Van Hove Hamiltonians—exactly solvable models of the infrared and ultraviolet problem

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Abstract

Quadratic bosonic Hamiltonians with a linear perturbation are studied. Depending on the infrared and ultraviolet behavior of the perturbation, their properties are described from the point of view of spectral and scattering theory.

1 Introduction

Some of the simplest exactly solvable models in classical and quantum physics are quadratic Hamiltonians with a linear perturbation. Following [Sch] we will call them van Hove Hamiltonians. They arise in classical and quantum field theory in many contexts. For example, the Hamiltonian of electrodynamics with prescribed external charges is a van Hove Hamiltonian.

It is well known that one can fully analyze the properties of van Hove Hamiltonians. These properties are quite interesting, both physically and mathematically. In fact, some types of van Hove Hamiltonians can be viewed as exactly solvable toy models of renormalization, both in the infrared and ultraviolet regime. Depending on the assumptions on the perturbation, one can distinguish several types of these Hamiltonians with distinct properties.

Let us describe some of the results of our paper in a somewhat informal language. We will restrict ourselves to the quantum case, since the classical case is very similar. (In the main part of the paper a different, more compact notation is used; moreover, both the classical and quatum case is treated).

Let $(K, \mathrm{d}k)$ be a space with a measure. (We use the notation $\mathrm{d}k$ to denote an arbitrary measure, not necessarily the Lebesgue measure). Let $K \ni k \mapsto a^*(k)$ and

$K \ni k \mapsto a(k)$ be the corresponding bosonic creation and annihilation operators. Let $K \ni k \mapsto h(k)$ be an almost everywhere positive measurable function, describing the dispersion relation. Finally, suppose that $K \ni k \mapsto z(k)$ is a complex function describing the interaction. We define two varieties of van Hove Hamiltonians. Van Hove Hamiltonians of the first kind are self-adjoint operator of the following form:

$$H_{\mathrm{I}} := \int \big(h(k)a^*(k)a(k)\mathrm{d}k + z(k)a^*(k)\mathrm{d}k + \overline{z}(k)a(k)\big)\,\mathrm{d}k. \tag{1.1}$$

Van Hove Hamiltonians of the second kind have the form

$$H_{\mathrm{II}} := \int h(k)\Big(a^*(k) + \frac{z(k)}{h(k)}\Big)\Big(a(k) + \frac{\overline{z}(k)}{h(k)}\Big)\mathrm{d}k. \tag{1.2}$$

It turns out that if we assume

$$\textstyle\int_{h(k)<1} |z(k)|^2\mathrm{d}k + \int_{h(k)\geq 1} \frac{|z(k)|^2}{h(k)^2}\mathrm{d}k < \infty, \tag{1.3}$$

then either H_{I} or H_{II} is well defined. More precisely,

$$\textstyle\int_{h(k)<1} |z(k)|^2\mathrm{d}k + \int_{h(k)\geq 1} \frac{|z(k)|^2}{h(k)}\mathrm{d}k < \infty, \tag{1.4}$$

guarantees that H_{I} is well defined; under the condition

$$\textstyle\int_{h(k)<1} \frac{|z(k)|^2}{h(k)}\mathrm{d}k + \int_{h(k)\geq 1} \frac{|z(k)|^2}{h(k)^2}\mathrm{d}k < \infty, \tag{1.5}$$

we can define H_{II}.

The conditions (1.4) and (1.5) are true at the same time iff

$$\int \frac{|z(k)|^2}{h(k)}\mathrm{d}k < \infty, \tag{1.6}$$

and then the two types of van Hove Hamiltonians differ by a constant:

$$H_{\mathrm{II}} = H_{\mathrm{I}} + \int \frac{|z(k)|^2}{h(k)}\mathrm{d}k. \tag{1.7}$$

Next we would like to describe various types of behavior of van Hove Hamiltonians. We will consider separately the case when the dispersion relation is separated away from zero and the case of a bounded dispersion relation. In the former case the infrared problem is absent and we can study the ultraviolet problem in its pure form. In the latter case the ultraviolet problem is absent and all the difficulties are due to the infrared problem. We will see that one can distinguish $3 \times 3 = 9$ distinct classes of van Hove Hamiltonians.

1.1 Ultraviolet problem

Assume that for some $0 < c_0$ we have $c_0 \leq h(k)$. We allow h to be unbounded. (In quantum field theory this behavior is typical for massive particles). Under this assumption, we can always define H_{II}. We can distinguish three cases:

(1) $\int |z(k)|^2 \mathrm{d}k < \infty$.
This is the most regular case. The perturbation is operator bounded. H_{I} and H_{II} are well defined by the Kato-Rellich theorem [Kato, RS2]. One can define a "dressing operator"

$$U := \exp\left(-\int \frac{z(k)}{h(k)} a^*(k)\mathrm{d}k + \int \frac{\overline{z}(k)}{h(k)} a(k)\mathrm{d}k\right) \tag{1.8}$$

which intertwines H_{II} with the quadratic Hamiltonian:

$$H_{\mathrm{II}} = U \int h(k) a^*(k) a(k) \mathrm{d}k U^*. \tag{1.9}$$

Because of (1.9), the Hamiltonians have a ground state.

(2) $\int \frac{|z(k)|^2}{h(k)} \mathrm{d}k < \infty, \qquad \int |z(k)|^2 \mathrm{d}k = \infty.$
This is also quite a regular case. The only difference with (1) is the fact that the perturbation is only form bounded and one needs to use the KLMN theorem to define H_{I} or H_{II} [Kato, RS2].

(3) $\int \frac{|z(k)|^2}{h(k)^2} \mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)} \mathrm{d}k = \infty.$
The operator H_{I} is not defined. This follows from the fact that the "counterterm" in (1.7) is infinite. But the dressing operator (1.8) is well defined, which can be used to define H_{II}.

Note that all the difficulty stems from the ultraviolet behavior of the interaction. Thus this case provides an example of "the ultraviolet renormalization": one has to subtract an infinite counterterm from H_{I} to define a Hamiltonian.

1.2 Infrared problem

Let us assume that the function $k \mapsto h(k)$ is bounded, but we allow h to have arbitrarily small positive values. (This is typical for zero-mass particles with an ultraviolet cut-off). Then the operator H_{I} is always well defined. We can distinguish the following three cases:

(1) $\int \frac{|z(k)|}{h(k)^2} \mathrm{d}k < \infty$.
Again, this is the most regular case. The perturbation is operator bounded, both H_{I} and H_{II} are well defined by the Kato-Rellich theorem [Kato, RS2]. One can define the dressing operator (1.8) and the Hamiltonians have a ground state.

(2) $\int \frac{|z(k)|}{h(k)}\mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)^2}\mathrm{d}k = \infty.$
In this case, the perturbation is operator bounded, but the dressing operator is not defined. H_{I}, and consequently also H_{II}, have no ground states, but are bounded from below. This is the case where the infrared problem manifests itself. In quantum field theory this case is typical for the photons (or scalar massless particles) interacting with an external classical source of non-zero charge, as noted by Kibble [Ki]. One can say that the vacuum "escapes from the Hilbert space", "soft photons are present in every state in the Hilbert space". The dressing operator (at least in the usual sense) is no longer well defined.

(3) $\int |z(k)|^2\mathrm{d}k < \infty, \qquad \int \frac{|z(k)|^2}{h(k)}\mathrm{d}k = \infty.$
This is the case where the infrared problem is the most severe. In order to define H_{I} one cannot use the Kato-Rellich nor the KLMN theorem. Instead one needs Nelson's commutator theorem [RS2]. H_{I} is unbounded from below and the dressing operator is not well defined. The operator H_{II} is not defined at all.

1.3 Scattering theory

Van Hove Hamiltonians are also interesting from the point of view of scattering theory. It turns out that the usual formalism of scattering theory, used in the context of Schrödinger operators and described for instance in [RS3], does not apply in the case of van Hove Hamiltonians. Instead, one needs to use some other versions to scattering theory. We have in fact a choice of at leat two approaches.

One of them is based on replacing the usual definition of wave operators with a definition that uses the so-called Abelian limit [Ya]. In order to define unitary wave operators one has to perform the so-called "renormalization of the wave function". This approach breaks down if we have the infrared problem (unless one is willing to divide by zero).

