An Extension of Seshadri's Identities for Brownian Motion

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Abstract

In this note we extend and clarify some identities in law for Brownian motion proved by V. Seshadri [8] using a new identity in law obtained by H. Matsumoto and M. Yor [6].

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1 Introduction

Let $B = (B_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion with $B_0 = 0$. For a real constant ν and $t \geq 0$, set

$$A_t^{(\nu)} = \int_0^t \exp(2(B_s + \nu s))\, ds \qquad \text{and} \qquad A_t = A_t^{(0)}.$$

Let $\mathbf{e}$ be a standard exponential random variable independent from B and let L_t denote the local time of B at 0.

Recently, Matsumoto and Yor [6] proved the following result concerning the joint law of (A_t, B_t).

Theorem 1.1 (Matsumoto-Yor) *For fixed $t > 0$, the following identity in law holds:*

$$(\mathbf{e}\, e^{-B_t} A_t, B_t) \stackrel{\text{law}}{=} (\cosh(|B_t| + L_t) - \cosh(B_t), B_t). \tag{1.1}$$

Our aim in this note is to show that this result helps us to find a nontrivial extension of some identities in law (see Theorem 2.1 below) first proved by V. Seshadri. Motivated by the aim to study the joint law of (A_t, B_t) Matsumoto and Yor [6] focused on the left-hand side of the identity (1.1). Here on the contrary we shall be mainly concerned with the right-hand side of this identity.

We also note that Donati-Martin et al. ([1], [2]) used the identity (1.1) in their computations to rederive the expression for the moments of $A_t^{(\nu)}$ earlier obtained by Dufresne [3].

2 Main Result

We begin by recalling Seshadri's identities in law [8], following closely the presentation given by M. Yor [10].

Theorem 2.1 (Seshadri) *For $t \geq 0$ given and fixed, the following identities in law hold:*

$$(|B_t| L_t, L_t - |B_t|) \stackrel{\text{law}}{=} (t\, \mathbf{e}/2, B_t) \tag{2.1}$$

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$$\Big(\frac{2|B_t|+L_t}{2}\cdot\frac{L_t}{2},|B_t|\Big)\overset{\text{law}}{=}\Big(\frac{2L_t+|B_t|}{2}\cdot\frac{|B_t|}{2},L_t\Big)\overset{\text{law}}{=}(t\,\mathbf{e}/2,|B_t|). \tag{2.2}$$

Remarks:

1) Seshadri's result (2.1) asserts that for a fixed $t>0$, the two variables $|B_t|L_t$ and $L_t-|B_t|$ are mutually independent, and $|B_t|L_t$ is exponentially distributed with parameter $\lambda=2/t$. A similar explanation goes for (2.2).

2) Note that $|B_t|$ and L_t play a symmetric role in (2.2).

To understand better the title of this note, we *reformulate* Matsumoto-Yor 's result as follows

$$\Big(\frac{1}{c}\sinh\big(\sqrt{c}\cdot\frac{2|B_t|+L_t}{2}\big)\cdot\sinh\big(\sqrt{c}\cdot\frac{L_t}{2}\big),|B_t|\Big)\overset{\text{law}}{=}\Big(\mathbf{e}/2\,e^{-\sqrt{c}B_t}\int_0^t e^{2\sqrt{c}B_s}ds,|B_t|\Big)$$

by means of the scaling property of B and simple hyperbolic identities, where $c>0$. Letting now c tend to zero, the result 2.2 of Seshadri follows.

Similarly we have:

Theorem 2.2 *For $t\geq 0$ given and fixed, the following identity in law holds for all $c>0$:*

$$\Big(\frac{1}{c}\sinh\big(\sqrt{c}|B_t|\big)\sinh\big(\sqrt{c}L_t\big),L_t-|B_t|\Big)\overset{\text{law}}{=}\Big(\mathbf{e}/2\,e^{-\sqrt{c}B_t}\int_0^t e^{2\sqrt{c}B_s}\,ds,B_t\Big) \tag{2.3}$$

