COMPARATIVE STUDY OF FIRST TOUCH DIGITALS: NORMAL INVERSE GAUSSIAN VS. GAUSSIAN MODELLING.

OLEG KUDRYAVTSEV* AND SERGEI LEVENDORSKIǏ†

*Department of Informatics,
Russian Customs Academy Rostov Branch,
Budennovskiy 20,Rostov-on-Don, 344011, Russia
e-mail: kudriav@rnd.runnet.ru

† Rostov State University of Economics,
B. Sadovaya 69, Rostov-on-Don, 344007, Russia
e-mail: leven@rnd.runnet.ru

Abstract. We calculate prices of first touch digitals under normal inverse gaussian (NIG) processes, and compare them to prices in the gaussian model with the same instantaneous variance. Numerical results are produced to show that for typical parameters values, the relative error of the gaussian approximation to NIG-price can be several dozen percent if the spot price is at the distance 0.05-0.2 from the barrier (normalized to one). A fast approximate pricing formula under NIG is derived.

1. Introduction

Pricing of contingent claims of European type and perpetual American options under non-gaussian processes is fairly well understood both from the theoretical viewpoint and numerical implementation of pricing formulas (see e.g. [2], [4], [10], [13], [5], [7] and the bibliography therein) but not much is known even about the behavior of prices of contingent claims of other types. In the paper, we study the behavior of non-gaussian prices not far from the boundary, where non-gaussian effects must be felt in the first place, and we start with the simplest case of first touch digitals. One should expect similar deviation from gaussian prices near barrier for barrier options but in the case of first touch digitals, this boundary effect is observed in the most pure form since all the value of the option comes from the other side of the barrier. In the paper, we show that non-gaussian prices can differ drastically from the gaussian approximation when the spot price is 5-20 percent from the barrier. This naturally explains why this is the region where barrier options become usually illiquid.

In the gaussian case, explicit formulas for first touch digitals are well-known (see e.g. [11]); these formulas are easy to implement in numerical calculations. Recently, general formulas for the case of *Regular Lévy Processes of Exponential Type* (RLPE) were obtained in [6] and [7] (class RLPE contains many families of Lévy processes used in theoretical and empirical studies of financial markets, normal inverse gaussian processes including), however, these formulas involve the double Fourier inversion (and one more integration needed to calculate the factor in the Wiener-Hopf factorization formula), and hence it is difficult to implement them in practice. In the paper, we construct a numerical method based on time discretization, and use the method to study typical errors of the gaussian approximation when a non-gaussian model (here, normal inverse gaussian model) is replaced by the gaussian model with the same instantaneous variance. Numerical examples demonstrate that in the region $S/H \leq 1.2$, the relative error can be several dozen percent. The error remains sizable farther from the barrier but becomes smaller.

The numerical method works rather slowly, however. By using the general formulas from [6] and [7], we derive an approximate formula, which is both fairly accurate and fast, for many parameters values. The formula provides a good approximation for S not very close to H, and under assumption that the tails of probability density decay sufficiently fast: the faster the decay is, the closer to the boundary the approximate formula works. For typical parameters values, this approximate formula works well if the spot price S differs from H by 1-3 percent or more and time to expiry is 5 days or more; the farther the spot price from the barrier and time to expiry larger, the better the approximation. The approximate formula and any numerical method can be used in conjunction since the former works better far from the expiry and barrier, where numerical methods are usually too slow and/or not quite reliable. In addition, the approximate formula can be used to check a concrete implementation of any numerical method.

The plan of the paper is as follows. In Section 2, we recall the definition of NIG and pricing formula for the first-touch digitals. In Section 3, an approximate formula is derived: first, in the general form, and then the formula is made explicit for the case of Normal Inverse Gaussian processes. In Section 4, numerical examples are provided to compare the pricing under NIG with the Gaussian pricing, and study the accuracy of the approximate formula. Section 5 concludes, and the description of the numerical method and technical calculations are delegated to the appendix.

2. Normal Inverse Gaussian Processes and exact pricing formula for first-touch digitals

Consider a model market of a riskless bond and a stock. We assume that the riskless rate $r > 0$ is constant, and the logarithm of the spot price of the stock $X_t = \ln S_t$ follows a Lévy process under the historic measure $\mathbf{P}$. If X is not a Brownian motion, an equivalent martingale measure (EMM) is typically non-unique, and it follows from the general result due to Delbaen and Schachermayer [8] (see also bibliography therein) that the no-arbitrage condition is satisfied for pricing under any EMM $\mathbf{Q}$, which is absolutely continuous w.r.t. $\mathbf{P}$.

Let ψ be the characteristic exponent of X under $\mathbf{Q}$. We use the following definition of the characteristic exponent, ψ, of the Lévy process X: $E[e^{i\xi X_t}] = e^{-t\psi(\xi)}$. Since $\mathbf{Q}$ is an EMM, both the bond and stock must be priced under $\mathbf{Q}$, and therefore ψ must admit the analytic continuation into a strip $\Im\xi \in (0, 1)$, and continuous continuation into the closed strip $\Im\xi \in [0, 1]$. Further, the following condition must hold

$$r + \psi(-i) = 0. \tag{2.1}$$

The normal inverse Gaussian distribution has proven to be a flexible and yet simple statistical model which fits empirical logreturns on all time scales extremely well. The family of normal inverse gaussian distributions was introduced by Barndorff-Nielsen [1], and later applied to financial data by Barndorff-Nielsen [2], Rydberg [15], Prause [14], and Bolviken and Benth [3], to mention only a few. The characteristic exponent of a NIG (see eg. [2]) is given by

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{2.2}$$

where $\delta > 0, \alpha > |\beta|$. From (2.2), it is clearly seen that a mixture of NIG-processes with the same α and β but different μ and δ is a NIG process with the same α and β. In other words, a subclass of normal inverse gaussian distributions with fixed α and β is closed under convolution, as the class of normal distributions is.

NIG have another important property: for any $t > 0$, the following explicit analytical formula for the probability density is available:

$$p_t(x) = \frac{\alpha}{\pi} \exp[t(\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu) + \beta x] \frac{K_1(\alpha\delta\langle(x/t - \mu)/\delta\rangle)}{\langle(x/t - \mu)/\delta\rangle}, \tag{2.3}$$

where $\langle y\rangle = (1+|y|^2)^{1/2}$, and K_1 denotes the modified Bessel function of the third kind with index 1. It is often of interest to consider alternative parameterizations of the normal inverse Gaussian laws. In particular, letting $\overline{\alpha} = \delta\alpha$ and $\overline{\beta} = \delta\beta$, we have that $\overline{\alpha}$ and $\overline{\beta}$ are invariant under location-scale changes. Several authors (see e.g. [3, 14]) used this alternative parameterization in the definition of NIG. We refer to [3, 14] for results related to the fitting parameters in the case of NIG distributions and references to relevant literature.

We assume that X is a NIG under an EMM $\mathbf{Q}$, chosen by the market, and we consider the first-touch digital (another name is a touch-and-out option) which pays \$ 1 the first time the stock price S crosses the level H from above. If the stock price never crosses the level H before time T, the claim expires worthless. Denote by $V_{\rm d}(H,T;S,t)$ the no-arbitrage price of such a contract. The formula for the value $V_{\rm u}(H,T;S,t)$ of the similar contract, which pays \$ 1 the first time the stock price crosses the level H from below, easily follow by the reflection of the real axis w.r.t. the origin.

Set $x = \ln(S/H)$, $u(x,t) = V_{\rm d}(H,T;S,t)$. Then for $t < T$ and $x \in \mathbf{R}$,

$$u(x,t) = E[e^{-rT'}\mathbf{1}_{T'\le T} \mid X_t = x], \tag{2.4}$$

where T' is the hitting time of $(-\infty, 0]$ by X. Denote by L the infinitesimal generator of process X, and set $\tau = T-t, v(x,\tau) = u(x,T-\tau)$. In [6, 7], the following theorem is proven.

Theorem 2.1. *The v is a solution to the problem*

$$\begin{aligned} (\partial_\tau + r - L)v(x,\tau) &= 0, \quad x>0,\ \tau>0, && (2.5)\\ v(x,\tau) &= 1, \quad x\le 0,\ \tau\ge 0, && (2.6)\\ v(x,0) &= 0, \quad x>0, && (2.7) \end{aligned}$$

in the class of bounded measurable functions.

