A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group

Uffe Haagerup*† and Steen Thorbjørnsen*‡

Abstract

In the process of developing the theory of free probability and free entropy, Voiculescu introduced in 1991 a random matrix model for a free semicircular system. Since then, random matrices have played a key role in von Neumann algebra theory (cf. [V8], [V9]). The main result of this paper is the following extension of Voiculescu's random matrix result: Let $(X_1^{(n)}, \ldots, X_r^{(n)})$ be a system of r stochastically independent $n \times n$ Gaussian self-adjoint random matrices as in Voiculescu's random matrix paper [V5], and let $(x_1, \ldots, x_r)$ be a semi-circular system in a C^*-probability space. Then for every polynomial p in r non-commuting variables

$$\lim_{n\to\infty} \left\| p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big) \right\| = \|p(x_1, \ldots, x_r)\|,$$

for almost all ω in the underlying probability space. We use the result to show that the Ext-invariant for the reduced C^*-algebra of the free group on 2 generators is not a group but only a semi-group. This problem has been open since Anderson in 1978 found the first example of a C^*-algebra $\mathcal{A}$ for which $\mathrm{Ext}(\mathcal{A})$ is not a group.

1 Introduction.

A random matrix X is a matrix whose entries are real or complex random variables on a probability space $(\Omega, \mathcal{F}, P)$. As in [T], we denote by $\mathrm{SGRM}(n, \sigma^2)$ the class of complex self-adjoint $n \times n$ random matrices

$$X = (X_{ij})_{i,j=1}^n,$$

for which $(X_{ii})_i$, $(\sqrt{2}\mathrm{Re}X_{ij})_{i<j}$, $(\sqrt{2}\mathrm{Im}X_{ij})_{i<j}$ are n^2 independent identically distributed (i.i.d.) Gaussian random variables with mean value 0 and variance σ^2. In the terminology of Mehta's book [Me], X is a Gaussian unitary ensemble (GUE). In the following we put $\sigma^2 = \frac{1}{n}$ which is the normalization used in Voiculescu's random matrix paper [V5]. We shall need the following basic definitions from free probability theory (cf. [V3]):

*Department of Mathematics and Computer Science, University of Southern Denmark.

†MaPhySto - Centre for Mathematical Physics and Stochastics, funded by The Danish National Research Foundation.

‡Supported by The Danish Natural Science Research Council.

a) A C^*-probability space is a pair $(\mathcal{B}, \tau)$ consisting of a unital C^*-algebra $\mathcal{B}$ and a state τ on $\mathcal{B}$.

b) A family of elements $(a_i)_{i\in I}$ in a C^*-probability space $(\mathcal{B}, \tau)$ is free if for all $n \in \mathbb{N}$ and all polynomials $p_1, \ldots, p_n \in \mathbb{C}[X]$, one has

$$\tau(p_1(a_{i_1}) \cdots p_n(a_{i_n})) = 0,$$

whenever $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$ and $\varphi(p_k(a_{i_k})) = 0$ for $k = 1, \ldots, n$.

c) A family $(x_i)_{i\in I}$ of elements in a C^*-probability space $(\mathcal{B}, \tau)$ is a semicircular family, if $(x_i)_{i\in I}$ is a free family, $x_i = x_i^*$ for all $i \in I$ and

$$\tau(x_i^k) = \frac{1}{2\pi} \int_{-2}^{2} t^k \sqrt{4 - t^2}\, \mathrm{d}t = \begin{cases} \frac{1}{k/2+1}\binom{k}{k/2}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

for all $k \in \mathbb{N}$ and $i \in I$.

We can now formulate Voiculescu's random matrix result from [V5]: Let, for each $n \in \mathbb{N}$, $(X_i^{(n)})_{i\in I}$ be a family of independent random matrices from the class $\mathrm{SGRM}(n, \frac{1}{n})$, and let $(x_i)_{i\in I}$ be a semicircular family in a C^*-probability space $(\mathcal{B}, \tau)$. Then for all $p \in \mathbb{N}$ and all $i_1, \ldots, i_p \in I$, we have

$$\lim_{n\to\infty} \mathbb{E}\big\{ \mathrm{tr}_n\big(X_{i_1}^{(n)} \cdots X_{i_p}^{(n)}\big) \big\} = \tau(x_{i_1} \cdots x_{i_p}), \tag{1.1}$$

where tr_n is the normalized trace on $M_n(\mathbb{C})$, i.e., $\mathrm{tr}_n = \frac{1}{n}\mathrm{Tr}_n$, where $\mathrm{Tr}_n(A)$ is the sum of the diagonal elements of A. Furthermore, $\mathbb{E}$ denotes expectation (or integration) w.r.t. the probability measure P.

The special case $|I| = 1$ is Wigner's semi-circle law (cf. [Wi], [Me]). The strong law corresponding to (1.1) also holds, i.e.,

$$\lim_{n\to\infty} \mathrm{tr}_n\big(X_{i_1}^{(n)}(\omega) \cdots X_{i_p}^{(n)}(\omega)\big) = \tau(x_{i_1} \cdots x_{i_p}), \tag{1.2}$$

for almost all $\omega \in \Omega$ (cf. [Ar] for the case $|I| = 1$ and [HP], [T, Cor. 3.9] for the general case). Voiculescu's result is actually more general than the one quoted above. It also involves sequences of non random diagonal matrices. We will, however, only consider the case, where there are no diagonal matrices. The main result of this paper is that the strong version (1.2) of Voiculescu's random matrix result also holds for the operator norm in the following sense:

Theorem A. *Let $r \in \mathbb{N}$ and, for each $n \in \mathbb{N}$, let $(X_1^{(n)}, \ldots, X_r^{(n)})$ be a set of r independent random matrices from the class $\mathrm{SGRM}(n, \frac{1}{n})$. Let further $(x_1, \ldots, x_r)$ be a semicircular system in a C^*-probability space $(\mathcal{B}, \tau)$ with a faithful state τ. Then there is a P-null set $N \subseteq \Omega$ such that for all $\omega \in \Omega\backslash N$ and all polynomials p in r non-commuting variables, we have*

$$\lim_{n\to\infty} \big\| p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big) \big\| = \|p(x_1, \ldots, x_r)\|. \tag{1.3}$$

The proof of Theorem A is given in Section 7. The special case

$$\lim_{n\to\infty} \left\| X_1^{(n)}(\omega) \right\| = \|x_1\| = 2$$

is well known (cf. [BY], [Ba, Thm. 2.12] or [HT1, Thm. 3.1]).

From Theorem A above, it is not hard to obtain the following result (cf. section 8).

Theorem B. *Let $r \in \mathbb{N} \cup \{\infty\}$, $r \geq 2$, let F_r denote the free group on r generators, and let $\lambda\colon F_r \to \mathcal{B}(\ell^2(F_r))$ be the left regular representation of F_r. Then there exists a sequence of unitary representations $\pi_n\colon F_r \to M_n(\mathbb{C})$ such that for all $h_1, \ldots, h_m \in F_r$ and $c_1, \ldots, c_n \in \mathbb{C}$:*

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{m} c_j \pi_n(h_j) \right\| = \left\| \sum_{j=1}^{m} c_j \lambda(h_j) \right\|.$$

The invariant $\mathrm{Ext}(\mathcal{A})$ for separable unital C^*-algebras $\mathcal{A}$ was introduced by Brown, Douglas and Fillmore in 1973 (cf. [BDF1], [BDF2]). $\mathrm{Ext}(\mathcal{A})$ is the set of equivalence classes $[\pi]$ of one-to-one $*$-homomorphisms $\pi\colon \mathcal{A} \to \mathcal{C}(\mathcal{H})$, where $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra for the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$. The equivalence relation is defined as follows:

$$\pi_1 \sim \pi_2 \iff \exists u \in \mathcal{U}(\mathcal{B}(\mathcal{H}))\ \forall a \in \mathcal{A}\colon \pi_2(a) = \rho(u)\pi_1(a)\rho(u)^*,$$

where $\mathcal{U}(\mathcal{B}(\mathcal{H}))$ denotes the unitary group of $\mathcal{B}(\mathcal{H})$ and $\rho\colon \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is the quotient map. Since $\mathcal{H} \oplus \mathcal{H} \simeq \mathcal{H}$, the map $(\pi_1, \pi_2) \to \pi_1 \oplus \pi_2$ defines a natural semi-group structure on $\mathrm{Ext}(\mathcal{A})$. By Choi and Effros [CE], $\mathrm{Ext}(\mathcal{A})$ is a group for every separable unital nuclear C^*-algebra and by Voiculescu [V2], $\mathrm{Ext}(\mathcal{A})$ is a unital semi-group for all separable unital C^*-algebras $\mathcal{A}$. Anderson [An] provided in 1978 the first example of a unital C^*-algebra $\mathcal{A}$ for which $\mathrm{Ext}(\mathcal{A})$ is not a group. The C^*-algebra $\mathcal{A}$ in [An] is generated by the reduced C^*-algebra $C^*_{\mathrm{red}}(F_2)$ of the free group F_2 on 2 generators and a projection $p \in \mathcal{B}(\ell^2(F_2))$. Since then, it has been an open problem whether $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is a group. In [V6, Sect. 5.14], Voiculescu shows that if one could prove Theorem B, then it would follow that $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ is not a group for any $r \geq 2$. Hence we have

Corollary 1. *Let $r \in \mathbb{N} \cup \{\infty\}$, $r \geq 2$. Then $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ is not a group.*

The problem of proving Corollary 1 has been considered by a number of mathematicians; see [V6, Section 5.11] for a more detailed discussion.

In Section 9 we extend Theorem A (resp. Theorem B) to polynomials (resp. linear combinations) with coefficients in an arbitrary unital exact C^*-algebra. The first of these two results is used to provide new proofs of two key results from our previous paper [HT2]: "Random matrices and K-theory for exact C^*-algebras". Moreover, we use the second

result to make an exact computation of the constants $C(r)$, $r \in \mathbb{N}$, introduced by Junge and Pisier [JP] in connection with their proof of

$$\mathcal{B}(\mathcal{H}) \underset{\max}{\otimes} \mathcal{B}(\mathcal{H}) \neq \mathcal{B}(\mathcal{H}) \underset{\min}{\otimes} \mathcal{B}(\mathcal{H}).$$

Specifically, we prove the following

Corollary 2. *Let $r \in \mathbb{N}$, $r \geq 2$ and let $C(r)$ be the infimum of all real numbers $C > 0$ with the following property: There exists a sequence of natural numbers $(n(m))_{m\in\mathbb{N}}$ and a sequence of r-tuples $(u_1^{(m)}, \ldots, u_r^{(m)})_{m\in\mathbb{N}}$ of $n(m) \times n(m)$ unitary matrices, such that*

$$\Big\| \sum_{i=1}^{r} u_i^{(m)} \otimes \bar{u}_i^{(m')} \Big\| \leq C,$$

whenever $m, m' \in \mathbb{N}$ and $m \neq m'$. Then $C(r) = 2\sqrt{r-1}$.

Pisier proved in [P3] that $C(r) \leq 2\sqrt{r-1}$ and Valette proved subsequently in [V] that $C(r) = 2\sqrt{r-1}$, when r is of the form $r = p + 1$ for an odd prime number p.

We end section 9 by using Theorem A to prove the following result on powers of "circular" random matrices (cf. Section 9):

Corollary 3. *Let Y be a random matrix in the class $\mathrm{GRM}(n, \frac{1}{n})$, i.e., the entries of Y are i.i.d. complex Gaussian random variables with density $z \mapsto \frac{n}{\pi}\mathrm{e}^{-n|z|^2}$, $z \in \mathbb{C}$. Then for every $p \in \mathbb{N}$ and almost all $\omega \in \Omega$,*

$$\lim_{n\to\infty} \big\| Y(\omega)^p \big\| = \left(\frac{(p+1)^{p+1}}{p^p} \right)^{\frac{1}{2}}.$$

Note that for $p = 1$, Corollary 3 follows from Geman's result [Ge].

In the remainder of this introduction, we sketch the main steps in the proof of Theorem A. Throughout the paper, we denote by $\mathcal{A}_{\mathrm{sa}}$ the real vector space of self-adjoint elements in a C^*-algebra $\mathcal{A}$. In Section 2 we prove the following "linearization trick":

Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras, and let $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ be operators in $\mathcal{A}_{\mathrm{sa}}$ and $\mathcal{B}_{\mathrm{sa}}$, respectively. Assume that for all $m \in \mathbb{N}$ and all matrices $a_0, \ldots, a_r$ in $M_m(\mathbb{C})_{\mathrm{sa}}$, we have

$$\mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^{r} a_i \otimes y_i\big) \subseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^{r} a_i \otimes x_i\big),$$

where $\mathrm{sp}(T)$ denotes the spectrum of an operator T, and where $\mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{B}}$ denote the units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then there exists a unital $*$-homomorphism

$$\Phi\colon C^*(x_1, \ldots, x_r, \mathbf{1}_{\mathcal{A}}) \to C^*(y_1, \ldots, y_r, \mathbf{1}_{\mathcal{B}}),$$

such that $\Phi(x_i) = y_i$, $i = 1, \ldots, r$. In particular,

$$\|p(y_1, \ldots, y_r)\| \leq \|p(x_1, \ldots, x_r)\|,$$

for every polynomium p in r non-commuting variables.

The linearization trick allows us to conclude (see Section 7):

Lemma 1. *In order to prove Theorem A, it is sufficient to prove the following: With $(X_1^{(n)}, \ldots, X_r^{(n)})$ and $(x_1, \ldots, x_r)$ as in Theorem A, one has for all $m \in \mathbb{N}$, all matrices $a_0, \ldots, a_r$ in $M_m(\mathbb{C})_{\mathrm{sa}}$ and all $\varepsilon > 0$ that*

$$\mathrm{sp}\big(a_0 \otimes \mathbf{1}_n + \textstyle\sum_{i=1}^r a_i \otimes X_i^{(n)}(\omega)\big) \subseteq \mathrm{sp}(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) +]-\varepsilon, \varepsilon[,$$

eventually as $n \to \infty$, for almost all $\omega \in \Omega$, and where $\mathbf{1}_n$ denotes the unit of $M_n(\mathbb{C})$.

In the rest of this section, $(X_1^{(n)}, \ldots, X_r^{(n)})$ and $(x_1, \ldots, x_r)$ are defined as in Theorem A. Moreover we let $a_0, \ldots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}$ and put

$$\begin{aligned} s &= a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i \otimes x_i \\ S_n &= a_0 \otimes \mathbf{1}_n + \sum_{i=1}^r a_i \otimes X_i^{(n)}, \quad n \in \mathbb{N}. \end{aligned}$$

It was proved by Lehner in [Le] that Voiculescu's R-transform of s with amalgamation over $M_m(\mathbb{C})$ is given by

$$\mathcal{R}_s(z) = a_0 + \sum_{i=1}^r a_i z a_i, \quad z \in M_m(\mathbb{C}). \tag{1.4}$$

For $\lambda \in M_m(\mathbb{C})$, we let $\mathrm{Im}\lambda$ denote the self-adjoint matrix $\mathrm{Im}\lambda = \frac{1}{2\mathrm{i}}(\lambda - \lambda^*)$, and we put

$$\mathcal{O} = \big\{\lambda \in M_m(\mathbb{C}) \mid \mathrm{Im}\lambda \text{ is positive definite}\big\}.$$

From (1.4) one gets (cf. Section 6) that the matrix-valued Stieltjes transform of s,

$$G(\lambda) = (\mathrm{id}_m \otimes \tau)\big[(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big] \in M_m(\mathbb{C}),$$

is defined for all $\lambda \in \mathcal{O}$, and satisfies the matrix equation

$$\sum_{i=1}^r a_i G(\lambda) a_i G(\lambda) + (a_0 - \lambda) G(\lambda) + \mathbf{1}_m = 0. \tag{1.5}$$

For $\lambda \in \mathcal{O}$, we let $H_n(\lambda)$ denote the $M_m(\mathbb{C})$-valued random variable

$$H_n(\lambda) = (\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big],$$

and we put

$$G_n(\lambda) = \mathbb{E}\big\{H_n(\lambda)\big\} \in M_m(\mathbb{C}).$$

Then the following analogy to (1.5) holds (cf. Section 3):

Lemma 2 (Master equation). *For all $\lambda \in \mathcal{O}$ and $n \in \mathbb{N}$:*

$$\mathbb{E}\Big\{ \sum_{i=1}^{r} a_i H_n(\lambda) a_i H_n(\lambda) + (a_0 - \lambda) H_n(\lambda) + \mathbf{1}_m \Big\} = 0. \tag{1.6}$$

The proof of (1.6) is completely different from the proof of (1.5). It is based on the simple observation that the density of the standard Gaussian distribution, $\varphi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$ satisfies the first order differential equation $\varphi'(x) + x\varphi(x) = 0$. In the special case of a single SGRM$(n, \frac{1}{n})$ random matrix (i.e., $r = m = 1$ and $a_0 = 0, a_1 = 1$), equation (1.6) occurs in a recent paper by Pastur (cf. [Pas, Formula (2.25)]). Next we use the so-called "Gaussian Poincaré inequality" (cf. Section 4) to estimate the norm of the difference

$$\mathbb{E}\Big\{ \sum_{i=1}^{r} a_i H_n(\lambda) a_i H_n(\lambda) \Big\} - \sum_{i=1}^{r} a_i \mathbb{E}\{H_n(\lambda)\} a_i \mathbb{E}\{H_n(\lambda)\},$$

and we obtain thereby (cf. Section 4):

Lemma 3 (Master inequality). *For all $\lambda \in \mathcal{O}$ and all $n \in \mathbb{N}$, we have*

$$\Big\| \sum_{i=1}^{r} a_i G_n(\lambda) a_i G_n(\lambda) - (a_0 - \lambda) G_n(\lambda) + \mathbf{1}_m \Big\| \leq \frac{C}{n^2} \big\| (\mathrm{Im}\lambda)^{-1} \big\|^4, \tag{1.7}$$

where $C = m^3 \big\| \sum_{i=1}^{r} a_i^2 \big\|^2$.

In Section 5, we deduce from (1.5) and (1.7) that

$$\|G_n(\lambda) - G(\lambda)\| \leq \frac{4C}{n^2} \big(K + \|\lambda\|\big)^2 \big\| (\mathrm{Im}\lambda)^{-1} \big\|^7, \tag{1.8}$$

where C is as above and $K = \|a_0\| + 4 \sum_{i=1}^{r} \|a_i\|$. The estimate (1.8) implies that for every $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$:

$$\mathbb{E}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n) \varphi(S_n) \big\} = (\mathrm{tr}_m \otimes \tau)(\varphi(s)) + O\big(\tfrac{1}{n^2}\big), \tag{1.9}$$

for $n \to \infty$ (cf. Section 6). Moreover, a second application of the Gaussian Poincaré inequality yields that

$$\mathbb{V}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n) \varphi(S_n) \big\} \leq \frac{1}{n^2} \mathbb{E}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n)(\varphi'(S_n)^2) \big\}, \tag{1.10}$$

where $\mathbb{V}$ denotes the variance. Let now ψ be a C^∞-function with values in $[0, 1]$, such that ψ vanishes on a neighbourhood of the spectrum $\mathrm{sp}(s)$ of s, and such that ψ is 1 on the complement of $\mathrm{sp}(s) +]-\varepsilon, \varepsilon[$.

By applying (1.9) and (1.10) to $\varphi = \psi - 1$, one gets

$$\begin{aligned} \mathbb{E}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n) \psi(S_n) \big\} &= O(n^{-2}), \\ \mathbb{V}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n) \psi(S_n) \big\} &= O(n^{-4}), \end{aligned}$$

and by a standard application of the Borel-Cantelli lemma, this implies that

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n(\omega)) = O(n^{-4/3}),$$

for almost all $\omega \in \Omega$. But the number of eigenvalues of $S_n(\omega)$ outside $\mathrm{sp}(s) +]-\varepsilon, \varepsilon[$ is dominated by $mn(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n(\omega))$, which is $O(n^{-1/3})$ for $n \to \infty$. Being an integer, this number must therefore vanish eventually as $n \to \infty$, which shows that for almost all $\omega \in \Omega$,

$$\mathrm{sp}(S_n(\omega)) \subseteq \mathrm{sp}(s) +]-\varepsilon, \varepsilon[,$$

eventually as $n \to \infty$, and Theorem A now follows from Lemma 1.

2 A linearization Trick.

Throughout this section we consider two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and self-adjoint elements $x_1, \ldots, x_r \in \mathcal{A}$, $y_1, \ldots, y_r \in \mathcal{B}$. We put

$$\mathcal{A}_0 = C^*(\mathbf{1}_{\mathcal{A}}, x_1, \ldots, x_r) \quad \text{and} \quad \mathcal{B}_0 = C^*(\mathbf{1}_{\mathcal{B}}, y_1, \ldots, y_r).$$

Note that since $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are self-adjoint, the complex linear spaces

$$E = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{\mathcal{A}}, x_1, \ldots, x_r, \textstyle\sum_{i=1}^r x_i^2\} \quad \text{and} \quad F = \mathrm{span}_{\mathbb{C}}\{\mathbf{1}_{\mathcal{B}}, y_1, \ldots, y_r, \textstyle\sum_{i=1}^r y_i^2\}$$

are both operator systems.

2.1 Lemma. *Assume that $u_0 \colon E \to F$ is a unital completely positive (linear) mapping, such that*

$$u_0(x_i) = y_i, \qquad i = 1, 2, \ldots, r,$$

and

$$u_0\big(\textstyle\sum_{i=1}^r x_i^2\big) = \textstyle\sum_{i=1}^r y_i^2.$$

Then there exists a surjective $*$-homomorphism $u \colon \mathcal{A}_0 \to \mathcal{B}_0$, such that

$$u_0 = u_{|E}.$$

Proof. The proof is inspired by Pisier's proof of [P2, Prop. 1.7]. We may assume that $\mathcal{B}$ is a unital sub-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Combining Stinespring's theorem ([Pau, Theorem 4.1]) with Arveson's extension theorem ([Pau, Corollary 6.6]), it follows then that there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a unital $*$-homomorphism $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$, such that

$$u_0(x) = p\pi(x)p \qquad (x \in E),$$

where p is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Note, in particular, that

(a) $u_0(\mathbf{1}_{\mathcal{A}}) = p\pi(\mathbf{1}_{\mathcal{A}})p = p = \mathbf{1}_{\mathcal{B}(\mathcal{H})}$,

(b) $y_i = u_0(x_i) = p\pi(x_i)p, \;\; i = 1, \dots, r$,

(c) $\sum_{i=1}^r y_i^2 = u_0\big(\sum_{i=1}^r x_i^2\big) = \sum_{i=1}^r p\pi(x_i)^2 p$.

From (b) and (c), it follows that p commutes with $\pi(x_i)$ for all i in $\{1, 2, \dots, r\}$. Indeed, using (b) and (c), we find that

$$\sum_{i=1}^r p\pi(x_i)p\pi(x_i)p = \sum_{i=1}^r y_i^2 = \sum_{i=1}^r p\pi(x_i)^2 p,$$

so that

$$\sum_{i=1}^r p\pi(x_i)\big(\mathbf{1}_{\mathcal{B}(\mathcal{K})} - p\big)\pi(x_i)p = 0.$$

Thus, putting $b_i = (\mathbf{1}_{\mathcal{B}(\mathcal{K})} - p)\pi(x_i)p$, $i = 1, \dots, r$, we have that $\sum_{i=1}^r b_i^* b_i = 0$, so that $b_1 = \cdots = b_r = 0$. Hence, for each i in $\{1, 2, \dots, r\}$, we have

$$[p, \pi(x_i)] = p\pi(x_i) - \pi(x_i)p = p\pi(x_i)(\mathbf{1}_{\mathcal{B}(\mathcal{K})} - p) - (\mathbf{1}_{\mathcal{B}(\mathcal{K})} - p)\pi(x_i)p = b_i^* - b_i = 0,$$

as desired. Since π is a unital $*$-homomorphism, we may conclude, further, that p commutes with all elements of the C^*-algebra $\pi(\mathcal{A}_0)$.

Now define the mapping $u\colon \mathcal{A}_0 \to \mathcal{B}(\mathcal{H})$ by

$$u(a) = p\pi(a)p, \quad (a \in \mathcal{A}_0).$$

Clearly $u(a^*) = u(a)^*$ for all a in $\mathcal{A}_0$, and, using (a) above, $u(\mathbf{1}_{\mathcal{A}}) = u_0(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$. Furthermore, since p commutes with $\pi(\mathcal{A}_0)$, we find for any a, b in $\mathcal{A}_0$ that

$$u(ab) = p\pi(ab)p = p\pi(a)\pi(b)p = p\pi(a)p\pi(b)p = u(a)u(b).$$

Thus, $u\colon \mathcal{A}_0 \to \mathcal{B}(\mathcal{H})$ is a unital $*$-homomorphism, which extends u_0, and $u(\mathcal{A}_0)$ is a C^*-sub-algebra of $\mathcal{B}(\mathcal{H})$. It remains to note that $u(\mathcal{A}_0)$ is generated, as a C^*-algebra, by the set $u(\{\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r\}) = \{\mathbf{1}_{\mathcal{B}}, y_1, \dots, y_r\}$, so that $u(\mathcal{A}_0) = C^*(\mathbf{1}_{\mathcal{B}}, y_1, \dots, y_r) = \mathcal{B}_0$, as desired. ■

For any element c of a C^*-algebra $\mathcal{C}$, we denote by $\mathrm{sp}(c)$ the spectrum of c, i.e.,

$$\mathrm{sp}(c) = \{\lambda \in \mathbb{C} \mid c - \lambda\mathbf{1}_{\mathcal{C}} \text{ is not invertible}\}.$$

2.2 Theorem. *Assume that the self-adjoint elements $x_1, \dots, x_r \in \mathcal{A}$ and $y_1, \dots, y_r \in \mathcal{B}$ satisfy the property:*

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall a_0, a_1, \dots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}: \\ &\qquad \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) \supseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i\big). \end{aligned} \tag{2.1}$$

Then there exists a unique surjective unital $*$-homomorphism $\varphi\colon \mathcal{A}_0 \to \mathcal{B}_0$, such that

$$\varphi(x_i) = y_i, \qquad i = 1, 2, \dots, r.$$

Before the proof of Theorem 2.2, we make a few observations:

2.3 Remark. (1) In connection with condition (2.1) above, let V be a subspace of $M_m(\mathbb{C})$ containing the unit $\mathbf{1}_m$. Then the condition:

$$\begin{aligned} &\forall a_0, a_1, \dots, a_r \in V: \\ &\qquad \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) \supseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i\big) \end{aligned} \tag{2.2}$$

is equivalent to the condition:

$$\begin{aligned} &\forall a_0, a_1, \dots, a_r \in V: \\ &\qquad a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i \text{ is invertible} \Longrightarrow a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i \text{ is invertible.} \end{aligned} \tag{2.3}$$

Indeed, it is clear that (2.2) implies (2.3), and the reverse implication follows by replacing, for any complex number λ, the matrix $a_0 \in V$ by $a_0 - \lambda \mathbf{1}_m \in V$.

(2) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and consider the Hilbert space direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Consider further the operator R in $\mathcal{B}(\mathcal{H})$ given in matrix form as

$$R = \begin{pmatrix} x & y \\ z & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)}, \end{pmatrix}$$

where $x \in \mathcal{B}(\mathcal{H}_1), y \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then R is invertible in $\mathcal{B}(\mathcal{H})$ if and only if $x - yz$ is invertible in $\mathcal{B}(\mathcal{H}_1)$.

