A discrete time model of investment under non-Gaussian shocks

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Abstract. A discrete time model of irreversible investment is explored. The investor is a risk-neutral, value maximizing competitive firm. The unit price of the firm's output follows a non-Gaussian stochastic process. Under weak restrictions on the process and the firm's production function, explicit formulas for the optimal trigger price of investment and the value of the firm are derived. Advantages of discrete time setting are exploit to reduce the computation to a relatively simple procedure. A new method of solving optimization problems under uncertainty based on the Wiener-Hopf factorization in the form natural for Economics is suggested.

Key words: Investment threshold, non-Gaussian processes, Wiener-Hopf factorization.

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1 Introduction

The real options approach to investment under uncertainty has provided many insights into capital budgeting decision-making. However, the most popular models in the theory of real options are either borrowed from continuous time Finance or set in discrete time and discrete state space. Continuous time models require too stringent and unrealistic assumptions concerning the underlying stochastic processes. We believe that discrete time models are more relevant for

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real options theory, because investment decisions are not made every instant, and the firm's manager contemplating new investment has in mind commodity prices aggregated over certain time periods rather than continuously changing prices, which are relevant for financial markets. In addition, in discrete time models, it is easier to fit a process from a chosen family to empirical data. On the other hand, discrete state space models are less tractable analytically, hence it is better to use continuous state space.

In this paper, we present a discrete time model of irreversible investment under uncertainty, with a continuous state space (the method of the present article admits a modification for the discrete-space models; the corresponding result will be published elsewhere). We consider a risk-neutral, competitive firm, which chooses the investment strategy in order to maximize its present value net of installation cost of capital. We assume that all uncertainty is on the demand side, i.e., the price of a unit of the firm's output, P, is stochastic; the method can be applied when both the price and cost depend on the same factor, similarly to the results in Boyarchenko and Levendorskiĭ (2002a,b,c) for general perpetual American options in discrete and continuous time. A standard assumption on $P = \{P_t\}$, a process for the price, is that it is log-normal: $P_t = \exp X_t$, where $X = \{X_t\}$ is a Gaussian process; more general Itô processes are also applied. See Øksendal (2000) for a continuous time model under Markov processes with continuous trajectories. It is well documented, though (see, for example Yang and Brorsen (1992) or Deaton and Laroque (1992)), that Gaussian models do not give very good fit to empirical data since the latter exhibit significant skewness and kurtosis, nothing to say about apparent fat tails of probability distribution functions.

Here we make fairly weak assumptions on the process P. We assume that the log-price follows the random walk but we do not suppose that the transition density is infinitely divisible so that it is possible to pass to the continuous time limit.

We find the optimal investment threshold and value of the firm. The major result of the existing models of irreversible investment under uncertainty is that irreversibility increases the hurdle that projects must clear in order to be profitably undertaken. The formula obtained in the paper shows that the investment threshold exceeds the one computed under the naive net present value (NPV) rule by a factor which is the ratio of two expected present values of the average revenue: the one in the numerator is calculated for the original price process, and the value in the denominator is computed for the infimum price process $\underline{P}_t = \min_{0\le s\le t} P_s$. The aforementioned factor incorporates the effect of cumulative losses, which are caused by the downward movements of the price and irreversibility of investment. Upward movements are not so important because the capital can be increased when necessary. In the result, the higher level of price is needed to trigger new investment in order to compensate for possible negative movements. A similar result is well-known in the continuous-time Gaussian model of investment (see Dixit and Pindyck (1996)), though the meaning of the correction factor is not clear.

The value of the firm is the sum of the value calculated under the naive NPV rule and the option value of investment opportunities. Similar decomposition was made in Abel et al. (1996) for a two-period model of partially reversible investment. In the paper, we factor out the contributions of the infimum process $\underline{P}_t$ and supremum process $\bar{P}_t = \max_{0\le s\le t} P_s$ to the marginal option value of capital. The marginal option value increases in downward uncertainty and decreases in upward uncertainty. The overall effect of uncertainty is ambiguous.

The method of this paper is straightforward and can be summarized as follows. As it is standard in the literature on investment under uncertainty and endogenous default (see Dixit and Pindyck (1996) and Hilberink and Rogers (2002), respectively), we assume that the optimal strategy is of the form: invest if the price crosses certain threshold. We fix a prospective candidate for the investment threshold and notice that the Bellman equation in the model is the Wiener-Hopf equation. The latter can be solved by the Wiener-Hopf factorization method; and the central point of the Wiener-Hopf method in the form suggested in the paper is that virtually each step of the solution of the problem and the final answer can be interpreted as the calculation of the expected present value of a certain stream of payoffs; expectations are taken under assumption that the price follows either the supremum process or infumum process (depending on the step of the proof). We believe that this technique is much simpler and more natural for Economics than the Itô calculus which is routinely used in continuous time models of real options.

The explicit formula for the solution having been obtained, we notice that the value function must satisfy a certain property, which leads to an equation for the threshold. This equation has a unique solution. We check that the solution satisfies the sufficient optimality condition. Finally, an explicit analytic formula for the investment threshold is obtained. Under additional assumptions on the firm's production function, we also derive an analytical formula for the option value of investment opportunity and value of the firm.

By using essentially the same Wiener-Hopf factorization technique but in a more technically involved form, we calculated the optimal exercise prices of the perpetual Bermudan options (Boyarchenko and Levendorskiĭ (2002c)), and of the perpetual American options (Boyarchenko and Levendorskiĭ (2000, 2002a)), as well as the investment threshold in a continuous time model of investment (Boyarchenko (2001)) for wide classes of Lévy processes (see also the monograph Boyarchenko and Levendorskiĭ (2002b)). The Wiener-Hopf method can be used in several analytical and stochastic forms. Unlike in the aforementioned papers, where mainly the analytical version was used, here we use a hybrid of the analytical and stochastic version, in a simple form natural for Economics. Purely analytical tools are used in the very end, when it becomes necessary to calculate the answer explicitly. Notice that Hilberink and Rogers (2002) used an essentially different version of the Wiener-Hopf method to solve a model of endogenous default.

The rest of the paper is organized as follows. In Section 2, we specify the model, and present the main results. In Section 3, the Wiener-Hopf method is described and applied to find the solution to the Wiener-Hopf equation. The solution is used to obtain the investment threshold and calculate the marginal option value of investment opportunity. In Section 4, the value of the firm is computed under additional technical assumptions on the firm's production function. In Section 5, we consider a simple example, when the probability density is modelled as an exponential polynomial and the production function is a Cobb-Douglas one. In this case, all the calculations can be made explicitly. Section 6 concludes, and in the appendix, technical results are presented.

2 Model specification and main results

2.1 Process specification

We consider only processes such that the price can move both up and down with positive probability. If the price process is non-decreasing almost surely, then irreversibility does not matter, and the firm's manager may follow the net present value rule also known as the Marshallian law. This rule prescribes to invest as long the NPV is non-negative (for a discussion of irreversibility and uncertainty issues, see Dixit and Pindyck (1996)). If the price process is non-increasing almost surely, then the firm, which is already on the market, will never increase the capital stock further since the current market conditions can only deteriorate.