In the case of NIG, $\psi(\xi)$ admits the analytic continuation w.r.t. ξ into the strip $\Im\xi \in (-\alpha+\beta, \alpha+\beta)$, and moreover, for each $q = i\lambda + r > 0$, there exist $\sigma_- < 0 < \sigma_+$ and $c_0 > 0$ such that

$$\Re(i\lambda + r + \psi(\xi)) > c_0, \quad \forall\ \Im\xi \in (\sigma_-, \sigma_+) \tag{2.8}$$

(see [6, 7]). The solution to the problem (2.5)-(2.7) is expressed in terms of the minus-factor $\phi_-(\lambda,\xi)$ in the Wiener-Hopf factorization formula

$$\frac{i\lambda + r}{i\lambda + r + \psi(\xi))} = \phi_+(\lambda,\xi)\phi_-(\lambda,\xi), \tag{2.9}$$

which holds for ξ in (2.8). The factor $\phi_-(\lambda,\xi)$, for ξ in the half-space $\Im\xi < \sigma_+$, is given by

$$\begin{aligned}\phi_-(\lambda,\xi) &= \exp\left[-\frac{1}{2\pi i}\int_{\Im\xi=\rho_+}\frac{\xi\ln(i\lambda+r+\psi(\eta))}{\eta(\xi-\eta)}d\eta\right] \qquad (2.10)\\ &= \exp\left[-\frac{1}{2\pi i}\int_{\Im\xi=\rho_+}\frac{\psi'(\eta)}{i\lambda+r+\psi(\eta)}\ln\frac{\eta-\xi}{\eta}d\eta\right], \qquad (2.11)\end{aligned}$$

where $\rho_+ \in (\Im\xi, \sigma_+)$ is arbitrary; in this paper, we do not need the formulas for the plus-factor. In [6, 7], by using the Laplace transform and the Wiener-Hopf factorization method, the following pricing formula was derived

$$v(x,\tau) = (2\pi)^{-2}\int_{-\infty+i\sigma}^{+\infty+i\sigma}\int_{-\infty+i\omega_+}^{+\infty+i\omega_+} e^{i(\tau\lambda+x\xi)}\phi_-(\lambda,\xi)(\lambda\xi)^{-1}d\xi d\lambda, \qquad (2.12)$$

where $\sigma < 0$, and $\omega_+ \in (0, \sigma_+)$ is sufficiently close to 0. The reader may notice that (2.12) is not computationally effective, though relatively short. Below, we derive an effective approximate formula; it is long, however.

3. Approximate formulas for the case of large steepness parameters

3.1. **General case.** In this section, we obtain an approximate formula for the pricing under NIG for the case

$$\alpha >> 1, \quad |\beta| << \alpha, \quad \alpha\delta\tau >> 1. \qquad (3.1)$$

We find the leading term of the asymptotic of the price $v(x,\tau)$ of the first-touch digital as $\alpha \to +\infty$. Notice that in the empirical studies of the financial markets (see e.g. [3], [14]), α is large: typically, of order 25-60, and sometimes up to 100 or more, whereas $|\beta/\alpha|$ is small.

For α in the middle range $30-50$, the approximate formula provides reasonable approximation if $\tau \geq 5$ days and $S/H \geq 1.03$; one business day corresponds to $\tau = 1/252$. In the region $\tau \geq 10$ days, the relative error is only several percent. Notice that the asymptotic formula works well for large τ and x, where the numerical methods are expected to produce serious errors. Thus, the approximate formula can be used to supplement any numerical method, and the formula works pretty fast: on ordinary PC, it takes about 1.25 sec. to calculate option values at 40 points by using Matlab. The formula in the Black-Scholes model takes much less time: 0.05 sec., but the error of the latter is several times larger than that of the former.

The asymptotic formula is an integral over an appropriate contour $\mathcal{L}$ in the $\lambda-$plane, and the integrand is expressed via the roots $\xi = \xi(\lambda)$ of the "characteristic equation"

$$i\lambda + r + \psi(\xi) = 0, \tag{3.2}$$

for $\lambda \in \mathcal{L}$. One can show (see [7]) that for the model classes of RLPE, for typical parameters' values, the equation (3.2) has exactly two roots for $\lambda = 0$, call them $-i\beta_-(0)$ and $-i\beta_+(0)$, in the upper and the lower half-plane, respectively. These roots are outside the cuts in the complex plane, of the form $[i\lambda_+, +i\infty)$ and $(-i\infty, i\lambda_-]$; in the case of NIG, $\lambda_+ = \alpha + \beta$, $\lambda_- = \beta - \alpha$, where $\lambda_- < 0 < \lambda_+$. It follows that in a sufficiently small neighborhood of 0 in the λ-plane, there exist $\omega_{min} < 0 < \omega_{max}$ and branches $-i\beta_-(\lambda)$ and $-i\beta_+(\lambda)$ of roots of (3.2) with the following property

$$-\Re\beta_+(\lambda) < \omega_{min} < 0 < \omega_{max} < -\Re\beta_-(\lambda). \tag{3.3}$$

Suppose that there exists a contour $\mathcal{L}$ with the following properties

a) the roots $-i\beta_\pm(\lambda)$ exist for each $\lambda \in \mathcal{L}$, they are simple, and condition (3.3) holds;
b) there exist $C_0 << \lambda_+$ and $C_1, c_1 > 0$ s.t. for all $\lambda \in \mathcal{L}$ satisfying $|\Re\lambda| \geq C_0$, we have

$$c_1|\Re\lambda| \leq \Im\lambda \leq C_1|\Re\lambda|, \tag{3.4}$$

and

$$c_1|\Re\lambda| \leq \Im(-i\beta_-(\lambda)). \tag{3.5}$$

Then we can push the contour of the integration $\Im\xi = i\omega_+$ up, and on crossing the root $-i\beta_-(\lambda)$, apply the residue theorem. Since

$$\phi_-(\lambda, \xi) = \frac{i\lambda + r}{(i\lambda + r + \psi(\xi))\phi_+(\lambda, \xi)},$$

and $\phi_+(\lambda, \xi)$ does not vanish in the upper half-plane $\Im\xi \geq 0$, we obtain the leading term of asymptotics in the form

$$v(x, \tau) = -(2\pi)^{-1} \int_{\mathcal{L}} e^{i\lambda\tau + \beta_-(\lambda)x} \frac{i\lambda + r}{i\lambda\psi'(-i\beta_-(\lambda))\phi_+(\lambda, -i\beta_-(\lambda))} d\lambda. \tag{3.6}$$

In the next subsection, we construct a contour $\mathcal{L}$ satisfying conditions (3.4)-(3.5), in the case on NIG. Similar construction can be made for other model classes of RLPE; in cases, when an infinite contour satisfying properties a) and b) is difficult to construct, one can be satisfied with the construction of $\mathcal{L}$ in the region $|\lambda| < C, C > 0$, and use (3.6) with the integration over such a finite contour: due to (3.1) and (3.4)-(3.5), the integration over the part of $\mathcal{L}$ outside the ball of radius λ_+ makes a small contribution to (3.6), anyway.

Before proceeding further, we rewrite formula (3.6) in a more convenient form, under assumption that ψ is holomorphic in the upper half-plane with the cut $[i\lambda_+, +i\infty)$, and for each $\lambda \in \mathcal{L}$, equation (3.2) has no roots hanging over $-i\beta_-(\lambda)$. The first condition is satisfied for all model classes of RLPE, in particular, for NIG. In the next subsection, it is shown that in the case of NIG, the latter condition is satisfied for all $\lambda \in \mathcal{L}$, provided $\mathcal{L}$ is constructed properly. Under certain conditions a universal construction of the contour $\mathcal{L}$ is suggested.

By transforming the line of the integration (2.11) into the integral over the banks of the cut $[i\lambda_+, +i\infty)$ and using the residue theorem, we obtain

$$\phi_-(\lambda, \xi) = \frac{-\beta_-(\lambda)}{-\beta_-(\lambda) + i\xi} \exp(\Phi_-(\lambda, \xi)), \tag{3.7}$$

where

$$\Phi_-(\lambda, \xi) = \frac{1}{2\pi} \int_{\lambda_+}^{+\infty} \left[\frac{\psi'(iz - 0)}{i\lambda + r + \psi(iz - 0)} - \frac{\psi'(iz + 0)}{i\lambda + r + \psi(iz + 0)} \right] \cdot \ln \frac{z + i\xi}{z} dz. \tag{3.8}$$

By using (3.7), we can rewrite (3.6) as

$$v(x, \tau) = \frac{1}{2\pi i} \int_{\mathcal{L}} e^{i\lambda\tau + \beta_-(\lambda)x} \lambda^{-1} \exp \Phi_-(\lambda, -i\beta_-(\lambda)) d\lambda. \tag{3.9}$$

Equations (3.9) and (3.8) give an approximate formula for $v(x, \tau)$, which can be simplified. It can be shown that $\Phi_-(\lambda, -i\beta_-(\lambda))$ is small if $|\lambda|$ is not too large (below, the detailed study is made for the case of NIG). Hence, if we approximate $\Phi_-(\lambda, -i\beta_-(\lambda))$ in (3.9) with a relatively small error, the resulting relative error in (3.9) will be smaller still. Due to (3.5), for $|\lambda| << \lambda_+$ and $z \geq \lambda_+$, we have

$$\ln \frac{z + \beta_-(\lambda)}{z} = \frac{\beta_-(\lambda)}{z} + O((\beta_-(\lambda)/z)^2),$$

therefore (3.9) can be simplified further

$$v(x, \tau) = \frac{1}{2\pi i} \int_{\mathcal{L}} e^{i\lambda\tau + \beta_-(\lambda)x + \mathcal{P}(\lambda)} \lambda^{-1} d\lambda, \tag{3.10}$$

where

$$\mathcal{P}(\lambda) = \frac{1}{2\pi} \int_{\lambda_+}^{+\infty} \left[\frac{\psi'(iz - 0)}{i\lambda + r + \psi(iz - 0)} - \frac{\psi'(iz + 0)}{i\lambda + r + \psi(iz + 0)} \right] \frac{\beta_-(\lambda)}{z} dz. \tag{3.11}$$

A pair of formulas (3.10)-(3.11) is the approximate pricing procedure for first-touch digitals; to apply it, one must choose an appropriate contour $\mathcal{L}$. In the next subsection, we provide this construction in the case of NIG, and explicitly calculate the integral (3.11). In the result, we obtain the pricing formula, which involves only one integration.

