An addendum to 'An Introduction to Malliavin Calculus with Applications to Economics'

- Exercises with solutions and miscellaneous notes -

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1 Introduction

In May 1998 professor Bernt Øksendal gave an advanced course in Malliavin Calculus with applications to economics at the Centre for Mathematical Physics and Stochastics (MaPhySto). The course consisted of lectures and tutorial classes.

This note contains the exercises with suggested solutions discussed in the classes. Notation and necessary theoretical background to understand the exercises can be found in the Lecture Notes, [Ø]. Also included is a section with some remarks and theoretical results discussed in the lectures, but not treated in [Ø].

2 Exercises

The "chapters" below refer to the Lecture Notes [Ø].

Chapter 1 The Wiener-Ito chaos expansion

Exercise 1. *Exercises 1.1–1.3 in the Lecture Notes.*

Exercise 2. *Find the chaos expansion of*

$$\int_0^T W(s)dW(s)$$

and calculate its variance.

Exercise 3. *Find the chaos expansion of*

$$\exp\left(W(t)-\frac{1}{2}t\right)$$

Exercise 4. *Calculate the $L^2(\Omega)$-norm of*

$$\exp\left(W(t)-\frac{1}{2}t\right)$$

Exercise 5. *Find the chaos expansion to the solution X_t of the stochastic differential equation*

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t, \quad t > 0 \\ X_0 &= x \end{aligned}$$

¿From its chaos expansion, calculate $E[X_t]$ and $E[X_t^2]$.

Chapter 2 The Skorohod integral

Exercise 1. *Exercise 2.1 in the notes.*

Exercise 2. *Use the definition of the Skorohod integral to calculate*

$$\int_0^T e^{\left(W(t)-\frac{1}{2}t\right)} \delta W(t)$$

Exercise 3. *Give conditions on $g_t(\cdot) \in L^2([0,T])$ such that*

$$X_t = \exp\left(\int_0^T g_t(s) dW(s) - \frac{1}{2}\int_0^T g_t^2(s) ds\right)$$

is $\mathcal{F}_t$-adapted.

Chapter 4 Differentiation

Exercise 1. *Exercise 4.1–4.3 in the Lecture Notes.*

Exercise 2. *Find the Malliavin derivative of X_t, where X_t solves the stochastic differential equations*

a) $dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x$

b) $dX_t = \mu X_t dt + \sigma dW_t, \quad X_0 = x$

Exercise 3. *Consider the Ornstein-Uhlenbeck process*

$$X_t = x + \mu \int_0^t X_u du + \int_0^t dW(u)$$

Apply the Malliavin derivative D_s directly to this equation to derive that

$$D_s X_t = \begin{cases} 0 & \text{if } s > t \\ e^{\mu(t-s)} & \text{if } s \le t \end{cases}$$

Exercise 4. *Consider the solution X_t of the stochastic differential equation*

$$X_t = x + \int_0^t b(X_u) du + \sigma \int_0^t dW(u)$$

Find $D_s X_t$ for $s \in [0,T]$. (You can assume that b is continuously differentiable)

Exercise 5. *Let u_s be a stochastic process. Show that*

$$D_t \int_0^T u_s \delta W(s) = u_t + \int_0^T D_t u_s \delta W(s)$$

You can assume that all the objects involved are well-defined.

Exercise 6. *Find the Malliavin derivative $D_s X_t$, where X_t is the solution of*

$$X_t = x + \int_0^t \sigma(X_u) dW(u)$$

You can assume that σ is continuously differentiable. (Hint: Make use of exercise 5 above.)

Apply your result to the case $\sigma(x) = \sigma x$ and $\sigma(x) = \sigma$. (Check your results with exercise 2).

Chapter 5 The Clark-Ocone formula and its generalization

Exercise 1. *Exercises 5.1–5.4 in the Lecture Notes.*

3 Solutions to exercises

Chapter 1 The Wiener-Ito Chaos Expansion

Exercise 1:

Solutions to the exercises can be found in the Lecture Notes.