Let $t_k,\ k = 0, 1, \dots,$ be the dates when investment can be made. We assume these dates to be equally spaced: $\Delta \equiv t_{j+1} - t_j$ is independent of j. Parameter Δ is normalized to 1. Let $q \in (0, 1)$ be a (fixed) discount factor. Assume that $\ln(P_1/P_0), \ln(P_2/P_1), \dots, \ln(P_{t+1}/P_t), \dots$ are independently and identically distributed random variables on the probability space Ω. We impose the following restriction on the price process:

$$\rho \equiv qE[P_1/P_0] < 1. \tag{2.1}$$

The intuition behind the last condition is as follows. Let K be the current capital stock of the firm. For K constant, the expected revenues grow each period by the factor $E[P_1/P_0]$, and they are discounted back by the factor q. Hence, at the initial price level P_0 and the capital stock K, the expected discounted revenue is given by

$$P_0 G(K) \sum_{t=0}^{\infty} \rho^t = \frac{P_0 G(K)}{1 - \rho}, \tag{2.2}$$

where $G(K)$ is the production function of the firm. For the series on the LHS of (2.2) to converge, it is necessary and sufficient that (2.1) holds. It follows automatically from (2.1) that

$$E[P_1/P_0] < \infty. \tag{2.3}$$

2.2 Firm's problem

We assume that $G(K)$ is differentiable, concave, and satisfies the Inada conditions. At each time period t, the firm receives $P_t G(K_t)$ from the sales of its product and suffers the installation cost $C \cdot (K_{t+1} - K_t)$, should it decide to increase the capital stock. The firm's objective is to chose an optimal investment strategy $\mathcal{K} = \{K_{t+1}(K_t, P_t)\}_{t\geq 1}, K_0 = K, P_0 = P$, which maximizes the NPV of the firm:

$$V(K,P) = \sup_{\mathcal{K}} E\left[\sum_{t\geq 0} q^t (P_t G(K_t) - C(K_{t+1} - K_t))|P_0 = P\right]. \tag{2.4}$$

Here we treat the current price P and capital stock K as state variables, and $\mathcal{K}$ as a sequence of control variables. Due to irreversibility of investment, $K_{t+1} \geq K_t, \ \forall t$.

In order that firm's value (2.4) be bounded, we impose a resource constraint: there exists $\bar{K} < \infty$, such that $K_t \leq \bar{K}, \ \forall t$. The resource constraint, condition (2.1), and properties of the production function ensure that the value function (2.4) is well defined.

Following the tradition in the literature, we are going to view the space of state variables as a (disjoint) union of two regions: inaction and action ones. For all pairs (K, P) belonging to the inaction region, it is optimal to keep the capital stock unchanged. In the action region, investment becomes optimal. We may assume that the inaction region is closed. Denote by Γ the boundary of the inaction region. Then the investment strategy defined by the choice of the inaction region can be viewed as follows:

(i) do not invest as long as $P_t \leq H(K)$, where $H(K)$ is defined by $(K, H(K)) \in \Gamma$;

(ii) invest when $P_t > H(K)$, and increase the capital stock up to the level $\Phi(P_t)$ defined by $(\Phi(P_t), P_t) \in \Gamma$.

Denote the set of H satisfying (i) and (ii) by $\mathcal{H}$. Every $H \in \mathcal{H}$ uniquely defines the boundary of the inaction region, call it Γ, by $(\Phi(P), P) \in \Gamma \ \Leftrightarrow \ \exists\, K : P \in [H(K-0), H(K)]$; and Γ together with the rules (i) and (ii) uniquely define the strategy.

Let $V(K, P; H)$ be the NPV of the firm when $H \in \mathcal{H}$ is chosen to define the boundary Γ of the inaction region, that is

$$V(K,P;H) = E\left[\sum_{t\geq 0} q^t (P_t G(K_t) - C \cdot (K_{t+1} - K_t)) \mid P_0 = P\right].$$

Then we can restate the original firm's problem in an equivalent way: find the optimal investment threshold $H^* \in \mathcal{H}$, which is characterized by

$$V(K,P;H^*) \geq V(K,P;H), \quad \forall\ P \text{ and } H \in \mathcal{H},$$

and $V(K, P; H^*) = V(K, P)$, where $V(K, P)$ is defined by (2.4).

2.3 Optimal investment rule and shadow value of capital

Here we present the main results of the paper in terms of the processes $\underline{P} = \{\underline{P}_t\}_{t\geq 0}$ and $\bar{P} = \{\bar{P}_t\}_{t\geq 0}$, where $\underline{P}_t = \min_{0\leq s\leq t} P_s$ and $\bar{P}_t = \max_{0\leq s\leq t} P_s$ We will call $\underline{P}$ and $\bar{P}$ the infimum and supremum supremum price processes respectively as their analogs in continuous time.

Theorem 2.1 *Let (2.1)-(2.3) hold. Then the investment threshold $H^*(K)$ is the solution to*

$$G'(K)E\left[\sum_{t=1}^{\infty} q^t \underline{P}_t \mid P_0 = H\right] = C. \tag{2.5}$$

The LHS in (2.5) is the expected present value of the marginal revenue product (MRP) of capital with the original price process P_t being replaced by the infimum price process $\underline{P}_t$ (recall that in discrete time models, a unit of capital installed today starts working only tomorrow). Equivalently, (2.5) can be written as

$$G'(K)H(K)E\left[\sum_{t=1}^{\infty} q^t \underline{P}_t \mid P_0 = 1\right] = C. \tag{2.6}$$

Equation (2.6) is by no means computationally effective, so we provide an analytical formula for the investment threshold in Section 3.

If there is no uncertainty and investment is reversible, then the investment threshold is given by

$$G'(K)H\sum_{t=1}^{\infty} q^t = C. \tag{2.7}$$

The strategy defined by (2.7) prescribes to invest when the discounted MRP equals the cost of a unit of capital. If one tries to introduce uncertainty in a naive way, then the natural generalization of (2.7) is

$$G'(K)E\left[\sum_{t=1}^{\infty} q^t P_t \mid P_0 = H\right] = C. \tag{2.8}$$

The counterparts of (2.7) and (2.8) in continuous time define investment thresholds which are known as the Jorgensonian and Marshallian thresholds respectively.

Let $H^*(K)$ be the trigger price of investment in our model, and $H_M(K)$ be the price which triggers new investment according to the Marshallian rule. To compare these two prices, introduce

$$\kappa = \frac{H^*(K)}{H_M(K)} = \frac{\sum_{t=1}^{\infty} q^t E[P_t \mid P_0 = 1]}{\sum_{t=1}^{\infty} q^t E[\underline{P}_t \mid P_0 = 1]}.$$

The Marshallian prescription (2.8) does not take into consideration the option-like nature of investment opportunities. The firm can increase capital stock later, therefore it has an option similar to the call option. The trigger price $H^*(K)$

computed for the case when the option to delay the investment, irreversibility, and uncertainty are taken into account properly exceeds the Marshallian investment threshold $H_M(K)$ by a factor κ. The factor κ is the ratio of two expected present values of the average revenue: the one in the numerator is calculated for the original price process, and the value in the denominator is computed for the infimum price process. Clearly, $P_t \geq \underline{P}_t$ for all realizations of the process, and if X is not non-decreasing, then $E[\underline{P}_t|P_0 = 1] < E[P_t|P_0 = 1]$, hence $\kappa > 1$. The factor κ incorporates the effect of cumulative losses, which are caused by the downward movements of the price and irreversibility of investment. Upward movements are not so important because the capital can be increased when necessary. In the result, the higher level of price is needed to trigger new investment in order to compensate for possible negative movements. A similar result is well-known in the continuous-time Gaussian model of investment (see Dixit and Pindyck (1996)). In that model, $\kappa = \beta/(\beta-1)$, where $\beta > 1$ is a positive root to the characteristic equation $k(k-1)\sigma^2/2 + \alpha k - r = 0$, and α, σ^2 are the drift and diffusion coefficient of the underlying Gaussian process for P. For the analog in the continuous time model under non-Gaussian shocks, see Boyarchenko (2001) and the monograph Boyarchenko and Levendorskiĭ (2002b).

