all $s, t \geq 0$.

The standard Brownian motion is a fractional Brownian motion with Hurst index $H = \frac{1}{2}$.

The process Z is self-similar and ergodic. It has p-variation index $p = \frac{1}{H}$.

1.2. **A maximal inequality.** For any process X denote by X^* the supremum process: $X_t^* = \sup_{s \leq t} |X_s|$.

From the self-similarity it follows for the supremum process Z^* that $Z_{at}^* \overset{d}{=} a^H Z_t^*$. Hence for any $p > 0$ we have then the following result using self-similarity:

Theorem 1.1. *Let $T > 0$ be a constant and Z a fractional Brownian motion with Hurst index H. Then*

$$E\left(Z_T^*\right)^p = K(p,H)T^{pH}, \tag{1.1}$$

where $K(p,H) = E\left(Z_1^*\right)^p$.

The value of the constant $K(p,H)$ is not at our disposal.

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1.3. **Burkholder-Davis-Gundy inequality for continuous martingales.** Let N be a continuous martingale. Then the Burkholder-Davis-Gundy inequalities are:

Proposition 1.1. *For any $p > 0$ and stopping time τ there exists constants $c_p, C_p > 0$ such that*

$$c_p E\langle N,N\rangle_\tau^{p/2} \le E\left((N_\tau^*)^p\right) \le C_p E\langle N,N\rangle_\tau^{p/2}. \tag{1.2}$$

For the proof we refer to [RY, Theorem IV.4.1]. For the special case of standard Brownian motion W we have

$$c_p E\tau^{\frac{p}{2}} \le E[(W_\tau^*)^p] \le C_p E\tau^{\frac{p}{2}}. \tag{1.3}$$

2. Transformations

2.1. **The Molchan martingale M.** We give some recently obtained integral representation between a fractional Brownian motion Z and the Molchan martingale M. T

The integrals below can be defined by integration by parts, where the singularities of the kernels do not cause problems, due to the Hölder continuity of the fractional Brownian motion Z (see [NVV, Lemma 2.1]).

Put $m(t,s) \doteq \frac{c}{C} s^{-\alpha}(t-s)^{-\alpha}$ for $s \in (0,t)$ and $m(t,s) = 0$ for $s > t$, where $\alpha \doteq H - \frac{1}{2}$,

$$C \doteq \sqrt{\frac{H}{\left(H-\frac{1}{2}\right) B(H-\frac{1}{2}, 2-2H)}}$$

and

$$c \doteq \frac{1}{B(H+\frac{1}{2}, \frac{3}{2}-H)},$$

where the beta coefficient $B(\mu,\nu)$ for $\mu,\nu > 0$ is defined by

$$B(\mu,\nu) \doteq \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

Proposition 2.1. *For $H \in (0,1)$ define M by*

$$M_t = \int_0^t m(t,s)\,dZ_s. \tag{2.1}$$

Then M is a Gaussian martingale, with $\mathbf{F}^M = \mathbf{F}^Z$ and variance $\langle M\rangle_t = \frac{C^2}{4H^2(2-2H)} t^{2-2H}$.

For the proof we refer to [NVV, Proposition 2.1].

2.2. **From Molchan martingale to fractional Brownian motion.** For later use denote by c_2 the constant

$$c_2 \doteq \sqrt{\frac{C^2}{4H^2(2-2H)}}.$$

Put $Y_t \doteq \int_0^t s^{\frac{1}{2}-H} dZ_s$. Then $Z_s = \int_0^t s^{H-\frac{1}{2}} dY_s$. Moreover, by definition

$$M_t = \frac{c}{C}\int_0^t (t-s)^{\frac{1}{2}-H} dY_s,$$

and by [NVV, Theorem 3.2]

$$Y_t = 2H\int_0^t (t-s)^{H-\frac{1}{2}} dM_s. \tag{2.2}$$

3. Burkholder-Davis-Gundy inequality for fractional Brownian motions

3.1. **The result.** The following generalizes (1.3) for fractional Brownian motions.

Theorem 3.1. *Let τ be a stopping time. Then for any $p > 0$ and $H \in [1/2, 1)$ we have that*

$$c(p,H)E\left(\tau^{pH}\right) \le E\left((Z_\tau^*)^p\right) \le C(p,H)E\left(\tau^{pH}\right), \tag{3.1}$$

and for any $p > 0$ and $H \in (0, 1/2)$ we have that

$$c(p,H)E\left(\tau^{pH}\right) \le E\left((Z_\tau^*)^p\right), \tag{3.2}$$

where the constants $c(p,H), C(p,H) > 0$ depend only from parameters p, H.

3.2. **Case $H > \frac{1}{2}$, upper bound.** We give the proof of the right-hand side inequality in (3.1).

We have that $Z_t = \int_0^t s^{H-\frac{1}{2}} dY_s$. Use integration by parts to get the upper estimate for Z^*:

$$Z_t^* \le 2t^\alpha Y_t^*.$$

For the process Y use the representation (2.2) to get the estimate

$$Y_t^* \le 4Ht^\alpha M_t^*. \tag{3.3}$$

From these two upper bounds we derive the following upper bound

$$Z_t \le 8HM_t^* t^{2\alpha}. \tag{3.4}$$

Note that (3.4) is valid for any stopping time τ:

$$Z_\tau^* \le 8HM_\tau^* \tau^{2\alpha}.$$

Hence for any $p > 0$ we have that

$$E\left(Z_\tau^*\right)^p \le (8H)^p E\left(\tau^{2\alpha p}\left(M_\tau^*\right)^p\right). \tag{3.5}$$

With Hölder's inequality, (3.5) with $q = \frac{H}{2\alpha} = \frac{H}{2H-1} > 1$, and $r = \frac{H}{1-H}$ we get

$$E\left(\tau^{2\alpha p}\left(M_\tau^*\right)^p\right) \le \left(E\tau^{2\alpha q p}\right)^{\frac{1}{q}}\left(E\left(M_\tau^*\right)^{pr}\right)^{\frac{1}{r}}. \tag{3.6}$$

Apply now (1.2) to obtain

$$E\left(M_\tau^*\right)^{pr} \leq c_2^p C_p E \tau^{\frac{(1-2\alpha)pr}{2}} = c_2^p C_p E \tau^{pH}. \tag{3.7}$$

To finish, note that the right hand side inequality in (3.1) for $H > 1/2$ follows from (3.6), and (3.7) with constant $C \doteq C(p,H) = (8H)^p \, c_2^p C_p$ and C_p is the constant in the Burkholder-Davis-Gundy inequality.

3.3. **Remarks.** The proof for the lower bound in the case of $H < \frac{1}{2}$ is similar to the proof in subsection 3.2. The proof of the lower bound in the case of $H > \frac{1}{2}$ is longer and we refer to [NV] for the proof.

We do not have an upper bound for the case $H < \frac{1}{2}$.

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STOCHASTIC DIFFERENCE EQUATIONS, DISCRETE FOKKER-PLANCK EQUATION AND FINITE PATH-INTEGRALS

IMME VAN DEN BERG

We consider an appropriate discretisation of the Fokker-Planck equation. Then the Feynman-Kač solution takes the form of a finite path-integral, along the paths of an associated stochastic difference equation. In some special cases, such as the Black-Scholes equation, the path-integral becomes a simple or double Riemann-sum. The transition to continuity is made using a higher-order De Moivre-Laplace theorem, and the nonstandard notion of shadow.

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