Exercise 2:

Writing $W(s) = \int_0^s dW(t_1)$ we get

$$\begin{aligned}\int_0^T W(s)dW(s) &= \int_0^T \int_0^{t_2} dW(t_1)dW(t_2) \\ &= J_2(f)\end{aligned}$$

where $f(t_1, t_2) = 1$. f is obviously symmetric. Using the identity $n!J_n(f_n) = I_n(f_n)$ for $n = 2$, we have the chaos representation

$$\int_0^T W(s)dW(s) = I_2(\frac{1}{2})$$

Since the expectation is zero the variance is given by the $L^2(P)$-norm of $\int_0^T W(s)dW(s)$. But by the Chaos Representation Theorem we have

$$\begin{aligned}||\int_0^T W(s)dW(s)||^2_{L^2(P)} &= 2!||\frac{1}{2}||^2_{L^2([0,T]^2)} \\ &= 2 \cdot \frac{1}{4} \cdot T^2 \\ &= \frac{1}{2}T^2\end{aligned}$$

Exercise 3:

Using Exercise 1.1.**d** in the Lecture Notes we have

$$\begin{aligned}\exp\left(W(t)-\frac{1}{2}t\right) &= \sum_{n=0}^{\infty}\frac{t^{n/2}}{n!}h_n\left(\frac{W(t)}{\sqrt{t}}\right)\\ &= \sum_{n=0}^{\infty}\int_0^T\int_0^{t_n}\cdots\int_0^{t_2}\mathbf{1}_{[0,t]}(t_1)\cdots\mathbf{1}_{[0,t]}(t_n)dW(t_1)\cdots dW(t_n)\end{aligned}$$

In the second equality we have used equation (1.14) in the Lecture Notes with $g(s)=\mathbf{1}_{[0,t]}(s)$. Let

$$f_n(t_1,\dots,t_n)=\mathbf{1}_{[0,t]}(t_1)\cdots\mathbf{1}_{[0,t]}(t_n)=\mathbf{1}_{[0,t]}^{\otimes n}(t_1,\dots,t_n)$$

The function f_n is symmetric. Thus

$$\begin{aligned}\exp\left(W(t)-\frac{1}{2}t\right) &= \sum_{n=0}^{\infty}J_n(f_n)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}I_n(f_n)\\ &= \sum_{n=0}^{\infty}I_n\left(\frac{\mathbf{1}_{[0,t]}^{\otimes n}}{n!}\right)\end{aligned}$$

Exercise 4:

Recall from Exercise 3 above that $\exp(W(t)-t/2)$ has chaos expansion

$$\exp\left(W(t)-\frac{1}{2}t\right)=\sum_{n=0}^{\infty}I_n\left(\frac{\mathbf{1}_{[0,t]}^{\otimes n}}{n!}\right)$$

The Chaos Representation Theorem gives us the $L^2(P)$-norm of this object:

$$\begin{aligned}\|\exp\left(W(t)-\frac{1}{2}t\right)\|_{L^2(P)}^2 &= \sum_{n=0}^{\infty}n!\left\|\frac{\mathbf{1}_{[0,t]}^{\otimes n}}{n!}\right\|_{L^2([0,T]^n)}^2\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\|\mathbf{1}_{[0,t]}^{\otimes n}\|_{L^2([0,T]^n)}^2\end{aligned}$$

Since $\|\mathbf{1}_{[0,t]}^{\otimes n}\|_{L^2([0,T]^n)}^2$ is the volume of the n-dimensional cube in $\mathbb{R}^n$ with edges of length t, we get

$$\|\exp\left(W(t)-\frac{1}{2}t\right)\|_{L^2(P)}^2=\sum_{n=0}^{\infty}\frac{1}{n!}t^n=e^t$$

Exercise 5:

We recognize the stochastic differential equation as geometric Brownian motion,

$$X_t=x\exp\left((\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)\right)$$

Write X_t like

$$X_t = x\exp(\mu t)\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right)$$

