Transient properties of many–server queues and related QBD's

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Abstract

The time $\tau(n)$ of first passage from queue length x to queue length $n > x$ in an MAP/M/c queue is considered. The mean and the Laplace transform is computed as solutions of systems of linear equations coming out by optional stopping of a martingale obtained as a stochastic integral of the exponential Wald martingale for Markov additive processes. Compared to existing techniques for QBD's, the approach has the advantage of being far more efficient for large n.

Keywords: birth–death process, buffer overflow, exponential martingale, first passage problem, heterogeneous servers, Kella–Whitt martingale, Laplace transform, Lévy process, MAP/M/c queue, Markov additive process, optional stopping.

1 Introduction

In this paper, we study transient (time–dependent or non–stationary) properties of the queue length process $\{Q(t)\}_{t\geq 0}$ in a M/M/c or, more generally, a MAP/M/c queue (MAP = Markovian arrival process, cf. [25], [26]). In particular, in a finite buffer system with buffer size n and $Q(0) < n$, we can interpret $\tau(n) = \inf\{t > 0 : Q(t) = n\}$ as the time of first buffer overflow. This motivates the interest in telecommunications, and accordingly random

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times of this type have been studied by many authors for a variety of queueing models.

The usual approach of the literature to this type of problems is asymptotics, stating in this case that for a stable ($\rho < 1$) queue (in our model $\rho = \beta/c\mu$ where $1/\mu$ is the mean of F) $\tau(n)$ can be normalized to have limiting exponential distributions as $n \to \infty$, whereas $\tau(n)/n \to 1/(\lambda - c\mu)$ when $\rho > 1$. The asymptotics for $\rho > 1$ is an easy consequence of the law of large numbers, whereas the case $\rho < 1$ is more delicate. Here one proceeds via the regenerative structure and decomposes $\tau(n)$ into the geometric sum of the cycles without overflow and the (asymptotically unimportant) initial part of the cycle with overflow.

Rare events in regenerative processes approached via geometric sums is a topic to which V. Kalashnikov has made profound contributions (see in particular [16], [17], [18]), and this is our motivation for presenting the present study in this volume[1]. However, rather than giving asymptotics (additional references are [19], [14], [20], [13], and the surveys in [2], [3]) we will instead derive algorithms for exact computation of the expected value and the Laplace transform of $\tau(n)$ in the specific setting of MAP/M/c queues.

The case $c = 1$ was investigated in [4] by exploiting optional stopping of martingales introduced in [22] and [6]. We develop the (non–trivial) extension of this approach to $c > 1$.

The paper is divided into two parts. The first part in Section 2–3 presents the adaptation of the martingales of [22], [6] to many–server queues. The main idea is exposed in Section 2 in the simplest (toy) setting of $\mathbb{E}_x \tau(n)$ in M/M/c. Section 3 then presents the extension to MAP/M/c. The results there are the main new ones of the paper.

The second part in Sections 4 and 5 discusses some alternative approaches. For the simplest M/M/c case, these are motivated from M/M/c being a special birth–death process, and classical results on level crossing times of birth–death processes as treated in [20]; the extension of this approach to level–dependent QBD's (quasi birth–death processes), of which the MAP/M/c queue length process is a special case, is given in [12]. The question we raise is whether level crossing problems for birth–death processes and QBD's can be put in a modern martingale framework; this largely inspired by the modern theory of diffusion processes (note that a birth–death process in view of its skip–free property can be seen as a discrete analogue of a diffusion), in particular the theory of the natural scale. Somewhat surprisingly, the conclusion is that there is a close analogue in the simple birth–death case but

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that the QBD case is more involved.

Finally, in Section 6 we compare our martingale approach and the recursive one of [12], and in Section 7 we exemplify the extension of our approach to heterogeneous servers in a simple setting.

Additional references on more or less related martingale techniques and their queueing applications are [8], [27], [29] and [28]. For different approaches to birth–death processes, see [9] and references there.

2 $\mathbb{E}_x \tau(n)$ in M/M/c

We consider the M/M/c queue with service intensity μ and write $\mathbb{E}_x = \mathbb{E}(\cdot|Q(0)=x)$. Let $(p_{ij})_{i,j\in\{0,1,\ldots\}}$ denote the transition matrix of the imbedded Markov chain, $p_{ij} = \beta/(\beta+i\mu)$ if $j=i+1$ and $p_{ij} = i\mu/(\beta+i\mu)$ if $j=i-1$ and zero otherwise. In order to conform to the setting of [22] we will write $Q(t) = x + X(t) + L(t)$ where $X(t)$ is a Lévy process defined as the independent difference between two Poisson processes $N^{(+)}(t)$, $N^{(-)}(t)$ with intensities β, resp. $c\mu$. Let $\{N_i(t)\}$, $1\le i\le c$, denote the Poisson process corresponding to the number of services performed by server i up to time t. It is then easy to see that we can use the representation

$$N^{(-)}(t) = N_1(t) + \cdots + N_c(t), \quad \mathrm{d}L(t) = \sum_{i=1}^{c-Q(t)} \mathrm{d}N_i(t)$$

(the epochs of $L(t)$ can be viewed as dummy service events performed by the idle servers). For easy reference, we state the following result from [22].

Let $\{\xi(t)\}$ be a Lévy process with Lévy exponent $\kappa(\alpha)$ and consider

$$\zeta(t) = \zeta(0) + \xi(t) + v(t)$$

where $\{v(t)\}$ is an adapted process of locally bounded variation with finitely many jumps in an interval. Then $\{M(t)\}$ defined by

$$M(t) = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha\zeta(s)}\mathrm{d}s \; + \mathrm{e}^{\alpha\zeta(0)} - \mathrm{e}^{\alpha\zeta(t)} + \alpha\int_0^t \mathrm{e}^{\alpha\zeta(s)}\mathrm{d}v^c(s) + \sum_{0\le s\le t} \mathrm{e}^{\alpha\zeta(s)}(1-\mathrm{e}^{-\alpha\Delta v(s)}) \tag{2.1}$$

is a zero mean martingale, where $\{v^c(t)\}$ is the continuous part of $\{v(t)\}$ and $\Delta v(s) = v(s) - v(s-)$. In $M(t)$, take $v(t) = L(t)$. Since $\{v(t)\}$ can increase when $0 \le Q(t) \le c-1$ and $v^c(t) = 0$, the martingale is

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha Q(s)}\mathrm{d}s + \mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha Q(t)} + (1-\mathrm{e}^{-\alpha})\sum_{i=0}^{c-1} \mathrm{e}^{\alpha i} L_i(t) \tag{2.2}$$

where $L_i(t)$ is the number of elements in the set

$$\mathcal{M}_i(t) = \{s \le t : Q(s) = i, L(s) \ne L(s-)\}$$

and $\kappa(\cdot)$ is the Lévy exponent of $\{X(t)\}$,

$$\kappa(\alpha) = \beta(\mathrm{e}^{\alpha} - 1) + c\mu(\mathrm{e}^{-\alpha} - 1).$$

We denote by γ the non–zero solution of $\kappa(\gamma) = 0$ which exists when $\beta/c\mu = \rho \ne 1$, $\gamma = -\log\rho$.

In the M/M/1 case in [4], taking $\alpha = \gamma$, optional stopping at $\tau(n)$ is justified according to [6] which may be used directly, since in the M/M/1 case $L(t)$ is just the local time at zero of the reflected Lévy process. We put $\alpha = \gamma$, apply optional stopping at $\tau(n)$ and get

$$0 = \mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma n} + \mathbb{E}_x L_0(\tau(n))(1 - \mathrm{e}^{-\gamma}). \tag{2.3}$$

From (2.3) and $\mathbb{E}_x X(\tau(n)) = (\beta - \mu)\mathbb{E}_x \tau(n)$ it follows that in the M/M/1 case

$$\mathbb{E}_x \tau(n) = \frac{(1 - \mathrm{e}^{-\gamma})(n - x) - (\mathrm{e}^{\gamma n} - \mathrm{e}^{\gamma x})}{(\beta - \mu)(1 - \mathrm{e}^{-\gamma})}.$$

In M/M/c, [6] does not directly apply so we have to first directly verify that optional stopping is justified; the details are, however, close to [6] but given in the Appendix for the sake of the paper to be self–contained. However, the more serious difficulty is that (2.3) takes the form

$$0 = \mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma n} + \mathbb{E}_x(1 - \mathrm{e}^{-\gamma}) \sum_{i=0}^{c-1} \mathrm{e}^{\gamma i} L_i(\tau(n)) \tag{2.4}$$

so that we have an additional $c-1$ unknowns $(\mathbb{E}_x L_1(\tau(n)), \ldots, \mathbb{E}_x L_{c-1}(\tau(n)))$, but no more equations. Define

$$\alpha_i = \mathbb{E}_x \int_0^{\tau(n)} I(Q(s) = i)\mathrm{d}s, \ i = 0, \ldots, c-1.$$

