INVARIANT SUBSPACES OF THE QUASINILPOTENT DT–OPERATOR

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Abstract. In [4] we introduced the class of DT–operators, which are modeled by certain upper triangular random matrices, and showed that if the spectrum of a DT–operator is not reduced to a single point, then it has a nontrivial, closed, hyperinvariant subspace. In this paper, we prove that also every DT–operator whose spectrum is concentrated on a single point has a nontrivial, closed, hyperinvariant subspace. In fact, each such operator has a one–parameter family of them. It follows that every DT–operator generates the von Neumann algebra $L(\mathbf{F}_2)$ of the free group on two generators.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{H})$. An *invariant subspace* of A is a subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ such that $A(\mathcal{H}_0) \subseteq \mathcal{H}_0$, and a *hyperinvariant subspace* of A is a subspace $\mathcal{H}_0$ of $\mathcal{H}$ that is invariant for every operator $B \in \mathcal{B}(\mathcal{H})$ that commutes with A. A subspace of $\mathcal{H}$ is said to be *nontrivial* if it is neither $\{0\}$ nor $\mathcal{H}$ itself. The famous *invariant subspace problem* for Hilbert space asks whether every operator in $\mathcal{B}(\mathcal{H})$ has a closed, nontrivial, invariant subspace, and the *hyperinvariant subspace problem* asks whether every operator in $\mathcal{B}(\mathcal{H})$ that is not a scalar multiple of the identity operator has a closed, nontrivial, hyperinvariant subspace.

On the other hand, if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ is *affiliated to* $\mathcal{M}$ if the projection p from $\mathcal{H}$ onto $\mathcal{H}_0$ belongs to $\mathcal{M}$. It is not difficult to show that every closed, hyperinvariant subspace of A is affiliated to the von Neumann algebra, $W^*(A)$, generated by A. The question of whether every element of a von Neumann algebra $\mathcal{M}$ has a nontrivial invariant subspace affiliated to $\mathcal{M}$ is called the invariant subspace problem *relative to* the von Neumann algebra $\mathcal{M}$.

In [3], we began using upper triangular random matrices to study invariant subspaces for certain operators arising in free probability theory, including Voiculescu's circular operator. In the sequel [4], we introduced the DT–operators; these form a class of operators including all those studied in [3]. (We note that the DT–operators were defined in terms of approximation by upper triangular random matrices, and have been shown in [6] to solve a maxmimization problem for free entropy.) We showed that DT–operators are decomposable in the sense of Foiaş, which entails that those DT–operators whose spectra contain more than one point have nontrivial, closed, hyperinvariant subspaces. In this paper, we show that also DT–operators whose spectra are singletons have (a continuum of) closed, nontrivial, hyperinvariant subspaces. These operators are all scalar translates of scalar multiples of a single operator, the $\mathrm{DT}(\delta_0, 1)$–operator, which we will denote by T.

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The free group factor $L(\mathbf{F}_2) \subseteq \mathcal{B}(\mathcal{H})$ is generated by a semicircular element X and a free copy of $L^\infty[0,1]$, embedded via a normal $*$–homomorphism $\lambda : L^\infty[0,1] \to L(\mathbf{F}_2)$ such that $\tau \circ \lambda(f) = \int_0^1 f(t)dt$, where τ is the tracial state on $L(\mathbf{F}_2)$. Thus X and the image of λ are free with respect to τ and together they generate $L(\mathbf{F}_2)$. As proved in [4, §4], the $\mathrm{DT}(\delta_0, 1)$–operator T can be obtained by using projections from $\lambda(L^\infty[0,1])$ to cut out the "upper triangular part" of X; in the notation of [4, §4], $T = \mathcal{UT}(X, \lambda)$. It is clear from this construction that each of the subspaces $\mathcal{H}_t = \lambda(1_{[0,t]})\mathcal{H}$ is an invariant subspace of T. We will show that each of these subspaces is affiliated to $W^*(T)$ by proving $D_0 \in W^*(T)$, where $D_0 = \lambda(\mathrm{id}_{[0,1]})$ and $\mathrm{id}_{[0,1]}$ is the identity function from $[0,1]$ to itself. Since $X = T + T^*$, this will also imply $W^*(T) = L(\mathbf{F}_2)$. We will then show that each $\mathcal{H}_t$ is actually a hyperinvariant subspace of T, by characterizing $\mathcal{H}_t$ as the set of vectors $\xi \in \mathcal{H}$ such that $\|T^k\xi\|$ has a certain asymptotic property as $k \to \infty$.

2. Preliminaries and statement of results

In [4, §8], we showed that the distribution of T^*T is the probability measure μ on $[0,e]$ given by

$$d\mu(x) = \varphi(x)dx$$

where $\varphi\colon\ (0,e) \to \mathbf{R}^+$ is the function given uniquely by

$$\varphi\left(\frac{\sin v}{v}\exp(v\cot v)\right) = \frac{1}{\pi}\sin v\exp(-v\cot v), \qquad 0 < v < \pi. \tag{2.1}$$

Proposition 2.1. *Let $F(x) = \int_0^x \varphi(t)dt, x \in [0,e]$. Then*

$$F\left(\frac{\sin v}{v}\exp(v\cot v)\right) = 1 - \frac{v}{\pi} + \frac{1}{\pi}\frac{\sin^2 v}{v}, \qquad 0 < v < \pi. \tag{2.2}$$

Proof. From the proof of [4, Thm. 8.9] we have that

$$\sigma\colon\ v \mapsto \frac{\sin v}{v}\exp(v\cot v) \tag{2.3}$$

is a decreasing bijection from $(0,\pi)$ onto $(0,e)$. Hence

$$\begin{aligned}
F(\sigma(v)) &= \int_0^{\sigma(v)} \varphi(t)dt = -\int_v^\pi \varphi(\sigma(u))\sigma'(u)du \\
&= -[\varphi(\sigma(u))\sigma(u)]_v^\pi + \int_v^\pi \left(\frac{d}{du}\varphi(\sigma(u))\right)\sigma(u)du \\
&= -\frac{1}{\pi}\left[\frac{\sin^2 u}{u}\right]_v^\pi + \frac{1}{\pi}\int_v^\pi \frac{u}{\sin u}\cdot\frac{\sin u}{u}\,du = \frac{1}{\pi}\frac{\sin^2 v}{v} + 1 - \frac{v}{\pi}. \qquad \square
\end{aligned}$$

The following is the central result of this paper.

Theorem 2.2. *Let $S_k = k((T^k)^*T^k)^{\frac{1}{k}}$, $k = 1, 2, \ldots$. Then $\sigma(S_k) = [0,e]$ for all $k \in \mathbf{N}$ and*

$$\lim_{k\to\infty} \|F(S_k) - D_0\|_2 = 0 \quad \textit{for} \quad k \to \infty.$$

In particular $D_0 \in W^*(T)$. Therefore $\mathcal{H}_t = 1_{[0,t]}(D_0)\mathcal{H} = \lambda(1_{[0,t]})\mathcal{H}$, $0 < t < 1$ is a one-parameter family of nontrivial, closed, T-invariant subspaces affiliated with $W^*(T)$.

Corollary 2.3. *$W^*(T) \cong L(\mathbf{F}_2)$. Moreover, if Z is any DT–operator, then $W^*(Z) \cong L(\mathbf{F}_2)$.*

Proof. As described in the introduction, with $T = \mathfrak{UT}(X,\lambda) \in W^*(X \cup \lambda(L^\infty[0,1])) = L(\mathbf{F}_2)$, from Theorem 2.2 we have $D_0 \in W^*(T)$. Since clearly $X \in W^*(T)$, we have $W^*(T) = L(\mathbf{F}_2)$. By [4, Thm. 4.4], Z can be realized as $Z = D + cT$ for some $D \in \lambda(L^\infty[0,1])$ and $c > 0$. By [4, Lem. 6.2], $T \in W^*(Z)$, so $W^*(Z) = L(\mathbf{F}_2)$. □

We now outline the proof of Theorem 2.2. Let M be a factor of type II_1 with tracial state tr, and let $A, B \in M_{sa}$. By [1, §1], there is a unique probability measure $\mu_{A,B}$ on $\sigma(A) \times \sigma(B)$, such that for all bounded Borel functions f, g on $\sigma(A)$ and $\sigma(B)$, respectively, one has

$$\mathrm{tr}(f(A)g(B)) = \iint\limits_{\sigma(A)\times\sigma(B)} f(x)g(y)d\mu_{A,B}(x). \tag{2.4}$$

The following lemma is a simple consequence of (2.4) (cf. [1, Proposition 1.1]).

Lemma 2.4. *Let A, B and $\mu_{A,B}$ be as above, then for all bounded Borel functions f and g on $\sigma(A)$ and $\sigma(B)$, respectively,*

$$\|f(A) - g(B)\|_2^2 = \iint\limits_{\sigma(A)\times\sigma(B)} |f(x) - g(y)|^2 \, d\mu_{A,B}(x, y). \tag{2.5}$$

We shall need the following key result of Śniady [7]. Strictly speaking, the results of [7] concern an operator that can be described as a generalized circular operator with a given variance matrix. It's not entirely obvious that the operator T studied in [4] and in the present article is actually of this form. A proof is supplied in Appendix A below.

Theorem 2.5. [7, Thm. 5] *Let $E_{\mathcal{D}}$ be the trace preserving conditional expectation of $W^*(D_0, T)$ onto $\mathcal{D} = W^*(D_0)$, which we identify with $L^\infty[0,1]$ as in [7]. Let $k \in \mathbf{N}$ and let $(P_{k,n})_{n=0}^\infty$ be the sequence of polynomials in a real variable x determined by:*

$$P_{k,0}(x) = 1 \tag{2.6}$$

$$P_{k,n}^{(k)}(x) = P_{k,n-1}(x+1), \qquad n = 1, 2, \dots \tag{2.7}$$

$$P_{k,n}(0) = P'_{k,n}(0) = \dots = P_{k,n}^{(k-1)}(0) = 0, \qquad n = 1, 2, \dots \tag{2.8}$$

where $P_{k,n}^{(\ell)}$ denotes the ℓth derivative of $P_{k,n}$. Then for all $k, n \in \mathbf{N}$,

$$E_{\mathcal{D}}(((T^k)^*T^k)^n)(x) = P_{k,n}(x), \qquad x \in [0,1].$$

Remark 2.6. The above Theorem is equivalent to [7, Thm. 5] because

$$E_{\mathcal{D}}(((T^k)^*T^k)^n)(x) = E_{\mathcal{D}}((T^k(T^k)^*)^n)(1-x), \qquad x \in [0,1].$$

Śniady used Theorem 2.5 to prove the following formula, which was conjectured in [4, §9].

