# PERIODIC ORNSTEIN-UHLENBECK PROCESSES

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Abstract. In this paper the class of periodic Ornstein-Uhlenbeck processes is defined. It is shown that periodic Ornstein-Uhlenbeck processes are stationary Markov random fields and the class of stationary distributions is characterized. In particular, any selfdecomposable distribution is the stationary distribution of some periodic Ornstein-Uhlenbeck process. As examples gamma periodic Ornstein-Uhlenbeck processes and Gaussian periodic Ornstein-Uhlenbeck processes are considered.

## 1. Introduction

An Ornstein-Uhlenbeck process $\{Y_t\}_{t\geqslant 0}$ satisfies $Y_t = e^{-\lambda t}Y_0 + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z_s$, where $Y_0$ is independent of $\{Z_t\}_{t\geqslant 0}$. Here $\lambda$ is positive and $\{Z_t\}_{t\geqslant 0}$ is the so-called background driving Lévy process. It is well known that $\{Y_t\}_{t\geqslant 0}$ is a Markov process which, in the stationary case, has a selfdecomposable stationary distribution. Recently (non-Gaussian) Ornstein-Uhlenbeck processes have been succesfully applied in finance, see Barndorff-Nielsen and Shephard (2001) and references therein.

In this paper we consider a process $\{X_t\}_{t\in[0,1]}$ defined by

$$X_t = e^{-\lambda t}X_0 + e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}Z_s, \quad t\in[0,1], \quad X_0 = X_1.$$

In view of the definition of an ordinary Ornstein-Uhlenbeck process $\{X_t\}_{t\in[0,1]}$ is called a *periodic Ornstein-Uhlenbeck process.* We show that it is a stationary Markov random field where the stationary distribution is the law of $\left(e^{\lambda}-1\right)^{-1}\int_0^1 e^{\lambda s}\mathrm{d}Z_s$. An important property is that any selfdecomposable distribution allows the latter representation. This enables us to construct a periodic Ornstein-Uhlenbeck process with a given selfdecomposable stationary distribution. As examples gamma and Gaussian periodic Ornstein-Uhlenbeck processes are considered.

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We show that when $\{Z_t\}_{t\in[0,1]}$ is a Wiener process then $\{X_t\}_{t\in[0,1]}$ is a so-called continuous time first-order Markov process. This process has has been widely used in statistical shape analysis, see Grenander (1993) and Hobolth and Vedel (2000). Further, in this case $\{X_t\}_{t\in[0,1]}$ solves a linear stochastic differential equation driven by a Wiener process and with a boundary condition. References to the literature are given in Section 4. Norris (1998) considered Gaussian periodic Ornstein-Uhlenbeck processes in a very general setting.

In Section 2 we provide some background material on selfdecomposable and infinitely divisible distributions, and we recall some basic facts on usual Ornstein-Uhlenbeck processes. Besides we analyse the class of distributions $\mu$ given by $\mu = \mathcal{L}\left(\int_0^1 e^{\lambda s}\mathrm{d}Z_s\right)$ for some Lévy process $\{Z_t\}_{t\in[0,1]}$. Section 3 contains the main results in the paper. We define the class of periodic Ornstein-Uhlenbeck processes and derive the properties of $\{X_t\}_{t\in[0,1]}$ mentioned above. In Section 4 we specialize to the Gaussian case.

## 2. PRELIMINARIES

For a distribution $\mu$ on $\mathbb{R}$ let $\widehat{\mu}$ denote the characteristic function of $\mu$, $\widehat{\mu}(z) = \int_{\mathbb{R}} e^{\mathrm{i}zx}\mu(\mathrm{d}x), z \in \mathbb{R}$. For a random vector $X$ on $\mathbb{R}^n$, $\mathcal{L}(X)$ denotes the law of $X$ on $\mathbb{R}^n$. By $X \overset{\mathrm{d}}{=} Y$ we mean $\mathcal{L}(X) = \mathcal{L}(Y)$. For two processes $\{X_t\}_{t\in[0,1]}$ and $\{Y_t\}_{t\in[0,1]}$ on $\mathbb{R}$ we write $\{X_t\}_{t\in[0,1]} \overset{\mathrm{d}}{=} \{Y_t\}_{t\in[0,1]}$ if they have a common set of finite-dimensional marginal distributions. For distributions $\mu_n$ and $\mu$ on a metric space $S$, $\mu_n \to \mu$ means weak convergence of $\mu_n$ to $\mu$. Let $D = [-1, 1]$.

First we recall some basic facts on Lévy processes, infinite divisibility and self-decomposability. The reader is referred to Sato (1999) for more information. A distribution $\mu_0$ on $\mathbb{R}$ is selfdecomposable if for any $b > 1$ there is a distribution $\mu_{0b}$ on $\mathbb{R}$ such that $\widehat{\mu_0}(z) = \widehat{\mu_0}(b^{-1}z)\widehat{\mu_{0b}}(z)$. Let $SD$ denote the class of selfdecomposable distributions on $\mathbb{R}$. Let $ID$ be the set of infinitely divisible distributions on $\mathbb{R}$. Then, a distribution $\mu_0$ on $\mathbb{R}$ is in $ID$ if and only if $\widehat{\mu_0}$ is given by

$$\widehat{\mu}_0(z) = \exp\left[-\frac{1}{2}\kappa^2 z^2 + \mathrm{i}\delta z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zx} - 1 - \mathrm{i}zx1_D(x)\right)\rho(\mathrm{d}x)\right], \quad z \in \mathbb{R}.$$

Here, $(\kappa^2, \rho, \delta)$ is the characteristic triplet of $\mu_0$, that is, $\kappa^2 \geqslant 0$ is the Gaussian variance, $\delta \in \mathbb{R}$ is a location parameter and $\rho$ is a measure on $\mathbb{R}$ satisfying $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\rho(\mathrm{d}x) < \infty$. If $\int_{|x|\leqslant 1}|x|\rho(\mathrm{d}x) < \infty$ we define the drift of $\mu_0$ as $\delta_0 = \delta - \int_{|x|\leqslant 1} x\rho(\mathrm{d}x)$. We have $SD \subseteq ID$.

There is a one-to-one correspondence between $ID$ and the set of Lévy process on $\mathbb{R}$, if processes with the same law are identified. Precisely, if $\mu_0 \in ID$, then there exists a Lévy process $\{Z_t\}_{t\geqslant 0}$ such that $\mu_0 = \mathcal{L}(Z_1)$. The law of $Z_t$ is $\mathcal{L}(Z_t) = \mu_0^t$. Conversely, if $\{Z_t\}_{t\geqslant 0}$ is a Lévy process on $\mathbb{R}$, then $\mu_0 := \mathcal{L}(Z_1)$ is infinitely divisible.