The second approach to scattering for van Hove Hamiltonians is based on the notion of asymptotic fields. It goes back to the so-called LSZ formalism. In this approach there is no need for renormalization and the infrared problem is manifested by the non-Fock property of asymptotic fields. An essentially the same formalism works in a much more complicated context, eg [HK, DG1, DG2].

Both approaches lead essentially to the same wave operator equal to the dressing operator. Unfortunately, the scattering operator turns out to be equal to one—so physically, scattering theory of van Hove Hamiltonians turns out to be trivial.

1.4 Remarks about the literature

Many of the ideas of this paper are contained in the literature in one form or another.

The analysis of the ultraviolet problem can be found eg. in the book of Berezin [Be], chapter III, § 7.4 and [Sch], following earlier papers [vH, EP, To, GS].

The understanding of the infrared problem can be traced back to the paper by Bloch and Nordsieck [BN], and was then analyzed in a series of papers of Kibble [Ki]. The infrared problem is related to the so-called coherent state representations, which were discussed already in [Fr].

In recent literature one can find analysis of operators similar to but more complicated than van Hove Hamiltonians—under the names of the spin-boson, the Pauli-Fierz or the non-relativistic QED Hamiltonian, see eg. [DG1, BFS1, DJ]. In fact, one can view van Hove Hamiltonians as a special, exactly solvable subclass of Pauli-Fierz Hamiltonians. We believe that it is useful to study van Hove Hamiltonians to gain intuition about properties of Pauli-Fierz Hamiltonians.

We could not find a complete treatment of van Hove Hamiltonians in the literature. We think that a careful analysis of van Hove Hamiltonian helps to understand some of the concepts of quantum field theory. It is also an instructive exercise in the theory of unbounded operators.

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2 Notation

2.1 Differentiation in a Hilbert space

Let $\mathcal{W}$ be a Hilbert space with the scalar product $(\cdot|\cdot)$.

Let $\mathrm{Dom}\, G$ be a subset of $\mathcal{W}$ and $G : \mathrm{Dom}\, G \to \mathbb{R}$. Let $w_0 \in \mathrm{Dom}\, G$ and $w \in \mathcal{W}$. We will say that the derivative of G at w_0 in the direction of w exists iff there exists $\epsilon > 0$ such that $\{w_0 + tw \ : \ |t| < \epsilon\} \subset \mathrm{Dom}\, G$ and there exists

$$\frac{\mathrm{d}}{\mathrm{d}t} G(w_0 + tw)\Big|_{t=0} =: \nabla_w G(w_0).$$

We will say that G is differentiable at w_0 iff

$$\mathcal{D} := \{w \in \mathcal{W} \ : \ \nabla_w G(w_0) \ \text{ exists } \}$$

is a dense linear subspace of $\mathcal{W}$ and

$$\mathcal{D} \ni w \mapsto \nabla_w G(w_0)$$

is a bounded linear functional. If this is the case, then the gradient of G at w_0 is denoted by $\nabla G(w_0) \in \mathcal{W}$ and defined by

$$\mathrm{Re}(w|\nabla G(w_0)) = \nabla_w G(w_0), \quad w \in \mathcal{W}.$$

2.2 Unbounded perators

Let h be a positive self-adjoint operator on $\mathcal{W}$. Let $\mathrm{Dom}\, h$ denote its domain and $(\mathrm{Dom}\, h)^*$ the space of bounded antilinear functionals on $\mathrm{Dom}\, h$. sp$h$ will denote the spectrum of h.

Clearly, $\mathcal{W}$ can be embedded in a natural way in $(\mathrm{Dom}\, h)^*$. The operator h extends to a map from $\mathcal{W}$ to $(\mathrm{Dom}\, h)^*$. We denote by $h\mathcal{W}$ the image of this extension. (Note in particular that if $h \geq c_0 > 0$, then $h\mathcal{W} = (\mathrm{Dom}\, h)^*$).

Clearly, if f is a function on the spectrum of h such that $|f(t)| \leq t$, $t \in \mathrm{sp}h$, then $f(h)\mathcal{W}$ can be treated as a subspace of $h\mathcal{W}$.

2.3 Fock spaces

[RS2, BR] If $\mathcal{X}$ is a vector space, then $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{X})$ will denote the algebraic symmetric Fock space over $\mathcal{X}$, that means the space of finite linear combinations of symmetric tensor products of elements of $\mathcal{X}$. Ω will denote the vacuum.

If $\mathcal{W}$ is a Hilbert space, then $\Gamma_{\mathrm{s}}(\mathcal{W})$ will denote the (complete) Fock space, that is the completion of $\overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{W})$.

If u is a contraction on $\mathcal{W}$, then $\Gamma(u)$ denotes the contraction on $\Gamma_{\mathrm{s}}(\mathcal{W})$ that on the n particle sector equals $u^{\otimes n}$.

If h is a self-adjoint operator, then $\mathrm{d}\Gamma(h)$ denotes the self-adjoint operator that on the n particle sector equals $h \otimes 1^{\otimes(n-1)} + \cdots + 1^{\otimes(n-1)} \otimes h$.

Let b be a sesquilinear form on $\mathcal{W}$ with the domain $\widetilde{\mathcal{W}}$. Abusing the notation, we will use the symbol $\mathrm{d}\Gamma(b)$ to denote the sequilinear form on $\Gamma_{\mathrm{s}}(\mathcal{W})$ with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$ that on the n particle sector equals $b \otimes 1^{\otimes(n-1)} + \cdots + 1^{\otimes(n-1)} \otimes b$.

2.4 Creation and annihilation operators

The notion of creation and annihillation operators is standard in the context of Fock spaces and can be found eg. in [BR, RS2]. Nevertheless, we will need slight generalizations of these concepts.

Let w be an antilinear form on $\mathcal{W}$ with the domain $\mathrm{Dom}w = \widetilde{\mathcal{W}} \subset \mathcal{W}$. We define the annihilation operator $\overline{w}(a)$ as an operator with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$ satisfying

$$\overline{w}(a)\, z^{\otimes n} := \sqrt{n}(w|z) z^{\otimes(n-1)}, \quad z \in \widetilde{\mathcal{W}}.$$

(Note that vectors of the form $z^{\otimes n}$ span $\overset{\circ}{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}})$).

The operator $\overline{w}(a)$ is closable iff $w \in \mathcal{W}$. If this is the case, we denote its closure by the same symbol. Its adjoint is called the creation operator and denoted

$$w(a^*) := \overline{w}(a)^*.$$

If $f \in \overset{\circ}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$, then $f(a^*)$ and $\overline{f}(a)$ have the obvious meaning as polynomials in a^* and a. For instance $z^{\otimes n}(a^*) = (z(a^*))^n$ and $\overline{z}^{\otimes n}(a) = (\overline{z}(a))^n$.

We also introduce the field operator

$$\phi(w) := \frac{1}{\sqrt{2}}(w(a^*) + \overline{w}(a)), \tag{2.10}$$

which is self-adjoint on $\mathrm{Dom}\,\overline{w}(a)$. We also introduce the Weyl operators

$$W(w) := \mathrm{e}^{\mathrm{i}\phi(w)}. \tag{2.11}$$

If $w \notin \mathcal{W}$, then the annihilation operator $\overline{w}(a)$ is not closable. In this case the creation operator $w(a^*)$ is not even densely defined and is not of much use. On the other hand, we can define the "creation form", denoted $w(a^{*\mathrm{f}})$:

$$\overset{\circ}{\Gamma}_{\mathrm{s}}(\widetilde{\mathcal{W}}) \times \Gamma_{\mathrm{s}}(\mathcal{W}) \ni (\Phi, \Psi) \mapsto (\Phi | w(a^{*\mathrm{f}})\Psi) := (\overline{w}(a)\Phi | \Psi).$$

Note that $w(a^{*\mathrm{f}})$ is a different object from $w(a^*)$. In what follows, however, we will sometimes abuse the notation and we will write $w(a^*)$ instead of $w(a^{*\mathrm{f}})$.