Then the representation $\mathrm{d}L(t) = \sum_{i=1}^{c-Q(t)} \mathrm{d}N_i(t)$ of the increments of $L(t)$ immediately gives us $\mathbb{E}_q L_i(\tau(n)) = (c - i)\mu\alpha_i$. We have:

Proposition 2.1 *The unknowns* α_i, $i = 1, \ldots, c-1$ *depend linearly upon* α_0, $\alpha_i = \alpha_i(\alpha_0) = r_i\alpha_0 + s_i$. *More precisely,* $r_0 = 1$, $s_0 = 0$,

$$r_1 = \beta/\mu, \ s_1 = -\delta_{0x}/\mu$$

and for $i = 2, \ldots, c-1$*:*

$$r_i = \frac{(\beta + (i-1)\mu)r_{i-1} - \beta r_{i-2}}{i\mu}, \ s_i = \frac{(\beta + (i-1)\mu)s_{i-1} - \beta s_{i-2} - \delta_{(i-1)x}}{i\mu}.$$

Proof. For $i = 0, \ldots, c-1$, let

$$\alpha'_i = \mathbb{E}_x \sum_{0<s\leq\tau(n)} I(Q(s-) \neq i, Q(s) = i) + \delta_{ix}.$$

By the skip–free property of paths of $\{Q(t)\}$ it follows that a jump to state i must have originated at either of the states $i-1$ and $i+1$. Thus conditioning upon the origin of a jump to i we have:

$$\begin{aligned}
\alpha'_i \;&=\; \delta_{ix} + \mathbb{E}_x \sum_{0<s\leq\tau(n)} I(Q(s-) = i-1, Q(s) = i) \\
&\quad + \mathbb{E}_x \sum_{0<s\leq\tau(n)} I(Q(s-) = i+1, Q(s) = i) \\
&= \delta_{ix} + \left(\mathbb{E}_x \sum_{0<s\leq\tau(n)} I(Q(s-) \neq i-1, Q(s) = i-1) + \delta_{(i-1)x} \right) p_{i-1,i} \\
&\quad + \left(\mathbb{E}_x \sum_{0<s\leq\tau(n)} I(Q(s-) \neq i+1, Q(s) = i+1) + \delta_{(i+1)x} \right) p_{i+1,i} \\
&= \delta_{ix} + \alpha'_{i+1} \frac{(i+1)\mu}{\beta + (i+1)\mu} + \alpha'_{i-1} \frac{\beta}{\beta + (i-1)\mu} I(i > 0),
\end{aligned}$$

which together with $\alpha_i = \alpha'_i/(\beta + i\mu)$ becomes

$$\alpha_i = \frac{(\beta + (i-1)\mu)\alpha_{i-1} - \delta_{(i-1)x} - I(i \geq 2)\beta\alpha_{i-2}}{i\mu}. \tag{2.5}$$

From (2.5) the desired result follows. □

Further:

Theorem 2.2 $\alpha_0 = (-\mathrm{e}^{\gamma x} + \mathrm{e}^{\gamma n} - s)/r$ *where*

$$s = (1 - \mathrm{e}^{-\gamma}) \sum_{i=0}^{c-1} \mathrm{e}^{\gamma i}(c-i)\mu s_i, \;\; r = (1 - \mathrm{e}^{-\gamma}) \sum_{i=0}^{c-1} \mathrm{e}^{\gamma i}(c-i)\mu r_i.$$

Proof. We apply optional stopping at $\tau(n)$ in (2.4). Having Proposition 2.1 in mind, we obtain

$$0 = \mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma n} + (1 - \mathrm{e}^{-\gamma}) \sum_{i=0}^{c-1} \mathrm{e}^{i\gamma}(c-i)\mu(r_i\alpha_0 + s_i)$$

as asserted. □

Corollary 2.3

$$\mathbb{E}_x\tau(n) = \frac{1}{\beta - c\mu}\left(n - x - \frac{-\mathrm{e}^{-\gamma x} + \mathrm{e}^{-\gamma n} - s}{r}\sum_{i=0}^{c-1}(c-i)\mu r_i + \sum_{i=0}^{c-1}(c-i)\mu s_i\right)$$

Proof. From the representation $Q(t) = x + X(t) + L(t)$, Wald's identity gives

$$n = x + (\beta - c\mu)\mathbb{E}_x\tau(n) + \sum_{i=0}^{c-1}(c-i)\mu\alpha_i.$$

If we apply Theorem 2.2 the desired result follows. □

3 $\tau(n)$ in MAP/M/c

Let $\{J(t)\}_{t\geq 0}$ denote the background Markov process, with p states, intensity matrix $\mathbf{\Lambda} = (\lambda_{ij})_{i,j\in\{1,\dots,p\}}$ and stationary row vector $\boldsymbol{\pi} = (\pi_i)_{i=1,\dots,p}$. We assume that the arrival rate is β_j and the service intensity is μ_j when $J(t) = j$. Furthermore, a transition from state k to state ℓ of $\{J(t)\}$ has probability $P_{k\ell}$ of causing an arrival to the system. We use the notation $\mathbf{\Delta}_{\boldsymbol{\mu}}$, $\mathbf{\Delta}_{\boldsymbol{\pi}}, \dots$ to denote the diagonal matrices with the μ_i, $\pi_i, \dots$ on the diagonal.

As in the previous section, we let $\{Q(t)\}_{t\geq 0}$ denote the queue length process. We have $Q(t) = x + X(t) + L(t)$, where $\{X(t)\}_{t\geq 0}$ is the independent difference between two (non–homogeneous) Poisson processes $N^{(+)}(t)$, $N^{(-)}(t)$ with intensities $\beta_{J(t)}$, resp. $c\mu_{J(t)}$. Define $N_i(t)$, $1 \leq i \leq c$, as the number of services performed by the ith server prior to time t. We use the representation

$$N^{(-)}(t) = N_1(t) + \cdots + N_c(t),\ \mathrm{d}L(t) = \sum_{k=1}^{c-Q(t)} \mathrm{d}N_k(t).$$

The mean drift of $\{X(t)\}$, $m = \lim_{t\to\infty} X(t)/t$ is

$$m = \sum_{i=1}^{p}\pi_i(\beta_i - c\mu_i) = \boldsymbol{\pi}(\mathbf{\Delta}_{\boldsymbol{\beta}} - c\mathbf{\Delta}_{\boldsymbol{\mu}})\mathbf{1}$$

where $\mathbf{1}$ is a column vector with all entries equal to one.

Let $\boldsymbol{F}(\alpha) = \mathbf{\Lambda} + \mathbf{\Delta}_{\boldsymbol{\beta}}(\mathrm{e}^{\alpha} - 1) + \mathbf{\Delta}_{\boldsymbol{\mu}}(\mathrm{e}^{-\alpha} - 1)$. We denote by $\kappa(\alpha)$ the eigenvalue of $\boldsymbol{F}(\alpha)$ with maximal real part and $\boldsymbol{h}^{(\alpha)}$ the corresponding right eigenvector. Let $\mathbf{1}_j$ denote the p–row vector with jth entry 1 and 0 otherwise. We shall need the following result, which follows by combining [22] and [5]:

Consider $\zeta(t) = \zeta(0) + \zeta(t) + v(t)$ where $\{v(t)\}$ is an adapted process of locally bounded variation with only finitely many jumps in a bounded interval. Then, for any $\alpha \in \mathbb{C}$,

$$\begin{aligned} &\int_0^t \mathrm{e}^{\alpha\zeta(s)} \mathbf{1}_{J(s)} \boldsymbol{F}(\alpha) \mathrm{d}s + \mathrm{e}^{\alpha\zeta(0)} \mathbf{1}_{J(0)} - \mathrm{e}^{\alpha\zeta(t)} \mathbf{1}_{J(t)} \\ +\alpha &\int_0^t \mathrm{e}^{\alpha\zeta(s)} \mathbf{1}_{J(s)} \mathrm{d}v^c(s) + \sum_{0 \le s \le t} \mathrm{e}^{\alpha\zeta(s)} \mathbf{1}_{J(s)} (1 - \mathrm{e}^{-\alpha \Delta v(s)}) \end{aligned} \tag{3.1}$$

is a zero mean martingale, where $\{v^c(t)\}$ is the continuous part of $\{v(t)\}$ and $\Delta v(s) = v(s) - v(s-)$.

By the form of $\boldsymbol{F}(\alpha)$, we have that

$$\det(\boldsymbol{F}(\alpha)) = \sum_{j=-p}^{p} f_j \mathrm{e}^{j\alpha},$$

from which it follows that $\boldsymbol{F}(\alpha)$ has (not necessarily distinct) $2p$ eigenvalues $\gamma_1, \ldots, \gamma_{2p}$ with corresponding eigenvectors $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(2p)}$. We use the normalization $\boldsymbol{\pi}\boldsymbol{h}^{(\alpha)} = 1$. Define

$$\boldsymbol{k} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \boldsymbol{h}^{(\alpha)} \bigg|_{\alpha=0}.$$

Then (see [4] for details), $\boldsymbol{k} = (\boldsymbol{\Lambda} - \mathbf{1})^{-1}(m\boldsymbol{I} + \boldsymbol{\Delta}_{\boldsymbol{\mu}} - \boldsymbol{\Delta}_{\boldsymbol{\lambda}})\mathbf{1}$.

