Corrections to Fluid Dynamics

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Abstract

We show that a Galilean invariant version of fluid dynamics can be derived by the methods of statistical dynamics using Maxwell's balance equations. The basic equation is non-local, and might replace the Boltzmann equation if the latter turns out not to have global smooth solutions in general. As an approximation, a local form of the equations of motion is derived. It turns out to be a version of the Navier-Stokes system, obeying the Stokes relation, and with the viscosity coefficient rising as $\Theta^{1/2}$ with temperature Θ. The new feature is the presence of the Dufour effect for a gas of a single component. This ensures that the principal symbol of the parabolic system is non-singular.

1 Introduction

A central problem for mathematical fluid dynamics is the derivation of the Navier-Stokes equations (*N-S*) starting from a reversible dynamical theory such as classical hard spheres or quantum mechanics. To achieve this, it seems essential to know that the *N-S* equations themselves possess smooth solutions for all times, for a large enough class of smooth initial values for the fields. Hence the latter problem is revealed as the key question, for one version of which a Clay Millenium prizes is offered. The prize version describes an incompressible liquid under isothermal conditions. Nash [31, 32] had shown that, given smooth initial conditions, there exists a unique smooth solution for a small enough time. The question of smooth global-in-time solutions remains open except for small initial conditions. To model driven systems such as Bénard convection, the isothermal condition must be relaxed; recent numerical studies [20] show qualitative agreement with experiment. In [20], the condition of incompressiblitiy, $\operatorname{div} \boldsymbol{u}=0$ is maintained, and the energy equation is modified by the addition of a bouyancy condition. This is expressed by requiring that the liquid in hot regions is less dense than in cold regions, a rather *ad hoc* procedure. We shall argue in Sect. (2) that *N-S* is at the boundary of a more regular class of models. In *N-S*, the pressure is infinite, but its place is taken by a surrogate pressure determined by the requirement of self-consistency. Thus in *N-S*, the pressure is a balancing item like petty cash, much beloved by accountants, which can be adjusted to cancel errors made elsewhere in the calculation.

In this paper, *C-N-S-T* will denote the system of five coupled non-linear partial differential equations known [27] as 'compressible Navier-Stokes with temperature'. This might or might not be an easier problem than *N-S*, but it is certainly more widely applicable. A difficulty with *C-N-S-T* is that the symbol of the elliptic operator is singular. To derive *N-S*, some authors divide the problem into two parts [6]; first, to show that reversible dynamics is well represented in some limit by a stochastic process; then to show that the Fokker-Planck equations of this process gives rise to the *N-S* equations. The latter is only partially achieved in [6]. Other authors *start* with some version of stochastic dynamics [39, 28, 40], and prove things; this approach will be adopted here, as it avoids the much harder first part. We shall adapt information geometry [21, 4, 25] to the

dynamics of a rare gas. Thus, the state of the system is not a point in phase space, but a measure μ on it, and the dynamics is a path in the space Σ of measures. In the usual treatment [3] the currents of the conserved variables are given exactly by expressions involving higher moments of the same random fields; the time derivatives of these higher moments involve yet higher moments. The whole system goes on for ever, and is called the *BBGKY* hierarchy. Inasmuch as the system is equivalent to classical mechanics, it is reversible and shows no dissipation. Artfulness is needed to 'close' the system in terms of the slow fields; that is, to write the time-derivatives of the slow fields in terms of themselves, by truncating the system. It has proved possible to get a variety of kinetic equations from the hierarchy, by taking a limit appropriate to the physical situation [3]. These limiting systems exhibit dissipation. It then remains to show that the solutions to the limiting system are limits of solutions to the *BBGKY* hierarchy. This programme has proved to be difficult to complete.

Information dynamics offers an alternative. In the case studied here, where the potential between the particles is zero outside the hard core, the states in *local thermodynamic equilibrium, LTE* can be computed. The *LTE* states are products over the lattice; the state at a site $\boldsymbol{x}$ is of the form

$$\mu(\omega_{\boldsymbol{x}}) = N_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{k}),$$

where p is Maxwellian. The set of such states make up the information manifold, $\mathcal{M}$. Any state $\mu \in \Sigma$, having finite means for the slow variables, has a reduced description, denoted μQ; here Q is the the non-linear projection onto $\mathcal{M}$, acting on the right, which maps μ to the state in $\mathcal{M}$ with the same means for the slow variables as μ. The Gibbs principle [23, 22] states that μQ is the state of maximum entropy having these means. In the original formulation of information dynamics [21], in the time interval $(0, t)$ the state μ evolves under the reversible dynamics of classical mechanics, to $\mu(t)$ say. This state has a much simpler description by the *LTE* state $\mu(t)Q$, which cannot be distinguished from $\mu(t)$ by measuring the slow variables. The orbit $\{\mu(t)Q : t \geq 0\}$ in $\mathcal{M}$ was intended as the thermodynamic evolution. It is clear that the entropy of $\mu(t)Q$ is not less than that of μ; there is a transfer of information into inaccessible degrees of freedom by the reversible motion [5]. It is not always true that entropy increases along the orbit, as is seen if the classical motion were periodic. Another version of information dynamics was adopted in [4]; there, the reversible motion took place for a very small time t, and the reduced description $\mu(t)Q$ was used, instead of $\mu(t)$, as the initial state of the next step. This gives a discrete-time semigroup, with increasing entropy; however the time-step cannot be taken to zero, without sending the rate of entropy production to zero as well [25, 4]. It is necessary to keep the time-step positive; it represents the relaxation time, and the map Q implements the thermalisation of the state $\mu(t)$. The challenge is to do this in a way that is invariant under the Galilean group (denoted by $\mathcal{G}$ below).

Information dynamics has been extended [36] to allow stochastic dynamics; then the time-step can go to zero, still giving a non-zero rate of entropy production. Another idea is to allow state-dependent transition rates [1]. With these changes, one may call the theory *statistical dynamics*. It is designed to obey both the first and the second laws of thermodynamics, but otherwise puts few constraints on the form of the dynamical equations. The choice made for the dynamics determines the nature of the system under discussion. At first sight, statistical dynamics has too much noise; a simple application is shown [37] to lead to mass diffusion and the Soret effect for an inert gas at rest, contrary to the literature [5]. Indeed, without a velocity field, the theory 'has not got off the ground' [26]. Truesdell [38] ironically says "results of this kind are described by kinetic theorists as 'corrections to hydrodynamics' ".

In this paper, we apply statistical dynamics to the case of an inert gas of a single type. We arrive at *C-N-S-T*, but with one extra term, a Dufour effect. Thus the intuitively attractive 'method of Maxwell', [29] in which we compute the gain and loss of particles in a small time interval at each point $\boldsymbol{x}$, is successful. The new idea is to postulate that in the state μ, some but not all the

particles are thermalised; those that are, are described by the *LTE* state $\overline{\mu}$. However, this state is NOT equal to μQ! By construction, our model is the the continuum limit of a non-linear Markov process on a lattice, with a bistochastic transition matrix. This might be a possible starting point for a proof that smooth solutions exist. The method of Maxwell is treated in [14], p 93, and in Balian [5], but it is abandoned as too primitive, in favour of the Boltzmann equation. This might have been too hasty, especially if the *BE* turns out not to have smooth global solutions.

In Sect. (2) we study a discrete model of hard spheres, for which the equilibrium state exactly factorises. Sect. (3) contains a discussion of the idea that the true state μ should be distinguished from its thermalised part, $\overline{\mu}$, using the analogy of aircraft in an airfreight company. The hopping rules of the nonlinear Markov chain are presented, and related to the mean free time. We also suggest a $\mathcal{G}$-invariant collision function. From this, we get an explicit formula (40) for the mean free time, t_ℓ. In Sect (4) we start with the fundamental relation (44) expressing the full state as a non-local integral of the thermalised state. The dynamics is expressed in terms of the *BBGKY* hierarchy. We obtain the Euler equations as the zero$^{\text{th}}$ approximation, and obtain a useful short version of these. In Sect. (5) we find the differences of the means of the slow variables in the states μ and $\overline{\mu}$, and show how the method can be extended to other variables. This enables us to compute the *BBGKY* moments in terms of the means in the full state and thus arrive at a version of *C-N-S-T*. This exhibits the Dufour effect, contrary to the literature [5].

2 The Thermostatics of an Inert Gas

We take space to be $\Lambda \subseteq (a\mathbf{Z})^3$, and suppose the length a, representing the diameter of a molecule, to be so small compared with the variation of the macroscopic fields that we can replace all sums over Λ by integrals. The possible configurations of the fluid are the points in the product sample space

$$\Omega = \prod_{\boldsymbol{x}\in\Lambda} \Omega_{\boldsymbol{x}},$$

so a configuration is specified by the collection $\{\omega_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\Lambda}$. For each $\boldsymbol{x}$,

$$\Omega_{\boldsymbol{x}} = \left\{\emptyset, (\epsilon\mathbf{Z})^3\right\}.$$

Here, ϵ is a small parameter having the dimension of momentum; for example, we could take $a\epsilon = h$, the semi-classical division of the phase-space of a particle into cubes of volume h^3. If the system is in a configuration ω, such that $\omega_{\boldsymbol{x}} = \emptyset$, then we say that the site $\boldsymbol{x}$ is empty. If $\omega_{\boldsymbol{x}} = \boldsymbol{k}$, we say that the site $\boldsymbol{x}$ is occupied, by a particle of momentum $\boldsymbol{k}$. This simple exclusion of more than one particle on each site incorporates the hard-core repulsion between the particles, which are thus hard spheres sitting at some of the points of Λ. The field point of view enables us to avoid the Gibbs paradox.