3 Van Hove Hamiltonians

3.1 Classical dynamics

Let $\mathcal{W}$ be a Hilbert space. We will say that $\alpha : \mathcal{W} \to \mathcal{W}$ preserves the scalar product iff

$$(\alpha(w_1) - \alpha(w_2) | \alpha(w_3) - \alpha(w_4)) = (w_1 - w_2 | w_3 - w_4), \quad w_1, \ldots, w_4 \in \mathcal{W}.$$

Suppose that h is a positive operator on $\mathcal{W}$. We will assume that $\mathrm{Ker} h = \{0\}$. Let

$$z \in \mathcal{W} + h\mathcal{W} \tag{3.12}$$

For $w \in \mathcal{W}$, $t \in \mathbb{R}$, we define

$$\alpha^t(w) := \mathrm{e}^{\mathrm{i}th} w + (\mathrm{e}^{\mathrm{i}th} - 1)h^{-1} z.$$

It is easy to see that $\mathbb{R} \ni t \mapsto \alpha^t$ is a 1-parameter group of transformations preserving the scalar product. Therefore, it preserves the real scalar product $\mathrm{Re}(\cdot|\cdot)$ and the symplectic form $\mathrm{Im}(\cdot|\cdot)$.

If $z \in \mathcal{W} + h^{1/2}\mathcal{W}$, then we define $\mathrm{Dom}\, G_{\mathrm{I}} := \mathrm{Dom}\, h^{1/2}$ and for $w \in \mathrm{Dom}\, G_{\mathrm{I}}$ we set

$$G_{\mathrm{I}}(w) := \frac{1}{2}\left((w|hw) + (z|w) + (w|z)\right).$$

If $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$, then we define $\mathrm{Dom}\, G_{\mathrm{II}} := \{w \in \mathcal{W} \ : \ h^{1/2}w + h^{-1/2}z \in \mathcal{W}\}$, and for $w \in \mathrm{Dom}\, G_{\mathrm{II}}$ we set

$$G_{\mathrm{II}}(w) := \frac{1}{2}\|h^{1/2}w + h^{-1/2}z\|^2.$$

Clearly, both $G_{\rm I}$ and $G_{\rm II}$ are well defined iff $z \in h^{1/2}\mathcal{W}$, and then

$$G_{\rm II} = G_{\rm I} + \frac{1}{2}(z|h^{-1}z).$$

Therefore, for any $z \in \mathcal{W} + h\mathcal{W}$, there exists a family of functions, which equals $\{G_{\rm I} + c \ : \ c \in \mathbb{R}\}$ for $z \in \mathcal{W} + h^{1/2}\mathcal{W}$ and $\{G_{\rm II} + c \ : \ c \in \mathbb{R}\}$ for $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. We will say that a function of this form is a Hamiltonian for α^t. This name will be justified by the theorem below.

Theorem 3.1 *Let $z \in \mathcal{W} + h\mathcal{W}$ and let G be a Hamiltonian for α^t. Then the following statements are true:*

(1) *The function G is differentiable at $w \in \mathcal{W}$ iff $hw + z$ belongs to $\mathcal{W}$. We then have*

$$\nabla G(w) = hw + z.$$

The dynamics $t \mapsto \alpha^t(w)$ is differentiable wrt t iff $hw + z \in \mathcal{W}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w) = \mathrm{i}(h\alpha^t(w) + z).$$

Thus we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w) = \mathrm{i}\nabla G(\alpha^t(w)),$$

which is equivalent to

$$\mathrm{Im}(w_1|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^t(w) = \mathrm{Re}(w_1|\nabla G(\alpha^t(w))), \quad w_1 \in \mathcal{W}. \tag{3.13}$$

Clearly, (3.13) is the Hamilton equation for the Hamiltonian G. This justifies calling G by the name of "a Hamiltonian of α^t".

Note also that α^t leaves $\mathrm{Dom}\, G$ *invariant and G is constant along the trajectories.*

(2) 0 *belongs to* $\mathrm{Dom}\, G$ *iff $z \in \mathcal{W} + h^{1/2}\mathcal{W}$. We have $G = G_{\rm I}$ iff $G_{\rm I}(0) = 0$.*

(3) *G is bounded from below iff $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. We have $G = G_{\rm II}$ iff* $\inf G = 0$.

(4) *G has a minimum iff $z \in h\mathcal{W}$. This minimum is at $-h^{-1}z$, and then*

$$G_{\rm II}(w) = \frac{1}{2}\left(w + h^{-1}z|h(w + h^{-1}z)\right).$$

3.2 Quantum dynamics

Let h, z be as above. Assume $z \in \mathcal{W} + h\mathcal{W}$. Set

$$V(t) := \Gamma(\mathrm{e}^{\mathrm{i}th}) \exp\left((1 - \mathrm{e}^{-\mathrm{i}th})h^{-1}z(a^*) - (1 - \mathrm{e}^{\mathrm{i}t\overline{h}})\overline{h}^{-1}\overline{z}(a)\right).$$

Define for $B \in \Gamma_{\rm s}(\mathcal{W})$

$$\beta^t(B) := V(t)BV(t)^*. \tag{3.14}$$

It is easy to check that β^t is a 1-parameter group of $*$-automorphisms of $B(\Gamma_{\rm s}(\mathcal{W}))$ continuous in the strong operator topology.

In order to make the relationship with the classical dynamics more clear, one can note that

$$\beta^t(w(a^*)) = ((\mathrm{e}^{\mathrm{i}th} - 1)h^{-1}z|w) + (\mathrm{e}^{\mathrm{i}th}w)(a^*).$$

For $z \in \mathcal{W} + h^{1/2}\mathcal{W}$, define

$$U_{\rm I}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}z|\mathrm{e}^{\mathrm{i}th}h^{-1}z) - \mathrm{i}t(z|h^{-1}z)}V(t).$$

For $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$, define

$$U_{\rm II}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Im}(h^{-1}z|\mathrm{e}^{\mathrm{i}th}h^{-1}z)}V(t).$$

We easily check that both $U_{\rm I}(t)$ and $U_{\rm II}(t)$ are 1-parameter strongly continuous unitary groups. Therefore, there exist self-adjoint operators $H_{\rm I}$ and $H_{\rm II}$ such that

$$U_{\rm I}(t) = \mathrm{e}^{\mathrm{i}tH_{\rm I}}, \quad U_{\rm II}(t) = \mathrm{e}^{\mathrm{i}tH_{\rm II}}.$$

Clearly, both $H_{\rm I}$ and $H_{\rm II}$ are well defined iff $z \in h^{1/2}\mathcal{W}$, and then

$$H_{\rm II} = H_{\rm I} + (z|h^{-1}z).$$

Therefore, for any $z \in \mathcal{W} + h\mathcal{W}$, there exists a family of self-adjoint operators which equals $\{H_{\rm I} + c \ : \ c \in \mathbb{R}\}$ for $z \in \mathcal{W} + h^{1/2}\mathcal{W}$ and $\{H_{\rm II} + c \ : \ c \in \mathbb{R}\}$ for $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. We will say that a function of this form is a Hamiltonian for β^t.

The operators H will be called van Hove Hamiltonians. As we saw above, there are two natural ways to fix the arbitrary constant in the definition of H. The operator $H_{\rm I}$ will be called the van Hove Hamiltonian of the first kind and $H_{\rm II}$ will be called the van Hove Hamiltonian of the second kind.

Theorem 3.2 *Let $z \in \mathcal{W} + h\mathcal{W}$ and let H be a Hamiltonian for β^t. Then the following statements are true:*

(1) *H implements β^t, that means*

$$\beta^t(B) = \mathrm{e}^{\mathrm{i}tH}B\mathrm{e}^{-\mathrm{i}tH}, \quad B \in B(\Gamma_{\rm s}(\mathcal{W})).$$

(2) *Ω belongs to $\mathrm{Dom}|H|^{1/2}$ (the form domain of H) iff $z \in \mathcal{W} + h^{1/2}\mathcal{W}$. Under this condition $H = H_{\rm I}$ iff $(\Omega|H\Omega) = 0$*

(3) *The operator H is bounded from below iff $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Under this condition $H = H_{\rm II}$ iff $\inf H = 0$.*

(4) *The operator H has a ground state ($\inf H$ is an eigenvalue of H, where $\inf H$ denotes the infimum of the spectrum of H) iff $z \in h\mathcal{W}$. Then we can define the "dressing operator"*

$$U := \exp\left(-h^{-1}z(a^*) + \overline{h}^{-1}\overline{z}(a)\right),$$

and

$$H_{\rm II} = U\mathrm{d}\Gamma(h)U^*. \tag{3.15}$$

Easy calculations show that, at least formally,

$$H_{\rm I} = {\rm d}\Gamma(h) + z(a^*) + \overline{z}(a), \tag{3.16}$$

$$H_{\rm II} = {\rm d}\Gamma(h) + z(a^*) + \overline{z}(a) + (z|h^{-1}z). \tag{3.17}$$

Below we will make these formulas precise.

