A Change-of-Variable Formula with Local Time on Curves

Goran Peskir

Abstract

Let $X=(X_t)_{t\ge 0}$ be a continuous semimartingale and let $b: I\!\!R_+ \rightarrow I\!\!R$ be a continuous function of bounded variation. Setting $C=\{\,(t,x)\in I\!\!R_+ \times I\!\!R \mid x<b(t)\,\}$ and $D=\{\,(t,x)\in I\!\!R_+ \times I\!\!R \mid x>b(t)\,\}$ suppose that a continuous function $F: I\!\!R_+ \times I\!\!R \rightarrow I\!\!R$ is given such that F is $C^{1,2}$ on $\overline{C}$ and F is $C^{1,2}$ on $\overline{D}$. Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(t,X_t) = F(0,X_0) &+ \int_0^t \frac{1}{2}\Big(F_t(s,X_s+)+F_t(s,X_s-)\Big)\,ds \\
&+ \int_0^t \frac{1}{2}\Big(F_x(s,X_s+)+F_x(s,X_s-)\Big)\,dX_s \\
&+ \frac{1}{2}\int_0^t F_{xx}(s,X_s)\, I(X_s \ne b(s))\; d\langle X,X\rangle_s \\
&+ \frac{1}{2}\int_0^t \Big(F_x(s,X_s+)-F_x(s,X_s-)\Big) I(X_s=b(s))\; d\ell_s^b(X)
\end{aligned}
$$

where $\ell_s^b(X)$ is the local time of X at the curve b given by:

$$
\ell_s^b(X) = I\!\!P-\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^s I(b(r)-\varepsilon<X_r<b(r)+\varepsilon)\; d\langle X,X\rangle_r
$$

and $d\ell_s^b(X)$ refers to the integration with respect to $s\mapsto \ell_s^b(X)$. The formula derived has found applications in free-boundary problems of optimal stopping.

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