Let us recall some facts on integration with respect to a Lévy process. Let $\{Z_t\}_{t\geqslant 0}$ be a Lévy process on $\mathbb{R}$ and $\mathcal{L}(Z_1)$ have characteristic triplet $(\kappa^2, \rho, \delta)$. Let $0 \leqslant a \leqslant b \leqslant \infty$ and $f\colon \mathbb{R}_+ \to \mathbb{R}$ be continuous. The integral $\int_a^b f(s)\mathrm{d}Z_s$ always exists when $b < \infty$. We have $\mathcal{L}(\int_a^b f(s)\mathrm{d}Z_s) \in ID$ and the characteristic triplet $(\sigma^2, \nu, \gamma)$ of $\mathcal{L}(\int_a^b f(s)\mathrm{d}Z_s)$ is

$$\sigma^2 = \kappa^2 \int_a^b f^2(s)\mathrm{d}s, \tag{2.1}$$

$$\nu(B) = \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_a^b 1_B(f(s)y)\mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}), \tag{2.2}$$

$$\gamma = \delta \int_a^b f(s)\mathrm{d}s + \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_a^b f(s)y\left(1_D(f(s)y) - 1_D(y)\right)\mathrm{d}s. \tag{2.3}$$

Define the integral $\int_a^\infty f(s)\mathrm{d}Z_s$ as $\int_a^\infty f(s)\mathrm{d}Z_s = \lim_{t\to\infty}\int_a^t f(s)\mathrm{d}Z_s$, provided that the limit exists in probability. If it exists, $\int_a^\infty f(s)\mathrm{d}Z_s$ is infinitely divisible and the characteristic triplet $(\sigma^2, \nu, \gamma)$ of $\mathcal{L}(\int_a^\infty f(s)\mathrm{d}Z_s)$ is given by (2.1)–(2.3) with $b = \infty$. For an account on integration we refer to Rocha-Arteaga and Sato (2001).

The following characterization of $SD$ shows that selfdecomposable distributions have an integral representation. For a proof see Sato (1999, Section 17).

**Proposition 2.1.** (i) *Let $\{Z_t\}_{t\geqslant 0}$ be a Lévy process such that the Lévy measure $\rho$ of $\mathcal{L}(Z_1)$ satisfies the condition*

$$\int_{|x|>2} \log|x|\rho(\mathrm{d}x) < \infty. \tag{2.4}$$

*Then, for every $\lambda > 0$, the integral $\int_0^\infty e^{-\lambda t}\mathrm{d}Z_t$ exists and $\mathcal{L}\left(\int_0^\infty e^{-\lambda t}\mathrm{d}Z_t\right) \in SD$.*

(ii) *Conversely, let $\mu \in SD$ and $\lambda > 0$. Then there exists a Lévy process $\{Z_t^\lambda\}_{t\geqslant 0}$ such that the Lévy measure $\rho_\lambda$ of $\mathcal{L}(Z_1^\lambda)$ satisfies* (2.4)*, with $\rho$ replaced with $\rho_\lambda$, and $\mu = \mathcal{L}(\int_0^\infty e^{-\lambda t}\mathrm{d}Z_t^\lambda)$.*

**Remark 2.2.** The relation between the Lévy measure $\nu$ of $\mu$ and the Lévy measure $\rho_\lambda$ of $\mathcal{L}(Z_1^\lambda)$ in Proposition 2.1 (ii) is

$$\nu(B) = \int_{\mathbb{R}} \rho_\lambda(\mathrm{d}y) \int_0^\infty 1_B(e^{-\lambda s}y)\mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}), \tag{2.5}$$

which follows from (2.2).

Now we turn to Ornstein-Uhlenbeck processes. Let $\{Z_t\}_{t\geqslant 0}$ denote a Lévy process on $\mathbb{R}$, let $\lambda > 0$ and $U$ be a random variable on $\mathbb{R}$ which is independent of $\{Z_t\}_{t\geqslant 0}$. Consider the equation

$$\mathrm{d}Y_t = -\lambda Y_t \mathrm{d}t + \mathrm{d}Z_t, \tag{2.6}$$

with initial condition $Y_0 = U$. Then

$$Y_t = e^{-\lambda t} Y_0 + e^{-\lambda t} \int_0^t e^{\lambda s} \mathrm{d}Z_s, \quad Y_0 = U, \tag{2.7}$$

and $\{Y_t\}_{t\geqslant 0}$ is called an *Ornstein-Uhlenbeck process with parameter* $\lambda > 0$ *and background driving Lévy process* $\{Z_t\}_{t\geqslant 0}$. We say that $\{Y_t\}_{t\geqslant 0}$ is stationary if $\mathcal{L}(Y_t) = \mathcal{L}(Y_0)$ for all $t \geqslant 0$. In this case, the probability measure on $\mathbb{R}$ defined by $\mu := \mathcal{L}(Y_0)$ is called the stationary distribution of $\{Y_t\}_{t\geqslant 0}$.

It is well known that a probability measure on $\mathbb{R}$ is the stationary distribution of some Ornstein-Uhlenbeck process with parameter $\lambda$ if and only if it is in $SD$. Indeed, let $\mu \in SD$. Let the initial condition satisfy $\mathcal{L}(U) = \mu$ and $\{Z_t\}_{t\geqslant 0}$ be the Lévy process $\{Z_t^\lambda\}_{t\geqslant 0}$ in Proposition 2.1 (ii). Then it is easily verified that $\{Y_t\}_{t\geqslant 0}$ in (2.6) satisfies $\mathcal{L}(Y_t) = \mu$ for all $t$.

It turns out that the stationary distribution of a periodic Ornstein-Uhlenbeck process, up to a constant, is described by the second term on the right-hand side of (2.7) with $t = 1$. This motivates the following definition.

**Definition 2.3.** Let $\lambda > 0$. Let $I(\lambda)$ be the class of distributions on $\mathbb{R}$ defined as follows. A distribution $\mu$ is in $I(\lambda)$ if and only if there exists a Lévy process $\{Z_t\}_{t\in[0,1]}$ on $\mathbb{R}$ such that

$$\mu = \mathcal{L}\left(\int_0^1 e^{\lambda s} \mathrm{d}Z_s\right). \tag{2.8}$$

Note that in Definition 2.3 we consider a Lévy process indexed by $t \in [0, 1]$, while in other cases we use Lévy processes indexed by $t \in [0, \infty[$.

For a Lévy measure $\rho$ on $\mathbb{R}$ define $G_\rho(v)$ as

$$G_\rho(v) = \begin{cases} \rho([v, \infty[) & \text{if } v > 0 \\ \rho(]-\infty, v]) & \text{if } v < 0. \end{cases}$$

Let Leb be the Lebesgue measure on $\mathbb{R}$.

**Proposition 2.4.** *Let* $\lambda > 0$ *and* $\mu$ *in* $I(\lambda)$ *be given by* (2.8). *Let* $\mu$ *have characteristic triplet* $(\sigma^2, \nu, \gamma)$ *and* $\mathcal{L}(Z_1)$ *have characteristic triplet* $(\kappa^2, \rho, \delta)$.

