Local Time-Space Calculus and Extensions of Itô's Formula

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Abstract

Let $X = (X_t)_{t \geq 0}$ be a continuous semimartingale and let $F : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 function. Then the following change-of-variable formula holds:

$$F(t, X_t) = F(0, X_0) + \int_0^t F_t(s, X_s)\, ds + \int_0^t F_x(s, X_s)\, dX_s - \frac{1}{2} \int_0^t \int_{\mathbb{R}} F_x(s, x)\, d\ell_s^x$$

where ℓ_s^x is the local time of X at x defined by:

$$\ell_s^x = \mathbb{P}-\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s I(x \leq X_r < x+\varepsilon)\, d\langle X, X \rangle_r$$

and $d\ell_s^x$ refers to an area integration with respect to $(s, x) \mapsto \ell_s^x$. Further extensions of this formula for non-smooth functions F are also briefly examined. The approach leads to a formal $d\ell_t^x$ calculus which appears useful in guessing a candidate formula for $F(t, X_t)$ before a rigorous proof is known or given.

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