3.2. **The case of NIG.** For NIG, (3.2) assumes the form

$$i\lambda + r - i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}] = 0. \tag{3.12}$$

Let condition (3.1) hold, in particular, let α be large. We introduce some simplifying notation, change the variable, and in the end, describe $\mathcal{L}$ with the properties (3.3)–(3.5). Set

$$\sigma_0 = r + \mu\beta - \delta(\alpha^2 - \beta^2)^{1/2},$$

$$\rho = \frac{\sigma_0}{\alpha\sqrt{\delta^2 + \mu^2}}, \quad u_1 = \frac{\mu}{\sqrt{\delta^2 + \mu^2}}, \quad v_1 = \frac{\delta}{\sqrt{\delta^2 + \mu^2}}; \tag{3.13}$$

$$U = \alpha(\sqrt{\delta^2 + \mu^2}\tau + u_1 x), \quad V = \alpha v_1 x; \tag{3.14}$$

then $U \in \mathbf{R}$, and $V > 0$. We calculate the roots of (3.12), change the variable $\lambda = \alpha\sqrt{\delta^2 + \mu^2}k + i\sigma_0$ in (3.10), and obtain

$$v(x, \tau) = \frac{e^{-\sigma_0\tau - \beta x}}{2\pi i}\int_{\mathcal{L}'}\frac{\exp[iUk - V\sqrt{1 + k^2} + \mathcal{P}_0(k)]}{k + i\rho}dk, \tag{3.15}$$

where $\mathcal{P}_0(k) = \mathcal{P}(\alpha\sqrt{\delta^2 + \mu^2}k + i\sigma_0)$, and the branch of the square root is determined by the condition $\Re\sqrt{1 + k^2} > 0$; the contour $\mathcal{L}'$ will be chosen so that $\Re(1 + k^2) \notin (-\infty, 0]$. For details, see the appendix.

Set $\beta' = \beta/\alpha, \beta'_\pm(k) = \beta_\pm(\alpha\sqrt{\delta^2 + \mu^2}k + i\sigma_0)/\alpha$. By calculating the integral in (3.11) (see the appendix), we obtain

$$\mathcal{P}_0(k) = \frac{\beta'_-(k)\mathcal{P}_+(k) + \beta'_+(k)\mathcal{P}_-(k) + I_0(k)}{2\pi\beta'_+(k)}, \tag{3.16}$$

where

$$\mathcal{P}_\pm(k) = i\ln((k \pm \sqrt{1 + k^2})(v_1 + iu_1)); \tag{3.17}$$

$$I_0(k) = \frac{4v_1(i\beta' k - u_1)(\pi/2 - \arctan\sqrt{\frac{1+\beta'}{1-\beta'}})}{\sqrt{1 - \beta'^2}}, \tag{3.18}$$

$$\beta'_\pm(k) = -\beta' + iu_1 k \pm v_1\sqrt{1 + k^2}. \tag{3.19}$$

Introduce new parameters

$$W = \sqrt{U^2 + V^2}, \quad u^* = U/W, \quad v^* = V/W, \quad R = \frac{e^{-\sigma_0\tau - \beta x}}{2\pi i}, \tag{3.20}$$

and functions

$$S(k) = iu^*k - v^*\sqrt{1+k^2}, \quad G(k) = \frac{\exp(\mathcal{P}_0(k))}{k+i\rho}.$$

In the new notation,

$$v(x,\tau) = R\int_{\mathcal{L}'} \exp(WS(k))G(k)dk. \tag{3.21}$$

We restrict ourselves to the cases

$$0 < -u_1 < 2\beta' < v_1 < -\rho, \tag{3.22}$$

and

$$0 < u_1 < -2\beta' < v_1 < -\rho, \tag{3.23}$$

which are natural under condition (3.1). Set

$$u_0 = \min\{v_1, -2\beta' u_1 + v_1\sqrt{1-4\beta'^2}, -u_1/\beta', -\rho - 0.05\}, \tag{3.24}$$

and for u satisfying

$$| u_1 | < u \le u_0, \tag{3.25}$$

define a contour $\mathcal{L}'(u)$ in k-plane by

$$\mathcal{L}'(u) = \{k = k_u(s) \mid k_u(s) = \sqrt{1-u^2}s + iu\sqrt{s^2+1}, s \in \mathbf{R}\}. \tag{3.26}$$

Usually $\beta' > 0$, and further, we will consider the case (3.22). The case $\beta' < 0$ ((3.23)) is a simple modification of the case (3.22).

It can be shown that the corresponding contour $\mathcal{L}(u)$ in λ-plane

$$\mathcal{L}(u) = \{\lambda = \lambda_u(s) | \lambda_u(s) = \alpha\sqrt{\delta^2+\mu^2}k_u(s) + i\sigma_0, s \in \mathbf{R}\}, \tag{3.27}$$

satisfies conditions (3.4)-(3.5), provided (3.1), (3.22) and (3.25) hold. See the appendix for details.

Fix $u \in (-u_1; u_0)$ and set $v = \sqrt{1-u^2}$. Now we write down the formula for $v(x,\tau)$ by deforming first the line of the outer integration in the formula (2.12) into $\mathcal{L}(u)$ given by (3.27), and then the line of the inner integration into an appropriate contour hanging over the upper root $-i\beta_-(\lambda)$. Notice that in the process of deformation the contour of outer integration never reaches the cuts (since $|u| < 1$), and does not cross the pole (since $u + \rho < 0$). Finally, we obtain

$$v(x,\tau) = R\int_{-\infty}^{+\infty} e^{-W(uu^*+vv^*)\sqrt{s^2+1}+iW(vu^*-uv^*)s} G(k_u(s))k_u'(s)ds, \tag{3.28}$$

where parameters ρ and R, W given by (3.13) and (3.20), respectively. In order that the integrand in (3.28) oscillates slowly, we choose u to minimize $|vu^* - uv^*|$. Usually, the choice $u = u^*$ is possible. In this case $\mathcal{L}(u)$ is the curve of steepest descent (cf. [9]).

To compute numerically the integral in (3.28), we need to cut off an appropriate neighbourhood $|s| \geq \Lambda$ of the infinity (and we will choose $\Lambda \geq 1$). Set

$$a = -\frac{u_1 v_1}{3\pi}; \tag{3.29}$$

$$b = \frac{1}{4}\left(\frac{\beta' + \sqrt{2}}{\sqrt{2}v_1^2} - \frac{2}{3}\right) + \frac{v_1}{2\sqrt{1-\beta'^2}}\left(\frac{\beta' v_1}{3} - \frac{u_1}{\beta'(2\sqrt{2}-1)}\right); \tag{3.30}$$

$$c = \frac{2\sqrt{2}|R|e^b}{v(W(uu^* + vv^*) - a)}. \tag{3.31}$$

Direct calculations in the appendix show that the integral over $|s| \geq \Lambda$ (we denote it by $I(\Lambda)$) admits an estimate via

$$\epsilon(\Lambda) := c\exp[(a - W(uu^* + vv^*))\sqrt{\Lambda^2+1}], \quad \Lambda \geq 1. \tag{3.32}$$

For any given ϵ, we choose $\Lambda, \Lambda \geq 1$, so large that $\epsilon(\Lambda) = \epsilon/2$. As soon as Λ is chosen, we use the trapezoid rule to compute the integral over $[-\Lambda; \Lambda]$. To determine the number of points for integration, it is difficult to use the usual estimate for the computational error of the trapezoid method; the so-called "doubling procedure" is more efficient. Each next approximation is obtained by doubling of the number of points for integration until the absolute difference between two last approximations is less then the desired value $\epsilon/2$.

While considering the integral over a finite segment $[-\Lambda, \Lambda]$, the pole of the integrand must be taken care of. It follows from the explicit expression for $k(s)$ that $k(0) + i\rho$ is small iff $u + \rho$ is. Numerical calculations show that in the neighbourhood of zero, the integrand in (3.28) assumes not very large values, provided $u + \rho > 0.05$.

3.3. **Algorithm.** In this subsection, we summarize the constructions above in a form suitable for immediate implementation. We normalize $K = 1$, fix $\epsilon > 0$, the computational error, and assume that NIG's parameters are given. After that we

1. define parameters u_1, v_1, ρ by (3.13), and a, b by (3.29)-(3.30);
2. fix $x = \ln S/H$ and $\tau = T - t$, and check conditions (3.1) and (3.22); if the former fails, then the approximate pricing formula may produce significant errors, and if the latter does then a different formula (not written here) must be used;
3. define W, u^*, v^*, R by (3.20), c by (3.31) and u_0 by (3.24);
4. if $u^* \in (-u_1; u_0]$ then set $u = u^*$; otherwise, set $u = u_0$;
5. set $v = \sqrt{1-u^2}$;

6. set

$$\Lambda = \max\left\{1; \left[\left(\frac{\ln(2c\epsilon^{-1})}{W(uu^* + vv^*) - a}\right)^2 - 1\right]^{1/2}\right\}$$

(cf. (3.32));

7. apply the trapezoid method of numerical integration and "doubling procedure" to (3.28), and calculate the integral over $[-\Lambda; \Lambda]$ until the computational error assumes the value $\epsilon/2$; $k_u(s)$ in (3.28) is defined in (3.26).