The *state* of the system is a probability on Ω, denoted by μ. We denote the set of states by Σ. The 'slow variables' of our model are the 5 extensive conserved random fields

$$\mathcal{N}_{\boldsymbol{x}}(\omega) = \begin{cases} 0 & \text{if } \omega_{\boldsymbol{x}} = \emptyset \\ 1 & \text{if } \omega_{\boldsymbol{x}} = \boldsymbol{k} \end{cases} \tag{1}$$

$$\mathcal{E}_{\boldsymbol{x}}(\omega) = \begin{cases} 0 & \text{if } \omega_{\boldsymbol{x}} = \emptyset \\ \boldsymbol{k}\cdot\boldsymbol{k}/2m + \Phi(\boldsymbol{x}) & \text{if } \omega_{\boldsymbol{x}} = \boldsymbol{k} \end{cases} \tag{2}$$

$$\boldsymbol{\mathcal{P}}_{\boldsymbol{x}}(\omega) = \begin{cases} 0 & \text{if } \omega_{\boldsymbol{x}} = \emptyset \\ \boldsymbol{k} & \text{if } \omega_{\boldsymbol{x}} = \boldsymbol{k} \end{cases} \tag{3}$$

Here, $\Phi(\boldsymbol{x})$ is the external potential energy per particle. The variables appearing in the *C-N-S-T* equations are simply related to the *mean fields* in the state μ:

$$N_{\boldsymbol{x}} = \mathbf{E}_\mu[\mathcal{N}_{\boldsymbol{x}}]; \qquad E_{\boldsymbol{x}} = \mathbf{E}_\mu[\mathcal{E}_{\boldsymbol{x}}]; \qquad \mathbf{\Pi}_{\boldsymbol{x}} = \mathbf{E}_\mu[\boldsymbol{\mathcal{P}}_{\boldsymbol{x}}]. \tag{4}$$

In information geometry, the specification of the slow variables determines the *information manifold* $\mathcal{M}$, which in the context of fluid dynamics consists of states in *LTE* (local thermodynamic equilibrium). Such a state is specified by five *canonical fields*, dual to the mean fields: $\beta_{\boldsymbol{x}}, \xi_{\boldsymbol{x}}, \boldsymbol{\zeta}_{\boldsymbol{x}}$, and has the form

$$\mu(\omega) = \prod_{\boldsymbol{x}\in\Lambda} \Xi_{\boldsymbol{x}}^{-1} \exp\left\{-\xi_{\boldsymbol{x}}\mathcal{N}_{\boldsymbol{x}}(\omega) - \beta_{\boldsymbol{x}}\mathcal{E}_{\boldsymbol{x}}(\omega) - \boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{\mathcal{P}}_{\boldsymbol{x}}(\omega)\right\}. \tag{5}$$

In finding the partition function

$$\Xi_{\boldsymbol{x}} = 1 + \epsilon^{-3}\left(\frac{2\pi m}{\beta_{\boldsymbol{x}}}\right)^{3/2} \exp\left\{-\xi_{\boldsymbol{x}} - \beta_{\boldsymbol{x}}\Phi(\boldsymbol{x}) + m\ \boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{\zeta}_{\boldsymbol{x}}/2\beta_{\boldsymbol{x}}\right\} \tag{6}$$

we have replaced the sum over the momentum lattice of size ϵ by a Gaussian integral. The product structure of an *LTE* state means that an observable at a point of Λ is independent of an observable at any other. The state μ can be written in Maxwell form

$$\mu = N_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{k}) = N_{\boldsymbol{x}} Z^{-1} \exp\left\{-\beta_{\boldsymbol{x}}\Phi(\boldsymbol{x}) - \beta_{\boldsymbol{x}}\boldsymbol{k} \cdot \boldsymbol{k}/(2m) - \boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{k}\right\}, \tag{7}$$

where

$$Z_x = \epsilon^{-3}\left(\frac{2\pi m}{\beta_{\boldsymbol{x}}}\right)^{3/2} \exp\left\{-\beta_{\boldsymbol{x}}\Phi(\boldsymbol{x}) + \frac{m\boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{\zeta}_{\boldsymbol{x}}}{2\beta_{\boldsymbol{x}}}\right\}. \tag{8}$$

We note the identity for each $\boldsymbol{x}$

$$\Xi = 1 + e^{-\xi} Z.$$

The external potential does not influence the local velocity distribution, as it is cancelled out by the partition function. The mean fields (4) are related to the canonical fields by

$$E_{\boldsymbol{x}} = -\frac{\partial}{\partial\beta_{\boldsymbol{x}}} \log \Xi_{\boldsymbol{x}} = N(\boldsymbol{x})\left(\Phi(\boldsymbol{x}) + \frac{3}{2\beta_{\boldsymbol{x}}} + \frac{m\boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{\zeta}_{\boldsymbol{x}}}{2\beta_{\boldsymbol{x}}^2}\right) \tag{9}$$

$$N_{\boldsymbol{x}} = -\frac{\partial}{\partial\xi_{\boldsymbol{x}}} \log \Xi_{\boldsymbol{x}} = \frac{\Xi_{\boldsymbol{x}} - 1}{\Xi_{\boldsymbol{x}}} = \frac{Ze^{-\xi\boldsymbol{x}}}{1 + Ze^{-\xi\boldsymbol{x}}} \tag{10}$$

$$\Pi^i_{\boldsymbol{x}} = -\frac{\partial}{\partial\zeta_i} \log \Xi_{\boldsymbol{x}} = -\frac{mN_{\boldsymbol{x}}\zeta^i_{\boldsymbol{x}}}{\beta_{\boldsymbol{x}}}. \tag{11}$$

The formalism breaks down if β is zero or infinity, or if N vanishes, but the case of a fluid at rest, $\mathbf{\Pi} = 0$, is within the information manifold, $\mathcal{M}$.

Historically, the intensive variables used in the N-S equations were the chemical potential $-\xi/\beta$, the velocity field $\boldsymbol{u} = -\boldsymbol{\zeta}/\beta$ and the temperature $\Theta = (k_{_B}\beta)^{-1}$. We shall eliminate ξ in favour of the mass-density $\rho = a^{-3}mN$ using (10), which leads to

$$e^{-\xi\boldsymbol{x}} = Z_{\boldsymbol{x}}^{-1} N_{\boldsymbol{x}} / \left(1 - N_{\boldsymbol{x}}\right). \tag{12}$$

The mean occupation per site $N_{\boldsymbol{x}}$ obeys $0 < N_{\boldsymbol{x}} < 1$. The (von Neumann) entropy of any state μ is

$$S(\mu) := -k_{_B} \sum_\omega \mu(\omega) \log \mu(\omega). \tag{13}$$

Gibbs knew that the state of maximum entropy, among all states with the given means of the total energy and number of particles, is the exponential state that he called the grand canonical state [23, 22, 25]. This is a simple exercise in Lagrange multipliers. If the mean fields depend on $\boldsymbol{x}$, then the state of maximum entropy has the same form, in which the canonical fields ξ, β and ζ now depend on $\boldsymbol{x}$.

In this section we study the system in equilibrium, and denote by E, N and $\mathbf{\Pi}$ the total values of the energy, number and momentum; then (13) gives for the entropy

$$\Theta S(\mu) = E + k_{_B}\Theta\xi N - \boldsymbol{u}\cdot\mathbf{\Pi} + k_{_B}\Theta\log\Xi. \tag{14}$$

Compare this with the thermostatic formula

$$\Theta S = E + k_{_B}\Theta\xi N - \boldsymbol{u}\cdot\mathbf{\Pi} + PV \tag{15}$$

(note that the term $\boldsymbol{u}\cdot\mathbf{\Pi}$ is omitted in [27], eq. (1.17)), where P is the pressure and V is the volume; we see that

$$P = k_{_B}\frac{|\Lambda|}{V}\Theta\log\Xi = k_{_B}\Theta a^{-3}\log\Xi. \tag{16}$$

If there are $N = \sum_{\boldsymbol{x}} N_{\boldsymbol{x}}$ particles, and V_0 is the smallest volume they can occupy (one per site), then $V_0 = a^3 N$ and $N_{\boldsymbol{x}} = V_0/V$. Also,

$$\Xi_{\boldsymbol{x}} = (1 - N_{\boldsymbol{x}})^{-1} = 1 + V_0/(V - V_0).$$

Thus at equilibrium, we have the equation of state

$$P = \frac{k_{_B}\Theta}{V_0}N\log\left(1 + \frac{V_0}{V - V_0}\right). \tag{17}$$

For small V_0/V this is close to the van der Waals gas

$$(P + A/V^2)(V - V_0) = Nk_{_B}\Theta \tag{18}$$

with $A = 0$. Unlike the case $A > 0$, this model shows no failure in convexity in its isothermals.

3 The Statistical Dynamics of the Gas

3.1 An Airfreight Model

Consider a gas of free particles in a box with reflecting walls, in equilibrium; then $\boldsymbol{u}$, Θ and ρ do not depend on $\boldsymbol{x}$. There is still a lot going on. In a volume d^3x around $\boldsymbol{x}$, a particle of momentum $\boldsymbol{k}$, which is present with probability $Np(\boldsymbol{k})$, moves in the direction of the unit vector $\hat{\boldsymbol{k}}$, to be replaced in time t by a particle with the same $\boldsymbol{k}$ arriving from the point $\boldsymbol{x} - \boldsymbol{k}t/m$. This replacement was present with exactly the same probability. The larger $\boldsymbol{k}$ is, the further away is the source of the replacement. In this picture, equilibrium is described by a huge game of musical chairs; only the indistinguishability of the particles prevents this from being detected. Now look at the same mechanism, but where ρ, $\mathbf{\Pi}$ and Θ depend on $\boldsymbol{x}$. That is, we now consider the Knudsen gas. There is no longer exact replacement of the lost particles at $\boldsymbol{x}$; the parameters $\rho\ldots$ change with time. It might seem that the system gets closer to equilibrium, since the parameters start to become more and more nearly constant. Of course, the entropy is constant in time, as the system is Hamiltonian, (free, even). The apparent increase in entropy associated with the slow variables is exactly matched by a reduction of entropy in inaccessible observables [4, 5]. Thus, taking the initial state to be in *LTE*, the random variables $\mathcal{N}\ldots$ had independent values at every point. But after some time,

the free motion introduces correlations between very far points; if $\boldsymbol{x}$ at time t has a particle with momentum $\boldsymbol{k}$, then $\boldsymbol{x} - \boldsymbol{k}t/m$ must have had a particle of momentum $\boldsymbol{k}$ at time 0; so it did not have a particle of momentum $\boldsymbol{k}' \neq \boldsymbol{k}$. Thus $\boldsymbol{x} + (\boldsymbol{k}' - \boldsymbol{k})t/m$ has no particle of momentum $\boldsymbol{k}'$ at time t, a statement correlated with the assumption, above, about $\boldsymbol{x}$ at time t. Correlations like these might at any time show up in behaviour quite unlike that of a system near equilibrium. For spin systems, the spin-echo effect is such an example [5]. In a real gas, we do not expect any surprises such as the spin-echo effect. This is due to the interactions, idealised by collisions, which remove correlations between distant points, and also help to redistribute energy and momentum. Collisions do not contribute to transport; on the contrary, they inhibit the free flow: the diffusion constant is inversely proportional to the collision cross-section.