Our next result is the formula for the marginal (or shadow) value of capital. Suppose that the investment threshold is chosen according to (2.1). Let T be an exponentially distributed random variable independent of $\{\ln(P_{t+1}/P_t)\}$, with the mean $q(1-q)^{-1}$. Then the shadow value of capital is given by

$$V_K(K+0, P) = \frac{PG'(K)}{1-\rho} + V_K^{\mathrm{opt}}(K+0, P), \tag{2.9}$$

where $V_K(K+0, P)$ is the right derivative of the value function w.r.t. its first argument (the existence of this derivative will be proved later on); and

$$\begin{aligned} V_K^{\mathrm{opt}}(K+0, P) \quad &= \quad -\rho G'(K) \sum_{t=1}^{\infty} q^t E[\underline{P}_t \mid P_0 = 1] \\ &\quad \times E\left[(\bar{P}_T - H^*(K))_+ \mid P_0 = P\right], \end{aligned} \tag{2.10}$$

where $(\bar{P}_T - H^*(K))_+ \equiv \max\{(\bar{P}_T - H^*(K)), 0\}$. Notice that the shadow value of capital is proportional to Tobin's marginal $q \equiv V_K(K+0, P)/C$. Equation (2.9) separates the marginal value of capital into two components. The first one is the expected present value of the marginal returns to capital given the capital stock remains constant at the level K in the future. The second component is the marginal option value of the future investment opportunities. This value is negative because investing extinguishes the option. Similar result was obtained by Abel et al. (1996) for a two-period model of partially reversible investment.

Formula (2.10) factors out contributions of the infimum and supremum price processes to the marginal option value of capital. The marginal option value (in absolute terms) is a product of three factors. The first factor, $\rho G'(K)$, is the expected present value of the marginal returns to the current capital stock, K, in a period from now, deflated by the current price. The second factor in (2.10) is the expected present value of the average revenue calculated under the assumption that prices follow the infimum process, and the initial price is normalized to one. This factor decreases if the probability of downward jumps in prices increases. The third factor is the price of the European call option on $\bar{P}_T$ with the exercise price $H^*(K)$ and random date of expiry T. The price of this option increases as the probability of upward jumps in prices increases. Hence the marginal option value of capital increases in downward uncertainty and decreases in upward uncertainty. The overall effect of uncertainty is ambiguous.

3 Proof of the main results

If one works with (geometric) Gaussian processes, the argument can be made equally easily by using differential equations on the half-line (the state space for the price processes) or on the line (the state space for the log-price process). In discrete time (and in the continuous time, if the process is non-Gaussian), the choice of the half-line unnecessarily complicates the technique. So, instead of characterizing the state by the pair (K, P) as in the previous sections, we use (K, x) ($x = \ln P \in \mathbf{R}$) as a generic state variable, and $h(K) = \ln H(K)$ as the investment threshold. The boundary which separates the action and inaction regions in the (K, x)-space is denoted by γ. Let $W(K, x; h) = V(K, P; H)$. We are looking for an h^* satisfying

$$W(K, x; h^*) \geq W(K, x; h), \quad \forall\ x \text{ and } h. \tag{3.1}$$

To find h^*, we fix h, a prospective candidate for the investment threshold, and derive the formula for $W(K, x; h)$ in several steps. First, we write the Bellman equation

$$W(K, x; h) = \max_{K' \geq K} [G(K)e^x - C(K' - K) + qE[W(K', X_1; h) \mid X_0 = x], \tag{3.2}$$

next, we reduce the Bellman equation to the Wiener-Hopf equation, and solve the latter. Then, by using a solution to the Wiener-Hopf equation for different h, and a natural hypothesis about the optimal threshold, we conjecture the formula for h^*. Finally, we verify (3.1), and obtain the firm's value $W(K, x; h^*) = W(K, x)$. Now we are going to describe each of the steps in details.

3.1 Reduction to the Wiener-Hopf equation

It follows from (i) and (ii), that h is non-decreasing and continuous a. e. with each point of discontinuity of the first kind, and $h(K + 0) = h(K)$, $\forall K$. Hence, $h'(K)$ exists for almost all K. Fix h, and establish some properties of the value function $W(K, x; h)$.

Lemma 3.1 *Let $h'(K)$ exist. Then $W_K(K + 0, x; h(K))$, the right derivative of W w.r.t. the first argument, exists for any x.*

Proof Consider (K, x) and (K^1, x) in the action region. From the Bellman equation,

$$W(K^1, x; h) - W(K, x; h) = (G(K') - G(K))e^x + C(K^1 - K),$$

therefore the derivative $W_K(K, x; h)$ exists, and

$$W_K(K, x; h) = G'(K)e^x + C, \; x > h(K). \tag{3.3}$$

Notice that (3.3) is just the envelope condition. The case $x \le h(K)$ will be considered in the appendix. Below we consider only K such that $h'(K)$ exists, and derive an analytic expression for $h(K)$. We will see that this analytic expression defines a differentiable function, hence it defines $h(K)$ for all K. We conclude that h is differentiable everywhere, and the result will be valid for all K.

Suppose that a pair (K, x) belongs to the inaction region. Then

$$W(K, x; h) = G(K)e^x + qE[W(K, X_1; h) \mid X_0 = x], \; x \le h(K). \tag{3.4}$$

We differentiate (3.4) w.r.t. K, in the region $x \le h(K)$:

$$W_K(K + 0, x; h) = e^x G'(K) + qE[W_K(K + 0, X_1; h) \mid X_0 = x]. \tag{3.5}$$

Set

$$u(x) \equiv W_K(K + 0, x + h(K); h) - G'(K)e^{x+h(K)} - C, \tag{3.6}$$
$$g(x) = \rho G'(K)e^{x+h(K)} - (1 - q)C. \tag{3.7}$$

Thus, u and g depends not only on x but on K and $h(K)$ as well; the latter couple is fixed in this section. We see that (3.3) and (3.5) become

$$u(x) = 0, \quad x > 0, \tag{3.8}$$
$$u(x) - qE[u(X_1) \mid X_0 = x] = g(x), \quad x \le 0. \tag{3.9}$$

(We have used the equality $E[1] = 1$). Similarly to the proof of Lemma 3.1 (see the appendix), it is possible to show that $W_K(K, x)$ is bounded on $(-\infty, h(K))$, hence it suffices to look for a solution of the problem (3.8)-(3.9) in $L^\infty(\mathbf{R})$. Since only the values of g on $\mathbf{R}_-$ matter, we may define g on $\mathbf{R}_+$ by arbitrary expression, say, $g(x) = 0$ for all $x > 0$. Then $g \in L^\infty(\mathbf{R})$, too.

For $t = 0, 1, \ldots$, define an operator P_t in $L^\infty(\mathbf{R})$ by $P_t u(x) = E[u(X_t) \mid X_0 = x]$, and set $P = P_1$. Clearly, all these operators have the norm 1: $||P_t|| = 1$, and since Y_j are i.i.d., the law of iterated expectations gives $P_t = P^t$. Rewrite (3.9) as

$$u(x) - q(Pu)(x) = g(x), \quad x \le 0. \tag{3.10}$$

Equation (3.10) subject to (3.8) is called the Wiener-Hopf equation.

3.2 Expected present value and resolvent

If (3.10) had been an equation on the whole axis, it could have been solved easily, by using the inverse to $I - qP$:

$$(I - qP)^{-1} = I + qP + (qP)^2 + \cdots . \tag{3.11}$$

The series converges since $||qP|| = q||P|| = q \in (0, 1)$, and applying (3.11) to equation (3.10) on the whole axis, we would have obtained

$$u = (I - qP)^{-1} f = \sum_{t=0}^{\infty} (qP)^t f.$$

Since $P^t f(x) = P_t f(x) = E[f(X_t) \mid X_0 = x]$, we can rewrite the last equation as

$$u = U_X^q g, \tag{3.12}$$

where U_X^q is the resolvent operator (another name: potential operator) of the process X, defined by

$$(U_X^q g)(x) \equiv E\left[\sum_{t=0}^{\infty} q^t g(X_t) \mid X_0 = x\right] = \sum_{t=0}^{\infty} q^t E[g(X_t) \mid X_0 = x].$$

Thus, the reader may regard the resolvent operator applied to g as the operation of calculation of the expected present value of the stochastic stream $g(X_t)$, and the name *expected present value (EPV)* operator seems to be natural in applications to Economics. The argument above shows that for the random walk X,

$$U_X^q (I - qP) = (I - qP) U_X^q = I \tag{3.13}$$

or

$$(I - qP)^{-1} = U_X^q, \quad (U_X^q)^{-1} = I - qP. \tag{3.14}$$

Therefore, to calculate the EPV $u(x) = (U_X^q g)(x)$ of the stream $g(X_t)$ it suffices to solve equation (3.10) on the whole axis, and vice versa.

