A Change-of-Variable Formula with Local Time on Curves

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Abstract

Let $X = (X_t)_{t \geq 0}$ be a continuous semimartingale and let $b : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function of bounded variation. Setting $C = \{ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x < b(t) \}$ and $D = \{ (t,x) \in \mathbb{R}_+ \times \mathbb{R} \mid x > b(t) \}$ suppose that a continuous function $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is given such that F is $C^{1,2}$ on $\overline{C}$ and F is $C^{1,2}$ on $\overline{D}$. Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(t,X_t) = \; & F(0,X_0) + \int_0^t \frac{1}{2} \Big(F_t(s,X_s+) + F_t(s,X_s-) \Big) \, ds \\
& + \int_0^t \frac{1}{2} \Big(F_x(s,X_s+) + F_x(s,X_s-) \Big) \, dX_s \\
& + \frac{1}{2} \int_0^t F_{xx}(s,X_s) \, I(X_s \neq b(s)) \; d\big\langle X,X \big\rangle_s \\
& + \frac{1}{2} \int_0^t \Big(F_x(s,X_s+) - F_x(s,X_s-) \Big) I(X_s = b(s)) \; d\ell_s^b(X)
\end{aligned}
$$

where $\ell_s^b(X)$ is the local time of X at the curve b given by:

$$
\ell_s^b(X) = \mathbb{P} - \lim_{\varepsilon \downarrow 0} \; \frac{1}{2\varepsilon} \int_0^s I(b(r) - \varepsilon < X_r < b(r) + \varepsilon) \; d\big\langle X,X \big\rangle_r
$$

and $d\ell_s^b(X)$ refers to the integration with respect to $s \mapsto \ell_s^b(X)$. The formula derived has found applications in free-boundary problems of optimal stopping.

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