The concept of whether a system is thermalised or not is independent of the Galilean frame of reference used in the description. This is expressed mathematically by the fact that the set of equilibrium states is mapped to itself by the group $\mathcal{G}$. The set $\mathcal{M}$ of states in *LTE* is also mapped to itself by $\mathcal{G}$, which we interpret as saying that the concept of partially thermalised systems is also invariant under $\mathcal{G}$. Physically, a gas consists of some (most) particles that are thermalised, and are described by an *LTE* state with means equal to the averages over the thermalised particles. A smaller number are not well described in this way; in particular, their correlations with other particles are underestimated by assuming *LTE*. Moreover, a particle that is thermalised at time t will move under its free motion to regions at different density and temperature, and so after some time (how short depends on the gradients) it will not be well described as being thermalised. On the other hand, particles left out of the count of thermalised particles make collisions during their relaxation time, and return to the thermal state. The mass, energy and momentum will on average be conserved, but a particle can leave the *LTE* state at one point, and another can return at another point, so the microscopic currents describing all particles may differ slightly from the currents of thermalised particles as described by the *LTE* state. This division of the state into thermalised particles and the rest differs from the division presented e.g. in [3], p. 160, where the *LTE* part of the state gives the same expectation values as the true state. On the contrary, in our division, the state $\overline{\mu}$ does not give the same means as the true state μ. A different division can be found in [18], p 229, where nF_1^σ is taken to be 'the density of particles that are not undergoing collisions at the given instant'. From now on, the variables $N, E, \boldsymbol{\Pi}$, etc., refer to the full state, and written with bars, they refer to the *LTE* state of the thermalised particles.

The dynamics of a classical gas of hard spheres is similar to that of an airfreight company, whose planes fly between airstrips arranged in a lattice Λ. In calculating the overall transport of goods, the company uses statistical methods; they have records only of the local averages, at each airstrip, of the number of planes, their velocities, and their kinetic energies, at time $t = 0$. Every plane is instructed to select a velocity from the Maxwell distribution at its airstrip, and to fly with this velocity until it meets another plane in its airspace. Both planes must then land very briefly, and record their presence, momentum and energy to the local computer. This recalculates the updated values of $N, \boldsymbol{\Pi}, E$ for this strip, and instructs them to take off with new velocities drawn from the updated Maxwellian. The problem is to find a theory which can predict the average transport of goods, $\mathcal{N}, \boldsymbol{\mathcal{P}}, \mathcal{E}$, without reading the local computers. To account for the transport of $N, \boldsymbol{\Pi}, E$, we must introduce accounting system non-local in space and time: we know that just after a landing and take-off, the distribution of velocities is Maxwellian. This simple fact leads us to the fundamental equation (44).

We want the dynamics to satisfy the second law of thermodynamics. This is ensured in a Markov chain if the transition matrix is bistochastic. Physically, if the gas has no velocity $\boldsymbol{u}$, the transition rate from $\boldsymbol{x}$ to $\boldsymbol{y}$ by a particle of momentum $\boldsymbol{k}$ is the same as the rate from $\boldsymbol{y}$ to $\boldsymbol{x}$ by a particle of momentum $-\boldsymbol{k}$. This expresses *time-reversal invariance* of the transition matrix. That is, if $\tau : \omega \mapsto \omega\tau$ is the time-reversal map on the sample space, we say a transition matrix T obeys

time reversal symmetry if

$$T(\omega|\omega') = T(\omega'\tau|\omega\tau) \qquad \text{for all } \omega, \omega' \in \Omega. \tag{19}$$

We note the following lemma, whose proof is simple:

Lemma 1 *Let T be a stochastic map obeying (19); then T is bistochastic.*

Our model dynamics will be given by a stochastic map obeying (19), and so, by the lemma, will be bistochastic, and so entropy-increasing.

3.2 The Hopping Rules

In this section, we give hopping rules for the case of zero external field, $\Phi = 0$. The dynamics will be invariant under $\mathcal{G}$. We start with a model in discrete space-time. The discrete dynamics will be given by specifying a hopping probability in one time-step. In the classical hard-sphere model, it is to be expected that on average, particles of different speed travel through the same amount of material before thermalising. Let $\ell(\boldsymbol{x}, \boldsymbol{k})$ denote the average distance travelled by a particle starting at $\boldsymbol{x}$ with momentum $\boldsymbol{k}$; to begin with, assume that $\boldsymbol{k}$ lies along one of the basis vectors of the lattice. ℓ is called the *mean free path*, and it generalises an idea going back to Clausius [16]; it is going to be the mean of a random distance r, the free path between collisions. We assume, as part of the model, that the particle thermalises on its first collision. The relaxation time t of a particular particle depends on its speed; for a particle travelling the free path r, t is the time taken, $rm/|\boldsymbol{k}|$. We therefore cannot choose a unique time-interval for the time step of all processes, and it seems difficult to construct a discrete-time stochastic process. We overcome this complication by noting the *rate* $|\boldsymbol{k}|/(rm)$ at which the process transfers mass, energy and momentum; we can then move to a continuous time process with the same rate. We shall work with t and its mean, t_ℓ, rather than with the free path r and its mean, ℓ; t_ℓ has the advantage of being the same in all inertial frames.

The dynamics must be such as to conserve the totals

$$\mathcal{N} := \sum_{\boldsymbol{x}\in\Lambda} \mathcal{N}_{\boldsymbol{x}}; \qquad \mathcal{E} := \sum_{\boldsymbol{x}\in\Lambda} \mathcal{E}_{\boldsymbol{x}}; \qquad \boldsymbol{\mathcal{P}} := \sum_{x\in\Lambda} \boldsymbol{\mathcal{P}}_{\boldsymbol{x}}.$$

These random variables divide Ω into simultaneous level sets, the mass-shells, energy-shells, and momentum shells, thus:

$$\Omega_{N,E,\boldsymbol{\Pi}} := \left\{ \omega \in \Omega : \mathcal{N}(\omega) = E, \mathcal{E}(\omega) = E, \boldsymbol{\mathcal{P}}(\omega) = \boldsymbol{\Pi} \right\}.$$

Clearly,

$$\Omega = \bigsqcup \Omega_{N,E,\boldsymbol{\Pi}}.$$

The dynamics, the Markov matrix T, must be chosen so that a point ω jumps to another point in the same shell. We cannot expect this to be given by a symmetric Markov transition matrix: the inverse process involves a change of sign for $\boldsymbol{k}$; however, we shall be able to construct a suitable bistochastic map. In fact, our Markov transition matrix T will be a convex mixture of permutations that move a configuration ω_1 to ω_2, where in ω_1 there is a particle at $\boldsymbol{x}$ and a hole at $\boldsymbol{y}$, and in ω_2 the opposite holds, with the mass, energy and momentum that was at $\boldsymbol{x}$ transported to $\boldsymbol{y}$. This move is only possible if all the points between $\boldsymbol{x}$ and $\boldsymbol{y}$ are empty. Moreover, to ensure that at the end of the flight the particle returns to the thermalised fold, the site one place past $\boldsymbol{y}$ must be occupied. We postulate that a particle moves in a straight line along empty sites until it meets a filled site; it then thermalises at the last empty site, $\boldsymbol{y}$ say, and dumps its mass, energy and

momentum there, which joins the mass, energy and momentum of the state $\mu_{\boldsymbol{y}}$. We postulate one such jump for each $r = sa$, where s is an integer, and for each momentum $\boldsymbol{k}$ at $\boldsymbol{x}$, and then for each $\boldsymbol{x}$.

Suppose that the site $\boldsymbol{x}$ is occupied, with momentum pointing in the direction of one of the basis vectors of the lattice, say $\boldsymbol{k} = |\boldsymbol{k}|\boldsymbol{e}$. The probability that the sites $\boldsymbol{x} + s'a\boldsymbol{e}$ be empty, $1 \leq s' \leq s$, and the site $\boldsymbol{x} + (s+1)a\boldsymbol{e}$ occupied, is

$$\prod_{s'=0}^{s} (1 - N_{\boldsymbol{x}+s'a\boldsymbol{e}})\, N_{\boldsymbol{x}+(s+1)a\boldsymbol{e}} \tag{20}$$

One can check that e.g. if $N_{\boldsymbol{y}} > 0$ is independent of $\boldsymbol{y}$ for large enough $|\boldsymbol{x} - \boldsymbol{y}|$, then

$$\sum_{s} \prod_{s'=0}^{s} (1 - N_{\boldsymbol{x}+s'a\boldsymbol{e}})\, N_{\boldsymbol{x}+(s+1)a\boldsymbol{e}} = 1. \tag{21}$$

This just expresses that with probability one, the number of holes adjacent to $\boldsymbol{x}$ on the line joining $\boldsymbol{x}$ to infinity along the direction $\boldsymbol{e}$ must be some integer. The product in (20) is a marginal probability of the state μ, and so is linear in μ; however, it is a polynomial of indefinite degree in the variables $N_{\boldsymbol{x}}$, in terms of which the equations of motion are to be written. In discussing the flow of mass, energy and momentum at the point $\boldsymbol{x}$, it is convenient to include this factor in the hopping probability, rather than in the initial state. We then get a Markov chain on the probability space of the two points at the ends of the path,

$$\Omega_{\boldsymbol{x}} \times \Omega_{\boldsymbol{y}}; \tag{22}$$

the transition probability depends on the state μ, but otherwise obeys the properties of a bistochastic map. This allows the continuum limit of (20) to be taken without leaving elementary probability theory. In this limit, we define the densities

$$\rho(\boldsymbol{x}) := ma^{-3} N_{\boldsymbol{x}}; \qquad\qquad \boldsymbol{\varpi} := a^{-3} \boldsymbol{\Pi}_{\boldsymbol{x}},$$

and get a damped exponential

$$\prod_{s'=0}^{s} (1 - N\boldsymbol{x} + s'a\boldsymbol{e}) \sim \exp\left\{ -\frac{a^2}{m} \int_0^r \rho(\boldsymbol{x} + r'\boldsymbol{e}) dr' \right\}, \qquad a \to 0.$$

We shall use an identity similar to (21) to find the mean free time, subject to two refinements. First we require that the site $\boldsymbol{x} + sa\hat{\boldsymbol{k}}$ be empty *at the time* $t(s')$ *that this point is reached* by our travelling particle. By the same argument, the free path is $r = sa$ if the site $\boldsymbol{x} + a(s+1)\boldsymbol{e}$ is occupied at the time at which the arriving particle reaches it. This refinement leads to a transition rate is non-local in the time; it will turn out that in the model we construct, we can replace these intermediate times by the current time with error of second order. With this done, the transition probability is invariant under time-reversal. It then follows from Lemma 1 that, by adjusting the stay-put probability so that the rows add up to one, we get a bistochastic map on the two-point sample space.

The second refinement comes from the requirement of $\mathcal{G}$-invariance in the continuum limit. We claim that the 'thermalised part' of a state should be a $\mathcal{G}$-invariant concept, and this will be achieved by (45). The Euler dynamics, which is close to the true dynamics, brings the state $\overline{\mu}$ out of *LTE* in any time interval. The part of the state not thermalised soon becomes thermalised by collisions, and it is this thermalisation that is involved in the dissipative part of the dynamics. It is thus essential that the non-thermalised part of the state should not be assumed to carry zero means

for the slow variables, since then their thermalisation would not cause any change in these means, and the dynamics would be non-dissipative. There is some ambiguity in the choice of splitting, because *all* the particles leaving $\boldsymbol{x}$ seem to be instantly dethermalised unless μ is at equilibrium. However, inasmuch as the gradients are small, some part of the thermal state at time t_0 might remain thermalised at time dt_0 later. How do we decide on how much? It is easy to agree about the rate at which particles thermalise at $\boldsymbol{x}$ at time t_0. We include those particles arriving at $\boldsymbol{x}$ having a momentum $\boldsymbol{k}$ and a relaxation time t, and originating at $\boldsymbol{x}-\boldsymbol{k}t/m$. The rate at which they transfer mass, momentum and energy is $1/t$ times the mass, momentum and energy they carry.