Remark 3.3 *By the spectral theorem, one can find a measure space $(K, {\rm d}k)$ such that $\mathcal{W}$ is isomorphic to $L^2(K, {\rm d}k)$ and h is a multiplication operator by a measurable function $K \ni k \mapsto h(k)$. Then we can introduce $K \ni k \mapsto a^*(k), a(k)$ and write $\int z(k)a^*(k){\rm d}k$, $\int \overline{z}(k)a(k)$, $\int h(k)a^*(k)a(k){\rm d}k$ instead of $z(a^*)$, $\overline{z}(a)$ and ${\rm d}\Gamma(h)$. We used this notation in the introduction. For example, Condition (1.3) of the introduction,*

$$\textstyle\int_{h(k)<1} |z(k)|^2 {\rm d}k + \int_{h(k)\geq 1} \frac{|z(k)|^2}{h(k)^2} {\rm d}k < \infty,$$

corresponds to the condition $z \in \mathcal{W} + h\mathcal{W}$.

The notation involving the operator valued measures $a^*(k)$ and $a(k)$ is very common and often convenient, but it depends on a non-canonical choice of the measure space K, and therefore we do not use it.

3.3 Separating the infrared from the ultraviolet part

Let p_1, p_2 be two complementary orthogonal projections commuting with h. For $i = 1, 2$, let $\mathcal{W}_i := {\rm Ran}p_i$. For $w \in \mathcal{W}$ set $w_i := p_i w \in \mathcal{W}_i$. Likewise, set $h_i := p_i h$, treated as a self-adjoint operator on $\mathcal{W}_i$

We clearly have $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and hence we have the identification $\Gamma_{\rm s}(\mathcal{W}) = \Gamma_{\rm s}(\mathcal{W}_1) \otimes \Gamma_{\rm s}(\mathcal{W}_2)$. The dynamics β^t factorizes

$$\beta^t(B_1 \otimes B_2) = \beta_1^t(B_1) \otimes \beta_2^t(B_2), \quad B_1 \in B(\Gamma_{\rm s}(\mathcal{W}_1),\ B_2 \in B(\Gamma_{\rm s}(\mathcal{W}_2),$$

Likewise, it is easy to see that

$$U_{\rm I}(t) = U_{{\rm I},1}(t) \otimes U_{{\rm I},2}(t),$$

$$U_{\rm II}(t) = U_{{\rm II},1}(t) \otimes U_{{\rm II},2}(t).$$

Therefore,

$$H_{\rm I} := H_{{\rm I},1} \otimes 1 + 1 \otimes H_{{\rm I},2},$$

$$H_{\rm II} := H_{{\rm II},1} \otimes 1 + 1 \otimes H_{{\rm II},2}$$

In particular, we can take $p_1 := 1_{[0,1]}(h)$, $p_2 := 1_{]1,\infty[}(h)$. Then h_1 is bounded and h_2 is bounded from below by a positive constant. In the case of h_1, the ultraviolet problem is absent, but the infrared problem can show up. In the case of h_2 we have the opposite situation: the infrared problem is absent, but we can face the ultraviolet problem.

In the next two subsections we will separately describe the properties of van Hove Hamiltonians for two types of 1-particle energies.

3.4 3 types of the ultraviolet problem

Theorem 3.4 *Assume that h is a self-adjoint (possibly unbounded) operator bounded from below by a positive constant. Then the dressing operator U and the van Hove Hamiltonian of the second kind H_{II} are well defined. H_{II} possesses a unique ground state at 0. Moreover, we can distinguish 3 cases:*

(1) *Let $z \in \mathcal{W}$.*
Then $z(a^*) + \overline{z}(a)$ is a $\mathrm{d}\Gamma(h)$-bounded operator with the infinitesimal bound. H_{I} and H_{II} are self-adjoint on $\operatorname{Dom}\mathrm{d}\Gamma(h)$ and can be defined by the formulas (3.16), (3.17) and by the Kato-Rellich theorem.

(2) *Let $z \in h^{1/2}\mathcal{W}\backslash\mathcal{W}$.*
Then $z(a^*)+\overline{z}(a)$ is not an operator. Instead of $z(a^*)+\overline{z}(a)$ we should consider the form with the domain $\overset{\circ}{\Gamma}_{\mathrm{s}}(\operatorname{Dom} z)$ equal to $z(a^{*\mathrm{f}}) + \overline{z}(a)$. This form is $\mathrm{d}\Gamma(h)$-form bounded with the infinitesimal bound. The operators H_{I} and H_{II} are bounded from below and their form domains equal $\operatorname{Dom}\mathrm{d}\Gamma(h)^{1/2}$. They can be defined by the formulas (3.16), (3.17) and by the KLMN theorem.

(3) *Let $z \in h\mathcal{W}\backslash h^{1/2}\mathcal{W}$.*
Then the form $z(a^{*\mathrm{f}}) + \overline{z}(a)$ is not $\mathrm{d}\Gamma(h)$-form bounded. H_{I} is not defined.

3.5 3 types of the infrared problem

Theorem 3.5 *Assume that h is a bounded positive operator. Then the formula (3.16) defines the operator H_{I} as an essentially self-adjoint operator on $\operatorname{Dom}\mathrm{d}\Gamma(h+1)$ by Nelson's commutator theorem. Moreover, we can distinguish the following three cases:*

(1) *Let $z \in h\mathcal{W}$*
Then $z(a^*) + \overline{z}(a)$ is $\mathrm{d}\Gamma(h)$-bounded with the infinitesimal bound. The operators H_{I} and H_{II} are self-adjoint on $\operatorname{Dom}\mathrm{d}\Gamma(h)$ and they can be defined by the formulas (3.16), (3.17) and the Kato-Rellich theorem. They have ground states and the dressing operator U is well defined.

(2) *Let $z \in h^{1/2}\mathcal{W}\backslash h\mathcal{W}$*
Then $z(a^*) + \overline{z}(a)$ is $\mathrm{d}\Gamma(h)$-bounded with the infinitesimal bound. Again, the operators H_{I} and H_{II} are self-adjoint on $\operatorname{Dom}\mathrm{d}\Gamma(h)$ and they can be defined by the formulas (3.16), (3.17) and the Kato-Rellich theorem. They are bounded from below but have no ground state and the dressing operator U is not defined.

(3) *Let $z \in \mathcal{W}\backslash h^{1/2}\mathcal{W}$.*
Then $z(a^*) + \overline{z}(a)$ is not $\mathrm{d}\Gamma(h)$-form bounded. H_{I} is not bounded from below and the operator H_{II} is not defined at all.

In the following subsections we will show various elements of the above theorems.