(i) *The triplet* $(\sigma^2, \nu, \gamma)$ *is*

$$\sigma^2 = \frac{\kappa^2(e^{2\lambda}-1)}{2\lambda}, \tag{2.9}$$

$$\nu(B) = \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_0^1 1_B\left(e^{\lambda s} y\right) \mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R}), \tag{2.10}$$

$$\gamma = \frac{\delta}{\lambda}(e^{\lambda}-1) + \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_0^1 e^{\lambda s} y \left(1_D\left(e^{\lambda s} y\right) - 1_D(y)\right) \mathrm{d}s. \tag{2.11}$$

(ii) *The Lévy measure* $\nu$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$. *Let the density be represented as* $\nu(\mathrm{d}v) = \frac{k(v)}{\lambda|v|}\mathrm{d}v$, *where* $k(v) \geqslant 0$. *Then* $k(v)$ *is determined by* $G_\rho$ *as follows*

$$k(v) = G_\rho(e^{-\lambda} v) - G_\rho(v), \quad \mathrm{Leb} - a.e.\ v. \tag{2.12}$$

*Conversely,* $k(v)$ *determines* $G_\rho$ *by the formula*

$$\sum_{j=1}^{\infty} k(e^{\lambda j} v) = G_\rho(v), \quad \mathrm{Leb} - a.e.\ v. \tag{2.13}$$

(iii) *The distribution* $\mu$ *determines* $\mathcal{L}(Z_1)$. *That is, if* $\{Z'_t\}_{t\in[0,1]}$ *is a Lévy process such that* $\mu = \mathcal{L}\left(\int_0^1 e^{\lambda s}\mathrm{d}Z_s\right) = \mathcal{L}\left(\int_0^1 e^{\lambda s}\mathrm{d}Z'_s\right)$, *then* $\mathcal{L}(Z_1) = \mathcal{L}(Z'_1)$.

*Proof.* (i) is an immediate consequence of (2.1)–(2.3).

(ii) Let $B \in \mathcal{B}(\mathbb{R})$. Using substitution $v = e^{\lambda s} y$ in (2.10) it follows that

$$\begin{aligned}
\nu(B) &= \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_y^{e^{\lambda} y} 1_B(v) \frac{\mathrm{d}v}{\lambda v} \\
&= \int_{\mathbb{R}} \rho(\mathrm{d}y) \int_{|v|\in[|y|, e^{\lambda}|y|]} 1_B(v) \frac{\mathrm{d}v}{\lambda |v|} \\
&= \int_{\mathbb{R}} 1_B(v) \frac{\mathrm{d}v}{\lambda|v|} \int_{|y|\in[e^{-\lambda}|v|, |v|]} \rho(\mathrm{d}y) \\
&= \int_{\mathbb{R}} 1_B(v) \left(G_\rho(e^{-\lambda} v) - G_\rho(v)\right) \frac{\mathrm{d}v}{\lambda|v|},
\end{aligned}$$

which implies (2.12).

Let $k(v)$ be defined by $k(v) = G_\rho(e^{-\lambda}v) - G_\rho(v)$ for all $v \in \mathbb{R} \setminus \{0\}$. Then,

$$k(e^{\lambda j} v) = G_\rho(e^{\lambda(j-1)} v) - G_\rho(e^{\lambda j} v), \quad j = 1, 2, \ldots$$

and

$$\sum_{j=1}^{n} k(e^{\lambda j} v) = G_\rho(v) - G_\rho(e^{\lambda n} v)$$

from which (2.13) follows by letting $n \to \infty$.

(iii) The Lévy measure $\nu$ determines $\rho$ by (2.13). Therefore the characteristic triplet of $\mu$ determines the characteristic triplet of $\mathcal{L}(Z_1)$ by (2.9)–(2.11). $\square$

**Example 2.5.** Let $\mathcal{L}(Z_1)$ be the Poisson distribution with mean 1. In this case $\rho$ in Proposition 2.4 is the Dirac measure at 1, which means that $G_\rho(v) = 1_{]0,1]}(v)$. Hence $k(v)$ in (2.12) is $k(v) = 1_{]1,e^\lambda]}(v)$. Then $\mu$ defined by (2.8) is in $I(\lambda)$ by definition, but it is not selfdecomposable. Indeed, in the case of selfdecomposability $k(v)$ is decreasing on $]0,\infty[$, see Sato (1999, Theorem 15.10).

**Proposition 2.6.** *Let $\lambda > 0$. Then,*

(i) *$I(\lambda)$ is stable under convolution. Moreover, if $\{Z_t\}_{t\in[0,1]}$ is a Lévy process, then $\mathcal{L}(a + b\int_0^1 e^{\lambda s}\mathrm{d}Z_s) \in I(\lambda)$ for all $a, b \in \mathbb{R}$;*

(ii) *$SD \subseteq I(\lambda)$;*

(iii) *for $n = 1, 2, \ldots$, $I(n\lambda) \subseteq I(\lambda)$.*

*Proof.* The proof of (i) is left to the reader.

(ii) Any Gaussian distribution is clearly represented as in (2.8) with $\{Z_t\}_{t\in[0,1]}$ a Gaussian Lévy process. Therefore it suffices to show that if $\mu$ is selfdecomposable with characteristic triplet $(0, \nu, \gamma)$, then we have the representation (2.10)–(2.11) of $(\nu, \gamma)$, where $\rho$ is a Lévy measure and $\delta \in \mathbb{R}$. But (2.11) is obtained with an appropriate choice of $\delta$, so we may and do concentrate on (2.10).

Recall the representation (2.5) of $\nu$, where $\rho_\lambda$ is a Lévy measure satisfying condition (2.4). For $j = 1, 2, \ldots$, let $\rho_{\lambda,j}$ be the Lévy measure defined by

$$\rho_{\lambda,j}(B) = \int_{\mathbb{R}} 1_B\left(e^{-\lambda j}y\right)\rho_\lambda(\mathrm{d}y), \quad B \in \mathcal{B}(\mathbb{R}). \tag{2.14}$$

Substituting $t = j - s$ we get

$$\begin{aligned} &\int_{\mathbb{R}} \rho_{\lambda,j}(\mathrm{d}y)\int_0^1 1_B\left(e^{\lambda s}y\right)\mathrm{d}s \\ = &\int_{\mathbb{R}} \rho_\lambda(\mathrm{d}y)\int_0^1 1_B\left(e^{-\lambda(j-s)}y\right)\mathrm{d}s = \int_{\mathbb{R}} \rho_\lambda(\mathrm{d}y)\int_{j-1}^{j} 1_B\left(e^{-\lambda t}y\right)\mathrm{d}t \end{aligned}$$

and hence

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}} \rho_{\lambda,j}(\mathrm{d}y)\int_0^1 1_B\left(e^{\lambda s}y\right)\mathrm{d}s = \int_{\mathbb{R}} \rho_\lambda(\mathrm{d}y)\int_{\mathbb{R}_+} 1_B\left(e^{-\lambda s}y\right)\mathrm{d}s = \nu(B),$$

which is (2.10) with $\rho = \sum_{j=1}^\infty \rho_{\lambda,j}$. To conclude the proof we show that $\sum_{j=1}^\infty \rho_{\lambda,j}$ is a Lévy measure. The inequality

$$e^{-2\lambda j}y^2 1_D\left(e^{-\lambda j}y\right) \leqslant \int_{j-1}^{j} e^{-2\lambda s}y^2 1_D\left(e^{-\lambda s}ye^{-\lambda}\right)\mathrm{d}s$$

yields

$$
\begin{aligned}
& \sum_{j=1}^{\infty} \int_{|y|\leqslant 1} y^2 \rho_{\lambda,j}(\mathrm{d}y) = \sum_{j=1}^{\infty} \int_{\mathbb{R}} y^2 1_D(y) \rho_{\lambda,j}(\mathrm{d}y) \\
= \;& \sum_{j=1}^{\infty} \int_{\mathbb{R}} e^{-2\lambda j} y^2 1_D \left(e^{-\lambda j} y\right) \rho_\lambda(\mathrm{d}y) \\
\leqslant \;& \int_{\mathbb{R}} \rho_\lambda(\mathrm{d}y) \int_{\mathbb{R}_+} e^{-2\lambda s} y^2 1_D \left(e^{-\lambda} e^{-\lambda s} y\right) \mathrm{d}s \\
= \;& \frac{1}{2\lambda} \int_{\mathbb{R}} \left(y^2 \wedge e^{2\lambda}\right) \rho_\lambda(\mathrm{d}y) < \infty.
\end{aligned}
$$

Similarly, since $\rho_\lambda$ satisfies condition (2.4) it follows that $\sum_{j=1}^{\infty} \rho_{\lambda,j}(\{|y|>1\}) < \infty$.