We note that the concept of thermalisation at a point $\boldsymbol{x}$ over a time-interval $(0,t)$ is not a $\mathcal{G}$-invariant concept. For suppose that in the inertial frame $\mathcal{O}$, various particles thermalise at $\boldsymbol{x}$ at times $0<t_1<\ldots<t_n<t$; then in an inertial frame $\mathcal{O}'$, moving relative to $\mathcal{O}$ with velocity $\boldsymbol{V}$, they will thermalise (at the same times) at the points $\boldsymbol{x}+\boldsymbol{V}t_i$. We therefore must consider the thermalisation (and dethermalisation) that occurs at t in a time-interval $(t,t+dt)$, for an infinitesimal dt; this has an invariant meaning. The number thermalising in any time interval of length dt is dt times the rate at which the thermalisation occurs.

The continuum analogue of (20) is a function $w(\boldsymbol{x},\boldsymbol{k},t_0;t)$ which is the probability density that a particle at $\boldsymbol{x}$ with momentum $\boldsymbol{k}$ will travel exactly a distance $r=|\boldsymbol{k}|t/m$ and then thermalise at $\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{k}t/m$ in the time interval (t_0+t,t_0+t+dt). Let W denote the probability density that a particle at $\boldsymbol{x}$ with momentum $\boldsymbol{k}$ has had no collision up to the point $\boldsymbol{y}$, and let $C(\boldsymbol{y},\boldsymbol{k},t+t_0)$ (for collisions) be the probability density that a particle at $\boldsymbol{y}$ with momentum $\boldsymbol{k}$ at time t_0+t will be thermalised in the tube of diameter a and length $dr=|\boldsymbol{k}|dt/m$. Then we have

$$w(\boldsymbol{x},\boldsymbol{k},t_0;t)=W(\boldsymbol{x},\boldsymbol{k},t_0;t)C(\boldsymbol{x}+\boldsymbol{k}t/m,t_0+t). \tag{23}$$

Then the analogue of (21) is

$$\int_0^\infty w(\boldsymbol{x},\boldsymbol{k},t_0;t)dt=1 \tag{24}$$

for all $\boldsymbol{x},\boldsymbol{k}$ and t_0. The mean relaxation time is defined to be

$$\int_0^\infty w(\boldsymbol{x},\boldsymbol{k},t_0;t)t\,dt=t_\ell(\boldsymbol{x},\boldsymbol{k},t_0). \tag{25}$$

We note that (24)

$$\int_0^\infty W(\boldsymbol{x},\boldsymbol{k},t_0;t)C(\boldsymbol{x}+\boldsymbol{k}t/m,\boldsymbol{k},t_0+t)dt=1 \tag{26}$$

can be solved in terms of C:

$$W(\boldsymbol{x},\boldsymbol{k},t_0;t)=\exp\left\{-\int_0^t C(\boldsymbol{x}+\boldsymbol{k}t_1/m,\boldsymbol{k},t_0+t_1)dt_1\right\}. \tag{27}$$

To see (27), differentiate to get

$$\frac{\partial W}{\partial t}=-WC,$$

and this is just minus the integrand in (26). Then one verifies (26):

$$\int_0^\infty W(\boldsymbol{x},\boldsymbol{k},t_0;t)C(\boldsymbol{x},\boldsymbol{k},t_0+t)=-\int_0^\infty\frac{\partial W}{\partial t}dt=-(W(\infty)-W(0))=1$$

assuming that ρ is bounded away from zero along the line $\boldsymbol{x}+\boldsymbol{k}t/m$; this is enough to ensure that the mean free path is finite. By construction, $w=-\partial_t W$ and another form for the mean free time is

$$t_\ell(\boldsymbol{x},\boldsymbol{k},t_0)=\int_0^\infty W(\boldsymbol{x},\boldsymbol{k},t_0;t)\,dt. \tag{28}$$

3.3 The Collision Function

We now find the collision function C appropriate for a hard-core gas. In the dynamics of the lattice model, particles hop from site to site with various rates. To implement invariance under $\mathcal{G}$, we extend the fields ρ, E, $\boldsymbol{\varpi}$ from Λ to its convex hull $\mathbf{R}^3$ as continuous piecewise linear functions, and consider a particle at $\boldsymbol{x}$ with momentum $\boldsymbol{k}$ not necessarily lying along a basis vector. Although the path of such a particle might not intersect Λ, we must assign a rotated version of w as the probability of a free time t, such that (24) holds. We can assume that the particle hops to the nearest site of Λ to $\boldsymbol{x}+\boldsymbol{k}t/m$. The continuum limit then makes sense, with densities replacing probabilities. Now divide the event, 'the free time is t' into the subevents 'the free time is t, and A collides with a particle B of momentum $\boldsymbol{q}$'. Let $w(\boldsymbol{x},\boldsymbol{k},\boldsymbol{q},t)$ be the probability density for this. Then

$$w(\boldsymbol{x},\boldsymbol{k},t) = \int d^3q\, w(\boldsymbol{x},\boldsymbol{k},\boldsymbol{q},t). \tag{29}$$

If t increases to $t+dt$, then the free path r increases to $r+dr$, where $dt = m\,dr/|\boldsymbol{k}|$. The probability that A meets B must be independent of the Galilean frame of reference. Consider the frame $\mathcal{O}'$ in which B is at rest. Let

$$\begin{aligned} (\boldsymbol{x},t) &\mapsto (\boldsymbol{x}',t') = (\boldsymbol{x}+\boldsymbol{q}t/m,t) \\ \boldsymbol{k} &\mapsto \boldsymbol{k}-\boldsymbol{q} \end{aligned} \tag{30}$$

be the Galilean transformation, and denote by $\rho', \boldsymbol{u}', \Theta', p'$ the *C-N-S-T* variables and the probability as observed in $\mathcal{O}'$. Then

$$\rho'(\boldsymbol{x}',t') = \rho(\boldsymbol{x},t) \tag{31}$$

$$\Theta'(\boldsymbol{x}',t') = \Theta(\boldsymbol{x},t) \tag{32}$$

$$\boldsymbol{u}'(\boldsymbol{x}',t') = \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{q}/m \tag{33}$$

$$p'(\boldsymbol{y}',\boldsymbol{q}',t') = p(\boldsymbol{y},\boldsymbol{q},t). \tag{34}$$

Then A has momentum $\boldsymbol{k}-\boldsymbol{q}$ and in time dt (which is the same in all Galilean frames) A sweeps out a region of volume $dV = \sigma|\boldsymbol{k}-\boldsymbol{q}|dt/m$, where σ is the cross-section. It meets a particle in this volume having momentum $\mathbf{0}$ with probability

$$m^{-1}dV\rho'(\boldsymbol{y}',t)p'(\boldsymbol{y}',\mathbf{0},t).$$

By invariance, this is also the probability of collision in the original frame, which is therefore

$$\sigma|\boldsymbol{k}-\boldsymbol{q}|(dt/m^2)\rho'(\boldsymbol{y}',t')p'(\boldsymbol{y}',\mathbf{0},t') = \sigma|\boldsymbol{k}-\boldsymbol{q}|(dt/m^2)\rho(\boldsymbol{y},t)p(\boldsymbol{y},\boldsymbol{q},t), \tag{35}$$

by (31) and (34). This suggests the choice of collision term

$$C(\boldsymbol{y},\boldsymbol{k},t_0+t) = \frac{\sigma}{m^2}\int d^3q|\boldsymbol{k}-\boldsymbol{q}|\rho(\boldsymbol{y},t+t_0)p(\boldsymbol{y},\boldsymbol{q},t_0+t). \tag{36}$$

Note that we include collisions in which the particle A is hit from behind by the particle B.

3.4 The mean free time

Our formula (36) for C gives for the mean free time

$$\begin{aligned} t_\ell(\boldsymbol{x},\boldsymbol{k}) = \int_0^\infty dt \exp\Big\{ &-\frac{\sigma}{m^2}\int_0^t dt_1\rho(\boldsymbol{x}+\boldsymbol{k}t_1/m,t_1) \\ &\int d^3q|\boldsymbol{k}-\boldsymbol{q}|p(\boldsymbol{x}+\boldsymbol{k}t_1/m,\boldsymbol{q},t_1)\Big\}. \end{aligned} \tag{37}$$

Comparing this with the the identity

$$t_\ell = \int_0^\infty dt\, e^{-t/t_\ell},$$

we see that if the integrand in the exponential in (37) had been independent of t_1, then we could have identified t_ℓ as

$$t_\ell = \frac{m^2}{\sigma}\left\{\rho(\boldsymbol{x})\int d^3q|\boldsymbol{k}-\boldsymbol{q}|p(\boldsymbol{x},\boldsymbol{q})\right\}^{-1}. \tag{38}$$

Since $e^{-16} \approx 10^{-7}$, only values of t less than $16t_\ell$ contribute significantly to the integral $\int ...dt$ in (37). By the mean-value theorem, we may write the argument of the exponential in (37) as $-t/t_2$, where t_2 is the expression (38) evaluated at $\boldsymbol{x}+\boldsymbol{k}t_3/m$ for the intermediate value $t_3 < 16t_\ell$. Then (given that $\partial t_\ell = O(t_\ell)$) the correction to (38) is $O(t_\ell^2)$, and so (38) can be taken as the mean free time. Now, $p = \overline{p} + O(t_\ell)$, so we may put $p = \overline{p}$ in (38). Let

$$R = \int d^3q|\boldsymbol{k}-\boldsymbol{q}|\overline{p}(\boldsymbol{x},\boldsymbol{q})/\overline{p}(\boldsymbol{x},\boldsymbol{k})$$

In the ratio R, the partition function cancels. The exponent in the Maxwellians is

$$\begin{aligned} -\frac{\beta}{2m}\boldsymbol{q}\cdot\boldsymbol{q} - \boldsymbol{q}\cdot\boldsymbol{\zeta} + \frac{\beta}{2m}\boldsymbol{k}\cdot\boldsymbol{k} \quad &+ \quad \boldsymbol{k}\cdot\boldsymbol{\zeta} = \\ -\frac{\beta}{2m}(\boldsymbol{q}-\boldsymbol{k})\cdot(\boldsymbol{q}-\boldsymbol{k}) \quad &- \quad (\boldsymbol{q}-\boldsymbol{k})\cdot((\beta/m)\boldsymbol{k}+\boldsymbol{\zeta}). \end{aligned}$$