3.6 Relative form boundedness of field operators

Proposition 3.6 *Let h be a positive operator on $\mathcal{W}$ and $z \in h^{1/2}\mathcal{W}$. Then*

(1)

$$\|\overline{z}(a)\Psi\|^2 \leq (z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi).$$

(2) *$z(a^{*\mathrm{f}}) + \overline{z}(a)$ is form bounded wrt $\mathrm{d}\Gamma(h)$ with the infinitesimal bound. More precisely, for any $\epsilon > 0$, we have*

$$|(\Psi|(z(a^{*\mathrm{f}}) + \overline{z}(a))\Psi)| \leq \epsilon(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi) + \epsilon^{-1}\|\Psi\|^2.$$

Proof. (1) If z is an antilinear functional on $\mathcal{W}$ with the domain $\mathrm{Dom}z$, then

$$\mathrm{Dom}\, z \times \mathrm{Dom}\, z \ni (w_1, w_2) \mapsto (w_1|z)(z|w_2)$$

defines a sesquilinear form, that we will denote by $|z)(z|$. Note that the following inequality is true:

$$|z)(z| \leq (z|h^{-1}z)h, \tag{3.18}$$

Clearly,

$$\|\overline{z}(a)\Psi\|^2 = (\Psi|\mathrm{d}\Gamma\left(|z)(z|\right)\Psi).$$

(3.18) implies

$$\mathrm{d}\Gamma\left(|z)(z|\right) \leq (z|h^{-1}z)\mathrm{d}\Gamma(h).$$

(2)

$$\begin{aligned}\pm(\Psi|(z(a^{*\mathrm{f}}) + \overline{z}(a))\Psi) &= \pm 2\mathrm{Re}\Big(\Psi|\overline{z}(a)\Psi\\ &\leq 2\|\Psi\|\|\overline{z}(a)\Psi\| \leq \epsilon^{-1}\|\Psi\|^2 + \epsilon\|\overline{z}(a)\Psi\|^2.\end{aligned}$$

□

Corollary 3.7 *If $z \in h^{1/2}\mathcal{W}$, then both (3.16) and (3.17) (with $z(a^*)$ replaced by $z(a^{*\mathrm{f}})$) define by the KLMN theorem the self-adjoint operators H_{I} and H_{II} with the form domains* $\mathrm{Dom}\,\mathrm{d}\Gamma(h)^{1/2}$.

3.7 Relative boundedness of field operators

Proposition 3.8 *Let h be a positive operator on $\mathcal{W}$ and $z \in h^{1/2}\mathcal{W} \cap \mathcal{W}$. Then*

(1)

$$\|(z(a^*) + \overline{z}(a))\Psi\|^2 \leq 4(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi) + 2\|z\|^2\|\Psi\|^2.$$

(2) *$z(a^*) + \overline{z}(a)$ is bounded wrt $\mathrm{d}\Gamma(h)$ with the infinitesimal bound. More precisely, for any $\epsilon > 0$, we have*

$$\|(z(a^*) + \overline{z}(a))\Psi\|^2 \leq 2\epsilon(z|h^{-1}z)\|\mathrm{d}\Gamma(h)\Psi\|^2 + \left(2\epsilon^{-1}(z|h^{-1}z) + 2\|z\|^2\right)\|\Psi\|^2.$$

Proof. (2) follows immediately from (1). To see (1) we compute using Proposition 3.6 (1):

$$\begin{aligned}\|(z(a^*)+\overline{z}(a))\Psi\|^2 &\leq 2\|z(a^*)\Psi\|^2+\|\overline{z}(a)\Psi\|^2\\ &= 4\|z(a^*)\Psi\|^2+2\|z\|^2\|\Psi\|^2\\ &\leq 4(z|h^{-1}z)(\Psi|\mathrm{d}\Gamma(h)\Psi)+2\|z\|^2\|\Psi\|^2.\end{aligned}$$

□

Corollary 3.9 *If $z\in\mathcal{W}\cap h^{1/2}\mathcal{W}$, then both (3.16) and (3.17) define by the Kato-Rellich theorem the self-adjoint operators H_{I} and H_{II} with the domains* $\mathrm{Dom}\,\mathrm{d}\Gamma(h)$.

3.8 The infimum of van Hove Hamiltonians

Proposition 3.10 *Assume $z\in\mathcal{W}+h^{1/2}\mathcal{W}$. Then the operator H_{I} satisfies*

$$\inf H_{\mathrm{I}}=-(z|h^{-1}z).$$

Proof. We drop I from H_{I}.
Step 1) To show that

$$\mathrm{d}\Gamma(h)+z(a^*)+\overline{z}(a)\geq-(z|h^{-1}z). \tag{3.19}$$

we set $\epsilon:=\pm(z|h^{-1}z)^{-1}$ in Proposition 3.6 (1).
Step 2) For any n define $\mathcal{W}^n:=1_{[\frac{1}{n},\infty[}(h)\mathcal{W}$, $h^n:=1_{[\frac{1}{n},\infty[}(h)h$, $z^n:=1_{[\frac{1}{n},\infty[}(h)z$, and the operators on $\Gamma_{\mathrm{s}}(\mathcal{W}^n)$, $H^n:=\mathrm{d}\Gamma(h^n)+z^n(a^*)+\overline{z^n}(a)$ and $U^n:=\exp(-h^{-1}z^n(a^*)+\overline{h}^{-1}\overline{z}^n(a))$. Then $U^{n*}H^nU^n=\mathrm{d}\Gamma(h^n)-(z^n|h^{-1}z^n)$. Clearly, $\inf\mathrm{d}\Gamma(h^n)=0$, hence

$$\inf H^n=-(z^n|h^{-1}z^n). \tag{3.20}$$

Step 3) Likewise, define $\mathcal{W}_n:=1_{[0,\frac{1}{n}[}(h)\mathcal{W}$, $h_n:=1_{[0,\frac{1}{n}[}(h)h$, $z_n:=1_{[0,\frac{1}{n}[}(h)z$, and the operator on $\Gamma_{\mathrm{s}}(\mathcal{W}_n)$, $H_n:=\mathrm{d}\Gamma(h_n)+z_n(a^*)+\overline{z}_n(a)$. We have $\Omega\in\mathrm{d}\Gamma(h_n)=\mathrm{Dom}H_n$ and $(\Omega|H_n\Omega)=0$. Hence,

$$\inf H_n\leq 0. \tag{3.21}$$

Step 4) $\Gamma_{\mathrm{s}}(\mathcal{W})$ can be identified with $\Gamma_{\mathrm{s}}(\mathcal{W}_n)\otimes\Gamma_{\mathrm{s}}(\mathcal{W}^n)$ and

$$H=H_n\otimes 1+1\otimes H^n. \tag{3.22}$$

Therefore,

$$\inf H=\inf H_n+\inf H^n. \tag{3.23}$$

It follows from (3.19), (3.20), (3.21) and (3.23) that

$$-(z|h^{-1}z)\leq\inf H\leq-(z^n|h^{-1}z^n).$$

For $n\to\infty$, the rhs goes to $-(z|h^{-1}z)$. □

3.9 Essential self-adjointness of van Hove Hamiltonians

Proposition 3.11 *Suppose that $z \in \mathcal{W}$. Then $H_{\rm I}$ is essentially self-adjoint on* $\mathrm{Dom}\, \mathrm{d}\Gamma(1+h)$.

Proof. First assume that h is bounded. We apply Nelson's commutator theorem with the comparison operator $B := 1 + \mathrm{d}\Gamma(1+h)$ [RS2]. In fact,

$$\|(z(a^*) + \overline{z}(a))\Psi\| \leq c\|N\Psi\| \leq c\|B\Psi)\|,$$

$$\|\mathrm{d}\Gamma(h)\Psi\| \leq \|B\Psi\|.$$

Moreover,

$$[B, (z(a^*) + \overline{z}(a))] = (1+h)z(a^*) + (1+\overline{h})\overline{z}(a).$$

Hence

$$|(\Psi|[B, (z(a^*) + \overline{z}(a))]\Psi)| \leq (\Psi|B\Psi).$$

Hence H is essentially self-adjoint on $\mathrm{Dom}B$.

Next consider an arbitrary h. As described in Subsection 3.3, we can split $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\mathcal{W}_1 = \mathrm{Ran}1_{[0,1]}(h)$. We can define the operator $H_{\rm I}$ on $\overset{\circ}{\Gamma}_{\rm s}(\mathcal{W}) = \overset{\circ}{\Gamma}_{\rm s}(\mathcal{W}_1) \otimes \overset{\circ}{\Gamma}_{\rm s}(\mathcal{W}_2)$ and it splits as

$$H_{\rm I} = H_{\rm I,1} \otimes 1 + 1 \otimes H_{\rm I,2},$$

with

$$H_{{\rm I},i} = \mathrm{d}\Gamma(h_i) + z_i(a^*) + \overline{z}_i(a).$$

We proved above that $H_{\rm I,1}$ is essentially self-adjoint on $\mathrm{Dom}\, \mathrm{d}\Gamma(h_1+1)$. By corollary 3.9, $H_{\rm I,2}$ is self-adjoint on $\mathrm{Dom}\, \mathrm{d}\Gamma(h_2)$. This implies that $H_{\rm I}$ is essentially self-adjoint on $\mathrm{Dom}\, \mathrm{d}\Gamma(h_1+1) \overset{\circ}{\otimes} \mathrm{Dom}\, \mathrm{d}\Gamma(h_1)$, which is dense in $\mathrm{Dom}\, \mathrm{d}\Gamma(h+1)$. ($\overset{\circ}{\otimes}$ denotes the algebraic tensor product). □

3.10 Absence of a ground state

Let us recall the following well known result about coherent states:

Theorem 3.12 *Suppose that $\widetilde{\mathcal{W}}$ is a dense subspace of $\mathcal{W}$ and f is an antilinear functional on $\widetilde{\mathcal{W}}$. Let $\Psi \in \Gamma_{\rm s}(\mathcal{W})$, for any $w \in \widetilde{\mathcal{W}}$, $\Psi \in \mathrm{Dom}\overline{h}(a)$ and*

$$\overline{h}(a)\Psi = (h|f)\Psi.$$

Then the following is true:

(1) *If $f \in \mathcal{W}$, then Ψ is proportional to* $\exp(f(a^*) - \overline{f}(a))\Omega$.

