A Connection between Free and Classical Infinite Divisibility

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Abstract

In this paper we continue our studies, initiated in [BT1],[BT2] and [BT3], of the connections between the classes of infinitely divisible probability measures in classical and in free probability. We show that the free cumulant transform of any freely infinitely divisible probability measure equals the classical cumulant transform of a certain classically infinitely divisible probability measure, and we give several characterizations of the latter measure, including an interpretation in terms of stochastic integration. We find, furthermore, an alternative definition of the Bercovici-Pata bijection, which passes directly from the classical to the free cumulant transform, without passing through the Lévy-Khintchine representations (classical and free, respectively). As a byproduct, of some independent interest, the derivation in the final section establishes the existence of a one-to-one mapping of the class of Levy measures into a subset of that class, whose elements have densities, the restrictions to $]-\infty,0[$ and $]0,\infty[$ of which are representable as Laplace transforms.

1 Introduction.

The classes $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ of probability distributions on $\mathbb{R}$ that are infinitely divisible in the classical and the free sense, respectively, are connected by a bijective mapping - the Bercovici-Pata bijection $\Lambda : \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$. This mapping has several useful algebraic and topological properties and preserves the properties of stability and selfdecomposability. Moreover, by suitable definition of the free cumulant transform, the connection between the free and classical Lévy-Khintchine representations of a probability law μ in $\mathcal{ID}(*)$ and its counterpart $\Lambda(\mu)$ in $\mathcal{ID}(\boxplus)$ is determined simply by μ and $\Lambda(\mu)$ having the same characteristic triplet (classical and free, respectively). These properties can be used to define free Lévy processes and integration with respect to such processes. See [BT1],[BT2],[BT3] and references given there.

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The present note establishes a further connection between free and classical infinite divisibility. We show that there exists a one-to-one mapping $\Upsilon : \mathcal{ID}(*) \to \mathcal{ID}(*)$ such that, for any μ in $\mathcal{ID}(*)$, the free cumulant transform $\mathcal{C}_{\Lambda(\mu)}(z)$ (defined as indicated above) of $\Lambda(\mu)$ is equal to the classical cumulant transform $\mathrm{C}_{\Upsilon(\mu)}(\zeta)$ of $\Upsilon(\mu)$, when $z = \mathrm{i}\zeta$ and $\zeta \leq 0$. The mapping Υ has algebraic and topological properties similar to those of Λ but, remarkably, $\Upsilon(\mathcal{ID}(*))$ is a proper subset of $\mathcal{ID}(*)$. Furthermore, the law $\Upsilon(\mu)$ is identifiable as that of a certain stochastic integral with respect to the (classical) Lévy process (Y_t) for which the law of Y_1 is equal to μ. We establish, further, a formula (see formula (4.1)) linking the classical cumulant transform C_μ of μ directly with the free cumulant transform $\mathcal{C}_{\Lambda(\mu)}$ of $\Lambda(\mu)$. This formula provides, thus, an alternative definition of Λ, which by-passes the intermediate step of the Lévy-Khintchine representations (classical and free, respectively).

Section 2 provides background material on infinite divisibility in free probability. In Section 3 the mapping Υ is introduced, and in Section 4 its relation to free infinite divisibility is established. In Section 5 we derive some algebraic properties of Υ, similar to those of Λ, and we note, as immediate consequences, that Υ preserves the notions of stability and selfdecomposability. Section 6 derives the stochastic integral interpretation of Υ, based on some initial considerations on stochastic integration w.r.t. (classical) Lévy processes. In the final Section 7, we establish that the Lévy measure for any probability measure in the range of Υ is absolutely continuous w.r.t. Lebesgue measure, and we give an explicit formula for the density. Furthermore, we use this formula to calculate some examples. The results in Section 7 do not, in fact, involve any notions from free probability and concern a Laplace-like one-to-one transformation of arbitrary Lévy measures to Lévy measures that, in particular, are absolutely continuous w.r.t. Lebesgue measure.

2 Background.

In this section we review briefly, for the readers convenience, the basic definitions from the theory of infinite divisibility in free probability. For a more detailed account of that theory we refer to [BT1] or [BT2].

A W^*-probability space is a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a von Neumann algebra (acting on some Hilbert space $\mathcal{H}$) and τ is a faithful normal tracial state on $\mathcal{A}$. Suppose a is a (possibly unbounded) selfadjoint operator in $\mathcal{H}$. Then a is *affiliated with* $\mathcal{A}$ if $f(a) \in \mathcal{A}$ for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$ (here $f(a)$ is defined in terms of spectral calculus).

Suppose a is a selfadjoint operator affiliated with $\mathcal{A}$. Then, by the Riesz representation theorem, there exists a unique probability measure μ_a on $(\mathbb{R}, \mathcal{B})$, satisfying that

$$\int_{\mathbb{R}} f(t)\ \mu_a(\mathrm{d}t) = \tau(f(a)), \tag{2.1}$$

for any bounded Borel function $f\colon \mathbb{R} \to \mathbb{R}$. We refer to the measure μ_a as the (spectral) distribution of a w.r.t. τ, and we denote it also by $L\{a\}$ (the law of a).

In the early 1980's, D.V. Voiculescu introduced the notion of *free independence* among non-commutative random variables:

2.1 Definition. Let $a_1, a_2, \ldots, a_r$ be selfadjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. We say then that $a_1, a_2, \ldots, a_r$ are *freely independent* w.r.t. τ, if

$$\tau\big\{[f_1(a_{i_1}) - \tau(f_1(a_{i_1}))][f_2(a_{i_2}) - \tau(f_2(a_{i_2}))] \cdots [f_p(a_{i_p}) - \tau(f_p(a_{i_p}))]\big\} = 0,$$

for any p in $\mathbb{N}$, any bounded Borel functions $f_1, f_2, \ldots, f_p \colon \mathbb{R} \to \mathbb{R}$ and any indices $i_1, i_2, \ldots, i_p$ in $\{1, 2, \ldots, r\}$ satisfying that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{p-1} \neq i_p$.

If a_1 and a_2 are two freely independent operators affiliated with $(\mathcal{A}, \tau)$, then their sum is again a selfadjoint operator affiliated with $(\mathcal{A}, \tau)$, and the distribution $L\{a_1 + a_2\}$ of $a_1 + a_2$ is uniquely determined by the marginals $L\{a_1\}$ and $L\{a_2\}$ (cf. [BV]). Hence, one may define the free convolution $L\{a_1\} \boxplus L\{a_2\}$ of $L\{a_1\}$ and $L\{a_2\}$ to be $L\{a_1 + a_2\}$. Furthermore, any given probability measures μ and ν on $\mathbb{R}$ can be realized as the marginals $L\{a_1\}$ and $L\{a_2\}$ for two freely independent selfadjoint operators a_1 and a_2 affiliated with a suitable W^*-probability space. Thus, free (additive) convolution $\boxplus$ is a well-defined binary operation on the class of all Borel probability measures on $\mathbb{R}$. Having defined free convolution, we may subsequently define free infinite divisibility exactly as in the classical case:

2.2 Definition. A probability measure μ on $\mathbb{R}$ is infinitely divisible w.r.t. free convolution (or just $\boxplus$-infinitely divisible), if there exists, for any $n \in \mathbb{N}$, a probability measure μ_n on $\mathbb{R}$, such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

We denote by $\mathcal{ID}(\boxplus)$ the class of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$.