We change the variables of integration to $\boldsymbol{q}-\boldsymbol{k}$, rewritten $\boldsymbol{q}$, to get

$$R = \int d^3q|\boldsymbol{q}|\exp\left\{-\frac{\beta\boldsymbol{q}\cdot\boldsymbol{q}}{2m} - \boldsymbol{q}\cdot\left(\frac{\beta}{m}\boldsymbol{k}+\boldsymbol{\zeta}\right)\right\}.$$

Put

$$\boldsymbol{\kappa} = \left(\frac{m}{\beta}\right)^{1/2}\left(\frac{\beta}{m}\boldsymbol{k}+\boldsymbol{\zeta}\right) = c^{-1}\left(\frac{\boldsymbol{k}}{m}-\boldsymbol{u}\right); \tag{39}$$

thus, $c\boldsymbol{\kappa}$ is the peculiar velocity [14], p. 27. Let $\boldsymbol{q}' = (\beta/m)^{1/2}\boldsymbol{q}$. Then

$$|\boldsymbol{q}|d^3q = (m/\beta)^2|\boldsymbol{q}'|\,d^3q'.$$

Dropping the dash, and choosing the q_3 axis along $\boldsymbol{\kappa}$, we get

$$\begin{aligned} R &= \int d^3q|\boldsymbol{q}|\frac{m^2}{\beta^2}\exp\left\{-\frac{1}{2}q^2 - \boldsymbol{q}\cdot\boldsymbol{\kappa}\right\} \\ &= \int_0^\infty q^3dq\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\varphi\frac{m^2}{\beta^2}\exp\left\{-\frac{1}{2}q^2 - q\kappa\cos\theta\right\} \\ &= 2\pi\int_0^\infty q^3\,dq\frac{m^2}{\beta^2}e^{-1/2q^2}(q\kappa)^{-1}\left(e^{q\kappa}-e^{-q\kappa}\right). \end{aligned}$$

Thus,

$$R = \frac{2\pi m^2}{\beta^2\kappa}e^{1/2\kappa^2}\left(I_2(-\kappa) - I_2(\kappa)\right)$$

where we use the functions I_n [17]

$$I_n(\kappa) = \int_0^\infty e^{-1/2(q+\kappa)^2}q^n dq.$$

According to [17],

$$I_n(-\kappa) - I_n(\kappa) = \frac{1}{2^{1/2}(\frac{1}{2})!} y_1(\kappa)$$

where

$$y_1(\kappa) = \kappa + \frac{1}{3!}(n-1)\kappa^3 + \frac{1}{5!}(n-1)(n-3)\kappa^5 \ldots .$$

Put

$$F(\kappa) = \kappa 2^{1/2}(\frac{1}{2})! \exp\{-\frac{1}{2}\kappa^2\} \left(y_1(\kappa)\right)^{-1}.$$

Then the mean free time is given by

$$\overline{\rho p} t_\ell = \frac{\beta^2}{2\pi\sigma} F(\kappa). \tag{40}$$

3.5 Galilean invariance

We now show that the total transition probability *rate* is invariant under $\mathcal{G}$. We have split up the time evolution into sub-processes, in one of which particle A at $\boldsymbol{x}$ with momentum $\boldsymbol{k}$ has a free path of length $r = sa$, and then collides with particle B of momentum $\boldsymbol{q}$ between r and $r + dr$. The rate of this process was taken to be the same as that of a process in which B was brought to rest by a change of inertial frame, and makes a collision between r' and $r' + dr'$, the positions as viewed by the observer moving with B. This is obviously necessary if the theory is to be $\mathcal{G}$- invariant. We now show that it is also sufficient: the *rate* of the one physical process, as viewed in two relatively moving frames, will be shown to be the same. The key is to remark that the time interval dt in which A collides with B after its free path is the same in all inertial frames, in contrast to the distance gone, the free path r and its increment dr, which depend on the speed of A.

Suppose that $\mathcal{O}, \mathcal{O}'$ are two inertial observers, with $\mathcal{O}'$ moving with velocity $-\boldsymbol{v}$ relative to $\mathcal{O}$, such that $t' = t$ and

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t. \tag{41}$$

The field variables as viewed by $\mathcal{O}'$ are p', $\rho' = ma^{-3}N'$, $\boldsymbol{k}'$, $\boldsymbol{q}'$, where

$$\begin{aligned} p'(\boldsymbol{x}', \boldsymbol{k}', t) &= p(\boldsymbol{x}, \boldsymbol{k}, t) \\ N'(\boldsymbol{x}', \boldsymbol{k}', t) &= N(\boldsymbol{x}, \boldsymbol{k}, t), \end{aligned} \tag{42}$$

but along a path of a moving particle, the probability densities w and w' must satisfy

$$w(\boldsymbol{x}, \boldsymbol{k}, \boldsymbol{q}, r)dr = w'(\boldsymbol{x}', \boldsymbol{k}', \boldsymbol{q}', r')dr'. \tag{43}$$

Here,

$$\boldsymbol{k}' = \boldsymbol{k} - m\boldsymbol{v} \qquad\qquad \boldsymbol{q}' = \boldsymbol{q} - m\boldsymbol{v}.$$

The point $\boldsymbol{x} + \hat{\boldsymbol{k}} r_1 = \boldsymbol{x} + \boldsymbol{k}t_1/m$ on the free path, is assigned the coordinate $\boldsymbol{x} + \boldsymbol{k}'t_1/m$ by $\mathcal{O}'$. The righthand side of (43) is calculated by $\mathcal{O}'$ using (36) and (27) to be

$$\exp\left\{-m^{-1}\sigma \int_{\boldsymbol{x}'}^{\boldsymbol{x}' + \hat{\boldsymbol{k}}' r'} \rho'(\boldsymbol{x}'(t_1), t_1) p'(\boldsymbol{x}'(t_1), \boldsymbol{q}', t_1) |\boldsymbol{k}' - \boldsymbol{q}'| \frac{dr_1'}{|\boldsymbol{k}'|}\right\}$$

$$N'(\boldsymbol{x}', 0) p'(\boldsymbol{x}', \boldsymbol{k}', 0) \sigma \rho' \left(\boldsymbol{x}' + \hat{\boldsymbol{k}}' r', t\right) |\boldsymbol{k}' - \boldsymbol{q}'| \, p'\left(\boldsymbol{x}'(t), \boldsymbol{q}', t\right) \frac{dr'}{m|\boldsymbol{k}|'}.$$

Here, $r_1' = |\boldsymbol{k}'|t_1/m$. Then, by using (42) and the remark that

$$\frac{dr_1'}{|\boldsymbol{k}'|} = \frac{dt_1}{m} = \frac{dr_1}{|\boldsymbol{k}|}, \qquad 0 \le r_1 \le r,$$

we see that $\mathcal{O}'$ and $\mathcal{O}$ assign the same probability to every event, so the integrals over $\boldsymbol{k}, \boldsymbol{q}$ are also the same.

4 Compressible Navier-Stokes with Temperature

4.1 The Fundamental Relation

The number of particles, thermalised plus unthermalised, is conserved *locally*; that is, in any region, however small, the loss in particles is the same as the integral of the current over the boundary. This local conservation law does not apply to the thermalised subset of particles. A particle can cease to be thermalised at $\boldsymbol{x}$ and collide at $\boldsymbol{y}$, thereby returning to the fold after a time in the unthermalised state. We now show how the total probability μ is related to the thermalised part, $\overline{\mu}$. At time t_0, any particle at $\boldsymbol{x}$ of momentum $\boldsymbol{k}$ must have been from a thermalised sample at some earlier time, $t_0 - t$, at the point $\boldsymbol{x} - \boldsymbol{k}t/m$, and remained unthermalised at $\boldsymbol{x}$, which it passes at time t_0. The probability of thermalising exactly at $\boldsymbol{x}$ is zero. It must thermalise at some later time, say after it has travelled for a free time $\tau = t'$. Then $t' > t$ must hold. We first compute the probability arising from a hop of fixed size $r' = |\boldsymbol{k}|t'/m$. The rate at which this occurs is

$$(1/t')w(\boldsymbol{x} - \boldsymbol{k}t/m, \boldsymbol{k}, t_0 - t, t').$$

In the interval of time from t to $t + dt$, the number of hops is *rate* $\times\, dt$, so the probability of a particle being at $\boldsymbol{x}$ at time t_0 with momentum $\boldsymbol{k}$, and having free time t', is, at time t_0,

$$\begin{aligned} P(t') &:= \mathrm{Prob}_\mu \{\omega : \mathcal{N}_{\boldsymbol{x}}(\omega) = 1 \cap \boldsymbol{\mathcal{P}}_{\boldsymbol{x}} = \boldsymbol{k} | \tau = t'\} = \\ &= \int_0^{t'} dt (t')^{-1} \overline{N}(\boldsymbol{x} - \boldsymbol{k}t/m, t_0 - t) \overline{p}(\boldsymbol{x} - \boldsymbol{k}t/m, \boldsymbol{k}, t_0 - t) \\ &\quad w(\boldsymbol{x} - \boldsymbol{k}t/m, \boldsymbol{k}, t_0 - t, t'). \end{aligned}$$

Now, w is a density of probability (of collision) as a function of free path size t', so the total contribution to μ due to hops from one side of $\boldsymbol{x}$ to the other along the line of $\boldsymbol{k}$ is

$$\begin{aligned} N(\boldsymbol{x}, t_0) p(\boldsymbol{x}, \boldsymbol{k}, t_0) &= \int_0^\infty P(t') dt' \\ &= \int_0^\infty \frac{dt'}{t'} \int_0^{t'} dt\, \overline{N}(\boldsymbol{x} - \boldsymbol{k}t/m, t_0 - t) \overline{p}(\boldsymbol{x} - \boldsymbol{k}t/m, \boldsymbol{k}, t_0 - t) \\ &\quad w(\boldsymbol{x} - \boldsymbol{k}t/m, \boldsymbol{k}, t_0 - t, t'). \end{aligned} \tag{44}$$

This is the fundamental relation. The logarithmic divergence at $t' = 0$ is only apparent, if the functions entering the integral are smooth enough. If so we can expand in Taylor series in t' up to $O(t_\ell)$ around the point $\boldsymbol{x}, \boldsymbol{k}, t_0$, at which the functions are evaluated:

$$\begin{aligned} Np &= \int_0^\infty \frac{dt'}{t'} \int_0^{t'} dt \left\{ \overline{N}\overline{p} w(t') - t(\boldsymbol{k} \cdot \boldsymbol{\partial}/m + \partial_0) \left(\overline{N}\overline{p} w(t') \right) \right\} \\ &= \overline{N}\overline{p} - \frac{1}{2} \left(\frac{\boldsymbol{k} \cdot \boldsymbol{\partial}}{m} + \partial_0 \right) \overline{N}\overline{p} t_\ell \end{aligned} \tag{45}$$

because of (24) and (25). Here, ∂_0 means $\partial/\partial t_0$. This is a $\mathcal{G}$-invariant splitting; for, the equation shows that Np and $\overline{N}\overline{p}$ differ by a quantity of order t_ℓ in smallness, so $\overline{N}\overline{p}$ transforms correctly up to first order. But t_ℓ is $\mathcal{G}$-invariant, and $\boldsymbol{k} \cdot \boldsymbol{\partial}/m + \partial_0$ is a $\mathcal{G}$-invariant operator (on fields that transform correctly), so $\overline{N}\overline{p}$ transforms correctly up to $O(t_\ell^2)$; and so on.