(iii) Let $\mu \in I(n\lambda)$ with Lévy measure $\nu$. Then, $\nu$ is given by (2.10) with $\lambda$ replaced by $n\lambda$. This is rewritten as

$$
\nu(B) = \sum_{i=0}^{n-1} \int_{\mathbb{R}} \rho_i(\mathrm{d}y) \int_0^1 1_B \left(e^{\lambda s} y\right) \mathrm{d}s, \quad B \in \mathcal{B}(\mathbb{R})
$$

with $\rho_i(B) = \frac{1}{n} \int_{\mathbb{R}} 1_B(e^{\lambda i} y) \rho(\mathrm{d}y)$. Since $\sum_{i=0}^{n-1} \rho_i$ is a Lévy measure, $\mu \in I(\lambda)$. $\square$

Note that by Example 2.5 the inclusion in (ii) above is strict.

## 3. Properties of periodic Ornstein-Uhlenbeck processes

Let $\{Z_t\}_{t\in[0,1]}$ be a Lévy process on $\mathbb{R}$ and let $\lambda > 0$. It follows from (3.3)–(3.4) below that there is one and only one process $\{X_t\}_{t\in[0,1]}$ satisfying the following two conditions (3.1) and (3.2), where

$$
\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \mathrm{d}Z_t, \qquad t \in [0,1], \tag{3.1}
$$

$$
X_0 = X_1. \tag{3.2}
$$

The process $\{X_t\}_{t\in[0,1]}$ is called a *periodic Ornstein-Uhlenbeck process with parameter* $\lambda$ *and background driving Lévy process* $\{Z_t\}_{t\in[0,1]}$.

To construct the periodic Ornstein-Uhlenbeck process note that (3.1) implies

$$
X_t = e^{-\lambda t} X_0 + e^{-\lambda t} \int_0^t e^{\lambda s} \mathrm{d}Z_s, \quad t \in [0,1]. \tag{3.3}
$$

Letting $t = 1$ and using the boundary condition (3.2) it follows that

$$
X_0 = X_1 = \frac{1}{e^\lambda - 1} \int_0^1 e^{\lambda s} \mathrm{d}Z_s. \tag{3.4}
$$

Thus, $\{X_t\}_{t\in[0,1]}$ exists and is uniquely determined by (3.3)–(3.4).

Now and then we need the periodic extension $\{X_t\}_{t\in\mathbb{R}}$ of $\{X_t\}_{t\in[0,1]}$; $X_{t+k} = X_t$ for $t \in [0,1]$ and $k \in \mathbb{Z}$. Recall that

$$x \mod 1 = \begin{cases} x+1 & \text{if } x \in [-1,0[ \\ x & \text{if } x \in [0,1[. \end{cases}$$

The next lemma gives an alternative representation of $X_t$.

**Lemma 3.1.** *Let $t^0 \in [0,1]$. Construct a process $\{Z_t^{t^0}\}_{t\in[0,1]}$ from the background driving Lévy process in the following way:*

$$Z_t^{t^0} = \begin{cases} Z_{t^0+t} - Z_{t^0} & \text{if } 0 \leqslant t \leqslant 1-t^0 \\ Z_{1-t^0}^{t^0} + Z_{t-(1-t^0)} & \text{if } 1-t^0 < t \leqslant 1. \end{cases} \tag{3.5}$$

*Then*

(i) $\{Z_t^{t^0}\}_{t\in[0,1]}$ *is a Lévy process such that* $\{Z_t^{t^0}\}_{t\in[0,1]} \stackrel{\mathrm{d}}{=} \{Z_t\}_{t\in[0,1]}$;

(ii) *for $t \in [0,1]$ we have*

$$X_{t^0+t} = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda((t-s) \mod 1)} \mathrm{d}Z_s^{t^0}. \tag{3.6}$$

*In particular,*

$$X_t = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda((t-s) \mod 1)} \mathrm{d}Z_s. \tag{3.7}$$

*Proof.* (i) is obvious by construction of $\{Z_t^{t^0}\}_{t\in[0,1]}$. (ii) First assume that $t + t^0 \leqslant 1$. By (3.3)–(3.4) we have

$$\begin{aligned}
& (1-e^{-\lambda})X_{t^0+t} \\
= & \int_0^1 e^{-\lambda(t^0+t)} e^{-\lambda} e^{\lambda s} \mathrm{d}Z_s + (1-e^{-\lambda}) \int_0^{t^0+t} e^{-\lambda(t^0+t)} e^{\lambda s} \mathrm{d}Z_s \\
= & \int_{t^0+t}^1 e^{-\lambda(t^0+t)} e^{-\lambda} e^{\lambda s} \mathrm{d}Z_s + \int_0^{t^0+t} e^{-\lambda(t^0+t)} e^{\lambda s} \mathrm{d}Z_s \\
= & \int_{t^0+t}^1 e^{-\lambda(t^0+t)} e^{-\lambda} e^{\lambda s} \mathrm{d}Z_s + \int_0^{t^0} e^{-\lambda(t^0+t)} e^{\lambda s} \mathrm{d}Z_s + \int_{t^0}^{t+t^0} e^{-\lambda(t^0+t)} e^{\lambda s} \mathrm{d}Z_s \\
= & \int_t^{1-t^0} e^{-\lambda(t^0+t)} e^{-\lambda} e^{\lambda(s+t^0)} \mathrm{d}Z_s^{t^0} + \int_{1-t^0}^1 e^{-\lambda(t^0+t)} e^{\lambda(s-(1-t^0))} \mathrm{d}Z_s^{t^0} + \int_0^t e^{-\lambda(t^0+t)} e^{\lambda(s+t^0)} \mathrm{d}Z_s^{t^0} \\
= & \int_0^1 e^{-\lambda((t-s) \mod 1)} \mathrm{d}Z_s^{t^0},
\end{aligned}$$

which gives (3.6). In the case $t \in [0,1]$ and $t + t^0 > 1$ we have $X_{t^0+t} = X_{t^0+t-1}$. Calculations as above result in (3.6). Finally, (3.7) follows from the fact that $Z_t = Z_t^0$ a.s. □

Recall that a process $\{Y_t\}_{t\in[0,1]}$ on $\mathbb{R}$ is called a Markov random field if, for all $0 \leqslant a < b \leqslant 1$, the conditional distribution of $\{Y_t\}_{t\in[a,b]}$ given $\{Y_t\}_{t\in[0,1]\setminus]a,b[}$ depends only on $(Y_a, Y_b)$. In mathematical terms this property is defined as follows. Let $\mathcal{F}_{ab}$ and $\mathcal{F}^{ab}$ be the $\sigma$-algebras $\mathcal{F}_{ab} = \sigma(Y_t\colon t \in [a,b])$, $\mathcal{F}^{ab} = \sigma(Y_t\colon t \in [0,1]\setminus]a,b[)$. Then, $\{Y_t\}_{t\in[0,1]}$ is a Markov random field if, for all $0 \leqslant a < b \leqslant 1$ and $\mathcal{F}_{ab}$-measurable bounded $H$, we have $E[H|\mathcal{F}^{ab}] = E[H|(Y_a,Y_b)]$. It is well known that the usual Markov property implies the Markov random field property.