In a similar fashion, one may introduce the classes of *stable* and *selfdecomposable* probability measures in free probability, simply by replacing classical convolution $*$ by free convolution $\boxplus$ in the classical definitions of these concepts. Thus, a probability measure μ on $\mathbb{R}$ belongs to the class $\mathcal{S}(\boxplus)$ of freely stable probability measures, if the set $\{\psi(\mu) \mid \psi \colon \mathbb{R} \to \mathbb{R} \text{ increasing affine function}\}$ of increasing affine transformations of μ is closed under $\boxplus$. Similarly, μ belongs to the class $\mathcal{L}(\boxplus)$ of freely selfdecomposable probability measures if there exists, for any c in $]0,1[$, a probability measure μ_c on $\mathbb{R}$, such that $\mu = D_c\mu \boxplus \mu_c$. Here, D_c denotes dilation by c, i.e., $D_c\mu(B) = \mu(c^{-1}B)$ for any Borel set B in $\mathbb{R}$.

If μ is a probability measure on $\mathbb{R}$, we denote by C_μ the (classical) cumulant transform of μ, i.e., $C_\mu = \log(f_\mu)$, where f_μ denotes the characteristic function (or Fourier transform) of μ. Recall that μ is infinitely divisible in the classical sense if and only if C_μ has the Lévy-Khintchine representation:

$$C_\mu(u) = \mathrm{i}\eta u - \tfrac{1}{2}au^2 + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}ut} - 1 - \mathrm{i}ut1_{[-1,1]}(t)\big)\, \rho(\mathrm{d}t), \quad (u \in \mathbb{R}),$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$, i.e.,

$$\rho(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, x^2\}\, \rho(\mathrm{d}x) < \infty.$$

The triplet (a, ρ, η) is uniquely determined, and it is called the generating triplet for μ.

The free analog of the cumulant transform, the so-called R-transform, was introduced by Voiculescu in [Vo]. Its main property is that it linearizes free convolution, just as the classical cumulant transform linearizes classical convolution. Furthermore, a probability measure is uniquely determined by its R-transform, and the free infinitely divisible measures can be characterized as those measures for which the R-transform has a certain representation, the free Lévy-Khintchine representation (cf. [BV]). We prefer, in this note, to work with a slight modification of the R-transform, which we refer to as the free cumulant transform. For the exact definitions of the R-transform and free cumulant transform we refer to [BT1] or [BT2]. In terms of the free cumulant transform, the free Lévy-Khintchine representation, mentioned above, takes the following form:

2.3 Theorem. *A probability measure ν on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there exist a non-negative number a, a real number η and a Lévy measure ρ, such that the free cumulant transform $\mathcal{C}_\nu$ has the representation:*

$$\mathcal{C}_\nu(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-tz} - 1 - tz1_{[-1,1]}(t)\Big)\,\rho(\mathrm{d}t), \quad (z \in \mathbb{C},\ \mathrm{Im}(z) < 0). \tag{2.2}$$

In that case, the triplet (a, ρ, η) is uniquely determined and is called the free generating triplet for ν.

The following bijection was introduced and studied by Bercovici and Pata in [BP]. The definition, we give below, is in terms of the generating triplets (classical and free, respectively), in contrast to the original definition given in [BP].

2.4 Definition. The Bercovici-Pata bijection is the mapping $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ defined in the following way: Suppose μ is in $\mathcal{ID}(*)$ and has generating triplet (a, ρ, η). Then $\Lambda(\mu)$ is the measure in $\mathcal{ID}(\boxplus)$ with free generating triplet (a, ρ, η).

At a first glance, the Bercovici-Pata bijection might seem as a very formal correspondence. The following result, however, shows that it is of a deeper nature:

2.5 Theorem. ([BT1]) *The Bercovici-Pata bijection $\Lambda\colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$, satisfies:*

(i) *If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.*

(ii) *If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.*

(iii) *For any c in $\mathbb{R}$, $\Lambda(\delta_c) = \delta_c$, where δ_c denotes the Dirac measure at c.*

(iv) *Λ is a homeomorphism w.r.t. weak convergence.*

For proofs of the statements (i)-(iv) above, we refer to [BT1].

3 The mapping Υ: $\mathcal{ID}(*) \to \mathcal{ID}(*)$.

Recall that for any measure μ on $\mathbb{R}$ and any positive constant c, we denote by $D_c\mu$ the *dilation* of μ by c, i.e. the measure on $\mathbb{R}$ given by:

$$D_c\mu(B) = \mu(c^{-1}B),$$

for any Borel set B in $\mathbb{R}$.

3.1 Definition. For any μ in $\mathcal{ID}(*)$, with generating triplet (a, ρ, η), we take (μ) to be the element of $\mathcal{ID}(*)$ whose generating triplet is $(2a, \tilde\rho, \tilde\eta)$ where

$$\tilde\eta = \eta + \int_0^\infty \Big(\int_{\mathbb{R}} t \left(1_{[-1,1]}(t) - 1_{[-x,x]}(t) \right) \, D_x\rho(\mathrm{d}t) \Big) \mathrm{e}^{-x} \, \mathrm{d}x \tag{3.1}$$

and

$$\tilde\rho = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\mathrm{d}x. \tag{3.2}$$

In a series of lemmas, we verify, next, that the above definition is meaningful, i.e., that the integral in (3.1) is well-defined and the measure $\tilde\rho$ is a Lévy measure.

3.2 Lemma. *Let ρ be a Lévy measure on $\mathbb{R}$ and consider the Markov kernel $(D_x\rho)_{x\in]0,\infty[}$. Then the mixed measure*

$$\tilde\rho = \int_0^\infty (D_x\rho)\mathrm{e}^{-x} \, \mathrm{d}x,$$

is again a Lévy measure.

Proof. We show first that $\int_{\mathbb{R}\setminus[-1,1]} 1 \, \tilde\rho(\mathrm{d}t) < \infty$. Note for this that

$$\begin{aligned} \int_{\mathbb{R}\setminus[-1,1]} 1 \, \rho(\mathrm{d}t) = \tilde\rho(\mathbb{R} \setminus [-1,1]) &= \int_0^\infty D_x\rho(\mathbb{R} \setminus [-1,1])\mathrm{e}^{-x} \, \mathrm{d}x \\ &= \int_0^\infty \rho(\mathbb{R} \setminus [-x^{-1}, x^{-1}])\mathrm{e}^{-x} \, \mathrm{d}x. \end{aligned}$$

If $0 < x \leq 1$, then

$$\rho(\mathbb{R} \setminus [-x^{-1}, x^{-1}]) \leq \rho(\mathbb{R} \setminus [-1,1]) \leq \int_{\mathbb{R}} \min\{1, t^2\} \, \rho(\mathrm{d}t),$$

and if $x > 1$,

$$\rho(\mathbb{R} \setminus [-x^{-1}, x^{-1}]) \leq \int_{[-1,1]\setminus[-x^{-1},x^{-1}]} x^2t^2 \, \rho(\mathrm{d}t) + \int_{\mathbb{R}\setminus[-1,1]} 1 \, \rho(\mathrm{d}t) \leq x^2 \int_{\mathbb{R}} \min\{1, t^2\} \, \rho(\mathrm{d}t).$$

We conclude, thus, that

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus[-1,1]} 1\,\tilde{\rho}(\mathrm{d}t) \\
&\qquad \leq \int_0^1 \Big(\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x + \int_1^\infty x^2\Big(\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \qquad (3.3)\\
&\qquad = \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\int_{\mathbb{R}} \max\{1,x^2\}\mathrm{e}^{-x}\,\mathrm{d}x < \infty.
\end{aligned}
$$

We show next that $\int_{-1}^1 t^2\,\tilde{\rho}(\mathrm{d}t) < \infty$. Note that

$$
\begin{aligned}
\int_{-1}^1 t^2\,\tilde{\rho}(\mathrm{d}t) &= \int_0^\infty \Big(\int_{-1}^1 t^2\,D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_0^\infty \Big(\int_{\mathbb{R}} x^2t^2 1_{[x^{-1},x^{-1}]}(t)\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x.
\end{aligned}
$$