Theorem 2.7. [7, Thm. 7] *For all $n, k \in \mathbf{N}$:*

$$\mathrm{tr}(((T^k)^*T^k)^n) = \frac{n^{nk}}{(nk+1)!}\ . \tag{2.9}$$

Śniady proved that Theorem 2.5 implies Theorem 2.7 by a tricky and clever combinatorial argument. In the course of proving Theorem 2.2, we also obtained a purely analytic proof of Thm. 2.5 ⇒ Thm. 2.7 (see (3.2) and Remark 4.3). Note that it follows from Theorem 2.7 that $S_k^k = k^k(T^k)^*T^k$ has the same moments as $(T^*T)^k$. Hence the distribution measures

μ_{S_k} and μ_{T^*T} in Prob($\mathbf{R}$) are equal. In particular their supports are equal. Hence, by [4, Thm. 8.9],

$$\sigma(S_k) = \sigma(T^*T) = [0, e]. \tag{2.10}$$

We will use Theorem 2.5 to derive in Theorem 2.8 an explicit formula for the measure μ_{D_0,S_k} defined in (2.4). The formula involves Lambert's W function, which is defined as the multivalued inverse function of the function $\mathbf{C} \ni z \mapsto ze^z$. We define a function ρ by

$$\rho(z) = -W_0(-z), \quad z \in \mathbf{C}\backslash[\tfrac{1}{e}, \infty), \tag{2.11}$$

where W_0 is the principal branch of Lambert's W–function. By [2, §4], ρ is an analytic bijection of $\mathbf{C}\backslash[\frac{1}{e}, \infty)$ onto

$$\Omega = \{x + iy \mid -\pi < y < \pi,\, x < y \cot y\},$$

where we have used the convention $0 \cot 0 = 1$. Moreover, ρ is the inverse function of the function f defined by

$$f(w) = we^{-w}, \quad w \in \Omega.$$

Note that f maps the boundary of Ω onto $[\frac{1}{e}, \infty)$, because

$$f(\theta \cot \theta \pm i\theta) = f\left(\frac{\theta}{\sin\theta} e^{\pm i\theta}\right) = \frac{\theta}{\sin\theta} e^{-\theta\cot\theta} \tag{2.12}$$

and $\theta \mapsto \frac{\sin\theta}{\theta} e^{\theta\cot\theta}$ is a bijection of $(0, \pi)$ onto $(0, e)$ (see [4, §8]). By (2.12), it also follows that if we define functions $\rho^+, \rho^- : [\frac{1}{e}, \infty) \to \mathbf{C}$ by

$$\rho^\pm\left(\frac{\theta}{\sin\theta} e^{-\theta\cot\theta}\right) = \theta\cot\theta \pm i\theta, \qquad 0 \le \theta < \pi, \tag{2.13}$$

then

$$\rho^\pm(x) = \lim_{y \downarrow 0} \rho(x \pm iy), \qquad x \in [\tfrac{1}{e}, \infty).$$

In particular $\rho^+\left(\frac{1}{e}\right) = \rho^-\left(\frac{1}{e}\right) = 1$.

Theorem 2.8. *Let $k \in \mathbf{N}$ be fixed. Define for $t > \frac{1}{e}$ and $j = 0, \ldots, k$ the functions $a_j(t)$, $c_j(t)$ by*

$$\begin{cases} a_0(t) = \rho^+(t) \\ a_j(t) = \rho\left(t \exp\left(i\frac{2\pi j}{k}\right)\right), \quad 1 \le j \le k-1 \\ a_k(t) = \rho^-(t) \end{cases} \tag{2.14}$$

and

$$c_j(t) = -k a_j(t) \prod_{\ell \ne j} \frac{a_\ell(t)}{a_\ell(t) - a_j(t)}\,. \tag{2.15}$$

Then the probability measure μ_{D_0,S_k} on $\sigma(D_0) \times \sigma(S_k) = [0, 1] \times [0, e]$ is absolutely continuous with respect to the 2-dimensional Lebesgue measure and, with φ as in (2.1), has density

$$\frac{d\mu_{D_0,S_k}(x, y)}{dxdy} = \varphi(y)\left(\sum_{j=0}^{k} c_j(y^{-1}) e^{k a_j(y^{-1}) x}\right) \tag{2.16}$$

for $x \in (0, 1)$ and $y \in (0, e)$.

We will prove Theorem 2.2 by combining Lemma 2.4 and Theorem 2.8 (see Section 6).

Finally, we will prove the following characterization of the subspaces $\mathcal{H}_t$ (see Section 7).

Theorem 2.9. *For every $t \in [0,1]$,*

$$\mathcal{H}_t = \{\xi \in \mathcal{H} \mid \limsup_{n\to\infty} \big(\frac{k}{e}\|T^k\xi\|^{2/k}\big) \le t\}. \tag{2.17}$$

In particular, $\mathcal{H}_t$ is a closed, hyperinvariant subspace of T.

3. Proof of Theorem 2.8 for $k = 1$

This section is devoted to the proof of Theorem 2.8 in the special case $k = 1$, which is somewhat easier than in the general case. For $k = 1$ it is easy to solve equations (2.6)–(2.8) explicitly to obtain

$$P_{1,n}(x) = \frac{1}{n!}x(x+n)^{n-1}, \qquad (n \ge 1). \tag{3.1}$$

From (3.1) one immediately gets (2.9) for $k = 1$, because

$$\mathrm{tr}((T^*T)^n) = \int_0^1 P_{1,n}(x)dx = \left[\frac{1}{(n+1)!}(x-1)(x+n)^n\right]_0^1 = \frac{n^n}{(n+1)!}\,. \tag{3.2}$$

Lemma 3.1. *For $x \in \mathbf{R}$ and $z \in \mathbf{C}$, $|z| < \frac{1}{e}$, one has*

$$\sum_{n=0}^{\infty} P_{1,n}(x)z^n = e^{\rho(z)x}$$

where $\rho\colon\ \mathbf{C}\backslash\left[\frac{1}{e},\infty\right) \to \mathbf{C}$ is the analytic function defined in §2.

Proof. Note that $\rho(0) = 0, \rho'(0) = 1$. Let $\rho(z) = \sum_{n=1}^{\infty}\gamma_n z^n$ be the power series expansion of ρ in $B\left(0,\frac{1}{e}\right)$. The convergence radius is $\frac{1}{e}$, because ρ is analytic in $B\left(0,\frac{1}{e}\right)$ and $\frac{1}{e}$ is a singular point for ρ. Hence for $|z| < \frac{1}{e}$ and $x \in \mathbf{C}$, the function $(z,x) \mapsto e^{\rho(z)x}$ has a power series expansion

$$e^{\rho(z)x} = \sum_{\ell,m=0}^{\infty} c_{\ell m} z^\ell x^m.$$

Since

$$e^{\rho(z)x} = \sum_{m=0}^{\infty}\frac{1}{m!}\rho(z)^m x^m$$

and since the first non-zero term in the power series for $\rho(z)^m$ is z^m, we have $c_{\ell m} = 0$ for $\ell < m$. Hence

$$e^{\rho(z)x} = \sum_{\ell=0}^{\infty} Q_\ell(x)z^\ell \tag{3.3}$$

where $Q_\ell(x)$ is the polynomial $\sum_{m=0}^{\ell} c_{\ell m}x^m$. Putting $z = 0$ in (3.3) we get $Q_0(x) = 1$ and putting $x = 0$ in (3.3) we get $Q_n(0) = 0$ for $n \ge 1$. Moreover since $\rho(z)e^{-\rho(z)} = z$ for $\mathbf{C}\backslash\left[\frac{1}{e},\infty\right)$, we get

$$\frac{d}{dx}(e^{\rho(z)x}) = \rho(z)e^{\rho(z)x} = \rho(z)e^{-\rho(z)}e^{\rho(z)(x+1)} = ze^{\rho(z)(x+1)}.$$

Hence differentiating (3.3), we get

$$\sum_{\ell=0}^{\infty} Q'_\ell(x) z^\ell = \sum_{\ell=0}^{\infty} Q_\ell(x+1) z^{\ell+1} = \sum_{\ell=1}^{\infty} Q_{\ell-1}(x+1) z^\ell, \qquad |z| < \tfrac{1}{e}.$$

Therefore $Q'_\ell(x) = Q_{\ell-1}(x+1)$ for $\ell \geq 1$. Together with $Q_0(x) = 1$, $Q_\ell(x) = 0$, $(\ell \geq 1)$, this proves that $Q_\ell(x) = P_{1,\ell}(x)$ for $\ell \geq 0$. □

Remark 3.2. From Lemma 3.1 and (3.1) we can find the power series expansion of $\rho(z)$, namely

$$\rho(z) = ze^{\rho(z)} = \sum_{n=0}^{\infty} P_{1,n}(1) z^{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^{n+1} = \sum_{n=1}^{\infty} \frac{n^{n-2}}{(n-1)!} z^n. \tag{3.4}$$

Similarly one gets

$$\frac{1}{\rho(z)} = \frac{1}{z} e^{-\rho(z)} = \sum_{n=0}^{\infty} P_{1,n}(-1) z^{n-1} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(n-1)^{n-1}}{n!} z^{n-1} = \frac{1}{z} - \sum_{n=0}^{\infty} \frac{n^n}{(n+1)!} z^n. \tag{3.5}$$

The latter formula was also found in [4, §8] by different means. Actually, both formulae can be obtained from the Lagrange Inversion Formula, (cf. [9, Example 5.44]).

Lemma 3.3. *For every $x \in [0,1]$ there is a unique probability measure ν_x on $[0,e]$ such that*

$$\int_0^e y^n \, d\nu_x(y) = P_{1,n}(x), \qquad n \in \mathbf{N}_0. \tag{3.6}$$

Proof. The uniqueness is clear by Weierstrass' approximation theorem. For existence, recall that $\sigma(D) = [0,1]$ and, by [4, §8], $\sigma(T^*T) = [0,e]$. Let now $\mu = \mu_{D_0,T^*T}$ denote the joint distribution of D_0 and T^*T in the sense of (2.4). For $x = 0$, $\nu_x = \delta_0$ (the Dirac measure at 0) is a solution of (3.6). Assume now that $x > 0$ and let $\varepsilon \in (0,x)$. Then for $n \in \mathbf{N}_0$,

$$\begin{aligned}\int_{x-\varepsilon}^{x} P_{1,n}(x') dx' &= \int_0^1 1_{[x-\varepsilon,x]}(x') P_{1,n}(x') dx' = \operatorname{tr}(1_{[x-\varepsilon,x]}(D) E_{\mathcal{D}}((T^*T)^n)) \\ &= \operatorname{tr}(1_{[x-\varepsilon,x]}(D)(T^*T)^n) = \iint\limits_{[0,1]\times[0,e]} 1_{[x-\varepsilon,x]}(x') y^n \, d\mu(x',y).\end{aligned}$$

Let $\nu_{\varepsilon,x}$ denote the Borel measure on $[0,e]$ given by $\nu_{\varepsilon,x}(B) = \frac{1}{\varepsilon}\mu([x-\varepsilon,x]\times B)$ for any Borel set B in $[0,e]$. Then by the above calculation,

$$\int_0^e y^n \, d\nu_{\varepsilon,x}(y) = \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x} P_{1,n}(x') dx', \qquad n \in \mathbf{N}_0. \tag{3.7}$$

Since $P_{1,0}(x') = 1$, $\nu_{\varepsilon,x}$ is a probability measure. By (3.7), $\nu_{\varepsilon,x}$ converges as $\varepsilon \to 0$ in the w^*-topology on $\operatorname{Prob}([0,e])$ to a measure ν_x satisfying (3.6). □

Lemma 3.4. *Let $x \in [0,1]$.*

(a) *For $\lambda \in \mathbf{C}\backslash[0,e]$, the Stieltjes transform (or Cauchy transform) of ν_x is given by*

$$G_x(\lambda) = \frac{1}{\lambda} \exp\left(\rho\left(\frac{1}{\lambda}\right) x\right). \tag{3.8}$$

(b) *If* $x \in (0,1], d\nu_x(y) = h_x(y)dy$, *where*

$$h_x(y) = \frac{1}{\pi y}\mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right), \qquad y \in (0,e]. \tag{3.9}$$