**Theorem 3.2.** *We have*

(i) $\{X_t\}_{t\in\mathbb{R}}$ *is a stationary process;*

(ii) $\{X_t\}_{t\in[0,1]}$ *is a Markov random field;*

(iii) *if* $\{X_t\}_{t\in\mathbb{R}}$ *is square integrable, then*

$$\operatorname{cov}(X_{t^0}, X_{t^0+t}) = \tau^2 \frac{e^{(t-1/2)\lambda} + e^{-(t-1/2)\lambda}}{e^{\lambda/2} + e^{-\lambda/2}}, \quad t^0 \in \mathbb{R}, \quad t \in [0,1], \tag{3.8}$$

*where* $\tau^2$ *is the variance of* $X_t$. *The latter is represented as*

$$\tau^2 = \operatorname{var}(X_t) = \frac{\operatorname{var}(\mathrm{Z}_1)}{2\lambda}\frac{e^{-\lambda/2} + e^{\lambda/2}}{e^{\lambda/2} - e^{-\lambda/2}}, \quad t \in \mathbb{R}. \tag{3.9}$$

*Proof.* (i) Let $t^0 \in [0,1]$. By (3.6),

$$X_{t^0+t} = \frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda((t-s) \mod 1)} \mathrm{d}Z_s^{t^0}, \quad t \in [0,1].$$

Hence by Lemma 3.1 (i) and (3.7)

$$\{X_{t^0+t}\}_{t\in[0,1]} \stackrel{\mathrm{d}}{=} \left\{\frac{1}{1-e^{-\lambda}} \int_0^1 e^{-\lambda((t-s) \mod 1)} \mathrm{d}Z_s\right\}_{t\in[0,1]} = \{X_t\}_{t\in[0,1]}.$$

This yields (i) by periodicity.

(ii) By periodicity and stationarity it suffices to show that for $0 < b < 1$ the conditional distribution of $\{X_t\}_{t\in[b,1]}$ given $\{X_t\}_{t\in[0,b]}$ depends on $(X_0, X_b)$ only. Let us introduce the process $\{X_t^0\}_{t\in[0,1]}$ which solves equation (3.1) with initial condition $X_0^0 = 0$. That is,

$$X_t^0 = e^{-\lambda t} \int_0^t e^{\lambda s} \mathrm{d}Z_s = X_t - e^{-\lambda t} X_0, \quad t \in [0,1]. \tag{3.10}$$

Using (3.10) and $X_0 = X_1$, it follows that $\{X_t\}_{t\in[b,1]}$ is in one-to-one correspondence with $\{X_t^0\}_{t\in[b,1]}$, that $\{X_t\}_{t\in[0,b]}$ is in one-to-one correspondence with $\{X_t^0\}_{t\in[0,b]}$ and that $(X_0, X_b)$ is in one-to-one correspondence with $(X_0^0, X_b^0)$. Since $\{X_t^0\}_{t\in[0,1]}$ is a Markov process, and hence also a Markov random field, the conditional distribution of $\{X_t\}_{t\in[b,1]}$ given $\{X_t\}_{t\in[0,b]}$ depends only on $(X_0, X_b)$.

(iii) By stationarity we may and do assume $t^0 = 0$. Then, (3.7) gives

$$\text{(3.11)} \qquad \begin{aligned} \operatorname{cov}(X_0, X_t) &= \frac{\operatorname{var}(Z_1)}{(1-e^{-\lambda})^2} \int_0^1 e^{-\lambda((t-s) \mod 1)} e^{-\lambda((0-s) \mod 1)} \mathrm{d}s \\ &= \frac{\operatorname{var}(Z_1)}{2\lambda} \frac{e^{\lambda(t-1/2)} + e^{-\lambda(t-1/2)}}{e^{\lambda/2} - e^{-\lambda/2}}. \end{aligned}$$

We get (3.9) by letting $t = 0$. The representation (3.8) is an immediate consequence of (3.11) and (3.9). □

**Remark 3.3.** *The stationary distribution.*

(i) Let $\{X_t\}_{t\in[0,1]}$ be a periodic Ornstein-Uhlenbeck process with parameter $\lambda$. Stationarity implies $\mathcal{L}(X_t) = \mathcal{L}(X_0)$ for all $t$. We call the common distribution $\mathcal{L}(X_0)$ *the stationary distribution of* $\{X_t\}_{t\in[0,1]}$. By (3.4) and Proposition 2.6 (i) a distribution $\mu$ is the stationary distribution of some periodic Ornstein-Uhlenbeck process with parameter $\lambda$ if and only if $\mu \in I(\lambda)$.

The set $I(\lambda)$ depends on $\lambda$. This is in contrast to usual Ornstein-Uhlenbeck processes where the set of stationary distributions is $SD$, irrespectively of the value of $\lambda$.

(ii) The law of a periodic Ornstein-Uhlenbeck process with parameter $\lambda$ and the law of the corresponding background driving Lévy process is determined by the stationary distribution. The proof is as follows. For $j = 1, 2$ let $\{X_t^j\}_{t\in[0,1]}$ be a periodic Ornstein-Uhlenbeck process with parameter $\lambda$, stationary distribution $\mu$ and background driving Lévy process $\{Z_t^j\}_{t\in[0,1]}$. Then, by Proposition 2.4 (iii) and (3.4), $\mathcal{L}(Z_1^1) = \mathcal{L}(Z_1^2)$ and (3.7) yields $\{X_t^1\}_{t\in[0,1]} \stackrel{\mathrm{d}}{=} \{X_t^2\}_{t\in[0,1]}$.