Use $g(s) = \sigma \mathbf{1}_{[0,t]}(s)$ in Exercise 1.1. **c** in the notes to derive

$$\exp\left(\sigma W(t) - \frac{1}{2}\sigma^2 t\right) = \sum_{n=0}^{\infty} \frac{||g||^n}{n!} h_n\left(\frac{\sigma W(t)}{||g||}\right)$$

But

$$||g|| = \left(\int_0^t \sigma^2 ds\right)^{1/2} = \sigma\sqrt{t}$$

Hence,

$$\begin{aligned} X_t &= x\exp(\mu t)\sum_{n=0}^{\infty} \frac{\sigma^n t^{n/2}}{n!} h_n\left(\frac{W(t)}{\sqrt{t}}\right) \\ &= x\exp(\mu t)\sum_{n=0}^{\infty} \sigma^n I_n\left(\frac{\mathbf{1}_{[0,t]}^{\otimes n}}{n!}\right) \\ &= \sum_{n=0}^{\infty} I_n\left(x\exp(\mu t)\frac{\sigma^n}{n!}\mathbf{1}_{[0,t]}^{\otimes n}\right) \end{aligned}$$

In the second equality we have made use of Exercise 3 above.

To calculate the expectation of X_t note that all the I_n's have zero expectation except I_0. Thus

$$E[X_t] = I_0\left(x\exp(\mu t)\right) = x\exp(\mu t)$$

The $L^2(P)$-norm of X_t is calculated by applying the Chaos Representation Theorem again:

$$\begin{aligned} E[X_t^2] = ||X_t||^2_{L^2(P)} &= \sum_{n=0}^{\infty} n!||x\exp(\mu t)\frac{\sigma^n}{n!}\mathbf{1}_{[0,t]}^{\otimes n}||^2_{L^2([0,T]^n)} \\ &= \sum_{n=0}^{\infty} n!\frac{x^2\exp(2\mu t)}{(n!)^2}\sigma^{2n}t^n \\ &= x^2\exp(2\mu t)\sum_{n=0}^{\infty}\frac{1}{n!}(\sigma^2 t)^n \\ &= x^2\exp(2\mu t + \sigma^2 t) \end{aligned}$$

Chapter 2 The Skorohod Integral

Exercise 1:

Solutions to the exercises can be found in the Lecture Notes.

Exercise 2:

¿From previous exercises we have seen that $\exp(W(t) - t/2)$ has chaos expansion,

$$\exp\left(W(t) - \frac{1}{2}t\right) = \sum_{n=0}^{\infty} I_n\left(\frac{\mathbf{1}_{[0,t]}^{\otimes n}}{n!}\right)$$

Put

$$f_n(t_1, \dots, t_n; t) := \frac{1}{n!}\mathbf{1}_{[0,t]}^{\otimes n}(t_1, \dots, t_n)$$

We calculate the symmetrization $\tilde{f}_n$ of f_n:

$$\tilde{f}_n(t_1, \dots, t_{n+1}) = \frac{1}{n+1}\left(f_n(t_1, \dots, t_n; t_{n+1}) + \dots + f_n(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n+1}; t_i) + \right.$$
$$\left. \dots + f_n(t_2, \dots, t_{n+1}; t_1)\right)$$

where

$$\begin{aligned} f_n(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n+1}; t_i) &= \mathbf{1}_{[0,t]}(t_1) \cdots \mathbf{1}_{[0,t]}(t_{i-1})\mathbf{1}_{[0,t]}(t_{i+1}) \cdots \mathbf{1}_{[0,t]}(t_{n+1}) \\ &= 1, \text{ iff } \; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n+1} < t_i \end{aligned}$$

Note that for a given tuple $t_1, \dots, t_{n+1}$ we can always find a t_i such that $t_i > t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n+1}$, namely $t_i = \max_{1 \le j \le n+1} t_j$. Thus

$$\tilde{f}_n(t_1, \dots, t_{n+1}) = \frac{1}{(n+1)!}$$

We use the definition of the Skorohod integral to obtain

$$\begin{aligned} \int_0^T \exp\left(W(t) - \frac{1}{2}t\right)\delta W(t) &= \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\tilde{f}_n) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} I_{n+1}(1) \\ &= \exp\left(W(t) - \frac{1}{2}t\right) - 1 \end{aligned}$$