Proof: A scaling argument shows that only the case $c=1$ need to be considered. Recalling that $|B_t|=\beta_t+L_t$ where $\beta_t=\int_0^t \operatorname{sgn}(B_t)\,dB_t$ (see e.g. [9]) we can rewrite (2.1) in the following manner:

$$(|B_t|\,(|B_t|-\beta_t),-\beta_t)=(|B_t\,|L_t,L_t-|B_t|)\overset{\text{law}}{=}(t\,\mathbf{e}/2,L_t-|B_t|)).$$

Recalling the well-known result concerning the joint law of (B_t,β_t) (see e.g. [9])

$$P(B_t\in dx;\beta_t\in dy)=\frac{2|x|-y}{\sqrt{2\pi t^3}}\exp\Big\{-\frac{(2|x|-y)^2}{2t}\Big\}\mathbf{1}_{\{y\leq|x|\}}\,dxdy$$

and consequently

$$(B_t\in dx;L_t\in du)=\frac{|x|+u}{\sqrt{2\pi t^3}}\exp\Big\{-\frac{(|x|+u)^2}{2t}\Big\}\mathbf{1}_{\{u\geq 0\}}\,dxdu$$

it follows that

$$P(B_t\in dx\,|\,\beta_t=y)=\frac{2|x|-y}{t}\exp\Big\{-\frac{(2|x|-y)^2}{2t}+\frac{y^2}{2t}\Big\}\mathbf{1}_{\{|x|\geq y\}}\,dx$$

for all $y\in\mathbf{R}$. Thus for every bounded Borel function f we have for all y by substituting $u=2|x|-y$ and using another well-known hyperbolic identity that

$$E[\,f\,(\sinh(|B_t|)\cdot\sinh(L_t))\mid|B_t|-L_t=y]=E[\,f\,(\sinh(|B_t|)\cdot\sinh(|B_t|-y))\mid\beta_t=y]$$

$$=\int_{|y|}^\infty\frac{u}{t}\exp\Big\{\frac{y^2}{2t}\Big\}\exp\Big\{-\frac{u^2}{2t}\Big\}f(\,(\cosh(u)-\cosh(y))/2\,)\,du$$

$$\int_0^\infty\frac{v+|y|}{t}\exp\Big\{\frac{y^2}{2t}\Big\}\exp\Big\{-\frac{(v+|y|)^2}{2t}\Big\}f(\,(\cosh(v+|y|)-\cosh(|y|))/2\,)\,dv$$

$$=E[f\,((\cosh(|B_t|+L_t)-\cosh(|B_t|)/2)\mid B_t=y]$$

which by (1.1) equals

$$E[f\,\big(\mathbf{e}/2\,e^{-B_t}A_t\big)\mid B_t=y]=E[f\,\Big(\mathbf{e}/2\,e^{-B_t}\int_0^t e^{2B_s}\,ds\Big)\mid B_t=y].$$

Alltogether we have proved that for every bounded Borel function f the following identity

$$E[\, f\,(\sinh(|B_t|)\cdot\sinh(L_t))\mid |B_t|-L_t=y\,]=E[f\left(\mathbf{e}/2\cdot e^{-B_t}\int_0^t e^{2B_s}\,ds\right)\mid B_t=y\,]$$

is true for all $y\in\mathbf{R}$ from which the result follows observing that B_t and $|B_t|-L_t$ are identically distributed.

3 Moments of $A_t^{(\nu)}$

In this section we compute moments of certain exponential Brownian functionals connected to the evaluation of Asian options. The techniques used are very simple compared to former proofs (see e.g. [4], [11]) of the same results and furthemore they can be applied in more general situations.

We shall compute all moments of the random variable $\int_0^t \exp((B_s+\nu s))ds$ i.e. we shall determine the numbers

$$E[(\int_0^t \exp((B_s+\nu s))ds)^n]$$

for all $n\geq 1$ with $\nu\in\mathbf{R}$ and $t>0$ given and fixed.

The computation will be based on the following well-known simple fact:

Lemma 3.1 *If (M_t) is a non-negative right-continuous martingale and (C_t) a continuous increasing process such that $C_0\equiv 0$, then*

$$E[\int_0^t M_s\,dC_s]=E[M_t\,C_t]$$

for all $t\geq 0$.

Since the arguments apply not only to the Brownian motion we will assume that we are given a probability space $(\Omega,\mathcal{F},P)$ and a right-continuous process $X=(X_s)_{0\leq s\leq T}$ defined on $(\Omega,\mathcal{F},P)$ that starts at 0 and has stationary independent increments (shortly called a Lévy process).