(2) *If $f \notin \mathcal{W}$, then $\Psi = 0$.*

Proof. By induction we show that for $w_1, \dots, w_n \in \widetilde{\mathcal{W}}$, $\overline{w}_{n-1}(a) \cdots \overline{w}_1(a)\Psi \in \mathrm{Dom}\overline{w}_n(a)$ and

$$\overline{w}_n(a) \cdots \overline{w}_1(a)\Psi = (w_1|f) \cdots (w_n|f)\Psi.$$

This implies

$$(w_1(a^*) \cdots w_n(a^*)\Omega|\Psi) = (w_1|f) \cdots (w_n|f)(\Omega|\Psi). \tag{3.24}$$

In particular,

$$(w|\Psi) = (w(a^*)\Omega|\Psi) = (w|f)(\Omega|\Psi), \quad w \in \widetilde{\mathcal{W}}.$$

Using the fact that $\widetilde{\mathcal{W}}$ is dense in $\mathcal{W}$ we see that $(\Omega|\Psi)f$ is a bounded functional on $\mathcal{W}$, hence it belongs to $\mathcal{W}$. Thus either $f \in \mathcal{W}$ or $(\Omega|\Psi) = 0$. In the latter case, (3.24) implies that $\Psi = 0$. □

Proposition 3.13 *Suppose that $z \notin h\mathcal{W}$. Then H_{I} has no ground state.*

Proof. We use the notation of the proof of Proposition 3.10. Let Ψ be a ground state of H. Then it is also a ground state of $H_n \otimes 1$ and of $1 \otimes H^n$. Being a ground state of $1 \otimes H^n$, it must be equal to $\Psi_n \otimes U^n\Omega$. Therefore, for $w \in \widetilde{\mathcal{W}} := \bigcup_{n=1}^{\infty} \mathcal{W}^n$

$$\overline{w}(a)\Psi = (w|h^{-1}z)\Psi.$$

But $\widetilde{\mathcal{W}}$ is dense in $\mathcal{W}$. By Theorem 3.12, this means that either $h^{-1}z \in \mathcal{W}$ or $\Psi = 0$. □

4 Scattering theory

4.1 The usual formalism

The most common setup for scattering theory starts with a pair of self-adjoint operators H_0 and H. The wave operators $\Omega^{\pm}$ are defined by the formulas

$$\Omega^{\pm} := \mathrm{s}\text{-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}. \tag{4.25}$$

Note that $\Omega^{\pm}$ are automatically isometric and

$$\Omega^{\pm}H_0 = H\Omega^{\pm}. \tag{4.26}$$

The scattering operator is defined as

$$S := \Omega^{+*}\Omega^{-}. \tag{4.27}$$

It satisfies

$$S = \mathrm{w}\text{-}\lim_{t_+, t_- \to \infty} \mathrm{e}^{\mathrm{i}t_+H_0}\mathrm{e}^{-\mathrm{i}(t_+ + t_-)H}\mathrm{e}^{\mathrm{i}t_-H_0} \tag{4.28}$$

and commutes with H_0. If we have

$$\mathrm{Ran}\Omega^{+} = \mathrm{Ran}\Omega^{-},$$

then S is unitary.

4.2 Wave operators defined by the Abelian limit—general formalism

The limits in (4.25) often do not exists. This happens, for instance, if H_0 has an eigenvector, which is not an eigenvector of H. This is the case of many models of quantum field theory, whose free vacuum (the unique eigenvector of H_0) is often different from the interacting vacuum (the unique eigenvector of H).

Nevertheless, sometimes even in this situation some kind of a scattering theory can be developed. In this subsection we will describe one of possible approaches to scattering theory, which, as we will see, works in the case of van Hove Hamiltonians. One can argue that this approach, or some its variation, is implicit in most textbook presentations of QFT.

Again, we start with a pair of self-adjoint operators H_0 and H. We suppose that there exists the Abelian limit

$$\Omega^{\pm}_{\mathrm{ur}} := \mathrm{s}-\lim_{\epsilon\downarrow 0} 2\epsilon \int_0^\infty \mathrm{e}^{\pm \mathrm{i}tH} \mathrm{e}^{\mp \mathrm{i}tH_0} \mathrm{e}^{-2\epsilon t}. \tag{4.29}$$

Note that there is no guarantee that $\Omega^{\pm}_{\mathrm{ur}}$ are isometric. One knows only that $\Omega^{\pm}_{\mathrm{ur}}$ are contractions. One can easily see that $\Omega^{\pm}_{\mathrm{ur}}$ have the intertwining property:

$$\Omega^{\pm}_{\mathrm{ur}} H_0 = H \Omega^{\pm}_{\mathrm{ur}}. \tag{4.30}$$

We will call $\Omega^{\pm}_{\mathrm{ur}}$ the "unrenormalized wave operators". Of course, it may happen that $\Omega^{\pm}_{\mathrm{ur}} = 0$.

Define the "renormalization of wave function operator"

$$Z^{\pm} := \Omega^{\pm *}_{\mathrm{ur}} \Omega^{\pm}_{\mathrm{ur}}.$$

It is easy to see hat $Z^{\pm}$ commutes with H_0. Assume that $\mathrm{Ker} Z^{\pm} = \{0\}$. Then we define the "renormalized wave operators"

$$\Omega^{\pm}_{\mathrm{rn}} := \Omega^{\pm}_{\mathrm{ur}} (Z^{\pm})^{-1/2}.$$

Note that $\Omega^{\pm}_{\mathrm{rn}}$ are isometric and have the intertwining property:

$$\Omega^{\pm}_{\mathrm{rn}} H_0 = H \Omega^{\pm}_{\mathrm{rn}}. \tag{4.31}$$

The unrenormalized scattering operator is defined as

$$S_{\mathrm{ur}} := \Omega^{+ *}_{\mathrm{ur}} \Omega^{-}_{\mathrm{ur}}. \tag{4.32}$$

Note that it can be also obtained as the following weak limit:

$$S_{\mathrm{ur}} = \mathrm{w}-\lim_{\epsilon_-,\epsilon_+\downarrow 0} 4\epsilon_-\epsilon_+ \int_0^\infty \mathrm{d}t_- \int_0^\infty \mathrm{d}t_+ \mathrm{e}^{\mathrm{i}t_+ H_0} \mathrm{e}^{-\mathrm{i}(t_-+t_+)H} \mathrm{e}^{\mathrm{i}t_- H_0} \mathrm{e}^{-2(\epsilon_-+2\epsilon_+)t}.$$

S_{ur} is a contraction that commutes with H_0.

We can define the renormalized scattering operator as

$$S_{\rm rn} := (Z^+)^{-1/2} S_{\rm ur} (Z^-)^{-1/2} = \Omega_{\rm rn}^{+*} \Omega_{\rm rn}^-.$$

$S_{\rm rn}$ also commutes with H_0 and if

$$\mathrm{Ran}\Omega_{\rm rn}^+ = \mathrm{Ran}\Omega_{\rm rn}^-,$$

then it is unitary.

We will see in the following subsection that van Hove Hamiltonians provide an example where the formalism based on the Abelian limit is applicable.