Exercise 3:

¿From the exercises in chapter 1 we know the chaos expansion of X_t,

$$X_t = \sum_{n=0}^{\infty} I_n\left(\frac{g_t^{\otimes n}}{n!}\right)$$

The adaptedness of X_t can be characterized via its chaos functions by Lemma 2.5 in the notes:

$$X_t \text{ adapted } \Leftrightarrow \frac{1}{n!}g_t^{\otimes n}(t_1, \dots, t_n) = 0, \text{ if } t < \max_{1 \le j \le n} t_i$$

But this is equivalent of saying that

$$g_t(t_1) \cdots g_t(t_n) = 0, \text{ if } t < \max_{1 \le j \le n} t_i$$

which again is equivalent to

$$g_t(s) = 0, \quad \text{if } s > t \tag{3.1}$$

Thus, to ensure adaptedness of X_t we need to impose the condition (3.1).

Chapter 4 Differentiation

Exercise 1:

Solutions to the exercises can be found in the Lecture Notes.

Exercise 2:

a) The solution of the stochastic differential equation is the geometric Brownian motion,

$$X(t) = x \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right)$$

Applying the Malliavin derivative D_s to $X(t)$ gives,

$$\begin{aligned} D_s X(t) &= x \exp((\mu - \frac{1}{2}\sigma^2)t) \cdot D_s \exp(\sigma W(t)) \\ &= x \exp((\mu - \frac{1}{2}\sigma^2)t) \exp(\sigma W(t)) \sigma D_s W(t) \\ &= \sigma X(t) \cdot \mathbf{1}_{[0,t]}(s) \end{aligned}$$

We see that the Malliavin derivative is

$$D_s X(t) = \begin{cases} \sigma X(t), & s \leq t \\ 0, & s > t \end{cases}$$

b) The solution $X(t)$ is the Ornstein-Uhlenbeck process:

$$X(t) = \exp(\mu t)\left(x + \sigma \int_0^t \exp(-\mu u) dW(u)\right)$$

For our purposes it is convenient to rewrite this as

$$X(t) = \exp(\mu t)\left(x + \sigma \int_0^T \mathbf{1}_{[0,t]}(u) \exp(-\mu u) dW(u)\right)$$

Application of the Malliavin derivative yields,

$$\begin{aligned} D_s X(t) &= \exp(\mu t)\left(\sigma D_s \int_0^T \mathbf{1}_{[0,t]}(u) \exp(-\mu u) dW(u)\right) \\ &= \exp(\mu t) \sigma \mathbf{1}_{[0,t]}(s) \exp(-\mu s) \\ &= \sigma \exp(\mu(t-s)) \mathbf{1}_{[0,t]}(s) \end{aligned}$$

Note that the Malliavin derivative is deterministic in the case of an Ornstein-Uhlenbeck process, while it is stochastic for the geometric Brownian motion.

Exercise 3:

By using the rules of the Malliavin derivative we get,

$$\begin{aligned} D_s X(t) &= D_s x + \mu \int_0^t D_s X(u) du + D_s \int_0^T \mathbf{1}_{[0,t]}(u) dW(u) \\ &= \mu \int_0^t D_s X(u) du + \mathbf{1}_{[0,t]}(s) \end{aligned}$$