The second term in (45) is responsible for the dissipation. Putting in (40) for t_ℓ, we see that the density cancels; so the conductivity and viscosity of a gas are independent of the density. This is Maxwell's famous result.

Taking the expectation values of $\chi = \mathcal{N}, \boldsymbol{\mathcal{P}}$ or $\mathcal{E}$ gives us the relation between the mean of the thermalised particles and the true means of all the particles. In this, we need to evaluate ∂_0 applied

to the thermalised variables. Here we can assume that these obey Euler's equations, since these hold up to first order, and the operator ∂_0 acts only on small quantities. The integral over w in (44) acts as a smoothing operator, so we expect μ to be more regular than $\overline{\mu}$.

4.2 The Euler equations

The current of the conserved variable χ is

$$\boldsymbol{j}_\chi := \int d^3k\, Np\boldsymbol{\Upsilon}\chi, \qquad \text{where } \boldsymbol{\Upsilon} = \boldsymbol{\mathcal{P}}/m. \tag{46}$$

This gives us the dynamics

$$\frac{d\langle\chi\rangle}{dt} = -\partial_j \int d^3k \chi(\boldsymbol{k}) N(\boldsymbol{x}) p(\boldsymbol{x},\boldsymbol{k}) \frac{k_j}{m} = -\partial_j \langle \chi \frac{k_j}{m} \rangle. \tag{47}$$

These equations, for χ running over the slow variables m, $\boldsymbol{k}$ and $\boldsymbol{k}\cdot\boldsymbol{k}/2$, can replace *C-N-S-T* for dilute inert gases. Putting $\chi = m$ gives the usual 'equation of continuity'

$$\frac{d\rho}{dt} + \partial_j(u_j\rho) = 0, \tag{48}$$

which is exact. We shall show how to compute the other equations up to order t_ℓ for our choice of C. Our strategy is to use (45) in (47), allowing us to evaluate the righthand side in terms of the means in $\overline{\mu}$; we then use (45) again to rewrite this in terms of the true means.

The zero$^{\text{th}}$ order approximation to (45), namely $Np = \overline{N}\overline{p}$, can be put in (47) and computed exactly: we get the Euler equations; this is shown very smoothly by using the cumulant generating function, $\log Z$.

For the momentum, put $\chi = \mathcal{P}^i_{\boldsymbol{x}}$. Then we have for each $\boldsymbol{x}$,

$$\begin{aligned} \frac{\partial(Nu^i)}{\partial t} &= -\partial_j \left(N \mathbf{E}_{\overline{p}}[\Upsilon^i \Upsilon^j] \right) \\ &= -\partial_j \left(N(\langle \Upsilon^i \Upsilon^j \rangle_T + u^i u^j) \right) \end{aligned}$$

Since

$$\langle \mathcal{P}^i \mathcal{P}^j \rangle_T = \frac{\partial^2 \log Z}{\partial\zeta_i \partial\zeta_j} = m k_{_B} \Theta \delta_{ij},$$

we get the Euler equation for momentum conservation:

$$\frac{\partial}{\partial t}(\rho u^i) + \partial_j(\rho u^i u^j) + \partial_i(\rho k_{_B}\Theta/m) = 0. \tag{49}$$

Finally, for the energy, put $\chi = \mathcal{E}_{\boldsymbol{x}} = \boldsymbol{k}\cdot\boldsymbol{k}/(2m)$ in (47), which then becomes

$$\dot{E} = -\text{div}\left(m^{-1} N \mathbf{E}_{\overline{p}}[\mathcal{E}\boldsymbol{\mathcal{P}}] \right). \tag{50}$$

Now, for each $\boldsymbol{x}$,

$$\begin{aligned} \mathbf{E}_{\overline{p}}[\mathcal{E}\mathcal{P}^j] &= \langle \mathcal{E}\mathcal{P}^j \rangle_T + \langle \mathcal{E} \rangle \langle \mathcal{P}^j \rangle \\ &= \frac{\partial^2 \log Z}{\partial\beta\partial\zeta_j} + NEmu^j \\ &= -\frac{m\zeta_j}{\beta^2} + mNu^j \left(k_{_B}\Theta + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \right) \end{aligned}$$

from (8). Since

$$\zeta^j = -\beta u^j \qquad \text{and } E = N(3k_{_B}\Theta + m\boldsymbol{u}\cdot\boldsymbol{u})/2$$

we can collect terms to get the Euler equation for energy conservation:

$$\frac{\partial}{\partial t}\left(\rho(3k_{_B}\Theta/m + \boldsymbol{u}\cdot\boldsymbol{u})\right)/2 + \text{div }\left(\rho\boldsymbol{u}(5k_{_B}\Theta/m + \boldsymbol{u}\cdot\boldsymbol{u})/2\right) = 0. \tag{51}$$

The pressure appearing in the usual form of Euler's equations is here replaced by $\rho k_{_B}\Theta/m$, the pressure for a perfect gas. This differs from the static pressure (17) by terms which vanish in the low density limit. We shall use the Euler equations, which are first-order PDE in space and time, to relate ∂_0 to a first-order gradient. Let $D := \boldsymbol{u}\cdot\boldsymbol{\partial} + \partial_0$ be the Lagrange material derivative. Then we have

Lemma 2 *The Euler equations (48), (49) and (51) imply the short Eulers:*

$$D\rho = -\rho\partial_j u^j \tag{52}$$

$$Du^i = -k_{_B}\rho^{-1}\partial_i\left(\rho\Theta/m\right) \tag{53}$$

$$D\Theta = -\frac{2}{3}\Theta\partial_j u^j. \tag{54}$$

Proof. 'It does not seem necessary to reproduce the details of this proof; the mathematician will be able to construct them for himself, while the physicist will probably not wish to be detained over them' [24].
The $\mathcal{G}$-invariance of this form is obvious.

4.3 Calculations

It is the full state μ, rather than the thermalised part, $\overline{\mu}$, that is usually measured in experiments. For example, a measurement of density can be made by noting the absorption of a laser passing through the gas. The scattering of photons with particles makes no distinction between thermalised and non-thermalised particles. Again, one can measure the temperature by a probing thermometer, which would tend to thermalise any particles that struck it, whether they were thermalised before or not. However, the means in the thermalised state are much easier to calculate; the state $\overline{p}$ is Gaussian, and the fields are independent at different points (at the same time, say t_0). This enables us to relate the extensive to the intensive variables. For μ, the intensive variables have not even been defined yet.

In the Boltzmann equation, authors write the phase-space density as a product $N(\boldsymbol{x})f(\boldsymbol{x},\boldsymbol{k})$ with some hesitation, as $\mathcal{N}$ and $\boldsymbol{\mathcal{P}}$ are not independent random variables, even in the Maxwellian. Not to worry. For the general state μ we define

$$N(\boldsymbol{x}) = \text{Prob}_\mu\left\{\omega : \mathcal{N}(\boldsymbol{x}) = 1\right\}, \tag{55}$$

and $p(\boldsymbol{x},\boldsymbol{k})$ is the conditional probability

$$p(\boldsymbol{x},\boldsymbol{k}) = \text{Prob}_\mu\left\{\boldsymbol{\mathcal{P}}_{\boldsymbol{x}} = \boldsymbol{k} | \mathcal{N}(\boldsymbol{x}) = 1\right\}. \tag{56}$$

By Bayes's definition,

$$\mu_{\boldsymbol{x}}(\omega_{\boldsymbol{x}}) = N(\boldsymbol{x})p(\boldsymbol{x},\boldsymbol{k}). \tag{57}$$

Then we may define $\rho = mN/a^3$. The momentum density also has a definition in terms of μ, which does not assume that p is Maxwellian, by

$$\boldsymbol{\varpi} = \boldsymbol{\Pi}/a^3 = \mathbf{E}_\mu[\boldsymbol{\mathcal{P}}]/a^3.$$

We can then define the velocity field, without recourse to information geometry, by

$$\boldsymbol{u} = \boldsymbol{\varpi}/\rho,$$

provided that the density never vanishes. This is indeed so, as we see from the basic equation (44). Finally, we define the *thermal energy per unit mass*, e, to be given in terms of a 'temperature' Θ at each $\boldsymbol{x}$ by

$$e = \frac{3}{2m} k_{_B} \Theta := m^{-1} \mathbf{E}_p[\mathcal{E}_{\boldsymbol{x}}] - \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}. \tag{58}$$

This definition of temperature is $\mathcal{G}$-invariant, as it can also be written as $m\mathbf{E}_p[(k_i/m - u_i)(k_i/m - u_i)/2]$ or $mc^2\kappa_i\kappa_i/2$.

Suppose that we know the fields ρ, Θ and $\boldsymbol{u}$, at time=t_0, and thus also their space gradients. We can use (45) and the short Eulers to find the barred parameters of the Maxwellian $\overline{p}$, to first order in t_ℓ. When any expression is multiplied by t_ℓ, we are able to replace any thermalised parameters by the above unbarred parameters, or *vice versa*, with only a second order error. So the analysis reduces to linear algebra.