4.3 Wave operators for the van Hove Hamiltonian

Let $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Let $H = H_{\rm II}$ be the van Hove Hamiltonian of the second kind and

$$H_0 := \mathrm{d}\Gamma(h).$$

It is easy to see that in the case of the van Hove Hamiltonian, the limit in (4.25) does not exist unless $H = H_0$. Hence the construction of wave operators cannot be based on the approach described in Subsection 4.1.

We will show, however, that the formalism of Subsection 4.2 works for van Hove Hamiltonians if h has an absolutely continuous spectrum. It will turn out that the two renormalized wave operator coincide with one another and are equal to the dressing operator U. The operators $Z_\pm =: Z$ coincide and are just constants. The renormalized scattering operator equals identity.

If $z \notin h\mathcal{W}$, then $Z = 0$. This is one of the manifestations of the infrared problem.

All these statements are described in the following theorem.

Theorem 4.1 *Suppose that h has absolutely continuous spectrum and $z \in h^{1/2}\mathcal{W} + h\mathcal{W}$. Then $\Omega_{\rm ur}^\pm$ exists and*

$$\Omega_{\rm ur}^\pm = Z^{1/2} U, \quad S_{\rm ur} = Z,$$

where

$$Z = \mathrm{e}^{-\|h^{-1}z\|^2}.$$

$Z \neq 0$ iff $z \in h\mathcal{W}$, and then

$$Z = (\Omega | U\Omega)^2.$$

We can then define the renormalized operators, which are equal

$$\Omega_{\rm rn}^\pm = U, \quad S_{\rm rn} = 1.$$

Proof. Let us assume that $z \in h\mathcal{W}$. (The general case can be obtained by the limiting argument). Set $g := h^{-1}z$.

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0} &= U\mathrm{e}^{\mathrm{i}tH_0}U^*\mathrm{e}^{-\mathrm{i}tH_0} \\ &= U\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*) + \mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a)) \\ &= U\mathrm{e}^{\frac{1}{2}\|g\|^2}\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}g(a)). \end{aligned} \tag{4.33}$$

Let P_m be the projection onto the states with $\leq m$ particles. Suppose that $P_m\Psi = \Psi$. Now

$$\begin{aligned} 2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a))\Psi\mathrm{d}t = \Psi \\ +2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\big(\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*)) - 1\big)\Psi\mathrm{d}t \\ +2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))\big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a)) - 1\big)\mathrm{d}t. \end{aligned} \tag{4.34}$$

The norm of the third term can be estimated from above by

$$2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\|\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))P_m\|\|\big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a)) - 1\big)\Psi\|\mathrm{d}t.$$

Clearly, $\|\exp(-\mathrm{e}^{\mathrm{i}th}g(a^*))P_m\|$ is bounded uniformly in time. Besides, $\|\big(\exp(\mathrm{e}^{-\mathrm{i}t\overline{h}}\overline{g}(a)) - 1\big)\Psi\| \to 0$, by the absolute continuity of h and the Riemann-Lebesgue lemma. Therefore, the third term of (4.34) goes to zero.

The second term equals

$$\begin{aligned} &\sum_{n=1}^\infty 2\epsilon\int_0^\infty \mathrm{e}^{-2\epsilon t}\tfrac{(-1)^n}{n!}(\mathrm{e}^{\mathrm{i}th}g)^{\otimes n}(a^*)\Psi\mathrm{d}t \\ &= \sum_{n=1}^\infty \tfrac{(-1)^n}{n!}2\epsilon\big((2\epsilon - \mathrm{id}\Gamma(h))^{-1}g^{\otimes n}\big)(a^*)\Psi. \end{aligned} \tag{4.35}$$

Note that the nth term on the right goes to zero as $\epsilon \searrow 0$ and can be estimated by

$$\frac{\sqrt{(m+1)\cdots(m+n)}}{n!}\|g\|^n\|\Psi\|. \tag{4.36}$$

The series with elements (4.36) is convergent. Hence by the dominated convergence theorem, (4.35) goes to zero as $\epsilon \searrow 0$.

This shows that, for a finite particle Ψ, the left hand side of (4.34) goes to Ψ. By density, we can extend this to all $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{W})$. □

4.4 Asymptotic fields—general formalism

There exists an alternative approach to scattering in quantum field theory. Instead of starting from wave operators, one looks at the limits of certain observables in

the interaction picture. There are various forms of this approach, some of them go under the name of the LSZ formalism, see eg. [Schwa].

Let us present the abstract framework of one of the versions of this approach developed in [HK] and used eg. in [DG2].

Suppose that H is a self-adjoint operator on the Fock space $\Gamma_{\rm s}(\mathcal{W})$ and h is a self-adjoint operator on $\mathcal{W}$. Assume that for some subspace $\mathcal{W}_1 \subset \mathcal{W}$ there exists

$$\mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-\mathrm{i}th}w)\mathrm{e}^{-\mathrm{i}tH} =: W^{\pm}(w), \quad w \in \mathcal{W}_1.$$

Then

$$\mathcal{W}_1 \ni w \mapsto W^{\pm}(w) \in U(\Gamma_{\rm s}(\mathcal{W})) \tag{4.37}$$

are two representations of Canonical Commutation Relations (CCR), that means

$$W^{\pm}(w_1)W^{\pm}(w_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(w_1|w_2)}W^{\pm}(w_1+w_2).$$

Moreover, they satisfy

$$\mathrm{e}^{\mathrm{i}tH} W^{\pm}(w)\mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(\mathrm{e}^{\mathrm{i}th}w).$$

Suppose that the representations (4.37) are unitarily equivalent to the Fock representation, which means that there exist unitary operators $\Omega^{\pm}$ such that

$$W^{\pm}(w) = \Omega^{\pm} W(w) \Omega^{\pm *}. \tag{4.38}$$

Then the operators $\Omega^{\pm}$ are defined up to a phase factor. They are called wave operators.

The scattering operator is defined as $S := \Omega^{+*}\Omega^{-}$. Again, the scattering operator is defined up to a phase factor.

Suppose that both the formalism of Subsection 4.2 and of Subsection 4.4 apply. One can ask whether the renormalized wave operators $\Omega^{\pm}_{\rm rn}$, defined as in Subsection 4.2, and the wave operator $\Omega^{\pm}$ defined in this section coincide up to a phase factor. In general, there seems to be no guarantee for this to hold. Nevertheless we will see that this is true in the case of van Hove Hamiltonians.

4.5 Asymptotic fields for van Hove Hamiltonians

The formalism of asymptotic fields works very well in the case of van Hove Hamiltonians.

Theorem 4.2 *Let h have an absolutely continuous spectrum, $0 \leq \beta \leq 1$ and $z \in h^{1-\beta}\mathcal{W} + h\mathcal{W}$. Let $w \in \mathrm{Dom}\, h^{-\beta}$.*

(1) *There exists the norm limit*

$$\lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-ith}w)\mathrm{e}^{itH} = W(w)\mathrm{e}^{\mathrm{i}2\mathrm{Re}(w|h^{-1}z)} =: W^{\rm as}(w).$$

(2) $\mathrm{Dom}\, h^{-\beta} \ni w \mapsto W^{\mathrm{as}}(w)$ *is a regular representation of CCR.*

(3) *This representation is unitarily equivalent to the Fock representation iff* $z \in h\mathcal{W}$*, and then*

$$W^{\mathrm{as}}(w) = UW(w)U^*,$$

where U *is the dressing operator.*

Proof.

$$\mathrm{e}^{\mathrm{i}tH} W(\mathrm{e}^{-ith} w)\mathrm{e}^{itH} = W(w) \exp\left(\mathrm{i}2\mathrm{Re}(w|(1-\mathrm{e}^{-\mathrm{i}th})h^{-1}z)\right).$$

Now $h^{-1}z \in h^{-\beta}\mathcal{W} + \mathcal{W}$ and $w \in \mathrm{Dom}\, h^{-\beta}$. Hence $\lim_{|t|\to\infty}(w|\mathrm{e}^{-\mathrm{i}th}h^{-1}z) = 0$ by the Riemann-Lebesgue lemma. This proves (1). □

5 Examples

5.1 Harmonic oscillators

In this section we describe van Hove Hamiltonians in a somewhat more concrete setting, typical for physical applications. We will restrict ourselves to the classical case, since it is parallel to the quantum case. We will describe a system of harmonic oscillators with a linear perturbing potential.