3.1 $\mathbb{E}_{ax}\tau(n)$

Let $\mathbb{E}_{ax} = \mathbb{E}(\cdot | J(0) = a, Q(0) = x)$. Also, let

$$\mathcal{M}_{jk}(t) = \{s \le t : J(s) = j, Q(s) = k, L(s) \ne L(s-)\},$$

let $L_{jk}(t)$ be the number of elements in $\mathcal{M}_{jk}(t)$ and

$$\alpha_{jk} = \mathbb{E}_{ax} \int_0^{\tau(n)} I(J(s) = j, Q(s) = k) \mathrm{d}s.$$

Then $\mathbb{E}_{ax} L_{jk}(\tau(n)) = (c-k)\mu_j \alpha_{jk}$ and

Proposition 3.1 *The pc unknowns α_{jk} depend linearly upon the p α_{j0};*

$$\alpha_{jk} = \alpha_{jk}(\alpha_{10}, \ldots, \alpha_{p0}) = r_{jk1}\alpha_{10} + \cdots + r_{jkp}\alpha_{p0} + s_{jk}.$$

More precisely: Let

$$a_{jk} = \beta_j + k\mu_j - \lambda_{jj},\ \ b_{jk} = (\beta_j + (k-1)\mu_j - \lambda_{jj})/(\beta_j + k\mu_j - \lambda_{jj}).$$

Then $r_{j0k} = \delta_{jk},\ s_{j0} = 0,$

$$r_{j1j} = \frac{a_{j0}}{\mu_j},\ \ r_{j1\ell} = -\lambda_{\ell j}(1-P_{\ell j})\ \ \ell \neq j,\ \ s_{j1} = \delta_{(j,0)(a,x)},$$

$$\begin{aligned} r_{jkm} &= \frac{1}{k\mu_j}\big(a_{j(k-1)}r_{j(k-1)m} - \beta_j r_{j(k-2)m} \\ &- \sum_{\ell \neq j}\big(\lambda_{\ell j}((1-P_{\ell j})r_{\ell(k-1)m} + P_{\ell j} r_{\ell(k-2)m})\big)\big), \\ s_{jk} &= \frac{1}{k\mu_j}\big(a_{j(k-1)}s_{j(k-1)} - \beta_j s_{j(k-2)} \\ &- \sum_{\ell \neq j}\big(\lambda_{\ell j}((1-P_{\ell j})s_{\ell(k-1)} + P_{\ell j} s_{\ell(k-2)})\big) - \delta_{(j,k-1)(a,x)}\big). \end{aligned}$$

Proof. Let

$$\alpha'_{jk} = \delta_{(j,k)(a,x)} + \mathbb{E}_{ax} \sum_{0<s\leq\tau(n)} I((J(s-),Q(s-)) \neq (j,k), (J(s),Q(s)) = (j,k)).$$

By Wald's equality, $\alpha_{jk} = \alpha'_{jk}/(\beta_j + k\mu_j + \sum_{\ell\neq j}\lambda_{j\ell}) = \alpha'_{jk}/(\beta_j + k\mu_j - \lambda_{jj})$. Conditioning upon the origin of a jump to state jk we get:

$$\alpha'_{j0} = \delta_{(j,0)(a,x)} + \frac{\mu_j}{\beta_j + \mu_j - \lambda_{jj}}\alpha'_{j1} + \sum_{\ell\neq j}\frac{\lambda_{\ell j}}{\beta_\ell - \lambda_{\ell\ell}}(1-P_{\ell j})\alpha'_{\ell 0}$$

and for $1 \leq k \leq c-1$

$$\begin{aligned} \alpha'_{jk} &= \delta_{(j,k)(a,x)} + \frac{(k+1)\mu_j}{\beta_j + (k+1)\mu_j - \lambda_{jj}}\alpha'_{j(k+1)} \\ &+ \frac{\beta_j}{\beta_j + (k-1)\mu_j - \lambda_{jj}}\alpha'_{j(k-1)} \\ &+ \sum_{\ell\neq j}\frac{\lambda_{\ell j}}{\beta_\ell + k\mu_\ell - \lambda_{\ell\ell}}\left[(1-P_{\ell j})\alpha'_{\ell k} + P_{\ell j}\alpha'_{\ell(k-1)}\right] \end{aligned}$$

and it follows that

$$\mu_j\alpha_{j1} = a_{j0}\alpha_{j0} - \delta_{(j,0)(a,x)} - \sum_{\ell\neq j}\lambda_{\ell j}(1-P_{\ell j})\alpha_{\ell 0}$$

and for $k = 2, \dots, c-1$:

$$k\mu_j\alpha_{jk} = a_{j(k-1)}\alpha_{j(k-1)} - \delta_{(j,k-1)(a,x)} - \beta_j\alpha_{j(k-2)} \\ - \sum_{\ell\neq j}(\lambda_{\ell j}(1-P_{\ell j})\alpha_{\ell(k-1)} + \lambda_{\ell j}P_{\ell j}a_{\ell(k-1)}\alpha_{\ell(k-2)}),$$

which gives the assertion. □

We now form an equivalent of Theorem 2.2.

Theorem 3.2 *Define $q_i = \mathbb{P}_{ax}(J(\tau(n)) = i)$. Then $q_1, \dots, q_p$, $\alpha_{10}, \dots, \alpha_{p0}$ can be computed as the solution of the $2p$ linear equations*

$$\mathrm{e}^{\gamma_i n}\sum_{j=1}^{p} h_j^{(i)} q_j = \\ \mathrm{e}^{\gamma_i x} h_a^{(i)}(1-\mathrm{e}^{-\gamma_i})\sum_{j=1}^{p}\mu_j h_j^{(i)}\sum_{k=0}^{c-1}\mathrm{e}^{\gamma_i k}(c-k)\left(\sum_{\ell=1}^{p} r_{jk\ell}\alpha_{\ell 0} + s_{jk}\right). \quad (3.2)$$

Furthermore,

$$\mathbb{E}_{ax}\tau(n) = \frac{\mathbb{E}_{ax}X(\tau(n))}{m} = \frac{n-x-\ell-c}{m} \quad (3.3)$$

where

$$\ell = \mathbb{E}_{ax}L(\tau(n)) = \sum_{i=0}^{c-1}(c-i)\sum_{j=1}^{p}\mu_j\alpha_{ji}, \;\; c = k_a - \mathbb{E}_{ax}k_{J(\tau(n))} = k_a - \sum_{i=1}^{p}k_i q_i.$$

Proof. In the martingale (3.1), take $\zeta(t) = X(t)$ and $\upsilon(t) = L(t)$. Now $\{\upsilon(t)\}$ can increase when $0 \le Q(t) \le c-1$ and the martingale is

$$\int_0^t \mathrm{e}^{\alpha Q(s)}\mathbf{1}_{J(s)}\boldsymbol{F}(\alpha)\mathrm{d}s + \mathrm{e}^{\alpha x}\mathbf{1}_a - \mathrm{e}^{\alpha Q(t)}\mathbf{1}_{J(t)} + (1-\mathrm{e}^{-\alpha})\sum_{j=1}^{p}\mathbf{1}_j\sum_{k=0}^{c-1}\mathrm{e}^{\alpha k}L_{jk}(t).$$

For $i = 1, \dots, 2p$, we put $\alpha = \gamma_i$, multiply $\boldsymbol{h}^{(i)}$ to the right and get a new martingale

$$\mathrm{e}^{\gamma_i x}h_a^{(i)} - \mathrm{e}^{\gamma_i Q(t)}h_{J(t)}^{(i)} + (1-\mathrm{e}^{-\gamma_i})\sum_{j=1}^{p}h_j^{(i)}\sum_{k=0}^{c-1}\mathrm{e}^{\gamma_i k}L_{jk}(t).$$