If $x \geq 1$, we find that

$$
\int_{\mathbb{R}} x^2t^2 1_{[-x^{-1},x^{-1}]}(t)\,\rho(\mathrm{d}t) \leq x^2\int_{\mathbb{R}} t^2 1_{[-1,1]}(t)\,\rho(\mathrm{d}t) \leq x^2\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t),
$$

and, if $0 < x < 1$,

$$
\begin{aligned}
\int_{\mathbb{R}} x^2t^2 1_{[-x^{-1},x^{-1}]}(t)\,\rho(\mathrm{d}t) &= x^2\int_{-1}^1 t^2\,\rho(\mathrm{d}t) + x^2\int_{\mathbb{R}} t^2 1_{[-x^{-1},x^{-1}]\setminus[-1,1]}(t)\,\rho(\mathrm{d}t) \\
&\leq x^2\int_{-1}^1 t^2\,\rho(\mathrm{d}t) + x^2\int_{\mathbb{R}} x^{-2} 1_{[-x^{-1},x^{-1}]\setminus[-1,1]}(t)\,\rho(\mathrm{d}t) \\
&\leq \int_{-1}^1 t^2\,\rho(\mathrm{d}t) + \int_{\mathbb{R}} 1_{\mathbb{R}\setminus[-1,1]}(t)\,\rho(\mathrm{d}t) \\
&= \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t).
\end{aligned}
$$

We conclude, thus, that

$$
\begin{aligned}
\int_{-1}^1 t^2\,\tilde{\rho}(\mathrm{d}t) &= \int_0^1 \Big(\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x + \int_1^\infty x^2\Big(\int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \min\{1,t^2\}\,\rho(\mathrm{d}t)\int_{\mathbb{R}} \max\{1,x^2\}\mathrm{e}^{-x}\,\mathrm{d}x < \infty.
\end{aligned}
\qquad (3.4)
$$

Combining (3.3) and (3.4), it follows that $\tilde{\rho}$ is a Lévy measure. ∎

3.3 Lemma. *Let ρ be a Lévy measure on $\mathbb{R}$. Then for any x in $]0,\infty[$, we have that*

$$\int_{\mathbb{R}} \big|ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big| \, \rho(\mathrm{d}u) < \infty.$$

Furthermore,

$$\int_0^\infty \Big(\int_{\mathbb{R}} \big|ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big| \, \rho(\mathrm{d}u)\Big) \mathrm{e}^{-x} \, \mathrm{d}x < \infty.$$

Proof. Note first that for any x in $]0,\infty[$ we have

$$\begin{aligned}
&\int_{\mathbb{R}} \big|ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big| \, \rho(\mathrm{d}u) \\
&\quad = \int_{\mathbb{R}} \big|ux \cdot \big(1_{[-x^{-1},x^{-1}]}(u) - 1_{[-1,1]}(u)\big)\big| \, \rho(\mathrm{d}u) \\
&\quad = \begin{cases} x \int_{\mathbb{R}} |u| \cdot 1_{[-x^{-1},x^{-1}]\setminus[-1,1]}(u) \, \rho(\mathrm{d}u), & \text{if } x \le 1, \\ x \int_{\mathbb{R}} |u| \cdot 1_{[-1,1]\setminus[-x^{-1},x^{-1}]}(u) \, \rho(\mathrm{d}u), & \text{if } x > 1. \end{cases}
\end{aligned}$$

Note then that whenever $0 < \epsilon < K$, we have

$$|u| \cdot 1_{[-K,K]\setminus[-\epsilon,\epsilon]}(u) \le \min\{K, \tfrac{u^2}{\epsilon}\} \le \max\{K, \epsilon^{-1}\} \min\{u^2, 1\},$$

for any u in $\mathbb{R}$. Hence, if $0 < x \le 1$, we find that

$$\begin{aligned}
x \int_{\mathbb{R}} \big|u \cdot \big(1_{[-x^{-1},x^{-1}]}(ux) - 1_{[-1,1]}(u)\big)\big| \, \rho(\mathrm{d}u) &\le x \max\{x^{-1}, 1\} \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) \\
&= \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) < \infty,
\end{aligned}$$

since ρ is a Lévy measure. Similarly, if $x \ge 1$,

$$\begin{aligned}
x \int_{\mathbb{R}} \big|u \cdot \big(1_{[-1,1]}(ux) - 1_{[-x^{-1},x^{-1}]}(ux)\big)\big| \, \rho(\mathrm{d}u) &\le x \max\{1, x\} \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) \\
&= x^2 \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) < \infty.
\end{aligned}$$

Altogether, we find that

$$\begin{aligned}
&\int_0^\infty \Big(\int_{\mathbb{R}} \big|ux \cdot \big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big| \, \rho(\mathrm{d}u)\Big) \mathrm{e}^{-x} \, \mathrm{d}x \\
&\qquad \le \int_{\mathbb{R}} \min\{u^2, 1\} \, \rho(\mathrm{d}u) \cdot \Big(\int_0^1 \mathrm{e}^{-x} \, \mathrm{d}x + \int_1^\infty x^2 \mathrm{e}^{-x} \, \mathrm{d}x\Big) < \infty,
\end{aligned}$$

as asserted. ■

3.4 Remark. In connection with (3.1), note that it follows from Lemma 3.3 above that the integral

$$\int_0^\infty \Big(\int_{\mathbb{R}} u\big(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\big)\, D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x,$$

is well-defined. Indeed,

$$\int_0^\infty \Big(\int_{\mathbb{R}} \big|u\big(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\big)\big|\, D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x$$
$$= \int_0^\infty \Big(\int_{\mathbb{R}} \big|ux\big(1_{[-1,1]}(ux) - 1_{[-x,x]}(ux)\big)\big|\, \rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x.$$

4 Relation to free probability.

The purpose of this section is to prove the following

4.1 Theorem. *For any $\mu \in \mathcal{ID}(*)$ we have*

$$C_{\Upsilon(\mu)}(\zeta) = \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) = \int_0^\infty C_\mu(\zeta x)\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (\zeta \in \,]-\infty,0[). \tag{4.1}$$

In particular, any free cumulant function of an element in $\mathcal{ID}(\boxplus)$ is, in fact, identical to a classical cumulant function of an element of $\mathcal{ID}(*)$. Furthermore, the second equality in (4.1) provides an alternative, more direct, way of passing from the measure μ to its free counterpart, $\Lambda(\mu)$, without passing through the Lévy-Khintchine representations. This way is often quite effective, when it comes to calculating $\Lambda(\mu)$ for specific examples of μ.

In order to prove Theorem 4.1, we first need the following technical result:

4.2 Lemma. *Let ρ be a Lévy measure on $\mathbb{R}$. Then for any number ζ in $]-\infty,0[$, we have that*

$$\int_0^\infty \Big(\int_{\mathbb{R}} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\, \rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x < \infty.$$

Proof. Let ζ from $]-\infty,0[$ and x in $[0,\infty[$ be given. Note first that

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\, \rho(\mathrm{d}t) &= \int_{\mathbb{R}\setminus[-1,1]} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1\big|\, \rho(\mathrm{d}t)\\ &\le 2\int_{\mathbb{R}\setminus[-1,1]} \min\{1,t^2\}\rho(\mathrm{d}t)\\ &\le 2\int_{\mathbb{R}} \min\{1,t^2\}\rho(\mathrm{d}t).\end{aligned}$$

To estimate $\int_{-1}^{1} |\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx|\,\rho(\mathrm{d}t)$, note that for any real number t, we have by second order Taylor expansion that