4. Numerical examples

Fix the riskless rate $r > 0$, and assume that under an EMM chosen by market, X is a NIG with parameters δ, α, β. When these parameters are chosen, the last parameter, μ, is fixed by the EMM requirement (2.1). In empirical studies of financial markets (see [3, 14]), typically, the instantaneous variance σ^2 is in the range (0.15, 0.5), and α is large: of order 25 - 60 or even more, and $|\beta/\alpha|$ is small. Notice that in the case of NIG

$$\sigma^2 := \psi''(0) = \frac{\delta}{\alpha(1-(\beta/\alpha)^2)^{3/2}},$$

and conditions

$$\alpha \geq 25, \tag{4.1}$$

$$0.15 < \sigma^2 < 0.5; \tag{4.2}$$

$$0 < \beta/\alpha << 1; \tag{4.3}$$

together with the EMM requirement define a surface in the (α, δ, β, μ) - space for the parameters of NIG, where good performance of the approximate formula derived in Section 3 can be guaranteed, as our numerical experiments show.

In Fig. 1 we plot the NIG-price $v_{NIG}(x,\tau)$ of the first touch digital for typical parameter's values, and in Fig. 2, we compare $v_{NIG}(x,\tau)$ with the price in the Black - Scholes model, $v_{BS}(x,\tau)$, calculated for the volatility $\sigma^2 = (\psi)''(0)$. Numerical examples show that the most significant difference $\Delta_{BS}(x,\tau) := v_{BS}(x,\tau) - v_{NIG}(x,\tau)$ is for x in the neighbourhood of $-\mu\tau$. The absolute error Δ_{BS} has the maximum fairly close to zero, and starting with $\tau \geq 5$, it decreases rather slowly as S/K increases. It remains sizable up to $S/K = 1.2$ and even farther, where the price itself is rather small. Hence, the study of relative errors is more natural. We introduce relative error functions for the asymptotic formula and Black-Scholes approximation

$$\epsilon_{asymp} = \frac{v_{asymp}(x,\tau) - v_{NIG}(x,\tau)}{v_{NIG}(x,\tau)}, \quad \epsilon_{BS} = \frac{v_{BS}(x,\tau) - v_{NIG}(x,\tau)}{v_{NIG}(x,\tau)},$$

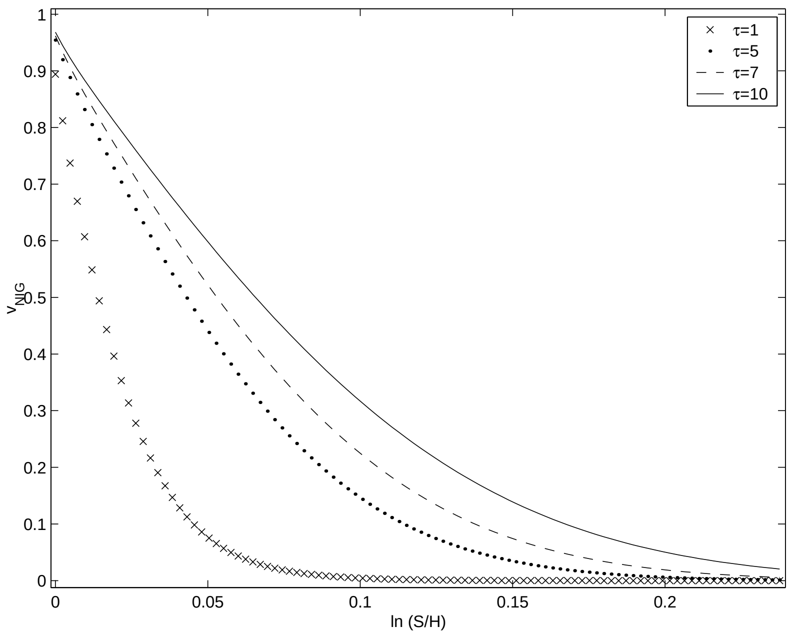

FIGURE 1. The NIG-price of the first touch digital at $\tau = 1, 5, 7, 10$ days to expiry. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6$

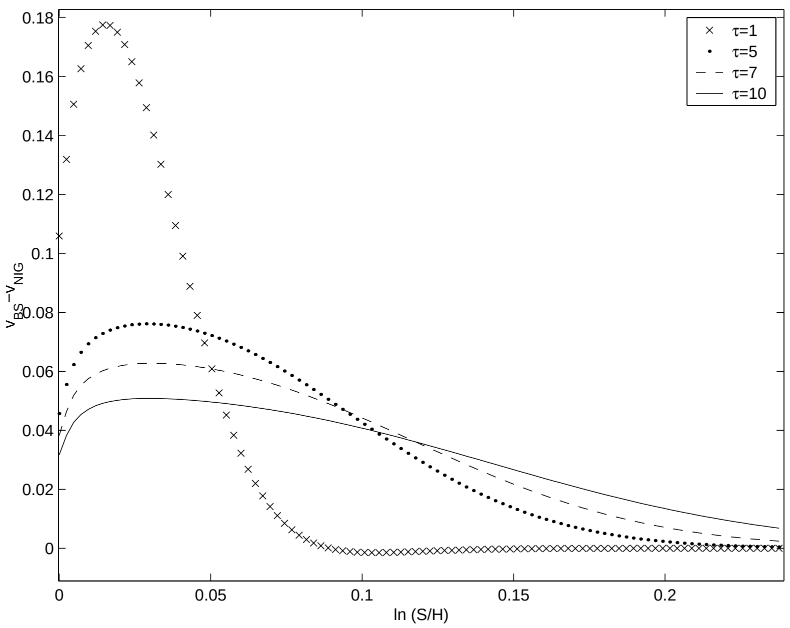

FIGURE 2. The difference of the Gaussian price and NIG-price, at $\tau = 1, 5, 7, 10$ days to expiry. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6$

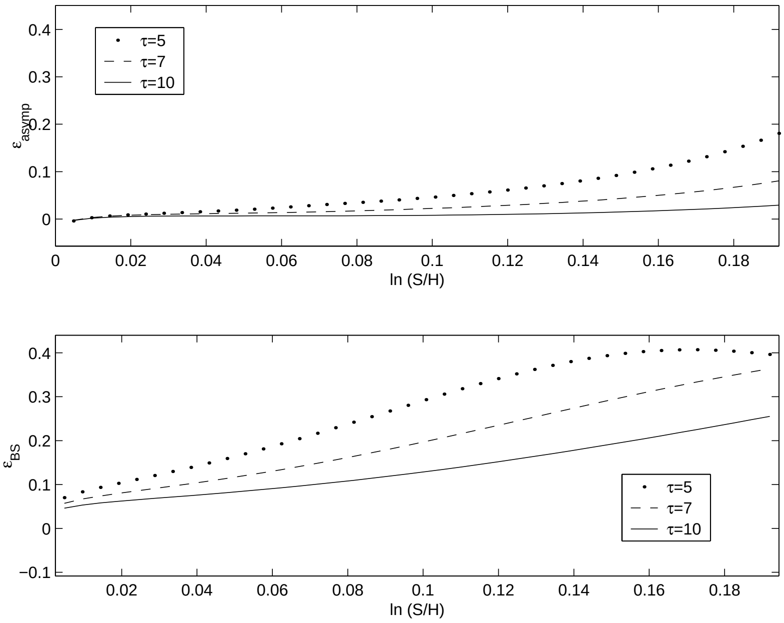

FIGURE 3. Relative error functions ϵ_{asymp} and ϵ_{BS}. Upper panel: approximate NIG; lower panel: Black-Scholes. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6, \sigma^2 = 0.2845$ at $\tau = 5, 7, 10$ days to expiry.

and

$$\epsilon^*_{asymp} = \frac{v_{asymp}(x,\tau) - v_{NIG}(x,\tau)}{1 - v_{NIG}(x,\tau)}, \quad \epsilon^*_{BS} = \frac{v_{BS}(x,\tau) - v_{NIG}(x,\tau)}{1 - v_{NIG}(x,\tau)},$$

which are plotted in Fig. 3 and Fig. 4, respectively, for the chosen parameter's values. The role of the relative errors ϵ^*_{asymp} and ϵ^*_{BS} is two-fold: first, near the boundary the leading term of the price v is 1, therefore the quality of an approximate formula for v can be better characterized by the relative error of calculation of $1 - v$ rather than v itself; second, qualitatively, near the boundary, $1 - v$ behaves as the barrier price, hence ϵ^*_{asymp} and ϵ^*_{BS} can be used as qualitative proxies for expected relative errors of the corresponding formulas for barrier options, near the barrier. Typically, the relative error ϵ_{asymp} is less than 5 per cent in the region $(0\ ;\ 0.2)$ for $\tau \geq 10$ days to expiry, and it decreases with increasing τ and α, δ. For smaller τ, ϵ_{asymp} is larger but still much less that ϵ_{BS}. If $\alpha \in (30, 50)$, then the relative error ϵ_{asymp} grows from -1% up to several per cent in the region, where $v_{NIG}(x,\tau) > 0.01$. Usually, $v_{NIG}(x,5) > 0.01$ and $v_{NIG}(x,10) > 0.01$ as $x < 0.15$ and $x < 0.2$, respectively (see Fig.1). In this region the