3.3 Infimum and supremum processes and the Wiener-Hopf factorization

Unfortunately, in our case, equation (3.10) holds on the half-axis only, and so (3.12) does not help. Nevertheless, the solution can be written in terms of resolvents: not of the random walk X but of the processes $N_t = \min_{0 \le s \le t} X_s$ and $M_t = \max_{0 \le s \le t} X_s$. The Wiener-Hopf factorization theorem (see the appendix) allows one to factorize U_X^q into the product of the resolvents of M and N:

$$(1 - q)^{-1} U_X^q = U_M^q U_N^q = U_N^q U_M^q. \tag{3.15}$$

By using the symbol 0 to denote the trivial process, we can write (3.15) in the form

$$U_0^q U_X^q = U_M^q U_N^q = U_N^q U_M^q.$$

Factorization formula (3.15) allows one to solve the Wiener-Hopf equation. If $u \in L^\infty(\mathbf{R})$ vanishes on $\mathbf{R}_\pm$, we write $u \in L^\infty(\mathbf{R}_\mp)$. Clearly, $L^\infty(\mathbf{R}_\mp) \subset L(\mathbf{R})$ is a subspace. Let $\mathbf{1}_{(-\infty,0]}$ be the indicator function of the interval $(-\infty, 0]$.

Theorem 3.2 *For any $g \in L^\infty(\mathbf{R}_-)$, the Wiener-Hopf equation (3.10) has a unique solution $u \in L^\infty(\mathbf{R}_-)$. The solution is given by*

$$u = (1-q)U^q_M \mathbf{1}_{(-\infty,0]} U^q_N g. \tag{3.16}$$

Let M' denote the process M killed on first leaving the interval $(-\infty, 0]$. Then (3.16) can be reexpressed in the form

$$u = (1-q)U^q_{M'} U^q_N g. \tag{3.17}$$

Equation (3.17) says that u can be calculated in three steps: first, starting with the stochastic stream $g(X_t)$, we replace X_t with the infimum process, and calculate the EPV of $g(X_t)$. Next, we regard the result of the first step, $U^q_N g$, as a new stochastic stream, replace the process with M', and compute the EPV of the stream $U^q_N g(M'_t)$. Finally, we multiply the last value by $1-q$.

3.4 Solution of the Wiener-Hopf equation

Problem (3.8), (3.10) is equivalent to the following problem: find $u \in L^\infty(R_-)$ and $g_1 \in L^\infty(\mathbf{R}_+)$ which satisfy the equation

$$(I - qP)u = g + g_1. \tag{3.18}$$

By using (3.14) and (3.15), we can rewrite (3.18) as

$$(U^q_N)^{-1}(U^q_M)^{-1}u = (1-q)(g + g_1). \tag{3.19}$$

Notice that to justify (3.19), we need to know that U^q_M and U^q_N are invertible. We prove the boundedness of the inverses to U^q_M and U^q_N as follows: first, $I - qP$ is bounded. Second, U^q_M and U^q_N are bounded (it suffices to notice that $|E[f(x + M_t)]| \le ||f||$, hence the norm of U^q_M is bounded by $1 + q + q^2 + \cdots = (1-q)^{-1}$, and the same holds with N instead of M), and finally, on the strength of (3.15), the inverses

$$(U^q_M)^{-1} = (1-q)(I - qP)U^q_N$$

and

$$(U^q_N)^{-1} = (1-q)(I - qP)U^q_M$$

are bounded as well. For the next step, we need the following lemma.

Lemma 3.3 *Let $q \in (0, 1)$. Then*

a) For any $f \in L^\infty(\mathbf{R}_-)$, we have $U^q_M f \in L^\infty(\mathbf{R}_-)$, and moreover, $U^q_M : L^\infty(\mathbf{R}_-) \to L^\infty(\mathbf{R}_-)$ is invertible;

b) For any $f \in L^\infty(\mathbf{R}_+)$, we have $U^q_N f \in L^\infty(\mathbf{R}_+)$, and moreover, $U^q_N : L^\infty(\mathbf{R}_+) \to L^\infty(\mathbf{R}_+)$ is invertible.

Proof a) Let $x > 0$. Then for each t, and each realization $M_t(\omega)$, $\omega \in \Omega$, of M_t we have $f(x + M_t(\omega)) = 0$, and hence $E[f(x + M_t)] = 0$. Thus, $U^q_M f(x) = 0$. To prove that $(U^q_M)^{-1} f(x) = 0$ as well, a more detailed study of the structure of U^q_M is needed (see the appendix).

b) is proved similarly. Now we can solve (3.19). We have $g_1 \in L^\infty(\mathbf{R}_+)$, hence by applying U^q_N to (3.19), we get

$$(U^q_M)^{-1} u = (1 - q) U^q_N g + g_2, \tag{3.20}$$

where $g_2 \in L^\infty(\mathbf{R}_+)$. By (3.8), $u \in L^\infty(\mathbf{R}_-)$, and on the strength of Lemma 3.3, the LHS in (3.20) belongs to $L^\infty(\mathbf{R}_-)$. Hence, multiplying (3.20) by $\mathbf{1}_{(-\infty,0]}$, we get

$$(U^q_M)^{-1} u = \mathbf{1}_{(-\infty,0]} (1 - q) U^q_N g,$$

and then applying U^q_M, we obtain (3.16).

By Lemma 3.3, $\mathbf{1}_{(-\infty,0]}(1-q)U^q_N g$ is independent of values of g on $\mathbf{R}_+$, hence u in (3.16) is, and we may define $g(x)$ by (3.7) for all x. Set

$$y(K; x) \equiv (1 - q) U^q_N (\rho G'(K) e^{\cdot} - (1 - q) C)(x), \tag{3.21}$$

and make the change of variables $x \to x - h(K)$ in (3.6) and (3.7) to obtain the marginal value of capital in the inaction region:

$$W_K(K + 0, x; h) = (U^q_M \mathbf{1}_{(-\infty,h]} y)(K; x) + G'(K) e^x + C. \tag{3.22}$$

Notice that (3.22) holds for $x > h(K)$, too (the first summand on the RHS of (3.22) vanishes there).

3.5 Optimal threshold and marginal value of capital

In this subsection, the investment threshold is obtained by "guess and verify" method, and the formula for the marginal option value of investment opportunity is derived. By using equations (2.2), (3.15), and the equality $(1 - q)U^q_M 1 = 1$, we obtain

$$\begin{aligned}(U^q_M y)(K, x) &= \rho G'(K) \left(U^q_M (1 - q) U^q_N e^{\cdot}\right)(x) - C \\ &= \frac{\rho G'(K) e^x}{1 - \rho} - C.\end{aligned}$$

Now we can rewrite (3.22) as

$$\begin{aligned}W_K(K + 0, x; h) &= \left((U^q_M - U^q_M \mathbf{1}_{[h,+\infty)})y\right)(K, x) + G'(K) e^x + C \\ &= \frac{G'(K) e^x}{1 - \rho} - (U^q_M \mathbf{1}_{[h,+\infty)} y)(K, x). \end{aligned} \tag{3.23}$$

The first term on the RHS is the expected present value of the marginal returns to capital given the capital stock remains constant in the future, and the second term, $-(U^q_M \mathbf{1}_{[h,+\infty)} y)(K, x)$, is the marginal option value of the future investment opportunities *given the chosen investment threshold, h.* If the choice is optimal, this value must be non-positive because investing extinguishes the option. Hence, the optimal h^* must be chosen so that $(U^q_M \mathbf{1}_{[h,+\infty)} y)(K, \cdot)$ is non-negative. Since the resolvent maps non-negative functions into non-negative ones, the natural guess is: h^* is the minimal one such that $y(K, x)$ is positive for all $x > h^*$. To show that such an h^* exists and find it, we need to rewrite (3.21).