By Theorem 3.1, we have $\alpha_{jk} = s_{jk} + \sum_{\ell=1}^{p} r_{jk\ell}\alpha_{\ell 0}$ and optional stopping at $\tau(n)$ gives us

$$0 = \mathrm{e}^{\gamma_i x} h_a^{(i)} - \mathrm{e}^{\gamma_i n}\mathbb{E}_{ax} h_{J(\tau(n))}^{(i)} + (1-\mathrm{e}^{-\gamma_i})\sum_{j=1}^{p}\mu_j h_j^{(i)}\sum_{k=0}^{c-1}\mathrm{e}^{\gamma_i k}(c-k)\left(\sum_{\ell=1}^{p} r_{jk\ell}\alpha_{\ell 0} + s_{jk}\right)$$

which is precisely (3.2). By Wald's identity for Markov additive processes (see e.g. [11])

$$\mathbb{E}_{ax} X(\tau(n)) = m\mathbb{E}_{ax}\tau(n) + \mathbb{E}_{ax} k_{J(0)} - \mathbb{E}_{ax} k_{J(\tau(n))} = m\mathbb{E}_{ax}\tau(n) + k_a - \mathbb{E}_{ax} k_{J(\tau(n))}$$

which together with

$$n = Q(\tau(n)) = x + X(\tau(n)) + L(\tau(n))$$

proves (3.3). □

3.2 $\mathbb{E}_{ax}\mathrm{e}^{\theta\tau(n)}$ in MAP/M/c

Define

$$\mathcal{A}'_{jk}(t) = \{0 < s \le t : (J(s-), Q(s-)) \ne (j,k),\ (J(s), Q(s)) = (j,k)\},$$

$$\mathcal{A}_{jk}(t) = \mathcal{A}'_{jk}(t) \text{ if } (j,k) \ne (a,x)$$

and

$$\mathcal{A}_{ax}(t) = \mathcal{A}'_{ax}(t) \cup \{0\}.$$

Then

Proposition 3.3 $\mathbb{E}_{ax}\sum_{t\in\mathcal{M}_{jk}(\tau(n))}\mathrm{e}^{\theta t} = \beta_{jk}(\theta)\mathbb{E}_{ax}\sum_{s\in\mathcal{A}_{jk}(\tau(n))}\mathrm{e}^{\theta s}$ *where*

$$\beta_{jk} = \frac{(c-k)\mu_j}{\beta_j + k\mu_j - \lambda_{jj} - \theta}$$

if $\Re\theta < \beta_j + k\mu_j - \lambda_{jj}$ *and* $+\infty$ *otherwise.*

Proof. If state (j,k) is entered at time t_x and we let $S = \{t_x \le s \le t_x + T_{jk} : L(s-) \ne L(s)\}$ where T_{jk} is the sojourn time in state (j,k), then

$$\mathbb{E}_{ax}\sum_{s\in S}\mathrm{e}^{\theta s} = \mathbb{E}_{ax}\mathrm{e}^{\theta t_x}\mathbb{E}_{jk}\sum_{s\in S-t_x}\mathrm{e}^{\theta s}.$$

Since $\mathrm{d}L(t) = (c-k)\mu_j \mathrm{d}t$, every instant v contributes to $\mathbb{E}_{jk} \sum_{s \in S - t_x} \mathrm{e}^{\theta s}$ by $\mathrm{e}^{\theta v}(c-k)\mu_j \mathrm{e}^{-(\lambda + k\mu - \lambda_{jj})v}\mathrm{d}v$ since, in order for an event at time v to make a contribution, we have to have $T_{jk} > v$ ($\mathbb{P}_{jk}(T_{jk} > v) = \mathrm{e}^{-(\lambda + k\mu_j - \lambda_{jj})v}$). Integration over all positive vs yields

$$\mathbb{E}_{jk} \sum_{s \in S - t_x} \mathrm{e}^{\theta s} = \int_0^\infty (c-k)\mu_j \mathrm{e}^{-(\lambda + k\mu_j - \lambda_{jj} - \theta)v} \mathrm{d}v = \frac{(c-k)\mu_j}{\lambda + k\mu_j - \lambda_{jj} - \theta}.$$

□

We put $\eta_{jk}(\theta) = \mathbb{E}_{ax} \sum_{s \in \mathcal{A}_{jk}(\tau(n))} \mathrm{e}^{\theta s}, \; 1 \le j \le p, \; 0 \le k \le c-1$.

Proposition 3.4 *For all θ, the pc unknowns $\eta_{jk}(\theta)$ depend linearly upon the p η_{j0};*

$$\eta_{jk} = \eta_{jk}(\eta_{10}, \dots, \eta_{p0}) = m_{jk1}\eta_{10} + \dots + m_{jkp}\eta_{p0} + n_{jk}.$$

More precisely: Define for θ with $\Re\theta < \min_1^p(\beta_i - \lambda_{ii})$,

$$c_{jk} = \frac{k\mu_j}{\beta_j + k\mu_j - \lambda_{jj} - \theta}, \; d_{jk} = \frac{\beta_j}{\beta_j + k\mu_j - \lambda_{jj} - \theta}$$

$$\text{and } e_{jk\ell} = \frac{\lambda_{\ell j}}{\beta_\ell + k\mu_j - \lambda_{\ell\ell} - \theta}.$$

Then

$$m_{j0k} = \delta_{jk}, \; s_{j0} = 0,$$

$$m_{j1j} = \frac{1}{c_{j1}}, \; m_{j1\ell} = \frac{-d_{\ell j}(1 - P_{\ell j})}{c_{j1}} \; \ell \ne j, \; n_{j1} = \frac{-\delta_{(j,0)(a,x)}}{c_{j1}},$$

$$m_{jkm} = \frac{1}{c_{jk}}\Big(m_{j(k-1)m} - d_{j(k-2)}m_{k(k-2)m}$$
$$- \sum_{\ell \ne j}\big(e_{j(k-1)\ell}(1 - P_{\ell j})m_{\ell(k-1)m} + e_{j(k-2)\ell}P_{\ell j}m_{\ell(k-2)m}\big)\Big)$$

$$n_{jk} = \frac{1}{c_{jk}}\Big(n_{j(k-1)} - d_{j(k-2)}n_{j(k-2)}$$
$$- \sum_{\ell \ne j}\big(e_{j(k-1)\ell}(1 - P_{\ell j})n_{\ell(k-1)} + e_{j(k-2)\ell}P_{\ell j}n_{\ell(k-2)}\big) - \delta_{(j,k)(a,x)}\Big).$$

Proof. Let

$$\begin{aligned}
A_{jk} &= \{(J(s+T_{jk}), Q(s+T_{jk})) = (j, k-1)\} \\
B_{jk} &= \{(J(s+T_{jk}), Q(s+T_{jk})) = (j, k+1)\} \\
C_{jk\ell} &= \{(J(s+T_{jk}), Q(s+T_{jk})) = (\ell, k)\} \\
D_{jk\ell} &= \{(J(s+T_{jk}), Q(s+T_{jk})) = (\ell, k+1)\}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{s\in\mathcal{A}_{jk}(\tau(n))} \mathrm{e}^{\theta s} \quad = & \\
& \delta_{(j,k)(a,x)} + \sum_{s\in\mathcal{A}_{j(k+1)}(\tau(n))} \mathrm{e}^{\theta(s+T_{j(k+1)})} I(A_{j(k+1)}) \\
& + I(k\neq 0) \sum_{s\in\mathcal{A}_{(i-1)j}(\tau(n))} \mathrm{e}^{\theta(s+T_{j(k-1)})} I(B_{j(k-1)}) \\
& + \sum_{\ell\neq j}\left[\sum_{s\in\mathcal{A}_{\ell k}(\tau(n))} \mathrm{e}^{\theta(s+T_{\ell k})} I(C_{\ell kj})\right] \\
& + I(k\neq 0)\sum_{\ell\neq j}\left[\sum_{s\in\mathcal{A}_{\ell(k-1)}(\tau(n))} \mathrm{e}^{\theta(s+T_{\ell(k-1)})} I(D_{\ell kj})\right].
\end{aligned}$$

If we take expectations of both sides we get

$$\begin{aligned}
\eta_{jk} = \delta_{(j,k)(a,x)} &+ \eta_{j(k+1)}\mathbb{E}_{j(k+1)}\mathrm{e}^{\theta T_{j(k+1)}} p_{j(k+1),jk} \\
&+ I(k\neq 0)\eta_{j(k-1)}\mathbb{E}_{j(k-1)}\mathrm{e}^{\theta T_{j(k-1)}} p_{j(k-1),jk} \\
+ \sum_{\ell\neq j} &\left[\eta_{\ell k}\mathbb{E}_{\ell k}\mathrm{e}^{\theta T_{\ell k}} p_{\ell k,jk} + I(k\neq 0)\eta_{\ell(k-1)}\mathbb{E}_{\ell(k-1)}\mathrm{e}^{\theta T_{\ell(k-1)}} p_{\ell(k-1),jk}\right]
\end{aligned}$$

where

$$\begin{aligned}
p_{j(k-1)j,jk} &= \beta_j/(\beta_j+(k-1)\mu_j-\lambda_{jj}) \\
p_{j(k+1),jk} &= (k+1)\mu_j/(\beta_j+(k+1)\mu_j-\lambda_{jj}) \\
p_{\ell k,jk} &= \lambda_{\ell j}(1-P_{\ell j})/(\beta_\ell+k\mu_\ell-\lambda_{\ell\ell}) \\
p_{\ell(k-1),jk} &= \lambda_{\ell j}P_{\ell j}/(\beta_\ell+(k-1)\mu_\ell-\lambda_{\ell\ell}) \\
\mathbb{E}_{jk}\mathrm{e}^{\theta T_{jk}} &= (\beta_j+k\mu_j-\lambda_{jj})/(\beta_j+k\mu_j-\lambda_{jj}-\theta).
\end{aligned}$$