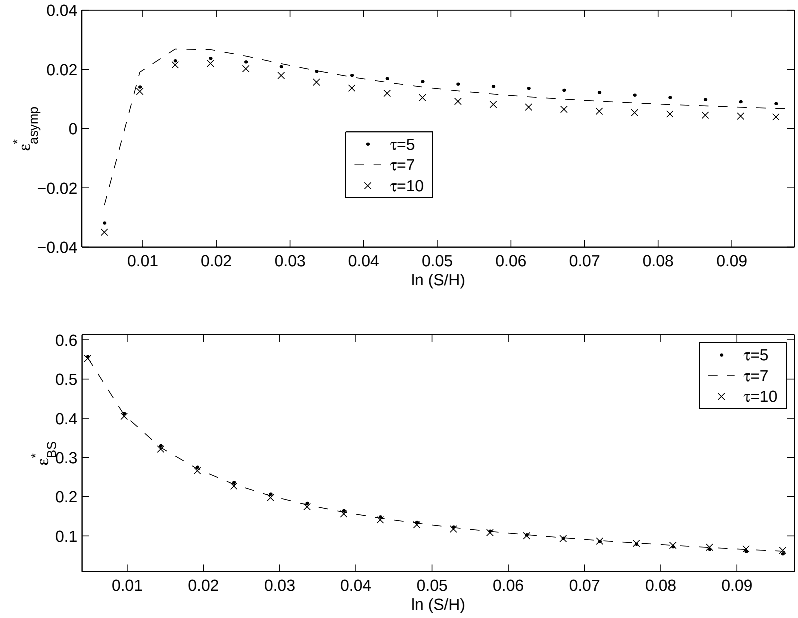

FIGURE 4. Relative error functions ϵ^*_{asymp} and ϵ^*_{BS}. Upper panel: approximate NIG; lower panel: Black-Scholes. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6, \sigma^2 = 0.2845$ at $\tau = 5, 7, 10$ days to expiry.

Black-Scholes error ϵ_{BS} is essentially larger than ϵ_{asymp}, and it can be several dozen percent; it weakly decreases as the volatility increases and α remains fixed. Fig. 3 demonstrates this effect. Comparing ϵ_{asymp} and ϵ_{BS}, we conclude that the approximation of $v_{NIG}(x, \tau)$ by the asymptotic formula $v_{asymp}(x, \tau)$ is much better than the approximation by the Gaussian price $v_{BS}(x, \tau)$, in the region $\alpha \in (30, 50)$, $\tau > 5$ days, $x \in (0, 0.2)$. For $\alpha \in (50, 80)$, $\tau > 5$ days, $x \in (0, 0.2)$, the relative error $(v_{BS}(x, \tau) - v_{asymp}(x, \tau))/v_{BS}(x, \tau)$ is 10 -25 percent, and we expect that ϵ_{BS} is of the same order of magnitude: the larger α, the better the asymptotic formula works, and the less ϵ_{asymp} is (for large α, the performance of the numerical procedure is rather poor, and so it is better to characterize the performance of the Black-Scholes model by using the approximate formula).

In Fig.4, we compare relative errors ϵ^*_{asymp} and ϵ^*_{BS}. Usually, $|\epsilon^*_{asymp}|$ is less than $4 - 5\%$ and $\epsilon^*_{BS} \in (0.2; 0.6)$ for $\ln(S/K) \in (0.005; 0.05)$. For larger $\ln(S/K)$, the relative advantage of the asymptotic formula is also clear.

Set $\kappa_- = 0.5 - \pi^{-1} \arctan(\mu/\delta)$. It is shown in [7] that near the barrier, for $\tau > 0$ fixed (and not too small) the difference $1 - v_{NIG}(x, \tau)$

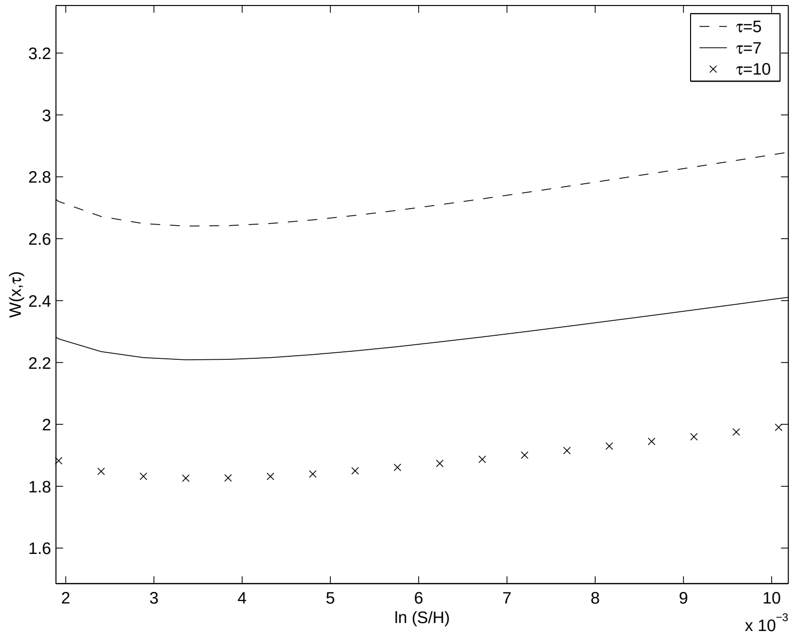

FIGURE 5. Graph of $W(x,\tau)$ near the barrier. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6, \kappa_- = 0.5505$ at $\tau = 5, 7, 10$ days to expiry.

behaves approximately like $const \cdot x^{\kappa_-}$. The numerical calculations show that this is really the case in the region $x \in (0; 0.01)$. Recall that in the Gaussian model, the price is smooth up to the boundary, and therefore, the difference $1 - v_{BS}(x,\tau)$ is approximately linear as a function of x, near the barrier. Notice that κ_- is approximately 0.5, provided parameters satisfy conditions (4.1)-(4.3). Thus, one should expect the significant difference between NIG and Gaussian prices in the neighbourhood of origin. This is confirmed in Fig.4, the lower panel. Further, we conclude that for a fixed $\tau > 0$, the function

$$W(x,\tau) := (1 - v_{NIG}(x,\tau))/x^{\kappa_-}$$

is approximately constant near the barrier, and Fig. 5 demonstrates this effect.

To estimate the deviation between $v_{NIG}(x,\tau)$ and the approximate formula $v_{asymp}(x,\tau)$, we introduce the absolute error function

$$\Delta_{asymp}(x,\tau) := v_{asymp}(x,\tau) - v_{NIG}(x,\tau),$$

and the similar function for the price in the Black-Scholes model. In a rather small neighborhood of the barrier, Δ_{asymp} is negative near the barrier, then it sharply grows till its positive maximum, and then

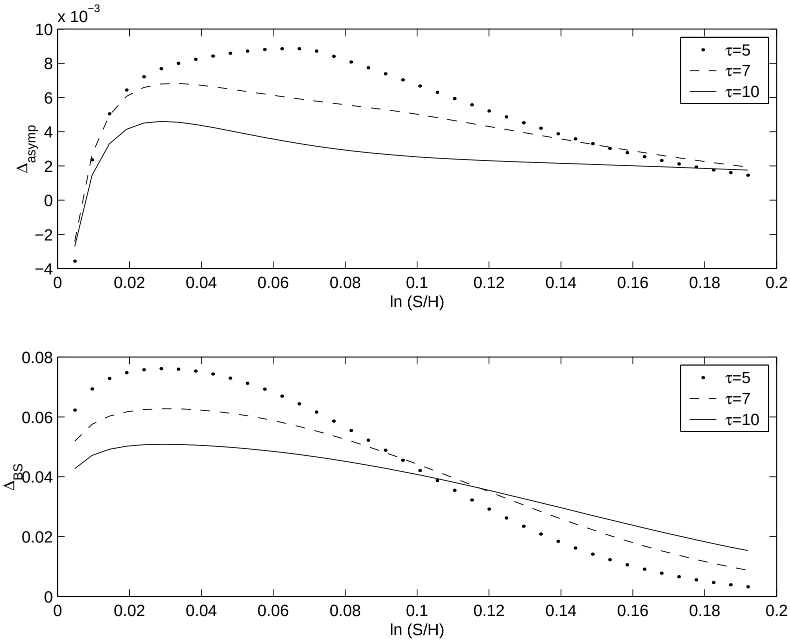

Figure 6. Absolute error functions. Upper panel: approximate NIG; lower panel: Black-Scholes. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6, \sigma^2 = 0.2845$ at $\tau = 5, 7, 10$ days to expiry.

it slowly decreases. Fig. 6 compares absolute errors $\Delta_{asymp}(x, \tau)$ and $\Delta_{BS}(x, \tau)$. In particular, $|\Delta_{BS}|$ is several times larger than $|\Delta_{asymp}(x, \tau)|$, and the larger x, the less $|\Delta_{BS}/\Delta_{asymp}|$. (Notice that in the upper graph, the vertical scale is 10 times less than in the lower one).

Numerical examples for typical parameters' values are presented in Table 1.