Proof. (a). Since $G_x(\lambda) = \int_0^e \frac{1}{\lambda - y} d\nu_x(y)$ is analytic in $\mathbf{C}\backslash[0,e]$, it is sufficient to check (3.8) for $|\lambda| > e$. In this case, we get from Lemma 3.3 and Lemma 3.1 that

$$G_x(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \int_0^e y^n \, d\nu_x(y) = \frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{-n} P_n(x) = \frac{1}{\lambda}\exp\left(\rho\left(\frac{1}{\lambda}\right)x\right).$$

(b). For $y \in (0,e]$, put

$$\begin{aligned} h_x(y) &= -\frac{1}{\pi}\lim_{z\to 0^+} \mathrm{Im}(G_x(y+iz)) = -\frac{1}{\pi y}\mathrm{Im}\left(\exp\left(\rho^-\left(\frac{1}{y}\right)x\right)\right) \\ &= \frac{1}{\pi y}\ \mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right). \end{aligned}$$

It is easy to see that the above convergence is uniform for y in compact subsets of $(0,e]$, so by the inverse Stieltjes transform, the restriction of ν_x to $(0,e]$ is absolutely continuous with respect to the Lebesgue measure and has density $h_x(y)$. It remains to be proved that $\nu_x(\{0\}) = 0$. But

$$\lim_{\lambda\to 0^-} \lambda G_x(\lambda) = \nu_x(\{0\}) + \lim_{\lambda\to 0^-}\left(\int\limits_{(0,e]} \frac{|\lambda|}{|\lambda|+y} d\nu_x(y)\right) = \nu_x(\{0\}).$$

However, $\lambda G_x(\lambda) = \exp\left(\rho\left(\frac{1}{\lambda}\right)x\right) \to 0$ as $\lambda \to 0^-$, because $x > 0$ and $\lim_{y\to-\infty}\rho(y) = -\infty$. Hence $\nu_x(\{0\}) = 0$, which completes the proof of (b). □

Proof of Theorem 2.8 for $k=1$. Put $\mu = \mu_{D_0, T^*T}$ as defined in (2.4). For $m,n \in \mathbf{N}_0$ we get from Lemma 3.3 and Lemma 3.4,

$$\begin{aligned} \iint\limits_{[0,1]\times[0,e]} x^m y^n \, d\mu(x,y) &= \mathrm{tr}(D_0^m (T^*T)^n) = \mathrm{tr}(D_0^m E_{\mathcal{D}}((T^*T)^n)) = \int_0^1 x^m P_{1,n}(x)dx \\ &= \int_0^1 x^m \int_0^e y^n \, d\nu_x(y)dx = \int_0^1 \left(\int_0^e x^m y^n h_x(y)dy\right)dx. \end{aligned}$$

Hence by the Stone–Weierstrass Theorem, μ is absolutely continuous with respect to the two dimensional Lebesgue measure on $[0,1]\times[0,e]$, and for $x \in (0,1)$, $y \in (0,e)$, we have

$$\frac{d\mu(x,y)}{dxdy} = h_x(y) = \frac{1}{\pi y}\ \mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right). \tag{3.10}$$

We now have to compare (3.10) with (2.16) in Theorem 2.8. Putting $k=1$ in (2.14) and (2.15) one gets for $t > \frac{1}{e}$,

$$a_0(t) = \rho^+(t), \quad a_1(t) = \overline{\rho^+(t)}$$

and

$$c_0(t) = \frac{|\rho^+(t)|^2}{2i\ \mathrm{Im}(\rho^+(t))}, \quad c_1(t) = -\frac{|\rho^+(t)|^2}{2i\ \mathrm{Im}(\rho^+(t))}.$$

Hence the RHS of (2.16) becomes

$$\varphi(y)c_0\left(\frac{1}{y}\right)\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)-\exp\left(\overline{\rho^+\left(\frac{1}{y}\right)}x\right)\right)=$$

$$=\frac{\varphi(y)\left|\rho^+\left(\frac{1}{y}\right)\right|^2}{\mathrm{Im}\ \rho^+\left(\frac{1}{y}\right)}\ \mathrm{Im}\left(\exp\left(\rho^+\left(\frac{1}{y}\right)x\right)\right).$$

Substituting now $y=\frac{\sin v}{v}e^{v\cot v}$ with $0<v<\pi$ as in (2.3), by (2.13) and (2.1) we get

$$\frac{\varphi(y)\left|\rho^+\left(\frac{1}{y}\right)\right|^2}{\mathrm{Im}\ \rho^+\left(\frac{1}{y}\right)}=\frac{1}{\pi v}\left(\sin v e^{-v\cot v}\cdot\frac{v^2}{\sin^2 v}\right)=\frac{1}{\pi y}\,. \tag{3.11}$$

Hence (3.10) coincides with (2.16) for $k=1$. □

4. A generating function for Śniady's polynomials for $k\geq 2$

Throughout this section and Section 5, k is a fixed integer, $k\geq 2$.

Lemma 4.1. *Let $\alpha_1,\ldots,\alpha_k$ be distinct complex numbers and put*

$$\gamma_j=\prod_{\ell\neq j}\frac{\alpha_\ell}{\alpha_\ell-\alpha_j},\qquad j=1,\ldots,n. \tag{4.1}$$

Then

$$\begin{cases}\displaystyle\sum_{j=1}^k\gamma_j=1\\ \displaystyle\sum_{j=1}^k\gamma_j\alpha_j^p=0\quad\text{for}\quad p=1,2,\ldots,k-1.\end{cases} \tag{4.2}$$

Proof. We can express (4.2) as

$$\begin{bmatrix}1&1&\ldots&1\\ \alpha_1&\alpha_2&&\alpha_k\\ \vdots&&&\vdots\\ \alpha_1^{k-1}&\ldots&\ldots&\alpha_k^{k-1}\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_k\end{bmatrix}=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix} \tag{4.3}$$

where the determinant of the coefficient matrix is non-zero (Vandermonde's determinant), so we just have to check that (4.1) is the unique solution to (4.3). Let A denote the coefficient matrix in (4.3). Then the solution to (4.3) is given by

$$\begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_k\end{bmatrix}=A^{-1}\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}.$$

Hence $\gamma_j = (-1)^{j+1}\frac{\det(A_{1j})}{\det(A)}$, where A_{1j} is the $(1, j)$th minor of A. By Vandermonde's formula,

$$\det A = \prod_{\ell<m}(a_m - a_\ell)$$

and

$$\det(A_{1j}) = (\alpha_1 \cdots \alpha_{j-1})(\alpha_{j+1} \cdots \alpha_k) \prod_{\substack{\ell<m \\ \ell,m\neq j}} (a_m - a_\ell).$$

Hence

$$\gamma_j = \frac{(-1)^{j+1} \prod_{\ell\neq j} \alpha_\ell}{\prod_{\ell<j}(\alpha_j - \alpha_\ell) \prod_{\ell>j}(\alpha_\ell - \alpha_j)} = \prod_{\ell\neq j} \frac{\alpha_\ell}{\alpha_\ell - \alpha_j}. \quad \square$$

We prove next a generalization of Lemma 3.1 to $k \geq 2$.

Proposition 4.2. *Let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials defined Theorem 2.5. For $z \in \mathbf{C}$, $|z| < \frac{1}{e}$ and $j = 1, \ldots, k$, put*

$$\alpha_j(z) = \rho(ze^{i\frac{2\pi j}{k}}) \tag{4.4}$$

$$\gamma_j(z) = \begin{cases} \prod_{\ell\neq j} \frac{\alpha_j(z)}{\alpha_\ell(z)-\alpha_j(z)}, & z \neq 0 \\ 1/k, & z = 0. \end{cases} \tag{4.5}$$

Then

$$\sum_{n=0}^{\infty}(kz)^{nk} P_{k,n}(x) = \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} \tag{4.6}$$

for all $z \in B\left(0, \frac{1}{e}\right)$ and all $x \in \mathbf{R}$.

Proof. Since ρ is analytic and one-to-one on $\mathbf{C}\setminus\left[\frac{1}{e}, \infty\right)$, it is clear that $\alpha_j(z)$ is analytic in $B\left(0, \frac{1}{e}\right)$ and $\gamma_j(z)$ is analytic in $B\left(0, \frac{1}{e}\right)\setminus\{0\}$. Using $\rho(0) = 0$ and $\rho'(0) = 1$, one gets

$$\lim_{z\to 0} \gamma_j(z) = \prod_{\ell\neq j} \frac{1}{1 - \exp\left(i\frac{2\pi(j-\ell)}{k}\right)} = \prod_{m=1}^{k-1}\left(1 - \exp\left(i\frac{2\pi m}{k}\right)\right)^{-1}.$$

But the numbers $\exp\left(i\frac{2\pi m}{k}\right)$, $m = 1, \ldots, k-1$ are precisely the $k-1$ roots of the polynomial

$$S(z) = \frac{z^k - 1}{z - 1} = z^{k-1} + z^{k-2} + \ldots + 1.$$

Hence

$$\lim_{z\to 0} \gamma_j(z) = \frac{1}{S(1)} = \frac{1}{k} = \gamma_j(0).$$

Thus γ_j is analytic in $B\left(0, \frac{1}{e}\right)$. The RHS of (4.6) is equal to

$$\sum_{\ell=0}^{\infty} \beta_\ell(z) x^\ell$$

where

$$\beta_\ell(z) = \sum_{j=1}^{k} \gamma_j(z) k^\ell \alpha_j(z)^\ell.$$

Since $\alpha_j(0) = 0$, the coefficients to $1, z, \ldots, z^{\ell-1}$ in the power series expansion of $\beta_\ell(z)$ are equal to 0. Hence

$$\sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} = \sum_{\ell,m=0}^{\infty} \beta_{\ell,m} x^\ell z^m \tag{4.7}$$

where $\beta_{\ell,m} = 0$ when $m < \ell$. But, by the definition of $\alpha_j(z)$ and $\gamma_j(z)$ the LHS of (4.7) is invariant under the transformation $z \to e^{i\frac{2\pi}{k}} z$. Hence $\beta_{\ell,m} = 0$ unless m is a multiple of k. Therefore

$$\sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} = \sum_{n=0}^{\infty} R_n(x) z^{nk} \tag{4.8}$$

where

$$R_n(x) = \sum_{\ell=0}^{nk} \beta_{\ell,nk} x^\ell \tag{4.9}$$

is a polynomial of degree at most nk. To complete the proof of Proposition 4.2, we now have to prove, that the sequence of polynomials

$$Q_n(x) = k^{-nk} R_n(x), \qquad n = 0, 1, 2, \ldots \tag{4.10}$$

satisfies the same three conditions (2.6)–(2.8) as $P_{k,n}$. Putting $z = 0$ in (4.8) we get

$$Q_0(x) = R_0(x) = \sum_{j=1}^{k} \gamma_j(0) = 1.$$

Moreover by (4.5)

$$\frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty} R_n(x) z^{nk}\right) = \sum_{j=1}^{k} \gamma_j(z) k^k \alpha_j(z)^k e^{k\alpha_j(z)x}.$$

By definition of ρ, $\rho(z)e^{-\rho(z)} = z$ for all $z \in \mathbf{C}\setminus\left(\frac{1}{e}, \infty\right)$. Hence

$$(\alpha_j(z)e^{-\alpha_j(z)})^k = (ze^{i\frac{2\pi}{k}j})^k = z^k, \qquad j = 1, \ldots, k.$$

Thus

$$\frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty} R_n(z) z^{nk}\right) = (kz)^k \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)(x+1)} = (kz)^k \sum_{n=0}^{\infty} R_n(x+1) z^{nk}$$
$$= k^k \sum_{n=1}^{\infty} R_{n-1}(x+1) z^{nk}$$

so differentiating termwise, we get

$$R_n^{(k)}(x) = k^k R_{n-1}(x+1), \qquad n \geq 1$$

and thus $Q_n^{(k)}(x) = Q_{n-1}(x+1)$ for all $n \geq 1$. We next check the last condition (2.8) for the Q_n, i.e.

$$Q_n(0) = Q_n'(0) = \ldots = Q_n^{(k-1)}(0) = 0, \qquad n \geq 1.$$

If we put $x = 0$ in (4.5), we get

$$\sum_{n=0}^{\infty} R_n(x) z^{nk} = \sum_{j=1}^{k} \gamma_j(z) = 1,$$

where the last equality follows from (4.2) in Lemma 4.1. Hence $Q_n(0) = R_n(0) = 0$ for $n \geq 1$. For $p = 1, \ldots, k-1$ we have

$$\sum_{n=0}^{\infty} R_n^{(p)}(0) z^{nk} = \frac{d^p}{dx^p}\left(\sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x}\right)\Bigg|_{x=0} = k^p \sum_{j=1}^{k} \gamma_j(z)\alpha_j(z)^p = 0,$$

where we again use (4.2) from Lemma 4.1. Hence $Q_n^{(p)}(0) = k^{-nk} R_n^{(p)}(0) = 0$ for all $n = 0, 1, 2, \ldots$ and $p = 1, \ldots, k-1$.