We get

$$\begin{aligned}
\eta_{jk} \quad = & \\
& \delta_{(j,k)(a,x)} + \eta_{j(k+1)}\frac{(k+1)\mu_j}{\beta_j+(k+1)\mu_j-\lambda_{jj}-\theta} \\
& + I(k\neq 0)\eta_{j(k-1)}\frac{\beta_j}{\beta_j+(k-1)\mu_j-\lambda_{jj}-\theta} \\
& + \sum_{\ell\neq j}\eta_{\ell k}\frac{\lambda_{\ell j}(1-P_{\ell j})}{\beta_\ell+k\mu_\ell-\lambda_{\ell\ell}-\theta} \\
& + I(k\neq 0)\sum_{\ell\neq j}\eta_{\ell(k-1)}\frac{\lambda_{\ell j}P_{\ell j}}{\beta_\ell+(k-1)\mu_\ell-\lambda_{\ell\ell}-\theta}.
\end{aligned}$$

If we let $k+1 \to k$ we get

$$c_{j1}\eta_{j1} = \eta_{j0} - \delta_{(j,0)(a,x)} - \sum_{\ell \neq j} d_{\ell j}(1-P_{\ell j})\eta_{\ell 0}$$

and for $k = 2, \ldots, c-1$:

$$c_{jk}\eta_{jk} = \eta_{j(k-1)} - \delta_{(j,k)(a,x)} - d_{j(k-2)}\eta_{j(k-2)} - \sum_{\ell \neq j}(e_{j(k-1)\ell}(1-P_{\ell j})\eta_{\ell(k-1)} + e_{j(k-2)\ell}P_{\ell j}\eta_{\ell(k-2)})$$

and we obtain the assertion. □

Theorem 3.5 *For each $\theta \leq 0$, define $\gamma_1(\theta), \ldots, \gamma_{2p}(\theta)$ as the roots of the equation $0 = \det(\boldsymbol{F}(\gamma) + \theta \boldsymbol{I})$ and let $\boldsymbol{h}^{i;\theta}$ be a non–zero column vector satisfying $(\boldsymbol{F}(\gamma) + \theta \boldsymbol{I})\boldsymbol{h}^{i;\theta} = \boldsymbol{0}$. Then one can compute $x = \mathbb{E}_{ax}\mathrm{e}^{\theta\tau(n)}$ as $x = x_1 + \cdots + x_p$ by solving the $2p$ linear equations (here $x_1, \ldots, x_p$ and $\eta_{10}, \ldots, \eta_{p0}$ are unknowns)*

$$\mathrm{e}^{\gamma_i x} h_a^{i;\theta} = \mathrm{e}^{\gamma_i n} \sum_{j=1}^{p} x_j h_j^{i;\theta} - (1-\mathrm{e}^{-\gamma_i}) \sum_{j=1}^{p} h_j^{i;\theta} \sum_{k=0}^{c-1} \mathrm{e}^{\gamma_i k} \beta_{jk}(\theta) \left(\sum_{\ell=1}^{p} m_{jk\ell}\eta_{\ell 0} + n_{jk} \right)$$

Proof. In the martingale (3.1), take $\xi(t) = X(t)$, $\zeta(0) = x$ and $v(t) = L(t) + t\theta/\alpha$. Then $\alpha\zeta(t) = \alpha Q(t) + \theta t$ and in particular $\alpha\zeta(\tau(n)) = \alpha n + \theta\tau(n)$. Also, we have added a continuous part to the martingale, and $\mathrm{d}v^c(t) = \theta \mathrm{d}t/\alpha$. Then

$$\sum_{0 \leq s \leq t} \mathrm{e}^{\alpha\zeta(s)} \mathbf{1}_{J(s)}(1 - \mathrm{e}^{-\alpha\Delta v(s)}) = (1-\mathrm{e}^{-\alpha}) \sum_{j=1}^{p} \mathbf{1}_j \sum_{k=0}^{c-1} \mathrm{e}^{\alpha k} \sum_{s \in \mathcal{M}_{jk}(t)} \mathrm{e}^{\theta s}.$$

The martingale becomes

$$(\kappa(\alpha) + \theta) \int_0^t \mathrm{e}^{\alpha Q(s)} \mathbf{1}_{J(s)} \mathrm{d}s + \mathrm{e}^{\alpha x} \mathbf{1}_a - \mathrm{e}^{\alpha Q(t) + \theta t} \mathbf{1}_{J(t)} + (1-\mathrm{e}^{-\alpha}) \sum_{j=1}^{p} \mathbf{1}_j \sum_{k=0}^{c-1} \mathrm{e}^{\alpha k} \sum_{s \in \mathcal{M}_{jk}(t)} \mathrm{e}^{\theta s}.$$

For $i = 1, \ldots, 2p$, we put $\alpha = \gamma_i(\theta)$, multiply $\boldsymbol{h}^{i;\theta}$ to the right and get the martingale

$$\mathrm{e}^{\gamma_i x} h_a^{i;\theta} - \mathrm{e}^{\gamma_i Q(t) + \theta t} h_{J(t)}^{i;\theta} + (1-\mathrm{e}^{-\gamma_i}) \sum_{j=1}^{p} h_j^{i;\theta} \sum_{k=0}^{c-1} \mathrm{e}^{\gamma_i k} \sum_{s \in \mathcal{M}_{jk}(t)} \mathrm{e}^{\theta s}.$$

We apply optional stopping at $\tau(n)$ and get

$$0 = \mathrm{e}^{\gamma_i x} h_a^{i;\theta} - \mathrm{e}^{\gamma_i n} \mathbb{E}_{ax} \mathrm{e}^{\theta \tau(n)} h_{J(\tau(n))}^{i;\theta} + (1 - \mathrm{e}^{-\gamma_i}) \sum_{j=1}^{p} h_j^{i;\theta} \sum_{k=0}^{c-1} \mathrm{e}^{\gamma_i k} \beta_{jk}(\theta) \left(\sum_{\ell=1}^{p} m_{jk\ell} \eta_{\ell 0} + n_{jk} \right)$$

where we used Proposition 3.3 and Proposition 3.4 to obtain the last term. If we put $x_i = \mathbb{E}_{ax}[\mathrm{e}^{\theta\tau(n)}; J(\tau(n)) = i]$ the desired result follows. The interpretation of the auxiliary variables η_{j0} is

$$\eta_{j0}(\theta) = \mathbb{E}_{ax} \sum_{s \in \mathcal{A}_{j0}(\tau(n))} \mathrm{e}^{\theta s}.$$

□

4 Characteristics of $\tau(n)$ in simple birth–death processes

We consider a simple birth–death process $\{Q(t)\}_{t \geq 0}$ on $\mathbb{N}$ with birth rate λ_n when $Q(t) = n$ and death rate μ_n when $Q(t) = n > 0$. As before, $\tau(n) = \inf\{t > 0 : Q(t) = n\}$.

The classical approach (see [20]) is to write $\tau(n)$ as the independent sum

$$\tau(n) \;=\; \tau(x, x+1) + \cdots + \tau(n-1, n) \tag{4.1}$$

where $x = Q(0)$ and

$$\tau(k, k+1) \;=\; \inf\{t > 0 : Q(t) = k+1 \,|\, Q(0) = k\},$$

and to derive recursions for characteristics of $\tau(n)$. Consider first $\mathbb{E}_x \tau(n)$. Then

$$\mathbb{E}\tau(k, k+1) \;=\; \frac{1}{\lambda_k + \mu_k} \;+\; \frac{\mu_k}{\lambda_k + \mu_k} [\mathbb{E}\tau(k-1, k) + \mathbb{E}\tau(k, k+1)]\,; \tag{4.2}$$

indeed, the first term is the expected time until the first jump (exit from state x), and the second is the additional contribution from the event that the first jump is downwards (note that $\mu_k/(\lambda_k + \mu_k)$ is the probability of this event). Moving the $\mathbb{E}\tau(k, k+1)$ on the r.h.s. to the l.h.s. yields a recursion which is easily solved to get

$$\mathbb{E}\tau(k,k+1) \;=\; \frac{1}{\lambda_k}+\frac{\mu_k}{\lambda_{k-1}\lambda_k}+\cdots+\frac{\mu_1\ldots\mu_k}{\lambda_0\ldots\lambda_k}. \tag{4.3}$$

If we use the notation of [20], $\pi_0=1$, $\pi_n=\lambda_0\ldots\lambda_{n-1}/\mu_1\ldots\mu_n$ (note that in the ergodic case, $\{\pi_n\}$ is proportional to the stationary distribution) this gives

$$\mathbb{E}_x\tau(n) \;=\; \sum_{k=x}^{n-1}\mathbb{E}\tau(k,k+1) \;=\; \sum_{k=x}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{i=0}^{k}\pi_i\,. \tag{4.4}$$

In [20], also an expression for $\mathbb{P}_x(Q(\omega(n))=n)$ is given where

$$\omega(n) \;=\; \inf\{t>0:\; Q(t)=0 \text{ or } n\}\,.$$

The proof is by similar recursions. Asmussen [3] gave an alternative derivation, determining a function $\varphi_0(\cdot)$ such that $\{\varphi_0(Q(t\wedge\omega(n)))\}$ is a martingale and using optional stopping. Proceeding similarly for $\mathbb{E}_x\tau(n)$ motivates:

Proposition 4.1 *For each set $\varphi_1(0),\ldots,\varphi_1(n)$ such that*

$$\{M(t)\} \;=\; \{\varphi_1(Q(t\wedge\tau(n)))-\varphi_1(Q(0))+t\wedge\tau(n)\} \tag{4.5}$$

is a martingale, we have $\mathbb{E}_x\tau(n)=\varphi_1(x)-\varphi_1(n)$. Further such a set exists, is unique up to an additive constant and the set satisfying $\varphi_1(n)=0$ is given by

$$\varphi_1(x) \;=\; \mathbb{E}_x\tau(n) \;=\; \sum_{k=x}^{n-1}\frac{1}{\lambda_k\pi_k}\sum_{i=0}^{k}\pi_i$$

Proof. Since (4.5) may be bounded by $\tau(n)+2\sup_0^n|\varphi_1(k)|$ and $\mathbb{E}_x\tau(n)<\infty$, optional stopping at $\tau(n)$ is permissible and yields

$$0 \;=\; M(0) \;=\; \mathbb{E}_x M(\tau(n)) \;=\; \varphi_1(n)-\varphi_1(x)+\mathbb{E}_x\tau(n)$$

and the first statement. The uniqueness statement is then clear.