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx &= (\cos(\zeta tx) - 1) + \mathrm{i}(\sin(\zeta tx) - \zeta tx)\\ &= -\frac{1}{2}\cos(\xi_1)(\zeta tx)^2 - \frac{1}{2}\mathrm{i}\sin(\xi_2)(\zeta tx)^2\\ &= -\frac{1}{2}\big[\cos(\xi_1) + \mathrm{i}\sin(\xi_2)\big](\zeta tx)^2,\end{aligned}$$

for suitable numbers ξ_1 and ξ_2 in the interval between 0 and ζtx. In particular, it follows that

$$\big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx\big| \leq \frac{1}{2}\big(1^2 + 1^2\big)^{1/2}(\zeta tx)^2 = \frac{1}{\sqrt{2}}(\zeta tx)^2,$$

for any real number t, and hence

$$\int_{-1}^{1} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx\big|\,\rho(\mathrm{d}t) \leq \frac{1}{\sqrt{2}}(\zeta x)^2 \int_{-1}^{1} t^2\,\rho(\mathrm{d}t) \leq \frac{1}{\sqrt{2}}(\zeta x)^2 \int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t).$$

Altogether, we find that for any number x in $[0, \infty[$,

$$\int_{\mathbb{R}} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\,\rho(\mathrm{d}t) \leq \Big(2 + \frac{1}{\sqrt{2}}(\zeta x)^2\Big) \int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t),$$

and therefore

$$\begin{aligned}&\int_0^\infty \Big(\int_{\mathbb{R}} \big|\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big|\,\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x\\ &\qquad \leq \int_{\mathbb{R}} \min\{1, t^2\}\,\rho(\mathrm{d}t) \int_0^\infty \Big(2 + \frac{1}{\sqrt{2}}(\zeta x)^2\Big)\mathrm{e}^{-x}\,\mathrm{d}x < \infty,\end{aligned}$$

as desired. ■

We are now ready to prove the key property of the mapping Υ.

Proof of Theorem 4.1. Recall first that for any $z \in \mathbb{C}$ with $\mathrm{Re}z < 1$ we have

$$\frac{1}{1-z} = \int_0^\infty \mathrm{e}^{zx}\mathrm{e}^{-x}\mathrm{d}x$$

implying that for ζ real with $\zeta \leq 0$

$$\frac{1}{1 - \mathrm{i}\zeta t} - 1 - \mathrm{i}\zeta t 1_{[-1,1]}(t) = \int_0^\infty \big(\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big)\,\mathrm{e}^{-x}\mathrm{d}x \tag{4.2}$$

Now, let μ from $\mathcal{ID}(*)$ be given and let (a, ρ, η) be the generating triplet for μ. Then by the definition of Λ and (4.2) above, we find for ζ in $]-\infty, 0[$ that

$$\begin{aligned}\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) &= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_{\mathbb{R}} \Big(\frac{1}{1 - \mathrm{i}\zeta t} - 1 - \mathrm{i}\zeta t 1_{[-1,1]}(t)\Big)\,\rho(\mathrm{d}t)\\ &= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_{\mathbb{R}}\int_0^\infty \big(\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big)\,\mathrm{e}^{-x}\mathrm{d}x\rho(\mathrm{d}t) \qquad (4.3)\\ &= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_0^\infty \mathrm{e}^{-x}\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta tx} - 1 - \mathrm{i}\zeta tx 1_{[-1,1]}(t)\big)\,\rho(\mathrm{d}t)\mathrm{d}x,\end{aligned}$$

where we have changed the order of integration in accordance with Lemma 4.2.

We first establish the second equation in (4.1). Recalling the formula

$$n! = \int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x, \qquad (n \in \mathbb{N}).$$

and using this for $n = 1$ and $n = 2$, we find that

$$\begin{aligned}
\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) &= \int_0^\infty \mathrm{e}^{-x}\Big(-\frac{1}{2}a\zeta^2 x^2 + \mathrm{i}\eta\zeta x + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta t x} - 1 - \mathrm{i}\zeta t x 1_{[-1,1]}(t)\big)\ \rho(\mathrm{d}t)\Big)\,\mathrm{d}x, \\
&= \int_0^\infty \mathrm{e}^{-x} C_\mu(\zeta x)\mathrm{d}x,
\end{aligned}$$

which proves the second equation in (4.1).

To prove the first equation in (4.1), we continue the calculation in (4.3) in another way:

$$\begin{aligned}
\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) &= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta t x} - 1 - \mathrm{i}\zeta t x 1_{[-1,1]}(t)\big)\ \rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x, \\
&= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta t x} - 1 - \mathrm{i}\zeta t x 1_{[-x,x]}(tx)\big)\ \rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x, \\
&= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-x,x]}(u)\big)\ D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\mathrm{d}x \\
&= -a\zeta^2 + \mathrm{i}\eta\zeta + \int_0^\infty \Big(\int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-1,1]}(u)\big)\ D_x\rho(\mathrm{d}u)\Big)\,\mathrm{e}^{-x}\mathrm{d}x \\
&\qquad + \mathrm{i}\zeta\Big[\int_0^\infty \Big(\int_{\mathbb{R}} u\big(1_{[-1,1]}(u) - 1_{[-x,x]}(u)\big)\, D_x\rho(\mathrm{d}u)\Big)\mathrm{e}^{-x}\,\mathrm{d}x\Big] \\
&= -a\zeta^2 + i\tilde{\eta}\zeta + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\zeta u} - 1 - \mathrm{i}\zeta u 1_{[-1,1]}(u)\big)\,\tilde{\rho}(\mathrm{d}u),
\end{aligned}$$

where (cf. Remark 3.4)

$$\tilde{\eta} = \eta + \int_0^\infty \Big(\int_{\mathbb{R}} t\,\big(1_{[-1,1]}(t) - 1_{[-x,x]}(t)\big)\ D_x\rho(\mathrm{d}t)\Big)\mathrm{e}^{-x}\,\mathrm{d}x$$

and

$$\tilde{\rho} = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\mathrm{d}x.$$

Comparing with Definition 3.1, the calculation above shows that $\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) = C_{\Upsilon(\mu)}(\zeta)$. This concludes the proof. ■

5 Properties of the mapping Υ.

The following proposition lists a number of properties of the mapping Υ. These properties are all consequences of the corresponding properties for Λ (cf. Theorem 2.5) together with the key property (4.1) of Υ.

5.1 Proposition. *The mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ has the following properties:*

(i) Υ is injective.

(ii) For any measures μ, ν in $\mathcal{ID}(*)$, $\Upsilon(\mu * \nu) = \Upsilon(\mu) * \Upsilon(\nu)$.

(iii) For any measure μ in $\mathcal{ID}(*)$ and any constant c in $\mathbb{R}$, $\Upsilon(D_c\mu) = D_c\Upsilon(\mu)$.

(iv) For any constant c in $\mathbb{R}$, $\Upsilon(\delta_c) = \delta_c$.

(v) Υ is continuous w.r.t. weak convergence.

Proof. (i) Assume that $\mu, \nu \in \mathcal{ID}(*)$ and that $\Upsilon(\mu) = \Upsilon(\nu)$. Then, by (4.1), $\mathcal{C}_{\Lambda(\mu)} = \mathcal{C}_{\Lambda(\nu)}$, and hence $\Lambda(\mu) = \Lambda(\nu)$ (cf. Section 2). Since Λ is one-to-one (cf. Section 2), this implies that $\mu = \nu$, as desired.