5. Conclusion

In the paper, typical behavior of prices of first-touch digitals in the non-gaussian NIG model is analyzed and compared to the behavior of prices in the Black-Scholes model with the same instantaneous variance. Starting from the barrier, the NIG-price sharply falls below the gaussian price, and it remains lower in the region, which depends on time to expiry. The smaller the τ, the sharper the peak of the difference, and the faster the decay of the latter. For typical parameters values, in the range 5-20 percent to barrier, the relative error of the

Table 1. Error functions

A

τ	5 days			7 days			10 days		
$\ln(S/K)$	0.05	0.1	0.15	0.1	0.15	0.2	0.1	0.15	0.2
ϵ_{asymp}	0.016	0.046	0.082	0.027	0.045	0.070	0.015	0.023	0.035
ϵ_{BS}	0.156	0.285	0.410	0.193	0.286	0.371	0.130	0.187	0.249
Δ_{asymp}	0.009	0.010	0.006	0.008	0.006	0.004	0.006	0.005	0.004
Δ_{BS}	0.082	0.062	0.031	0.059	0.039	0.022	0.052	0.041	0.029

B

τ	5 days			7 days			10 days		
$\ln(S/K)$	0.05	0.1	0.15	0.1	0.15	0.2	0.1	0.15	0.2
ϵ_{asymp}	0.007	0.033	0.108	0.003	0.032	0.126	-0.014	-0.008	0.024
ϵ_{BS}	0.153	0.255	0.256	0.177	0.229	0.257	0.112	0.157	0.204
Δ_{asymp}	0.003	0.003	0.002	0.001	0.001	0.001	-0.003	-0.001	0.001
Δ_{BS}	0.061	0.023	0.004	0.028	0.009	0.002	0.027	0.014	0.006

C

τ	5 days			7 days			10 days		
$\ln(S/K)$	0.05	0.1	0.15	0.1	0.15	0.2	0.1	0.15	0.2
ϵ_{asymp}	0.019	0.047	0.092	0.023	0.043	0.081	0.008	0.015	0.029
ϵ_{BS}	0.159	0.293	0.394	0.198	0.291	0.364	0.129	0.190	0.255
Δ_{asymp}	0.009	0.007	0.003	0.005	0.003	0.002	0.003	0.002	0.002
Δ_{BS}	0.073	0.042	0.014	0.044	0.022	0.009	0.041	0.027	0.015

D

τ	5 days			7 days			10 days		
$\ln(S/K)$	0.05	0.1	0.15	0.1	0.15	0.2	0.1	0.15	0.2
ϵ_{asymp}	0.029	0.087	0.211	0.043	0.100	0.238	0.018	0.041	0.093
ϵ_{BS}	0.234	0.492	0.581	0.336	0.511	0.622	0.214	0.350	0.494
Δ_{asymp}	0.011	0.006	0.002	0.006	0.003	0.001	0.004	0.003	0.002
Δ_{BS}	0.084	0.034	0.006	0.043	0.014	0.003	0.045	0.024	0.010

E

τ	5 days			7 days			10 days		
$\ln(S/K)$	0.05	0.1	0.15	0.1	0.15	0.2	0.1	0.15	0.2
ϵ_{asymp}	0.035	0.094	0.183	0.019	0.062	0.145	0.004	0.020	0.053
ϵ_{BS}	0.247	0.406	0.385	0.253	0.316	0.301	0.174	0.236	0.276
Δ_{asymp}	0.015	0.012	0.006	0.004	0.004	0.003	0.001	0.003	0.003
Δ_{BS}	0.103	0.050	0.012	0.051	0.022	0.007	0.051	0.030	0.015

Panel A. Parameters: $r = 0.05, \delta = 12, \alpha = 30, \beta = 5, \mu = -2.19, \sigma^2 = 0.42$.
Panel B. Parameters: $r = 0.05, \delta = 10, \alpha = 50, \beta = 5, \mu = -1.06, \sigma^2 = 0.2$.
Panel C. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6, \mu = -1.76, \sigma^2 = 0.29$.
Panel D. Parameters: $r = 0.05, \delta = 7.2, \alpha = 40, \beta = 8, \mu = -1.5, \sigma^2 = 0.19$.
Panel E. Parameters: $r = 0.05, \delta = 8, \alpha = 30, \beta = 3, \mu = -0.89, \sigma^2 = 0.27$.
ϵ_{asymp} (Δ_{asymp}) and ϵ_{BS} (Δ_{BS}) are relative (absolute) error functions for the approximate formula in the NIG model, and the Black-Scholes model with the same instantaneous variance, respectively.

gaussian approximation is dozens percent, if the time to expiry, τ, is less than 10 days.

A fast efficient approximate pricing formula is derived. The formula is the leading term of the asymptotics of the exact pricing formula (the latter is too complex for numerical implementation) as the steepness parameter α in the definition of a NIG process tends to infinity. The α characterizes the rate of the exponential decay of the tails of probability densities (provided the asymmetry parameter β is relatively small which is usually the case); thus, the approximate formula works better when the tails decay fast. It is shown that for typical parameters values, the formula performs well at 5 days to expiry and more, and at the relative distance to the barrier 3 percent or more; the larger the time to expiry and distance to barrier are, the better the performance of the formula. Typically, the relative error of the asymptotic formula is several times less than that of the Black-Scholes approximation.

Appendix A. Numerical procedure

We approximate $V_{\mathrm{d}}(T; X_t, t)$ by the price $f(X_t, t)$ of the touch and out option in the corresponding discrete time model with equally spaced dates t_k, $k = 0, 1, \dots, m$, where $t_0 = 0$, $t_m = T$. Set $\Delta\tau := T/m$. We have

$$f(x, t_m) = 0, \ x > 0, \tag{A.1}$$

and for all k,

$$f(x, t_k) = 1, \ x \le 0. \tag{A.2}$$

For $k = m-1, m-2, \dots$, and $x > 0$, the price $f(x, t_k)$ can be found as the price of the European option with the terminal payoff $f(X_{t_{k+1}})$ and the expiry date t_{k+1}:

$$f(x, t_k) = E[e^{-r\Delta\tau} f(X_{t_{k+1}}) \mid X_{t_k} = x], \ x > 0. \tag{A.3}$$

In the case of NIG, an explicit formula for the probability density $p_{\Delta\tau}$ of X under EMM is known (see (2.3)), and we can use it to write (A.3) in the form

$$f(x, t_k) = e^{-r\Delta\tau} \int_{-\infty}^{-x} p_{\Delta\tau}(y) dy + e^{-r\Delta\tau} \int_{-x}^{+\infty} p_{\Delta\tau}(y) f(x + y, t_{k+1}) dy. \tag{A.4}$$

Denote by $I_1(-x)$ and $I_2(x, k)$ the first and the second summands in the R.H.S. of (A.4), respectively.

In the general case, $p_{\Delta\tau}$ can be expressed in terms of the characteristic exponent $\psi(\xi)$, by using the Fourier transform

$$p_{\Delta\tau}(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi - \Delta\tau\psi(\xi)} d\xi, \tag{A.5}$$

and $I_1(-x)$ assumes the form

$$I_1(-x) = (2\pi i)^{-1} \int_{-\infty+i\sigma}^{+\infty+i\sigma} e^{ix\xi-\Delta\tau(r+\psi(\xi))}\xi^{-1}d\xi, \qquad (A.6)$$

where $\sigma \in (-\alpha+\beta, 0)$ is arbitrary.

Now we turn to $I_2(x,k)$. We choose large $\Lambda > 0$ and N, so that $h := \Lambda/N$ is small, and construct the grid $x_j = \pm jh,\ j = 1,2,\dots,N$. For a chosen step of numerical integration, h, with the weight function $e^{-r\Delta\tau}p_{\Delta\tau}(x)$, we can express $I_2(x_l,k)$ as follows.

$$I_2(x_l,k) \approx \sum_{j=-l}^{N-1} c_j f(x_j + x_l, t_{k+1}), \qquad (A.7)$$

where

$$c_j = \int_{x_j}^{x_{j+1}} e^{-r\Delta\tau}p_{\Delta\tau}(y)dy.$$

(When $|j+l| > N$, we set the corresponding term to be zero). Notice that the integrals $I_1(x_l)$ and coefficients c_j can be found with the desired relative error by using a simple modification of the fast option pricing methods (cf. [7], [12]). Denote by $I_1^*(x_l)$ and c_j^* their numerical values, respectively. On the first step, we set

$$f(x_l, T-\Delta\tau) = I_1^*(x_{-l}), l = 0,1,\dots,N.$$

On the step k, $k > 1$, we obtain an approximation for $f(x_l, T-k\Delta\tau), l = 0,1,\dots,N$

$$f(x_l, T-k\Delta\tau) \approx I_1^*(x_{-l}) + \sum_{-l}^{N-1} c_j^* f(x_l + x_j, T-(k-1)\Delta\tau). \qquad (A.8)$$

We increase Λ and decrease h till the relative difference between two results becomes less than 0.001. To test the program we use the trapezoid rule for the integrals $I_2(x,k)$. Numerical calculations show that the maximum difference between the results of the two programs is about 0.01, mostly much smaller, and relative difference is less than four percent even at τ =10 days to expiry; for smaller τ, the relative difference is smaller. The difference between the two numerical procedures for typical parameters values is shown on Fig. 7. We see that the error of the numerical procedure makes relatively insignificant impact even on the quantitative results in Section 4, nothing to say about qualitative description of behavior of prices and performance of the approximate formula and the formula in the Black-Scholes model.