**Remark 3.4.** *The bivariate distributions.* Let $\{X_t\}_{t\in[0,1]}$ be a periodic Ornstein-Uhlenbeck process with parameter $\lambda$. The finite-dimensional marginals of $\{X_t\}_{t\in[0,1]}$ are then easily derived. For simplicity we shall concentrate on the bivariate distributions and by stationarity it suffices to consider $X_0$ and $X_t$ with $t \in ]0, 1[$. Equations (3.3)–(3.4) yield

$$\text{(3.12)} \qquad X_0 = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^t e^{\lambda s} \mathrm{d}Z_s + \frac{e^{-\lambda} e^{\lambda t}}{1-e^{-\lambda}} \int_t^1 e^{\lambda(s-t)} \mathrm{d}Z_s,$$

$$\text{(3.13)} \qquad X_t = \frac{e^{-\lambda t}}{1-e^{-\lambda}} \int_0^t e^{\lambda s} \mathrm{d}Z_s + \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_t^1 e^{\lambda(s-t)} \mathrm{d}Z_s.$$

Note that $\int_0^t e^{\lambda s} \mathrm{d}Z_s$ and $\int_t^1 e^{\lambda(s-t)} \mathrm{d}Z_s$ are independent. The corresponding characteristic triplets are easily derived using (2.1)–(2.3). This specifies the joint distribution of $(X_0, X_t)$ completely. In the case $t = 1/2$ stationarity and the property $X_0 = X_1$

implies $\mathcal{L}((X_0, X_{1/2})) = \mathcal{L}((X_{1/2}, X_0))$, i.e. $X_0$ and $X_{1/2}$ are exchangeable. The representation (3.12)–(3.13) reduces to

$$X_0 = \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_0^{1/2} e^{\lambda s} \mathrm{d}Z_s + \frac{e^{-\lambda/2}}{1-e^{-\lambda}} \int_{1/2}^{1} e^{\lambda(s-1/2)} \mathrm{d}Z_s, \tag{3.14}$$

$$X_{1/2} = \frac{e^{-\lambda/2}}{1-e^{-\lambda}} \int_0^{1/2} e^{\lambda s} \mathrm{d}Z_s + \frac{e^{-\lambda}}{1-e^{-\lambda}} \int_{1/2}^{1} e^{\lambda(s-1/2)} \mathrm{d}Z_s, \tag{3.15}$$

where $\mathcal{L}(\int_0^{1/2} e^{\lambda s} \mathrm{d}Z_s) = \mathcal{L}(\int_{1/2}^{1} e^{\lambda(s-1/2)} \mathrm{d}Z_s)$.

**Remark 3.5.** *A selfdecomposable stationary distribution.* Let $\mu \in SD$ and $\lambda > 0$. Recall from Proposition 2.1 (ii) that $\mu$ is represented as $\mu = \mathcal{L}\left(\int_0^\infty e^{-\lambda s} \mathrm{d}Z_s^\lambda\right)$. That is, $\{Z_t^\lambda\}_{t \geqslant 0}$ is the background driving Lévy process of an ordinary Ornstein-Uhlenbeck process with parameter $\lambda$ and stationary distribution $\mu$. By Proposition 2.6 $\mu$ is in $I(\lambda)$ as well. We use $\{Z_t^\lambda\}_{t \geqslant 0}$ to construct the background driving Lévy process $\{Z_t\}_{t\in[0,1]}$ of a periodic Ornstein-Uhlenbeck process with parameter $\lambda$ and stationary distribution $\mu$.

Define a sequence of independent Lévy processes $\{Z_t^{\lambda,j}\}_{t\in[0,1]}$, $j = 1, 2, \ldots$ by

$$Z_t^{\lambda,j} = e^{-\lambda j}(Z_{t+(j-1)}^\lambda - Z_{j-1}^\lambda), \quad t \in [0,1], \quad j = 1, 2, \ldots\,. \tag{3.16}$$

The Lévy measure of $\mathcal{L}(Z_1^{\lambda,j})$ is $\rho_{\lambda,j}$ in equation (2.14), where $\rho_\lambda$ is the Lévy measure of $\mathcal{L}(Z_1^\lambda)$. Define $\{Z_t\}_{t\in[0,1]}$ by

$$Z_t = \left(e^\lambda - 1\right) \sum_{j=1}^\infty Z_t^{\lambda,j}, \quad t \in [0,1], \tag{3.17}$$

and note that $\mathcal{L}\left(\left(e^\lambda - 1\right)^{-1} Z_1\right)$ has Lévy measure $\sum_{j=1}^\infty \rho_{\lambda,j}$. (It was shown in the proof of Proposition 2.6 that this is indeed a Lévy measure). Let $\{X_t\}_{t\in[0,1]}$ be the periodic Ornstein-Uhlenbeck process with parameter $\lambda > 0$ and background driving Lévy process $\{Z_t\}_{t\in[0,1]}$. Then the stationary distribution of $\{X_t\}_{t\in[0,1]}$ is $\mu$. Indeed, by (3.4),

$$X_0 = \left(e^\lambda - 1\right)^{-1} \int_0^1 e^{\lambda s} \mathrm{d}Z_s = \sum_{j=1}^\infty \int_0^1 e^{\lambda s} \mathrm{d}Z_s^{\lambda,j} = \sum_{j=1}^\infty \int_{j-1}^{j} e^{-\lambda(2j-1-s)} \mathrm{d}Z_s^\lambda.$$

Using (2.1)–(2.3) it is readily seen that $\int_{j-1}^{j} e^{-\lambda(2j-1-s)} \mathrm{d}Z_s^\lambda \stackrel{\mathrm{d}}{=} \int_{j-1}^{j} e^{-\lambda s} \mathrm{d}Z_s^\lambda$. Consequently $X_0 \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-\lambda s} \mathrm{d}Z_s^\lambda$, which shows that $\mu = \mathcal{L}(X_0)$.

For $t \in ]0,1[$ the distribution of $(X_0, X_t)$ is a bivariate $\mu$-distribution in the sense that $\mathcal{L}(X_0) = \mathcal{L}(X_t) = \mu$. Further, $X_0$ and $X_{1/2}$ are exchangeable and the correlation of $X_0$ and $X_{1/2}$ is $1/\cosh(\lambda/2)$, if it exists. When $\lambda$ varies from 0 to infinity the correlation of $X_0$ and $X_{1/2}$ varies from $1/2$ to 0.

In contrast to $X_0$ and $X_{1/2}$ the bivariate distributions of an ordinary Ornstein-Uhlenbeck process are usually not exchangeable. That is, if $\{Y_t\}_{t\geqslant 0}$ is an Ornstein-Uhlenbeck process with stationary distribution $\mu$ (see (2.6)), then $Y_0$ and $Y_t$ are exchangeable only under strict assumptions (such as Gaussianity) on $\mu$.

**Example 3.6.** *Gamma periodic Ornstein-Uhlenbeck processes.* Let $\mu$ be the gamma distribution $\Gamma(\alpha,\beta)$ with density $x \to \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}, x>0$. The Lévy measure $\nu$ of $\mu$ is $\nu(\mathrm{d}x) = \alpha x^{-1}e^{-\beta x}\mathrm{d}x, x>0$, and $\mu$ has zero drift. By selfdecomposability there exists a periodic Ornstein-Uhlenbeck process with stationary distribution $\mu$, which we call a *gamma periodic Ornstein-Uhlenbeck process.*

Gamma periodic Ornstein-Uhlenbeck processes can be constructed using the preceding remark. Indeed, the Lévy measure $\rho_\lambda$ of $\mathcal{L}(Z_1^\lambda)$ in Proposition 2.1 (ii) is $\rho_\lambda(\mathrm{d}x) = \alpha\beta\lambda e^{-\beta x}\mathrm{d}x, x>0$, which follows by verification in (2.5), and $\mathcal{L}(Z_1^\lambda)$ has drift 0. Hence, $\{Z_t^\lambda\}_{t\geqslant 0}$ is a compound Poisson process represented as $Z_t^\lambda = \sum_{n=1}^{N_t} Y_n$, where $\{N_t\}_{t\geqslant 0}$ is a Poisson process with $E[N_t] = \alpha\lambda t$, independent of $\{Y_n\}_{n\geqslant 1}$, which is an iid sequence with $\mathcal{L}(Y_n) = \Gamma(1,\beta)$. Let $\{X_t\}_{t\in[0,1]}$ be the periodic Ornstein-Uhlenbeck process with parameter $\lambda$ and background driving Lévy process $\{Z_t\}_{t\in[0,1]}$, where $Z_t$ is defined in (3.17). By Remark 3.5 $\{X_t\}_{t\in[0,1]}$ is a gamma periodic Ornstein-Uhlenbeck process with stationary distribution $\mu$. In particular, $(X_0, X_t)$ is a bivariate gamma-distribution which is exchangeable when $t = 1/2$.