(ii) Consider measures μ, ν from $\mathcal{ID}(*)$. Then for any negative number ζ,

$$\begin{aligned} C_{\Upsilon(\mu*\nu)}(\zeta) &= \mathcal{C}_{\Lambda(\mu*\nu)}(\mathrm{i}\zeta) = \mathcal{C}_{\Lambda(\mu)\boxplus\Lambda(\nu)}(\mathrm{i}\zeta) = \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta) + \mathcal{C}_{\Lambda(\nu)}(\mathrm{i}\zeta) \\ &= C_{\Upsilon(\mu)}(\zeta) + C_{\Upsilon(\nu)}(\zeta) = C_{\Upsilon(\mu)*\Upsilon(\nu)}(\zeta), \end{aligned}$$

and hence, denoting by f_σ the characteristic function of a probability measure σ,

$$f_{\Upsilon(\mu*\nu)}(\zeta) = \exp(C_{\Upsilon(\mu*\nu)}(\zeta)) = \exp(C_{\Upsilon(\mu)*\Upsilon(\nu)}(\zeta)) = f_{\Upsilon(\mu)*\Upsilon(\nu)}(\zeta).$$

If $\zeta > 0$, then

$$f_{\Upsilon(\mu*\nu)}(\zeta) = \overline{f_{\Upsilon(\mu*\nu)}(-\zeta)} = \overline{f_{\Upsilon(\mu)*\Upsilon(\nu)}(-\zeta)} = f_{\Upsilon(\mu)*\Upsilon(\nu)}(\zeta).$$

Thus, the characteristic functions $f_{\Upsilon(\mu*\nu)}$ and $f_{\Upsilon(\mu)*\Upsilon(\nu)}$ coincide on $\mathbb{R}\setminus\{0\}$, and, of course, also at 0. This implies that $\Upsilon(\mu * \nu) = \Upsilon(\mu) * \Upsilon(\nu)$, as desired.

(iii) Consider a measure μ in $\mathcal{ID}(*)$ and a constant c in $\mathbb{R}$. Then for any negative number ζ,

$$C_{\Upsilon(D_c\mu)}(\zeta) = \mathcal{C}_{\Lambda(D_c\mu)}(\mathrm{i}\zeta) = \mathcal{C}_{D_c\Lambda(\mu)}(\mathrm{i}\zeta) = \mathcal{C}_{\Lambda(\mu)}(ic\zeta) = C_{\Upsilon(\mu)}(c\zeta) = C_{D_c\Upsilon(\mu)}(\zeta).$$

Arguing now exactly as in the proof of (ii), we may conclude that $\Upsilon(D_c\mu) = D_c\Upsilon(\mu)$, as desired.

(iv) Let c be a real constant. Then δ_c has generating triplet $(0, 0, c)$, and it follows immediately from the definition of Υ, that $\Upsilon(\delta_c)$ has generating triplet $(0, 0, c)$ too. Hence $\Upsilon(\delta_c) = \delta_c$.

(v) Let $\mu, \mu_1, \mu_2, \mu_3, \ldots$, be probability measures in $\mathcal{ID}(*)$, such that $\mu_n \xrightarrow{\mathrm{w}} \mu$, as $n \to \infty$. We need to show that $\Upsilon(\mu_n) \xrightarrow{\mathrm{w}} \Upsilon(\mu)$ as $n \to \infty$. Since Λ is continuous w.r.t. weak convergence, $\Lambda(\mu_n) \xrightarrow{\mathrm{w}} \Lambda(\mu)$, as $n \to \infty$, and this implies that $\mathcal{C}_{\Lambda(\mu_n)}(\mathrm{i}\zeta) \to \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}\zeta)$, as $n \to \infty$, for any ζ in $]-\infty, 0[$ (cf. e.g. [BT1, Theorem 3.8]). Thus,

$$C_{\Upsilon(\mu_n)}(\zeta) = \mathcal{C}_{\Lambda(\mu_n)}(\mathrm{i}\zeta) \underset{n\to\infty}{\longrightarrow} \mathcal{C}_{\mu}(\mathrm{i}\zeta) = C_{\Upsilon(\mu)}(\zeta),$$

for any negative number ζ, and hence also $f_{\Upsilon(\mu_n)}(\zeta) = \exp(C_{\Upsilon(\mu_n)}(\zeta)) \to \exp(C_{\Upsilon(\mu)}(\zeta)) = f_{\Upsilon(\mu)}(\zeta)$, as $n \to \infty$, for such ζ. Applying now complex conjugation, as in the proof of (ii), it follows that $f_{\Upsilon(\mu_n)}(\zeta) \to f_{\Upsilon(\mu)}(\zeta)$, as $n \to \infty$, for any (non-zero) ζ, and this means that $\Upsilon(\mu_n) \xrightarrow{\text{w}} \Upsilon(\mu)$, as $n \to \infty$. ■

5.2 Corollary. *The mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ preserves stability and selfdecomposability. In other words, if $\mathcal{S}(*)$ and $\mathcal{L}(*)$ denote, respectively, the classes of stable and of selfdecomposable probability measures on $\mathbb{R}$ (in the classical sense), then*

$$\Upsilon(\mathcal{S}(*)) \subseteq \mathcal{S}(*) \quad \textit{and} \quad \Upsilon(\mathcal{L}(*)) \subseteq \mathcal{L}(*).$$

Proof. Suppose $\mu \in \mathcal{S}(*)$ and that $c, c' > 0$ and $d, d' \in \mathbb{R}$. Then

$$(D_c\mu * \delta_d) * (D_{c'}\mu * \delta_{d'}) = D_{c''}\mu * \delta_{d''},$$

for suitable c'' in $]0,\infty[$ and d'' in $\mathbb{R}$. Using now (ii)-(iv) of Proposition 5.1, we find that

$$\begin{aligned}\big(D_c\Upsilon(\mu) * \delta_d\big) * \big(D_{c'}\Upsilon(\mu) * \delta_{d'}\big) &= \big(\Upsilon(D_c\mu) * \Upsilon(\delta_d)\big) * \big(\Upsilon(D_{c'}\mu) * \Upsilon(\delta_{d'})\big)\\ &= \Upsilon(D_c\mu * \delta_d) * \Upsilon(D_{c'}\mu * \delta_{d'})\\ &= \Upsilon\big((D_c\mu * \delta_d) * (D_{c'}\mu * \delta_{d'})\big)\\ &= \Upsilon\big(D_{c''}\mu * \delta_{d''})\big)\\ &= D_{c''}\Upsilon(\mu) * \delta_{d''},\end{aligned}$$

which shows that $\Upsilon(\mu) \in \mathcal{S}(*)$.

Assume next that $\mu \in \mathcal{L}(*)$. Then for any c in $\mathbb{R}$, there exists a measure μ_c in $\mathcal{ID}(*)$, such that $\mu = D_c\mu * \mu_c$. Using now (ii)-(iii) of Proposition 5.1, we find that

$$\Upsilon(\mu) = \Upsilon(D_c\mu * \mu_c) = \Upsilon(D_c\mu) * \Upsilon(\mu_c) = D_c\Upsilon(\mu) * \Upsilon(\mu_c),$$

which shows that $\Upsilon(\mu) \in \mathcal{L}(*)$. ■

5.3 Remark. The mapping $\Upsilon\colon \mathcal{ID}(*) \to \mathcal{ID}(*)$ *is not surjective*. For example, the (classical) Poisson distributions are not in the image of Υ. Indeed, the generating triplet for the Poisson distribution with mean $c > 0$ is $(0, c\delta_1, c)$, so it suffices to verify that δ_1 cannot appear as the Lévy measure for any distribution in the image of Υ. This can easily be seen directly, but it follows also immediately from Section 7 below, where we show, in particular, that the Lévy measure of a distribution in the image of Υ is always absolutely continuous w.r.t. Lebesgue measure, with a C^∞ density on $\mathbb{R} \setminus \{0\}$.

6 Stochastic interpretation of Υ.

The purpose of this section is to show that for any measure μ in $\mathcal{ID}(*)$, the measure $\Upsilon(\mu)$ can be realized as the distribution of a canonical stochastic integral w.r.t. to the (classical) Lévy process corresponding to μ.

We start with a general discussion of the existence of stochastic integrals w.r.t. (classical) Lévy processes, and their associated cumulant functions. Some related results are given in [CS] and [Sa2], but they do not fully cover the situation considered below.

Throughout this section, we shall use the notation $C\{u \ddagger X\}$ to denote the (classical) cumulant function of (the distribution of) the random variable X, evaluated at the real number u.