This follows immediately by writing

$$\begin{pmatrix} x & y \\ z & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & y \\ 0 & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix} \cdot \begin{pmatrix} x - yz & 0 \\ 0 & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & 0 \\ z & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix},$$

where the first and last matrix on the right hand side are invertible with inverses given by:

$$\begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & y \\ 0 & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & -y \\ 0 & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & 0 \\ z & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{1}_{\mathcal{B}(\mathcal{H}_1)} & 0 \\ -z & \mathbf{1}_{\mathcal{B}(\mathcal{H}_2)} \end{pmatrix}.$$

Proof of Theorem 2.2. By Lemma 2.1, our objective is to prove the existence of a unital completely positive map $u_0 \colon E \to F$, satisfying that $u_0(x_i) = y_i$, $i = 1, 2, \dots, r$ and $u_0(\sum_{i=1}^r x_i^2) = \sum_{i=1}^r y_i^2$.

Step I. We show first that the assumption (2.1) is equivalent to the seemingly stronger condition:

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall a_0, a_1, \dots, a_r \in M_m(\mathbb{C}): \\ &\qquad \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) \supseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i\big). \end{aligned} \tag{2.4}$$

Indeed, let $a_0, a_1, \dots, a_r$ be arbitrary matrices in $M_m(\mathbb{C})$ and consider then the self-adjoint matrices $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_r$ in $M_{2m}(\mathbb{C})$ given by:

$$\tilde{a}_i = \begin{pmatrix} 0 & a_i^* \\ a_i & 0 \end{pmatrix}, \qquad i = 0, 1, \dots, r.$$

Note then that

$$\tilde{a}_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^{r} \tilde{a}_i \otimes x_i = \begin{pmatrix} 0 & a_0^* \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i^* \otimes x_i \\ a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbf{1}_{\mathcal{A}} \\ \mathbf{1}_{\mathcal{A}} & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i & 0 \\ 0 & a_0^* \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i^* \otimes x_i \end{pmatrix}.$$

Therefore, $\tilde{a}_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r \tilde{a}_i \otimes x_i$ is invertible in $M_{2m}(\mathcal{A})$ if and only if $a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i$ is invertible in $M_m(\mathcal{A})$, and similarly, of course, $\tilde{a}_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r \tilde{a}_i \otimes y_i$ is invertible in $M_{2m}(\mathcal{B})$ if and only if $a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i \otimes y_i$ is invertible in $M_m(\mathcal{B})$. It follows that

$$\begin{aligned} a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i \text{ is invertible} &\iff \tilde{a}_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r \tilde{a}_i \otimes x_i \text{ is invertible} \\ &\implies \tilde{a}_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r \tilde{a}_i \otimes y_i \text{ is invertible} \\ &\iff a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i \otimes y_i \text{ is invertible,} \end{aligned}$$

where the second implication follows from the assumption (2.1). Since the argument above holds for arbitrary matrices $a_0, a_1, \ldots, a_r$ in $M_m(\mathbb{C})$, it follows from Remark 2.3(1) that condition (2.4) is satisfied.

Step II. We prove next that the assumption (2.1) implies the condition:

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall a_0, a_1, \ldots, a_r, a_{r+1} \in M_m(\mathbb{C}): \\ &\qquad \operatorname{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i + a_{r+1} \otimes \sum_{i=1}^r x_i^2\big) \\ &\qquad\qquad \supseteq \operatorname{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i + a_{r+1} \otimes \sum_{i=1}^r y_i^2\big). \end{aligned} \tag{2.5}$$

Using Remark 2.3(1), we have to show, given m in $\mathbb{N}$ and $a_0, a_1, \ldots, a_{r+1}$ in $M_m(\mathbb{C})$, that invertibility of $a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes x_i + a_{r+1} \otimes \sum_{i=1}^r x_i^2$ in $M_m(\mathcal{A})$ implies invertibility of $a_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r a_i \otimes y_i + a_{r+1} \otimes \sum_{i=1}^r y_i^2$ in $M_m(\mathcal{B})$. For this, consider the matrices:

$$S = \begin{pmatrix} a_0 \otimes \mathbf{1}_{\mathcal{A}} & -\mathbf{1}_m \otimes x_1 & -\mathbf{1}_m \otimes x_2 & \cdots & -\mathbf{1}_m \otimes x_r \\ a_1 \otimes \mathbf{1}_{\mathcal{A}} + a_{r+1} \otimes x_1 & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} & & & O \\ a_2 \otimes \mathbf{1}_{\mathcal{A}} + a_{r+1} \otimes x_2 & & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} & & \\ \vdots & & & \ddots & \\ a_r \otimes \mathbf{1}_{\mathcal{A}} + a_{r+1} \otimes x_r & O & & & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{A}} \end{pmatrix} \in M_{(r+1)m}(\mathcal{A})$$

and

$$T = \begin{pmatrix} a_0 \otimes \mathbf{1}_{\mathcal{B}} & -\mathbf{1}_m \otimes y_1 & -\mathbf{1}_m \otimes y_2 & \cdots & -\mathbf{1}_m \otimes y_r \\ a_1 \otimes \mathbf{1}_{\mathcal{B}} + a_{r+1} \otimes y_1 & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}} & & & O \\ a_2 \otimes \mathbf{1}_{\mathcal{B}} + a_{r+1} \otimes y_2 & & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}} & & \\ \vdots & & & \ddots & \\ a_r \otimes \mathbf{1}_{\mathcal{B}} + a_{r+1} \otimes y_r & O & & & \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}} \end{pmatrix} \in M_{(r+1)m}(\mathcal{B}).$$

By Remark 2.3(2), invertibility of S in $M_{(r+1)m}(\mathcal{A})$ is equivalent to invertibility of

$$\begin{aligned} a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r (\mathbf{1}_m \otimes x_i) \cdot (a_i \otimes \mathbf{1}_{\mathcal{A}} + a_{r+1} \otimes x_i) \\ = a_0 \otimes \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i \otimes x_i + a_{r+1} \otimes \sum_{i=1}^r x_i^2 \end{aligned}$$

in $M_m(\mathcal{A})$. Similarly, T is invertible in $M_{(r+1)m}(\mathcal{B})$ if and only if

$$a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes y_i + a_{r+1} \otimes \sum_{i=1}^r y_i^2$$

is invertible in $M_m(\mathcal{B})$. It remains, thus, to show that invertibility of S implies that of T. This, however, follows immediately from Step I, since we may write S and T in the form:

$$S = b_0 \otimes \mathbf{1}_{\mathcal{A}} + \sum_{i=1}^r b_i \otimes x_i \quad \text{and} \quad T = b_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r b_i \otimes y_i,$$

for suitable matrices $b_0, b_1, \ldots, b_r$ in $M_{(r+1)m}(\mathbb{C})$; namely

$$b_0 = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & \mathbf{1}_m & & & \mathrm{O} \\ a_2 & & \mathbf{1}_m & & \\ \vdots & & & \ddots & \\ a_r & \mathrm{O} & & & \mathbf{1}_m \end{pmatrix}$$

and

$$b_i = \begin{pmatrix} 0 & \cdots & 0 & -\mathbf{1}_m & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & & & & & & \\ a_{r+1} & & & \mathrm{O} & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix}, \qquad i = 1, 2, \ldots, r.$$

For i in $\{1, 2, \ldots, r\}$, the (possible) non-zero entries in b_i are at positions $(1, i+1)$ and $(i+1, 1)$. This concludes Step II.

Step III. We show, finally, the existence of a unital completely positive mapping $u_0 \colon E \to F$, satisfying that $u_0(x_i) = y_i$, $i = 1, 2, \ldots, r$ and $u_0(\sum_{i=1}^r x_i^2) = \sum_{i=1}^r y_i^2$.

Using Step II in the case $m = 1$, it follows that for any complex numbers $a_0, a_1, \ldots, a_{r+1}$, we have that

$$\mathrm{sp}\big(a_0 \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i x_i + a_{r+1} \sum_{i=1}^r x_i^2\big) \supseteq \mathrm{sp}\big(a_0 \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i y_i + a_{r+1} \sum_{i=1}^r y_i^2\big). \tag{2.6}$$

If $a_0, a_1, \ldots, a_{r+1}$ are real numbers, then the operators

$$a_0 \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i x_i + a_{r+1} \sum_{i=1}^r x_i^2 \qquad \text{and} \qquad a_0 \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^r a_i y_i + a_{r+1} \sum_{i=1}^r y_i^2$$

are self-adjoint, since $x_1, \dots, x_r$ and $y_1, \dots, y_r$ are self-adjoint. Hence (2.6) implies that

$$\begin{aligned} &\forall a_0, \dots, a_{r+1} \in \mathbb{R}: \\ &\quad \big\| a_0 \mathbf{1}_{\mathcal{A}} + \textstyle\sum_{i=1}^r a_i x_i + a_{r+1} \sum_{i=1}^r x_i^2 \big\| \geq \big\| a_0 \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i y_i + a_{r+1} \sum_{i=1}^r y_i^2 \big\|. \end{aligned} \tag{2.7}$$

Let E' and F' denote, respectively, the $\mathbb{R}$-linear span of $\{\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r, \sum_{i=1}^r x_i^2\}$ and $\{\mathbf{1}_{\mathcal{B}}, y_1, \dots, y_r, \sum_{i=1}^r y_i^2\}$:

$$E' = \operatorname{span}_{\mathbb{R}}\{\mathbf{1}_{\mathcal{A}}, x_1, \dots, x_r, \textstyle\sum_{i=1}^r x_i^2\} \quad \text{and} \quad F' = \operatorname{span}_{\mathbb{R}}\{\mathbf{1}_{\mathcal{B}}, y_1, \dots, y_r, \textstyle\sum_{i=1}^r y_i^2\}.$$

It follows then from (2.7) that there is a (well-defined) $\mathbb{R}$-linear mapping $u_0' \colon E' \to F'$ satisfying that $u_0'(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$, $u_0'(x_i) = y_i$, $i = 1, 2, \dots, r$ and $u_0'(\sum_{i=1}^r x_i^2) = \sum_{i=1}^r y_i^2$. For an arbitrary element x in E, note that $\operatorname{Re}(x) = \frac{1}{2}(x + x^*) \in E'$ and $\operatorname{Im}(x) = \frac{1}{2\mathrm{i}}(x - x^*) \in E'$. Hence, we may define a mapping $u_0 \colon E \to F$ by setting:

$$u_0(x) = u_0'(\operatorname{Re}(x)) + \mathrm{i} u_0'(\operatorname{Im}(x)), \qquad (x \in E).$$

It is straightforward, then, to check that u_0 is a $\mathbb{C}$-linear mapping from E onto F, which extends u_0'.

Finally, it follows immediately from Step II that for all m in $\mathbb{N}$, the mapping $\mathrm{id}_{M_m(\mathbb{C})} \otimes u_0$ preserves positivity. In other words, u_0 is a completely positive mapping. This concludes the proof. ∎

In Section 7, we shall need the following strengthening of Theorem 2.2:

2.4 Theorem. *Assume that the self adjoint elements $x_1, \dots, x_r \in \mathcal{A}$, $y_1, \dots, y_r \in \mathcal{B}$ satisfy the property*

$$\begin{aligned} &\forall m \in \mathbb{N}\ \forall a_0, \dots, a_r \in M_m(\mathbb{Q} + \mathrm{i}\mathbb{Q})_{\mathrm{sa}} : \\ &\quad \operatorname{sp}\big(a_0 \otimes \mathbf{1}_A + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) \supseteq \operatorname{sp}\big(a_0 \otimes 1_B + \textstyle\sum_{i=1}^r a_i \otimes y_i\big). \end{aligned} \tag{2.8}$$

Then there exists a unique surjective unital $*$-homomorphism $\varphi \colon A_0 \to B_0$ such that $\varphi(x_i) = y_i$, $i = 1, \dots, r$.

Proof. By Theorem 2.2, it suffices to prove that condition (2.8) is equivalent to condition (2.1) of that theorem. Clearly (2.1) $\Rightarrow$ (2.8). It remains to be proved that (2.8) $\Rightarrow$ (2.1). Let $d_H(K, L)$ denote the Hausdorff distance between two subsets K, L of $\mathbb{C}$:

$$d_H(K, L) = \max\Big\{ \sup_{x \in K} d(x, L),\ \sup_{y \in L} d(y, K) \Big\}. \tag{2.9}$$

For normal operators A, B in $M_m(\mathbb{C})$ or $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space) one has

$$d_H(\operatorname{sp}(A), \operatorname{sp}(B)) \leq \|A - B\| \tag{2.10}$$

(cf. [Da, Prop. 2.1]). Assume now that (2.8) is satisfied, let $m \in \mathbb{N}$, $b_0, \dots, b_r \in M_m(\mathbb{C})$ and let $\varepsilon > 0$.

Since $M_m(\mathbb{Q}+\mathrm{i}\mathbb{Q})_{\mathrm{sa}}$ is dense in $M_m(\mathbb{C})_{\mathrm{sa}}$, we can choose $a_0,\ldots,a_r \in M_m(\mathbb{Q}+\mathrm{i}\mathbb{Q})_{\mathrm{sa}}$ such that

$$\|a_0-b_0\|+\sum_{i=1}^{r}\|a_i-b_i\|\|x_i\|<\varepsilon$$

and

$$\|a_0-b_0\|+\sum_{i=1}^{r}\|a_i-b_i\|\|y_i\|<\varepsilon.$$

Hence, by (2.10),

$$d_H\big(\mathrm{sp}\big(a_0\otimes 1+\textstyle\sum_{i=1}^r a_i\otimes x_i\big),\mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes x_i\big)\big)<\varepsilon$$

and

$$d_H\big(\mathrm{sp}\big(a_0\otimes 1+\textstyle\sum_{i=1}^r a_i\otimes y_i\big),\mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes y_i\big)\big)<\varepsilon.$$

By these two inequalities and (2.8) we get

$$\begin{aligned}\mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes y_i\big) &\subseteq \mathrm{sp}\big(a_0\otimes 1+\textstyle\sum_{i=1}^r a_i\otimes y_i\big)+\,]-\varepsilon,\varepsilon[\\ &\subseteq \mathrm{sp}\big(a_0\otimes 1+\textstyle\sum_{i=1}^r a_i\otimes x_i\big)+\,]-\varepsilon,\varepsilon[\\ &\subseteq \mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes x_i\big)+\,]-2\varepsilon,2\varepsilon[.\end{aligned}$$

Since $\mathrm{sp}(b_0\otimes 1+\sum_{i=1}^r b_i\otimes y_i)$ is compact and $\varepsilon>0$ is arbitrary, it follows that

$$\mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes y_i\big)\subseteq \mathrm{sp}\big(b_0\otimes 1+\textstyle\sum_{i=1}^r b_i\otimes x_i\big),$$

for all $m\in\mathbb{N}$ and all $b_0,\ldots,b_r\in M_m(\mathbb{C})_{\mathrm{sa}}$, i.e. (2.1) holds. This completes the proof of Theorem 2.4. ∎

3 The master equation.

Let $\mathcal{H}$ be a Hilbert space. For $T\in\mathcal{B}(\mathcal{H})$ we let $\mathrm{Im}T$ denote the self adjoint operator $\mathrm{Im}T=\frac{1}{2\mathrm{i}}(T-T^*)$. We say that a matrix T in $M_m(\mathbb{C})_{\mathrm{sa}}$ is positive definite if all its eigenvalues are strictly positive, and we denote by $\lambda_{\max}(T)$ and $\lambda_{\min}(T)$ the largest and smallest eigenvalues of T, respectively.

3.1 Lemma. *(i) Let $\mathcal{H}$ be a Hilbert space and let T be an operator in $\mathcal{B}(\mathcal{H})$, such that the imaginary part $\mathrm{Im}T$ satisfies one of the two conditions:*

$$\mathrm{Im}T\geq\varepsilon\mathbf{1}_{\mathcal{B}(\mathcal{H})}\qquad\textit{or}\qquad \mathrm{Im}T\leq-\varepsilon\mathbf{1}_{\mathcal{B}(\mathcal{H})},$$

for some ε in $]0,\infty[$. Then T is invertible and $\|T^{-1}\|\leq\frac{1}{\varepsilon}$.

(ii) Let T be a matrix in $M_m(\mathbb{C})$ and assume that $\mathrm{Im}T$ is positive definite. Then T is invertible and $\|T^{-1}\| \leq \|(\mathrm{Im}T)^{-1}\|$.

Proof. Note first that (ii) is a special case of (i). Indeed, since $\mathrm{Im}T$ is self-adjoint, we have that $\mathrm{Im}T \geq \lambda_{\min}(\mathrm{Im}T)\mathbf{1}_m$. Since $\mathrm{Im}T$ is positive definite, $\lambda_{\min}(\mathrm{Im}T) > 0$, and hence (i) applies. Thus, T is invertible and, furthermore,

$$\|T^{-1}\| \leq \frac{1}{\lambda_{\min}(\mathrm{Im}T)} = \lambda_{\max}\big((\mathrm{Im}T)^{-1}\big) = \|(\mathrm{Im}T)^{-1}\|,$$

since $(\mathrm{Im}T)^{-1}$ is positive.

To prove (i), note first that by replacing, if necessary, T by $-T$, it suffices to consider the case where $\mathrm{Im}T \geq \varepsilon\mathbf{1}_{\mathcal{B}(\mathcal{H})}$. Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the norm and the inner product on $\mathcal{H}$. Then, for any unit vector ξ in $\mathcal{H}$, we have

$$\|T\xi\|^2 = \|T\xi\|^2\|\xi\|^2 \geq |\langle T\xi,\xi\rangle|^2 = \big|\langle \mathrm{Re}(T)\xi,\xi\rangle + \mathrm{i}\langle \mathrm{Im}T\xi,\xi\rangle\big|^2 \geq \langle \mathrm{Im}T\xi,\xi\rangle^2 \geq \varepsilon^2\|\xi\|^2,$$

where we used that $\langle \mathrm{Re}(T)\xi,\xi\rangle, \langle \mathrm{Im}T\xi,\xi\rangle \in \mathbb{R}$. Note further, for any unit vector ξ in $\mathcal{H}$, that

$$\|T^*\xi\|^2 \geq |\langle T^*\xi,\xi\rangle|^2 = |\langle T\xi,\xi\rangle|^2 \geq \varepsilon^2\|\xi\|^2.$$

Altogether, we have verified that $\|T\xi\| \geq \varepsilon\|\xi\|$ and that $\|T^*\xi\| \geq \varepsilon\|\xi\|$ for any (unit) vector ξ in $\mathcal{H}$, and by [Pe, Prop. 3.2.6] this implies that T is invertible and that $\|T^{-1}\| \leq \frac{1}{\varepsilon}$. ∎

3.2 Lemma. *Let $\mathcal{A}$ be a unital C^*-algebra and denote by $\mathrm{GL}(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$. Let, further, $A\colon I \to \mathrm{GL}(\mathcal{A})$ be a mapping from an open interval I in $\mathbb{R}$ into $\mathrm{GL}(\mathcal{A})$, and assume that A is differentiable, in the sense that*

$$A'(t_0) := \lim_{t\to t_0} \frac{1}{t-t_0}\big(A(t) - A(t_0)\big)$$

exists in the operator norm, for any t_0 in I. Then the mapping $t \mapsto A(t)^{-1}$ is also differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)^{-1} = -A(t)^{-1}A'(t)A(t)^{-1}, \qquad (t \in I).$$

Proof. The lemma is well known. For the reader's convenience we include a proof. For any t, t_0 in I, we have

$$\begin{aligned}\frac{1}{t-t_0}\big(A(t)^{-1} - A(t_0)^{-1}\big) &= \frac{1}{t-t_0}A(t)^{-1}\big(A(t_0) - A(t)\big)A(t_0)^{-1} \\ &= -A(t)^{-1}\Big(\frac{1}{t-t_0}\big(A(t) - A(t_0)\big)\Big)A(t_0)^{-1} \\ &\xrightarrow[t\to t_0]{} -A(t_0)^{-1}A'(t_0)A(t_0)^{-1},\end{aligned}$$

where the limit is taken in the operator norm, and we use that the mapping $B \mapsto B^{-1}$ is a homeomorphism of $\mathrm{GL}(\mathcal{A})$ w.r.t. the operator norm. ∎

3.3 Lemma. *Let σ be a positive number, let N be a positive integer and let $\gamma_1, \dots, \gamma_N$ be N independent identically distributed real valued random variables with distribution $N(0, \sigma^2)$, defined on the same probability space $(\Omega, \mathcal{F}, P)$. Consider, further, a finite dimensional vector space E and a C^1-mapping:*

$$(x_1, \dots, x_N) \mapsto F(x_1, \dots, x_N) \colon \mathbb{R}^N \to E,$$

satisfying that F and all its first order partial derivatives $\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_N}$ are polynomially bounded. For any j in $\{1, 2, \dots, N\}$, we then have

$$\mathbb{E}\big\{\gamma_j F(\gamma_1, \dots, \gamma_N)\big\} = \sigma^2 \mathbb{E}\big\{\tfrac{\partial F}{\partial x_j}(\gamma_1, \dots, \gamma_N)\big\},$$

where $\mathbb{E}$ denotes expectation w.r.t. P.

Proof. Clearly it is sufficient to treat the case $E = \mathbb{C}$. The joint distribution of $\gamma_1, \dots, \gamma_N$ is given by the density function

$$\varphi(x_1, \dots, x_N) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\big(-\tfrac{1}{2\sigma^2} \textstyle\sum_{i=1}^N x_i^2\big), \qquad (x_1, \dots, x_N) \in \mathbb{R}^N.$$

Since

$$\frac{\partial \varphi}{\partial x_j}(x_1, \dots, x_N) = -\frac{1}{\sigma^2} x_j \varphi(x_1, \dots, x_N),$$

we get by partial integration in the variable x_j,

$$\begin{aligned}
\mathbb{E}\big\{\gamma_j F(\gamma_1, \dots, \gamma_N)\big\} &= \int_{\mathbb{R}^N} F(x_1, \dots, x_N) x_j \varphi(x_1, \dots, x_N)\, \mathrm{d}x_1, \dots, \mathrm{d}x_N \\
&= -\sigma^2 \int_{\mathbb{R}^N} F(x_1, \dots, x_N) \frac{\partial \varphi}{\partial x_j}(x_1, \dots, x_N)\, \mathrm{d}x_1, \dots, \mathrm{d}x_N \\
&= \sigma^2 \int_{\mathbb{R}^N} \frac{\partial F}{\partial x_j}(x_1, \dots, x_N) \varphi(x_1, \dots, x_N)\, \mathrm{d}x_1, \dots, \mathrm{d}x_N \\
&= \sigma^2 \mathbb{E}\bigg\{\frac{\partial F}{\partial x_j}(\gamma_1, \dots, \gamma_N)\bigg\}. \qquad \blacksquare
\end{aligned}$$

Let r and n be positive integers. In the following we denote by $\mathcal{E}_{r,n}$ the *real* vector space $(M_n(\mathbb{C})_{\mathrm{sa}})^r$. Note that $\mathcal{E}_{r,n}$ is a Euclidean space with inner product $\langle \cdot, \cdot \rangle_e$ given by

$$\langle (A_1, \dots, A_r), (B_1, \dots, B_r) \rangle_e = \mathrm{Tr}_n\Big(\sum_{j=1}^r A_j B_j\Big), \qquad ((A_1, \dots, A_r), (B_1, \dots, B_r) \in \mathcal{E}_{r,n}),$$

and with norm given by

$$\|(A_1, \dots, A_r)\|_e^2 = \mathrm{Tr}_n\Big(\sum_{j=1}^r A_j^2\Big) = \sum_{j=1}^r \|A_j\|_{2,\mathrm{Tr}_n}^2, \qquad ((A_, \dots, A_r) \in \mathcal{E}_{r,n}).$$

Finally, we shall denote by $S_1(\mathcal{E}_{r,n})$ the unit sphere of $\mathcal{E}_{r,n}$ w.r.t. $\|\cdot\|_e$.

3.4 Remark. Let r, n be positive integers, and consider the linear isomorphism Ψ_0 between $M_n(\mathbb{C})_{\mathrm{sa}}$ and $\mathbb{R}^{n^2}$ given by

$$\Psi_0((a_{kl})_{1\leq k,l\leq n}) = \big((a_{kk})_{1\leq k\leq n}, (\sqrt{2}\mathrm{Re}(a_{kl}))_{1\leq k<l\leq n}, (\sqrt{2}\mathrm{Im}(a_{kl}))_{1\leq k<l\leq n}\big), \tag{3.1}$$

for $(a_{kl})_{1\leq k,l\leq n}$ in $M_n(\mathbb{C})_{\mathrm{sa}}$. We denote, further, by Ψ the natural extension of Ψ_0 to a linear isomorphism between $\mathcal{E}_{r,n}$ and $\mathbb{R}^{rn^2}$:

$$\Psi(A_1, \dots, A_r) = (\Psi_0(A_1), \dots, \Psi_0(A_r)), \qquad (A_1, \dots, A_r \in M_n(\mathbb{C})_{\mathrm{sa}}).$$

We shall identify $\mathcal{E}_{r,n}$ with $\mathbb{R}^{rn^2}$ via the isomorphism Ψ. Note that under this identification, the norm $\|\cdot\|_e$ on $\mathcal{E}_{r,n}$ corresponds to the usual Euclidean norm on $\mathbb{R}^{rn^2}$. In other words, Ψ is an isometry.

Consider next independent random matrices $X_1^{(n)}, \dots, X_r^{(n)}$ from $\mathrm{SGRM}(n, \frac{1}{n})$ as defined in the introduction. Then $\mathbb{X} = (X_1^{(n)}, \dots, X_r^{(n)})$ is a random variable taking values in $\mathcal{E}_{r,n}$, so that $\mathbb{Y} = \Psi(\mathbb{X})$ is a random variable taking values in $\mathbb{R}^{rn^2}$. From the definition of $\mathrm{SGRM}(n, \frac{1}{n})$ and the fact that $X_1^{(n)}, \dots, X_r^{(n)}$ are independent, it is easily seen that the distribution of $\mathbb{Y}$ on $\mathbb{R}^{rn^2}$ is the product measure $\mu = \nu \otimes \nu \otimes \cdots \otimes \nu$ (rn^2 terms), where ν is the Gaussian distribution with mean 0 and variance $\frac{1}{n}$.

In the following, we consider a given family $a_0, \dots, a_r$ of matrices in $M_m(\mathbb{C})_{\mathrm{sa}}$, and, for each n in $\mathbb{N}$, a family $X_1^{(n)}, \dots, X_r^{(n)}$ of independent random matrices in $\mathrm{SGRM}(n, \frac{1}{n})$. Furthermore, we consider the following random variable with values in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$:

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^{r} a_i \otimes X_i^{(n)}. \tag{3.2}$$

3.5 Lemma. *For each n in $\mathbb{N}$, let S_n be as above. For any matrix λ in $M_m(\mathbb{C})$, for which* $\mathrm{Im}\lambda$ *is positive definite, we define a random variable with values in $M_m(\mathbb{C})$ by (cf. Lemma 3.1),*

$$H_n(\lambda) = (\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big].$$

Then, for any j in $\{1, 2, \dots, r\}$, we have

$$\mathbb{E}\big\{H_n(\lambda) a_j H_n(\lambda)\big\} = \mathbb{E}\big\{(\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\mathbf{1}_m \otimes X_j^{(n)}) \cdot (\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\}.$$

Proof. Let λ be a fixed matrix in $M_m(\mathbb{C})$, such that $\mathrm{Im}\lambda$ is positive definite. Consider the canonical isomorphism $\Psi\colon \mathcal{E}_{r,n} \to \mathbb{R}^{rn^2}$, introduced in Remark 3.4, and then define the mappings $\tilde{F}\colon \mathcal{E}_{r,n} \to M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and $F\colon \mathbb{R}^{rn^2} \to M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ by (cf. Lemma 3.1

$$\tilde{F}(v_1, \dots, v_r) = \big(\lambda \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^{r} a_i \otimes v_i\big)^{-1}, \qquad (v_1, \dots, v_r \in M_n(\mathbb{C})_{\mathrm{sa}}),$$

and

$$F = \tilde{F} \circ \Psi^{-1}.$$

Note then that

$$\big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1} = F(\Psi(X_1^{(n)}, \dots, X_r^{(n)})),$$

where $\mathbb{Y} = \Psi(X_1^{(n)}, \dots, X_r^{(n)})$ is a random variable taking values in $\mathbb{R}^{rn^2}$, and the distribution of $\mathbb{Y}$ equals that of a tuple $(\gamma_1, \dots, \gamma_{rn^2})$ of rn^2 independent identically $N(0, \frac{1}{n})$-distributed real-valued random variables.