Since $(1-q)U^q_N 1 = 1$, and

$$\begin{aligned}
(1-q)(U^q_N e^{\cdot})(x) &= (1-q)\sum_{t=1}^{\infty} q^t E[e^{N_t} \mid N_0 = x] \\
&= (1-q)\sum_{t=0}^{\infty} q^t E[e^{N_t + x} \mid N_0 = 0] \\
&= (1-q)e^x \sum_{t=0}^{\infty} q^t E[e^{N_t} \mid N_0 = 0] \\
&= \phi^-(q, -i)e^x,
\end{aligned}$$

where $\phi^-(q, -i) \equiv (1-q)(U^q_N e^{\cdot})|_{x=0}$ is a positive constant (for the explanation of the notation $\phi^-(q, -i)$, see (A.6)), we obtain

$$\begin{aligned}
y(K; x) &\equiv (1-q)(\rho G'(K)(U^q_N e^{\cdot})(x) - C) && (3.24) \\
&= \phi^-(q, -i)\rho G'(K)e^x - (1-q)C. && (3.25)
\end{aligned}$$

From (3.24)-(3.25), we see that $y(K, x)$ is positive for $x > h^*$, and negative for $x < h^*$, where $h^* = h^*(K)$ is the solution to the equation

$$\rho G'(K)(U^q_N e^{\cdot})(h) - C = 0,$$

or equivalently,

$$\rho G'(K)e^h \sum_{t=0}^{\infty} q^t E[e^{N_t} \mid N_0 = 0] - C = 0. \tag{3.26}$$

Now we check that the solution to (3.26) satisfies the optimality condition (3.1). Since so far only the formula for the marginal value of capital is available, we formulate a sufficient condition in terms of W_K.

Lemma 3.4 *Let h^* satisfy the following property:*

$$W_K(K+0, x; h^*) \le W_K(K+0, x; h), \ \forall \ h, \tag{3.27}$$

for all x and K such that $W_K(K+0, x; h^*)$ and $W_K(K+0, x; h)$ exist.

Then $h^* = h^*(K)$ is the investment threshold.

Proof Fix h and x. Since $W_K(K+0, x; h^*)$ and $W_K(K+0, x; h)$ exist for almost all K, we have

$$\begin{aligned}
W(K, x; h) &= -\int_K^{\bar{K}} W_K(K'+0, x; h)dK' \\
&\le -\int_K^{\bar{K}} W_K(K'+0, x; h^*)dK' = W(K, x; h^*).
\end{aligned}$$

By using (3.22), we obtain for any h

$$W_K(K+0,\cdot;h^*) - W_K(K+0,\cdot;h) = U^q_M \left\{\mathbf{1}_{(-\infty,h^*]} - \mathbf{1}_{(-\infty,h]}\right\} y(K;\cdot).$$

The function $y(K;x)$ is negative for $x < h^*$, and positive for $x > h^*$. Hence, if $h \neq h^*$, then the function $\left\{\mathbf{1}_{(-\infty,h^*]} - \mathbf{1}_{(-\infty,h]}\right\} y(K;\cdot)$ is non-positive. Since the resolvent U^q_M maps non-positive functions into non-positive ones, (3.27) holds, and therefore, $h^* = h^*(K)$ is the investment threshold. It is obvious, that (3.26) is equivalent to (2.5) presented in Section 2; this completes the proof of the first main result.

To calculate the marginal option value of investment opportunity, we use (3.26) to write

$$y(K,x) = \rho G'(K)(e^x - e^{h^*})(1-q)\sum_{t=0}^{\infty} q^t E[e^{N_t} \mid N_0 = 0].$$

Substituting the last formula into (3.23) and using the definition of U^q_M, we arrive at

$$\begin{aligned} W_K(K+0,x) \quad &= \quad \frac{G'(K)e^x}{1-\rho} - \rho G'(K)\sum_{t=0}^{\infty} q^t E[e^{N_t} \mid N_0 = 0] \\ &\quad \times (1-q)\sum_{t=0}^{\infty} q^t E[(e^{M_t} - e^{h^*})_+ \mid M_0 = x]. \end{aligned}$$

It suffices to rewrite the last equation in terms of prices (as opposed to log-prices) to get (2.9) and (2.10). Thus the second main result obtains.

3.6 Explicit formulas

From (3.26), we obtain the formula for the investment threshold

$$H^*(K) = \frac{(1-q)C}{\phi^-(q,-i)\rho G'(K)} \tag{3.28}$$

(for general analytical formulas for $\phi^-(q,-i)$, see (A.6) and Section 5). Formulas for firm's NPV and the option value of possible capital expansion will be obtained in Sections 4 and 5, under additional conditions on the production function.

3.7 The failure of the "smooth pasting condition", and the "continuous pasting" principle

Notice that to obtain the candidate for the investment threshold, h, we never resorted to the "smooth pasting condition". In fact, it fails here. (We are grateful to Avinash Dixit for pointing out to us that this principle may not hold in a discrete time model.) Let $\mu(dx)$ be the probability distribution of $\ln(P_1/P_0)$. Suppose that $\hat{\mu} \in L_1(\mathbf{R})$ (here $\hat{\mu}$ is the Fourier transform of μ), then $W_K(K+0,x)$ is not smooth w.r.t. x at the investment threshold. Instead of the smooth pasting condition we have a continuous pasting condition: the investment threshold, h, is determined by the requirement that $W_K(K+0,\cdot)$ is continuous at h. This can be proved as similar results in continuous time non-Gaussian Lévy models - see Boyarchenko and Levendorskiǐ (2000, 2002a,b).

4 The option value and the value of the firm

Let the investment threshold $h^* = h^*(K)$ be determined from (3.26). We fix $x < h^*(K)$, and use (2.9) to calculate the value of the firm $W(K,x)$:

$$W(K+0,x) = \frac{G(K)e^x}{1-\rho} + W^{\text{opt}}(K,x), \tag{4.1}$$

where

$$W^{\text{opt}}(K,x) = -\int_K^{\bar{K}} W_K^{\text{opt}}(K'+0,x)dK' \tag{4.2}$$

is the option value of investment opportunity. To obtain an explicit formula for the integrand, we need the representation $(1-q)U_M^q = \phi^+(q,D)$; here $\phi^+(q,D)$ denotes the pseudo-differential operator (PDO) with the symbol $\phi^+(q,\xi) = E[e^{i\xi M_T}]$, and $i = \sqrt{-1}$ (see the appendix). The PDO $\phi^+(q,D)$ acts as follows:

$$\phi^+(q,D)u(x) = (2\pi)^{-1}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} e^{ix\xi}\phi^+(q,\xi)\hat{u}(\xi)d\xi,$$

where $\hat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}u(x)dx,$$

and the line of integration $\Im\xi = \omega_-$ is chosen so that the integral converges.