The martingale property will hold if and only if $\mathbb{E}_k[\varphi_1(Q(h))+h]=\varphi_1(k)+\mathrm{o}(h)$, $k=0,\ldots,n-1$, which for $k\neq 0$ gives

$$(\lambda_k+\mu_k)\varphi_1(k)+\mu_k\varphi_1(k-1)+\lambda_k\varphi_1(k+1)+1 \;=\; 0,$$

i.e. if $\Delta_{k+1}=-1/\lambda_k+(\mu_k/\lambda_k)\Delta_k$ where $\Delta_k=\varphi_1(k)-\varphi_1(k-1)$. Similarly, $-\lambda_0\varphi_1(0)+\lambda_0\varphi_1(1)+1=0$. It follows that $\varphi_1(0)$ may be taken arbitrary and that then there is a unique solution for the remaining $\varphi_1(k)$. The rest is easy algebra. $\square$

Remark 4.2 The existence part in Proposition 4.1 can alternatively be obtained by noting that $\mathbb{E}[\tau(n) \mid \mathcal{F}_t]$ is a martingale where $\mathcal{F}_t = \sigma(Q(s) : 0 \leq s \leq t)$. Indeed,

$$\mathbb{E}[\tau(n) \mid \mathcal{F}_t] \;=\; t \wedge \tau(n) + \mathbb{E}_{Q(t\wedge\tau(n))}\tau(n)$$

(note that $\mathbb{E}_{Q(\tau(n))}\tau(n) = \mathbb{E}_n\tau(n) = 0$) so that taking $\varphi_1(k) = \mathbb{E}_k\tau(n)$ makes $\{\varphi_1(Q(t\wedge\tau(n))) + t\wedge\tau(n)\}$ and hence also

$$\{\varphi_1(Q(t\wedge\tau(n))) - \varphi_1(Q(0)) + t\wedge\tau(n)\}$$

a martingale. Note incidentally that this is precisely the martingale used by Aldous & Shepp [1] (in the setting of general finite absorbing Markov processes).

The ideas sketched above have of course been applied in many similar situations. The essence is for a given Markov process $\{Y(t)\}$ to look for functions f, g such that $\mathcal{G}f(Y(t)) = 0$ or $\mathcal{G}^*f(Y(t),t) = 0$ where $\mathcal{G}$, $\mathcal{G}^*$ are the generators of $\{Y(t)\}$, resp. the space–time process $\{(Y(t),t)\}$. Then Dynkin's identity and suitable integrability conditions ensure that $\{f(Y(t))\}$, $\{g(Y(t),t)\}$ are martingales. For examples of this technique, see e.g. Dassios & Embrechts [10], Kella & Stadje [21] and Jacobsen [15].

Note that an f making $\{f(Y(t))\}$ a martingale is a discrete analogue of the natural scale for a diffusion. $\square$

Similarly, for the Laplace transform $\mathbb{E}_x\mathrm{e}^{\theta\tau(n)}$, the same reasoning that lead to (4.2) yields

$$\mathbb{E}\mathrm{e}^{\theta\tau(k,k+1)} \;=\; \frac{\lambda_k}{-\theta + \lambda_k + \mu_k - \mu_k\mathbb{E}\mathrm{e}^{\theta\tau(k-1,k)}}.$$

This together with

$$\mathbb{E}\mathrm{e}^{\theta\tau(0,1)} \;=\; \frac{\lambda_0}{-\theta + \lambda_0}$$

gives us recursions for determining $\mathbb{E}_x\mathrm{e}^{\theta\tau(n)}$, since

$$\mathbb{E}_x\mathrm{e}^{\theta\tau(n)} \;=\; \mathbb{E}\mathrm{e}^{\theta\tau(x,x+1)}\mathbb{E}\mathrm{e}^{\theta\tau(x+1,x+2)}\ldots\mathbb{E}\mathrm{e}^{\theta\tau(n-1,n)},$$

and we obtain the following parallel of Proposition 4.1:

Proposition 4.3 *For each set $\varphi_2(0),\ldots,\varphi_2(n)$ and every θ such that*

$$\{M'(t)\} = \{\mathrm{e}^{\theta(t\wedge\tau(n))}\varphi_2(Q(t\wedge\tau(n)))/\varphi_2(Q(0))\}$$

is a martingale and $\mathbb{E}\mathrm{e}^{\theta\tau(n)} < \infty$, it holds that $\mathbb{E}_x\mathrm{e}^{\theta\tau(n)} = \varphi_2(x)/\varphi_2(n)$. Such a set exists and is unique up to a multiplicative constant. Also, we may find θ with $\Re\theta > 0$ such that $\mathbb{E}\mathrm{e}^{\theta\tau(n)} < \infty$.

Proof. Since $\mathbb{E}\mathrm{e}^{\theta\tau(n)} < \infty$ and $\mathbb{P}(\tau(n) < \infty) = 1$, optional stopping at $\tau(n)$ is permissible and gives

$$1 \;=\; M'(0) \;=\; \mathbb{E}_x M'(\tau(n)) \;=\; \varphi_2(n)\mathbb{E}\mathrm{e}^{\theta\tau(n)}/\varphi_2(x)$$

from which the first assertion and the uniqueness follow. The martingale property holds if and only if $\mathbb{E}_k[\varphi_2(Q_h)\mathrm{e}^{\theta h}] = \varphi_2(k) + \mathrm{o}(h),\ k = 0, \ldots, n-1$, which for $k \neq 0$ yields

$$\theta\varphi_2(k) - (\lambda_k + \mu_k)\varphi_2(k) + \lambda_k\varphi_2(k+1) + \mu_k\varphi_2(k-1) = 0$$

i.e. if $\Delta_{k+1} = -(\theta\varphi_2(k))/\lambda_k + (\mu_k/\lambda_k)\Delta_k$. Similarly

$$\theta\varphi_2(0) + \lambda_0\Delta_1 = 0.$$

Thus $\varphi_2(0)$ may be chosen arbitrarily and then there is a unique solution for the remaining $\varphi_2(k)$. For the last part, see Keilson [20]. □

5 Level–dependent QBD's

We now consider a level–dependent QBD (see [24] for basic definitions), that is, a Markov process $\{(J(t), Q(t))\}$ such that $Q(t) \in \mathbb{N}$ and $J(t) \in E_\ell$ when $Q(t) = \ell$ and that the intensity matrix is of the form

$$\boldsymbol{Q} \;=\; \begin{pmatrix} \boldsymbol{B}(0) & \boldsymbol{A}(0) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \boldsymbol{C}(1) & \boldsymbol{B}(1) & \boldsymbol{A}(1) & \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{C}(2) & \boldsymbol{B}(2) & \boldsymbol{A}(2) & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{C}(3) & \boldsymbol{B}(3) & \boldsymbol{A}(3) & \ldots \\ \vdots & & & & & \ddots \end{pmatrix} \tag{5.1}$$

where the dimensions are

$$\boldsymbol{C}(\ell) :\ E_\ell \times E_{\ell-1}, \quad \boldsymbol{B}(\ell) :\ E_\ell \times E_\ell, \quad \boldsymbol{A}(\ell) :\ E_\ell \times E_{\ell+1}.$$

We assume that $\boldsymbol{Q}$ is irreducible. As above, $\tau(n) = \inf\{t > 0 :\ Q(t) = n\}$, and we want to compute $\mathbb{E}_{ax}\tau(n)$ or possibly $\mathbb{E}_{ax}[\tau(n);\ J(\tau(n)) = b]$.

For $a \in E_\ell, b \in E_{\ell+1},$, write

$$p_{ab}(\ell) = \mathbb{P}_{a\ell}(J(\tau(\ell+1)) = b), \quad m_{ab}(\ell) = \mathbb{E}_{a\ell}[\tau(\ell+1);\ J(\tau(\ell+1)) = b],$$

collected in matrix form as $\boldsymbol{P}(\ell) = (p_{ab}(\ell))$, $\boldsymbol{M}(\ell) = (m_{ab}(\ell))$.