Consider the case when $s > t$: We prove that $D_sX(t) = 0$ (confer Corollary 5.7 in the notes for a general statement). The equation becomes

$$D_sX(t) = \mu \int_0^t D_sX(u)du$$

Let $\rho_s(t) = D_sX(t)$ for $t \in [0, s)$. It is easily seen that $\rho_s(t)$ satisfies the linear differential equation

$$\frac{d\rho_s(t)}{dt} = \mu\rho_s(t)$$

with initial condition $\rho_s(0) = 0$. This implies

$$\rho_s(t) = 0, \text{ for all } t \in [0, s)$$

and we conclude

$$D_sX(t) = 0, s > t$$

Consider now the case $s \leq t$: The equation for the Malliavin derivative becomes

$$\begin{aligned} D_sX(t) &= 1 + \mu \int_0^t D_sX(u)du \\ &= 1 + \int_s^t D_sX(u)du \end{aligned}$$

by using that $D_sX(u) = 0$ whenever $s > u$. Introduce again the function $\rho_s(t) = D_sX(t)$, but this time for $t \geq s$. We see that $\rho_s(t)$ is the solution of the differential equation

$$\frac{d\rho_s(t)}{dt} = \mu\rho_s(t), t \geq s$$

with initial condition $\rho_s(s) = 1$. The solution of this differential equation is known to be

$$\rho_s(t) = \exp(\mu(t - s))$$

which gives

$$D_sX(t) = \exp(\mu(t - s)), t \geq s$$

Exercise 4:

First of all note that since $X(t)$ is adapted, $D_sX(t)$ is zero when $s > t$ by Corollary 5.7 in the notes. Consider $s \leq t$ and apply the Malliavin derivative directly to the stochastic differential equation:

$$\begin{aligned} D_sX(t) &= D_sx + \int_0^t b'(X(u))D_sX(u)du + D_s \int_0^T \mathbf{1}_{[0,t]}(u)\sigma dW(u) \\ &= \int_s^t b'(X(u))D_sX(u)du + \sigma\mathbf{1}_{[0,t]}(s) \end{aligned}$$

Define

$$\rho_s(t) = D_sX(t)$$

It is easily seen that $\rho_s(s) = \sigma$ and

$$\frac{d\rho_s(t)}{dt} = b'(X(t))\rho_s(t), \ t > s$$

This differential equation has the solution

$$\rho_s(t) = \sigma \exp\left(\int_s^t b'(X(u))du\right)$$

Thus we have proved that the Malliavin derivative $D_s X(t)$ is

$$D_s X(t) = \begin{cases} \sigma \exp(\int_s^t b'(X(u))du) & , \quad s \leq t \\ 0 & , \quad s > t \end{cases}$$

Exercise 5:

The proof of the result can be found in the Lecture Notes, Theorem 5.12. The following argument is, however, a direct calculation showing the relation: Let

$$u_t = \sum_{n=0}^{\infty} I_n(u_n(\cdot;t))$$

¿From the definition of the Malliavin derivative and the Skorohod integral, we have

$$\begin{aligned} D_t \int_0^T u_s \delta W(s) &= D_t \left(\sum_{n=0}^{\infty} I_{n+1}(\tilde{u}_n) \right) \\ &= \sum_{n=0}^{\infty} (n+1) I_n(\tilde{u}_n(\cdot, t)) \\ &= \sum_{n=0}^{\infty} (n+1) I_n \left(\frac{1}{n+1} \{u_n(t_1, \ldots, t_n; t) + \ldots + u_n(t_2, \ldots, t_n, t; t_1)\} \right) \\ &= \sum_{n=0}^{\infty} I_n(u_n(\cdot;t)) + \sum_{n=0}^{\infty} I_n \left(\sum_{i=1}^{n} u_n(t_t, \ldots, \hat{t}_i, \ldots, t_n, t; t_i) \right) \\ &= u_t + \sum_{n=0}^{\infty} n I_n(\text{symm } u_n(\cdot, t; \cdot)) \end{aligned}$$

where the notation $\hat{t}_i$ means that t_i is removed and symm $u_n(\cdot, t; \cdot)$ is the symmetrization of the function $u_n(t_1, \ldots, t_{n-1}, t; t_n)$ for fixed t. Consider $\int_0^T D_t u_s \delta W(s)$:

$$\begin{aligned} \int_0^T D_t u_s \delta W(s) &= \int_0^T \left(\sum_{n=0}^{\infty} n I_{n-1}(u_n(\cdot, t; s)) \right) \delta W(s) \\ &= \sum_{n=0}^{\infty} n I_n(\text{symm } u_n(\cdot, t; \cdot)) \end{aligned}$$

which proves the relation.