For any local random variable χ, slow or not, let

$$\langle \chi \rangle := \int d^3k\, m^{-1} \rho(\boldsymbol{x}) p(\boldsymbol{x}, \boldsymbol{k}) \chi(\boldsymbol{k}) \qquad \langle \overline{\chi} \rangle := \int d^3k\, \overline{\rho}(\boldsymbol{x}) \overline{p}(\boldsymbol{x}, \boldsymbol{k}) \chi(\boldsymbol{k}).$$

To help in the evaluation of various derivatives of integrals that occur here, we note that while $F(\kappa)$ is a complicated function of $\boldsymbol{u}$ and β, the integrals arising can be evaluated if we change the variable of integration from $\boldsymbol{k}$ to $\boldsymbol{\kappa}$, given in (39). We do this for each $\boldsymbol{x}$, and it is valid provided that we keep the derivatives $\boldsymbol{\partial}$ and ∂_0 to the left of the expression. We note that $d^3k = (m/\beta)^{3/2}\, d^3\kappa$. Then using (45) and (40) we have to order t_ℓ:

$$\begin{aligned}
\langle \chi \rangle &= \langle \overline{\chi} \rangle - \frac{1}{4\pi\sigma} \int d^3k \left(\frac{\boldsymbol{k} \cdot \boldsymbol{\partial}}{m} + \partial_0 \right) F(\kappa) \beta^2 \chi(\boldsymbol{k}) \\
&= \langle \overline{\chi} \rangle - \frac{1}{4\pi\sigma} \left(\boldsymbol{u} \cdot \boldsymbol{\partial} + \partial_0 \right) \int d^3k\, \beta^2 \chi(\boldsymbol{k}) - \frac{1}{4\pi\sigma} \int d^3k\, \boldsymbol{\kappa} \cdot \boldsymbol{\partial} c \beta^2 \chi(\boldsymbol{k}) \\
&= \langle \overline{\chi} \rangle - \frac{m}{4\pi\sigma} \left(\frac{m}{k_{_B}} \right)^{1/2} D\Theta^{-1/2} \int d^3\kappa F(\kappa) \chi[m(c\boldsymbol{\kappa} + \boldsymbol{u})] \\
&\quad - \frac{m}{4\pi\sigma} \partial_i \int d^3\kappa \kappa_i F(\kappa) \chi[m(c\boldsymbol{\kappa} + \boldsymbol{u})].
\end{aligned} \tag{59}$$

Let us put

$$\lambda_n := \frac{m}{\sigma} \left(\frac{m}{k_{_B}} \right)^{1/2} \int_0^\infty \kappa^{2n} F(\kappa) d\kappa, \qquad n = 1, 2, 3. \tag{60}$$

On putting $\chi = m$, the second small term is zero, as it is odd in $\boldsymbol{\kappa}$, and we get

$$\rho = \overline{\rho} - \lambda_1 m D \Theta^{-1/2}. \tag{61}$$

We now put $\chi = \boldsymbol{k}$, and $\hat{\boldsymbol{\kappa}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi)$. In the following calculations, we use that

$$\int \sin\theta\, d\theta\, d\varphi\, \hat{\kappa}_i \hat{\kappa}_j = \frac{4\pi}{3} \delta_{ij}. \tag{62}$$

Then (59) gives

$$\begin{aligned}
\varpi_i &= \overline{\varpi}_i - \frac{m^2}{4\pi\sigma} \int_0^\infty F(\kappa) \kappa^2 d\kappa \left(\frac{m}{k_{_B}} \right)^{1/2} \Theta^{-1/2} u_i \int \sin\theta\, d\theta\, d\varphi \\
&\quad - \frac{m^2}{4\pi\sigma} \left(\frac{k_{_B}}{m} \right)^{1/2} \partial_j \int \kappa^4 F(\kappa) d\kappa \int \sin\theta\, d\theta\, d\varphi\, \hat{\kappa}_i \hat{\kappa}_j \Theta^{1/2} \\
&= \overline{\varpi}_i - m\lambda_1 D \left(u_i \Theta^{-1/2} \right) - \frac{k_{_B}}{3} \lambda_2 \partial_i \Theta^{1/2}.
\end{aligned} \tag{63}$$

We now put $\chi = \boldsymbol{k}\cdot\boldsymbol{k}/(2m)$; we get

$$\begin{aligned} E &= \overline{E} - \frac{m^2}{8\pi\sigma}\left(\frac{m}{k_{_B}}\right)^{1/2} D\int F(\kappa) d^3\kappa \Theta^{-1/2}\left(c\kappa_j + u_j\right)\left(c\kappa_j + u_j\right) \\ &- \frac{m^2}{8\pi\sigma}\partial_i \int d^3\kappa\, \kappa_i F(\kappa)\left(c\kappa_j + u_j\right)\left(c\kappa_j + u_j\right), \text{ so} \\ E - \overline{E} &= -\frac{k_{_B}}{2}\lambda_2 D\Theta^{1/2} - \frac{m}{2}\lambda_1 D\left(\Theta^{-1/2}\boldsymbol{u}\cdot\boldsymbol{u}\right) - \frac{k_{_B}}{3}\lambda_2\partial_j\left(u_j\Theta^{1/2}\right). \end{aligned} \tag{64}$$

We can use these results to relate $\langle\chi\rangle$ and $\langle\chi\rangle$ for any polynomial. Let

$$\delta\chi = \langle\chi\rangle - \langle\overline{\chi}\rangle. \tag{65}$$

Then up to $O(t_\ell)$, δ is a derivation, and we have for example,

$$\delta(\rho\boldsymbol{u}\cdot\boldsymbol{u}) = 2u_j\delta(\rho u_j) - \boldsymbol{u}\cdot\boldsymbol{u}\delta\rho. \tag{66}$$

Lemma 3 *We have the relation*

$$\delta(\rho\Theta) = -\frac{2}{9}m\lambda_2\Theta^{1/2}\partial_j u_j - \frac{1}{3}m\lambda_2 D\Theta^{1/2} = -\frac{m\lambda_2}{9}\Theta^{1/2}\partial_j u_j = \frac{m\lambda_2}{3}D\Theta^{1/2}. \tag{67}$$

Proof.

$$\begin{aligned} \frac{3}{2}k_{_B}\delta(\rho\Theta/m) &= \frac{3}{2}\frac{k_{_B}}{m}(\rho\Theta - \overline{\rho}\overline{\Theta}) \\ &= \delta E - u_i\delta\varpi_i + u_i u_i\delta\rho/2 \\ &= -\frac{k_{_B}\lambda_2}{2}D\Theta^{1/2} - \frac{m\lambda_1}{2}D\left(\Theta^{-1/2}u_i u_i\right) - \frac{k_{_B}\lambda_2}{3}\partial_i\left(u_i\Theta^{1/2}\right) \\ &+ u_i m\lambda_1 D\left(u_i\Theta^{-1/2}\right) + \frac{mk_{_B}\lambda_2}{3}u_i\partial_i\Theta^{1/2} - \frac{m\lambda_1}{2}u_i u_i D\Theta^{-1/2} \\ &= -\frac{1}{2}k_{_B}\lambda_2 D\Theta^{1/2} - \frac{k_{_B}\lambda_2}{3}\Theta^{1/2}\partial_i u_i, \end{aligned}$$

making use of (61), (63) and (64). This gives the result, using Lemma (2).

4.4 Equations of Motion

The equations of motion express the rate of change of the slow variables (the conserved quantities) in terms of the mean of their microscopic currents; we use (47).

4.4.1 Viscosity Terms

The choice $\chi = k_i$ in (47) gives, from (59),

$$\begin{aligned} \frac{d\varpi_i}{dt} &= -\partial_j\int d^3k\frac{k_i k_j}{m}\rho p \\ &= \overline{F}_i + \frac{m^2}{4\pi\sigma}\partial_j D\Theta^{-1/2}\left(\frac{m}{k_{_B}}\right)^{1/2}\int\left(c\kappa_i + u_i\right)\left(c\kappa_j + u_j\right)F(\kappa)\,d^3\kappa \\ &+ \frac{m^2}{4\pi\sigma}\partial_j\partial_k\int d^3\kappa\,\kappa_k F(\kappa)\left(c\kappa_i + u_i\right)\left(c\kappa_j + u_j\right), \end{aligned}$$

where

$$\overline{F}_i = -\partial_j\left(\overline{\rho u}_i \overline{u}_j\right) - \partial_i\left(\overline{\rho} k_{_B} \overline{\Theta}/m\right) \tag{68}$$

is the righthand side of the (free) Euler equation in terms of $\overline{\mu}$. Noting that odd powers of $\boldsymbol{\kappa}$ integrate to zero, this reduces to

$$\begin{aligned}\frac{d\varpi_i}{dt} &= \overline{F}_i + \frac{k_{_B}}{3}\lambda_2\partial_i D\Theta^{1/2} + m\lambda_1\partial_j D\left(\Theta^{-1/2}u_iu_j\right)\\ &+ \frac{k_{_B}}{3}\lambda_2\partial_j\left(\partial_i u_j\Theta^{1/2} + \partial_j u_i\Theta^{1/2}\right).\end{aligned}$$

Note that derivatives act on all functions to their right. We get a special case of the Navier-Stokes equation by relating $\overline{F}_i$ to F_i:

$$\begin{aligned}\overline{F}_i - F_i &= \partial_j u_j\delta(\varpi_i) + \partial_j u_i\delta(\varpi_j) - \partial_j\left(u_iu_j\delta\rho\right) - \frac{2}{9}k_{_B}\lambda_2\partial_i\Theta^{1/2}\partial_j u_j - \frac{k_{_B}}{3}\lambda_2\partial_i D\Theta^{1/2}\\ &= \partial_j\left[u_j\left(-m\lambda_1 D(\Theta^{-1/2}u_i) - \frac{k_{_B}}{3}\lambda_2\partial_i\Theta^{1/2}\right)\right]\\ &+ \partial_j\left[u_i\left(-m\lambda_1 D(\Theta^{-1/2}u_j) - \frac{k_{_B}}{3}\lambda_2\partial_j\Theta^{1/2}\right)\right]\\ &+ \partial_j\left[u_iu_jm\lambda_1 D\Theta^{-1/2}\right] - \frac{2k_{_B}}{9}\lambda_2\partial_i\left(\Theta^{1/2}\partial_j u_j\right) - \frac{k_{_B}}{3}\lambda_2\partial_i D\Theta^{1/2}\end{aligned}$$

We can now get the final form for the equation of motion:

$$\begin{aligned}\frac{d\varpi_i}{dt} &= F_i - m\lambda_1\partial_j\left[u_jD\left(\Theta^{-1/2}u_i\right)\right] - \frac{k_{_B}}{3}\lambda_2\partial_j\left[u_j\partial_i\Theta^{1/2}\right]\\ &- m\lambda_1\partial_j\left[u_iD\left(\Theta^{-1/2}u_j\right)\right] - \frac{k_{_B}}{3}\lambda_2\partial_j\left[u_i\partial_j\Theta^{1/2}\right]\\ &+ m\lambda_1\partial_j\left[u_iu_jD\Theta^{-1/2}\right] - \frac{2k_{_B}}{9}\lambda_2\partial_i\left[\Theta^{1/2}\partial_j u_j\right] - \frac{k_{_B}}{3}\lambda_2\partial_i D\Theta^{1/2}\\ &+ \frac{k_{_B}}{3}\lambda_2\partial_i D\Theta^{1/2} + m\lambda_1\partial_j D\left(\Theta^{-1/2}u_iu_j\right)\\ &+ \frac{k_{_B}}{3}\lambda_2\partial_j\left[\partial_i\left(\Theta^{1/2}u_j\right) + \partial_j\left(\Theta^{1/2}u_i\right)\right]\\ &= F_i + \mu\partial_j\left[\Theta^{1/2}\left(\partial_j u_i + \partial_i u_j\right)\right] + \lambda\partial_i\left[\Theta^{1/2}\partial_j u_j\right].\end{aligned} \tag{69}$$

Here, $\mu = k_{_B}\lambda_2/3$ and $\lambda = -2k_{_B}\lambda_2/9$ in the notation of [15], except that we have the extra factor $\Theta^{1/2}$. The relation $\lambda + 2\mu/3 = 0$ holds, [35], so we get a very special case of the general equations, at the edge of possible values. It is believed that the Stokes relation only occurs when, as here, there is no interaction whatsoever between the particles outside the hard core.