Now, let j in $\{1, 2, \dots, r\}$ be fixed, and then define

$$\begin{aligned} X_{j,k,k}^{(n)} &= (X_j^{(n)})_{kk}, \qquad (1 \le k \le n), \\ Y_{j,k,l}^{(n)} &= \sqrt{2}\mathrm{Re}(X_j^{(n)})_{k,l}, \qquad (1 \le k < l \le n), \\ Z_{j,kl}^{(n)} &= \sqrt{2}\mathrm{Im}(X_j^{(n)})_{k,l}, \qquad (1 \le k < l \le n). \end{aligned}$$

Note that $\big((X_{j,k,k}^{(n)})_{1\le k\le n}, (Y_{j,k,l}^{(n)})_{1\le k<l\le n}, (Z_{j,k,l}^{(n)})_{1\le k<l\le n}\big) = \Psi_0(X_j^{(n)})$, where Ψ_0 is the mapping defined in (3.1) of Remark 3.4. Note also that the standard orthonormal basis for $\mathbb{R}^{n^2}$ corresponds, via Ψ_0, to the following orthonormal basis for $M_n(\mathbb{C})_{\mathrm{sa}}$:

$$\begin{aligned} & e_{k,k}^{(n)}, \qquad (1 \le k \le n) \\ & f_{k,l}^{(n)} = \tfrac{1}{\sqrt{2}}\big(e_{k,l}^{(n)} + e_{l,k}^{(n)}\big) \qquad (1 \le k < l \le n), \\ & g_{k,l}^{(n)} = \tfrac{\mathrm{i}}{\sqrt{2}}\big(e_{k,l}^{(n)} - e_{l,k}^{(n)}\big) \qquad (1 \le k < l \le n). \end{aligned} \tag{3.3}$$

In other words, $\big((X_{j,k,k}^{(n)})_{1\le k\le n}, (Y_{j,k,l}^{(n)})_{1\le k<l\le n}, (Z_{j,k,l}^{(n)})_{1\le k<l\le n}\big)$ are the coefficients of $X_j^{(n)}$ w.r.t. the orthonormal basis set out in (3.3).

Combining, now, the above observations with Lemma 3.3, it follows that

$$\begin{aligned} \frac{1}{n}\mathbb{E}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\lambda \otimes \mathbf{1}_n - S_n - ta_j \otimes e_{k,k}^{(n)}\big)^{-1}\Big\} &= \mathbb{E}\big\{X_{j,k,k}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\}, \\ \frac{1}{n}\mathbb{E}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\lambda \otimes \mathbf{1}_n - S_n - ta_j \otimes f_{k,l}^{(n)}\big)^{-1}\Big\} &= \mathbb{E}\big\{Y_{j,k,l}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\}, \\ \frac{1}{n}\mathbb{E}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\lambda \otimes \mathbf{1}_n - S_n - ta_j \otimes g_{k,l}^{(n)}\big)^{-1}\Big\} &= \mathbb{E}\big\{Z_{j,k,l}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\}, \end{aligned}$$

for all values of k, l in $\{1, 2, \dots, n\}$ such that $k < l$. On the other hand, it follows from Lemma 3.2 that for any vector v in $M_n(\mathbb{C})_{\mathrm{sa}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\lambda \otimes \mathbf{1}_n - S_n - ta_j \otimes v\big)^{-1} = (\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes v)(\lambda \otimes \mathbf{1}_n - S_n)^{-1},$$

and we obtain, thus, the identities:

$$\mathbb{E}\big\{X_{j,k,k}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\} = \tfrac{1}{n}\mathbb{E}\big\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{k,k}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\} \tag{3.4}$$

$$\mathbb{E}\big\{Y_{j,k,l}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\} = \tfrac{1}{n}\mathbb{E}\big\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes f_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\} \tag{3.5}$$

$$\mathbb{E}\big\{Z_{j,k,l}^{(n)} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\} = \tfrac{1}{n}\mathbb{E}\big\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes g_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\} \tag{3.6}$$

for all relevant values of k, l, $k < l$. Note next that for $k < l$, we have

$$\begin{aligned}
(X_j^{(n)})_{k,l} &= \tfrac{1}{\sqrt{2}}\big(Y_{j,k,l}^{(n)} + \mathrm{i}Z_{j,k,l}^{(n)}\big),\\
(X_j^{(n)})_{l,k} &= \tfrac{1}{\sqrt{2}}\big(Y_{j,k,l}^{(n)} - \mathrm{i}Z_{j,k,l}^{(n)}\big),\\
e_{k,l}^{(n)} &= \tfrac{1}{\sqrt{2}}\big(f_{k,l}^{(n)} - \mathrm{i}g_{k,l}^{(n)}\big),\\
e_{l,k}^{(n)} &= \tfrac{1}{\sqrt{2}}\big(f_{k,l}^{(n)} + \mathrm{i}g_{k,l}^{(n)}\big),
\end{aligned}$$

and combining this with (3.5)-(3.6), it follows that

$$\mathbb{E}\big\{(X_j^{(n)})_{k,l} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\} = \frac{1}{n}\mathbb{E}\big\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{l,k}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\}, \tag{3.7}$$

and that

$$\mathbb{E}\big\{(X_j^{(n)})_{l,k} \cdot \big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1}\big\} = \frac{1}{n}\mathbb{E}\big\{(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\}, \tag{3.8}$$

for all k, l, $k < l$. Taking also (3.4) into account, it follows that (3.7) actually holds for all k, l in $\{1, 2, \dots, n\}$. By adding the equation (3.7) for all values of k, l and by recalling that

$$X_j^{(n)} = \sum_{1 \le k,l \le n} (X_j^{(n)})_{k,l} e_{k,l}^{(n)},$$

we conclude that

$$\begin{aligned}
&\mathbb{E}\big\{(\mathbf{1}_m \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\}\\
&\qquad = \frac{1}{n}\sum_{1 \le k,l \le n} \mathbb{E}\big\{(\mathbf{1}_m \otimes e_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{l,k}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\}.
\end{aligned} \tag{3.9}$$

To calculate the right hand side of (3.9), we write

$$\big(\lambda \otimes \mathbf{1}_n - S_n\big)^{-1} = \sum_{1 \le u,v \le n} F_{u,v} \otimes e_{u,v},$$

where, for all u, v in $\{1, 2, \dots, n\}$, $F_{u,v} \colon \Omega \to M_m(\mathbb{C})$ is an $M_m(\mathbb{C})$-valued random variable. Recall, then, that for any k, l, u, v in $\{1, 2, \dots, n\}$,

$$e_{k,l}^{(n)} \cdot e_{u,v}^{(n)} = \begin{cases} e_{k,v}, & \text{if } l = u,\\ 0, & \text{if } l \ne u. \end{cases}$$

For any fixed u, v in $\{1, 2, \dots, n\}$, it follows, thus, that

$$\sum_{1 \le k,l \le n} (\mathbf{1}_m \otimes e_{k,l}^{(n)})(F_{u,v} \otimes e_{u,v}^{(n)})(a_j \otimes e_{l,k}^{(n)}) = \begin{cases} (F_{u,u} \cdot a_j) \otimes \mathbf{1}_n, & \text{if } u = v,\\ 0, & \text{if } u \ne v. \end{cases} \tag{3.10}$$

Adding the equation (3.10) for all values of u, v in $\{1, 2, \dots, n\}$, it follows that

$$\sum_{1\le k,l\le n} (\mathbf{1}_m \otimes e_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{l,k}^{(n)}) = \big(\textstyle\sum_{u=1}^n F_{u,u} a_j\big) \otimes \mathbf{1}_n.$$

Note here that

$$\sum_{u=1}^n F_{u,u} = n \cdot \mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big] = n \cdot H_n(\lambda),$$

so that

$$\sum_{1\le k,l\le n} (\mathbf{1}_m \otimes e_{k,l}^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(a_j \otimes e_{l,k}^{(n)}) = nH_n(\lambda)a_j \otimes \mathbf{1}_n.$$

Combining this with (3.9), we find that

$$\mathbb{E}\big\{(\mathbf{1}_m \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\} = \mathbb{E}\big\{(H_n(\lambda)a_j \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\}. \tag{3.11}$$

Applying, finally, $\mathrm{id}_m \otimes \mathrm{tr}_n$ to both sides of (3.11), we conclude that

$$\begin{aligned}\mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\mathbf{1}_m \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\} &= \mathbb{E}\big\{H_n(\lambda)a_j \cdot \mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\}\\ &= \mathbb{E}\big\{H_n(\lambda)a_jH_n(\lambda)\big\},\end{aligned}$$

which is the desired formula. ∎

3.6 Theorem. (Master equation) *Let, for each n in $\mathbb{N}$, S_n be the random matrix introduced in* (3.2)*, and let λ be a matrix in $M_m(\mathbb{C})$ such that* $\mathrm{Im}(\lambda)$ *is positive definite. Then with*

$$H_n(\lambda) = (\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]$$

(cf. Lemma 3.1), we have the formula

$$\mathbb{E}\Big\{\sum_{i=1}^r a_iH_n(\lambda)a_iH_n(\lambda) + (a_0 - \lambda)H_n(\lambda) + \mathbf{1}_m\Big\} = 0, \tag{3.12}$$

as an $M_m(\mathbb{C})$-valued expectation.

Proof. By application of Lemma 3.5, we find that

$$\begin{aligned}\mathbb{E}\Big\{\sum_{j=1}^r a_jH_n(\lambda)a_jH_n(\lambda)\Big\} &= \sum_{j=1}^r a_j\mathbb{E}\big\{H_n(\lambda)a_jH_n(\lambda)\big\}\\ &= \sum_{j=1}^r a_j\mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\mathbf{1}_m \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\}\\ &= \sum_{j=1}^r \mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[(a_j \otimes \mathbf{1}_n)(\mathbf{1}_m \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\}\\ &= \sum_{j=1}^r \mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[(a_j \otimes X_j^{(n)})(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\{a_0 H_n(\lambda)\} &= \mathbb{E}\{a_0(\mathrm{id}_m \otimes \mathrm{tr}_n)((\lambda \otimes \mathbf{1}_n - S_n)^{-1})\} \\ &= \mathbb{E}\{(\mathrm{id}_m \otimes \mathrm{tr}_n)((a_0 \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\Big\{a_0 H_n(\lambda) + \sum_{i=1}^{r} a_j H_n(\lambda) a_j H_n(\lambda)\Big\} \\ &\qquad = \mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[S_n(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\} \\ &\qquad = \mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[\big(\lambda \otimes \mathbf{1}_n - (\lambda \otimes \mathbf{1}_n - S_n)\big)(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\} \\ &\qquad = \mathbb{E}\big\{\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1} - \mathbf{1}_m \otimes \mathbf{1}_n\big]\big\} \\ &\qquad = \mathbb{E}\big\{\lambda H_n(\lambda) - \mathbf{1}_m\big\},\end{aligned}$$

from which (3.12) follows readily. ∎

4 Variance estimates.

Let K be a positive integer. Then we denote by $\|\cdot\|$ the usual Euclidean norm $\mathbb{C}^K$, i.e.,

$$\|(\zeta_1, \dots, \zeta_K)\| = \big(|\zeta_1|^2 + \cdots + |\zeta_K|^2\big)^{1/2}, \qquad (\zeta_1, \dots, \zeta_K \in \mathbb{C}).$$

Furthermore, we denote by $\|\cdot\|_{2,\mathrm{Tr}_K}$ the Hilbert-Schmidt norm on $M_K(\mathbb{C})$, i.e.,

$$\|T\|_{2,\mathrm{Tr}_K} = \big(\mathrm{Tr}_K(T^*T)\big)^{1/2}, \qquad (T \in M_K(\mathbb{C})).$$

We shall also, occasionally, consider the norm $\|\cdot\|_{2,\mathrm{tr}_k}$ given by:

$$\|T\|_{2,\mathrm{tr}_K} = \big(\mathrm{tr}_K(T^*T)\big)^{1/2} = K^{-1/2}\|T\|_{2,\mathrm{Tr}_K}, \qquad (T \in M_K(\mathbb{C})).$$

4.1 Proposition. (Gaussian Poincaré inequality) *Let N be a positive integer and equip $\mathbb{R}^N$ with the probability measure $\mu = \nu \otimes \nu \otimes \cdots \otimes \nu$ (N terms), where ν is the Gaussian distribution on $\mathbb{R}$ with mean 0 and variance 1. Let $f\colon \mathbb{R}^N \to \mathbb{C}$ be a C^1-function, such that $\mathbb{E}\{|f|^2\} < \infty$. Then with $\mathbb{V}\{f\} = \mathbb{E}\{|f - \mathbb{E}\{f\}|^2\}$, we have*

$$\mathbb{V}\{f\} \le \mathbb{E}\big\{\|\mathrm{grad}(f)\|^2\big\}.$$

Proof. See [Cn, Theorem 2.1]. ∎

The Gaussian Poincaré inequality is a folklore result which goes back to the 30's (cf. Beckner [Be]). It was rediscovered by Chernoff [Cf] in 1981 in the case $N = 1$ and by Chen [Cn] in 1982 for general N. The original proof as well as Chernoff's proof is based on an expansion of f in Hermite polynomials (or tensor products of Hermite polynomials in

the case $N \geq 2$). Chen gives in [Cn] a self-contained proof which does not rely on Hermite polynomials. In a preliminary version of this paper, we proved the slightly weaker inequality: $\mathbb{V}\{f\} \leq \frac{\pi^2}{8}\mathbb{E}\{\|\mathrm{grad} f\|^2\}$ using the method of proof of [P1, Lemma 4.7]. We wish to thank Gilles Pisier for bringing the papers by Bechner, Chernoff and Chen to our attention.

4.2 Corollary. *Let $N \in \mathbb{N}$, and let $Z_1, \ldots, Z_N$ be N i.i.d. real Gaussian random variables with mean zero and variance σ^2 and let $f\colon \mathbb{R}^N \to \mathbb{C}$ be a C^1-function, such that f and* $\mathrm{grad}(f)$ *are both polynomially bounded. Then*

$$\mathbb{V}\big\{ f(Z_1, \ldots, Z_N)\big\} \leq \sigma^2 \mathbb{E}\big\{ \|(\mathrm{grad} f)(Z_1, \ldots, Z_N)\|^2 \big\}.$$

Proof. In the case $\sigma = 1$, this is an immediate consequence of Proposition 4.1. In the general case, put $Y_j = \frac{1}{\sigma} Z_j$, $j = 1, \ldots, N$, and define $g \in C^1(\mathbb{R}^N)$ by

$$g(y) = f(\sigma y), \qquad (y \in \mathbb{R}^N). \tag{4.1}$$

Then

$$(\mathrm{grad} g)(y) = \sigma (\mathrm{grad} f)(\sigma y), \qquad (y \in \mathbb{R}^N). \tag{4.2}$$

Since $Y_1, \ldots, Y_N$ are independent standard Gaussian distributed random variables, we have from Proposition 4.1 that

$$\mathbb{V}\big\{ g(Y_1, \ldots, Y_N)\big\} \leq \mathbb{E}\big\{ \|(\mathrm{grad} g)(Y_1, \ldots, Y_N)\|^2 \big\}. \tag{4.3}$$

Since $Z_j = \sigma Y_j$, $j = 1, \ldots, N$, it follows from (4.1), (4.2), and (4.3) that

$$\mathbb{V}\big\{ f(Z_1, \ldots, Z_N)\big\} \leq \sigma^2 \mathbb{E}\big\{ \|(\mathrm{grad} f)(Z_1, \ldots, Z_N)\|^2 \big\}. \qquad \blacksquare$$

4.3 Remark. Consider the canonical isomorphism $\Psi\colon \mathcal{E}_{r,n} \to \mathbb{R}^{rn^2}$ introduced in Remark 3.4. Consider, further, independent random matrices $X_1^{(n)}, \ldots, X_r^{(n)}$ from $\mathrm{SGRM}(n, \frac{1}{n})$. Then $\mathbb{X} = (X_1^{(n)}, \ldots, X_r^{(n)})$ is a random variable taking values in $\mathcal{E}_{r,n}$, so that $\mathbb{Y} = \Psi(\mathbb{X})$ is a random variable taking values in $\mathbb{R}^{rn^2}$. As mentioned in Remark 3.4, it is easily seen that the distribution of $\mathbb{Y}$ on $\mathbb{R}^{rn^2}$ is the product measure $\mu = \nu \otimes \nu \otimes \cdots \otimes \nu$ (rn^2 terms), where ν is the Gaussian distribution with mean 0 and variance $\frac{1}{n}$. Now, let $\tilde{f}\colon \mathbb{R}^{rn^2} \to \mathbb{C}$ be a C^1-function, such that $\tilde{f}$ and $\mathrm{grad}\tilde{f}$ are both polynomially bounded, and consider, further, the C^1-function $f\colon \mathcal{E}_{r,n} \to \mathbb{C}$ given by $f = \tilde{f} \circ \Psi$. T Since Ψ is a linear isometry (i.e., an orthogonal transformation), it follows from Corollary 4.2 that

$$\mathbb{V}\big\{ f(\mathbb{X})\big\} \leq \frac{1}{n} \mathbb{E}\big\{ \big\|\mathrm{grad} f(\mathbb{X})\big\|_e^2 \big\}. \tag{4.4}$$

4.4 Lemma. *Let m, n be positive integers, and assume that $a_1, \ldots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}$ and $w_1, \ldots, w_r \in M_n(\mathbb{C})$. Then*

$$\Big\| \sum_{i=1}^r a_i \otimes w_i \Big\|_{2, \mathrm{Tr}_m \otimes \mathrm{Tr}_n} \leq m^{1/2} \Big\| \sum_{i=1}^r a_i^2 \Big\|^{1/2} \Big(\sum_{i=1}^r \|w_i\|_{2, \mathrm{Tr}_n}^2 \Big)^{1/2}$$

Proof. We find that

$$\begin{aligned}\big\|\textstyle\sum_{i=1}^r a_i\otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n} &\le \textstyle\sum_{i=1}^r \|a_i\otimes w_i\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}\\ &= \textstyle\sum_{i=1}^r \|a_i\|_{2,\mathrm{Tr}_m}\cdot\|w_i\|_{2,\mathrm{Tr}_n}\\ &\le \big(\textstyle\sum_{i=1}^r \|a_i\|_{2,\mathrm{Tr}_m}^2\big)^{1/2}\big(\textstyle\sum_{i=1}^r \|w_i\|_{2,\mathrm{Tr}_n}^2\big)^{1/2}\\ &= \big(\mathrm{Tr}_m\big(\textstyle\sum_{i=1}^r a_i^2\big)\big)^{1/2}\cdot\big(\textstyle\sum_{i=1}^r \|w_i\|_{2,\mathrm{Tr}_n}^2\big)^{1/2}\\ &\le m^{1/2}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|^{1/2}\cdot\big(\textstyle\sum_{i=1}^r \|w_i\|_{2,\mathrm{Tr}_n}^2\big)^{1/2}. \qquad \blacksquare\end{aligned}$$

Note, in particular, that if $w_1,\dots,w_r\in M_n(\mathbb{C})_{\mathrm{sa}}$, then Lemma 4.4 provides the estimate:

$$\big\|\textstyle\sum_{i=1}^r a_i\otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}\le m^{1/2}\big(\textstyle\sum_{i=1}^r\|a_i\|^2\big)^{1/2}\cdot\big\|(w_1,\dots,w_r)\big\|_e.$$

4.5 Theorem. (Master inequality) *Let λ be a matrix in $M_m(\mathbb{C})$ such that $\mathrm{Im}(\lambda)$ is positive definite. Consider, further, the random matrix $H_n(\lambda)$ introduced in Theorem 3.6 and put*

$$G_n(\lambda)=\mathbb{E}\big\{H_n(\lambda)\big\}\in M_m(\mathbb{C}).$$

Then

$$\Big\|\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\lambda)G_n(\lambda)+\mathbf{1}_m\Big\|\le\frac{C}{n^2}\big\|(\mathrm{Im}(\lambda))^{-1}\big\|^4,$$

where $C=m^3\|\sum_{i=1}^r a_i^2\|^2$.

Proof. We put

$$K_n(\lambda)=H_n(\lambda)-G_n(\lambda)=H_n(\lambda)-\mathbb{E}\big\{H_n(\lambda)\big\}.$$

Then, by Theorem 3.6, we have

$$\begin{aligned}&\mathbb{E}\Big\{\sum_{i=1}^r a_iK_n(\lambda)a_iK_n(\lambda)\Big\}\\ &\qquad=\mathbb{E}\Big\{\sum_{i=1}^r a_i\big(H_n(\lambda)-G_n(\lambda)\big)a_i\big(H_n(\lambda)-G_n(\lambda)\big)\Big\}\\ &\qquad=\mathbb{E}\Big\{\sum_{i=1}^r a_iH_n(\lambda)a_iH_n(\lambda)\Big\}-\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)\\ &\qquad=\Big(-(a_0-\lambda)\mathbb{E}\big\{H_n(\lambda)\big\}-\mathbf{1}_m\Big)-\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)\\ &\qquad=-\Big(\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\lambda)G_n(\lambda)+\mathbf{1}_m\Big).\end{aligned}$$

Hence, we can make the following estimates

$$\begin{aligned}\Big\| \sum_{i=1}^r a_i G_n(\lambda) a_i G_n(\lambda) + (a_0 - \lambda) G_n(\lambda) + \mathbf{1}_m \Big\| &= \Big\| \mathbb{E}\Big\{ \sum_{i=1}^r a_i K_n(\lambda) a_i K_n(\lambda) \Big\} \Big\| \\ &\leq \mathbb{E}\Big\{ \Big\| \sum_{i=1}^r a_i K_n(\lambda) a_i K_n(\lambda) \Big\| \Big\} \\ &\leq \mathbb{E}\Big\{ \Big\| \sum_{i=1}^r a_i K_n(\lambda) a_i \Big\| \cdot \big\| K_n(\lambda) \big\| \Big\}.\end{aligned}$$

Note here that since $a_1, \dots, a_r$ are self-adjoint, the mapping $v \mapsto \sum_{i=1}^r a_i v a_i \colon M_m(\mathbb{C}) \to M_m(\mathbb{C})$ is completely positive. Therefore, it attains its norm at the unit $\mathbf{1}_m$, and the norm is $\| \sum_{i=1}^r a_i^2 \|$. Using this in the estimates above, we find that

$$\begin{aligned}\Big\| \sum_{i=1}^r a_i G_n(\lambda) a_i G_n(\lambda) + (a_0 - \lambda) G_n(\lambda) + \mathbf{1}_m \Big\| &\leq \Big\| \sum_{i=1}^r a_i^2 \Big\| \cdot \mathbb{E}\Big\{ \big\| K_n(\lambda) \big\|^2 \Big\} \\ &\leq \Big\| \sum_{i=1}^r a_i^2 \Big\| \cdot \mathbb{E}\Big\{ \big\| K_n(\lambda) \big\|_{2,\mathrm{Tr}_m}^2 \Big\},\end{aligned} \tag{4.5}$$

where the last inequality uses that the operator norm of a matrix is always dominated by the Hilbert-Schmidt norm. It remains to estimate $\mathbb{E}\{\|K_n(\lambda)\|_{2,\mathrm{Tr}_m}^2\}$. For this, let $H_{n,j,k}(\lambda)$, $(1 \leq j, k \leq n)$ denote the entries of $H_n(\lambda)$, i.e.,

$$H_n(\lambda) = \sum_{j,k=1}^m H_{n,j,k}(\lambda) e(m,j,k), \tag{4.6}$$

where $e(m,j,k)$, $(1 \leq j, k \leq m)$ are the usual $m \times m$ matrix units. Let, correspondingly, $K_{n,j,k}(\lambda)$ denote the entries of $K_n(\lambda)$. Then $K_{n,j,k}(\lambda) = H_{n,j,k}(\lambda) - \mathbb{E}\{H_{n,j,k}(\lambda)\}$, for all j, k, so that $\mathbb{V}\{H_{n,j,k}(\lambda)\} = \mathbb{E}\{|K_{n,j,k}(\lambda)|^2\}$. It follows, thus, that

$$\mathbb{E}\Big\{ \big\| K_n(\lambda) \big\|_{2,\mathrm{Tr}_m}^2 \Big\} = \mathbb{E}\Big\{ \sum_{j,k=1}^m |K_{n,j,k}(\lambda)|^2 \Big\} = \sum_{j,k=1}^m \mathbb{V}\big\{ H_{n,j,k}(\lambda) \big\}. \tag{4.7}$$

Note further that by (4.6)

$$\begin{aligned}H_{n,j,k}(\lambda) &= \mathrm{Tr}_m\big(e(m,k,j) H_n(\lambda) \big) \\ &= m \cdot \mathrm{tr}_m\big(e(m,k,j) \cdot (\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1} \big] \big) \\ &= m \cdot \mathrm{tr}_m \otimes \mathrm{tr}_n\big[(e(m,j,k) \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n)^{-1} \big].\end{aligned}$$

For any j, k in $\{1, 2, \dots, m\}$, consider, next, the mapping $f_{n,j,k} \colon \mathcal{E}_{r,n} \to \mathbb{C}$ given by:

$$f_{n,j,k}(v_1, \dots, v_r) = m \cdot (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[(e(m,k,j) \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes v_i)^{-1} \big],$$

for all $v_1, \dots, v_r$ in $M_n(\mathbb{C})_{\mathrm{sa}}$. Note then that

$$H_{n,j,k}(\lambda) = f_{n,j,k}(X_1^{(n)}, \dots, X_r^{(n)}),$$

for all j, k. Using now the "concentration estimate" (4.4) in Remark 4.3, it follows that for all j, k,

$$\mathbb{V}\big\{H_{n,j,k}(\lambda)\big\} \leq \frac{1}{n}\mathbb{E}\Big\{\big\|\mathrm{grad} f_{n,j,k}(X_1^{(n)}, \dots, X_r^{(n)})\big\|_e^2\Big\}. \tag{4.8}$$

For fixed j, k in $\{1, 2, \dots, m\}$ and $v = (v_1, \dots, v_r)$ in $\mathcal{E}_{r,n}$, note that $\mathrm{grad} f_{n,j,k}(v)$ is the vector in $\mathcal{E}_{r,n}$, characterized by the property that

$$\big\langle \mathrm{grad} f_{n,j,k}(v), w\big\rangle_e = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,j,k}(v + tw),$$

for any vector $w = (w_1, \dots, w_r)$ in $\mathcal{E}_{r,n}$. It follows, thus, that

$$\big\|\mathrm{grad} f_{n,j,k}(v)\big\|_e^2 = \max_{w \in S_1(\mathcal{E}_{r,n})} \big|\big\langle \mathrm{grad} f_{n,j,k}(v), w\big\rangle_e\big|^2 = \max_{w \in S_1(\mathcal{E}_{r,n})} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,j,k}(v + tw)\Big|^2. \tag{4.9}$$