6.1 Lemma. *Let $(X_{n,m})_{n,m\in\mathbb{N}}$ be a family of random variables indexed by $\mathbb{N}\times\mathbb{N}$ and all defined on the same probability space $(\Omega,\mathcal{F},P)$. Assume that*

$$\forall u\in\mathbb{R}\colon \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu}\, L\{X_{n,m}\}(\mathrm{d}t) \to 1, \quad \text{as } n,m\to\infty. \tag{6.1}$$

Then $X_{n,m} \xrightarrow{\mathrm{p}} 0$, as $n,m\to\infty$, in the sense that

$$\forall \epsilon>0\colon P(|X_{n,m}|>\epsilon)\to 0, \quad \text{as } n,m\to\infty. \tag{6.2}$$

Proof. This is, of course, a variant of the usual continuity theorem for characteristic functions. For completeness, we include a proof.

To prove (6.2), it suffices, by a standard argument, to prove that $L\{X_{n,m}\} \xrightarrow{\mathrm{w}} \delta_0$, as $n,m\to\infty$, i.e. that

$$\forall f\in C_b(\mathbb{R})\colon \int_{\mathbb{R}} f(t)\, L\{X_{n,m}\}(\mathrm{d}t) \longrightarrow \int_{\mathbb{R}} f(t)\,\delta_0(\mathrm{d}t) = f(0), \quad \text{as } n,m\to\infty, \tag{6.3}$$

where $C_b(\mathbb{R})$ denotes the space of continuous bounded functions $f\colon\mathbb{R}\to\mathbb{R}$.

So assume that (6.3) is not satisfied. Then we may choose f in $C_b(\mathbb{R})$ and ϵ in $]0,\infty[$ such that

$$\forall N\in\mathbb{N}\ \exists n,m\geq N\colon \Big|\int_{\mathbb{R}} f(t)\, L\{X_{n,m}\}(\mathrm{d}t) - f(0)\Big| \geq \epsilon.$$

By an inductive argument, we may choose a sequence $n_1<n_2<n_3<\cdots$, of positive integers, such that

$$\forall k\in\mathbb{N}\colon \Big|\int_{\mathbb{R}} f(t)\, L\{X_{n_{2k},n_{2k-1}}\}(\mathrm{d}t) - f(0)\Big| \geq \epsilon. \tag{6.4}$$

On the other hand, it follows from (6.1) that

$$\forall u\in\mathbb{R}\colon \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tu}\, L\{X_{n_{2k},n_{2k-1}}\}(\mathrm{d}t) \to 1, \quad \text{as } k\to\infty,$$

so by the usual continuity theorem for characteristic functions, $L\{X_{n_{2k},n_{2k-1}}\} \xrightarrow{\mathrm{w}} \delta_0$. But this contradicts (6.4). ■

6.2 Lemma. *Assume that $0\leq a<b<\infty$, and let $f\colon[a,b]\to\mathbb{R}$ be a continuous function. Let, further, $(X_t)_{t\geq 0}$ be a (classical) Lévy process, and put $\mu = L\{X_1\}$. Then* *the stochastic integral $\int_a^b f(t)\,\mathrm{d}X_t$ exists as the limit, in probability, of approximating Riemann sums. Furthermore, $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$, and*

$$C\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\} = \int_a^b C_\mu(uf(t))\,\mathrm{d}t,$$

for all u in $\mathbb{R}$.

Proof. This is essentially well-known, but, for completeness, we sketch the proof: By definition (cf. [Lu2]), $\int_a^b f(t)\,\mathrm{d}X_t$ is the limit in probability of the Riemann sums:

$$R_n := \sum_{j=1}^n f(t_j^{(n)})\big(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}}\big),$$

where, for each n, $a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b$ is a subdivision of $[a,b]$, such that $\max_{j=1,2,\ldots,n}(t_j^{(n)} - t_{j-1}^{(n)}) \to 0$ as $n \to \infty$. Since (X_t) has stationary, independent increments, it follows that for any u in $\mathbb{R}$,

$$\begin{aligned} C\{u \ddagger R_n\} &= \sum_{j=1}^n C\big\{f(t_j^{(n)})u \ddagger \big(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}}\big)\big\} = \sum_{j=1}^n C\big\{f(t_j^{(n)})u \ddagger X_{t_j^{(n)} - t_{j-1}^{(n)}}\big\} \\ &= \sum_{j=1}^n C_\mu\big(f(t_j^{(n)})u\big)\cdot(t_j^{(n)} - t_{j-1}^{(n)}), \end{aligned}$$

where, in the last equality, we used [Sa1, Theorem 7.10]. Since C_μ and f are both continuous, it follows that

$$C\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\} = \lim_{n\to\infty}\displaystyle\sum_{j=1}^n C_\mu\big(f(t_j^{(n)})u\big)\cdot(t_j^{(n)} - t_{j-1}^{(n)}) = \int_a^b C_\mu(f(t)u)\,\mathrm{d}t,$$

for any u in $\mathbb{R}$. ■

The following result extends Lemma 6.2 to continuous functions $f\colon]a,b[\to \mathbb{R}$, where $0 \le a < b \le \infty$. In particular, we no longer require that f be bounded at the endpoints a and b. Moreover, b is allowed to equal ∞.

6.3 Proposition. *Assume that $0 \le a < b \le \infty$, and let $f\colon]a,b[\to \mathbb{R}$ be a continuous function. Let, further, $(X_t)_{t\ge 0}$ be a classical Lévy process, and put $\mu = L\{X_1\}$. Assume that*

$$\forall u \in \mathbb{R}\colon \int_a^b \big|C_\mu(uf(t))\big|\,\mathrm{d}t < \infty.$$

Then the stochastic integral $\int_a^b f(t)\,\mathrm{d}X_t$ exists as the limit, in probability, of the sequence $(\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t)_{n\in\mathbb{N}}$, where (a_n) and (b_n) are arbitrary sequences in $]a,b[$ such that $a_n \le b_n$ for all n and $a_n \searrow a$ and $b_n \nearrow b$ as $n \to \infty$.

Furthermore, $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$ *and*

$$C\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\} = \int_a^b C_\mu(uf(t))\,\mathrm{d}t, \tag{6.5}$$

for all u *in* $\mathbb{R}$.

Proof. Let (a_n) and (b_n) be arbitrary sequences in $]a,b[$, such that $a_n \le b_n$ for all n and $a_n \searrow a$ and $b_n \nearrow b$ as $n \to \infty$. Then, for each n, consider the stochastic integral $\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$. Since the topology corresponding to convergence in probability is complete (cf. [BT1]), the convergence of the sequence $(\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t)_{n\in\mathbb{N}}$ will follow, once we have verified that it is a Cauchy sequence. Towards this end, note that whenever $n > m$ we have that

$$\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t - \int_{a_m}^{b_m} f(t)\,\mathrm{d}X_t = \int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t + \int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t,$$

so it suffices to show that

$$\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t \xrightarrow{\text{p}} 0 \quad \text{and} \quad \int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t \xrightarrow{\text{p}} 0, \quad \text{as } n,m \to \infty.$$

By Lemma 6.1, this, in turn, will follow if we prove that

$$\forall u \in \mathbb{R}\colon C\{u \ddagger \textstyle\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t\} \longrightarrow 0, \qquad \text{as } n,m \to \infty, \tag{6.6}$$

and

$$\forall u \in \mathbb{R}\colon C\{u \ddagger \textstyle\int_{b_m}^{b_n} f(t)\,\mathrm{d}X_t\} \longrightarrow 0, \qquad \text{as } n,m \to \infty. \tag{6.7}$$

But for n,m in $\mathbb{N}$, $m < n$, it follows from Lemma 6.2 that

$$\left|C\{u \ddagger \textstyle\int_{a_n}^{a_m} f(t)\,\mathrm{d}X_t\}\right| \le \int_{a_n}^{a_m} \left|C_\mu(uf(t))\right|\mathrm{d}t, \tag{6.8}$$

and since $\int_a^a |C_\mu(uf(t))|\,\mathrm{d}t < \infty$, the right hand side of (6.8) tends to 0 as $n,m \to \infty$. Statement (6.7) follows similarly.