Up to now, we assumed that our system is described by phase space $\mathcal{W}$. There was no need to introduce the configuration space. For a system of harmonic oscillators it is however natural to start from a configuration space, which will be described by a real Hilbert space $\mathcal{X}$ with the scalar product denoted by the dot. The preliminary phase space is $\mathcal{X} \oplus \mathcal{X}$. It has the structure of a symplectic space with the symplectic form

$$(x_1, \xi_1)\omega(x_2, \xi_2) = x_1 \cdot \xi_2 - x_2 \cdot \xi_1. \tag{5.39}$$

Note, however, that we will have to take a slightly different phase space.

Let r denote a positive operator on $\mathcal{X}$ and q is a linear functional on $\mathcal{X}$ (possibly unbounded and not everywhere defined). A system of harmonic oscillators with a linear perturbing potential is described by the (classical) Hamiltonian

$$G(x, \xi) = \frac{1}{2}|rx|^2 + \frac{1}{2}|\xi|^2 + q \cdot x,$$

defined for $x \ni \mathrm{Dom}\, r \cap \mathrm{Dom}\, q$, $\xi \in \mathcal{X}$. It is easy to see that $\mathcal{X} \oplus \mathcal{X}$ is not an appropriate space for the Hamiltonian G. It is natural to replace it by the space $\mathcal{W} := r^{-1/2}\mathcal{X} \oplus r^{1/2}\mathcal{X}$. We keep the symplectic form (5.39)

We equip $\mathcal{W}$ with the complex structure

$$\mathrm{i}(x, \xi) := (-r^{-1}\xi, rx).$$

We can view $\mathcal{W}$ as a complex Hilbert space:

$$(w_1|w_2) = r^{1/2}x_1 \cdot r^{1/2}x_2 + r^{-1/2}\xi_1 \cdot r^{-1/2}\xi_2 + \mathrm{i}x_1 \cdot \xi_2 - \mathrm{i}x_2 \cdot \xi_1,$$

$$w_1 = (x_1, \xi_1), \quad w_2 = (x_2, \xi_2).$$

Note that the symplectic form (5.39) is the imaginary part of the scalar product.

Introduce $z := (r^{-1/2}q, 0)$ and a positive self-adjoint operator h on $\mathcal{W}$ defined by $h := r \oplus r$. Then we can rewrite G as

$$G(w) = \frac{1}{2}\big((w|hw) + (z|w) + (w|z)\big).$$

Note that the infrared and ultraviolet conditions expressed in terms of q instead of z have their power shifted by $1/2$. More precisely, $z \in h^\alpha \mathcal{W}$ iff $q \in r^{1/2+\alpha}\mathcal{X}$.

5.2 Scalar massless field theory

Suppose that $\mathcal{X} = L^2(\mathbb{R}^d)$, $r = |\mathrm{i}\nabla|$. Then the Hamiltonian $\frac{1}{2}|rx|^2 + \frac{1}{2}|\xi|^2$ describes the so called scalar massless field theory. After taking the Fourier transformation, the operator r becomes the multiplication by $|\xi|$, where ξ is the momentum variable.

Suppose that we add a linear perturbation given by $q \in \mathcal{S}(\mathbb{R}^d)$. After taking the Fourier transformation we get $\hat{q} \in \mathcal{S}(\mathbb{R}^d)$ and we see that the ultraviolet problem is absent. The infrared problem will depend on whether $\hat{q}(0)$ equals zero or not. $\hat{q}(0)$ equals the integral of q over the whole configuration space. Since in some physical examples q can be interpreted as the density of the charge, we will call $\hat{q}(0)$ the total charge. Note that if $\hat{q}(0) = 0$, then $|\hat{q}(\xi)| = O(|\xi|)$ around zero.

Concerning the type of the infrared behavior, we easily get the following table (the number in the round brackets corresponds to the part of Theorem 3.5):

Dimension of configuration space	Nonzero total charge	Zero total charge
$d = 1$	Hamiltonian undefined	(2)
$d = 2$	(3)	(1)
$d = 3$	(2)	(1)
$d \geq 4$	(1)	(1)

Remark 5.1 *As we see from the table, in dimension 3, in the nonzero charge case we get the infrared behavior of type (2). Thus the Hamiltonian is bounded from below, but the ground state is absent. This is the type of the infrared problem widely discussed in the literature [Ki].*

Some authors say, however, that the type (2) behavior is an artifact of the model and disappears if one takes a more physical Hamiltonian. In fact, in [BFS2] it is proven that the (ultraviolet cut-off) Hamiltonian of QED with a confining potential possesses a ground state. This is related to the fact that in above considerations we considered a scalar field, whereas photons in QED have spin one and are coupled to the charge by the minimal coupling prescription.

6 Time dependent van Hove Hamiltonians

In this section we describe a certain class of strongly continuous dynamics on the Fock space. One can say that these are the dynamics generated by all possible time dependent families of van Hove Hamiltonians.

Let $\mathbb{R} \ni t \mapsto g^t \in \mathcal{W}$ be a continuous vector valued function and $\mathbb{R} \ni t \mapsto u^t \in U(\mathcal{W})$ be a strongly continuous function with values in unitary operators. We assume that $g^0 = 0$ and $u^0 = 1$. Set

$$V(t) := \Gamma(u^t) \exp\bigl(\mathrm{i}g^t(a^*) + \mathrm{i}\overline{g}^t(a)\bigr).$$

For $A \in B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ we set

$$\beta^t(A) := V(t)AV(t)^*.$$

Note that

$$\beta^t(w(a^*)) = u^t w(a^*) + \mathrm{i}(g^t|w).$$

Theorem 6.1 (1) *$V(t)$ is a strongly continuous family of unitary operators on $\Gamma_{\mathrm{s}}(\mathcal{W})$ such that $V(0) = 1$.*

(2) *β^t is a pointwise strongly continuous family of $*$-automorphisms of $B(\Gamma_{\mathrm{s}}(\mathcal{W}))$ such that β^0 is the identity.*

(3) *$V(t)$ is the distinguished implementation of β^t in the following sense: if $\tilde{V}(t)$ is a family of unitary operators such that $\beta^t(A) = \tilde{V}(t)A\tilde{V}(t)^*$ and $(\Omega|\tilde{V}(t)\Omega) > 0$, then $\tilde{V}(t) = V(t)$.*

One can ask what is the time-dependent generator of $V(t)$. To answer this question we proceed formally, without worrying about the exact meaning of various objects involved in our formulas.

Suppose that the dot denotes the temporal derivative. It is easy to check the following identities:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}g^t(a^*)+\mathrm{i}\overline{g}^t(a)} &= \bigl(\tfrac{\mathrm{i}}{2}\mathrm{Im}(\dot{g}^t|g^t) + \mathrm{i}\dot{g}^t(a^*) + \mathrm{i}\dot{\overline{g}}^t(a)\bigr)\mathrm{e}^{\mathrm{i}g^t(a^*)+\mathrm{i}\overline{g}^t(a)}, \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\Gamma(u^t) &= \mathrm{d}\Gamma(\dot{u}^t u^{t*})\Gamma(u^t). \end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = \mathrm{i}\Bigl(\frac{1}{2}\mathrm{Im}(\dot{g}^t|g^t) + u^t\dot{g}^t(a^*) + \overline{u}^t\dot{\overline{g}}^t(a) - \mathrm{i}\mathrm{d}\Gamma(\dot{u}^t u^{t*})\Bigr)V(t)$$

Now suppose that $t \mapsto z^t$ is a family of vectors and $t \mapsto h^t$ is a family of self-adjoint operators. Suppose that u^t is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}u^t = \mathrm{i}h^t u^t, \quad u^0 = 1;$$

and

$$g^t := \int_0^t u^{s*} z^s \mathrm{d}s,$$
$$\sigma^t := \tfrac{1}{2}\mathrm{Im} \int_0^t (z^t | u^t u^{s*} z^s) \mathrm{d}s.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t) = \mathrm{i} H(t) V(t),$$

where

$$H(t) := \mathrm{d}\Gamma(h^t) + z^t(a^*) + \overline{z}^t(a) + \sigma^t.$$

Thus, at least on a formal level, $V(t)$ is generated by van Hove Hamiltonians.

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