Exercise 6:

Corollary 5.7 gives that $D_s X(t) = 0$ whenever $s > t$ since $X(t)$ is adapted. Consider the case $s \leq t$. Applying the Malliavin derivative directly to the stochastic differential equation and then using the relation from exercise 5 above, we get

$$\begin{aligned} D_s X(t) &= D_s x + D_s \int_0^T \mathbf{1}_{[0,t]}(u)\sigma(X(u))dW(u) \\ &= \sigma(X(s))\mathbf{1}_{[0,t]}(s) + \int_0^T D_s \left(\mathbf{1}_{[0,t]}(u)\sigma(X(u))\right)\delta W(u) \\ &= \sigma(X(s)) + \int_0^t \sigma'(X(u))D_s X(u)\delta W(u) \\ &= \sigma(X(s)) + \int_s^t \sigma'(X(u))D_s X(u)\delta W(u) \end{aligned}$$

By corollary 5.7 again, we know that $D_s X(u)$ is adapted with respect to the variable u. Thus

$$D_s X(t) = \sigma(X(s)) + \int_s^t \sigma'(X(u))D_s X(u)dW(u)$$

We observe that the equation for $D_s X(t)$ is a linear stochastic differential equation with a stochastic coefficient and initial variable. Moreover, $\sigma(X(s))$ is independent of $W(u)$ when $u \geq s$. Hence we get

$$D_s X(t) = \sigma(X(s)) \exp\left(-\frac{1}{2}\int_s^t \sigma'(X(u))^2 du + \int_s^t \sigma'(X(u))dW(u)\right)$$

Letting $\sigma(x) = \sigma x$ or $\sigma(x) = \sigma$ we reobtain the results from exercise 2 above.

Chapter 5 The Clark-Ocone Formula and its Generalization

Exercise 1:

Solutions to the exercises can be found in the Lecture Notes.

4 Miscellaneous Results and Remarks presented in the lectures

Proof of Ito's identity between n-fold Wiener integrals and Hermite polynomials (eq. (1.14) in the Lecture Notes.)

The following argument was suggested by F. Oertel from University of Bonn, Germany.

Define the "normalized" Hermite polynomial

$$H_n(x, \lambda) := \lambda^{n/2} h_n\left(\frac{x}{\sqrt{\lambda}}\right)$$

¿From the series representation

$$\exp(tx - t\lambda^2/2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, \lambda)$$

we derive the following properties:

$$\frac{\partial}{\partial x} H_n(x,\lambda) = nH_{n-1}(x,\lambda)$$
$$\frac{\partial^2}{\partial x^2} H_n(x,\lambda) + \frac{\partial}{\partial \lambda} H_n(x,\lambda) = 0$$

The proof of the identity

$$I_n(g^{\otimes n}) = H_n(\int_0^T g(s)dW(s), ||g||^2)$$

goes by induction on n:

$$\begin{aligned} I_{n+1}(g^{\otimes n+1}) &= (n+1)! J_{n+1}(g^{\otimes n+1}|S_{n+1}) \\ &= (n+1)\int_0^T g(s) I_n(g^{\otimes n}) dW(s) \end{aligned}$$

where we have restricted the I_n inside the integral to $[0,s]^n$. Using the Induction Assumption, we have

$$\begin{aligned} I_{n+1}(g^{\otimes n+1}) &= (n+1)\int_0^T g(s) H_n(\int_0^s g(u)dW(u), ||g||^2_{L^2([0,s])})dW(s) \\ &= \int_0^T \frac{\partial}{\partial x} H_{n+1}(\int_0^s g(u)dW(u), ||g||^2_{L^2([0,s])})dW(s) \\ &= H_{n+1}(\int_0^T g(s)dW(s), ||g||^2) \end{aligned}$$

In the last equality we have used the Itos Formula and the second property of H_n stated above.