It is interesting that λ_1 does not occur in the answer, and that all the terms in D cancel out without recourse to the short Euler equations, Lemma (2). The viscosity coefficient increases as $\Theta^{1/2}$ with temperature, like Enskog's and Chapman's prediction from Boltzmann's equation.

4.4.2 Equation of motion for the energy

Putting $\chi = k_ik_i/(2m)$ in (47) gives

$$\frac{dE}{dt} = \overline{F_{_E}} + \partial_j D[A] + \partial_j\partial_\ell[B], \tag{70}$$

where

$$\begin{aligned} A &= \frac{m^2\Theta^{-1/2}}{8\pi\sigma}\left(\frac{m}{k_{_B}}\right)^{1/2}\int d^3\kappa F(\kappa)\left(c\kappa_i+u_i\right)\left(c\kappa_i+u_i\right)\left(c\kappa_j+u_j\right) \\ &= \frac{m\lambda_1}{2}\Theta^{-1/2}u_iu_iu_j+\frac{5k_{_B}\lambda_2}{6}\Theta^{1/2}u_j, \end{aligned} \tag{71}$$

$$\begin{aligned} B &= \frac{m^2\Theta^{-1/2}}{8\pi\sigma}\left(\frac{m}{k_{_B}}\right)^{1/2}\int d^3\kappa F(\kappa)c\kappa_\ell\left(c\kappa_i+u_i\right)\left(c\kappa_i+u_i\right)\left(c\kappa_j+u_j\right) \\ &= \frac{k_{_B}^2\lambda_3}{6m}\delta_{j\ell}\Theta^{3/2}+\frac{k_{_B}\lambda_2}{3}\Theta^{1/2}u_ju_\ell+\frac{k_{_B}\lambda_2}{6}\delta_{j\ell}\Theta^{1/2}u_iu_i, \end{aligned} \tag{72}$$

and $F_{_E}$ is the Euler term

$$F_{_E}:=-\partial_j\left[\rho u_j\left(\frac{5k_{_B}}{2m}\Theta+\frac{1}{2}u_iu_i\right)\right].$$

Then $\overline{F_{_E}}-F_{_E}=-\delta F_{_E}$ has six terms:

$$\begin{aligned} -\delta F_{_E} &= \partial_j\left[\frac{5k_{_B}}{2m}\left(\Theta\delta\varpi_j+u_j\delta(\rho\Theta)-\Theta u_j\delta\rho\right)\right] \\ &+ \partial_j\left[\frac{1}{2}u_iu_i\delta\varpi_j+u_iu_j\delta\varpi_i-u_iu_iu_j\delta\rho\right] \\ &= -\partial_j\left[\frac{5k_{_B}}{2m}\Theta\left(m\lambda_1D(u_j\Theta^{-1/2})+\frac{k_{_B}}{3}\lambda_2\partial_j\Theta^{1/2}\right)\right]+\partial_j\left[\frac{5k_{_B}}{2m}u_j\left(\frac{m}{3}\lambda_2D\Theta^{1/2}\right)\right] \\ &+ \partial_j\left[\frac{5k_{_B}}{2m}u_j\Theta\left(m\lambda_1D\Theta^{-1/2}\right)\right]-\partial_j\left[\frac{1}{2}u_iu_i\left(m\lambda_1D(u_j\Theta^{-1/2})+\frac{k_{_B}}{3}\lambda_2\partial_j\Theta^{1/2}\right)\right] \\ &- \partial_j\left[u_iu_j\left(m\lambda_1D(u_i\Theta^{-1/2})+\frac{k_{_B}}{3}\lambda_2\partial_i\Theta^{1/2}\right)\right]+\partial_j\left[u_iu_iu_j\lambda_1mD\Theta^{-1/2}\right]. \end{aligned} \tag{73}$$

Collecting up A, B, and $-\delta F_{_E}$, using the same method of proof as in Lemma (2), the heat equation reduces to

$$\begin{aligned} \frac{dE}{dt} &= F_{_E}+\frac{k_{_B}^2}{m}\left(\frac{\lambda_3}{4}-\frac{5\lambda_2}{4}+\frac{5\lambda_1}{2}\right)\partial_j\left[\Theta^{1/2}\partial_j\Theta\right] \\ &+ \frac{5k_{_B}^2}{2m}\left(\lambda_1-\frac{\lambda_2}{3}\right)\partial_j\left[\Theta^{3/2}\partial_j\log\rho\right]-\frac{2k_{_B}}{9}\lambda_2\partial_j\left[\Theta^{1/2}u_j\partial_iu_i\right] \\ &+ \frac{k_{_B}}{3}\lambda_2\partial_j\left[\Theta^{1/2}u_i\partial_iu_j\right]+\frac{k_{_B}}{3}\lambda_2\partial_j\left[\Theta^{1/2}\partial_ju_iu_i/2\right]. \end{aligned} \tag{74}$$

The coefficient of the Fourier term is positive, since

$$\frac{5}{2}-\frac{5}{4}\kappa^2+\frac{1}{4}\kappa^4>0.$$

The new term involves the logarithmic derivative of the density, whose presence in a gas of a single component is denied in the literature. It means that a gradient in the density contributes to the heat current. This is the Dufour effect, also called the diffusive thermal effect. The sign of the term does not need to be definite. Some authors invoke Onsager symmetry to eliminate this term without having to evaluate it, since its Onsager dual, the Soret effect in the continuity equation for the mass, is absent. However Onsager duality is not true here, because the state p is not in *LTE* and the transition rate depends on $\boldsymbol{x}$. The present work does suggest that the effect should be looked for experimentally, in say Helium, but this is quite delicate since the Dufour effect is transitory, and becomes less pronounced, and is masked by heat conduction and convection, as time goes by. The other terms have appeared in the literature [15]. One can check that the system of equations is invariant under $\mathcal{G}$: 'in the tradition of British applied mathematics, it is not considered gentlemanly to press a colleague for a proof' (G. Pistone).

5 Conclusions

We have shown that the 'method of Maxwell' [29] can be made $\mathcal{G}$-invariant, and gives *C-N-S-T* with a Dufour term. The fluid equations we get are the following:

$$\frac{d\rho}{dt} = -\partial_j(\rho u_j) \tag{75}$$

$$\frac{d\varpi_i}{dt} = -\partial_j(\rho u_i u_j) - \partial_i P + \lambda\partial_i\left(\Theta^{1/2}\partial_j u_j\right) + \mu\partial_j\left[\Theta^{1/2}\left(\partial_j u_i + \partial_i u_j\right)\right] \tag{76}$$

$$\begin{aligned}\frac{d}{dt}[\rho(e + u_i u_i/2)] &= -\partial_j\left(\rho u_j(e + u_i u_i/2) + u_j P\right) + \lambda_4\partial_j\left(\Theta^{1/2}\partial_j\Theta\right) + \\ &+ \lambda_5\partial_j\left(\Theta^{3/2}\partial_j\log\rho\right) + \lambda\partial_i\left(\Theta^{1/2}u_i\partial_j u_j\right) + \\ &+ \mu\partial_j\left[\Theta^{1/2}u_i\left(\partial_j u_i + \partial_i u_j\right)\right].\end{aligned} \tag{77}$$

The transport coefficients are independent of density, as found by Maxwell. Our starting point is not the Boltzmann equation, but a non-local integral equation, (44). The presence of coefficient λ_5 is at variance with the results of Chapman and Cowling [14]. The Boltzmann equation suffers from the Hilbert paradox [5], II, p. 348. Namely, the state in Boltzmann's equation is parametrized by the initial distribution f, which is a general integrable function of six variables, whereas the hydrodynamic solutions are parametrized by five functions of three variables; the set of hydrodynamic solutions cannot describe the most general solution. To show that they nevertheless provide a good approximate solution is an extra burden if one uses the Boltzmann equation as the starting point. In [14] this question is discussed but not solved on p 120; the authors refer the reader to [18]. Chapman himself has said that reading his book is like 'chewing glass' [13]. The fact is that the Boltzmann distribution f is too detailed a description for an easy move to thermodynamical variables. Hilbert's paradox also shows up as follows. If the initial state happens to be in *LTE*, then the collision term in *BE* is zero, and (at that instant) the thermodynamic variables follow the Euler equation, and the instantaneous rate of entropy production is zero. However, fluids following the equations supposedly derived from the *BE* do not at any instant follow the Euler equations, or possess a zero rate of entropy production, except in the special states with Θ and $\boldsymbol{u}$ independent of $\boldsymbol{x}$. This paradox lasts a very brief time, after which the stirring due to the Euler convection spoils *LTE*. For consistency between the *BE* and the fluid equations, we are not allowed to choose an arbitrary initial state for the *BE*; the small deviations from *LTE* must be related to the fluxes of the theory [3], p. 160. Our approach avoids the paradox: an initial state μ can be *LTE*, and then we would modify (44) in the obvious way.

The point of view of the present paper differs from the usual one, such as [33]. We assume that just *after* a collision, which we prefer to call a thermalisation, the particle is well described as being in *LTE*, and almost independent of its neighbours. This is a good time 'to carve Nature at it joints' [2], p 341. It is during the free propagation that the state loses its *LTE* property, since then particles from regions of different density and temperature come together. They are likely to be independent because 'they have a different history' [34]. Concerning the other phase of the dynamics, our collision term involves the density at different space-time points, whereas the Boltzmann kernel is local. In the models in [33], several collisions are needed before a particle is close to thermalised. In the present model, a particle thermalised after one collision. Our result shows that the details of how many collisions are needed does not affect the qualitative results, though it changes the relationship between geometric cross-section and mean free path. It is difficult to believe that the simplification made here is responsible for the Dufour effect.

Our equations (77) are less singular than *C-N-S-T* since the presence of the Dufour term means that the symmetrised part of the operator ∂_0 has a principal symbol of full rank, at least in general

position. It might be that *C-N-S-T* has no global smooth solutions, or, even if it has some, it might be too hard for anyone to prove it. Whatever the case, it might be easier to show that (77) has smooth global solutions.

We can generalise in various ways. If there is an external potential, Φ, it does not affect the local state, because it cancels out in $\bar{\mu}$; however, it does affect the hopping rates, and thus appears in the equations of motion. In a paper [19] we find the equations of motion for a fluid moving in a potential, in a non-galilean model. The same method can be applied to the present model. It is possible to extend both models to the case of inter-particle potentials by following a suggestion of Biler and collaborators [7, 8, 9, 10, 11, 12, 30]. This gives a macroscopic dynamics in which the rate of change of energy at a point $\boldsymbol{x}$ is governed by the mean field of all the other particles. It seems unlikely that making the model more realistic by including interactions will exactly cancel out the prediction of the Dufour effect; thus this should be looked for in Helium or Argon.

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