Note that by approximating $Z_t$ by $\left(e^\lambda - 1\right)\sum_{j=1}^K Z_t^{\lambda,j}$ the variance of the error term $Z_t - \left(e^\lambda - 1\right)\sum_{j=1}^K Z_t^{\lambda,j}$ converges to zero exponentially fast as $K$ tends to infinity.

## 4. Gaussian periodic Ornstein-Uhlenbeck processes

In this section we collect some properties of Gaussian periodic Ornstein-Uhlenbeck processes. These processes have been studied both in probability theory and statistics.

Let $\{Z_t\}_{t\in[0,1]}$ be given by $Z_t = \kappa W_t$, where $\kappa > 0$ and $\{W_t\}_{t\in[0,1]}$ is a standard Wiener process on $\mathbb{R}$. The Gaussian periodic Ornstein-Uhlenbeck process $\{X_t\}_{t\in[0,1]}$ with parameter $\lambda > 0$ and background driving Lévy process $\{Z_t\}_{t\in[0,1]}$ is

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \kappa \mathrm{d}W_t, \quad t\in[0,1], \quad X_0 = X_1. \tag{4.1}$$

The variance $\tau^2$ of $X_t$ is, by (3.9),

$$\operatorname{var}(X_t) = \tau^2 = \frac{\kappa^2}{2\lambda} \frac{e^{-\lambda/2} + e^{\lambda/2}}{e^{\lambda/2} - e^{-\lambda/2}}. \tag{4.2}$$

Equation (4.1) is a simple example of a stochastic differential equation with a boundary condition. See Ocone and Pardoux (1989), Alabert, Ferrante and Nualart (1995) and Nualart (1995) for results in a more general setting. The Markov random field property of $\{X_t\}_{t\in[0,1]}$ in Theorem 3.2 (ii) was given also in Alabert et al. (1995, Theorem 5.2). Norris (1998) studied more general Gaussian periodic Ornstein-Uhlenbeck processes.

In statistical shape analysis Gaussian periodic processes are often used to model the boundary of a random solid object in the plane, and $\{X_t\}_{t\in[0,1]}$ has been widely used for this purpose, see Grenander (1993), Hobolth and Jensen (2000) and references therein. In these references the representation (4.1) was not recognized. Instead $\{X_t\}_{t\in[0,1]}$ was defined as the zero mean stationary periodic Gaussian process with variance $\tau^2$ and covariance (3.8). Grenander (1993) realized the representation (3.7) of $\{X_t\}_{t\in[0,1]}$ with $\{Z_t\}_{t\in[0,1]}$ a Wiener process. Hobolth and Vedel (2000) referred to $\{X_t\}_{t\in[0,1]}$ as a (continuous time) first-order Markov process. The reason is that it appears as the limit of discrete time first-order Markov processes as we now describe.

Let $(\alpha, \beta)$ be the parameters given by

$$\lambda^2 = \alpha/\beta, \quad \kappa^2 = 1/\beta \tag{4.3}$$

and $\operatorname{circ}(a_1, \dots, a_n)$ denote the $n \times n$ circulant matrix whose first row is $(a_1, \dots, a_n)$. The following result is due to Grenander (1993, pp. 476–480).

**Proposition 4.1.** *Let $\{X_t^n\}_{t\in[0,1]}$ be the periodic Gaussian process defined by*

$$\mathcal{L}(X_{t_0}^n, \dots, X_{t_{n-1}}^n) = N_n(0, \Sigma_n) \tag{4.4}$$

*for $t_i = i/n, i = 0, \dots, n$, with linear interpolation between $t_i$ and $t_{i+1}$ and with $X_0^n = X_1^n$. Here $\Sigma_n$ is the $n \times n$ matrix with*

$$\Sigma_n^{-1} = \operatorname{circ}(\alpha/n + 2\beta n, -\beta n, 0, \dots, 0, -\beta n). \tag{4.5}$$

*Then $\{X_t^n\}_{t\in[0,1]}$ converges weakly to $\{X_t\}_{t\in[0,1]}$ as $n \to \infty$.*

This convergence takes place on $C_p[0,1]$, the set of periodic continuous paths. That is, $\{x_t\}_{t\in[0,1]}$ is in $C_p[0,1]$ if $t \to x_t$ is continuous from $[0,1]$ into $\mathbb{R}$ with $x_0 = x_1$. Equip $C_p[0,1]$ with the topology corresponding to uniform convergence on $[0,1]$.

The likelihood function for $(\lambda, \kappa^2)$ based on observations of $\{X_t^n\}_{t\in[0,1]}$ is well known. The following is the analogous result for $\{X_t\}_{t\in[0,1]}$.

Let $P_{\lambda,\kappa^2}$, $\lambda, \kappa^2 > 0$, be the law of $\{X_t\}_{t\in[0,1]}$ in (4.1) on the space $C_p[0,1]$.

**Proposition 4.2.** *Let $\lambda, \tilde{\lambda}, \kappa^2, \tilde{\kappa}^2 > 0$.*

(i) *If $\kappa^2 \neq \tilde{\kappa}^2$, then $P_{\lambda,\kappa^2}$ and $P_{\tilde{\lambda},\tilde{\kappa}^2}$ are singular.*

(ii) *If $\kappa^2 = \tilde{\kappa}^2$, then $P_{\lambda,\kappa^2}$ and $P_{\tilde{\lambda},\kappa^2}$ are equivalent, and*

$$\frac{\mathrm{d}P_{\tilde{\lambda},\kappa^2}}{\mathrm{d}P_{\lambda,\kappa^2}} = a(\tilde{\lambda}, \lambda; \kappa^2) \exp\left(-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2} \int_0^1 X_t^2 \mathrm{d}t\right), \tag{4.6}$$

*where $a(\tilde{\lambda}, \lambda; \kappa^2)$ is a norming constant.*

In the last statement $\{X_t\}_{t\in[0,1]}$ is taken to be the coordinate process on $C_p[0,1]$.

*Proof.* Throughout the proof let $\{X_t\}_{t\in[0,1]}$ be the coordinate process on $C_p[0,1]$.