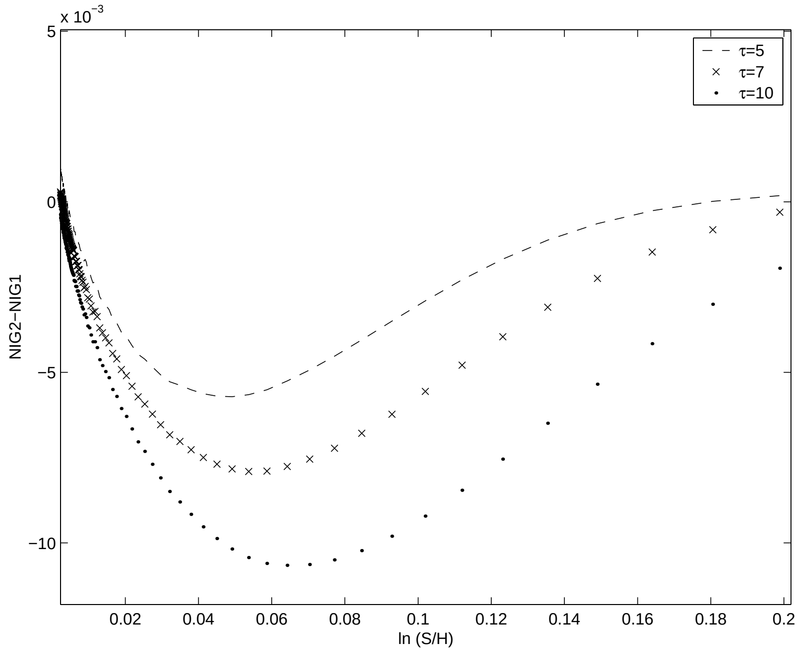

FIGURE 7. Difference between results of two numerical procedures. Parameters: $r = 0.05, \delta = 11, \alpha = 40, \beta = 6$, at $\tau = 5, 7, 10$ days to expiry.

APPENDIX B. TECHNICAL DETAILS

B.1. **Computation of $\mathcal{P}(\lambda)$.** We make the change of variable $z = \beta + \alpha w$, and introduce the shorthand notation $\lambda' = \lambda/\alpha,\ \sigma'_0 = \sigma_0/\alpha,\ \beta' = \beta/\alpha$. The direct albeit lengthy calculations in (3.11) lead to

$$\begin{aligned}\mathcal{P}(\lambda) &= \frac{\delta\beta_-(\lambda)}{\pi\alpha}\int_1^\infty \frac{\mu(w^2-1) - w(i\lambda' + \sigma' + \mu w)}{(i\lambda' + r' + \mu\beta' + \mu w - \delta\sqrt{1-(\beta')^2})^2 + \delta^2(w^2-1)} \\ &\qquad \cdot \frac{dw}{(w+\beta')\sqrt{w^2-1}} \\ &= -\frac{\delta\beta_-(\lambda)}{\pi\alpha(\delta^2+\mu^2)}\int_1^\infty \frac{\mu + w(i\lambda' + \sigma')}{(w - w_+(\lambda))(w - w_-(\lambda))}\frac{dw}{(w+\beta')\sqrt{w^2-1}},\end{aligned}$$

where

$$w_\pm = w_\pm(\lambda) := -\frac{\beta_\pm(\lambda) + \beta}{\alpha} = -iu_1 k \mp v_1\sqrt{1+k^2}, \tag{B.1}$$

Recall that $\lambda = \alpha\sqrt{\delta^2+\mu^2}k + i\sigma_0, s \in \mathbf{R}$. By calculating the integral, we obtain the following explicit formula for $\mathcal{P}(\lambda)$:

$$\mathcal{P}(\lambda) = c_+(\lambda)\mathcal{P}_1(w_+(\lambda)) + c_-(\lambda)\mathcal{P}_1(w_-(\lambda)), \tag{B.2}$$

where

$$c_\pm(\lambda) = \frac{2\delta}{\delta^2+\mu^2} \cdot \frac{\mu + w_\pm(\lambda)(i\lambda' + \sigma')}{w_\pm(\lambda) - w_\mp(\lambda)} = iv_1 k \mp u_1\sqrt{1+k^2}; \qquad \text{(B.3)}$$

$$\begin{aligned} \mathcal{P}_1(w_\pm(\lambda)) &= -\frac{\beta_-(\lambda)}{2\pi\alpha}\int_1^\infty \frac{dw}{(w-w_\pm(\lambda))(w+\beta')\sqrt{w^2-1}} \\ &= \frac{\beta_-(\lambda)}{2\pi\beta_\pm(\lambda)}(\mathcal{P}_2(w_\pm(\lambda)) - I_1); \qquad \text{(B.4)} \end{aligned}$$

$$I_1 = \frac{2}{\sqrt{1-(\beta/\alpha)^2}}(\pi/2 - \arctan\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}), \qquad \text{(B.5)}$$

and

$$\begin{aligned} \mathcal{P}_2(w_\pm(\lambda)) &= \int_1^{+\infty} \frac{dw}{(w-w_\pm(\lambda))\sqrt{w^2-1}} \\ &= \frac{1}{i\sqrt{1-w_\pm(\lambda)^2}}\ln(-w_\pm(\lambda) + i\sqrt{1-w_\pm(\lambda)^2}) \\ &= -\frac{\ln((k\pm\sqrt{1+k^2})(v_1+iu))}{i(iv_1k \mp u_1\sqrt{1+k^2})}. \qquad \text{(B.6)} \end{aligned}$$

Gathering (B.2)-(B.6), we obtain the explicit formula (3.16) for $\mathcal{P}_0(k)$.

B.2. **Verification of condition (3.3).** The roots $-i\beta_\pm(\lambda)$ of equation (3.12) are

$$\begin{aligned} \beta_-(\lambda) &= -\beta + \frac{\mu(\sigma_0 + i\lambda) - \delta\sqrt{(\delta^2+\mu^2)\alpha^2 - (i\lambda+\sigma_0)^2}}{(\delta^2+\mu^2)}, \qquad \text{(B.7)} \\ \beta_+(\lambda) &= -\beta + \frac{\mu(\sigma_0 + i\lambda) + \delta\sqrt{(\delta^2+\mu^2)\alpha^2 - (i\lambda+\sigma_0)^2}}{(\delta^2+\mu^2)}. \qquad \text{(B.8)} \end{aligned}$$

For $\lambda = \lambda_u(s)$ (see (3.27)), we obtain

$$-\Re\beta_-(\lambda) = \beta - \alpha(uu_1 + vv_1)\sqrt{s^2+1};$$

$$-\Re\beta_+(\lambda) = \beta - \alpha(vv_1 - uu_1)\sqrt{s^2+1}.$$

Note that on the strength of (3.22) and (3.25), $uu_1 + vv_1 \geq 0$ and $\alpha(vv_1 - uu_1) \geq 2\beta$. Thus, (3.3) holds.

B.3. **Derivation of an estimate for $I(\Lambda)$.** For a fixed $\Lambda \ge 1$, we have

$$|I(\Lambda)| \le 2|R| \int_{\Lambda}^{+\infty} \frac{e^{-W(uu^*+vv^*)\sqrt{s^2+1}} e^{\mathcal{P}_0(k_u(s))} |k_u'(s)|}{|k_u(s)+i\rho|} ds. \tag{B.9}$$

Then we need estimates for $\frac{|k_u'(s)|}{|k_u(s)+i\rho|}$ and $e^{\mathcal{P}_0(k_u(s))}$ in the region $|s| \ge 1$.

$$\frac{|k_u'(s)|}{|k_u(s)+i\rho|} \le \left[\frac{s^2+v^2}{(s^2+1)(v^2s^2+(u\sqrt{s^2+1}+\rho)^2)}\right]^{1/2} \le \frac{1}{v}. \tag{B.10}$$

To obtain an estimate for $e^{\mathcal{P}_0(k_u(s))}$, we use (3.16)-(3.19), and the standard technique of calculus. Notice that

$$\beta_\pm'(k_u(s)) = -\beta' - (u_1 u \mp v_1 v)\sqrt{s^2+1} + i(u_1 v \pm v_1 u)s$$

and

$$\mathcal{P}_\pm(k_u(s)) = \pm i \ln(\sqrt{s^2+1}+s) - \theta_\pm,$$

where

$$\theta_+ = \pi/2 - \arctan\frac{vv_1 - uu_1}{vu_1 + uv_1}, \quad \theta_- = \pi/2 + \arctan\frac{vv_1 + uu_1}{uv_1 - vu_1}.$$

It follows from (3.25) that

$$0 < \theta_+ < \frac{\pi}{2}, \quad \frac{\pi}{2} < \theta_- < \pi. \tag{B.11}$$

We obtain

$$\begin{aligned} &\Re\frac{\beta_-'(k_u(s))\mathcal{P}_+(k_u(s)) + \beta_+'(k_u(s))\mathcal{P}_-(k_u(s))}{2\pi\beta_+'(k_u(s))} \qquad\qquad (B.12)\\ = \;& \frac{\ln(\sqrt{s^2+1}+s)}{2\pi}\Re\frac{\beta_-'(k_u(s)) - \beta_+'(k_u(s))}{\beta_+'(k_u(s))}i - \frac{\theta_+}{2\pi}\Re\frac{\beta_-'(k_u(s))}{\beta_+'(k_u(s))} - \frac{\theta_-}{2\pi}. \end{aligned}$$