Let $v = (v_1, \dots, v_n)$ be a fixed vector in $\mathcal{E}_{r,n}$, and put $\Sigma = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^r a_i \otimes v_i$. Let, further, $w = (w_1, \dots, w_n)$ be a fixed vector in $S_1(\mathcal{E}_{r,n})$. It follows then by Lemma 3.2 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f_{n,j,k}(v + tw) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} m \cdot (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[(e(m,k,j) \otimes \mathbf{1}_n)\big(\lambda \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes (v_i + tw_i)\big)^{-1}\big] \\
&= m \cdot (\mathrm{tr}_m \otimes \mathrm{tr}_n)\Big[(e(m,k,j) \otimes \mathbf{1}_n)\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\lambda \otimes \mathbf{1}_n - a_0 \otimes \mathbf{1}_n - \textstyle\sum_{i=1}^r a_i \otimes (v_i + tw_i)\big)^{-1}\Big] \\
&= m \cdot (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[(e(m,k,j) \otimes \mathbf{1}_n)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i \otimes w_i\big)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big].
\end{aligned} \tag{4.10}$$

Using next the Cauchy-Schwartz inequality for $\mathrm{Tr}_n \otimes \mathrm{Tr}_m$, we find that

$$\begin{aligned}
&m^2\big|(\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[e(m,k,j) \otimes \mathbf{1}_n \cdot \big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i \otimes w_i\big)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big]\big|^2 \\
&= \frac{1}{n^2}\big|(\mathrm{Tr}_m \otimes \mathrm{Tr}_n)\big[e(m,k,j) \otimes \mathbf{1}_n \cdot \big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i \otimes w_i\big)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big]\big|^2 \\
&\leq \frac{1}{n^2}\big\|e(m,j,k) \otimes \mathbf{1}_n\big\|_{2,\mathrm{Tr}_m \otimes \mathrm{Tr}_n}^2 \cdot \big\|\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i \otimes w_i\big)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big\|_{2,\mathrm{Tr}_m \otimes \mathrm{Tr}_n}^2 \\
&= \frac{1}{n}\big\|\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i \otimes w_i\big)\big(\lambda \otimes \mathbf{1}_n - \Sigma\big)^{-1}\big\|_{2,\mathrm{Tr}_m \otimes \mathrm{Tr}_n}^2.
\end{aligned} \tag{4.11}$$

Note here that

$$\begin{aligned}
&\big\|\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i\otimes w_i\big)\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2\\
&\qquad\le\big\|\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|^2\cdot\big\|\textstyle\sum_{i=1}^r a_i\otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2\cdot\big\|\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|^2\\
&\qquad\le\big\|\textstyle\sum_{i=1}^r a_i\otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4,
\end{aligned}$$

where the last inequality uses Lemma 3.1 and the fact that Σ is self-adjoint:

$$\big\|\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|\le\big\|\big(\mathrm{Im}(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|=\big\|\big(\mathrm{Im}(\lambda\otimes\mathbf{1}_n)\big)^{-1}\big\|=\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|.$$

Note further that by Lemma 4.4, $\|\sum_{i=1}^r a_i\otimes w_i\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}\le m^{1/2}\big\|\sum_{i=1}^r a_i^2\big\|^{1/2}$, since $w=(w_1,\dots,w_r)\in S_1(\mathcal{E}_{r,n})$. We conclude, thus, that

$$\big\|\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big(\textstyle\sum_{i=1}^r a_i\otimes w_i\big)\big(\lambda\otimes\mathbf{1}_n-\Sigma\big)^{-1}\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2\le m\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4. \tag{4.12}$$

Combining now formulas (4.10)-(4.12), it follows that for any j,k in $\{1,2,\dots,m\}$, any vector $v=(v_1,\dots,v_r)$ in $\mathcal{E}_{r,n}$ and any unit vector $w=(w_1,\dots,w_r)$ in $\mathcal{E}_{r,n}$, we have that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}f_{n,j,k}(v+tw)\Big|^2\le\frac{m}{n}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4,$$

and hence, by (4.9),

$$\big\|\mathrm{grad}f_{n,j,k}(v)\big\|_e^2\le\frac{m}{n}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4.$$

Note that this estimate holds at any point $v=(v_1,\dots,v_r)$ in $\mathcal{E}_{r,n}$. Using this in conjunction with (4.8), we may, thus, conclude that

$$\mathbb{V}\big\{H_{n,j,k}(\lambda)\big\}\le\frac{m}{n^2}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4,$$

for any j,k in $\{1,2\dots,m\}$, and hence, by (4.7)

$$\mathbb{E}\Big\{\big\|K_n(\lambda)\big\|_{2,\mathrm{Tr}_m}^2\Big\}\le\frac{m^3}{n^2}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4. \tag{4.13}$$

Inserting, finally, (4.13) into (4.5), we find that

$$\Big\|\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\lambda)G_n(\lambda)+\mathbf{1}_m\Big\|\le\frac{m^3}{n^2}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|^2\cdot\big\|\big(\mathrm{Im}(\lambda)\big)^{-1}\big\|^4,$$

and this is the desired estimate ■

4.6 Lemma. *Let N be a positive integer, let I be an open interval in $\mathbb{R}$, and let $t\mapsto a(t)\colon I\to M_N(\mathbb{C})_{\mathrm{sa}}$ be a C^1-function. Consider further a function φ in $C^1(\mathbb{R})$. Then the function $t\mapsto\mathrm{tr}_N[\varphi(a(t))]$ is C^1-function on I, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}_N\big[\varphi(a(t))\big]=\mathrm{tr}_N\big[\varphi'(a(t))\cdot a'(t)\big].$$

Proof. This is well known. For the reader's convenience we include a proof: Note first that for any k in $\mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(a(t)^k\big) = \sum_{j=0}^{k-1} a(t)^j a'(t) a(t)^{k-j-1}.$$

Hence, by the trace property $\mathrm{tr}_N(xy) = \mathrm{tr}_N(yx)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{tr}_N(a(t)^k) = \mathrm{tr}_N(ka(t)^{k-1}a'(t)).$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}_N(p(a(t))) = \mathrm{tr}_N(p'(a(t))a'(t))$$

for all polynomials $p \in \mathbb{C}[X]$. The general case $\varphi \in C^1(I)$ follows easily from this by choosing a sequence of polynomials $p_n \in \mathbb{C}[X]$, such that $p_n \to \varphi$ and $p_n' \to \varphi'$ uniformly on compact subsets of I, as $n \to \infty$. ∎

4.7 Proposition. *Let $a_0, a_1, \dots, a_r$ be matrices in $M_m(\mathbb{C})_{\mathrm{sa}}$ and put as in* (3.1)

$$S_n = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^{r} a_i \otimes X_i^{(n)}.$$

Let, further, $\varphi\colon \mathbb{R} \to \mathbb{C}$ be a C^1-function with compact support, and consider the random matrices $\varphi(S_n)$ and $\varphi'(S_n)$ obtained by applying the spectral mapping associated to the self-adjoint (random) matrix S_n. We then have:

$$\mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[\varphi(S_n)]\big\} \le \frac{1}{n^2}\Big\|\sum_{i=1}^{r} a_i^2\Big\|^2 \mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[|\varphi'|^2(S_n)\big]\big\}.$$

Proof. Consider the mappings $g\colon \mathcal{E}_{r,n} \to M_{nm}(\mathbb{C})_{\mathrm{sa}}$ and $f\colon \mathcal{E}_{r,n} \to \mathbb{C}$ given by

$$g(v_1, \dots, v_r) = a_0 \otimes \mathbf{1}_n + \sum_{i=1}^{r} a_i \otimes v_i, \qquad (v_1, \dots, v_r \in M_n(\mathbb{C})_{\mathrm{sa}}),$$

and

$$f(v_1, \dots, v_r) = (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[\varphi(g(v_1, \dots, v_r))\big], \qquad (v_1, \dots, v_r \in M_m(\mathbb{C})_{\mathrm{sa}}),$$

Note then that $S_n = g(X_1^{(n)}, \dots, X_r^{(n)})$ and that $(\mathrm{tr}_m \otimes \mathrm{tr}_n)[\varphi(S_n)] = f(X_1^{(n)}, \dots, X_r^{(n)})$. Note also that f is a bounded function on $M_n(\mathbb{C})_{\mathrm{sa}}$, and, by Lemma 4.6, it has bounded continuous partial derivatives. Hence, we obtain from (4.4) in Remark 4.3 that

$$\mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[\varphi(S_n)]\big\} \le \frac{1}{n}\mathbb{E}\Big\{\big\|\mathrm{grad} f(X_1^{(n)}, \dots, X_r^{(n)})\big\|_e^2\Big\}. \tag{4.14}$$

Recall next that for any v in $\mathcal{E}_{r,n}$, $\mathrm{grad} f(v)$ is the vector in $\mathcal{E}_{r,n}$, characterized by the property that

$$\big\langle \mathrm{grad} f(v), w\big\rangle_e = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw),$$

for any vector $w = (w_1, \dots, w_r)$ in $\mathcal{E}_{r,n}$. It follows, thus, that

$$\big\|\mathrm{grad} f(v)\big\|_e^2 = \max_{w\in S_1(\mathcal{E}_{r,n})} \big|\big\langle \mathrm{grad} f(v), w\big\rangle_e\big|^2 = \max_{w\in S_1(\mathcal{E}_{r,n})} \Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\Big|^2, \tag{4.15}$$

at any point $v = (v_1, \dots, v_r)$ of $\mathcal{E}_{r,n}$. Now, let $v = (v_1, \dots, v_r)$ be a fixed point in $\mathcal{E}_{r,n}$ and let $w = (w_1, \dots, w_r)$ be a fixed point in $S_1(\mathcal{E}_{r,n})$. By Lemma 4.6, we have then that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[\varphi(g(v+tw))\big] \\ &= (\mathrm{tr}_m \otimes \mathrm{tr}_n)\Big[\varphi'(g(v)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(v+tw)\Big] \\ &= (\mathrm{tr}_m \otimes \mathrm{tr}_n)\big[\varphi'(g(v)) \cdot \textstyle\sum_{i=1}^r a_i \otimes w_i\big].\end{aligned}$$

Using then the Cauchy-Schwartz inequality for $\mathrm{Tr}_m \otimes \mathrm{Tr}_n$, we find that

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\Big|^2 &= \frac{1}{m^2n^2}\Big|(\mathrm{Tr}_m \otimes \mathrm{Tr}_n)\big[\varphi'(g(v)) \cdot \textstyle\sum_{i=1}^r a_i \otimes w_i\big]\Big|^2 \\ &= \frac{1}{n^2m^2}\big\|\overline{\varphi}'(g(v))\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2 \cdot \big\|\textstyle\sum_{i=1}^r a_i \otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2.\end{aligned}$$

Note here that

$$\big\|\overline{\varphi}'(g(v))\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2 = \mathrm{Tr}_m \otimes \mathrm{Tr}_n\big[|\varphi'|^2(g(v))\big] = mn \cdot \mathrm{tr}_m \otimes \mathrm{tr}_n\big[|\varphi'|^2(g(v))\big],$$

and that, by Lemma 4.4,

$$\big\|\textstyle\sum_{i=1}^r a_i \otimes w_i\big\|_{2,\mathrm{Tr}_m\otimes\mathrm{Tr}_n}^2 \le m\big\|\sum_{i=1}^r a_i^2\big\|,$$

since w is a unit vector w.r.t. $\|\cdot\|_e$. We find, thus, that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(v+tw)\Big|^2 \le \frac{1}{n}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\mathrm{tr}_m \otimes \mathrm{tr}_n\big[|\varphi'|^2(g(v))\big].$$

Since this estimate holds for any unit vector w in $\mathcal{E}_{r,n}$, we conclude, using (4.15), that

$$\big\|\mathrm{grad} f(v)\big\|_e^2 \le \frac{1}{n}\big\|\textstyle\sum_{i=1}^r a_i^2\big\|\mathrm{tr}_m \otimes \mathrm{tr}_n\big[|\varphi'|^2(g(v))\big],$$

for any point v in $\mathcal{E}_{r,n}$. Combining this with (4.14), we obtain the desired estimate. ∎

5 Estimation of $\|G_n(\lambda) - G(\lambda)\|$.

5.1 Lemma. *For each n in $\mathbb{N}$, let X_n be a random matrix in* $\mathrm{SGRM}(n, \frac{1}{n})$. *Then*

$$\mathbb{E}\big\{\|X_n\|\big\} \leq 2 + 2\sqrt{\frac{\log(2n)}{2n}}, \qquad (n \in \mathbb{N}). \tag{5.1}$$

In particular, it follows that

$$\mathbb{E}\big\{\|X_n\|\big\} \leq 4, \tag{5.2}$$

for all n in $\mathbb{N}$.

Proof. In [HT1, Proof of Lemma 3.3] it was proved that for any n in $\mathbb{N}$ and any positive number t, we have

$$\mathbb{E}\big\{\mathrm{Tr}_n(\exp(tX_n))\big\} \leq n \exp\big(2t + \tfrac{t^2}{2n}\big). \tag{5.3}$$

Let $\lambda_{\max}(X_n)$ and $\lambda_{\min}(X_n)$ denote the largest and smallest eigenvalue of X_n as functions of $\omega \in \Omega$. Then

$$\begin{aligned}\exp(t\|X_n\|) &= \max\{\exp(t\lambda_{\max}(X_n)), \exp(-t\lambda_{\min}(X_n))\} \\ &\leq \exp(t\lambda_{\max}(X_n)) + \exp(-t\lambda_{\min}(X_n)) \leq \mathrm{Tr}_n\big(\exp(tX_n) + \exp(-tX_n)\big).\end{aligned}$$

Using this in connection with Jensen's inequality, we find that

$$\begin{aligned}\exp\big(t\mathbb{E}\{\|X_n\|\}\big) &\leq \mathbb{E}\big\{\exp(t\|X_n\|)\big\} \leq \mathbb{E}\big\{\mathrm{Tr}_n(\exp(tX_n))\big\} + \mathbb{E}\big\{\mathrm{Tr}_n(\exp(-tX_n))\big\} \\ &= 2\mathbb{E}\big\{\mathrm{Tr}_n(\exp(tX_n))\big\},\end{aligned} \tag{5.4}$$

where the last equality is due to the fact that $-X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ too. Combining (5.3) and (5.4) we obtain the estimate

$$\exp\big(t\mathbb{E}\{\|X_n\|\}\big) \leq 2n \exp\big(2t + \tfrac{t^2}{2n}\big),$$

and hence, after taking logarithms and dividing by t,

$$\mathbb{E}\{\|X_n\|\} \leq \frac{\log(2n)}{t} + 2 + \frac{t}{2n}. \tag{5.5}$$

This estimate holds for all positive numbers t. As a function of t, the right hand side of (5.5) attains its minimal value at $t_0 = \sqrt{2n\log(2n)}$ and the minimal value is $2 + 2\sqrt{\log(2n)/2n}$. Combining this with (5.5) we obtain (5.1). The estimate (5.2) follows, subsequently, by noting that the function $t \mapsto \log(t)/t$ $(t > 0)$ attains its maximal value at $t = \mathrm{e}$, and thus $2 + 2\sqrt{\log(t)/t} \leq 2 + 2\sqrt{1/\mathrm{e}} \approx 3.21$ for all positive numbers t. ∎

In the following we consider a fixed positive integer m and fixed self-adjoint matrices $a_0, \ldots, a_r$ in $M_m(\mathbb{C})_{\mathrm{sa}}$. We consider further, for each positive integer n, independent random matrices $X_1^{(n)}, \ldots, X_r^{(n)}$ in $\mathrm{SGRM}(n, \frac{1}{n})$. As in sections 3 and 4, we define

$$S_n = a_0 + \sum_{i=1}^{r} a_i \otimes X_i^{(n)}.$$

and, for any matrix λ in $M_m(\mathbb{C})$ such that $\mathrm{Im}(\lambda)$ is positive definite, we put

$$H_n(\lambda) = (\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big],$$

and

$$G_n(\lambda) = \mathbb{E}\{H_n(\lambda)\}.$$

5.2 Proposition. *Let λ be a matrix in $M_m(\mathbb{C})$ such that $\mathrm{Im}(\lambda)$ is positive definite. Then $G_n(\lambda)$ is invertible and*

$$\big\|G_n(\lambda)^{-1}\big\| \leq \big(\|\lambda\| + K\big)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|,$$

where $K = \|a_0\| + 4\sum_{i=1}^r \|a_i\|$.

Proof. We note first that

$$\begin{aligned}
&\mathrm{Im}\big((\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big) \\
&\quad = \frac{1}{2\mathrm{i}}\big((\lambda \otimes \mathbf{1}_n - S_n)^{-1} - (\lambda^* \otimes \mathbf{1}_n - S_n)^{-1}\big) \\
&\quad = \frac{1}{2\mathrm{i}}\big((\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big((\lambda^* \otimes \mathbf{1}_n - S_n) - (\lambda \otimes \mathbf{1}_n - S_n)\big)(\lambda^* \otimes \mathbf{1}_n - S_n)^{-1}\big) \\
&\quad = -(\lambda \otimes \mathbf{1}_n - S_n)^{-1}(\mathrm{Im}(\lambda) \otimes \mathbf{1}_n)(\lambda^* \otimes \mathbf{1}_n - S_n)^{-1}.
\end{aligned}$$

From this it follows that $-\mathrm{Im}((\lambda \otimes \mathbf{1}_n - S_n)^{-1})$ is positive definite at any ω in Ω, and the inverse is given by

$$\big(-\mathrm{Im}((\lambda \otimes \mathbf{1}_n - S_n)^{-1})\big)^{-1} = (\lambda^* \otimes \mathbf{1}_n - S_n)((\mathrm{Im}\lambda)^{-1} \otimes \mathbf{1}_n)(\lambda \otimes \mathbf{1}_n - S_n).$$

In particular, it follows that

$$0 \leq \big(-\mathrm{Im}((\lambda \otimes \mathbf{1}_n - S_n)^{-1})\big)^{-1} \leq \big\|\lambda \otimes \mathbf{1}_n - S_n\big\|^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\| \cdot \mathbf{1}_m \otimes \mathbf{1}_n,$$

and this implies that

$$-\mathrm{Im}\big((\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big) \geq \frac{1}{\|\lambda \otimes \mathbf{1}_n - S_n\|^2 \|(\mathrm{Im}\lambda)^{-1}\|} \cdot \mathbf{1}_m \otimes \mathbf{1}_n.$$

Since the slice map $\mathrm{id}_m \otimes \mathrm{tr}_n$ is positive, we have, thus, established that

$$-\mathrm{Im}H_n(\lambda) \geq \frac{1}{\|\lambda \otimes \mathbf{1}_n - S_n\|^2 \|(\mathrm{Im}\lambda)^{-1}\|} \cdot \mathbf{1}_m \geq \frac{1}{(\|\lambda\| + \|S_n\|)^2 \|(\mathrm{Im}\lambda)^{-1}\|} \cdot \mathbf{1}_m.$$

so that

$$-\mathrm{Im}G_n(\lambda) = \mathbb{E}\{-\mathrm{Im}H_n(\lambda)\} \geq \frac{1}{\|(\mathrm{Im}\lambda)^{-1}\|}\mathbb{E}\Big\{\frac{1}{(\|\lambda\| + \|S_n\|)^2}\Big\}\mathbf{1}_m.$$

Note here that the function $t \mapsto \frac{1}{(\|\lambda\|+t)^2}$ is convex, so applying Jensen's inequality to the random variable $\|S_n\|$, yields the estimate

$$\mathbb{E}\Big\{\frac{1}{(\|\lambda\|+\|S_n\|)^2}\Big\} \geq \frac{1}{(\|\lambda\|+\mathbb{E}\{\|S_n\|\})^2},$$

where

$$\mathbb{E}\{\|S_n\|\} \leq \mathbb{E}\Big\{\|a_0\|+\sum_{i=1}^r \|a_i\|\cdot\|X_i^{(n)}\|\Big\} = \|a_0\|+\sum_{i=1}^r \|a_i\|\cdot\mathbb{E}\big\{\|X_i^{(n)}\|\big\} \leq \|a_0\|+4\sum_{i=1}^r \|a_i\|,$$

by application of Lemma 5.1. Putting $K = 4\sum_{i=1}^r \|a_i\|$, we may, thus, conclude that

$$-\mathrm{Im}G_n(\lambda) \geq \frac{1}{\|(\mathrm{Im}\lambda)^{-1}\|}\frac{1}{(\|\lambda\|+K)^2}\mathbf{1}_m.$$

By Lemma 3.1, this implies that $G_n(\lambda)$ is invertible and that

$$\big\|G_n(\lambda)^{-1}\big\| \leq (\|\lambda\|+K)^2\cdot\big\|(\mathrm{Im}\lambda)^{-1}\big\|,$$

as desired. ∎

5.3 Corollary. *Let λ be a matrix in $M_m(\mathbb{C})$ such that $\mathrm{Im}\lambda$ is positive definite. Then*

$$\Big\|a_0+\sum_{i=1}^r a_iG_n(\lambda)a_i+G_n(\lambda)^{-1}-\lambda\Big\| \leq \frac{C}{n^2}(K+\|\lambda\|)^2\big\|(\mathrm{Im}\lambda)^{-1}\big\|^5, \tag{5.6}$$

where, as before, $C = m^3\|\sum_{i=1}^r a_i^2\|^2$ and $K = \|a_0\|+4\sum_{i=1}^r\|a_i\|$.

Proof. Note that

$$a_0+\sum_{i=1}^r a_iG_n(\lambda)a_i+G_n(\lambda)^{-1}-\lambda = \Big(\sum_{i=1}^r a_iG_n(\lambda)a_iG_n(\lambda)+(a_0-\lambda)G_n(\lambda)+\mathbf{1}_m\Big)G_n(\lambda)^{-1}.$$

Hence, (5.6) follows by combining Theorem 4.5 with Proposition 5.2. ∎

In addition to the given matrices $a_0,\ldots,a_r$ in $M_m(\mathbb{C})_{\mathrm{sa}}$, we consider next, as replacement for the random matrices $X_1^{(n)},\ldots,X_r^{(n)}$, free self-adjoint operators $x_1,\ldots,x_r$ in some C^*-probability space $(\mathcal{B},\tau)$. We assume that $x_1,\ldots,x_r$ are identically semi-circular distributed, such that $\tau(x_i)=0$ and $\tau(x_i^2)=1$ for all i. Then put

$$s = a_0\otimes\mathbf{1}_{\mathcal{B}}+\sum_{i=1}^r a_i\otimes x_i \in M_m(\mathbb{C})\otimes\mathcal{B}. \tag{5.7}$$

Consider, further, the subset $\mathcal{O}$ of $M_m(\mathbb{C})$, given by

$$\mathcal{O} = \{\lambda\in M_m(\mathbb{C}) \mid \mathrm{Im}(\lambda) \text{ is positive definite}\} = \{\lambda\in M_m(\mathbb{C}) \mid \lambda_{\min}(\mathrm{Im}\lambda)>0\} \tag{5.8}$$

and for each positive number δ, put

$$\mathcal{O}_\delta = \{\lambda \in \mathcal{O} \mid \|(\mathrm{Im}\lambda)^{-1}\| < \delta\} = \{\lambda \in \mathcal{O} \mid \lambda_{\min}(\mathrm{Im}\lambda) > \delta^{-1}\}. \tag{5.9}$$

Note that $\mathcal{O}$ and $\mathcal{O}_\delta$ are open subsets of $M_m(\mathbb{C})$.

If $\lambda \in \mathcal{O}$, then it follows from Lemma 3.1 that $\lambda \otimes \mathbf{1}_{\mathcal{B}} - s$ is invertible, since s is self-adjoint. Hence, for each λ in $\mathcal{O}$, we may define

$$G(\lambda) = \mathrm{id}_m \otimes \tau\big[(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big].$$

As in the proof of Lemma 5.2, it follows that $G(\lambda)$ is invertible for any λ in $\mathcal{O}$. Indeed, for λ in $\mathcal{O}$, we have

$$\begin{aligned}
&\mathrm{Im}\big((\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big) \\
&\quad = \frac{1}{2\mathrm{i}}\big((\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big((\lambda^* \otimes \mathbf{1}_{\mathcal{B}} - s) - (\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)\big)(\lambda^* \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big) \\
&\quad = -(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}(\mathrm{Im}(\lambda) \otimes \mathbf{1}_{\mathcal{B}})(\lambda^* \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1},
\end{aligned}$$

which shows that $-\mathrm{Im}((\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1})$ is positive definite and that

$$\begin{aligned}
0 \leq \big(-\mathrm{Im}((\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1})\big)^{-1} &= (\lambda^* \otimes \mathbf{1}_{\mathcal{B}} - s)((\mathrm{Im}\lambda)^{-1} \otimes \mathbf{1}_{\mathcal{B}})(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s) \\
&\leq \big\|\lambda \otimes \mathbf{1}_{\mathcal{B}} - s\big\|^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\| \cdot \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}}.
\end{aligned}$$

Consequently,

$$-\mathrm{Im}\big((\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big) \geq \frac{1}{\|\lambda \otimes \mathbf{1}_{\mathcal{B}} - s\|^2 \|(\mathrm{Im}\lambda)^{-1}\|} \cdot \mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}},$$

so that

$$-\mathrm{Im}G(\lambda) \geq \frac{1}{\|\lambda \otimes \mathbf{1}_{\mathcal{B}} - s\|^2 \|(\mathrm{Im}\lambda)^{-1}\|} \cdot \mathbf{1}_m.$$

By Lemma 3.1, this implies that $G(\lambda)$ is invertible and that

$$\big\|G(\lambda)^{-1}\big\| \leq \big\|(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)\big\|^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|.$$

The following lemma shows that the estimate (5.6) in Corollary 5.3 becomes an exact equation, when $G_n(\lambda)$ is replaced by $G(\lambda)$.

5.4 Lemma. *With $\mathcal{O}$ and $G(\lambda)$ defined as above, we have that*

$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \lambda,$$

for all λ in $\mathcal{O}$.