Altogether we have shown that $(Q_n(x))_{n=0}^{\infty}$ satisfies the defining relations (2.6)–(2.8) for $P_{k,n}(x)$, and hence $Q_n(x) = P_{k,n}(x)$ for all n and. This proves (4.6). $\square$

Remark 4.3. *Based on Proposition 4.2, we give a new proof of the implication Theorem 2.5 $\Rightarrow$ Theorem 2.7. Put*

$$s_{k,n} = tr(((T^k)^* T^k)^n) = \int_0^1 P_{k,n}(x)dx.$$

Then by (4.6)

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = \sum_{j=1}^{k} \gamma_j(k) \int_0^1 e^{k\alpha_j(z)x} dx \tag{4.11}$$

for all $z \in B\left(0, \frac{1}{e}\right)$. By definition, the function ρ satisfies

$$\rho(s)e^{-\rho(s)} = s, \quad s \in \mathbf{C}\backslash[\tfrac{1}{e}, \infty).$$

Therefore,

$$\alpha_j(z)^k e^{-k\alpha_j(z)} = (ze^{i\frac{2\pi j}{k}})^k = z^k$$

for all $z \in B\left(0, \frac{1}{e}\right)$. Hence for $z \in B\left(0, \frac{1}{e}\right) \backslash \{0\}$,

$$\int_0^1 e^{k\alpha_j(z)x} dx = \frac{1}{k\alpha_j(z)}(e^{k\alpha_j(z)} - 1) = \frac{1}{kz^k}\alpha_j(z)^{k-1} - \frac{1}{k\alpha_j(z)}\ .$$

By Lemma 4.1, we have $\sum_{j=0}^{k} \gamma_j(z)\alpha_j(z)^{k-1} = 0$. Hence by (4.11),

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = -\frac{1}{k}\sum_{j=1}^{k} \frac{\gamma_j(z)}{\alpha_j(z)}\ . \tag{4.12}$$

To compute the right hand side of (4.12)*, we apply the residue theorem to the rational function $f(s) = \frac{1}{s^2}\prod_{\ell=1}^{k} \frac{\alpha_\ell}{\alpha_\ell - s}$, $s \in \mathbf{C}\backslash\{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}$. In the following computation z is fixed, so let us put $\alpha_j = \alpha_j(z)$, $\gamma_j = \gamma_j(z)$. Note that f has simple poles at $\alpha_1, \ldots, \alpha_k$ and*

$$Res(f; \alpha_j) = -\frac{1}{\alpha_j}\prod_{\ell \neq j} \frac{\alpha_\ell}{\alpha_\ell - \alpha_j} = -\frac{\gamma_j}{\alpha_j}\ .$$

Moreover f has a second order pole at 0 and Res$(f;0)$ is the coefficient of s in the power series expansion of $s^2 f(s) = \prod_{\ell=1}^{k}(1-\frac{s}{\alpha_\ell})^{-1}$ i.e.

$$\mathit{Res}(f;0) = \sum_{j=1}^{\ell} \frac{1}{\alpha_j}\ .$$

Since $f(s) = O(|s|^{-(k+2)})$ as $|s| \to \infty$, we have

$$\lim_{R\to\infty} \int_{\partial B(0,R)} f(s)ds = 0.$$

Hence, by the residue Theorem, Res$(f;0) + \sum_{j=1}^{k}$ Res$(f;\alpha_j) = 0$, giving

$$\sum_{j=1}^{k} \frac{\gamma_j}{\alpha_j} = \sum_{j=1}^{k} \alpha_j^{-1}. \tag{4.13}$$

Thus, by (4.12), we get

$$\sum_{n=0}^{\infty} s_{k,n}(kz)^{nk} = -\frac{1}{k}\sum_{j=1}^{k} \alpha_j(z)^{-1} = -\frac{1}{k}\sum_{j=1}^{k} \rho(ze^{i\frac{2\pi j}{k}})^{-1}. \tag{4.14}$$

By (3.5), $\rho(z)^{-1} = \frac{1}{z} - \sum_{m=0}^{\infty} \frac{m^m}{(m+1)!} z^m$ whenever $0 < |z| < \frac{1}{e}$. Hence

$$\sum_{j=1}^{k} \rho(ze^{i\frac{2\pi j}{k}})^{-1} = -k\sum_{k\,|\,m} \frac{m^m}{(m+1)!}z^m = -k\sum_{n=0}^{\infty} \frac{(nk)^{nk}}{(nk+1)!}z^{nk}\ . \tag{4.15}$$

So by comparing the terms in (4.14) and (4.15), we get $s_{kn} = \frac{n^{nk}}{(nk+1)!}$ as desired. $\square$

5. Proof of Theorem 2.8 for $k \geq 2$

Lemma 5.1. *Put $\Omega_k = \{z \in \mathbf{C} \mid z^k \notin [e^{-k}, \infty)\}$ and define $\alpha_j(z)$, $\gamma_j(z)$, $j = 1, \ldots, k$ by (4.4) and (4.5) for all $z \in \Omega_k$. Then for every $x \in \mathbf{R}$, the function*

$$M_x(z) = \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} \tag{5.1}$$

is analytic in Ω_k and for every $t \in \left[\frac{1}{e}, \infty\right)$, the following two limits exist:

$$M_x^+(t) = \lim_{\substack{z\to t \\ \mathrm{Im}\ z>0}} M_x(z), \quad M_x^-(t) = \lim_{\substack{z\to t \\ \mathrm{Im}\ z<0}} M_x(z).$$

Let $a_j(t)$ and $c_j(t)$ for $t > \frac{1}{e}$ and $j = 0, \ldots, k$ be as in Theorem 2.8. Then for $t > \frac{1}{e}$,

$$\mathrm{Im}\ M_x^+(t) = \frac{\mathrm{Im}\ \rho^+(t)}{k|\rho^+(t)|^2} \sum_{j=0}^{k} c_j(t) e^{ka_j(t)x}. \tag{5.2}$$

Proof. Since $\rho\colon\ \mathbf{C}\backslash\left[\frac{1}{e},\infty\right)\to\mathbf{C}$ is one–to–one and analytic, it is clear, that M_x is defined and analytic on Ω_k. Moreover for $t\geq\frac{1}{e}$,

$$\lim_{\substack{z\to t\\ \text{Im } z>0}}\alpha_j(z)=\begin{cases}\rho(te^{i\frac{2\pi j}{k}}), & j=1,\ldots,k-1\\ \rho^+(t), & j=k\end{cases}$$
$$=\begin{cases}a_j(t), & j=1,\ldots,k-1\\ a_0(t), & j=k\end{cases}$$

and similarly

$$\lim_{\substack{z\to t\\ \text{Im } z<0}}\alpha_j(z)=a_j(t),\qquad j=1,\ldots,k.$$

Moreover

$$\lim_{\substack{z\to t\\ \text{Im } z>0}}\gamma_j(z)=\begin{cases}\displaystyle\prod_{\substack{0\leq\ell\leq k-1\\ \ell\neq j}}\tfrac{a_\ell(t)}{a_\ell(t)-a_j(t)}, & j=1,\ldots,k-1\\ \displaystyle\prod_{\substack{0\leq\ell\leq k-1\\ \ell\neq 0}}\tfrac{a_\ell(t)}{a_\ell(t)-a_j(t)}, & j=k\end{cases}$$
$$\lim_{\substack{z\to t\\ \text{Im } z<0}}\gamma_j(z)=\prod_{\substack{1\leq\ell\leq k\\ \ell\neq j}}\frac{a_\ell(t)}{a_\ell(t)-a_j(t)},\qquad j,\ldots,k.$$

Hence the two limits $M_x^+(t)$ and $M_x^-(t)$ are well defined and by relabeling the kth term to be the 0th term in case of $M_x^+(t)$ one gets:

$$M_\lambda^+(t)=\sum_{j=0}^{k-1}\left(\prod_{\substack{0\leq\ell\leq k-1\\ \ell\neq j}}\frac{a_\ell(t)}{a_\ell(t)-a_j(t)}\right)e^{ka_j(t)x}\tag{5.3}$$
$$M_\lambda^-(t)=\sum_{j=1}^{k}\left(\prod_{\substack{1\leq\ell\leq k\\ \ell\neq j}}\frac{a_\ell(t)}{a_\ell(t)-a_j(t)}\right)e^{ka_j(t)x}.\tag{5.4}$$

It is clear, that $M_x(\bar{z})=\overline{M_x(z)}$, $z\in\Omega_k$. Therefore $M_\lambda^-(t)=\overline{M_\lambda^+(t)}$ and

$$\text{Im } M_\lambda^+(t)=\frac{1}{2i}(M_\lambda^+(t)-M_\lambda^-(t)).$$

Hence for $t>\frac{1}{e}$,

$$\text{Im } M_\lambda^+(t)=\sum_{j=0}^{k}b_j(t)e^{ka_j(t)x}$$

where

$$
\begin{aligned}
b_0(t) &= \frac{1}{2i} \prod_{1 \le \ell \le k-1} \frac{a_\ell(t)}{a_\ell(t) - a_0(t)} \\
b_j(t) &= \frac{1}{2i} \left(\frac{a_0(t)}{a_0(t) - a_j(t)} - \frac{a_k(t)}{a_k(t) - a_j(t)} \right) \prod_{\substack{1 \le \ell \le k-1 \\ \ell \ne j}} \frac{a_\ell(t)}{a_\ell(t) - a_0(t)} \\
b_k(t) &= -\frac{1}{2i} \prod_{1 \le \ell \le k-1} \frac{a_\ell(t)}{a_\ell(t) - a_k(t)}.
\end{aligned}
$$

Using (2.15) and the identity

$$
\frac{a_0(t)}{a_0(t) - a_j(t)} - \frac{a_k(t)}{a_k(t) - a_j(t)} = \frac{a_j(t)(a_k(t) - a_0(t))}{(a_0(t) - a_j(t))(a_k(t) - a_j(t))} \ ,
$$

one observes that for all $j \in \{0, 1, \ldots, k\}$

$$
b_j(t) = \frac{1}{2i} \frac{a_0(t) - a_k(t)}{k a_0(t) a_k(t)} c_j(t) = \frac{\mathrm{Im}\ \rho^+(t)}{k |\rho^+(t)|^2} c_j(t) \ .
$$

This proves (5.2). □

We next prove results analogous to Lemma 3.3 and Lemma 3.4 for $k \ge 2$.