Set $y_1(x) = \mathbf{1}_{(0,+\infty)}(x)(e^x-1)$. By using (3.28), we rewrite (3.23) as

$$W_K^{\text{opt}}(K'+0,x) = \frac{G'(K')e^x}{1-\rho} + W_K^{\text{opt}}(K'+0,x),$$

where

$$\begin{aligned} W_K^{\text{opt}}(K'+0,x) &= -(U_M^q\mathbf{1}_{[h,+\infty)}y)(K',x) \\ &\equiv -C(1-q)(U_M^q y_1)(x-h^*(K')) \\ &= -C(\phi^+(q,D)y_1)(x-h^*(K')). \end{aligned} \tag{4.3}$$

The Fourier transform of y_1 is well-defined in the half-space $\Im\xi < -1$:

$$\hat{y}_1(\xi) = \frac{1}{(-i\xi)(1-i\xi)}.$$

Suppose that $\phi^+(q,\xi)$ is well-defined and differentiable w.r.t. $\xi = \sigma + i\tau$ in a half-plane $\tau = \Im\xi > \sigma_-$, where $\sigma_- < -1$ (for sufficient conditions, see the appendix). Take $\omega_- \in (\sigma_-, -1)$, and use the definition of PDO to make (4.3) explicit:

$$W_K^{\rm opt}(K'+0,x) = -\frac{C}{2\pi}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \frac{e^{i(x-h^*(K'))\xi}\phi^+(q,\xi)}{(-i\xi)(1-i\xi)}d\xi. \qquad (4.4)$$

Use (3.26) once again to compute

$$e^{ih^*(K)\xi - ih^*(K')\xi} = (G'(K')/G'(K))^{i\xi},$$

and substitute into (4.4):

$$W_K^{\rm opt}(K'+0,x) = -\frac{C}{2\pi}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\left(\frac{G'(K')}{G'(K)}\right)^{i\xi} \frac{e^{i(x-h^*(K))\xi}\phi_+(q,\xi)}{-i\xi(1-i\xi)}d\xi. \quad (4.5)$$

Even though we derived the investment threshold under the resource constraint assumption $K \le \bar{K} < +\infty$, the threshold is independent of $\bar{K}$. Assume that $\bar{K}$ is very large indeed, and notice that under an additional condition imposed below on the firm's production function, the integral in (4.5) over a very large finite interval $[K, \bar{K}]$ can be approximated by the integral over a semi-infinite interval. In other words, we want to use the formula

$$W^{\rm opt}(K,x) = -\int_K^{+\infty} W_K^{\rm opt}(K',x)dK',$$

and to ensure that the firm's value is finite, we assume that there exists $\gamma > 1/(-\sigma_-)$ such that for $K' > K$,

$$\frac{G'(K')}{G'(K)} \le \left(\frac{K'}{K}\right)^{-\gamma}. \qquad (4.6)$$

Since $\gamma\sigma_- < -1$ and $\omega_- \in (\sigma_-, -1)$ is arbitrary, we can choose ω_- so that $\gamma\omega_- < -1$. Then the inequality (4.6) implies that for any K, the integral

$$\Phi(K,\xi) \equiv \int_K^{+\infty}\left(\frac{G'(K')}{G'(K)}\right)^{i\xi} dK' \qquad (4.7)$$

converges absolutely and uniformly w.r.t. ξ in the half-space $\Im\xi \le \omega_-$; hence, $\Phi(K,\xi)$ is well-defined and uniformly bounded w.r.t. ξ in this half-space. Therefore, for any K, x,

$$W^{\rm opt}(K,x) = \frac{C}{2\pi}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-} \Phi(K,\xi)\frac{e^{i(x-h^*(K))\xi}\phi^+(q,\xi)}{-i\xi(1-i\xi)}d\xi. \qquad (4.8)$$

In the case of a Cobb-Douglas production function, $G(K) = AK^\theta,\ A > 0,\ \theta \in (0,1)$, we have $G'(K) = A\theta K^{\theta-1}$, therefore if we assume that $1-\theta > -1/\sigma_-$, then (4.6) holds,

$$\Phi(K,\xi) = \frac{K}{(1-\theta)i\xi - 1},$$

and (4.8) simplifies to

$$W^{\mathrm{opt}}(K,x)=\frac{CK}{2\pi}\int_{-\infty+i\omega_-}^{+\infty+i\omega_-}\frac{e^{i(x-h^*(K))\xi}\phi^+(q,\xi)}{((1-\theta)i\xi-1)(-i\xi)(1-i\xi)}d\xi. \tag{4.9}$$

In the next Section, we show how to calculate the option value when the probability density of $\mu(dx)$ is given by exponential polynomials on each half-axis.

5 Example: the case of exponential polynomials

The discrete time model admits an approximation of the empirical probability density so that the factors $\phi^\pm(q,\xi)$ in the Wiener-Hopf factorization formula can be computed relatively easily. Here we consider the simplest approximation by exponential polynomials (for more sophisticated approximation, see the appendix). Let

$$p_\pm(x)=\frac{\mp\lambda_\mp}{2}\mathbf{1}_{\mathbf{R}_\pm}(x)e^{\lambda_\mp x},$$

where $\lambda_-<-1<0<\lambda_+$. First, we compute for η in the half-plane $\Im\eta<\lambda_+$

$$\hat{p}_-(-\eta)=\frac{1}{2}\lambda_+\int_{-\infty}^{0}e^{ix\eta+\lambda_+x}dx=\frac{\lambda_+}{2(\lambda_++i\eta)},$$

and for η in the half-plane $\Im\eta>\lambda_-$,

$$\hat{p}_+(-\eta)=\frac{1}{2}\lambda_+\int_{0}^{+\infty}e^{ix\eta+\lambda_-x}dx=\frac{-\lambda_-}{2(-\lambda_--i\eta)}.$$

Next, we obtain

$$\begin{aligned}1-q\hat{\mu}(-\eta)&=1-\frac{q}{2}\left[\frac{-\lambda_-}{-\lambda_--i\eta}+\frac{\lambda_+}{\lambda_++i\eta}\right]\\&=\frac{-2(1-q)\lambda_-\lambda_+-(2-q)(\lambda_++\lambda_-)i\eta+2\eta^2}{2(\lambda_++i\eta)(-\lambda_--i\eta)}.\end{aligned}$$

Denote by $-i\beta_-$ (respectively, $-i\beta_+$) the root of the numerator in the upper half-plane (respectively, lower half-plane). Then

$$\phi^-(q,\xi)=\frac{(\lambda_++i\xi)(-\beta_-)}{\lambda_+(-\beta_-+i\xi)},\quad \phi^+(q,\xi)=\frac{(\lambda_-+i\xi)(-\beta_+)}{\lambda_-(-\beta_++i\xi)}.$$

Now we can explicitly calculate $\phi^-(q,-i)$, and find the optimal investment threshold from (3.28):

$$H^*(K)=\frac{(1-q)\lambda_+(1-\beta_-)}{G'(K)(\lambda_++1)(-\beta_-)}.$$

Similar but lengthier calculations (the residue theorem is needed) allow one to derive the option value from (4.9):

$$W^{\mathrm{opt}}(K,\ln P)=CK\left(\frac{P}{H^*(K)}\right)^{\beta_+}\frac{\lambda_-+\beta_+}{((1-\theta)\beta_+-1)(\beta_+-1)\lambda_-}.$$

Notice that to obtain the analytical value for the investment threshold and firm's value function, one has only to solve a quadratic equation for $-i\beta_\pm$, and find $\beta_\pm = [(1-q/2)(\lambda_+ + \lambda_-) \pm \sqrt{D}]/2$, where $D \equiv (1-q/2)^2(\lambda_+ + \lambda_-)^2 - 4(1-q)\lambda_+\lambda_-$.

For generic p, the computation of the factors $\phi^\pm(q,\xi)$ is a non-trivial computational task. However, for many families of probability distributions used in empricial studies (for instance, normal inverse gaussian distributions), it is possible to obtain efficient approximate formulas – see Boyarchenko and Levendorskiĭ (2002c).

6 Conclusion

We have constructed a discrete time model of irreversible investment under uncertainty. The main tool employed in the paper is the Wiener-Hopf method. We present the method and the formulas for the threshold and the value of the firm in the form meaningful from the point of view of Economics. We believe that this form is suitable for applications in different fields of Economics.