Direct calculations show that under conditions (3.22) and (3.25)

$$\begin{aligned} &\frac{\ln(\sqrt{s^2+1}+s)}{2\pi}\Re\frac{\beta_-'(k_u(s)) - \beta_+'(k_u(s))}{\beta_+'(k_u(s))}i \\ = \;& -\frac{v_1(\beta' u + u_1)s\sqrt{s^2+1}\ln(\sqrt{s^2+1}+s)}{\pi|\beta_+'(k_u(s))|^2} \\ < \;& -\frac{u_1 v_1}{\pi} F_1(s) F_2(s) \sqrt{s^2+1}, \qquad\qquad (B.13) \end{aligned}$$

where for $|s| \geq 1$,

$$\begin{aligned} F_1(s) &:= \frac{s^2}{|\beta'_+(k_u(s))|^2} \\ &\leq \frac{s^2}{(-\beta' + (v_1 v - u_1 u)\sqrt{s^2+1})^2} \\ &\leq \frac{\beta'^2}{(v_1 v - u_1 u)^2 - \beta'^2)} \leq \frac{1}{3}, \end{aligned} \tag{B.14}$$

and

$$F_2(s) := \frac{\ln(\sqrt{s^2+1} + s)}{s} \leq \ln(1+\sqrt{2}) < 1. \tag{B.15}$$

(All the estimates below are also written for $|s| \geq 1$). Substituting (B.11) and (B.14)-(B.15) into (B.13) yields

$$\frac{\ln(\sqrt{s^2+1}+s)}{2\pi} \Re \frac{\beta'_-(k_u(s)) - \beta'_+(k_u(s))}{\beta'_+(k_u(s))} i < -\frac{u_1 v_1}{3\pi}\sqrt{s^2+1}, \tag{B.16}$$

Then we have

$$-\Re \frac{\beta'_-(k_u(s))}{\beta'_+(k_u(s))} = F_3(s) + F_4(s), \tag{B.17}$$

where

$$\begin{aligned} F_3(s) &:= \frac{(v_1 v\sqrt{s^2+1})^2 - (\beta' + u_1 u\sqrt{s^2+1})^2}{|\beta'_+(k_u(s))|^2} \\ &\leq \frac{\beta' + (v_1 v + u u_1)\sqrt{s^2+1}}{-\beta' + (v_1 v - u u_1)\sqrt{s^2+1}} \leq \frac{\beta' + \sqrt{2}(v_1 v + u u_1)}{-\beta' + \sqrt{2}(v_1 v - u u_1)} \\ &\leq \frac{\beta' + \sqrt{2}}{-\beta' + \sqrt{2}(v_1^2 - u_1^2)} < \frac{\beta' + \sqrt{2}}{\sqrt{2} v_1^2}; \end{aligned} \tag{B.18}$$

$$F_4(s) := \frac{(v_1^2 u^2 - u_1^2 v^2) s^2}{|\beta'_+(k_u(s))|^2} \leq \frac{v_1^2 u^2 - u_1^2 v^2}{3} < \frac{1}{3}. \tag{B.19}$$

By using (B.11) and (B.17)-(B.19), we derive the estimate for the second term in (B.12):

$$-\frac{\theta_+}{2\pi} \Re \frac{\beta'_-(k_u(s))}{\beta'_+(k_u(s))} < \frac{1}{4}\left(\frac{\beta' + \sqrt{2}}{\sqrt{2} v_1^2} + \frac{1}{3}\right). \tag{B.20}$$

Substitute (B.11) and (B.16), (B.20) into (B.12) and obtain

$$\begin{aligned} &\Re\frac{\beta'_-(k_u(s))\mathcal{P}_+(k_u(s)) + \beta'_+(k_u(s))\mathcal{P}_-(k_u(s))}{2\pi\beta'_+(k_u(s))} \\ < \ & \frac{1}{4}\left(\frac{\beta' + \sqrt{2}}{\sqrt{2}v_1^2} - \frac{2}{3}\right) - \frac{u_1 v_1}{3\pi}\sqrt{s^2+1}. \end{aligned} \tag{B.21}$$

Now we derive an estimate for $\Re[I_0(k_u(s))/(2\pi\beta'_+(k_u(s)))]$

$$\begin{aligned} \Re\frac{I_0(k_u(s))}{2\pi\beta'_+(k_u(s))} &= \frac{v_1 I_1}{\pi}\left(-u_1\Re\frac{1}{\beta'_+(k_u(s))} + \beta'\Re\frac{ik_u(s)}{\beta'_+(k_u(s))}\right) \\ &< \frac{v_1}{2\sqrt{1-\beta'^2}}(F_5(s) + F_6(s)), \end{aligned} \tag{B.22}$$

where

$$\begin{aligned} F_5(s) &:= -\frac{u_1}{-\beta' + (v_1 v - u u_1)\sqrt{s^2+1}} \\ &\le -\frac{u_1}{-\beta' + \sqrt{2}(v_1 v - u u_1)} < -\frac{u_1}{\beta'(2\sqrt{2}-1)}; \end{aligned} \tag{B.23}$$

$$\begin{aligned} F_6(s) &:= \frac{\beta' v(v_1 u + u_1 v)s^2}{(-\beta' + (v_1 v - u u_1)\sqrt{s^2+1})^2} \\ &\le \frac{\beta' v(v_1 u + u_1 v)}{3} < \frac{\beta' v_1}{3}. \end{aligned} \tag{B.24}$$

Using (B.22)-(B.24), we have

$$\Re\frac{I_0(k_u(s))}{2\pi\beta'_+(k_u(s))} < \frac{v_1}{2\sqrt{1-\beta'^2}}\left(\frac{\beta' v_1}{3} - \frac{u_1}{\beta'(2\sqrt{2}-1)}\right). \tag{B.25}$$

It follows from (3.16), (B.21) and (B.25) that in the region $|s| \ge 1$

$$\exp(\Re\mathcal{P}_0(k_u(s))) < \exp(a\sqrt{s^2+1} + b), \tag{B.26}$$

where a, b are defined in (3.29)-(3.30). Substitute (B.10) and (B.26) into (B.9):

$$\begin{aligned} |I(\Lambda)| &\le 2|R|v^{-1}e^b \int_\Lambda^{+\infty} e^{(a - W(uu^* + vv^*))\sqrt{s^2+1}} ds \\ &\le 2\sqrt{2}|R|v^{-1}e^b \int_\Lambda^{+\infty} \frac{e^{(a - W(uu^* + vv^*))\sqrt{s^2+1}} s ds}{\sqrt{s^2+1}} \\ &= \epsilon(\Lambda). \end{aligned} \tag{B.27}$$

Note that a is small and integral in (B.27) converges, provided $W(uu^* + vv^*)$ is large (see (3.1)).

References

[1] O.E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proc. Roy. Soc. London A 353 (1977), 401-419.

[2] O.E. Barndorff-Nielsen, *Processes of Normal Inverse Gaussian Type*, Finance and Stochastics, 2 (1998), 41–68.

[3] E. Bolviken and F. E. Benth, *Quatification of risk in norwegian stocks via the normal inverse Gaussian distribution*, In R. Norberg, P.Embrechts, C. Hipp, P. Lillevold, J.Paulsen, E. Shiu and D.Wilkie (eds.), Proceedings of the AFIR 2000 Colloquium, Tromso, Norway, (2000), 87–98,

[4] J-P. Bouchaud and M. Potters, *Theory of Financial Risk. Paris*, Aléa-Saclay, Eurolles, 1997.

[5] S. I. Boyarchenko and S. Z. Levendorskiĭ, *Perpetual American options under Lévy processes*, SIAM Journ. Control and Optimization, 40 (2002), 1663–1696.

[6] S.I. Boyarchenko and S.Z. Levendorskiĭ, *Barrier options and touch-and-out options under regular Lévy processes of exponential type*, Annals of Applied Probability, (2002)

[7] Svetlana Boyarchenko and Sergei Levendorskiĭ, Non-Gaussian Merton-Black-Scholes theory World Scientific, Singapore, 2002

[8] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Annalen, 300 (1994), 463–520.

[9] M.V. Fedoryuk, *The saddle point method*, Moscow, Nauka 1988 (In Russian)

[10] E. Eberlein, U. Keller and K. Prause, *New insights into smile, mispricing and value at risk: The hyperbolic model*, Journ. of Business, 71 (1998), 371–406.

[11] J.E. Ingersoll, *Digital contracts: Simple tools for pricing complex derivatives*, Journal of Business, 73(1), (2000), 67–88.

[12] S.Z. Levendorskiĭ and V.M. Zherder, *Fast Option Pricing Under Regular Lévy Processes of Exponential type*, submitted to Computational Finance

[13] D. B. Madan, P. Carr and E. C. Chang, *The variance Gamma process and option pricing*, European Finance Review, 2 (1998), 79–105.

[14] K. Prause, *Modelling financial data using generalized hyperbolic distributions*, FDM Preprint 48. University of Freiburg. (1997)

[15] T. Rydberg, *The normal inverse Gaussian Lévy processes: simulation and approximation*, Commun. Statist.-Stochastic Models. 13(4) (1997), 887-910.