Proof. We start by recalling the definition of the R-transform $\mathcal{R}_s$ of (the distribution of) s with amalgamation over $M_m(\mathbb{C})$: It can be shown (cf. [V7]) that the expression

$$G(\lambda) = \mathrm{id}_m \otimes \tau\big[(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big],$$

gives rise to a well-defined and bijective mapping on a region of the form

$$\mathcal{U}_\delta = \big\{\lambda \in M_m(\mathbb{C}) \mid \lambda \text{ is invertible and } \|\lambda^{-1}\| < \delta\big\},$$

where δ is a (suitably small) positive number. Denoting by $G^{\langle -1\rangle}$ the inverse of the mapping $\lambda \mapsto G(\lambda)$ ($\lambda \in \mathcal{U}_\delta$), the R-transform $\mathcal{R}_s$ of s with amalgamation over $M_m(\mathbb{C})$ is defined as

$$\mathcal{R}_s(\rho) = G^{\langle -1\rangle}(\rho) - \rho^{-1}, \qquad (\rho \in G(\mathcal{U}_\delta)).$$

In [Le] it was proved that

$$\mathcal{R}_s(\rho) = a_0 + \sum_{i=1}^{r} a_i \rho a_i,$$

so that

$$G^{\langle -1\rangle}(\rho) = a_0 + \sum_{i=1}^{r} a_i \rho a_i + \rho^{-1}, \qquad (\rho \in G(\mathcal{U}_\delta)),$$

and hence

$$a_0 + \sum_{i=1}^{r} a_i G(\lambda) a_i + G(\lambda)^{-1} = \lambda, \qquad (\lambda \in \mathcal{U}_\delta). \tag{5.10}$$

Note now that by Lemma 3.1, the set $\mathcal{O}_\delta$, defined in (5.9), is a subset of $\mathcal{U}_\delta$, and hence (5.10) holds, in particular, for λ in $\mathcal{O}_\delta$. Since $\mathcal{O}_\delta$ is an open, non-empty subset of $\mathcal{O}$ (defined in (5.8)) and since $\mathcal{O}$ is a non-empty connected (even convex) subset of $M_m(\mathbb{C})$, it follows, then, from the principle of uniqueness of analytic continuation (for analytical functions in m^2 complex variables) that formula (5.10) actually holds for all λ in $\mathcal{O}$, as desired. ∎

For n in $\mathbb{N}$ and λ in the set $\mathcal{O}$ (defined in (5.8)), we introduce, further, the following notation:

$$\Lambda_n(\lambda) = a_0 + \sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1}, \tag{5.11}$$

$$\varepsilon(\lambda) = \frac{1}{\|(\mathrm{Im}\lambda)^{-1}\|} = \lambda_{\min}(\mathrm{Im}\lambda), \tag{5.12}$$

$$\mathcal{O}'_n = \big\{\lambda \in \mathcal{O} \;\big|\; \tfrac{C}{n^2}(K + \|\lambda\|)^2 \varepsilon(\lambda)^{-6} < \tfrac{1}{2}\big\}, \tag{5.13}$$

where, as before, $C = \frac{\pi^2}{8} m^3 \|\sum_{i=1}^{r} a_i^2\|^2$ and $K = \|a_0\| + 4\sum_{i=1}^{r} \|a_i\|$. Note that $\mathcal{O}'_n$ is an open subset of $M_m(\mathbb{C})$, since the mapping $\lambda \mapsto \varepsilon(\lambda)$ is continuous on $\mathcal{O}$. With the above notation we have the following

5.5 Lemma. *For any positive integer n and any matrix λ in $\mathcal{O}'_n$ we have*

$$\mathrm{Im}\Lambda_n(\lambda) \geq \frac{\varepsilon(\lambda)}{2}\mathbf{1}_m. \tag{5.14}$$

In particular, $\Lambda_n(\lambda) \in \mathcal{O}$. Moreover

$$a_0 + \sum_{i=1}^{r} a_i G(\Lambda_n(\lambda)) a_i + G(\Lambda_n(\lambda))^{-1} = a_0 + \sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1}, \tag{5.15}$$

for any λ in $\mathcal{O}'_n$.

Proof. Note that the right hand side of (5.15) is nothing else than $\Lambda_n(\lambda)$. Therefore, (5.15) follows from Lemma 5.4, once we have established that $\Lambda_n(\lambda) \in \mathcal{O}$ for all λ in $\mathcal{O}'_n$. This, in turn, is an immediate consequence of (5.14). It suffices, thus, to verify (5.14). Note, first, that for any λ in $\mathcal{O}$, we have by Corollary 5.3 that

$$\begin{aligned}\big\|\mathrm{Im}\Lambda_n(\lambda) - \mathrm{Im}\lambda\big\| \leq \big\|\Lambda_n(\lambda) - \lambda\big\| &= \Big\|a_0 + \sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1} - \lambda\Big\| \\ &\leq \frac{C}{n^2}(K + \|\lambda\|)^2 \varepsilon(\lambda)^{-5}.\end{aligned}$$

In particular, $\mathrm{Im}\Lambda_n(\lambda) - \mathrm{Im}\lambda \geq -\frac{C}{n^2}(K + \|\lambda\|)^2\varepsilon(\lambda)^{-5}\mathbf{1}_m$, and since, also, $\mathrm{Im}\lambda \geq \varepsilon(\lambda)\mathbf{1}_m$, by definition of $\varepsilon(\lambda)$, we conclude that

$$\mathrm{Im}\Lambda_n(\lambda) = \mathrm{Im}\lambda + (\mathrm{Im}\Lambda_n(\lambda) - \mathrm{Im}\lambda) \geq \big(\varepsilon(\lambda) - \tfrac{C}{n^2}(K + \|\lambda\|)^2\varepsilon(\lambda)^{-5}\big)\mathbf{1}_m, \tag{5.16}$$

for any λ in $\mathcal{O}$. Assume now that $\lambda \in \mathcal{O}'_n$. Then $\frac{C}{n^2}(K + \|\lambda\|)^2\varepsilon(\lambda)^{-5} < \frac{1}{2}\varepsilon(\lambda)$, and inserting this in (5.16), we find that

$$\mathrm{Im}\Lambda_n(\lambda) \geq \tfrac{1}{2}\varepsilon(\lambda)\mathbf{1}_m,$$

as desired. ∎

5.6 Proposition. *Let n be a positive integer. Then with G, G_n and $\mathcal{O}'_n$ as defined above, we have that*

$$G(\Lambda_n(\lambda)) = G_n(\lambda),$$

for all λ in $\mathcal{O}'_n$.

Proof. Note first that the functions $\lambda \mapsto G_n(\lambda)$ and $\lambda \mapsto G(\Lambda_n(\lambda))$ are both analytical functions (of m^2 complex variables) defined on $\mathcal{O}'_n$ and taking values in $M_m(\mathbb{C})$. Applying the principle of uniqueness of analytic continuation, it suffices, thus, to prove the following two assertions:

(a) The set $\mathcal{O}'_n$ is an open connected subset of $M_m(\mathbb{C})$.

(b) The formula $G(\Lambda_n(\lambda)) = G_n(\lambda)$ holds for all λ in some open, non-empty subset $\mathcal{O}_n''$ of $\mathcal{O}_n'$.

Proof of (a): We have already noted that $\mathcal{O}_n'$ is open. Consider the subset I_n of $\mathbb{R}$ given by:

$$I_n = \left\{t \in]0,\infty[\,\middle|\, \tfrac{C}{n^2}(K+t)^2 t^{-6} < \tfrac{1}{2}\right\},$$

with C and K as above. Note that since the function $t \mapsto (K+t)^2 t^{-6}$ $(t > 0)$ is continuous and strictly decreasing, I_n has the form: $I_n =]t_n, \infty[$, where t_n is uniquely determined by the equation: $\frac{C}{n^2}(K+t)^2 t^{-6} = \frac{1}{2}$. Note, further, that for any t in I_n, $\mathrm{i}t\mathbf{1}_m \in \mathcal{O}_n'$, and hence the set

$$\mathcal{I}_n = \{\mathrm{i}t\mathbf{1}_m \mid t \in I_n\},$$

is an arc-wise connected subset of $\mathcal{O}_n'$. To prove (a), it suffices, then, to show that any λ in $\mathcal{O}_n'$ is connected to some point in $\mathcal{I}_n$ via a continuous curve γ_λ, which is entirely contained in $\mathcal{O}_n'$. So let λ from $\mathcal{O}_n'$ be given, and note that $0 \leq \varepsilon(\lambda) = \lambda_{\min}(\mathrm{Im}\lambda) \leq \|\lambda\|$. Thus,

$$\frac{C}{n^2}(K+\varepsilon(\lambda))^2\varepsilon(\lambda)^{-6} \leq \frac{C}{n^2}(K+\|\lambda\|)^2\varepsilon(\lambda)^{-6} < \frac{1}{2},$$

and therefore $\varepsilon(\lambda) \in I_n$ and $\mathrm{i}\varepsilon(\lambda)\mathbf{1}_m \in \mathcal{I}_n$. Now, let $\gamma_\lambda\colon [0,1] \to M_m(\mathbb{C})$ be the straight line from $\mathrm{i}\varepsilon(\lambda)\mathbf{1}_m$ to λ, i.e.,

$$\gamma_\lambda(t) = (1-t)\mathrm{i}\varepsilon(\lambda)\mathbf{1}_m + t\lambda, \qquad (t \in [0,1]).$$

We show that $\gamma_\lambda(t) \in \mathcal{O}_n'$ for all t in $[0,1]$. Note, for this, that

$$\mathrm{Im}\gamma_\lambda(t) = (1-t)\varepsilon(\lambda)\mathbf{1}_m + t\mathrm{Im}\lambda, \qquad (t \in [0,1]),$$

so obviously $\gamma_\lambda(t) \in \mathcal{O}$ for all t in $[0,1]$. Furthermore, if $0 \leq r_1 \leq r_2 \leq \cdots \leq r_m$ denote the eigenvalues of $\mathrm{Im}(\lambda)$, then, for each t in $[0,1]$, $(1-t)\varepsilon(\lambda) + tr_j$ $(j = 1, 2, \ldots, m)$ are the eigenvalues of $\mathrm{Im}\gamma_\lambda(t)$. In particular, since $r_1 = \varepsilon(\lambda)$, $\varepsilon(\gamma_\lambda(t)) = \lambda_{\min}(\mathrm{Im}\gamma_\lambda(t)) = \varepsilon(\lambda)$ for all t in $[0,1]$. Note also that

$$\|\gamma_\lambda(t)\| \leq (1-t)\varepsilon(\lambda) + t\|\lambda\| \leq (1-t)\|\lambda\| + t\|\lambda\| = \|\lambda\|,$$

for all t in $[0,1]$. Altogether, we conclude that

$$\frac{C}{n^2}(K+\|\gamma_\lambda(t)\|)^2\varepsilon(\gamma_\lambda(t))^{-6} \leq \frac{C}{n^2}(K+\|\lambda\|)^2\varepsilon(\lambda)^{-6} < \frac{1}{2},$$

and hence $\gamma_\lambda(t) \in \mathcal{O}_n'$ for all t in $[0,1]$, as desired.

Proof of (b): Consider, for the moment, a fixed matrix λ from $\mathcal{O}_n'$, and put $\zeta = G_n(\lambda)$ and $\upsilon = G(\Lambda_n(\lambda))$. Then Lemma 5.5 asserts that

$$a_0 + \sum_{i=1}^{r} a_i \upsilon a_i + \upsilon^{-1} = a_0 + \sum_{i=1}^{r} a_i \zeta a_i + \zeta^{-1},$$

so that

$$\upsilon\Big(\sum_{i=1}^{r} a_i \upsilon a_i + \upsilon^{-1}\Big)\zeta = \upsilon\Big(\sum_{i=1}^{r} a_i \zeta a_i + \zeta^{-1}\Big)\zeta.$$

and hence

$$\sum_{i=1}^{r} \upsilon a_i(\upsilon - \zeta)a_i\zeta = \upsilon - \zeta.$$

In particular, it follows that

$$\Big(\|\upsilon\|\|\zeta\|\sum_{i=1}^{r}\|a_i\|^2\Big)\|\upsilon - \zeta\| \geq \|\upsilon - \zeta\|. \tag{5.17}$$

Note here that by Lemma 3.1,

$$\begin{aligned}\|\zeta\| = \|G_n(\lambda)\| &= \big\|\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\| \\ &\leq \big\|(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big\| \leq \big\|(\mathrm{Im}\lambda)^{-1}\big\| = \frac{1}{\varepsilon(\lambda)}.\end{aligned} \tag{5.18}$$

Similarly, it follows that

$$\|\upsilon\| = \|G(\Lambda_n(\lambda))\| \leq \big\|(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big\| \leq \big\|(\mathrm{Im}\Lambda_n(\lambda))^{-1}\big\| \leq \frac{2}{\varepsilon(\lambda)}, \tag{5.19}$$

where the last inequality follows from (5.14) in Lemma 5.5. Combining (5.17)-(5.19), it follows that

$$\Big(\frac{2}{\varepsilon(\lambda)^2}\sum_{i=1}^{r}\|a_i\|^2\Big)\|\upsilon - \zeta\| \geq \|\upsilon - \zeta\|. \tag{5.20}$$

This estimate holds for all λ in $\mathcal{O}'_n$. If λ satisfies, in addition, that $\frac{2}{\varepsilon(\lambda)^2}\sum_{i=1}^{r}\|a_i\|^2 < 1$, then (5.20) implies that $\zeta = \upsilon$, i.e., $G_n(\lambda) = G(\Lambda_n(\lambda))$. Thus, if we put

$$\mathcal{O}''_n = \big\{\lambda \in \mathcal{O}'_n \;\big|\; \varepsilon(\lambda) > \sqrt{2\textstyle\sum_{i=1}^{r}\|a_i\|^2}\big\},$$

we have established that $G_n(\lambda) = G(\Lambda_n(\lambda))$ for all λ in $\mathcal{O}''_n$. Since $\varepsilon(\lambda)$ is a continuous function of λ, $\mathcal{O}''_n$ is clearly an open subset of $\mathcal{O}'_n$, and it remains to check that $\mathcal{O}''_n$ is non-empty. Note, however, that for any positive number t, the matrix $\mathrm{i}t\mathbf{1}_m$ is in $\mathcal{O}$ and it satisfies that $\|\mathrm{i}t\mathbf{1}_m\| = \varepsilon(\mathrm{i}t\mathbf{1}_m) = t$. From this, it follows easily that $\mathrm{i}t\mathbf{1}_m \in \mathcal{O}''_n$ for all sufficiently large positive numbers t. This concludes the proof of (b) and hence the proof of Proposition 5.6. ∎

5.7 Theorem. *Let r, m be positive integers, let $a_1, \ldots, a_r$ be self-adjoint matrices in $M_m(\mathbb{C})$ and, for each positive integer n, let $X_1^{(n)}, \ldots, X_r^{(n)}$ be independent random matrices in* $\mathrm{SGRM}(n, \frac{1}{n})$. *Consider, further, free self-adjoint identically semi-circular distributed operators $x_1, \ldots, x_r$ in some C^*-probability space $(\mathcal{B}, \tau)$, and normalized such*

that $\tau(x_i) = 0$ and $\tau(x_i^2) = 1$ for all i. Then put as in (3.2) and (5.7):

$$\begin{aligned} s &= a_0 \otimes \mathbf{1}_{\mathcal{B}} + \sum_{i=1}^{r} a_i \otimes x_i \in M_m(\mathbb{C}) \otimes \mathcal{B} \\ S_n &= a_0 \otimes \mathbf{1}_n + \sum_{i=1}^{r} a_i \otimes X_i^{(n)} \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C}), \quad (n \in \mathbb{N}), \end{aligned}$$

and for λ in $\mathcal{O} = \{\lambda \in M_m(\mathbb{C}) \mid \mathrm{Im}(\lambda)$ is positive definite$\}$ define

$$\begin{aligned} G_n(\lambda) &= \mathbb{E}\big\{(\mathrm{id}_m \otimes \mathrm{tr}_n)\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\} \\ G(\lambda) &= (\mathrm{id}_m \otimes \tau)\big[(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big]. \end{aligned}$$

Then, for any λ in $\mathcal{O}$ and any positive integer n, we have

$$\big\|G_n(\lambda) - G(\lambda)\big\| \leq \frac{4C}{n^2}(K + \|\lambda\|)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|^7, \tag{5.21}$$

where $C = m^3 \|\sum_{i=1}^r a_i^2\|^2$ and $K = \|a_0\| + 4\sum_{i=1}^r \|a_i\|$.

Proof. Let n in $\mathbb{N}$ be fixed, and assume, first, that λ is in the set $\mathcal{O}_n'$ defined in (5.13). Then, by Proposition 5.6, we have

$$\begin{aligned} \big\|G_n(\lambda) - G(\lambda)\big\| &= \big\|G(\Lambda_n(\lambda)) - G(\lambda)\big\| \\ &= \big\|\mathrm{id}_m \otimes \tau\big[(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1} - (\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big]\big\| \\ &\leq \big\|(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1} - (\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big\|. \end{aligned}$$

Note here that

$$(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1} - (\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1} = (\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big((\lambda - \Lambda_n(\lambda) \otimes \mathbf{1}_n\big)(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1},$$

and therefore, taking Lemma 3.1 into account,

$$\begin{aligned} \big\|G_n(\lambda) - G(\lambda)\big\| &\leq \big\|(\Lambda_n(\lambda) \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big\| \cdot \big\|\lambda - \Lambda_n(\lambda)\big\| \cdot \big\|(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big\| \\ &\leq \big\|(\mathrm{Im}\Lambda_n(\lambda))^{-1}\big\| \cdot \big\|\lambda - \Lambda_n(\lambda)\big\| \cdot \big\|(\mathrm{Im}\lambda)^{-1}\big\|. \end{aligned}$$

Now, $\|(\mathrm{Im}\lambda)^{-1}\| = 1/\varepsilon(\lambda)$ (cf. (5.12)) and hence, by (5.14) in Lemma 5.5, $\|(\mathrm{Im}\Lambda_n(\lambda))^{-1}\| \leq 2/\varepsilon(\lambda) = 2\|(\mathrm{Im}\lambda)^{-1}\|$. Furthermore, by (5.11) and Corollary 5.3,

$$\big\|\Lambda_n(\lambda) - \lambda\big\| = \Big\|a_0 + \sum_{i=1}^{r} a_i G_n(\lambda) a_i + G_n(\lambda)^{-1} - \lambda\Big\| \leq \frac{C}{n^2}(K + \|\lambda\|)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|^5.$$

Thus, we conclude that

$$\big\|G_n(\lambda) - G(\lambda)\big\| \leq \frac{2C}{n^2}(K + \|\lambda\|)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|^7,$$

which shows, in particular, that (5.21) holds for all λ in $\mathcal{O}_n'$.

Assume, next, that $\lambda \in \mathcal{O} \setminus \mathcal{O}'_n$, so that

$$\frac{C}{n^2}(K + \|\lambda\|)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|^6 = \frac{C}{n^2}(K + \|\lambda\|)^2 \varepsilon(\lambda)^{-6} \geq \frac{1}{2}. \tag{5.22}$$

By application of Lemma 3.1, it follows that

$$\big\|G(\lambda)\big\| \leq \big\|(\lambda \otimes \mathbf{1}_{\mathcal{B}} - s)^{-1}\big\| \leq \big\|(\mathrm{Im}\lambda)^{-1}\big\|, \tag{5.23}$$

and similarly we find that

$$\big\|\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n - S_n(\omega))^{-1}\big]\big\| \leq \big\|(\mathrm{Im}\lambda)^{-1}\big\|,$$

at all points ω in Ω. Hence, after integrating w.r.t. ω and using Jensen's inequality,

$$\|G_n(\lambda)\| \leq \mathbb{E}\big\{\big\|\mathrm{id}_m \otimes \mathrm{tr}_n\big[(\lambda \otimes \mathbf{1}_n - S_n)^{-1}\big]\big\|\big\} \leq \big\|(\mathrm{Im}\lambda)^{-1}\big\|. \tag{5.24}$$

Combining (5.22)-(5.24), we find that

$$\big\|G_n(\lambda) - G(\lambda)\big\| \leq 2\big\|(\mathrm{Im}\lambda)^{-1}\big\| = \frac{1}{2} \cdot 4\big\|(\mathrm{Im}\lambda)^{-1}\big\| \leq \frac{4C}{n^2}(K + \|\lambda\|)^2 \big\|(\mathrm{Im}\lambda)^{-1}\big\|^7,$$

verifying that (5.21) holds for λ in $\mathcal{O} \setminus \mathcal{O}'_n$ too. ∎

6 The spectrum of S_n.

Let $r, m \in \mathbb{N}$, let $a_0, \ldots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}$ and for each $n \in \mathbb{N}$, let $X_1^{(n)}, \ldots, X_r^{(n)}$ be r independent random matrices in $\mathrm{SGRM}(n, \frac{1}{n})$. Let, further, $x_1, \ldots, x_r$ be a semi-circular family in a C^*-probability space $(\mathcal{B}, \tau)$, and define S_n, s, $G_n(\lambda)$ and $G(\lambda)$ as in Theorem 5.7.

6.1 Lemma. *For $\lambda \in \mathbb{C}$ with $\mathrm{Im}\lambda > 0$, put*

$$g_n(\lambda) = \mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)[(\lambda(\mathbf{1}_m \otimes \mathbf{1}_{\mathcal{B}}) - S_n)^{-1}]\big\} \tag{6.1}$$

and

$$g(\lambda) = (\mathrm{tr}_m \otimes \tau)\big[(\lambda \mathbf{1}_{mn} - s)^{-1}\big]. \tag{6.2}$$

Then

$$|g_n(\lambda) - g(\lambda)| \leq \frac{4C}{n^2}\big(K + |\lambda|\big)^2 (\mathrm{Im}\lambda)^{-7} \tag{6.3}$$

where C, K are the constants defined in Theorem 5.7.

Proof. This is immediate from Theorem 5.7 because

$$g_n(\lambda) = \mathrm{tr}_m(G_n(\lambda \mathbf{1}_m))$$

and

$$g(\lambda) = \mathrm{tr}_m(G(\lambda \mathbf{1}_m)). \qquad \blacksquare$$

Let Prob($\mathbb{R}$) denote the set of Borel probability measures on $\mathbb{R}$. We equip Prob($\mathbb{R}$) with the weak*-topology given by $C_0(\mathbb{R})$, i.e., a net $(\mu_\alpha)_{\alpha\in A}$ in Prob($\mathbb{R}$) converges in w^*-topology to $\mu \in \mathrm{Prob}(\mathbb{R})$, if and only if

$$\lim_\alpha \left(\int_{\mathbb{R}} \varphi \, \mathrm{d}\mu_\alpha \right) = \int_{\mathbb{R}} \varphi \, \mathrm{d}\mu$$

for all $\varphi \in C_0(\mathbb{R})$.

Since S_n and s are self-adjoint, there are, by Riesz' representation theorem, unique probability measures μ_n, $n = 1, 2, \ldots$ and μ on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}\mu_n = \mathbb{E}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n) \big\} \tag{6.4}$$

$$\int_{\mathbb{R}} \varphi \, \mathrm{d}\mu = (\mathrm{tr}_m \otimes \tau)\varphi(s) \tag{6.5}$$

for all $\varphi \in C_0(\mathbb{R})$. Note that μ is compactly supported while μ_n, in general, is not compactly supported.

6.2 Theorem. *Let S_n and s be given by (3.2) and (5.7), and let $C = \frac{\pi^2}{8} m^3 \| \sum_{i=1}^r a_i^2 \|^2$ and $K = \|a_0\| + 4\sum_{i=1}^r \|a_i\|$. Then for all $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$;*

$$\mathbb{E}\big\{ (\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n) \big\} = (\mathrm{tr}_m \otimes \tau)\varphi(s) + R_n \tag{6.6}$$

where

$$|R_n| \le \frac{4C}{315\pi n^2} \int_{\mathbb{R}} \big| ((1+D)^8 \varphi)(x) \big| \big(K + 2 + |x| \big)^2 \, \mathrm{d}x \tag{6.7}$$

and $D = \frac{\mathrm{d}}{\mathrm{d}x}$. In particular $R_n = O(\frac{1}{n^2})$ for $n \to \infty$.

Proof. Let g_n, g, μ_n, μ be as in (6.1), (6.2), (6.4) and (6.5). Then for any complex number λ, such that $\mathrm{Im}(\lambda) > 0$, we have

$$g_n(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x} \, \mathrm{d}\mu_n(x) \tag{6.8}$$

$$g(\lambda) = \int_{\mathbb{R}} \frac{1}{\lambda - x} \, \mathrm{d}\mu(x). \tag{6.9}$$

Hence g_n and g are the Stieltjes transforms (or Cauchy transforms, in the terminology of [VDN]) of μ_n and μ in the half plane $\mathrm{Im}\lambda > 0$. Hence, by the inverse Stieltjes transform,

$$\mu_n = \lim_{y \to 0^+} \Big(-\frac{1}{\pi} \mathrm{Im}(g_n(x + \mathrm{i}y)) \, \mathrm{d}x \Big)$$

where the limit is taken in the weak*-topology on Prob($\mathbb{R}$). In particular, for all φ in $C_c^\infty(\mathbb{R},\mathbb{R})$:

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu_n(x) = \lim_{y\to 0^+}\Big[-\frac{1}{\pi}\mathrm{Im}\Big(\int_{\mathbb{R}} \varphi(x) g_n(x+\mathrm{i}y)\,\mathrm{d}x\Big)\Big]. \tag{6.10}$$

In the same way we get for $\varphi \in C_c^\infty(\mathbb{R},\mathbb{R})$:

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu(x) = \lim_{y\to 0^+}\Big[-\frac{1}{\pi}\mathrm{Im}\int_{\mathbb{R}} \varphi(x) g(x+\mathrm{i}y)\,\mathrm{d}x\Big]. \tag{6.11}$$

In the rest of the proof, $n \in \mathbb{N}$ is fixed, and we put $h(\lambda) = g_n(\lambda) - g(\lambda)$. Then by (6.10) and (6.11)

$$\Big|\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu_n(x) - \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu(x)\Big| \le \frac{1}{\pi}\limsup_{y\to 0^+}\Big|\int_{\mathbb{R}} \varphi(x) h(x+\mathrm{i}y)\,\mathrm{d}x\Big|. \tag{6.12}$$

For $\mathrm{Im}\lambda > 0$ and $p \in \mathbb{N}$, put

$$I_p(\lambda) = \frac{1}{(p-1)!}\int_0^\infty h(\lambda+t)t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{6.13}$$

Note that $I_p(\lambda)$ is well defined because, by (6.8) and (6.9), $h(\lambda)$ is uniformly bounded in any half-plane of the form $\mathrm{Im}\lambda \ge \varepsilon$, where $\varepsilon > 0$. Also, it is easy to check that $I_p(\lambda)$ is an analytic function of λ, and its first derivative is given by

$$I_p'(\lambda) = \frac{1}{(p-1)!}\int_0^\infty h'(\lambda+t)t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t \tag{6.14}$$

where $h' = \frac{\mathrm{d}h}{\mathrm{d}\lambda}$. We claim that

$$I_1(\lambda) - I_1'(\lambda) = h(\lambda) \tag{6.15}$$

$$I_p(\lambda) - I_p'(\lambda) = I_{p-1}(\lambda), \quad p \ge 2. \tag{6.16}$$

Indeed, by (6.14) and partial integration we get

$$\begin{aligned} I_1'(\lambda) &= \big[h(\lambda+t)\mathrm{e}^{-t}\big]_0^\infty + \int_0^\infty h(\lambda+t)\mathrm{e}^{-t}\,\mathrm{d}t \\ &= -h(\lambda) + I_1(\lambda), \end{aligned}$$

which proves (6.15) and in the same way we get for $p \ge 2$,

$$\begin{aligned} I_p'(\lambda) &= \frac{1}{(p-1)!}\int_0^\infty h'(\lambda+t)t^{p-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &= -\frac{1}{(p-1)!}\int_0^\infty h(\lambda+t)((p-1)t^{p-2} - t^{p-1})\mathrm{e}^{-t}\,\mathrm{d}t \\ &= -I_{p-1}(\lambda) + I_p(\lambda), \end{aligned}$$

which proves (6.16). Assume now that $\varphi \in C_c^\infty(\mathbb{R},\mathbb{R})$ and that $y > 0$. Then, by (6.15) and partial integration, we have

$$\begin{aligned}\int_{\mathbb{R}} \varphi(x)h(x+\mathrm{i}y)\,\mathrm{d}x &= \int_{\mathbb{R}} \varphi(x)I_1(x+\mathrm{i}y)\,\mathrm{d}x - \int_{\mathbb{R}} \varphi(x)I_1'(x+\mathrm{i}y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi(x)I_1(x+\mathrm{i}y)\,\mathrm{d}x + \int_{\mathbb{R}} \varphi'(x)I_1(x+\mathrm{i}y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} ((1+D)\varphi)(x)\cdot I_1(x+\mathrm{i}y)\,\mathrm{d}x,\end{aligned}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Using (6.16), we can continue to perform partial integrations, and after p steps, we obtain

$$\int_{\mathbb{R}} \varphi(x)h(x+\mathrm{i}y)\,\mathrm{d}x = \int_{\mathbb{R}} ((1+D)^p\varphi)(x)\cdot I_p(x+\mathrm{i}y)\,\mathrm{d}x.$$