Lemma 5.2. *For every $x \in [0, 1]$, there is a unique probability measure ν_x on $[0, e^k]$, such that*

$$
\int_0^{e^k} u^n \ d\nu_x(u) = k^{nk} P_{k,n}(x), \qquad n \in \mathbf{N}_0. \tag{5.5}
$$

For $\lambda \in \mathbf{C} \backslash [0, e^k]$, the Cauchy transform of ν_x is given by

$$
G_x(\lambda) = \frac{1}{\lambda} \sum_{j=1}^{k} \gamma_j(\lambda^{-\frac{1}{k}}) e^{k \alpha_j (\lambda^{-\frac{1}{k}}) x} \tag{5.6}
$$

where α_j, γ_j are given by (4.4) and (4.5) and $\lambda^{-1/k}$ is the principal value of $(\sqrt[k]{\lambda})^{-1}$. Moreover, the restriction of ν_x to $(0, e^k]$ is absolutely continuous with respect to Lebesgue measure, and its density is given by

$$
\frac{d\nu_x(u)}{du} = \frac{u^{\frac{1}{k}-1} \varphi(u^{1/k})}{k} \sum_{j=0}^{k} c_j(u^{-1/k}) e^{k a_j (u^{-1/k}) x} \tag{5.7}
$$

for $u \in (0, e^k)$.

Proof. By Theorem 2.5

$$
k^{nk} P_{k,n}(x) = E_D(k^{nk}((T^k)^* T^k)^n)(x) = E_D(S_k^{nk})(x)), \qquad x \in [0, 1].
$$

Moreover $\sigma(S_k^k) = \sigma(S_k)^k = [0, e^k]$ by (2.10). Hence the existence and uniqueness of ν_x can be proved exactly as in Lemma 3.3. From Proposition 4.2, we get that for $|\lambda| > e^k$, the Stieltjes transform $G_x(\lambda)$ of ν_x is given by

$$
G_x(\lambda) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^{-n} k^{nk} P_{k,n}(x) = \frac{1}{\lambda} \sum_{j=1}^{k} \gamma_j(\lambda^{-\frac{1}{k}}) e^{k \alpha_j (\lambda^{-\frac{1}{k}}) x}.
$$

Let $M_x(z), z \in \Omega_k$ and $M_x^+(t), M_x^-(t),\ t \geq 1/e$ be as in Lemma 5.1. Then it is easy to see that the function

$$\widetilde{M}_x(z) = \begin{cases} M_x(z), & z \in \Omega_K \\ M_x^-(z), & z \in [1/e, \infty) \end{cases}$$

is a continuous function on the set

$$\left\{ x + iy \mid x \geq 0, \frac{-1}{ke} \leq y \leq 0 \right\}.$$

Hence, by applying the inverse Stieltjes transform, we get that the restriction of ν_x to $(0, e^k]$ is absolutely continuous with respect to the Lebesgue measure with density

$$\begin{aligned} h_x(u) &= -\frac{1}{\pi} \lim_{v \to 0^+} \operatorname{Im}(G_x(u + iv)) = -\frac{1}{\pi u} \lim_{\substack{z \to u^{-1/k} \\ \operatorname{Im} z < 0}} \left(\operatorname{Im} \sum_{j=1}^{k} \gamma_j(z) e^{k\alpha_j(z)x} \right) \\ &= -\frac{1}{\pi u} \operatorname{Im} M_x^-(u^{-1/k}) = \frac{1}{\pi u} \operatorname{Im} M_x^+(u^{-1/k}). \end{aligned}$$

Hence, by Lemma 5.1 we get that for $u \in (0, e^k)$,

$$h_x(u) = \frac{1}{\pi u} \frac{\operatorname{Im}\,(\rho^+(u^{-1/k}))}{k|\rho^+(u^{-1/k})|^2} \sum_{j=0}^{k} c_j(u^{-1/k}) e^{ka_j(u^{-1/k})x}.$$

By (3.11),

$$\varphi(y) = \frac{1}{\pi y} \frac{\operatorname{Im}\,(\rho^+(1/y))}{|\rho^+(1/y)|^2}, \qquad 0 < y < e.$$

Hence

$$h_x(u) = \frac{u^{\frac{1}{k}-1} \varphi(u^{1/k})}{k} \sum_{j=0}^{k} c_j(u^{-1/k}) e^{ka_j(u^{-1/k})x}. \tag{5.8}$$

□

Remark 5.3. In order to derive Theorem 2.8 from Lemma 5.2, we will have to prove $\nu_x(\{0\}) = 0$ for almost all $x \in [0, 1]$ w.r.t. Lebesgue measure. This is done in the proof of Lemma 5.4 below. Actually it can be proved that $\nu_x(\{0\}) = 0$ for *all* $x > 0$. This can be obtained from the formula

$$\nu_x(\{0\}) = \lim_{\lambda \to 0^-} \lambda G_x(\lambda)$$

(cf. proof of Lemma 3.4) together with the following asymptotic formula for $\rho(z)$ for large values of $|z|$:

$$\rho(z) = -\log(-z) + \log(\log(-z)) + O\left(\frac{\log(\log|z|))}{\log|z|} \right),$$

where $\log(-z)$ is the principal value of the logarithm. The latter formula can also be obtained from [2, pp. 347–350] using (2.11).

Lemma 5.4. *Let $\nu = \mu_{D_0, S_k^k}$ be the measure on $[0, 1] \times [0, e^k]$ defined in (2.4). Then ν is absolutely continuous with respect to the Lebesgue measure, and its density is given by*

$$\frac{d\nu(x, u)}{dxdu} = h_x(u), \quad x \in (0, 1), \quad u \in (0, e^k),$$

where $h_x(u)$ is given by (5.8).

Proof. For $m, n \in \mathbf{N}_0$ we have from Lemma 5.2 and Theorem 2.5 that

$$\iint\limits_{[0,1]\times[0,e^k]} x^m u^n \, d\nu(x,u) = \mathrm{tr}(D_0^m S_k^{kn}) = \mathrm{tr}(D_0^m E_D(S_k^{kn})) \tag{5.9}$$

$$= \int_0^t x^m (k^{nk} P_{k,n}(x))dx = \int_0^1 x^m \left(\int_e^{e^k} u^n \, d\nu_x(u) \right) dx.$$

Put $g(x) = \nu_x(\{0\})$, $x \in [0,1]$. From the definition of ν_x it is clear that $x \to \nu_x$ is a w^*-continuous function from $[0,1]$ to $\mathrm{Prob}([0,e^k])$, i.e.

$$x \to \int_0^{e^k} f(u) \, d\nu_x(u), \qquad x \in [0,1]$$

is continuous for all $f \in C([0,e^k])$. Put for $j \in \mathbf{N}$,

$$f_j(u) = \begin{cases} j, & 0 \le u \le 1/j \\ 0, & u > 1/j. \end{cases}$$

Then $g(x) = \lim\limits_{j\to\infty} \left(\int_0^{e^k} f_j(u) d\nu_x(u) \right)$, and hence g is a Borel function on $[0,1]$. Putting now $m = 0$ in (5.9) we get

$$\mathrm{tr}(S_k^{kn}) = \int_0^1 \left(\int_0^{e^k} u^n h_x(u) du \right) dx, \qquad n = 1, 2, \dots \tag{5.10}$$

and for $n = 0$ we get

$$1 = \int_0^1 g(x)dx + \int_0^1 \left(\int_0^{e^k} h_x(u)du \right) dx. \tag{5.11}$$

Let $\lambda \in \mathrm{Prob}([0,e^k])$ be the distribution of S_k^k. Then

$$\int_0^{e^k} u^n \, d\lambda(u) = \mathrm{tr}(S_k^{kn})$$

so by (5.10) and (5.11), $\lambda(\{0\}) = \int_0^1 g(x)dx$ and λ is absolutely continuous on $(0, e^k]$ w.r.t. Lebesgue measure, with density $u \to \int_0^1 h_x(u)dx$, $u \in (0, e^k)$. However by (2.9) S_k^k and $(T^*T)^k$ have the same moments. Thus S_k^k and $(T^*T)^k$ have the same distribution measure. By ([4, §8]), $\ker(T^*T) = \ker(T) = \{0\}$. Hence $\lambda(\{0\}) = 0$, which implies that $g(x) = 0$ for almost all $x \in [0,1]$. Thus, using (5.9), we have for all $m, n \in \mathbf{N}_0$

$$\int\limits_{[0,1]\times[0,e^k]} x^m u^n \, d\nu(x,u) = \int_0^1 x^m \left(\int_0^{e^k} u^n h_x(u) \, du \right) dx.$$

Hence by Stone–Weierstrass Theorem, ν is absolutely continuous w.r.t. two dimensional Lebesgue measure, and

$$\frac{d\nu(x,u)}{dx \, du} = h_x(u), \qquad x \in (0,1), \quad u \in (0, e^k). \quad \square$$

Proof of Theorem 2.8 for $k \geq 2$. Let f, g be bounded Borel functions on $[0,1]$ and $[0,e]$ respectively, and put

$$g_1(u) = g(u^{1/k}), \qquad u \in [0, e^k].$$

By Lemma 5.4,

$$\begin{aligned}\operatorname{tr}(f(D_0)g(S_k)) = \operatorname{tr}(f(D_0)g_1(S_k^k)) &= \iint\limits_{[0,1]\times[0,e^k]} f(x)g_1(u)h_x(u)dxdu \\ &= \iint\limits_{[0,1]\times[0,e]} f(x)g(y)h_x(y^k)ky^{k-1}dxdy\end{aligned}$$

where the last equality is obtained by substituting $u = y^k$, $y \in [0, e]$. Hence the measure μ_{D_0,S_k} is absolutely continuous with respect to the two dimensional Lebesgue measure, and by (5.8) the density is given by

$$h_x(y^k)ky^{k-1} = \varphi(y)\sum_{j=0}^{\infty} c_j\big(\tfrac{1}{y}\big)e^{ka_j(\frac{1}{y})x}$$

for $x \in (0,1)$, $y \in (0,e)$. □

6. Proof of Theorem 2.8 ⇒ Theorem 2.2

Lemma 6.1. *Let $k \in \mathbf{N}$ band let $a_0, \dots, a_k$ be distinct numbers in $\mathbb{C}\backslash\{0\}$ and put*

$$b_j = \prod_{\substack{\ell=0\\ \ell\neq j}}^{k} \frac{a_\ell}{a_\ell - a_j}.$$

Then

$$\sum_{j=0}^{k} b_j a_j^p = 0 \quad p = 1, 2, \dots, k \tag{6.1}$$

$$\sum_{j=0}^{k} b_j = 1 \tag{6.2}$$

$$\sum_{j=0}^{k} b_j a_j^{-1} = \sum_{j=0}^{k} a_j^{-1} \tag{6.3}$$

$$\sum_{j=0}^{k} b_j a_j^{-2} = \sum_{0\leq i\leq j\leq k} (a_i a_j)^{-1}. \tag{6.4}$$

Proof. By applying Lemma 4.1 to the $k+1$ numbers $a_0, \dots, a_k$, we get (6.1) and (6.2). Moreover, (6.3) follows from the residue calculus argument in Remark 4.3 (cf. (4.13)), and (6.4) follows by a similar argument. Indeed, letting g be the rational function

$$g(s) = \frac{1}{s^3}\prod_{\ell=0}^{k}\left(\frac{a_\ell}{a_\ell - s}\right), \qquad s \in \mathbf{C}\backslash\{0, a_0, \dots, a_k\},$$

we have $\mathrm{Res}(g; a_j) = -\frac{1}{a_j^2}\prod_{\ell\neq j}\frac{a_\ell}{a_\ell - a_j} = -b_j a_j^{-2}$ and $\mathrm{Res}(g; 0)$ is the coefficient of s^2 in the power series expansion of

$$s^3 g(s) = \prod_{\ell=0}^{k}\left(1 - \frac{s}{a_\ell}\right)^{-1} = \prod_{\ell=0}^{k}\left(1 + \frac{s}{a_\ell} + \frac{s^2}{a_\ell^2} + \dots\right).$$