One of the straightforward applications of the model presented in the paper is effective capital budgeting which is important for corporate survival. Current real option models are not widely used in corporate decision making. Among one of the primary reasons for that, Lander and Pinches (1999) point out that many of the required modeling assumptions are often violated in practical real options applications. In particular, this concerns the choice of the stochastic process for the underlying variable. As we already mentioned it in the Introduction, even though the normality of the process is rejected by empirical evidence, Gaussian processes are often used in the investment literature. The model presented in the paper is much more realistic and flexible.

The second reason for theoretical models being rarely used by practitioners is technical involvement of the models. The task of fitting the parameters of a model process to the data by no means makes life easier. The model presented in the paper can be used for practical purposes. One of the advantages of discrete time setting is that the probability distribution can be approximated by exponential polynomials with desired accuracy and simplicity (although there is certainly a tradeoff between these two). After that, to obtain the analytical expression for the investment threshold one has only to find roots of a polynomial - a task easy for any practitioner. In cases when a simple exponential-polynomial approximation fits the data poorly, efficient approximate numerical procedures can be developed for the calculation of the factors in the Wiener-Hopf factorization formula, hence, of the investment threshold and value of the firm.

A Technicalities

A.1 Proof of Lemma 3.1, cont-d

Let $x < h(K)$ and $K_1 > K$. Denote by $\tau(a)$ the first entrance time of X_t into $(a, +\infty)$: $\tau(a) \equiv \min\{t \mid X_t > a\}$, and set $\tau = \tau(h(K)), \tau_1 = \tau(h(K^1))$. Clearly, if $\{K_t\}$ is the strategy which is determined by h and the initial state (K, x), and $\{K_t^1\}$ is the strategy determined by h and the initial state (K^1, x), then $K_t^1 = K_t$ for all $t \geq \tau_1$. Hence,

$$W(K^1, x; h) - W(K, x; h) = E\left[\sum_{t=0}^{\tau-1} q^t e^{X_t}(G(K^1) - G(K))|X_0 = x\right]$$

$$+E\left[\sum_{t=\tau}^{\tau_1-1} q^t(e^{X_t}(G(K^1) - G(K_t)) - C(K_{t+1} - K_t))|X_0 = x\right].$$

Denote the terms on the RHS by $W^1(K^1, K, x)$ and $W^2(K^1, K, x)$, with the usual convention that the last term, $W^2(K^1, K, x)$, is zero if $\tau = \tau_1$. Since G is differentiable, the limit

$$\lim_{K^1 \downarrow K} \frac{W^1(K^1, K, x)}{K^1 - K} = E\left[\sum_{t=0}^{\tau-1} q^t e^{X_t} G'(K) \mid X_0 = x\right]$$

exists, and it is finite. To show that the same limit with W^2 in place of W^1 exists (and equals zero, in fact), we first notice that the expression under the expectation sign in the formula for W^2 is $O(K^1 - K)$, hence after we divide by $K^1 - K$ under the E sign, we obtain an estimate

$$\frac{W^1(K^1, K, x)}{K^1 - K} \leq C_1(K^1)e^{h(K^1)}E\left[\sum_t q^t \mathbf{1}_{[h(K),h(K^1)]}(X_t) \mid X_0 = x\right],$$

where $C_1(K^1) > 0$ is a constant. It remains to show that

$$U_X^q \mathbf{1}_{[h(K),h(K^1)]} \to 0. \tag{A.1}$$

Since $h'(K)$ exists, we have $h(K^1) - h(K) \sim h'(K)(K^1 - K) \to 0$, as $K^1 \downarrow K$, and since $\mu(dx)$ is absolutely continuous, we conclude that (A.1) holds. Lemma 3.1 has been proved.

A.2 The Wiener-Hopf factorization

Let $z \in (0, 1)$, let $Y_1, Y_2, \cdots$ be i.i.d. random variables with the probability distribution $\mu(dx)$. Let $X_t = X_0 + Y_1 + \cdots + Y_t$ be the random walk started at 0: $X_0 = 0$, and denote by $\mu_t(dx)$ the probability distribution of X_t. Let T be a random variable independent of X and taking values in $\{0, 1, \ldots\}$, with $P(T = t) = (1 - z)z^t$. Consider the random variable X_T.

Theorem A.1 *(Spitzer (1964))*

$$E\left[e^{i\xi X_T}\right] = E\left[e^{i\xi M_T}\right] E\left[e^{i\xi N_T}\right]. \tag{A.2}$$

Moreover, we have the Spitzer identities

$$E\left[e^{i\xi M_T}\right] = \exp\left[\sum_{t=1}^{\infty} \frac{z^t}{t} \int_0^{\infty} (e^{ix\xi} - 1)\mu_t(dx)\right], \tag{A.3}$$

and

$$E\left[e^{i\xi N_T}\right] = \exp\left[\sum_{t=1}^{\infty} \frac{z^t}{t} \int_{-\infty}^{0} (e^{ix\xi} - 1)\mu_t(dx)\right]. \tag{A.4}$$

Set

$$\phi^+(z,\xi) \equiv E\left[e^{i\xi M_T}\right] \equiv (1-z)U_M^z(e^{ix\xi})|_{x=0}, \tag{A.5}$$

$$\phi^-(z,\xi) \equiv E\left[e^{i\xi N_T}\right] \equiv (1-z)U_N^z(e^{ix\xi})|_{x=0}. \tag{A.6}$$

Denote by $\hat{\mu}(\xi)$ the Fourier transform of $\mu(dx)$:

$$\hat{\mu}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi}\mu(dx).$$

By using (A.5) and (A.6), one can rewrite (A.2) as

$$(1-z)/[1 - z\hat{\mu}(-\xi)] = \phi^+(z,\xi)\phi^-(z,\xi). \tag{A.7}$$

Fix $z \in (0,1)$, and allow $\xi = \sigma + i\tau$ to be a complex number. It is easily seen that the expression $\Phi^+(z,\xi)$ under the exponent sign in (A.3) is well-defined and bounded in the half-plane $\tau = \Im\xi > 0$ and continuous up to the boundary of the half-plane. The derivative of $\Phi^+(z,\xi)$ w.r.t. ξ is defined in the open half-plane as well (one says that $\Phi^+(z,\xi)$ is *analytic* in the half-plane $\Im\xi > 0$). Hence, $\phi^+(z,\xi)$ and $1/\phi^+(z,\xi)$ are analytic and bounded in the half-plane $\Im\xi > 0$, and continuous up to the boundary. Similarly, $\phi^-(z,\xi)$ and $1/\phi^-(z,\xi)$ its are analytic and bounded in the half-plane $\Im\xi < 0$, and continuous up to the boundary.