Hence by (6.12) we have for all $p \in \mathbb{N}$:

$$\Big|\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu_n(x) - \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\mu(x)\Big| \le \frac{1}{\pi}\limsup_{y\to 0^+}\Big|\int_{\mathbb{R}} ((1+D)^p\varphi)(x)\cdot I_p(x+\mathrm{i}y)\,\mathrm{d}x\Big|. \quad (6.17)$$

Next, we use (6.3) to show that for $p = 8$ and $\mathrm{Im}\lambda > 0$ one has

$$|I_8(\lambda)| \le \frac{4C(K+2+|\lambda|)^2}{315n^2}. \quad (6.18)$$

To prove (6.18), we apply Cauchy's integral theorem to the function

$$F(z) = \frac{1}{7!}h(\lambda+z)z^7\mathrm{e}^{-z},$$

which is analytic in the half-plane $\mathrm{Im}z > -\mathrm{Im}\lambda$. Hence for $r > 0$

$$\int_{[0,r]} F(z)\,\mathrm{d}z + \int_{[r,r+\mathrm{i}r]} F(z)\,\mathrm{d}z + \int_{[r+\mathrm{i}r,0]} F(z)\,\mathrm{d}z = 0$$

where $[\alpha,\beta]$ denotes the line segment connecting α and β in $\mathbb{C}$ oriented from α to β. Put

$$M(\lambda) = \sup\big\{|h(w)| \;\big|\; \mathrm{Im}w \ge \mathrm{Im}\lambda\big\}.$$

Then by (6.8) and (6.9), $M(\lambda) \le \frac{2}{|\mathrm{Im}\lambda|} < \infty$. Hence

$$\begin{aligned}\Big|\int_{[r,r+\mathrm{i}r]} F(z)\,\mathrm{d}z\Big| &\le \frac{M(\lambda)}{7!}\int_0^r |\lambda+r+\mathrm{i}t|^7\mathrm{e}^{-r}\,\mathrm{d}t \\ &\le \frac{M(\lambda)}{7!}(|\lambda|+2r)^7 r\cdot\mathrm{e}^{-r} \\ &\to 0, \qquad \text{for } r\to\infty.\end{aligned}$$

Therefore,

$$\begin{aligned} I_8(\lambda) &= \frac{1}{7!}\int_0^\infty h(\lambda+t)t^7\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \lim_{r\to\infty}\int_{[0,r]} F(z)\,\mathrm{d}z \\ &= \lim_{r\to\infty}\int_{[0,r+\mathrm{i}r]} F(z)\,\mathrm{d}z \\ &= \frac{1}{7!}\int_0^\infty h(\lambda+(1+\mathrm{i})t)((1+\mathrm{i})t)^7\mathrm{e}^{-(1+\mathrm{i})t}(1+\mathrm{i})\,\mathrm{d}t. \end{aligned} \tag{6.19}$$

By (6.3),

$$|h(w)| \le \frac{4C}{n^2}(K+|w|)^2(\mathrm{Im}w)^{-7}, \qquad \mathrm{Im}w>0.$$

Inserting this in (6.19) we get

$$\begin{aligned} |I_8(\lambda)| &\le \frac{4C}{7!n^2}\int_0^\infty \frac{\big(K+|\lambda|+\sqrt{2}t\big)^2}{(\mathrm{Im}\lambda+t)^7}(\sqrt{2}t)^7\mathrm{e}^{-t}\sqrt{2}\,\mathrm{d}t \\ &\le \frac{2^6C}{7!n^2}\int_0^\infty \big(K+|\lambda|+\sqrt{2}t\big)^2\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \frac{4C}{315n^2}\big((K+|\lambda|)^2+2\sqrt{2}(K+|\lambda|)+4\big) \\ &\le \frac{4C}{315n^2}(K+|\lambda|+2)^2. \end{aligned}$$

This proves (6.18). Now, combining (6.17) and (6.18), we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}\mu_n(x)-\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}\mu(x)\Big| \\ &\qquad\le \frac{4C}{315\pi n^2}\limsup_{y\to0^+}\int_{\mathbb{R}}\big|((1+D)^8\varphi)(x)\big|\big(K+2+|x+\mathrm{i}y|\big)^2\,\mathrm{d}x \\ &\qquad= \frac{4C}{315\pi n^2}\int_{\mathbb{R}}\big|((1+D)^8\varphi)(x)\big|\big(K+2+|x|\big)^2\,\mathrm{d}x \end{aligned}$$

for all $\varphi\in C_c^\infty(\mathbb{R},\mathbb{R})$. Together with (6.4) and (6.5) this proves Theorem 6.2. ■

6.3 Lemma. *Let S_n and s be given by (3.2) and (5.7), and let $\varphi\colon\mathbb{R}\to\mathbb{R}$ be a C^∞-function which is constant outside a compact subset of $\mathbb{R}$. Assume further, that*

$$\mathrm{supp}(\varphi)\cap\mathrm{sp}(s)=\emptyset. \tag{6.20}$$

Then

$$\mathbb{E}\big\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)\varphi(S_n)\big\} = O\big(\tfrac{1}{n^2}\big), \qquad \text{for } n\to\infty \tag{6.21}$$

$$\mathbb{V}\big\{(\mathrm{tr}_m\otimes\mathrm{tr}_n)\varphi(S_n)\big\} = O\big(\tfrac{1}{n^4}\big), \qquad \text{for } n\to\infty \tag{6.22}$$

where $\mathbb{V}$ is the absolute variance of a complex random variable (cf. Section 4). Moreover

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n(\omega)) = O(n^{-4/3}) \tag{6.23}$$

for almost all ω in the underlying probability space Ω.

Proof. By the assumptions, $\varphi = \psi + c$, for some ψ in $C_c^\infty(\mathbb{R}, \mathbb{R})$ and some constant c in $\mathbb{R}$. By Theorem 6.2

$$\mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\big\} = (\mathrm{tr}_m \otimes \tau)\psi(s) + O\big(\tfrac{1}{n^2}\big), \qquad \text{for } n \to \infty,$$

and hence also

$$\mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n)\big\} = (\mathrm{tr}_m \otimes \tau)\varphi(s) + O\big(\tfrac{1}{n^2}\big), \qquad \text{for } n \to \infty.$$

But since φ vanishes on $\mathrm{sp}(s)$, we have $\varphi(s) = 0$. This proves (6.21). Moreover, applying Proposition 4.7 to $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\big\} \leq \frac{1}{n^2}\Big\|\sum_{i=1}^r a_i^2\Big\|^2 \mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)(\psi'(S_n))^2\big\}. \tag{6.24}$$

By (6.20), $\psi' = \varphi'$ also vanishes on $\mathrm{sp}(s)$. Hence, by Theorem 6.2

$$\mathbb{E}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)|\psi'(S_n)|^2\big\} = O\big(\tfrac{1}{n^2}\big), \qquad \text{as } n \to \infty.$$

Therefore, by (6.24)

$$\mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\big\} = O\big(\tfrac{1}{n^4}\big), \qquad \text{as } n \to \infty.$$

Since $\varphi(S_n) = \psi(S_n) + c\mathbf{1}_{mn}$, $\mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n)\big\} = \mathbb{V}\big\{(\mathrm{tr}_m \otimes \mathrm{tr}_n)\psi(S_n)\big\}$. This proves (6.22). Now put

$$\begin{aligned} Z_n &= (\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n) \\ \Omega_n &= \big\{\omega \in \Omega \;\big|\; |Z_n(\omega)| \geq n^{-4/3}\big\}. \end{aligned}$$

By (6.21) and (6.22)

$$\mathbb{E}\big\{|Z_n|^2\big\} = |\mathbb{E}\{Z_n\}|^2 + \mathbb{V}\{Z_n\} = O\big(\tfrac{1}{n^4}\big), \qquad \text{for } n \to \infty.$$

Hence

$$P(\Omega_n) = \int_{\Omega_n} \mathrm{d}P(\omega) \leq \int_{\Omega_n} \big|n^{4/3} Z_n(\omega)\big|^2 \,\mathrm{d}P(\omega) \leq n^{8/3}\mathbb{E}\big\{|Z_n|^2\big\} = O(n^{-4/3}), \tag{6.25}$$

for $n \to \infty$. In particular $\sum_{n=1}^\infty P(\Omega_n) < \infty$. Therefore, by the Borel-Cantelli lemma (see e.g. [Bre]), $\omega \notin \Omega_n$ eventually, as $n \to \infty$, for almost all $\omega \in \Omega$, i.e., $|Z_n(\omega)| < n^{-4/3}$ eventually, as $n \to \infty$, for almost all $\omega \in \Omega$. This proves (6.23). ∎

6.4 Theorem. *Let $m \in \mathbb{N}$ and let $a_0, \ldots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}$, S_n and s be as in Theorem 5.7. Then for any $\varepsilon > 0$ and for almost all $\omega \in \Omega$,*

$$\mathrm{sp}(S_n(\omega)) \subseteq \mathrm{sp}(s) +]-\varepsilon, \varepsilon[,$$

eventually as $n \to \infty$.

Proof. Put

$$\begin{aligned} K &= \mathrm{sp}(s) + \big[-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}\big] \\ F &= \big\{t \in \mathbb{R} \mid d(t, \mathrm{sp}(s)) \geq \varepsilon\big\}. \end{aligned}$$

Then K is compact, F is closed and $K \cap F = \emptyset$. Hence there exists $\varphi \in C^\infty(\mathbb{R})$, such that $0 \leq \varphi \leq 1$, $\varphi(t) = 0$ for $t \in K$ and $\varphi(t) = 1$ for $t \in F$ (cf. [F, (8.18) p. 237]). Since $\mathbb{C}\backslash F$ is a bounded set, φ satisfies the requirements of lemma 6.3. Hence by (6.23), there exists a P-null set $N \subseteq \Omega$, such that for all $\omega \in \Omega\backslash N$:

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)\varphi(S_n(\omega)) = O(n^{-4/3}), \qquad \text{as } n \to \infty.$$

Since $\varphi \geq 1_F$, it follows that

$$(\mathrm{tr}_m \otimes \mathrm{tr}_n)1_F(S_n(\omega)) = O(n^{-4/3}), \qquad \text{as } n \to \infty.$$

But for fixed $\omega \in \Omega\backslash N$, the number of eigenvalues (counted with multiplicity) of the matrix $S_n(\omega)$ in the set F is equal to $mn(\mathrm{tr}_m \otimes \mathrm{tr}_n)1_F(S_n(\omega))$, which is $O(n^{-1/3})$ as $n \to \infty$. However, for each $n \in \mathbb{N}$ the above number is an integer. Hence, the number of eigenvalues of $S_n(\omega)$ in F is zero eventually as $n \to \infty$. This shows that

$$\mathrm{sp}(S_n(\omega)) \subseteq \mathbb{C}\backslash F = \mathrm{sp}(s) +]-\varepsilon, \varepsilon[$$

eventually as $n \to \infty$, when $\omega \in \Omega\backslash N$. ∎

7 Proof of the main Theorem.

Throughout this section, $r \in \mathbb{N} \cup \{\infty\}$, and, for each n in $\mathbb{N}$, we let $(X_i^{(n)})_{i=1}^r$ denote a finite or countable set of independent random matrices from $\mathrm{SGRM}(n, \frac{1}{n})$, defined on the same probability space $(\Omega, \mathcal{F}, P)$. In addition, we let $(x_i)_{i=1}^r$ denote a corresponding semi-circular family in a C^*-probability space $(\mathcal{B}, \tau)$, where τ is a faithful state on $\mathcal{B}$. Furthermore, as in [VDN], we let $\mathbb{C}\langle (X_i)_{i=1}^r \rangle$ denote the algebra of all polynomials in r non-commuting variables. Note that $\mathbb{C}\langle (X_i)_{i=1}^r \rangle$ is a unital $*$-algebra, with the $*$-operation given by:

$$(cX_{i_1}X_{i_2}\cdots X_{i_k})^* = \overline{c}X_{i_k}X_{i_{k-1}}\cdots X_{i_2}X_{i_1},$$

for c in $\mathbb{C}$, k in $\mathbb{N}$ and $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, r\}$, when r is finite, and in $\mathbb{N}$ when $r = \infty$. The purpose of this section is to conclude the proof of the main theorem (Theorem 7.1 below) by combining the results of the previous sections.

7.1 Theorem. *Let r be in $\mathbb{N} \cup \{\infty\}$. Then there exists a P-null-set $N \subseteq \Omega$, such that for all p in $\mathbb{C}\langle (X_i)_{i=1}^r \rangle$ and all ω in $\Omega \setminus N$, we have*

$$\lim_{n\to\infty} \big\|p\big((X_i^{(n)}(\omega))_{i=1}^r\big)\big\| = \big\|p\big((x_i)_{i=1}^r\big)\big\|.$$

We start by proving the following

7.2 Lemma. *Assume that $r \in \mathbb{N}$. Then there exists a P-null set $N_1 \subseteq \Omega$, such that for all p in $\mathbb{C}\langle (X_i)_{i=1}^r \rangle$ and all ω in $\Omega \backslash N_1$:*

$$\liminf_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\big\| \geq \|p(x_1, \ldots, x_r)\|. \tag{7.1}$$

Proof. We first prove that for each p in $\mathbb{C}\langle X_1, \ldots, X_r \rangle$, there exists a P-null-set $N(p)$, depending on p, such that (7.1) holds for all ω in $\Omega \setminus N(p)$. This assertion is actually a special case of [T, Prop. 4.5], but for the readers convenience, we include a more direct proof: Consider first a fixed $p \in \mathbb{C}\langle X_1, \ldots, X_r \rangle$. Let $k \in \mathbb{N}$ and put $q = (p^*p)^k$. By [T, Cor. 3.9] or [HP],

$$\lim_{n\to\infty} \operatorname{tr}_n\big(q(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega))\big) = \tau\big(q(x_1, \ldots, x_r)\big), \tag{7.2}$$

for almost all $\omega \in \Omega$. For $s \geq 1$, $Z \in M_n(\mathbb{C})$ and $z \in \mathcal{B}$, put $\|Z\|_s = \operatorname{tr}_n(|Z|^s)^{1/s}$ and $\|z\|_s = \tau(|z|^s)^{1/s}$. Then (7.2) can be rewritten as

$$\lim_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\big\|_{2k}^{2k} = \big\|p(x_1, \ldots, x_r)\big\|_{2k}^{2k} \tag{7.3}$$

for $\omega \in \Omega \backslash N(p)$, where $N(p)$ is a P-null-set. Since $\mathbb{N}$ is a countable set, we can assume that $N(p)$ does not depend on $k \in \mathbb{N}$. For every bounded Borel function f on a probability space, one has

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p, \tag{7.4}$$

(cf. [F, Exercise 7, p. 179]). Put $a = p(x_1, \ldots, x_r)$, and let $\Gamma\colon \mathcal{D} \to C(\hat{\mathcal{D}})$ be the Gelfand transform of the Abelian C^*-algebra $\mathcal{D}$ generated by a^*a and $\mathbf{1}_{\mathcal{B}}$, and let μ be the probability measure on $\hat{\mathcal{D}}$ corresponding to $\tau_{|\mathcal{D}}$. Since τ is faithful, $\operatorname{supp}(\mu) = \hat{\mathcal{D}}$. Hence, $\|\Gamma(a^*a)\|_\infty = \|\Gamma(a^*a)\|_{\sup} = \|a^*a\|$. Applying then (7.4) to the function $f = \Gamma(a^*a)$, we find that

$$\|a\| = \|a^*a\|^{1/2} = \lim_{p\to\infty} \|a^*a\|_p^{1/2} = \lim_{p\to\infty} \|a\|_{2p}. \tag{7.5}$$

Let $\varepsilon > 0$. By (7.5), we can choose k in $\mathbb{N}$, such that

$$\|p(x_1, \ldots, x_r)\|_{2k} > \|p(x_1, \ldots, x_r)\| - \varepsilon.$$

Since $\|Z\|_s \leq \|Z\|$ for all $s \geq 1$ and all $Z \in M_n(\mathbb{C})$, we have by (7.3)

$$\liminf_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\big\| \geq \|p(x_1, \ldots, x_r)\|_{2k} > \|p(x_1, \ldots, x_r)\| - \varepsilon,$$

for all $\omega \in \Omega\backslash N(p)$, and since $N(p)$ does not depend on ε, it follows that (7.1) holds for all $\omega \in \Omega\backslash N(p)$. Now put $N' = \bigcup_{p\in\mathcal{P}} N(p)$, where $\mathcal{P}$ is the set of non-commutative polynomials with rational coefficients. Then N' is again a null set, and (7.1) holds for all $p \in \mathcal{P}$ and all $\omega \in \Omega\backslash N'$.

By [Ba, Thm. 2.12] or [HT1, Thm. 3.1], $\lim_{n\to\infty} \|X_i^{(n)}(\omega)\| = 2$, $i = 1, \ldots, r$, for almost all $\omega \in \Omega$. In particular

$$\sup_{n\in\mathbb{N}} \|X_i^{(n)}(\omega)\| < \infty, \quad i = 1, \ldots, r, \tag{7.6}$$

for almost all $\omega \in \Omega$. Let $N'' \subseteq \Omega$ be the set of $\omega \in \Omega$ for which (7.6) fails for some $i \in \{1, \ldots, r\}$. Then $N_1 = N' \cup N''$ is a null set, and a simple approximation argument shows that (7.1) holds for all p in $\mathbb{C}\langle X_1, \ldots, X_r\rangle$, when $\omega \in \Omega\backslash N_1$. ∎

In order to complete the proof of Theorem 7.1, we have to prove

7.3 Proposition. *Assume that $r \in \mathbb{N}$. Then there is a P-null set $N_2 \subseteq \Omega$, such that for all polynomials p in r non-commuting variables and all $\omega \in \Omega\backslash N_2$,*

$$\limsup_{n\to\infty} \left\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\right\| \leq \|p(x_1, \ldots, x_r)\|.$$

The proof of Proposition 7.3 relies on Theorem 6.4 combined with the linearization trick in the form of Theorem 2.4. Following the notation of [BK] we put

$$\prod_n M_n(\mathbb{C}) = \big\{(Z_n)_{n=1}^\infty \;\big|\; Z_n \in M_n(\mathbb{C}),\ \sup_{n\in\mathbb{N}}\|Z_n\| < \infty\big\}$$

and

$$\sum_n M_n(\mathbb{C}) = \big\{(Z_n)_{n=1}^\infty \;\big|\; Z_n \in M_n(\mathbb{C}),\ \lim_{n\to\infty}\|Z_n\| = 0\big\},$$

and we let $\mathcal{C}$ denote the quotient C^*-algebra

$$\mathcal{C} = \prod_n M_n(\mathbb{C}) \Big/ \sum_n M_n(\mathbb{C}). \tag{7.7}$$

Moreover, we let $\rho : \prod_n M_n(\mathbb{C}) \to \mathcal{C}$ denote the quotient map. By [RLL, Lemma 6.13], the quotient norm in $\mathcal{C}$ is given by

$$\left\|\rho\big((Z_n)_{n=1}^\infty\big)\right\| = \limsup_{n\to\infty} \|Z_n\|, \tag{7.8}$$

for $(Z_n)_{n=1}^\infty \in \prod M_n(\mathbb{C})$.

Let $m \in \mathbb{N}$. Then we can identify $M_m(\mathbb{C}) \otimes \mathcal{C}$ with $\prod_n M_{mn}(\mathbb{C}) \;/\; \sum M_{mn}(\mathbb{C})$, where $\prod_n M_{mn}(\mathbb{C})$ and $\sum_n M_{mn}(\mathbb{C})$ are defined as $\prod_n M_n(\mathbb{C})$ and $\sum_n M_n(\mathbb{C})$, but with $Z_n \in M_{mn}(\mathbb{C})$ instead of $Z_n \in M_n(\mathbb{C})$. Moreover, for $(Z_n)_{n=1}^\infty \in \prod_n M_{mn}(\mathbb{C})$, we have, again by [RLL, Lemma 6.13],

$$\left\|(\mathrm{id}_m \otimes \rho)\big((Z_n)_{n=1}^\infty\big)\right\| = \limsup_{n\to\infty} \|Z_n\|. \tag{7.9}$$

7.4 Lemma. *Let $m \in \mathbb{N}$ and let $Z = (Z_n)_{n=1}^{\infty} \in \prod_n M_{mn}(\mathbb{C})$, such that each Z_n is normal. Then for all $k \in \mathbb{N}$*

$$\mathrm{sp}\big((\mathrm{id}_m \otimes \rho)(Z)\big) \subseteq \overline{\bigcup_{n=k}^{\infty} \mathrm{sp}(Z_n)}.$$

Proof. Assume $\lambda \in \mathbb{C}$ is not in the closure of $\bigcup_{n=k}^{\infty} \mathrm{sp}(Z_n)$. Then there exists an $\varepsilon > 0$, such that $d(\lambda, \mathrm{sp}(Z_n)) \geq \varepsilon$ for all $n \geq k$. Since Z_n is normal, it follows that $\|(\lambda \mathbf{1}_{mn} - Z_n)^{-1}\| \leq \frac{1}{\varepsilon}$ for all $n \geq k$. Now put

$$y_n = \begin{cases} 0, & \text{if } 1 \leq n \leq k-1, \\ (\lambda \mathbf{1}_{mn} - Z_n)^{-1}, & \text{if } n \geq k. \end{cases}$$

Then $y = (y_n)_{n=1}^{\infty} \in \prod_n M_{mn}(\mathbb{C})$, and one checks easily that $\lambda \mathbf{1}_{M_m(\mathbb{C}) \otimes \mathcal{C}} - (\mathrm{id}_m \otimes \rho)(Z)$ is invertible in $M_m(\mathbb{C}) \otimes \mathcal{C} = \prod_n M_{mn}(\mathbb{C}) \ / \ \sum_n M_{mn}(\mathbb{C})$ with inverse $(\mathrm{id}_m \otimes \rho)y$. Hence $\lambda \notin \mathrm{sp}((\mathrm{id}_m \otimes \rho)(Z))$. ∎

Proof of Proposition 7.3 and Theorem 7.1. Assume first that $r \in \mathbb{N}$. Put

$$\Omega_0 = \big\{\omega \in \Omega \ \big| \ \sup_{n \in \mathbb{N}} \|X_i^{(n)}(\omega)\| < \infty, \ i = 1, \ldots, r\big\}.$$

By (7.6), $\Omega \backslash \Omega_0$ is a P-null set. For every $\omega \in \Omega_0$, we define

$$y_i(\omega) \in \mathcal{C} = \prod_n M_n(\mathbb{C}) \Big/ \sum_n M_n(\mathbb{C})$$

by

$$y_i(\omega) = \rho\big((X_i^{(n)}(\omega))_{n=1}^{\infty}\big), \quad i = 1, \ldots, r. \tag{7.10}$$

Then for every non-commutative polynomial $p \in \mathbb{C}\langle X_1, \ldots, X_r\rangle$ and every ω in Ω_0, we get by (7.8) that

$$\big\|p(y_1(\omega), \ldots, y_r(\omega))\big\| = \limsup_{n \to \infty} \big\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\big\|. \tag{7.11}$$

Let $j \in \mathbb{N}$ and $a_0, a_1, \ldots, a_r \in M_m(\mathbb{C})_{\mathrm{sa}}$. Then by Theorem 6.4 there exists a null set $N(m, j, a_0, \ldots, a_r)$, such that for

$$\mathrm{sp}\big(a_0 \otimes \mathbf{1}_n + \textstyle\sum_{i=1}^r a_i \otimes X_i^{(n)}(\omega)\big) \subseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) \ + \ \big]-\tfrac{1}{j}, \tfrac{1}{j}\big[,$$

eventually, as $n \to \infty$, for all $\omega \in \Omega \backslash N(m, j, a_0, \ldots, a_r)$. Let $N_0 = \bigcup N(m, j, a_0, \ldots, a_r)$, where the union is taken over all $m, j \in \mathbb{N}$ and $a_0, \ldots, a_r \in M_n(\mathbb{Q} + \mathrm{i}\mathbb{Q})_{\mathrm{sa}}$. This is a countable union. Hence N_0 is again a P-null set, and by Lemma 7.4

$$\mathrm{sp}\big(a_0 \otimes \mathbf{1}_n + \textstyle\sum_{i=1}^r a_i \otimes y_i(\omega)\big) \subseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big) + \big[-\tfrac{1}{j}, \tfrac{1}{j}\big],$$

for all $\omega \in \Omega_0 \backslash N_0$, all $m, j \in \mathbb{N}$ and all $a_0, \ldots, a_r \in M_n(\mathbb{Q} + \mathrm{i}\mathbb{Q})_{\mathrm{sa}}$. Taking intersection over $j \in \mathbb{N}$ on the right hand side, we get

$$\mathrm{sp}\big(a_0 \otimes \mathbf{1}_n + \textstyle\sum_{i=1}^r a_i \otimes y_i(\omega)\big) \subseteq \mathrm{sp}\big(a_0 \otimes \mathbf{1}_{\mathcal{B}} + \textstyle\sum_{i=1}^r a_i \otimes x_i\big),$$

for $\omega \in \Omega_0 \backslash N_0$, $m \in \mathbb{N}$ and $a_0, \ldots, a_r \in M_n(\mathbb{Q} + \mathrm{i}\mathbb{Q})_{\mathrm{sa}}$. Hence, by Theorem 2.4,

$$\big\|p(y_1(\omega), \ldots, y_r(\omega))\big\| \leq \|p(x_1, \ldots, x_r)\|,$$

for all $p \in \mathbb{C}\langle X_1, \ldots, X_r\rangle$ and all $\omega \in \Omega_0 \backslash N_0$, which, by (7.11), implies that

$$\limsup_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)\big)\big\| \leq \|p(x_1, \ldots, x_r)\|,$$

for all $p \in \mathbb{C}\langle X_1, \ldots, X_r\rangle$ and all $\omega \in \Omega_0 \backslash N_0$. This proves Proposition 7.3, which, together with Lemma 7.2, proves Theorem 7.1 in the case $r \in \mathbb{N}$. The case $r = \infty$ follows from the case $r \in \mathbb{N}$, because $\mathbb{C}\langle (X_i)_{i=1}^\infty\rangle = \cup_{r=1}^\infty \mathbb{C}\langle (X_i)_{i=1}^r\rangle$. ∎

8 $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ is not a group.

We start this section by translating Theorem 7.1 into a corresponding result, where the self-adjoint Gaussian random matrices are replaced by random unitary matrices and the semi-circular system is replaced by a free family of Haar-unitaries.

Define C^1-functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R} \to \mathbb{C}$ by

$$\varphi(t) = \begin{cases} -\pi, & \text{if} \quad t \leq -2, \\ \int_0^t \sqrt{4 - s^2}\,\mathrm{d}s, & \text{if} \quad -2 < t < 2, \\ \pi, & \text{if} \quad t \geq 2. \end{cases} \tag{8.1}$$

and

$$\psi(t) = \mathrm{e}^{\mathrm{i}\varphi(t)}, \qquad (t \in \mathbb{R}). \tag{8.2}$$

Let μ be the standard semi-circle distribution on $\mathbb{R}$:

$$\mathrm{d}\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2} \cdot 1_{[-2,2]}(t)\,\mathrm{d}t,$$

and let $\varphi(\mu)$ denote the push-forward measure of μ by φ, i.e., $\varphi(\mu)(B) = \mu(\varphi^{-1}(B))$ for any Borel subset B of $\mathbb{R}$. Since $\varphi'(t) = \sqrt{4 - t^2} \cdot 1_{[-2,2]}(t)$ for all t in $\mathbb{R}$, it follows that $\varphi(\mu)$ is the uniform distribution on $[-\pi, \pi]$, and, hence, $\psi(\mu)$ is the Haar measure on the unit circle $\mathbb{T}$ in $\mathbb{C}$.