The following algorithm was proposed by Gaver, Jacobs & Latouche [12] and is basically a Markov–modulated version of (4.1)–(4.4). Define

$$g_{i,j}^{(\ell)}(x) \;=\; \mathbb{P}_{i(\ell-1)}[\tau(\ell) \le x, J(\tau(\ell)) = j], \quad \ell \ge 1.$$

Let $G_{i,j}^{(\ell)}(\xi)$ be the Laplace–Stieltjes transform of $g_{i,j}^{(\ell)}(x)$

$$G_{i,j}^{(\ell)}(\xi) = \int_0^\infty \mathrm{e}^{-\xi x}\left[\frac{\mathrm{d}}{\mathrm{d}x} g_{i,j}^{(\ell)}(x)\right]\mathrm{d}x$$

and $\boldsymbol{G}^{(\ell)}(\xi) = (G_{i,j}^{(\ell)}(\xi))$. Then we have (a proof of this may be found in [12])

Proposition 5.1 *$\boldsymbol{P}(\ell) = -\boldsymbol{D}(\ell)^{-1}\boldsymbol{A}(\ell)$ where $\boldsymbol{D}(0) = \boldsymbol{B}(0)$ and $\boldsymbol{D}(\ell) = \boldsymbol{B}(\ell) + \boldsymbol{C}(\ell)(-\boldsymbol{D}(\ell-1)^{-1})\boldsymbol{A}(\ell-1)$, $\ell \ge 1$. Also*

$$\begin{aligned}
\boldsymbol{G}^{(1)}(\xi) &= \boldsymbol{E}_0(\xi)\boldsymbol{A}(0)\\
\boldsymbol{G}^{(\ell)}(\xi) &= \boldsymbol{E}_{\ell-1}(\xi)\boldsymbol{A}(\ell-1)
\end{aligned}$$

where $\boldsymbol{E}_0(\xi) = (\xi\boldsymbol{I} - \boldsymbol{B}(0))^{-1}$ and $\boldsymbol{E}_\ell(\xi) = (\xi\boldsymbol{I} - \boldsymbol{B}(\ell) - \boldsymbol{C}(\ell)\boldsymbol{G}^{(\ell)}(\xi))^{-1}$. In particular $\boldsymbol{M}(\ell) = -\mathrm{d}\boldsymbol{G}^{(\ell+1)}(\xi)/\mathrm{d}\xi|_{\xi=0}$, and it then follows

$$\begin{aligned}
\boldsymbol{M}(0) &= (\boldsymbol{D}_0^{-1})^2\boldsymbol{A}(0)\\
\boldsymbol{M}(\ell) &= \boldsymbol{D}_\ell^{-1}(\boldsymbol{I} + \boldsymbol{C}(\ell)\boldsymbol{M}(\ell-1))\boldsymbol{D}_\ell^{-1}\boldsymbol{A}(\ell),\ \ell \ge 1.
\end{aligned}$$

If we are only interested in $\boldsymbol{u}(\ell) = (\mathbb{E}_{a\ell}\tau(\ell+1)) = \boldsymbol{M}(\ell)\mathbf{1}$, we get

$$\begin{aligned}
\boldsymbol{u}(0) &= -\boldsymbol{D}(0)^{-1}\mathbf{1}\\
\boldsymbol{u}(\ell) &= -\boldsymbol{D}(\ell)^{-1}(\mathbf{1} + \boldsymbol{C}(\ell)\boldsymbol{u}(\ell-1)),\ \ell \ge 1.
\end{aligned}$$

Also, $\mathbb{E}_{ax}[\tau(n); J(\tau(n)) = b]$ is the abth element of the matrix

$$\sum_{k=x}^{n-1}\boldsymbol{R}(k)\boldsymbol{M}(k)\boldsymbol{S}(k+1) \quad \text{where} \quad \boldsymbol{R}(k) \;=\; \prod_{\ell=x}^{k-1}\boldsymbol{P}(\ell), \quad \boldsymbol{S}(k) \;=\; \prod_{\ell=k}^{n-1}\boldsymbol{P}(\ell) \tag{5.2}$$

(To see this, note that $\boldsymbol{R}(k)(i,j) = \mathbb{P}_{ix}(J(\tau(k)) = j)$ and $\boldsymbol{S}(k)(i,j) = \mathbb{P}_{ik}(J(\tau(n)) = j)$.)

In our MAP/M/c model we have for $i \ne j$ that

$$\begin{aligned}
\boldsymbol{A}(\ell)_{jj} &= \beta_j, \quad \boldsymbol{A}(\ell)_{ij} \;=\; \lambda_{ij}P_{ij}\\
\boldsymbol{B}(\ell)_{ij} &= \lambda_{ij}(1 - P_{ij}),\\
\boldsymbol{C}(\ell)_{jj} &= \ell\mu_j,\ \text{if } \ell \le c, \quad \boldsymbol{C}(\ell)_{jj} \;=\; c\mu_j,\ \text{if } \ell > c, \quad \boldsymbol{C}(\ell)_{ij} \;=\; 0.
\end{aligned}$$

Note that $\boldsymbol{B}(0)_{jj}$ and $\boldsymbol{B}(\ell)_{jj}$ are determined by the requirements $\boldsymbol{B}(0)\mathbf{1} + \boldsymbol{A}(0)\mathbf{1} = \mathbf{0}$ and $\boldsymbol{C}(\ell)\mathbf{1} + \boldsymbol{B}(\ell)\mathbf{1} + \boldsymbol{A}(\ell)\mathbf{1} = \mathbf{0}$. Thus we may use (5.2) and the results of [12] to compute $\mathbb{E}_{ax}\tau(n)$ in MAP/M/c as the ath component of $(\sum_{k=x}^{n-1} \boldsymbol{R}(k)\boldsymbol{M}(k)\boldsymbol{S}(k+1))\mathbf{1}$.

The obvious question now arises whether it is possible to find an equivalent to Proposition 4.1, i.e. is it to find $\varphi_3 : \cup_{i=0}^{n} i \times E_i \to \mathbb{R}$ such that

$$\{\varphi_3(J(t \wedge \tau(n)), Q(t \wedge \tau(n))) - \varphi_3(J(0), Q(0)) + t \wedge \tau(n)\} \tag{5.3}$$

is a martingale?

Proposition 5.2 *Let $\{(J(t), Q(t))\}$ be an irreducible QBD with intensity matrix $\boldsymbol{Q}$.*
(i) *For each set $\varphi_3(0,1), \ldots, \varphi_3(0, |E_0|), \ldots, \varphi_3(n,1), \ldots, \varphi_3(n, |E_n|)$ such that (5.3) is a martingale, we have*

$$\mathbb{E}_{ax}\tau(n) \;=\; \varphi_3(a,x) - \sum_{i \in E_n} \varphi_3(i,n)\mathbb{P}_{ax}(J(\tau(n)) = i);$$

(ii) *Such a set always exists, namely $\varphi_3(a,x) = \mathbb{E}_{ax}\tau(n)$; If all $\boldsymbol{A}(\ell)$ are quadratic with dimension $|E| \times |E|$ and non–singular, let $\boldsymbol{\varphi}_3(\ell) = (\varphi_3(\ell,1), \ldots, \varphi_3(\ell, |E|))^T$. Then we may choose an arbitrary $\boldsymbol{\varphi}_3(0)$ and then there is a unique solution for the remaining $\boldsymbol{\varphi}_3(\ell)$.*

Proof. (5.3) is bounded by

$$\tau(n) + 2 \sup_{0 \le j \le n} \sup_{1 \le i \le |E_j|} |\varphi_3(i,j)|$$

and $\mathbb{E}_{ax}\tau(n) < \infty$. Thus optional stopping at $\tau(n)$ is permissible and gives

$$\begin{aligned} 0 &= M(0) = \mathbb{E}_{ax} M(\tau(n)) \\ &= \sum_{i \in E_n} \mathbb{E}_{ax}(\varphi_3(J(\tau(n)), Q(\tau(n)))|J(\tau(n)) = i)\mathbb{P}_{ax}(J(\tau(n)) = i) - \varphi_3(a,x) + \mathbb{E}_{ax}\tau(n) \\ &= \sum_{i \in E_n} \varphi_3(i,n)\mathbb{P}_{ax}(J(\tau(n)) = i) - \varphi_3(a,x) + \mathbb{E}_{ax}\tau(n), \end{aligned}$$

showing part (i). Part (ii) follows just as in the first part of remark 4.2.