To prove that $\lim_{n\to\infty} \int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$ does not depend on the choice of sequences (a_n) and (b_n), let (a'_n) and (b'_n) be sequences in $]a,b[$, also satisfying that $a'_n \le b'_n$ for all n, and that $a'_n \searrow a$ and $b'_n \nearrow b$ as $n \to \infty$. We may then, by an inductive argument, choose sequences $n_1 < n_2 < n_3 < \cdots$ and $m_1 < m_2 < m_3 \cdots$ of positive integers, such that

$$a_{n_1} > a'_{m_1} > a_{n_2} > a'_{m_2} > \cdots, \quad \text{and} \quad b_{n_1} < b'_{m_1} < b_{n_2} < b'_{m_2} < \cdots.$$

Consider then the sequences (a''_k) and (b''_k) given by:

$$a''_{2k-1} = a_{n_k},\ a''_{2k} = a'_{m_k}, \quad \text{and} \quad b''_{2k-1} = b_{n_k},\ b''_{2k} = b'_{m_k}, \qquad (k \in \mathbb{N}).$$

Then $a''_k \le b''_k$ for all k, and $a''_k \searrow a$ and $b''_k \nearrow b$ as $k \to \infty$. Thus, by the argument given above, all of the following limits exist (in probability), and, by "sub-sequence considerations", they have to be equal:

$$\begin{aligned}\lim_{n\to\infty}\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t &= \lim_{k\to\infty}\int_{a_{n_k}}^{b_{n_k}} f(t)\,\mathrm{d}X_t = \lim_{k\to\infty}\int_{a''_{2k-1}}^{b''_{2k-1}} f(t)\,\mathrm{d}X_t \\ &= \lim_{k\to\infty}\int_{a''_k}^{b''_k} f(t)\,\mathrm{d}X_t = \lim_{k\to\infty}\int_{a''_{2k}}^{b''_{2k}} f(t)\,\mathrm{d}X_t = \lim_{k\to\infty}\int_{a'_{m_k}}^{b'_{m_k}} f(t)\,\mathrm{d}X_t \\ &= \lim_{n\to\infty}\int_{a'_n}^{b'_n} f(t)\,\mathrm{d}X_t,\end{aligned}$$

as desired.

To verify, finally, the last statements of the proposition, let (a_n) and (b_n) be sequences as above, so that, by definition, $\int_a^b f(t)\,\mathrm{d}X_t = \lim_{n\to\infty}\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t$ in probability. Since $\mathcal{ID}(*)$ is closed under weak convergence, this implies that $L\{\int_a^b f(t)\,\mathrm{d}X_t\} \in \mathcal{ID}(*)$. To prove (6.5), we use Gnedenko's theorem (cf. [GK, §19, Theorem 1]), which expresses weak convergence of measures in $\mathcal{ID}(*)$ in terms of their generating triplets[1]. This result implies, in particular, that weak convergence of measures in $\mathcal{ID}(*)$ implies point-wise convergence of their cumulant transforms. Combining the latter result with Lemma 6.2, we find that

$$\begin{aligned}C\big\{u \ddagger \textstyle\int_a^b f(t)\,\mathrm{d}X_t\big\} &= \lim_{n\to\infty} C\big\{u \ddagger \textstyle\int_{a_n}^{b_n} f(t)\,\mathrm{d}X_t\big\} \\ &= \lim_{n\to\infty}\int_{a_n}^{b_n} C_\mu(uf(t))\,\mathrm{d}t = \int_a^b C_\mu(uf(t))\,\mathrm{d}t,\end{aligned}$$

for any u in $\mathbb{R}$, and where the last equality follows from the assumption that $\int_a^b |C_\mu(uf(t))|\,\mathrm{d}t < \infty$. This concludes the proof. ■

6.4 Theorem. *Let μ be an arbitrary measure in $\mathcal{ID}(*)$, and let (X_t) be a (classical) Lévy process (in law), such that $L\{X_1\} = \mu$. Then the stochastic integral*

$$Z = \int_0^1 -\log(1-t)\,\mathrm{d}X_t$$

exists, as the limit in probability, of the stochastic integrals $\int_0^{1-1/n} -\log(1-t)\,\mathrm{d}X_t$, as $n \to \infty$. Furthermore, the distribution of Z is exactly $\Upsilon(\mu)$.

Proof. The existence of the stochastic integral $\int_0^1 -\log(1-t)\,\mathrm{d}X_t$ follows from Proposition 6.3, once we have verified that $\int_0^1 |C_\mu(-u\log(1-t))|\,\mathrm{d}t < \infty$, for any u in $\mathbb{R}$. Using the change of variable: $t = 1-\mathrm{e}^{-x}$, $x \in \mathbb{R}$, we find that

$$\int_0^1 \big|C_\mu(-u\log(1-t))\big|\,\mathrm{d}t = \int_0^\infty \big|C_\mu(ux)\big|\mathrm{e}^{-x}\,\mathrm{d}x,$$

[1] [GK, §19, Theorem 1] is formulated in terms of generating pairs rather than generating triplets.

and here the right hand side is finite, according to Lemma 4.2.

Combining, next, Proposition 6.3 and Theorem 4.1, we find for any u in $\mathbb{R}$ that

$$C_{L\{Z\}}(u) = \int_0^1 C_\mu(-u\log(1-t))\,\mathrm{d}t = \int_0^\infty C_\mu(ux)\mathrm{e}^{-x}\,\mathrm{d}x = C_{\Upsilon(\mu)}(u),$$

which implies that $L\{Z\} = \Upsilon(\mu)$, as desired. ■

Theorem 6.4 together with Theorem 4.1 show that for any measure μ in $\mathcal{ID}(*)$, the free cumulant transform of the measure $\Lambda(\mu)$ is, up to multiplication of the variable by i, equal to the classical cumulant transform of the stochastic integral $\int_0^1 -\log(1-t)\,\mathrm{d}X_t$, where (X_t) is a classical Lévy process (in law), such that $L\{X_1\} = \mu$.

7 Laplace transform characterisation of the Lévy measure for $\Upsilon(\mu)$.

We proceed to show that for any Lévy measure ρ of a distribution μ in $\mathcal{ID}(*)$ the Lévy measure $\tilde{\rho}$ of $\Upsilon(\mu)$ is absolutely continuous with a density $\tilde{r}$, which is a C^∞-function on $\mathbb{R}\setminus\{0\}$. We prove, in fact, that the restrictions $\tilde{r}_{|]-\infty,0[}$ and $\tilde{r}_{|]0,\infty[}$ can be represented as the Laplace transforms of measures concentrated on $]-\infty,0[$ and $]0,\infty[$, respectively (cf. Theorem 7.2 below). The proof we give does not involve concepts of free infinite divisibility and the result is of some independent interest as it establishes an injective transform of Lévy measures to Lévy measures, which has smoothing properties akin to those of the ordinary Laplace transform.