The following lemma is a simple application of Voiculescu's results in [V3].

8.1 Lemma. *Let $r \in \mathbb{N} \cup \{\infty\}$ and let $(x_i)_{i=1}^r$ be a semi-circular system in a C^*-probability space $(\mathcal{B}, \tau)$, where τ is a faithful state on $\mathcal{B}$. Let $\psi\colon \mathbb{R} \to \mathbb{T}$ be the function defined in (8.2), and then put*

$$u_i = \psi(x_i), \qquad (i = 1, \ldots, r).$$

Then there is a (surjective) $*$-isomorphism $\Phi\colon C^*_{\mathrm{red}}(F_r) \to C^*((u_i)_{i=1}^r)$, such that

$$\Phi\big(\lambda(g_i)\big) = u_i, \qquad (i = 1, \dots, r),$$

where $g_1, \dots, g_r$ are the generators of the free group F_r, and $\lambda\colon F_r \to \mathcal{B}(\ell^2(F_r))$ is the left regular representation of F_r on $\ell^2(F_r)$.

Proof. Recall that $C^*_{\mathrm{red}}(F_r)$ is, by definition, the C^*-algebra in $\mathcal{B}(\ell^2(F_r))$ generated by $\lambda(g_1), \dots, \lambda(g_r)$. Let e denote the unit in F_r and let $\delta_e \in \ell^2(F_r)$ denote the indicator function for $\{e\}$. Recall, then, that the vector state $\eta = \langle \cdot\, \delta_e, \delta_e \rangle\colon \mathcal{B}(\ell^2(F_r)) \to \mathbb{C}$, corresponding to δ_e, is faithful on $C^*_{\mathrm{red}}(F_r)$. We recall, further, from [V3] that $\lambda(g_1), \dots, \lambda(g_r)$ are $*$-free operators w.r.t. η, and that each $\lambda(g_i)$ is a Haar unitary, i.e.,

$$\eta(\lambda(g_i)^n) = \begin{cases} 1, & \text{if} \quad n = 0, \\ 0, & \text{if} \quad n \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Now, since $(x_i)_{i=1}^r$ are free self-adjoint operators in $(\mathcal{B}, \tau)$, $(u_i)_{i=1}^r$ are $*$-free unitaries in $(\mathcal{B}, \tau)$, and since, as noted above, $\psi(\mu)$ is the Haar measure on $\mathbb{T}$, all the u_i's are Haar unitaries as well. Thus, the $*$-distribution of $(\lambda(g_i))_{i=1}^r$ w.r.t. η (in the sense of [V3]) equals that of $(u_i)_{i=1}^r$ w.r.t. τ. Since η and τ are both faithful, the existence of a $*$-isomorhism Φ, with the properties set out in the lemma, follows from [V3, Remark 1.8]. ∎

Let $r \in \mathbb{N} \cup \{\infty\}$. As in Theorem 7.1, we consider next, for each n in $\mathbb{N}$, independent random matrices $(X_i^{(n)})_{i=1}^r$ in $\mathrm{SGRM}(n, \frac{1}{n})$. We then define, for each n, random unitary $n \times n$ matrices $(U_i^{(n)})_{i=1}^r$, by setting

$$U_i^{(n)}(\omega) = \psi(X_i^{(n)}(\omega)), \qquad (i = 1, 2, \dots, r), \tag{8.3}$$

where $\psi\colon \mathbb{R} \to \mathbb{T}$ is the function defined in (8.2). Consider, further, the (free) generators $(g_i)_{i=1}^r$ of F_r. Then, by the universal property of a free group, there exists, for each n in $\mathbb{N}$ and each ω in Ω, a unique group homomorphism:

$$\pi_{n,\omega}\colon F_r \to \mathcal{U}(n) = \mathcal{U}(M_n(\mathbb{C})),$$

satisfying

$$\pi_{n,\omega}(g_i) = U_i^{(n)}(\omega), \qquad (i = 1, 2, \dots, r). \tag{8.4}$$

8.2 Theorem. *Let $r \in \mathbb{N} \cup \{\infty\}$ and let, for each n in $\mathbb{N}$, $(U_i^{(n)})_{i=1}^r$ be the random unitaries given by (8.3). Let, further, for each n in $\mathbb{N}$ and each ω in Ω, $\pi_{n,\omega}\colon F_r \to \mathcal{U}(n)$ be the group homomorphism given by (8.4).*

Then there exists a P-null set $N \subseteq \Omega$, such that for all ω in $\Omega \setminus N$ and all functions $f\colon F_r \to \mathbb{C}$ with finite support, we have

$$\lim_{n\to\infty} \Big\| \sum_{\gamma \in F_r} f(\gamma)\pi_{n,\omega}(\gamma) \Big\| = \Big\| \sum_{\gamma \in F_r} f(\gamma)\lambda(\gamma) \Big\|,$$

where, as above, λ is the left regular representation of F_r on $\ell^2(F_r)$.

Proof. In the proof we shall need the following simple observation: If $a_1, \dots, a_s, b_1, \dots, b_s$ are $2s$ operators on a Hilbert space $\mathcal{K}$, such that $\|a_i\|, \|b_i\| \leq 1$ for all i in $\{1, 2, \dots, s\}$, then

$$\|a_1 a_2 \cdots a_s - b_1 b_2 \cdots b_s\| \leq \sum_{i=1}^{s} \|a_i - b_i\|. \tag{8.5}$$

We shall need, further, that for any positive ε there exists a polynomial q in one variable, such that

$$|q(t)| \leq 1, \qquad (t \in [-3, 3]), \tag{8.6}$$

and

$$|\psi(t) - q(t)| \leq \varepsilon, \qquad (t \in [-3, 3]). \tag{8.7}$$

Indeed, by Weierstrass' approximation theorem, we may choose a polynomial q_0 in one variable, such that

$$|\psi(t) - q_0(t)| \leq \varepsilon/2, \qquad (t \in [-3, 3]). \tag{8.8}$$

Then put $q = (1 + \varepsilon/2)^{-1} q_0$ and note that since $|\psi(t)| = 1$ for all t in $\mathbb{R}$, it follows from (8.8) that (8.6) holds. Furthermore,

$$|q_0(t) - q(t)| \leq \tfrac{\varepsilon}{2} |q(t)| \leq \tfrac{\varepsilon}{2}, \qquad (t \in [-3, 3]),$$

which, combined with (8.8), shows that (8.7) holds.

After these preparations, we start by proving the theorem in the case $r \in \mathbb{N}$. For each n in $\mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices in $\mathrm{SGRM}(n, \frac{1}{n})$ defined on $(\Omega, \mathcal{F}, P)$, and define the random unitaries $U_1^{(n)}, \dots, U_r^{(n)}$ as in (8.3). Then let N be a P-null set as in the main theorem (Theorem 7.1). By considering, for each i in $\{1, 2, \dots, r\}$, the polynomial $p(X_1, \dots, X_r) = X_i$, it follows then from the main theorem that

$$\lim_{n \to \infty} \big\| X_i^{(n)}(\omega) \big\| = 2,$$

for all ω in $\Omega \setminus N$. In particular, for each ω in $\Omega \setminus N$, there exists an n_ω in $\mathbb{N}$, such that

$$\big\| X_i^{(n)}(\omega) \big\| \leq 3, \qquad \text{whenever } n \geq n_\omega \text{ and } i \in \{1, 2, \dots, r\}.$$

Considering, then, the polynomial q introduced above, it follows from (8.6) and (8.7) that for all ω in $\Omega \setminus N$, we have

$$\big\| q\big(X_i^{(n)}(\omega)\big) \big\| \leq 1, \qquad \text{whenever } n \geq n_\omega \text{ and } i \in \{1, 2, \dots, r\}, \tag{8.9}$$

and

$$\big\| U_i^{(n)}(\omega) - q\big(X_i^{(n)}(\omega)\big) \big\| \leq \varepsilon, \qquad \text{whenever } n \geq n_\omega \text{ and } i \in \{1, 2, \dots, r\}. \tag{8.10}$$

Next, if $\gamma \in F_r \setminus \{e\}$, then γ can be written (unambiguesly) as a reduced word: $\gamma = \gamma_1\gamma_2\cdots\gamma_s$, where $\gamma_j \in \{g_1, g_2, \dots, g_r, g_1^{-1}, g_2^{-1}, \dots, g_r^{-1}\}$ for each j in $\{1, 2, \dots, s\}$, and where $s = |\gamma|$ is the length of the reduced word for γ. It follows then, by (8.4), that $\pi_{n,\omega}(\gamma) = a_1a_2\cdots a_s$, where

$$a_j = \pi_{n,\omega}(\gamma_j) \in \big\{U_1^{(n)}(\omega), \dots, U_r^{(n)}(\omega), U_1^{(n)}(\omega)^*, \dots, U_r^{(n)}(\omega)^*\big\}, \qquad (j = 1, 2, \dots, s).$$

Combining now (8.5), (8.9) and (8.10), it follows that for any γ in $F_r \setminus \{e\}$, there exists a polynomial p_γ in $\mathbb{C}\langle X_1, \dots, X_r\rangle$, such that

$$\big\|\pi_{n,\omega}(\gamma) - p_\gamma\big(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega)\big)\big\| \le |\gamma|\varepsilon, \qquad \text{whenever } n \ge n_\omega \text{ and } \omega \in \Omega \setminus N. \tag{8.11}$$

Now, let $\{x_1, \dots, x_r\}$ be a semi-circular system in a C^*-probability space $(\mathcal{B}, \tau)$, and put $u_i = \psi(x_i)$, $i = 1, 2, \dots, r$. Then, by Lemma 8.1, there is a surjective $*$-isomorphism $\Phi\colon C^*_{\mathrm{red}}(F_r) \to C^*(u_1, \dots, u_r)$, such that $(\Phi \circ \lambda)(g_i) = u_i$, $i = 1, 2, \dots, r$. Since $\|x_i\| \le 3$, $i = 1, 2, \dots, r$, the arguments that lead to (8.11) show also that for any γ in $F_r \setminus \{e\}$,

$$\big\|(\Phi \circ \lambda)(\gamma) - p_\gamma(x_1, \dots, x_r)\big\| \le |\gamma|\varepsilon, \tag{8.12}$$

where p_γ is the same polynomial as in (8.11). Note that (8.11) and (8.12) also hold in the case $\gamma = e$, if we put $p_e(X_1, \dots, X_r) = 1$, and $|e| = 0$.

Consider now an arbitrary function $f\colon F_r \to \mathbb{C}$ with finite support, and then define the polynomial p in $\mathbb{C}\langle X_1, \dots, X_r\rangle$, by: $p = \sum_{\gamma\in F_r} f(\gamma)p_\gamma$. Then, for any ω in $\Omega \setminus N$ and any $n \ge n_\omega$, we have

$$\Big\|\sum_{\gamma\in F_r} f(\gamma)\pi_{n,\omega}(\gamma) - p\big(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega)\big)\Big\| \le \Big(\sum_{\gamma\in F_r} |f(\gamma)| \cdot |\gamma|\Big)\varepsilon, \tag{8.13}$$

and

$$\Big\|\sum_{\gamma\in F_r} f(\gamma) \cdot (\Phi \circ \lambda)(\gamma) - p(x_1, \dots, x_r)\Big\| \le \Big(\sum_{\gamma\in F_r} |f(\gamma)| \cdot |\gamma|\Big)\varepsilon, \tag{8.14}$$

Taking also Theorem 7.1 into account, we may, on the basis of (8.13) and (8.14), conclude that for any ω in $\Omega \setminus N$, we have

$$\limsup_{n\to\infty} \left| \Big\|\sum_{\gamma\in F_r} f(\gamma)\pi_{n,\omega}(\gamma)\Big\| - \Big\|\sum_{\gamma\in F_r} f(\gamma) \cdot (\Phi \circ \lambda)(\gamma)\Big\| \right| \le 2\varepsilon\Big(\sum_{\gamma\in F_r} |f(\gamma)| \cdot |\gamma|\Big).$$

Since $\varepsilon > 0$ is arbitrary, it follows that for any ω in $\Omega \setminus N$,

$$\lim_{n\to\infty} \Big\|\sum_{\gamma\in F_r} f(\gamma)\pi_{n,\omega}(\gamma)\Big\| = \Big\|\sum_{\gamma\in F_r} f(\gamma) \cdot (\Phi \circ \lambda)(\gamma)\Big\| = \Big\|\sum_{\gamma\in F_r} f(\gamma)\lambda(\gamma)\Big\|,$$

where the last equation follows from the fact that Φ is a $*$-isomorphism. This proves Theorem 8.2 in the case where $r \in \mathbb{N}$. The case $r = \infty$ follows by trivial modifications of the above argument. ∎

8.3 Remark. The distributions of the random unitaries $U_1^{(n)}, \dots, U_r^{(n)}$ in Theorem 8.2 are quite complicated. For instance, it is easily seen that for all n in $\mathbb{N}$,

$$P\big(\{\omega \in \Omega \mid U_1^{(n)}(\omega) = -\mathbf{1}_n\}\big) > 0.$$

It would be interesting to know whether Theorem 8.2 also holds, if, for each n in $\mathbb{N}$, $U_1^{(n)}, \dots, U_r^{(n)}$ are replaced be stochastically independent random unitaries $V_1^{(n)}, \dots, V_r^{(n)}$, which are all distributed according to the normalized Haar measure on $\mathcal{U}(n)$.

8.4 Corollary. *For any r in $\mathbb{N} \cup \{\infty\}$, the C^*-algebra $C^*_{\mathrm{red}}(F_r)$ has a unital embedding into the quotient C^*-algebra*

$$\mathcal{C} = \prod_n M_n(\mathbb{C}) \Big/ \sum_n M_n(\mathbb{C}),$$

introduced in Section 7. In particular, $C^*_{\mathrm{red}}(F_r)$ is an MF-algebra in the sense of Blackadar and Kirchberg (cf. [BK]).

Proof. This follows immediately from Theorem 8.2 and formula (7.8). In fact, one only needs the existence of one ω in Ω for which the convergence in Theorem 8.2 holds! ∎

We remark that Corollary 8.4 could also have been proved, directly, from the main theorem (Theorem 7.1) together with Lemma 8.1.

8.5 Corollary. *For any r in $\{2, 3, \dots\} \cup \{\infty\}$, the semi-group $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ is not a group.*

Proof. In Section 5.14 of Voiculescu's paper [V6], it is proved that $\mathrm{Ext}(C^*_{\mathrm{red}}(F_r))$ cannot be a group, if there exists a sequence $(\pi_n)_{n\in\mathbb{N}}$ of unitary representations $\pi_n \colon F_r \to \mathcal{U}(n)$, with the property that

$$\lim_{n\to\infty} \Big\| \sum_{\gamma\in F_r} f(\gamma)\pi_n(\gamma) \Big\| = \Big\| \sum_{\gamma\in F_r} f(\gamma)\lambda(\gamma) \Big\|, \tag{8.15}$$

for any function $f \colon F_r \to \mathbb{C}$ with finite support.

For any $r \in \{2, 3, \dots\} \cup \{\infty\}$, the existence of such a sequence $(\pi_n)_{n\in\mathbb{N}}$ follows immediately from Theorem 8.2, by considering one single ω from the sure event $\Omega \setminus N$ appearing in that theorem. ∎

8.6 Remark. Let us briefly outline Voiculescu's argument in [V7] for the fact that (8.15) implies Corollary 8.5. It is obtained by combining the following two results of Rosenberg [Ro] and Voiculescu [V6], respectively:

(i) If Γ is a discrete countable non-amenable group, then $C^*_{\mathrm{red}}(\Gamma)$ is not quasi-diagonal ([Ro]).

(ii) A separable unital C^*-algebra $\mathcal{A}$ is quasi-diagonal if and only if there exists a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of completely positive unital maps $\varphi_k : \mathcal{A} \to M_{n_k}(\mathbb{C})$, such that $\lim_{k\to\infty} \|\varphi_k(a)\| = \|a\|$ and $\lim_{k\to\infty} \|\varphi_k(ab) - \varphi_k(a)\varphi_k(b)\| = 0$ for all $a, b \in \mathcal{A}$ ([V6]).

Let $\mathcal{A}$ be a separable unital C^*-algebra. Then, as mentioned in the introduction, $\mathrm{Ext}(\mathcal{A})$ is the set of equivalence classes $[\pi]$ of one-to-one unital $*$-homomorphisms π of $\mathcal{A}$ into the Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ over a separable infinite dimensional Hilbert space $\mathcal{H}$. Two such $*$-homomorphisms are equivalent if they are equal up to a unitary transformation of $\mathcal{H}$. $\mathrm{Ext}(\mathcal{A})$ has a natural semi-group structure and $[\pi]$ is invertible in $\mathrm{Ext}(\mathcal{A})$ if and only if π has a unital completely positive lifting: $\psi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (cf. [Arv]). Let now $\mathcal{A} = C^*_{\mathrm{red}}(F_r)$, where $r \in \{2, 3, \dots\} \cup \{\infty\}$. Moreover, let $\pi_n\colon F_r \to \mathcal{U}_n$, $n \in \mathbb{N}$, be a sequence of unitary representations satisfying (8.15) and let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{C}^n$. Clearly, $\prod_n M_n(\mathbb{C})/\sum_n M_n(\mathbb{C})$ embeds naturally into Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Hence, there exists a one-to-one $*$-homomorphism $\pi\colon \mathcal{A} \to \mathcal{C}(\mathcal{H})$, such that

$$\pi(\lambda(h)) = \rho \begin{pmatrix} \pi_1(h) & & 0 \\ & \pi_2(h) & \\ 0 & & \ddots \end{pmatrix},$$

for all $h \in F_r$ (here ρ denotes the quotient map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{C}(\mathcal{H})$). Assume $[\pi]$ is invertible in $\mathrm{Ext}(\mathcal{A})$. Then π has a unital completely positive lifting $\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Put $\varphi_n(a) = p_n\varphi(a)p_n$, $a \in \mathcal{A}$, where $p_n \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto the component $\mathbb{C}^n$ of $\mathcal{H}$. Then each φ_n is a unital completely positive map from $\mathcal{A}$ to $M_n(\mathbb{C})$, and it is easy to check that

$$\lim_{n\to\infty} \|\varphi_n(\lambda(h)) - \pi_n(h)\| = 0, \qquad (h \in F_r).$$

From this it follows that

$$\lim_{n\to\infty} \|\varphi_n(a)\| = \|a\| \quad \text{and} \quad \lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0, \qquad (a, b \in \mathcal{A})$$

so by (ii), $\mathcal{A} = C^*_{\mathrm{red}}(F_r)$ is quasi-diagonal. But since F_r is not amenable for $r \geq 2$, this contradicts (i). Hence $[\pi]$ is not invertible in $\mathrm{Ext}(\mathcal{A})$.

8.7 Remark. let $\mathcal{A}$ be a separable unital C^*-algebra and let $\pi\colon \mathcal{A} \to \mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be a one-to-one *-homomorphism. Then π gives rise to an extension of $\mathcal{A}$ by the compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$, i.e., a C^*-algebra $\mathcal{B}$ together with a short exact sequence of *-homomorphisms

$$0 \to \mathcal{K} \xrightarrow{\iota} \mathcal{B} \xrightarrow{q} \mathcal{A} \to 0.$$

Specifically, with $\rho\colon \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ the quotient map, $\mathcal{B} = \rho^{-1}(\pi(\mathcal{A}))$, ι is the inclusion map of $\mathcal{K}$ into $\mathcal{B}$ and $q = \pi^{-1} \circ \rho$. Let now $\mathcal{A} = C^*_{\mathrm{red}}(F_r)$, let $\pi\colon \mathcal{A} \to \mathcal{C}(\mathcal{H})$ be the one-to-one unital *-homomorphism from Remark 8.6, and let $\mathcal{B}$ be the compact extension of $\mathcal{A}$ constructed above. We then have

a) $\mathcal{A} = C^*_{\mathrm{red}}(F_r)$ is an exact C^*-algebra, but the compact extension $\mathcal{B}$ of $\mathcal{A}$ is not exact.

b) $\mathcal{A} = C^*_{\mathrm{red}}(F_r)$ is not quasi-diagonal but the compact extension $\mathcal{B}$ of $\mathcal{A}$ is quasi-diagonal.

To prove a), note that $C^*_{\mathrm{red}}(F_r)$ is exact by [DH, Cor. 3.12] or [Ki2, p. 453, l. 1–3]. Assume $\mathcal{B}$ is also exact. Then, in particular, $\mathcal{B}$ is locally reflexive (cf. [Ki2]). Hence by the lifting theorem in [EH] and the nuclearity of $\mathcal{K}$, the identity map $\mathcal{A} \to \mathcal{A}$ has a unital completely positive lifting $\varphi\colon \mathcal{A} \to \mathcal{B}$. If we consider φ as a map from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$, it is a unital completely positive lifting of $\pi\colon \mathcal{A} \to \mathcal{C}(\mathcal{H})$, which contradicts that $[\pi]$ is not invertible in $\mathrm{Ext}(\mathcal{A})$. To prove b), note that by Rosenberg's result, quoted in (i) above, $C^*_{\mathrm{red}}(F_r)$ is not quasi-diagonal. On the other hand, by the definition of π in Remark 8.6, every $x \in \mathcal{B}$ is a compact perturbation of an operator of the form

$$y = \begin{pmatrix} y_1 & & 0 \\ & y_2 & \\ 0 & & \ddots \end{pmatrix},$$

where $y_n \in M_n(\mathbb{C})$, $n \in \mathbb{N}$. Hence $\mathcal{B}$ is quasi-diagonal.

9 Other applications.

Recall that a C^*-algebra $\mathcal{A}$ is called exact if, for every pair $(\mathcal{B}, \mathcal{J})$ consisting of a C^*-algebra $\mathcal{B}$ and closed two-sided ideal $\mathcal{J}$ in $\mathcal{B}$, the sequence

$$0 \to \mathcal{A} \underset{\min}{\otimes} \mathcal{J} \to \mathcal{A} \underset{\min}{\otimes} \mathcal{B} \to \mathcal{A} \underset{\min}{\otimes} (\mathcal{B}/\mathcal{J}) \to 0 \tag{9.1}$$

is exact (cf. [Ki1]). In generalization of the construction described in the paragraph preceding Lemma 7.4, we may, for any sequence $(\mathcal{A}_n)_{n=1}^\infty$ of C^*-algebras, define two C^*-algebras

$$\begin{aligned} \prod_n \mathcal{A}_n &= \big\{(a_n)_{n=1}^\infty \mid a_n \in \mathcal{A}_n,\ \sup_{n\in\mathbb{N}}\|a_n\| < \infty\big\} \\ \sum_n \mathcal{A}_n &= \big\{(a_n)_{n=1}^\infty \mid a_n \in \mathcal{A}_n,\ \lim_{n\to\infty}\|a_n\| = 0\big\}. \end{aligned}$$

The latter C^*-algebra is a closed two-sided ideal in the first, and the norm in the quotient C^*-algebra $\prod_n \mathcal{A}_n / \sum_n \mathcal{A}_n$ is given by

$$\big\|\rho\big((x_n)_{n=1}^\infty\big)\big\| = \limsup_{n\to\infty} \|x_n\|, \tag{9.2}$$

where ρ is the quotient map (cf. [RLL, lemma 6.13]) . In the following we let $\mathcal{A}$ denote an exact C^*-algebra. By (9.1) we have the following natural identification of C^*-algebras

$$\mathcal{A} \underset{\min}{\otimes} \Big(\prod_n M_n(\mathbb{C}) \Big/ \sum_n M_n(\mathbb{C})\Big) = \Big(\mathcal{A} \underset{\min}{\otimes} \prod_n M_n(\mathbb{C})\Big) \Big/ \Big(\mathcal{A} \underset{\min}{\otimes} \sum_n M_n(\mathbb{C})\Big).$$

Moreover, we have (without assuming exactness) the following natural identification

$$\mathcal{A} \underset{\min}{\otimes} \sum_n M_n(\mathbb{C}) = \sum_n M_n(\mathcal{A})$$

and the natural inclusion

$$\mathcal{A} \underset{\min}{\otimes} \prod_n M_n(\mathbb{C}) \subseteq \prod_n M_n(\mathcal{A}).$$

If $\dim(\mathcal{A}) < \infty$, the inclusion becomes an identity, but in general the inclusion is proper. Altogether we have for all exact C^*-algebras $\mathcal{A}$ a natural inclusion

$$\mathcal{A} \underset{\min}{\otimes} \Big(\prod_n M_n(\mathbb{C}) \Big/ \sum_n M_n(\mathbb{C}) \Big) \subseteq \prod_n M_n(\mathcal{A}) \Big/ \sum_n M_n(\mathcal{A}). \tag{9.3}$$

Similarly, if $n_1 < n_2 < n_3 < \cdots$, are natural numbers, then

$$\mathcal{A} \underset{\min}{\otimes} \Big(\prod_k M_{n_k}(\mathbb{C}) \Big/ \sum_k M_{k_n}(\mathbb{C}) \Big) \subseteq \prod_k M_{n_k}(\mathcal{A}) \Big/ \sum_k M_{n_k}(\mathcal{A}). \tag{9.4}$$

After these preparations we can now prove the following generalizations of Theorems 7.1 and 8.2.