For the explicit calculation of the value of the firm, it is convenient to know that $\phi^+(z,\xi)$ is analytic in a wider half-plane, and we show this as follows. From (2.1), we conclude that

$$\hat{\mu}(-\xi) = \int_{-\infty}^{+\infty} e^{i\xi x}\mu(dx) = \int_{-\infty}^{+\infty} e^{(-\tau+i\sigma)x}\mu(dx)$$

is analytic in the strip $\Im\xi \in (-1,0)$ and continuous up to the boundary of the strip. Moreover, the real part of $1 - z\hat{\mu}(-\xi)$ is positive in the closed strip $\Im\xi \in [-1,0]$ (the proof is the same as of equation (2.8) in Boyarchenko and Levendorskiĭ (2002a)). Assume that $\hat{\mu}(-\xi)$ is analytic in a wider strip $(\lambda_-,0)$, where $\lambda_- < -1$. Then by continuity, there exists $\sigma_- \in (\lambda_-,-1)$ such that the real part of $1-z\hat{\mu}(-\xi)$ is positive in the closed strip $\Im\xi \in [\sigma_-,0]$. Hence, $\phi^+(z,\xi)$ can be extended into this strip by using (A.7):

$$\phi^+(z,\xi) = (1-z)(1-z\hat{\mu}(-\xi))^{-1}\phi^-(z,\xi)^{-1}. \tag{A.8}$$

A.3 Resolvents as PDO

Let u be a sufficiently regular function, say, $u \in \mathcal{S}(\mathbf{R})$ (that is, $u(x)$ and each of its derivatives decay at infinity faster than any power of x). By the Fourier inversion formula,

$$u(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{u}(\xi) d\xi, \tag{A.9}$$

therefore

$$\begin{aligned}
(U_X^z u)(x) &= E\left[\sum_{t=0}^{\infty} z^t u(X_t) \mid X_0 = x\right] \\
&= E\left[\sum_{t=0}^{\infty} z^t (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{iX_t\xi} \hat{u}(\xi) d\xi \mid X_0 = x\right] \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \sum_{t=0}^{\infty} z^t E[e^{iX_t\xi}] \hat{u}(\xi) d\xi \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} \sum_{t=0}^{\infty} z^t \hat{\mu}(-\xi)^t \hat{u}(\xi) d\xi \\
&= (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} (1 - z\hat{\mu}(-\xi))^{-1} \hat{u}(\xi) d\xi.
\end{aligned}$$

Let an operator A be defined by

$$Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi.$$

Then one says that A is a pseudo-differential operator (PDO) with the symbol a and writes $A = a(D)$ (in some cases, the integration along a different line $\Im\xi = \sigma$ in the complex plane must be used). Thus, the resolvent U_X^z is a PDO with the symbol $(1 - z\hat{\mu}(-\xi))^{-1}$:

$$U_X^z = (1 - z\hat{\mu}(-D))^{-1}.$$

By using (A.5) and (A.6), we similarly conclude that

$$(1 - z)U_M^q = \phi^+(z, D), \quad (1 - z)U_N^q = \phi^-(z, D). \tag{A.10}$$

A.4 Proof of (3.15)

Now we can rewrite (A.7) as

$$(1 - z)(1 - z\hat{\mu}(-D))^{-1} = \phi^+(z, D)\phi^-(z, D),$$

or equivalently,

$$(1 - z)U_X^z = (1 - z)U_M^z (1 - z)U_N^z.$$

This gives (3.15).

A.5 Proof of Lemma 3.3, cont-d

To finish the proof for U^q_M, we have to show that for any $f \in L^\infty(\mathbf{R}_-)$ and any $g \in C_0^\infty((0, +\infty))$,

$$((U^z_M)^{-1} f, g) \equiv \int_{-\infty}^{+\infty} (U^z_M)^{-1} f(x) g(x) dx = 0.$$

Let $-M$ be the infimum process for the dual process $-X$; then

$$\int_{-\infty}^{+\infty} ((U^z_M)^{-1} f)(x) g(x) dx = \int_{-\infty}^{+\infty} f(x)((U^z_{-M})^{-1} g)(x),$$

therefore it suffices to show that for any $x < 0$,

$$(1 - z)^{-1}((U^z_{-M})^{-1} g)(x) \equiv (\tilde{\phi}^-(z, D)^{-1} g)(x) = 0,$$

where $\tilde{\phi}^-$ is the minus-factor in the Wiener-Hopf factorization formula for the resolvent of the process $-X$. By using the definition of PDO, we have

$$\tilde{\phi}^-(z, D)^{-1} g(x) = (2\pi)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{ix\xi} \tilde{\phi}^-(z, \xi)^{-1} \hat{g}(\xi) d\xi, \qquad \text{(A.11)}$$

where $\sigma = 0$. Since $g \in C_0^\infty((0, +\infty))$, its Fourier transform admits the analytic continuation into the half-space $\Im\xi < 0$, and in the closed half-plane, it satisfies an estimate

$$|\hat{g}(\xi)| \le C_N (1 + |\xi|)^{-N}, \qquad \text{(A.12)}$$

for any N, where C_N depends on N but not on ξ. But $\tilde{\phi}^-(z, \xi)^{-1}$ is bounded in the same closed half-plane, therefore the integrand in (A.11) admits the estimate (A.12). By the Cauchy theorem, we may push the line of integration in (A.11) down: $\sigma \to -\infty$; in the limit, the integral (A.11) vanishes, and we are done.

A.6 Analytical formulas for the factors

The Spitzer identities (A.3)-(A.4) are by no means computationally effective. More convenient formulas can be obtained under additional conditions on $\mu(dx)$. For instance, if $\hat{\mu} \in L_1(\mathbf{R})$, then for ξ in the half-plane $\Im\xi > 0$,

$$\phi^+(q, \xi) = \exp\left[(2\pi i)^{-1} \int_{-\infty}^{+\infty} \frac{\xi \ln(1 - z\hat{\mu}(-\eta))}{\eta(\xi - \eta)} d\eta\right],$$

and for ξ in the lower half-plane,

$$\phi^-(q, \xi) = \exp\left[-(2\pi i)^{-1} \int_{-\infty}^{+\infty} \frac{\xi \ln(1 - z\hat{\mu}(-\eta))}{\eta(\xi - \eta)} d\eta\right]$$

(the proof is essentially the same as of similar formulas in the continuous time model - see equations (3.10) and (3.12) in Boyarchenko and Levendorskiǐ

(2002a)). Simpler still is the calculation of the factors $\phi^{\pm}(z,\xi)$ if $\mu(dx)$ is absolutely continuous: $\mu(dx) = p(x)dx$, and $p_{\pm} = \mathbf{1}_{\mathbf{R}_{\pm}} p$ are exponential polynomials:

$$p_-(x) = \sum_{j=0}^{m^+} c_j^+ e^{\lambda_+ x} |x|^j \mathbf{1}_{\mathbf{R}_-}(x), \tag{A.13}$$

where m^+ is a non-negative integer, $\lambda_+ > 0$ and $c_j^+ \in \mathbf{R}$, and

$$p_+(x) = \sum_{j=0}^{m^-} c_j^- e^{\lambda_- x} x^j \mathbf{1}_{\mathbf{R}_+}(x), \tag{A.14}$$

where m^- is a non-negative integer, and $\lambda_- < 0$ and $c_j^- \in \mathbf{R}$. In this case, both $1 - z\hat{\mu}(-\xi)$ and $\phi^{\pm}(z,\xi)$ are rational functions (see e.g. Borovkov (1976), p. 106-107), and so the formulas for $\phi^{\pm}(z,\xi)$ can easily be guessed from the formula for $1 - z\hat{\mu}(-\xi)$. Direct calculations show that $1 - z\hat{\mu}(-\xi) = P(z,\xi)/Q(\xi)$, where $Q(\xi) = (\lambda_+ + i\xi)^{m_+ + 1}(-\lambda_- - i\xi)^{m_- + 1}$ and $P(z,\xi)$ is a polynomial in ξ. By factorizing $P(z,\xi)$ ($Q(\xi)$ is already factorized), and picking factors which do not vanish in the lower half-plane, we obtain the following formula for $\phi^-(z,\xi)$:

$$\phi^-(z,\xi) = \left(\frac{\lambda_+ + i\xi}{\lambda_+}\right)^{m_+ + 1} \prod_j \frac{-\beta_j^-}{-\beta_j^- + i\xi}, \tag{A.15}$$

where $\{-i\beta_j^-\}$ are all the zeroes of P in the upper half-plane. Additional constant factors are needed in order to satisfy the normalization $\phi^-(z,0) = 1$. Similarly, the formula for $\phi^+(z,\xi)$ obtains. Now we find the optimal investment threshold from (3.28):

$$H^*(K) = \frac{(1-q)C}{\rho G'(K)} \left(\frac{\lambda_+}{\lambda_+ + 1}\right)^{m_+ + 1} \prod_j \frac{1 - \beta_j^-}{-\beta_j^-}.$$

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