For (5.3) to be a martingale, it is necessary and sufficient that $\mathbb{E}_{jk}[\varphi_3(J(h), Q(h)) + h] = \varphi(j,k) + \mathrm{o}(h)$ which becomes

$$\begin{aligned} \mathbf{1} &= \boldsymbol{B}(0)\boldsymbol{\varphi}_3(0) + \boldsymbol{A}(0)\boldsymbol{\varphi}_3(1) \\ \mathbf{1} &= \boldsymbol{B}(\ell)\boldsymbol{\varphi}_3(\ell) + \boldsymbol{A}(\ell)\boldsymbol{\varphi}_3(\ell+1) + \boldsymbol{C}(\ell)\boldsymbol{\varphi}_3(\ell-1),\ 1 \le \ell < n. \end{aligned}$$

If we pick an arbitrary $\boldsymbol{\varphi}_3(0)$, then there is a unique solution for the remaining $\boldsymbol{\varphi}_3(k)$. $\square$

The conclusion is that whereas φ_3 always exists, the recursive computation is not always feasible. Indeed, we need the $\boldsymbol{A}(\ell)$ to be non–singular, which does not hold in many specific cases (for example if the arrival process is Markov–modulated Poisson with some $\beta_i = 0$ or renewal with Erlang interarrival times).

6 Complexity

We now compare the complexities of the two different methods used, the one based upon the martingale (3.1) in Section 3 and the recursive approach in Section 5. For simplicity, we consider only the case of $\mathbb{E}_{ax}\tau(n)$. In the first approach we first need to use Proposition 3.1 to reduce the number of unknowns. A careful examination of the relevant equations shows that this costs precisely $(c-1)p(8p+2)$ flops, since we need to calculate various constants as well as solving the actual equations. Further, in Proposition 3.2, we need to solve a system of $2p$ linear equations which costs roughly $16p^3/3$, and calculate m and $\boldsymbol{k}$ which in all costs around $4p^3 + 6p^2$. This means that the method of Section 3.1 costs around $10p^3 + (8c-2)p^2 + (2c-1)p$ flops.

The second approach from Section 5 requires knowledge of all $\boldsymbol{M}(\ell)$ and $\boldsymbol{E}(\ell)$ $0 \le \ell < n$. This costs around $12np^3 + 2np^2$. In addition, we need to do $(n-1-x)^2$ multiplications to be able to compute the sum

$$\sum_{k=x}^{n-1} \boldsymbol{R}(k)\boldsymbol{M}(k)\boldsymbol{S}(k+1).$$

This costs about $2(n-1-x)^2p^3$ flops in addition to the $(n-1-x)p^2$ flops it costs to perform the summation. Thus this method requires something in the vicinity of $(2(n-1-x)^2 + 12n)p^3 + (2n-x)p^2$ flops.

An obvious conclusion is that the first method, though a bit less straightforward, is much more efficient than the second method when n is large, as will be the case in rare events problems.

7 $\mathbb{E}_x\tau(n)$ in M/M/2–heterogeneous

Consider the situation where we have a M/M/2 queueing system with arrival rate β and with the two independent servers, server a and server b, having different service intensities, μ_a and μ_b, with $\mu_b > \mu_a$. As usual, let

$Q(t)$ denote the queue length at time t. If a customer arrives at an empty system, we assume that he always chooses server b. Otherwise, the first vacant server is chosen. (Other schemes are possible, see e.g. [7].) Let $A(s) = \{\text{server } a \text{ idle at time } s\}$ and $B(s) = \{\text{server } b \text{ idle at time } s\}$. Define

$$\begin{aligned}
\alpha_i &= \mathbb{E}_x \int_0^{\tau(n)} I(Q(s) = i)\mathrm{d}s, \ i = 0 \text{ and } 2 \\
\alpha_1^a &= \mathbb{E}_x \int_0^{\tau(n)} I(Q(s) = 1, A(s))\mathrm{d}s \\
\alpha_1^b &= \mathbb{E}_x \int_0^{\tau(n)} I(Q(s) = 1, B(s))\mathrm{d}s \\
\alpha_i{}' &= \delta_{ix} + \mathbb{E}_x \sum_{0<s\le\tau(n)} I(Q(s-) \neq i, Q(s) = i), \ i = 0 \text{ and } 2 \\
\alpha_1^{a'} &= \delta_{1q} + \mathbb{E}_x \sum_{0<s\le\tau(n)} I(Q(s-) \neq 1, Q(s) = 1, A(s)) \\
\alpha_1^{b'} &= \mathbb{E}_x \sum_{0<s\le\tau(n)} I(Q(s-) \neq 1, Q(s) = 1, B(s)).
\end{aligned}$$

An argument like that of the proof of Proposition 2.1 gives that

$$\begin{aligned}
\alpha_0^{'} &= \delta_{0x} + \alpha_1^{a'} \frac{\mu_b}{\beta + \mu_b} + \alpha_1^{b'} \frac{\mu_a}{\beta + \mu_a} \\
\alpha_1^{a'} &= \delta_{1x} + \alpha_0^{'} + \alpha_2^{'} \frac{\mu_a}{\beta + \mu_a + \mu_b} \\
\alpha_1^{b'} &= \alpha_2^{'} \frac{\mu_b}{\beta + \mu_a + \mu_b}
\end{aligned}$$

which, together with

$$\alpha_0 = \frac{\alpha_0^{'}}{\beta}, \ \alpha_2 = \frac{\alpha_2^{'}}{\beta + \mu_a + \mu_b}, \ \alpha_1^a = \frac{\alpha_1^{a'}}{\beta + \mu_b}, \ \alpha_1^b = \frac{\alpha_1^{b'}}{\beta + \mu_a},$$

may be reduced to

$$\alpha_1^a \mu_b + \alpha_1^b \mu_a = \alpha_0 \beta - \delta_{0x} \tag{7.1}$$

$$\alpha_1^a(\beta + \mu_b) - \alpha_1^b \frac{\mu_a(\beta + \mu_a)}{\mu_b} = \alpha_0 \beta + \delta_{1x}. \tag{7.2}$$

(7.1) and (7.2) have solution

$$\alpha_1^a = \frac{\beta(\beta + \mu_a + \mu_b)\alpha_0 - (\beta + \mu_a)\delta_{0x} + \mu_b \delta_{1x}}{\mu_b(2\beta + \mu_a + \mu_b)} \tag{7.3}$$

$$\alpha_1^b = \frac{\beta^2 \alpha_0 - (\beta + \mu_b)\delta_{0x} - \mu_b \delta_{1x}}{\mu_a(2\beta + \mu_a + \mu_b)} \tag{7.4}$$

We have the representation

$$Q(t) = x + X(t) + L(t)$$

where $\{X(t)\}$ is a Lévy process defined as $X(t) = N^{(+)}(t) - N^{(-)}(t)$ where $N^{(+)}(t)$ and $N^{(-)}(t)$ are independent Poisson processes with intensities β resp. $\mu_a + \mu_b$. If we let $\kappa(\cdot)$ denote the Lévy exponent of $X(t)$ it is immediate that $\kappa(\alpha) = \beta(\mathrm{e}^{\alpha} - 1) + (\mu_a + \mu_b)(\mathrm{e}^{-\alpha} - 1)$. Let γ be the non–zero root of the equation $\kappa(\gamma) = 0$. It is easily seen that $\gamma = -\log(\beta/(\mu_a + \mu_b))$. If we copy the proof of Theorem 2.2, we get

$$0 = \mathrm{e}^{\gamma x} - \mathrm{e}^{\gamma n} + (1 - \mathrm{e}^{-\gamma})((\mu_a + \mu_b)\alpha_0 + \mathrm{e}^{\gamma}(\mu_a \alpha_1^a + \mu_b \alpha_1^b)).$$

From this, (7.3) and (7.4) we may calculate α_0. This and $n = x + (\beta - (\mu_a + \mu_b))\mathbb{E}_x \tau(n) + (\mu_a + \mu_b)\alpha_0 + \mu_a \alpha_1^a + \mu_b \alpha_1^b$ give us $\mathbb{E}_x \tau(n)$.

A Justification of optional stopping at $\tau(n)$

In M/M/c and MAP/M/c, $\mathbb{E}\tau(n) < \infty$ follows from the general theory of regenerative processes (see [3]). Since in the M/M/c case $\{L(t)\}$ is bounded in stochastic order by a Poisson process $\{L_{\max}(t)\}$ with intensity $c \max_1^c \mu_i$, $\mathbb{E}L_{\max}(\tau(n)) = c \max_1^c \mu_i \mathbb{E}\tau(n) < \infty$ and obviously $Q(t) \le n$ on $0 \le t \le \tau(n)$ we have that $\mathbb{E}M(\tau(n)) = 0$ follows from $\mathbb{E}M(t \wedge \tau(n)) = 0$ by dominated convergence of (2.2) (note that in (2.2) the first term is bounded by $\kappa(\alpha)\mathrm{e}^{\alpha n}\tau(n)$ and the third by $\mathrm{e}^{\alpha n}$). The same argument applies in MAP/M/c (i.e. in the proofs of Theorem 3.2 and Theorem 3.5), since we consider $\mathbb{E}\mathrm{e}^{\theta\tau(n)}$ for $\theta \le 0$ only and in the MAP/M/c case $\{L(t)\}$ is bounded in stochastic order by a Poisson process $\{L^*_{\max}(t)\}$ with intensity $c \max_1^p \mu_i$.

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