9.1 Theorem. *Let $(\Omega, \mathcal{F}, P)$, N, $(X_i^{(n)})_{i=1}^r$ and $(x_i)_{i=1}^r$ be as in Theorem 7.1, and let $\mathcal{A}$ be a unital exact C^*-algebra. Then for all polynomials p in r non-commuting variables and with coefficients in $\mathcal{A}$ (i.e., p is in the algebraic tensor product $\mathcal{A} \otimes \mathbb{C}\langle (X_i)_{i=1}^r \rangle$), and all $\omega \in \Omega \backslash N$,*

$$\lim_{n\to\infty} \big\| p\big((X_i^{(n)}(w))_{i=1}^r \big) \big\|_{M_n(\mathcal{A})} = \big\| p\big((x_i)_{i=1}^r \big) \big\|_{\mathcal{A} \otimes_{\min} C^*((x_i)_{i=1}^r, \mathbf{1}_{\mathcal{B}})}. \tag{9.5}$$

Proof. We consider only the case $r \in \mathbb{N}$. The case $r = \infty$ is proved similarly. By Theorem 7.1 we can for each $\omega \in \Omega \backslash N$ define a unital embedding π_ω of $C^*(x_1, \ldots, x_r, \mathbf{1}_{\mathcal{B}})$ into $\prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C})$, such that

$$\pi_\omega(x_i) = \rho\big(\big(X_i^{(n)}(\omega) \big)_{n=1}^\infty \big), \quad i = 1, \ldots, r,$$

where $\rho \colon \prod_n M_n(\mathbb{C}) \to \prod_n M_n(\mathbb{C}) / \sum_n M_n(\mathbb{C})$ is the quotient map. Since $\mathcal{A}$ is exact, we can, by (9.3), consider $\mathrm{id}_{\mathcal{A}} \otimes \pi_\omega$ as a unital embedding of $\mathcal{A} \otimes_{\min} C^*(x_1, \ldots, x_r, \mathbf{1}_{\mathcal{B}})$ into $\prod_n M_n(\mathcal{A}) / \sum_n M_n(\mathcal{A})$, for which

$$(\mathrm{id}_{\mathcal{A}} \otimes \pi_\omega)(a \otimes x_i) = \tilde{\rho}\big(\big(a \otimes X_i^{(n)}(\omega) \big)_{n=1}^\infty \big), \quad i = 1, \ldots, r,$$

where $\tilde{\rho} \colon \prod_n M_n(\mathcal{A}) \to \prod_n M_n(\mathcal{A}) / \sum M_n(\mathcal{A})$ is the quotient map. Hence, for every p in $\mathcal{A} \otimes \mathbb{C}\langle X_1, \ldots, X_r \rangle$,

$$(\mathrm{id}_{\mathcal{A}} \otimes \pi_\omega)\big(p(x_1, \ldots, x_r) \big) = \tilde{\rho}\big(\big(p(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega)) \big)_{n=1}^\infty \big).$$

By (9.2) it follows that for all $\omega \in \Omega / N$, and every p in $\mathcal{A} \otimes \mathbb{C}\langle X_1, \ldots, X_r \rangle$,

$$\big\| p(x_1, \ldots, x_r) \big\|_{\mathcal{A} \otimes_{\min} C^*(x_1, \ldots, x_r, \mathbf{1}_{\mathcal{B}})} = \limsup_{n\to\infty} \big\| p\big(X_1^{(n)}(\omega), \ldots, X_r^{(n)}(\omega) \big) \big\|_{M_n(\mathcal{A})}.$$

Consider now a fixed $\omega \in \Omega\backslash N$. Put

$$\alpha = \liminf_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega)\big)\big\|_{M_n(\mathcal{A})},$$

and choose natural numbers $n_1 < n_2 < n_3 < \cdots$, such that

$$\alpha = \lim_{k\to\infty} \big\|p\big(X_1^{(n_k)}(\omega), \dots, X_r^{(n_k)}(\omega)\big)\big\|_{M_n(\mathcal{A})}.$$

By Theorem 7.1 there is a unital embedding π'_ω of $C^*(x_1, \dots, x_r, \mathbf{1}_{\mathcal{B}})$ into the quotient $\prod_k M_{n_k}(\mathbb{C})/\sum_k M_{n_k}(\mathbb{C})$, such that

$$\pi'_\omega(x_i) = \rho'\big(\big(X_i^{(n_k)}(\omega)\big)_{k=1}^\infty\big), \quad i = 1, \dots, r,$$

where $\rho'\colon \prod_k M_{n_k}(\mathbb{C}) \to \prod_k M_{n_k}(\mathbb{C})/\sum_k M_{n_k}(\mathbb{C})$ is the quotient map. Using (9.4) instead of (9.3), we get, as above, that

$$\begin{aligned}
\|p(x_1, \dots, x_r)\|_{\mathcal{A}\otimes_{\min} C^*(x_1,\dots,x_r,\mathbf{1}_{\mathcal{B}})} &= \limsup_{k\to\infty} \big\|p\big(X_1^{(n_k)}(\omega), \dots, X_r^{(n_k)}(\omega)\big)\big\|_{M_n(\mathcal{A})} \\
&= \alpha \\
&= \liminf_{n\to\infty} \big\|p\big(X_1^{(n)}(\omega), \dots, X_r^{(n)}(\omega)\big)\big\|_{M_n(\mathcal{A})}.
\end{aligned}$$

This completes the proof of (9.5). ∎

9.2 Theorem. *Let $(\Omega, \mathcal{F}, P)$, $(U_i^{(n)})_{i=1}^r$, $\pi_{n,\omega}, \lambda$ and N be as in Theorem 8.2. Then for every unital exact C^*-algebra $\mathcal{A}$, every function $f\colon F_r \to \mathcal{A}$ with finite support (i.e. f is in the algebraic tensor product $\mathcal{A} \otimes \mathbb{C}F_r$), and for every $\omega \in \Omega\backslash N$*

$$\lim_{n\to\infty} \Big\| \sum_{\gamma\in F_r} f(\gamma) \otimes \pi_{n,\omega}(\gamma)\Big\|_{M_n(\mathcal{A})} = \Big\| \sum_{\gamma\in F_r} f(\gamma) \otimes \lambda(\gamma)\Big\|_{\mathcal{A}\otimes_{\min} C^*_{\mathrm{red}}(F_r)}.$$

Proof. This follows from Theorem 8.2 in the same way as Theorem 9.1 follows from Theorem 7.1, so we leave the details of the proof to the reader. ∎

In Corollary 9.3 below we use Theorem 9.1 to give new proofs of two key results from our previous paper [HT2]. As in [HT2] we denote by $\mathrm{GRM}(n, n, \sigma^2)$ or $\mathrm{GRM}(n, \sigma^2)$ the class of $n \times n$ random matrices $y = (y_{ij})_{1\le i,j\le n}$, whose entries y_{ij}, $1 \le i, j \le n$, are n^2 i.i.d. complex Gaussian random variables with density $(\pi\sigma^2)^{-1}\exp(-|z|^2/\sigma^2)$, $z \in \mathbb{C}$. It is elementary to check that Y is in $\mathrm{GRM}(n, \sigma^2)$ if and only if $Y = \frac{1}{\sqrt{2}}(X_1 + \mathrm{i}X_2)$ where

$$X_1 = \frac{1}{\sqrt{2}}(Y + Y^*), \quad X_2 = \frac{1}{\mathrm{i}\sqrt{2}}(Y - Y^*)$$

are two stochastically independent self-adjoint random matrices from the class $\mathrm{SGRM}(n, \sigma^2)$.

9.3 Corollary. [HT2, Thm. 4.5 and Thm. 8.7] *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let $c > 0$, let $r \in \mathbb{N}$ and let $a_1, \dots, a_r \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\Big\| \sum_{i=1}^r a_i^* a_i \Big\| \le c \quad \textit{and} \quad \Big\| \sum_{i=1}^r a_i a_i^* \Big\| \le 1,$$

and such that $\{a_i^* a_j \mid i,j = 1,\dots,r\} \cup \{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$ *generates an exact* C^**-algebra* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. *Assume further that* $Y_1^{(n)},\dots,Y_r^{(n)}$ *are stochastically independent random matrices from the class* $\mathrm{GRM}(n,\frac{1}{n})$, *and put* $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$. *Then for almost all* ω *in the underlying probability space* Ω,

$$\limsup_{n\to\infty} \max\left\{\mathrm{sp}(S_n(\omega)^* S_n(\omega))\right\} \le (\sqrt{c}+1)^2. \tag{9.6}$$

If, furthermore, $c > 1$ *and* $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, *then*

$$\liminf_{n\to\infty} \min\left\{\mathrm{sp}(S_n(\omega)^* S_n(\omega))\right\} \ge (\sqrt{c}-1)^2. \tag{9.7}$$

Proof. By the comments preceding Corollary 9.3, we can write

$$Y_i^{(n)} = \frac{1}{\sqrt{2}}(X_{2i-1}^{(n)} + \mathrm{i}X_{2i}^{(n)}), \quad i = 1,\dots,r,$$

where $X_1^{(n)},\dots,X_{2r}^{(n)}$ are independent self-adjoint random matrices from $\mathrm{SGRM}(n,\frac{1}{n})$. Hence $S_n^* S_n$ is a second order polynomial in $(X_1^{(n)},\dots,X_{2r}^{(n)})$ with coefficient in the exact unital C^*-algebra $\mathcal{A}$ generated by $\{a_i^* a_j \mid i,j = 1,\dots,r\} \cup \{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$. Hence, by Theorem 9.1, there is a P-null set N in the underlying probability space $(\Omega,\mathcal{F},P)$ such that

$$\lim_{n\to\infty} \|S_n^*(\omega)S_n(\omega)\| = \Big\|\Big(\sum_{i=1}^r a_i \otimes y_i\Big)^*\Big(\sum_{i=1}^r a_i \otimes y_i\Big)\Big\|,$$

where $y_i = \frac{1}{\sqrt{2}}(x_{2i-1} + \mathrm{i}x_{2i})$ and $(x_1,\dots,x_{2r})$ is any semicircular system in a C^*-probability space $(\mathcal{B},\tau)$ with τ faithful. Hence, in the terminology of [V3], $(y_1,\dots,y_r)$ is a circular system with the normalization $\tau(y_i^* y_i) = 1$, $i = 1,\dots,r$. By [V3], a concrete model for such a circular system is

$$y_i = \ell_{2i-1} + \ell_{2i}^*, \quad i = 1,\dots,r$$

where $\ell_1,\dots,\ell_{2r}$ are the creation operators on the full Fock space

$$\mathcal{T} = \mathcal{T}(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \cdots$$

over a Hilbert space $\mathcal{L}$ of dimension $2r$, and τ is the vector state given by the unit vector $1 \in \mathbb{C} \subseteq \mathcal{T}(\mathcal{L})$. Moreover, τ is a faithful trace on the C^*-algebra $\mathcal{B} = C^*(y_1,\dots,y_{2r},\mathbf{1}_{\mathcal{B}(\mathcal{T}(\mathcal{L}))})$. The creation operators $\ell_1,\dots,\ell_{2r}$ satisfy

$$\ell_i^* \ell_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \ne j. \end{cases}$$

Hence, we get

$$\sum_{i=1}^r a_i \otimes y_i = \Big(\sum_{i=1}^r a_i \otimes \ell_{2i-1}\Big) + \Big(\sum_{i=1}^r a_i \otimes \ell_{2i}^*\Big) = z + w,$$

where

$$z^*z = \Big(\sum_{i=1}^r a_i^*a_i\Big) \otimes \mathbf{1}_{\mathcal{B}(\mathcal{T})} \quad \text{and} \quad ww^* = \Big(\sum_{i=1}^r a_ia_i^*\Big) \otimes \mathbf{1}_{\mathcal{B}(\mathcal{T})}.$$

Hence

$$\Big\|\sum_{i=1}^r a_i \otimes y_i\Big\| \le \|z\| + \|w\| \le \Big\|\sum_{i=1}^r a_i^*a_i\Big\|^{\frac12} + \Big\|\sum_{i=1}^r a_ia_i^*\Big\|^{\frac12} \le \sqrt{c}+1.$$

This proves (9.5). If, furthermore, $c > 1$ and $\sum_{i=1}^r a_i^*a_i = c\cdot\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, then $z^*z = c\mathbf{1}_{\mathcal{A}\otimes\mathcal{B}(\mathcal{T})}$ and, as before, $\|w\| \le 1$. Thus, for all $\xi \in \mathcal{H}\otimes\mathcal{T}$, $\|z\xi\| = \sqrt{c}\|\xi\|$ and $\|w\xi\| \le \|\xi\|$. Hence

$$(\sqrt{c}-1)\|\xi\| \le \|(z+w)\xi\| \le (\sqrt{c}+1)\|\xi\|, \quad (\xi \in \mathcal{H}\otimes\mathcal{T}),$$

which is equivalent to

$$(\sqrt{c}-1)^2\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})} \le (z+w)^*(z+w) \le (\sqrt{c}+1)^2\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})}.$$

Hence

$$-2\sqrt{c}\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})} \le (z+w)^*(z+w) - (c+1)\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})} \le 2\sqrt{c}\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})},$$

and therefore

$$\big\|(z+w)^*(z+w) - (c+1)\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})}\big\| \le 2\sqrt{c}. \tag{9.8}$$

Since $S_n^*S_n$ is a second order polynomial in $(X_1^{(n)},\dots,X_{2r}^{(n)})$ with coefficients in $\mathcal{A}$, the same holds for $S_n^*S_n - (c+1)\mathbf{1}_{M_n(\mathcal{A})}$. Hence, by Theorem 9.1 and (9.8),

$$\begin{aligned}\lim_{n\to\infty}\big\|S_n(\omega)^*S_n(\omega) - (c+1)\mathbf{1}_{M_n(\mathcal{A})}\big\| &= \Big\|\Big(\sum_{i=1}^r a_i\otimes y_i\Big)^*\Big(\sum_{i=1}^r a_i\otimes y_i\Big) - (c+1)\mathbf{1}_{\mathcal{B}(\mathcal{H}\otimes\mathcal{T})}\Big\| \\ &\le 2\sqrt{c}.\end{aligned}$$

Therefore, $\liminf_{n\to\infty}\min\{\mathrm{sp}(S_n(\omega)^*S_n(\omega))\} \ge (c+1) - 2\sqrt{c}$, which proves (9.7). ∎

9.4 Remark. The condition that $\{a_i^*a_j \mid i,j = 1,\dots,r\} \cup \{\mathbf{1}_{\mathcal{B}(\mathcal{H})}\}$ generates an exact C^*-algebra is essential for Corollary 9.3 and hence also for Theorem 9.1. Both (9.6) and (9.7) are false in the general non-exact case (cf. [HT2, Prop. 4.9] and [HT3]).

We turn, next, to a result about the constants $C(r)$, $r \in \mathbb{N}$, introduced by Junge and Pisier in connection with their proof of

$$\mathcal{B}(\mathcal{H}) \underset{\max}{\otimes} \mathcal{B}(\mathcal{H}) \ne \mathcal{B}(\mathcal{H}) \underset{\min}{\otimes} \mathcal{B}(\mathcal{H}). \tag{9.9}$$

9.5 Definition. [JP] For $r \in \mathbb{N}$, let $C(r)$ denote the infimum of all $C \in \mathbb{R}_+$ for which there exists a sequence of natural numbers $(n(m))_{m=1}^{\infty}$ and a sequence of r-tuples of $n(m) \times n(m)$ unitary matrices

$$(u_1^{(m)}, \dots, u_r^{(m)})_{m=1}^{\infty}$$

such that for all $m, m' \in \mathbb{N}$, $m \neq m'$

$$\Big\| \sum_{i=1}^{r} u_i^{(m)} \otimes \bar{u}_i^{(m')} \Big\| \leq C, \tag{9.10}$$

where $\bar{u}_i^{(m')}$ is the unitary matrix obtained by complex conjugation of the entries of $u_i^{(m')}$.

To obtain (9.9), Junge and Pisier proved that $\lim_{r\to\infty} \frac{C(r)}{r} = 0$. Subsequently, Pisier [P3] proved that $C(r) \geq 2\sqrt{r-1}$ for all $r \geq 2$. Moreover, using Ramanujan graphs, Valette [V] proved that $C(r) = 2\sqrt{r-1}$, when $r = p+1$ for an odd prime number p. From Theorem 9.2 we obtain

9.6 Corollary. *$C(r) = 2\sqrt{r-1}$ for all $r \in \mathbb{N}$, $r \geq 2$.*

Proof. Let $r \geq 2$, and let $g_1, \dots, g_r$ be the free generators of F_r and let λ denote the left regular representation of F_r on $\ell^2(F_r)$. Recall from [P3, Formulas (4) and (7)] that

$$\Big\| \sum_{i=1}^{r} \lambda(g_i) \otimes v_i \Big\| = 2\sqrt{r-1} \tag{9.11}$$

for all unitaries $v_1, \dots, v_r$ on a Hilbert space $\mathcal{H}$. Let $C > 2\sqrt{r-1}$. We will construct natural numbers $(n(m))_{m=1}^{\infty}$ and r-tuples of $n(m) \times n(m)$ unitary matrices

$$(u_1^{(m)}, \dots, u_r^{(m)})_{m=1}^{\infty}$$

such that (9.10) holds for $m, m' \in \mathbb{N}$, $m \neq m'$. Note that by symmetry it is sufficient to check (9.10) for $m' < m$. Put first

$$n(1) = 1 \quad \text{and} \quad u_1^{(1)} = \cdots = u_r^{(1)} = 1.$$

Proceeding by induction, let $M \in \mathbb{N}$ and assume that we have found $n(m) \in \mathbb{N}$ and r-tuples of $n(m) \times m(n)$ unitaries $(u_1^{(m)}, \dots, u_r^{(m)})$ for $2 \leq m \leq M$, such that (9.10) holds for $1 \leq m' < m \leq M$. By (9.11),

$$\Big\| \sum_{i=1}^{r} \lambda(g_i) \otimes \bar{u}_i^{(m)} \Big\| = 2\sqrt{r-1},$$

for $m = 1, 2, \dots, M$. Applying Theorem 9.2 to the exact C^*-algebras $\mathcal{A}_{m'} = M_{n(m')}(\mathbb{C})$, $m' = 1, \dots, M$, we have

$$\lim_{n\to\infty} \Big\| \sum_{i=1}^{r} \pi_{n,\omega}(g_i) \otimes \bar{u}_i^{(m')} \Big\| = 2\sqrt{r-1} < C, \quad (m' = 1, 2, \dots, M),$$

where $\pi_{n,\omega}\colon F_r \to \mathcal{U}(n)$ are the group homomorphisms given by (8.4). Hence, we can choose $n \in \mathbb{N}$ such that

$$\Big\| \sum_{i=1}^{r} \pi_{n,\omega}(g_i) \otimes \bar{u}_i^{(m')} \Big\| < C, \quad m' = 1, \dots, M.$$

Put $n(M+1) = n$ and $u_i^{(M+1)} = \pi_{n,\omega}(g_i)$, $i = 1, \dots, r$. Then (9.10) is satisfied for all m, m' for which $1 \le m' < m \le M+1$. Hence, by induction, we get the desired sequence of numbers $n(m)$ and r-tuples of $n(m) \times n(m)$ unitary matrices. ∎

We close this section with an application of Theorem 7.1 to powers of random matrices:

9.7 Corollary. *Let for each $n \in \mathbb{N}$ Y_n be a random matrix in the class* $\mathrm{GRM}(n, \frac{1}{n})$*, i.e., the entries of Y_n are n^2 i.i.d. complex Gaussian variables with density $\frac{n}{\pi}\mathrm{e}^{-n|z|^2}$, $z \in \mathbb{C}$. Then for all $p \in \mathbb{N}$*

$$\lim_{n\to\infty} \|Y_n(\omega)^p\| = \left(\frac{(p+1)^{p+1}}{p^p}\right)^{\frac{1}{2}},$$

for almost all ω in the underlying probability space Ω.

Proof. By the remarks preceding Corollary 9.3, we have

$$(Y_n)^p = \left(\frac{1}{\sqrt{2}}\big(X_1^{(n)} + \mathrm{i}X_2^{(n)}\big)\right)^p,$$

where, for each $n \in \mathbb{N}$, $X_1^{(n)}, X_2^{(n)}$ are two independent random matrices from $\mathrm{SGRM}(n, \frac{1}{n})$. Hence, by Theorem 7.1, we have for almost all $\omega \in \Omega$:

$$\lim_{n\to\infty} \|Y_n(\omega)^p\| = \|y^p\|,$$

where $y = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)$, and $\{x_1, x_2\}$ is a semicircular system in a C^*-probability space $(\mathcal{B}, \tau)$ with τ faithful. Hence, y is a circular element in $\mathcal{B}$ with the standard normalization $\tau(y^*y) = 1$. By [La, Proposition 4.1], we therefore have $\|y^p\| = ((p+1)^{p+1}/p^p)^{\frac{1}{2}}$. ∎

9.8 Remark. For $p = 1$, Corollary 9.7 is just the complex version of Geman's result [Ge] for square matrices (see [Ba, Thm. 2.16] or [HT1, Thm. 7.1]), but for $p \ge 2$ the result is new. In [We, Example 1, p.125], Wegmann proved that the empirical eigenvalue distribution of $(Y_n^p)^*Y_n^p$ converges almost surely to a probability measure μ_p on $\mathbb{R}$ with

$$\max(\mathrm{supp}(\mu_p)) = \frac{(p+1)^{p+1}}{p^p}.$$

This implies that for all $\varepsilon > 0$, the number of eigenvalues of $(Y_n^p)^*Y_n^p$, which are larger than $(p+1)^{p+1}/p^p + \varepsilon$, grows slower than n, as $n \to \infty$ (almost surely). Corollary 9.7 shows that this number is, in fact, eventually 0 as $n \to \infty$ (almost surely).

References

[An] J. Anderson, A C^*-algebra A for which $\mathrm{Ext}(A)$ is not a group, *Annals Math.* 107 (1978), 455–458.

[Ar] L. Arnold, On the asymptotic distribution of the eigenvalues of random matrices, *Journ. Math. Anal. Appl.* 20 (1967), 262–268.

[Arv] W. Arveson, Notes on extensions of C^*-algebras, *Duke Math. Journ.* 44 (1977), 329–355.

[Ba] Z.D. Bai, Methodologies in spectral analysis of large dimensional random matrices, A review, *Statistica Sinica* 9 (1999), 611–677.

[BDF1] L.G. Brown, R.G. Douglas and P.A. Fillmore, Unitary equivalence modulo the compact operators and extensions of C^*-algebras, *Proc. Conf. on Operator Theory, Lecture Notes in Mathematics* 345, Springer Verlag (1973), 58–128.

[BDF2] L.G. Brown, R.G. Douglas and P.A. Fillmore, Extensions of C^*-algebras and K-homology, *Ann. of Math.* 105 (1977), 265–324.

[Be] W. Beckner, A generalized Poincaré inequality for Gaussian measures.

[BK] B. Blackadar, E. Kirchberg, Generalized inductive limits of finite dimensional C^*-algebras, *Math. Ann.* 307 (1997), 343–380.

[Bre] L. Breiman, Probability, *SIAM Classics in Applied Mathematics* (1992).

[BY] Z.D. Bai and Y.Q. Yin, Neccesary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix, *Ann. of Probab.* **16** (1988), 1729–1741.

[CE] M.D. Choi and E. Effros, The completely positive lifting problem for C^*-algebras, *Ann. of Math.* **104** (1976), 585–609.

[Cf] H. Chernoff, A note on an inequality involving the normal distribution, *Ann. Probab.* 9 (1981), 533–535.

[Cn] L.H.Y. Chen, An inequality for the multivariate normal distribution, *Journ. Multivariate Anal.* 12 (1982), 306–315.

[Da] K.H. Davidson, The distance between unitary orbits of normal operators, *Acta Sci. Math.* **50** (1986), 213–223. 585–609.

[DH] J. De Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, *Amer. J. Math.* 107 (1984), 455–500.

[EH] E. Effros and U. Haagerup, Lifting problems and local reflexivity for C^*-algebras, Duke Math. Journ. 52 (1985), 103–128.

[F] G.B. Folland, Real analysis, modern techniques and their applications, John Wiley and Sons (1984).

[Ge] S. Geman, A limit theorem for the norm of random matrices, *Annals Prob.* 8 (1980) 252–261.

[HP] F. Hiai and D. Petz, Asymptotic freeness almost everywhere for random matrices, *Acta Sci. Math.* (Szeged) 66 (2000), 809–834.

[HT1] U. Haagerup and S. Thorbjørnsen, Random matrices with complex Gaussian entries. Preprint 1998, `http://www.imada.sdu.dk/Research/Preprints/1998.html`

[HT2] U. Haagerup and S. Thorbjørnsen, Random matrices and K-theory for exact C^*-algebras. *Documenta Math.* 4 (1999), 330–441.

[HT3] U. Haagerup and S. Thorbjørnsen, Random Matrices and non-exact C^*-algebras, "C^*-algebras" (J. Cuntz, S. Echterhoff ed.), Springer Verlag (2000), 71–91.

[JP] M. Junge and G. Pisier, Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$, *Geom. Funct. Analysis* 5 (1995), 329–363.

[Ki1] E. Kirchberg, The Fubini Theorem for Exact C^*-algebras, *Journ. Operator Theory* 10 (1983), 3–8.

[Ki2] E. Kirchberg, On non-semisplit extensions, tensor products and exactness of group C^*-algebras, *Invent. Math.* 112 (1993), 449–489.

[KR] R.V. Kadison and J. Ringrose, Fundamentals of the theory of operator algebras, Vol. II, *Academic Press* 1986.

[La] F. Larsen, Powers of R-diagonal elements, preprint, Odense 1999. To appear in J. Operator Theory.

[Le] F. Lehner, Computing norms of free operators with matrix coefficients. *Amer. J. Math.* 121 (1999), 453–486.

[Me] M.L. Mehta, Random matrices, second edition, *Academic Press* (1991).

[Pas] L. Pastur, A simple Approach to global regime of random matrices, "Mathematical results in Statistical Mechanics" (S. Miracle-Sole, J. Ruiz and V. Zagrebnov, Eds.), World Science, Singapore (1999), 429-455.

[Pau] V. Paulsen, Completely positive maps and Dilations, *Pitman Research Notes in Mathematics* 146, Longman Scientific & Technical (1986).

[Pe] G.K. Pedersen, Analysis Now, *Springer Verlag* (1989).

[P1] G. Pisier, The volume of convex bodies and Banach space geometry, *Cambridge University Press* (1989).

[P2] G. Pisier, A simple proof of a Theorem of Kirchberg and related results on C^*-norms. *Journ. Operator Theory* 35 (1996), 317–335.

[P3] G. Pisier, Quadratic forms in unitary operators, *Linear algebra and its Appl.* 267 (1997),125–137.

[RLL] M. Rørdam, F. Larsen and N.J. Laustsen, An introduction to K-theory for C^*-algebras, *Cambridge Univ. Press* 2000.

[Ro] J. Rosenberg, Quasidiagonality and Nuclearity (appendix to strongly quasidiagonal operators by D. Hadwin), *Journ. Operator Theory* 18 (1987), 15–18.

[T] S. Thorbjørnsen, Mixed moments of Voiculescu's Gaussian random matrices, *Journ. Funct. Anal.* 176 (2000), 213–246.

[V] A. Valette, An application of Ramanujan graphs to C^*-algebra tensor products, *Discrete Math.* 167 (1997), 597–603.

[V1] D. Voiculescu, A non commutative Weyl-von Neumann Theorem, *Rev. Roum. Pures et Appl.* 21 (1976), 97–113.

[V2] D. Voiculescu, Symmetries of some reduced free group C^*-algebras, "Operator Algebras and Their Connections with Topology and Ergodic Theory", *Lecture Notes in Math.* Vol. 1132, Springer-Verlag 1985, 556–588.

[V3] D. Voiculescu, Circular and semicircular systems and free product factors, "Operator Algebras, Unitary representations, Algebras, and Invariant Theory", *Progress in Math.* Vol. 92, Birkhäuser, 1990, 45–60.

[V4] D. Voiculescu, Limit laws for random matrices and free products, *Inventiones Math.* 104 (1991), 202–220.

[V5] D. Voiculescu, A note on quasi-diagonal C^*-algebras and homotopy. *Duke Math. Journ.* 62 (1991), 267–271.

[V6] D. Voiculescu, Around quasidiagonal operators. *Integral Equations and Operator Theory* 17 (1993), 137–149.

[V7] D. Voiculescu, Operations on certain non-commutative random variables, "Recent advances in operator algebras, Orleans 1992" *Asterisque* 232 (1995), 243–275.

[V8] D. Voiculescu, Free Probability Theory: Random Matrices and von Neumann algebras, *Proceedings of the International Congress of Mathematicians*, Zürich 1994, Vol. 1. Birkhäuser Verlag (1995), 227–241.

[V9] D. Voiculescu, Free Entropy. Bull. London Math. Soc. 34 (2002), 257–278.

[VDN] D. Voiculescu, K. Dykema and A. Nica, Free Random Variables, CMR Monograph Series 1, *American Mathematical Society*, 1992.

[We] R. Wegmann, The Asymptotic Eigenvalue-Distribution for a Certain Class of Random Matrices, *J. Math. Anal. and Appl.* 56, 113–132 (1976)

[Wi] E. Wigner, Characterictic vectors of boardered matrices with infinite dimensions, *Ann. Math.* 62 (1955), 548–564.