Here we assume that the Lévy exponent ψ of X defined by

$$E[\exp(aX_t)]=\exp(t\psi(a))$$

for $t\in[0,T]$ and $a\in\mathbf{R}$ is finite. In the case when X is a standard Brownian motion we have $\psi(a)=a^2/2$.

Straightforward calulations show that

$$(M_s):=(\exp(X_s-s\psi(1))_{0\leq s\leq T}$$

is a non-negative right-continuous martingale starting at 1 and that $(X_s)_{0\leq s\leq T}$ is a Lévy process on $[0,T]$ under $\widetilde{P}$, where $\widetilde{P}$ denotes the probability measure on $(\Omega,\mathcal{F})$ defined by

$$d\widetilde{P}:=M_T\,dP.$$

The corresponding Lévy exponent $\widetilde{\psi}$ is easily seen to be given by

$$\widetilde{\psi}(a)=\psi(a+1)-\psi(1)\quad(\text{for } a\in\mathbf{R}).$$

Theorem 3.1 *Let $(X_s)_{0\leq s\leq T}$ be a Lévy process on $[0,T]$ wih exponent ψ. Define for $n\geq 1$, $t\in[0,T]$ and $\upsilon\in\mathbf{R}$*

$$C_n(t,\upsilon,\psi)=E[\Big(\int_0^t \exp(X_s+\upsilon s)\,ds\Big)^n]$$

Then for $n \geq 2$ we have the following recursive relation:

$$C_n(t,\upsilon,\psi) = n\int_0^t C_{n-1}(s,\upsilon,\widetilde{\psi})\,\exp(\psi(1)s+\upsilon s)\,ds \tag{3.1}$$

for all $t \in [0,T]$ and all $\upsilon \in \mathbf{R}$.

Proof: Using Lemma 3.1 and the integration by parts formula we obtain for $n \geq 2$:

$$\begin{aligned}
C_n(t,\upsilon,\psi) &= E\Big[\Big(\int_0^t \exp(X_s+\upsilon s)\,ds\Big)^n\Big] \\
&= n\cdot E\Big[\int_0^t \Big(\int_0^s exp(X_u+\upsilon u)\,du\Big)^{n-1}\exp(X_s+\upsilon s)\,ds\Big] \\
&= n\cdot E\Big[\int_0^t M_s\Big(\int_0^s exp(X_u+\upsilon u)\,du\Big)^{n-1}\exp(\psi(1)s+\upsilon s)\,ds\Big] \\
&= n\cdot E\Big[\,M_t\int_0^t \Big(\int_0^s exp(X_u+\upsilon u)\,du\Big)^{n-1}\exp(\psi(1)s+\upsilon s)\,ds\Big] \\
&= n\int_0^t \widetilde{E}\Big[\Big(\int_0^s exp(X_u+\upsilon u))du\Big)^{n-1}\Big]\exp(\psi(1)s+\upsilon s)\,ds
\end{aligned}$$

i.e.

$$C_n(t,\upsilon,\psi) = n\int_0^t C_{n-1}(t,\upsilon,\widetilde{\psi})\,\exp(\psi(1)s+\upsilon s)\,ds.$$

□

Using induction in (3.1) the recursive formula for $(C_n(t,\upsilon,\psi))_{n\geq 1}$ can be found, and in the Brownian case we obtain the following closed expression.

Corollary 3.1 *For all $n \geq 1$ and $t \geq 0$ we have:*

$$C_n(t,\upsilon,a^2/2) = E\Big[\Big(\int_0^t \exp(B_s+\upsilon s)\,ds\Big)^n\Big] = n!\sum_{j=0}^{n}\frac{1}{\prod_{i=0,\,i\neq j}^{n}(a_j^\upsilon - a_i^\upsilon)}\exp(ta_j^\upsilon)$$

where for each $0 \leq i \leq n$

$$a_i^\upsilon = \psi(i) + i\upsilon = \frac{i^2}{2} + i\upsilon.$$

Remark:

A negative answer to the long time unsolved question of whether or not the law of $A_t^{(\nu)}$ is determined by its moments has recently been given by A. Nikeghbali [7].

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