Hence $\mathrm{Res}(g; 0) = \sum_{0\le i\le j\le k}(a_i a_j)^{-1}$. Since $g(s) = O(|s|^{-(k+4)})$ as $|s| \to \infty$, as in Remark 4.3 we get

$$\mathrm{Res}(g; 0) + \sum_{j=0}^{k}\mathrm{Res}(g; a_j) = 0.$$

This proves (6.4). □

Lemma 6.2. *Let $k \in \mathbf{N}$ be fixed and let $a_j(t)$, $c_j(t)$ for $t \in \left(\frac{1}{e}, \infty\right)$ and $j = 0, \dots, k$ be defined as in* (2.14) *and* (2.15). *Put*

$$H(x, t) = \sum_{j=0}^{k} c_j(t)e^{ka_j(t)x}, \qquad x \in \mathbf{R}, \quad t > 1/e, \tag{6.5}$$

$$m(t) = -\frac{1}{k}\sum_{j=0}^{k} a_j(t)^{-1}, \tag{6.6}$$

$$v(t) = \frac{1}{k^2}\sum_{j=0}^{k} a_j(t)^{-2}. \tag{6.7}$$

Then

$$\int_0^1 H(x, t)dx = 1. \tag{6.8}$$

Moreover, if $k \geq 2$, then

$$\int_0^1 xH(x, t)dx = m(t) \tag{6.9}$$

and if $k \geq 3$, then

$$\int_0^1 x^2 H(x, t)dx = m(t)^2 + v(t). \tag{6.10}$$

Proof. For a fixed $t \in \left(\frac{1}{e}, \infty\right)$, we will apply Lemma 6.1 to the numbers $a_j(t)$, $j = 0, \dots, k$ and

$$b_j(t) = \prod_{\ell\neq j}\frac{a_\ell(t)}{a_\ell(t) - a_j(t)}. \tag{6.11}$$

Note that by (2.15)

$$c_j(t) = -ka_j(t)b_j(t). \tag{6.12}$$

Since t is fixed, we will drop the t in $a_j(t)$, $b_j(t)$ and $c_j(t)$ in the rest of this proof. We have

$$\int_0^1 H(x, t)dx = \sum_{j=0}^{k}\frac{c_j}{ka_j}(e^{ka_j} - 1) = \sum_{j=0}^{k} b_j(1 - e^{ka_j}). \tag{6.13}$$

Recall that

$$\begin{cases} a_0 = \rho^+(t) \\ a_j = \rho(te^{i\frac{2\pi j}{k}}), \quad 1 \le j \le n \\ a_k = \rho^-(t) \end{cases}$$

where $t \in \left(\frac{1}{e}, \infty\right)$. Since $\rho(z)e^{-\rho(z)} = z$ for $z \in \mathbf{C}\setminus\left[\frac{1}{e}, \infty\right)$ we get in the limit $z \to t$ with $\operatorname{Im} z > 0$, respectively $\operatorname{Im} z < 0$, that also

$$\rho^+(t)e^{-\rho^+(t)} = \rho^-(t)e^{-\rho^-(t)} = t.$$

Hence

$$(a_j e^{-a_j})^k = (te^{i\frac{2\pi j}{k}})^k = t^k, \qquad j = 0, \dots, k,$$

which shows

$$e^{ka_j} = \left(\frac{a_j}{t}\right)^k, \qquad j = 0, \dots, k. \tag{6.14}$$

Hence by (6.13), (6.1) and (6.2) we get

$$\int_0^1 H(x,t)dx = \sum_{j=0}^k b_j - \frac{1}{t^k}\sum_{j=0}^k b_j a_j^k = 1,$$

which proves (6.8). Moreover,

$$\int_0^1 xH(x,t)dx = \sum_{j=0}^k (-ka_jb_j)\left[x\frac{e^{ka_jx}}{ka_j} - \frac{e^{ka_jx}}{(ka_j)^2}\right]_0^1.$$

Using (6.14), (6.1) and (6.3) we get

$$\int_0^1 xH(x,t)dx = -\frac{1}{t^k}\sum_{j=0}^k b_j a_j^k + \frac{1}{kt^k}\sum_{j=0}^k b_j a_j^{k-1} - \frac{1}{k}\sum_{j=0}^k \frac{b_j}{a_j} = -\frac{1}{k}\sum_{j=0}^k \frac{1}{a_j} = m(t)$$

provided $k \ge 2$. This proves (6.9). Similarly

$$\begin{aligned}\int_0^1 x^2H(x,t)dx &= \sum_{j=0}^k (-ka_jb_j)\left[x^2\frac{e^{ka_jx}}{ka_j} 2x\frac{e^{ka_jx}}{(ka_j)^2} + 2\frac{e^{ka_jx}}{(ka_j)^3}\right]_0^1 \\ &= -\frac{1}{t^k}\sum_{j=0}^k b_j a_j^k + \frac{2}{kt^k}\sum_{j=0}^k b_j a_k^{k-1} - \frac{2}{k^2t^k}\sum_{j=0}^k b_j a_j^{k-2} + \frac{2}{k^2}\sum_{j=0}^k \frac{b_j}{a_j^2}\,.\end{aligned}$$

Hence by (6.1) and (6.4), we get for $k \ge 3$

$$\int_0^1 x^2H(x,t)dx = \frac{2}{k^2}\sum_{0\le i\le j\le k}(a_ia_j)^{-1} = \frac{1}{k^2}\left(\left(\sum_{j=0}^k a_j^{-1}\right)^2 + \sum_{j=0}^k a_j^{-2}\right) = m(t)^2 + v(t).$$

□

The functions H, m, v, a_j, c_j in Lemma 5.2 depend on $k \in \mathbf{N}$. Therefore we will in the rest of this section rename them $H_k, m_k, v_k, a_{kj}, c_{kj}$. Let $F(y) = \int_0^y \varphi(u)du$, $y \in [0, e]$ as in Proposition 2.1. Since φ is the density of a probability measure on $[0, e]$, we have

$$0 \le F(y) \le 1, \qquad y \in [0, e]. \tag{6.15}$$

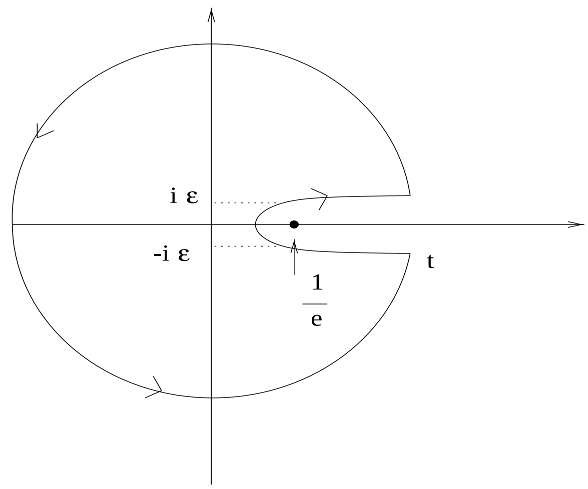

FIGURE 1. The contour C_ϵ.

Lemma 6.3. *For* $t \in \left(\frac{1}{e}, \infty\right)$,

$$\lim_{k\to\infty} m_k(t) = F\left(\frac{1}{t}\right) \tag{6.16}$$

$$\lim_{k\to\infty} v_k(t) = 0. \tag{6.17}$$

Proof.

$$m_k(t) = -\frac{1}{k}\sum_{j=0}^{k} a_{kj}(t)^{-1} = -\frac{1}{k}\left(\sum_{j=0}^{k} f\left(\frac{j}{k}\right)\right),$$

where $f\colon\ [0,1] \to \mathbf{C}$ is the continuous function

$$f(u) = \begin{cases} \rho^+(t)^{-1}, & u = 0 \\ \rho(te^{i2\pi u})^{-1}, & 0 < u < 1 \\ \rho^-(t)^{-1}, & u = 1. \end{cases}$$

Hence

$$\lim_{k\to\infty} m_k(t) = -\int_0^1 f(u)du = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\rho(te^{i\theta})}d\theta = -\frac{1}{2\pi i}\int_{\partial B(0,t)} \frac{1}{z\rho(z)}dz. \tag{6.18}$$

To evaluate the RHS of (6.18) we apply the residue theorem to compute the integral of $(z\rho(z))^{-1}$ along the closed path C_ε, $0 < \varepsilon < \frac{1}{e}$, which is drawn in Figure 1.
Since $\rho(z) \neq 0$ when $z \neq 0$ we have

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{dz}{z\rho(z)} = \operatorname{Res}\left(\frac{1}{z\rho(z)}; 0\right)$$

and by (3.5), $\operatorname{Res}\left(\frac{1}{z\rho(z)}, 0\right) = -1$. Thus, taking the limit $\varepsilon \to 0^+$, we get

$$\frac{1}{2\pi i}\left(\int_{1/e}^{t} \frac{dt}{t\rho^+(t)} + \int_{\partial B(0,t)} \frac{dz}{z\rho(z)} + \int_{t}^{1/e} \frac{dt}{t\rho^-(t)}\right) = -1.$$

Since $\rho^-(t) = \overline{\rho^+(t)}$, we get by (3.11)

$$\frac{1}{2\pi i}\int\limits_{\partial B(0,t)} \frac{dz}{z\rho(z)} = -1 - \frac{1}{\pi}\int_{1/e}^{t} \frac{1}{s}\mathrm{Im}\left(\frac{1}{\rho^+(s)}\right) ds = -1 + \frac{1}{\pi}\int_{1/e}^{t} \frac{\mathrm{Im}\ \rho^+(s)}{s|\rho^+(s)|^2} ds$$
$$= -1 + \int_{1/e}^{t} \frac{1}{s^2}\varphi\left(\frac{1}{s}\right) ds = -1 + \int_{1/t}^{e} \varphi(u)du$$
$$= -1 + F(1) - F(1/t) = -F(1/t).$$

Hence (6.16) follows from (6.18). In the same way we get

$$v_k(t) = \frac{1}{k^2}\sum_{j=0}^{k} f\left(\frac{j}{k}\right)^2.$$

Hence

$$\lim_{k\to\infty} k v_k(t) = \int_0^1 f(u)^2 du,$$

so in particular

$$\lim_{k\to\infty} v_k(t) = 0. \quad \square$$

Proof of Theorem 2.2. By Lemma 2.4, Theorem 2.8 and (6.5),

$$\|D_0 - F(S_k)\|_2^2 = \iint\limits_{[0,1]\times[0,e]} |x - F(y)|^2 \varphi(y) H_k\left(x, \tfrac{1}{y}\right) dxdy.$$

Moreover by (6.8)–(6.10) we have for $y \in (0,e)$ and $k \geq 3$,

$$\int_0^1 (x - F(y))^2 H_k(x, \tfrac{1}{y})dx = (v_k(\tfrac{1}{y}) + m_k(\tfrac{1}{y})^2) - 2m_k(\tfrac{1}{y})F(y) + F(y)^2$$
$$= (m_k(\tfrac{1}{y}) - F(y))^2 + v_k(\tfrac{1}{y}).$$

Hence for $k \geq 3$

$$\|D_0 - F(S_k)\|_2^2 = \int_0^e \left((m_k(\tfrac{1}{y}) - F(y))^2 + v_k(\tfrac{1}{y})\right)\varphi(y)dy.$$

Since $\varphi(y)H_k(x, \frac{1}{y})$ is a continuous density function for the probability measure $\mu_{D_0 S_k}$ on $(0,1) \times (0,e)$, and since $\varphi(y) > 0$, $0 < y < e$, we have $H_k(x,t) \geq 0$ for all $x \in (0,1)$ and $t \in (\frac{1}{e}, \infty)$. Thus by (6.8)–(6.10), $m_k(t)$ and $v_k(t)$ are the mean and variance of a probability measure on $(0,1)$. In particular $0 \leq m_k(t) \leq 1$ and $0 \leq v_k(t) \leq 1$ for all $t > 1/e$. Hence by (6.16), (6.17) and Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty} \|D_0 - F(S_k)\|_2^2 = 0.$$

Hence $D_0 \in W^*(T)$. For $0 < t < 1$, the subspace $\mathcal{H}_t = 1_{[0,t]}(D_0)\mathcal{H}$ is clearly T-invariant, and since $D_0 \in W^*(T)$, $\mathcal{H}_t$ is affiliated with $W^*(T)$. $\square$

7. Hyperinvariant subspaces for T

In this section, we prove Theorem 2.9. The proof relies on the following four results. Lemma 7.2 is probably well known, but we include a proof for convenience.