Remark to the chaos representation theorem

In the Chaos Representation Theorem it was proved that every $\phi \in L^2(P)$ has a representation

$$\phi = \sum_{n=0}^{\infty} I_n(f_n)$$

where $f_n(\cdot) \in \hat{L}^2([0,T]^n)$. We here prove that this representation is *unique*:

Proof. Let $\phi = \sum_{n=0}^{\infty} I_n(g_n)$ be another expansion of ϕ. Then by linearity of the n-fold Wiener integrals we have

$$0 = \sum_{n=0}^{\infty} I_n(f_n - g_n)$$

But this expansion will have $L^2(P)$-norm equal to

$$0 = \sum_{n=0}^{\infty} n! ||f_n - g_n||^2_{L^2([0,T]^n)}$$

Hence, $f_n = g_n$ a.e. □

How to calculate the functions in the chaos expansion

The following characterization of the chaos functions is taken from Ustunel, [U]: Let $\phi \in L^2(P)$ have the chaos representation $\phi = \sum_{n=0}^{\infty} I_n(f_n)$. Then

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} \mathrm{E}\left[D_{t_1} \cdots D_{t_n} \phi\right]$$

D_t denotes the Malliavin Derivative.

On the adjointness of Malliavin differentiation and Skorohod integration

The Malliavin derivative transforms random variables into stochastic processes, while the Skorohod integral maps stochastic processes into random variables,

$$\begin{aligned} D :& L^2(\Omega) \to L^2([0,T] \times \Omega) \\ \delta :& L^2([0,T] \times \Omega) \to L^2(\Omega) \end{aligned}$$

(we suppress the domain of definition to illustrate the action of the two operators). We have the following adjointness relation (see e.g. Nualart, [N]): If $F \in \mathcal{D}_{1,2}$ and $u_t \in \mathrm{Dom}(\delta)$,

$$\mathrm{E}\left[F\delta(u)\right] = \mathrm{E}\left[(D_t F, u_t)_{L^2([0,T])}\right]$$

where $(\cdot, \cdot)_{L^2([0,T])}$ is the inner product of $L^2([0,T])$.

Generalization of the Clark-Ocone Formula

The following extension of the Clark-Ocone Formula has been proved by Aase, Øksendal and Ubøe, [AaØU]:

Define the spaces $\mathcal{G}_\lambda$ for $\lambda \in I\!R$ as the set of random variables $F \in L^2(\Omega)$ where

$$||F||^2_{2,\lambda} := \sum_{n=0}^{\infty} n! e^{2\lambda n} ||f_n||^2_{L^2([0,T]^n)}$$

The f_n's are the chaos functions of F. Defining

$$\mathcal{G} := \cap_{\lambda \geq 0} \mathcal{G}_\lambda$$

and

$$\mathcal{G}^* := \cup_{\lambda \geq 0} \mathcal{G}_{-\lambda}$$

we have the following triplet of smooth and generalized random variables:

$$\mathcal{G} \subset L^2(\mu) \subset \mathcal{G}^*$$

Note that $\mathcal{G}$ and $\mathcal{G}^*$ are dual spaces, where $\mathcal{G}$ is equipped with the projective limit topology and $\mathcal{G}^*$ the inductive limit topology.

The Malliavin derivative is extended to the space $\mathcal{G}^*$ such that the following holds:

$$D_t \Phi = \sum_{n=1}^{\infty} n I_{n-1}(\Phi_n(\cdot, t))$$

whenever $D_t\Phi \in \mathcal{G}^*$. In the sense of random distributions, the Clark-Ocone formula is proved for Malliavin differentiable elements in $\mathcal{G}^*$:

$$\Phi = \mathrm{E}[\Phi] + \int_0^T \mathrm{E}[D_t\Phi|\mathcal{F}_t]dW(t)$$

The formula also has a multi-dimensional analogue. Conditional expectation and Wiener integration can be defined for distributions in $\mathcal{G}^*$. We refer to [AaØU] for details.

More information about the spaces $\mathcal{G}$ and $\mathcal{G}^*$ can be found in [PT].

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