Throughout this section we consider a Lévy measure ρ on $\mathbb{R}$, and, as before, we put

$$\tilde{\rho} = \int_0^\infty (D_x\rho)\mathrm{e}^{-x}\,\mathrm{d}x. \tag{7.1}$$

Furthermore, we let α denote the transformation of ρ by the mapping $x \mapsto x^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$ (recall that $\rho(\{0\}) = 0$). Note that α satisfies the properties:

$$\alpha(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, s^{-2}\}\,\alpha(\mathrm{d}s) < \infty. \tag{7.2}$$

7.1 Proposition. *With the notation introduced above, we have*

$$\tilde{\rho}([t,\infty[) = \int_0^\infty \mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), \qquad (t \in]0,\infty[), \tag{7.3}$$

and

$$\tilde{\rho}(]-\infty,t]) = \int_{-\infty}^0 \mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), \qquad (t \in]-\infty,0[). \tag{7.4}$$

Proof. Let t in $]0,\infty[$ be given. Using the change of variables $x = ts$, $s > 0$, we find then that

$$\begin{aligned}\tilde{\rho}([t,\infty[) &= \int_0^\infty D_x\rho([t,\infty[)\mathrm{e}^{-x}\,\mathrm{d}x = \int_0^\infty \rho([x^{-1}t,\infty[)\mathrm{e}^{-x}\,\mathrm{d}x \\ &= t\int_0^\infty \rho([s^{-1},\infty[)\mathrm{e}^{-ts}\,\mathrm{d}s = t\int_0^\infty \alpha(]0,s])\mathrm{e}^{-ts}\,\mathrm{d}s.\end{aligned}$$

Next, using partial integration for Stieltjes integrals, we obtain

$$\tilde{\rho}([t,\infty[) = \Big[-\alpha(]0,s])\mathrm{e}^{-ts}\Big]_0^\infty + \int_0^\infty \mathrm{e}^{-ts}\,\alpha(\mathrm{d}s). \tag{7.5}$$

Note here that

$$\alpha(]0,s])\mathrm{e}^{-ts} = \mathrm{e}^{-\frac{1}{2}ts}\int_{]0,s]} \mathrm{e}^{-\frac{1}{2}ts}\alpha(\mathrm{d}u) \le \mathrm{e}^{-\frac{1}{2}ts}\int_{]0,s]} \mathrm{e}^{-\frac{1}{2}tu}\alpha(\mathrm{d}u) \longrightarrow 0, \quad \text{as } s\to\infty,$$

since $\int_{]0,\infty[}\mathrm{e}^{-\frac{1}{2}tu}\alpha(\mathrm{d}u) < \infty$ (cf. (7.2)). Note also that

$$\alpha(]0,s])\mathrm{e}^{-ts} \le \alpha(]0,s]) \longrightarrow 0, \quad \text{as } s \to 0,$$

since $\alpha(]0,1]) < \infty$. Altogether, $[-\alpha(]0,s])\mathrm{e}^{-ts}]_0^\infty = 0$, and inserting this in (7.5), we obtain (7.3). Formula (7.4) can be proved similarly or by applying (7.3) to the Lévy measure $D_{-1}\rho$. ■

7.2 Theorem. *For any Lévy measure ρ on $\mathbb{R}$, the Lévy measure $\tilde{\rho}$, given by (7.1), is absolutely continuous w.r.t. Lebesgue measure. The density $\tilde{r}$ is the C^∞-function on $\mathbb{R}\setminus\{0\}$ given by*

$$\tilde{r}(t) = \begin{cases} \int_0^\infty s\mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), & \text{if } t > 0,\\ \int_{-\infty}^0 -s\mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), & \text{if } t < 0,\end{cases}$$

where α is the transformation of ρ by the mapping $x \mapsto x^{-1}\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$.

Proof. Since $\tilde{\rho}$ is σ-finite, being a Lévy measure, it suffices to prove, for example, that

$$\tilde{\rho}([t_1,t_2[) = \int_{t_1}^{t_2} \tilde{r}(s)\,\mathrm{d}s, \tag{7.6}$$

for any t_1, t_2 in $\mathbb{R}$, satisfying either that $0 < t_1 < t_2$ or that $t_1 < t_2 < 0$. Note next that it follows from (7.2) that

$$\int_0^\infty s^p\mathrm{e}^{-ts}\,\alpha(\mathrm{d}s) < \infty \quad \text{and} \quad \int_{-\infty}^0 |s|^p\mathrm{e}^{ts}\,\alpha(\mathrm{d}s) < \infty,$$

for any t in $]0,\infty[$ and any $p \in \mathbb{N}$. Thus, by the well-known theorem on differentiation under the integral sign, the functions (cf. Proposition 7.1)

$$t \mapsto \tilde{\rho}([t,\infty[) = \int_0^\infty \mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), \quad (t \in]0,\infty[),$$

and

$$t \mapsto \tilde{\rho}(]-\infty,t]) = \int_{-\infty}^{0} \mathrm{e}^{-ts}\,\alpha(\mathrm{d}s), \quad (t \in]-\infty,0[),$$

are both C^∞-functions with first derivatives $-\tilde{r}(t)$ and $\tilde{r}(t)$, respectively. Now, if $t_1 < t_2 < 0$, we find that

$$\int_{t_1}^{t_2} \tilde{r}(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(]-\infty,t])\,\mathrm{d}t = \tilde{\rho}(]-\infty,t_2]) - \tilde{\rho}(]-\infty,t_1]) = \tilde{\rho}(]t_1,t_2]). \tag{7.7}$$

Since $\int_{t_1}^{t_2} \tilde{r}(t)\,\mathrm{d}t$ is continuous in t_1 and t_2, it follows from (7.7) that (7.6) holds too. If $0 < t_1 < t_2$, formula (7.6) can be established similarly. ■

7.3 Corollary. *Suppose ρ is a Lévy measure on $\mathbb{R}$ with density r w.r.t. Lebesgue measure. Then the Lévy measure $\tilde{\rho}$, given by* (7.1)*, has density $\tilde{r}(t)$ given by*

$$\tilde{r}(t) = \begin{cases} \int_0^\infty s^{-1} r(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s, & \text{if } t > 0, \\ \int_{-\infty}^0 -s^{-1} r(s^{-1})\mathrm{e}^{-ts}\,\mathrm{d}s, & \text{if } t < 0. \end{cases}$$

Proof. This follows immediately from Theorem 7.2 together with the fact that the measure α has density

$$s \mapsto s^{-2} r(s^{-1}), \qquad (s \in \mathbb{R} \setminus \{0\}),$$

w.r.t. Lebesgue measure. ■

7.4 Example. Suppose ρ is concentrated on the positive half-line $]0,\infty[$, and that it has the following density w.r.t. Lebesgue measure:

$$r(s) = s^{-1-\alpha}\mathrm{e}^{-\beta s}, \qquad (s > 0),$$

where $0 \le \alpha < 2$ and $0 \le \beta$. For the moment, let both α and β be positive. Then the density of $\tilde{\rho}$ is

$$\tilde{r}(t) = 2t^{-\frac{1+\alpha}{2}}\beta^{\frac{1+\alpha}{2}} K_{1+\alpha}(2\sqrt{\beta t}),$$

where K_v denotes a Bessel function. Since, for $\nu > 0$,

$$K_\nu(x) \sim \begin{cases} \Gamma(\nu)2^{\nu-1}x^{-\nu}, & \text{for } x \searrow 0, \\ \sqrt{\frac{\pi}{2}}x^{-1/2}\mathrm{e}^{-x}, & \text{for } x \to \infty, \end{cases}$$

we find that

$$\tilde{r}(t) \sim \begin{cases} \Gamma(\frac{1+\alpha}{2})t^{-1-\alpha}, & \text{for } \beta t \searrow 0, \\ \sqrt{\pi}\beta^{\frac{1+2\alpha}{4}} t^{-\frac{3+2\alpha}{4}}\mathrm{e}^{-\sqrt{\beta t}}, & \text{for } \beta t \to \infty. \end{cases}$$

Furthermore, if $\alpha > 0$ and $\beta = 0$, then

$$\tilde{r}(t) = \Gamma\Big(\frac{1+\alpha}{2}\Big)t^{-1-\alpha},$$

while, if $\alpha = 0$ and $\beta > 0$,
$$\tilde{r}(t) = 2t^{-\frac{1}{2}}\beta^{\frac{1}{2}}K_1(2\sqrt{\beta t}).$$
Thus, in particular, Υ maps any of the positive α-stable laws to a positive α-stable law.

An immediate extension of the arguments given above shows, furthermore, that if ρ is the Lévy measure of a general α-stable law, then so is $\tilde{\rho}$.

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