Lemma 7.1. *For every $k \in \mathbf{N}$, $\|T^k\| = (\frac{e}{k})^{k/2}$.*

Proof. By (2.10), $\|T^k\|^2 = \|(T^*)^k T^k\| = k^{-k}\|S^k\| = (\frac{e}{k})^k$. □

Lemma 7.2. *Let $(S_\lambda)_{\lambda\in\Lambda}$ be a bounded net of selfadjoint operators on a Hilbert space $\mathcal{H}$ which converges in strong operator topology to the selfadjoint operator $S \in \mathcal{B}(\mathcal{H})$, and let $\sigma_p(S)$ denote the set of eigenvalues of S. Then for all $t \in \mathbf{R}\backslash\sigma_p(S)$, we have*

$$\lim_{\lambda\in\Lambda} 1_{(-\infty,t]}(S_\lambda) = 1_{(-\infty,t]}(S), \tag{7.1}$$

where the limit is in strong operator topology.

Proof. There is a compact interval $[a, b]$ such that $\sigma(S_\lambda) \subseteq [a, b]$ for all λ and $\sigma(S) \subseteq [a, b]$. Therefore, given a continuous function $\phi : \mathbf{R} \to \mathbf{R}$, approximating by polynomials we get

$$\lim_{\lambda\in\Lambda} \phi(S_\lambda) = \phi(S),$$

in strong operator topology. Let $t \in \mathbf{R}$, let $\epsilon > 0$ and choose a continuous function $\phi : \mathbf{R} \to \mathbf{R}$ such that $0 \le \phi \le 1$, $\phi(x) = 1$ for $x \le t - \epsilon$ and $\phi(x) = 0$ for $x \ge t$. Then for every $\xi \in \mathcal{H}$

$$\langle 1_{(-\infty,t-\epsilon]}(S)\xi, \xi\rangle \le \langle \phi(S)\xi, \xi\rangle = \lim_{\lambda\in\Lambda}\langle \phi(S_\lambda)\xi, \xi\rangle \le \liminf_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi, \xi\rangle.$$

Hence taking the limit as $\epsilon \to 0^+$, we get

$$\langle 1_{(-\infty,t)}(S)\xi, \xi\rangle \le \liminf_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi, \xi\rangle. \tag{7.2}$$

Similarly, by using a continuous function $\psi : \mathbf{R} \to \mathbf{R}$ satisfying $\psi(x) = 1$ for $x \le t$ and $\psi(x) = 0$ for $x \ge t + \epsilon$, we get

$$\langle 1_{(-\infty,t]}(S)\xi, \xi\rangle \ge \limsup_{\lambda\in\Lambda}\langle 1_{(-\infty,t]}(S_\lambda)\xi, \xi\rangle. \tag{7.3}$$

If $t \notin \sigma_p(S)$, then $1_{(-\infty,t)}(S) = 1_{(-\infty,t]}(S)$, and thus by (7.2) and (7.3), we have

$$\lim_{\lambda\in\Lambda} 1_{(-\infty,t]}(S_\lambda) = 1_{(-\infty,t]}(S), \tag{7.4}$$

with convergence in weak operator topology. However, the weak and strong operator topologies coincide on the set of projections in $\mathcal{B}(\mathcal{H})$. Hence we have convergence (7.1) in strong operator topology, as desired. □

Proposition 7.3. *Let $F : [0, e] \to [0, 1]$ be the increasing function defined in Proposition 2.1 and fix $t \in [0, 1]$. Let*

$$\mathcal{L}_t = \{\xi \in \mathcal{H} \mid \exists \xi_k \in \mathcal{H},\ \lim_{k\to\infty} \|\xi_k - \xi\| = 0,\ \limsup_{k\to\infty}(\tfrac{k}{e}\|T^k\xi_k\|^{2/k}) \le t\}.$$

Then $\mathcal{L}_t = \mathcal{H}_{F(et)}$.

Proof. For $t = 1$, we have by Lemma 7.1 that $\mathcal{L}_1 = \mathcal{H} = \mathcal{H}_1 = \mathcal{H}_{F(e)}$. Assume now $0 \le t < 1$, and let $\xi \in \mathcal{H}_{F(et)} = 1_{[0,F(et)]}(D_0)\mathcal{H} = 1_{[0,et]}(F(D_0))\mathcal{H}$. Since $\sigma_p(D_0) = \emptyset$ and since F is one–to–one, we also have $\sigma_p(F(D_0)) = \emptyset$. Hence, by Theorem 2.8 and Lemma 7.2,

$$\lim_{k\to\infty} 1_{[0,et]}(S_k)\xi = 1_{[0,et]}(F(D_0))\xi = \xi.$$

Let $\xi_k = 1_{[0,et]}(S_k)\xi$. Then as we just showed, $\lim_{k\to\infty}\|\xi-\xi_k\| = 0$. Moreover, since $(T^*)^kT^k = k^{-k}S_k^k$, we have

$$\|T^k\xi_k\|^2 = k^{-k}\langle S_k^k\xi_k,\xi_k\rangle \le k^{-k}(et)^k\|\xi_k\|^2 \le \big(\frac{et}{k}\big)^k\|\xi\|^2.$$

Hence $\limsup_{k\to\infty}(\frac{k}{e}\|T^k\xi_k\|^{2/k}) \le t$, which proves $\mathcal{H}_{F(et)} \subseteq \mathcal{L}_t$. To prove the reverse inclusion, let $\xi\in\mathcal{L}_t$ and choose $\xi_k\in\mathcal{H}$ such that

$$\lim_{k\to\infty}\|\xi_k-\xi\| = 0, \qquad \limsup_{k\to\infty}\big(\frac{k}{e}\|T^k\xi_k\|^{2/k}\big) \le t. \tag{7.5}$$

By (2.10), $\sigma(S_k) = [0,e]$. Let E_k be the spectral measure of S_k and let

$$\gamma_k(B) = \langle E_k(B)\xi_k,\xi_k\rangle$$

for every Borel set $B\subseteq[0,e]$. Then γ_k is a finite Borel measure on $[0,e]$ of total mass $\gamma_k([0,e]) = \|\xi_k\|^2$ and for all bounded Borel functions $f:[0,e]\to\mathbf{C}$, we have

$$\langle f(S_k)\xi_k,\xi_k\rangle = \int_0^e f\,d\gamma_k. \tag{7.6}$$

In particular,

$$\langle S_k^k\xi_k,\xi_k\rangle = \int_0^e x^k\,d\gamma_k(x).$$

Let $0<\epsilon<1-t$. By (7.5), there exists $k_0\in\mathbf{N}$ such that $\frac{k}{e}\|T^k\xi_k\|^{2/k} \le t+\frac{\epsilon}{2}$ for all $k\ge k_0$. Thus,

$$\int_0^e x^k\,d\gamma_k(x) = \langle S_k^k\xi_k,\xi_k\rangle = k^k\|T^k\xi_k\|^2 \le (e(t+\tfrac{\epsilon}{2}))^k, \qquad (k\ge k_0).$$

Since $(\frac{x}{e(t+\epsilon)})^k \ge 1$ for $x\in[e(t+\epsilon),e]$, we have

$$\gamma_k([e(t+\epsilon),e]) \le \int_0^e \left(\frac{x}{e(t+\epsilon)}\right)^k d\gamma_k(x) \le \left(\frac{t+\frac{\epsilon}{2}}{t+\epsilon}\right)^k\|\xi_k\|^2.$$

Hence, by (7.6),

$$\|1_{(e(t+\epsilon),\infty)}(S_k)\xi_k\|^2 = \langle 1_{(e(t+\epsilon),\infty)}(S_k)\xi_k,\xi_k\rangle \le \left(\frac{t+\frac{\epsilon}{2}}{t+\epsilon}\right)^k\|\xi_k\|^2,$$

which tends to zero as $k\to\infty$. Since $\|\xi_k-\xi\|\to 0$ as $k\to\infty$, we get

$$\lim_{k\to\infty}\|1_{(e(t+\epsilon),\infty)}(S_k)\xi\| = 0,$$

which is equivalent to

$$\lim_{k\to\infty} 1_{[0,e(t+\epsilon)]}(S_k)\xi = \xi.$$

Hence, by Theorem 2.8 and Lemma 7.2,

$$1_{[0,F(e(t+\epsilon))]}(D_0)\xi = 1_{[0,e(t+\epsilon)]}(F(D_0))\xi = \xi,$$

i.e. $\xi\in\mathcal{H}_{F(e(t+\epsilon))}$ for all $\epsilon\in(0,1-t)$. Since

$$\mathcal{H}_{F(et)} = \bigcap_{s\in(F(et),1)}\mathcal{H}_s,$$

it follows that $\mathcal{L}_t\subseteq\mathcal{H}_{F(et)}$, which completes the proof of the proposition. ☐

Lemma 7.4. *Let $t \in (0,1)$ and define $(a_n)_{n=1}^\infty$ recursively by*

$$a_1 = F(et) \tag{7.7}$$

$$a_{n+1} = a_n F\left(\frac{et}{a_n}\right). \tag{7.8}$$

Then $(a_n)_{n=1}^\infty$ is a strictly decreasing sequence in $[0,1]$ and $\lim_{n\to\infty} a_n = t$.

Proof. The function $x \mapsto F(ex)$ is a strictly increasing, continuous bijection of $[0,1]$ onto itself. By definition, the restriction of F to $(0,e)$ is differentiable with continuous derivative

$$F'(x) = \phi(x), \quad x \in (0,e),$$

where ϕ is uniquely determined by

$$\phi\left(\frac{\sin v}{v}\exp(v\cot v)\right) = \frac{1}{\pi}\sin v \exp(-v\cot v).$$

As observed in the proof of [4, Thm. 8.9], the map $v \mapsto \frac{\sin v}{v}\exp(v\cot v)$ is a strictly decreasing bijection from $(0,\pi)$ onto $(0,e)$. Moreover,

$$\frac{d}{dv}(\sin v \exp(-v\cot v)) = \frac{v}{\sin v}\exp(-v\cot v) > 0$$

for $v \in (0,\pi)$. Hence ϕ is a strictly decreasing function on $(0,e)$, which implies that F is strictly convex on $[0,e]$. Hence

$$F(ex) > (1-x)F(0) + xF(e) = x, \qquad x \in (0,1). \tag{7.9}$$

With $t \in (0,1)$ and with $(a_n)_{n=1}^\infty$ defined by (7.7) and (7.8), from (7.9) we have $a_1 = F(et) \in (t,1)$. If $a \in (t,1)$ and if $a' = aF(\frac{et}{a})$, then clearly $a' < a$. Moreover, by (7.9),

$$a' = aF\left(\frac{et}{a}\right) > a \cdot \frac{t}{a} = t.$$

Hence $(a_n)_{n=1}^\infty$ is a strictly decreasing sequence in $(t,1)$ and therefore converges. Let $a_\infty = \lim_{n\to\infty} a_n$. Then by the continuity of F on $[0,e]$, we have

$$a_\infty = a_\infty F\left(\frac{et}{a_\infty}\right).$$

Hence $F(\frac{et}{a_\infty}) = 1$, which implies $a_\infty = t$. □

Proof of Theorem 2.9. Let $T = \mathcal{UT}(X,\lambda)$ be constructed using [4, §4], as described in the introduction. For $t \in [0,1]$, let

$$\mathcal{K}_t = \{\xi \in \mathcal{H} \mid \limsup_{n\to\infty}\left(\frac{k}{e}\|T^k\xi\|^{2/k}\right) \le t\}. \tag{7.10}$$

We will show

$$\mathcal{H}_t \subseteq \mathcal{K}_t \subseteq \mathcal{H}_{F(et)}, \qquad t \in [0,1]. \tag{7.11}$$

The second inclusion in (7.11) follows immediately from Proposition 7.3. The first inclusion is trivial for $t = 0$, so we can assume $t > 0$. Letting $P_t = 1_{[0,t]}(D_0)$ be the projection onto $\mathcal{H}_t$, from [4, Lemma 4.10] we have

$$T_t \overset{\text{def}}{=} \frac{1}{\sqrt{t}} T\restriction_{\mathcal{H}_t} = P_t T P_t = \mathcal{UT}(\frac{1}{\sqrt{t}}P_t X P_t, \lambda_t), \tag{7.12}$$

where $\lambda_t : L^\infty[0,1] \to P_t L(\mathbf{F}_2) P_t$ is the injective, normal $*$–homomorphism given by $\lambda_t(f) = \lambda(f_t)$, where

$$f_t(s) = \begin{cases} f(s/t) & \text{if } s \in [0,t] \\ 0 & \text{if } s \in (t,1]. \end{cases}$$

Therefore, T_t is itself a DT$(\delta_0, 1)$–operator in $(P_t \mathcal{M} P_t, t^{-1}\tau\restriction_{P_t\mathcal{M}P_t})$. Hence, by Lemma 7.1 applied to T_t, we have, for all $\xi \in \mathcal{H}_t$,

$$\|T^k \xi\| = t^{k/2} \|T_t^k \xi\| \le \left(\frac{te}{k}\right)^{k/2} \|\xi\|.$$

Therefore, $\limsup_{k\to\infty}(\frac{k}{e}\|T^k\xi\|^{2/k}) \le t$ and $\xi \in \mathcal{K}_t$. This completes the proof of (7.11).

From (7.11), we have in particular $\mathcal{K}_0 = \mathcal{H}_0 = \{0\}$ and $\mathcal{K}_1 = \mathcal{H}_1 = \mathcal{H}$. Let $t \in (0,1)$ and let $(a_n)_{n=1}^\infty$ be the sequence defined by Lemma 7.4. We will prove by induction on n that $\mathcal{K}_t \subseteq \mathcal{H}_{a_n}$. By (7.11), $\mathcal{K}_t \subseteq \mathcal{H}_{a_1}$. Let $n \in \mathbf{N}$ and assume $\mathcal{K}_t \subseteq \mathcal{H}_{a_n}$. Then

$$\mathcal{K}_t = \{\xi \in \mathcal{H}_{a_n} \mid \limsup_{k\to\infty} \left(\frac{k}{e}\|T^k\xi\|^{2/k}\right) \le t\} \tag{7.13}$$

$$= \{\xi \in \mathcal{H}_{a_n} \mid \limsup_{k\to\infty} \left(\frac{k}{e}\|T_{a_n}^k\xi\|^{2/k}\right) \le \frac{t}{a_n}\}. \tag{7.14}$$

But the space (7.14) is the analogue of $\mathcal{K}_{t/a_n}$ for the operator T_{a_n}. By (7.11) applied to the operator T_{a_n}, we have that $\mathcal{K}_t$ is contained in the analogue of $\mathcal{H}_{F(et/a_n)}$ for T_{a_n}. Using (7.12) (with a_n instead of t), we see that this latter space is

$$\lambda_{a_n}(1_{[0,F(et/a_n)]})\mathcal{H}_{a_n} = \lambda(1_{[0,a_nF(et/a_n)]})\mathcal{H}_{a_n} = \lambda(1_{[0,a_{n+1}]})\mathcal{H}_{a_n} = \mathcal{H}_{a_{n+1}}.$$

Thus $\mathcal{K}_t \subseteq \mathcal{H}_{a_{n+1}}$ and the induction argument is complete.

Now applying Lemma 7.4, we get $\mathcal{K}_t \subseteq \bigcap_{n=1}^\infty \mathcal{H}_{a_n} = \mathcal{H}_t$, as desired. □

Appendix A. $\mathcal{D}$–Gaussianity of T, T^*

The operator T was defined in [4] as the limit in $*$–moments of upper triangular Gaussian random matrices, and it was shown in [4] that T can be constructed as $T = \mathcal{UT}(X,\lambda)$ in a von Neumann algebra $\mathcal{M}$ equipped with a normal, faithful, tracial state τ, from a semicircular element $X \in \mathcal{M}$ with $\tau(X) = 0$ and $\tau(X^2) = 1$ and an injective, unital, normal $*$–homomorphism $\lambda : L^\infty[0,1] \to \mathcal{M}$ such that $\{X\}$ and $\lambda(L^\infty[0,1])$ are free with respect to τ and $\tau \circ \lambda(f) = \int_0^1 f(t)dt$. (See the description in the introduction and [4, §4].) Let $\mathcal{D} = \lambda(L^\infty[0,1])$ and let $E_\mathcal{D} : \mathcal{M} \to \mathcal{D}$ be the τ–preserving conditional expectation onto $\mathcal{D}$.

In [7], it was asserted that T is a generalized circular element with respect to $E_\mathcal{D}$ and with a particular variance. It is the purpose of this appendix to provide a proof.

Lemma A.1. *Let $f \in L^\infty[0,1]$. Then*

$$E_\mathcal{D}(T\lambda(f)T^*) = \lambda(g), \tag{A.1}$$

$$E_\mathcal{D}(T^*\lambda(f)T) = \lambda(h), \tag{A.2}$$

$$E_\mathcal{D}(T\lambda(f)T) = 0, \tag{A.3}$$

$$E_\mathcal{D}(T^*\lambda(f)T^*) = 0, \tag{A.4}$$

where

$$g(x) = \int_x^1 f(t)dt, \qquad h(x) = \int_0^x f(t)dt. \tag{A.5}$$

Moreover,

$$E_{\mathcal{D}}(T) = 0. \tag{A.6}$$

Proof. From [4, §4], $\lim_{n\to\infty} \|T - T_n\| = 0$, where

$$T_n = \sum_{j=1}^{2^n-1} p[\tfrac{j-1}{2^n}, \tfrac{j}{2^n}]Xp[\tfrac{j}{2^n}, 1]$$

and $p[a, b] = \lambda(1_{[a,b]})$. Therefore,

$$\lim_{n\to\infty} \|E_{\mathcal{D}}(T\lambda(f)T^*) - E_{\mathcal{D}}(T_n\lambda(f)T_n^*)\| = 0.$$

We have

$$E_{\mathcal{D}}(T_n\lambda(f)T_n^*) = \sum_{j=1}^{2^n-1} p[\tfrac{j-1}{2^n}, \tfrac{j}{2^n}]E_{\mathcal{D}}(Xp[\tfrac{j}{2^n}, 1]\lambda(f)X).$$

Fixing n and letting $a = \int_{j/2^n}^1 f(t)dt$, we have

$$Xp[\tfrac{j}{2^n}, 1]\lambda(f)X = X(p[\tfrac{j}{2^n}, 1]\lambda(f) - a)X + a(X^2 - 1) + a,$$

and from this we see that $E_{\mathcal{D}}(Xp[\frac{j}{2^n}, 1]\lambda(f)X)$ is the constant $\int_{j/2^n}^1 f(t)dt$. Therefore, we get $E_{\mathcal{D}}(T_n\lambda(f)T_n^*) = \lambda(g_n)$, where

$$g_n(x) = \begin{cases} \int_{j/2^n}^1 f(t)dt & \text{if } \frac{j-1}{2^n} \le x \le \frac{j}{2^n},\ j \in \{1, \dots, 2^n - 1\} \\ 0 & \text{if } \frac{2^n-1}{2^n} \le x \le 1. \end{cases}$$

Letting $n \to \infty$, we obtain (A.1) with g as in (A.5).

Equations (A.2)–(A.4) and (A.6) are obtained similarly. □

Comparing Śniady's definition of a generalized circular element (with respect to $\mathcal{D}$) in [7] with Speicher's algorithm for passing from $\mathcal{D}$–cummulants to $\mathcal{D}$–moments in [8, §2.1 and §3.2], we see that an operator $S \in L(\mathbf{F}_2)$ is generalized circular if and only if all $\mathcal{D}$–cummulants of order $k \neq 2$ for the pair (S, S^*) vanish. Hence S is generalized circular if and only if the pair (S, S^*) is $\mathcal{D}$–Gaussian in the sense of [8, Def. 4.2.3]. Thus, in order to prove that T has the properties used in [7], it suffices to prove the following.

Proposition A.2. *The distribution of the pair T, T^* with respect to $E_{\mathcal{D}}$ is a $\mathcal{D}$–Gaussian distribution with covariance matrix determined by* (A.1)–(A.6).

Proof. Take $X_1, X_2, \ldots \in \mathcal{M}$, each a $(0,1)$–semicircular element such that

$$\mathcal{D},\ \big(\{X_j\}\big)_{j=1}^{\infty}$$

is a free family of sets of random variables. Then the family

$$\big(W^*(\mathcal{D} \cup \{X_j\})\big)_{j=1}^{\infty}$$

of $*$–subalgebras of $\mathcal{M}$ is free (over $\mathcal{D}$) with respect to $E_{\mathcal{D}}$. Let $T_j = \mathcal{UT}(X_j, \lambda)$. Then each T_j has $\mathcal{D}$–valued $*$–distribution (with respect to $E_{\mathcal{D}}$) the same as T. Therefore, by Speicher's $\mathcal{D}$–valued free central limit theorem [8, Thm. 4.2.4], the $\mathcal{D}$–valued $*$–distribution of $\frac{T_1+\cdots+T_n}{\sqrt{n}}$

converges as $n \to \infty$ to a $\mathcal{D}$–Gaussian $*$–distribution with the correct covariance. However, $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ is a $(0,1)$–semicircular element that is free from $\mathcal{D}$, and

$$\frac{T_1+\cdots+T_n}{\sqrt{n}} = \mathcal{UT}\big(\frac{X_1+\cdots+X_n}{\sqrt{n}}, \lambda\big).$$

Thus $\frac{T_1+\cdots+T_n}{\sqrt{n}}$ itself has the same $\mathcal{D}$–valued $*$–distribution as T. $\square$

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