(i) We have

$$X_t = X_0 - \int_0^t \lambda X_s \mathrm{d}s + \kappa W_t, \tag{4.7}$$

where $\{W_t\}_{t\in[0,1]}$ is a standard Wiener process under $P_{\lambda,\kappa^2}$. Since the integral on the right-hand side is of bounded variation the quadratic variation of $\{X_t\}_{t\in[0,1]}$ equals the quadratic variation of $\{\kappa W_t\}_{t\in[0,1]}$. Thus,

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(X_{i/2^n} - X_{(i-1)/2^n}\right)^2 = \kappa^2 \quad \text{under } P_{\lambda,\kappa^2}, \tag{4.8}$$

where the limit exists in probability. Similarly, under $P_{\tilde{\lambda},\tilde{\kappa}^2}$ the quadratic variation is $\tilde{\kappa}^2$, which shows that $P_{\lambda,\kappa^2}$ and $P_{\tilde{\lambda},\tilde{\kappa}^2}$ are singular.

(ii) Throughout the proof fix $\kappa^2 > 0$. Let the process $\{X_t^n\}_{t\in[0,1]}$ in Proposition 4.1 be defined on a measurable space $(\Omega^n, \mathcal{F}^n)$ where for simplicity we let $\mathcal{F}^n = \sigma(X_t^n\colon t \in [0,1])$. On $(\Omega^n, \mathcal{F}^n)$ define a family of probability measures $\{P^n_{\lambda,\kappa^2}\colon \lambda > 0\}$ such that $\{X_t^n\}_{t\in[0,1]}$ has the distribution specified by (4.4)–(4.5) under $P^n_{\lambda,\kappa^2}$. From Hobolth and Jensen (2000, p. 354) we have that $\{P^n_{\lambda,\kappa^2}\colon \lambda > 0\}$ are equivalent with $\frac{\mathrm{d}P^n_{\tilde{\lambda},\kappa^2}}{\mathrm{d}P^n_{\lambda,\kappa^2}} = a_n(\tilde{\lambda}, \lambda; \kappa) h_n(\tilde{\lambda}, \lambda; \kappa^2)$, $\lambda, \tilde{\lambda} > 0$, where

$$\begin{aligned}
h_n(\tilde{\lambda}, \lambda; \kappa^2) &= \exp\left(-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2} n^{-1} \sum_{i=0}^{n-1} \left(X^n_{i/n}\right)^2\right), \\
a_n(\tilde{\lambda}, \lambda; \kappa)^{-1} &= E_{\lambda,\kappa^2}\left[\exp\left(-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2} n^{-1} \sum_{i=0}^{n-1} \left(X^n_{i/n}\right)^2\right)\right]
\end{aligned}$$

and $E_{\lambda,\kappa}$ denotes expectation under $P^n_{\lambda,\kappa^2}$. Define similarly

$$\begin{aligned} h(\tilde{\lambda}, \lambda; \kappa^2) &= \exp\left(-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2}\int_0^1 X_t^2 dt\right) \\ a(\tilde{\lambda}, \lambda; \kappa)^{-1} &= E_{\lambda,\kappa}\left[\exp\left(-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2}\int_0^1 X_t^2 \mathrm{d}t\right)\right], \end{aligned}$$

where $E_{\lambda,\kappa^2}$ denotes expectation under $P_{\lambda,\kappa^2}$. Let $\lambda, \tilde{\lambda} > 0$ with $\lambda < \tilde{\lambda}$. It follows from Proposition 4.1 and the continuous mapping theorem that

$$\mathcal{L}_{\lambda,\kappa^2}\left(\{X_t^n\}_{t\in[0,1]}, h_n(\tilde{\lambda}, \lambda; \kappa^2)\right) \to \mathcal{L}_{\lambda,\kappa^2}\left(\{X_t\}_{t\in[0,1]}, h(\tilde{\lambda}, \lambda; \kappa^2)\right), \tag{4.9}$$

where $\mathcal{L}_{\lambda,\kappa^2}$ denotes the distribution under $P^n_{\lambda,\kappa^2}$ on the left-hand side and under $P_{\lambda,\kappa^2}$ on the right-hand side. These distributions are defined on $C_p[0,1] \times \mathbb{R}$ equipped with the product topology. Note that

$$-\frac{\tilde{\lambda}^2 - \lambda^2}{2\kappa^2}\sum_{i=0}^{n-1}(X^n_{i/n})^2 \leqslant 0. \tag{4.10}$$

Hence, from (4.9) we get

$$a_n(\tilde{\lambda}, \lambda; \kappa) \to a(\tilde{\lambda}, \lambda; \kappa). \tag{4.11}$$

For $f\colon C_p[0,1] \to \mathbb{R}$ continuous and bounded (4.9)–(4.11) yield

$$E_{\lambda,\kappa^2}\left[f\left(\{X_t^n\}_{t\in[0,1]}\right) h_n(\tilde{\lambda}, \lambda; \kappa^2) a_n(\tilde{\lambda}, \lambda; \kappa)\right] \to E_{\lambda,\kappa^2}\left[f\left(\{X_t\}_{t\in[0,1]}\right) h(\tilde{\lambda}, \lambda; \kappa^2) a(\tilde{\lambda}, \lambda; \kappa)\right].$$

On the other hand, since (4.9) also holds with $\lambda$ replaced by $\tilde{\lambda}$ it follows that

$$E_{\lambda,\kappa^2}\left[f\left(\{X_t^n\}_{t\in[0,1]}\right) h_n(\tilde{\lambda}, \lambda; \kappa^2) a_n(\tilde{\lambda}, \lambda; \kappa)\right] = E_{\tilde{\lambda},\kappa^2}\left[f\left(\{X_t^n\}_{t\in[0,1]}\right)\right] \to E_{\tilde{\lambda},\kappa^2}\left[f\left(\{X_t\}_{t\in[0,1]}\right)\right].$$

Thus, $P_{\tilde{\lambda},\kappa^2}$ and $P_{\lambda,\kappa^2}$ are equivalent with $\frac{\mathrm{d}P_{\tilde{\lambda},\kappa^2}}{\mathrm{d}P_{\lambda,\kappa^2}} = a(\tilde{\lambda}, \lambda; \kappa) h(\tilde{\lambda}, \lambda; \kappa^2)$. □

**Remark 4.3.** There are several other ways to establish the representation (4.6).

(i) A more direct approach relies on a Girsanov theorem for non-adapted processes, see e.g. Theorem 4.1.2 in Nualart (1995). By this result we have

$$\frac{\mathrm{d}P_{\tilde{\lambda},\kappa^2}}{\mathrm{d}P_{\lambda,\kappa^2}} = c\exp\left(-\int_0^1 \frac{\tilde{\lambda} - \lambda}{\kappa} X_t \mathrm{d}W_t - \frac{1}{2}\int_0^1 \left(\frac{\tilde{\lambda} - \lambda}{\kappa} X_t\right)^2 \mathrm{d}t\right), \tag{4.12}$$

where $\{W_t\}_{t\in[0,1]}$ is a standard Wiener processes under $P_{\lambda,\kappa^2}$ and we use the representation (4.7) of $\{X_t\}_{t\in[0,1]}$. Further, $\mathrm{d}W$ denotes the Skorohod integral with respect to $\{W_t\}_{t\in[0,1]}$ and $c$ is a norming constant (the Carleman-Fredholm determinant in Theorem 4.1.2 in Nualart (1995), which in our case is deterministic). After some manipulations of $\int_0^1 \frac{\tilde{\lambda}-\lambda}{\kappa} X_t \mathrm{d}W_t$ it follows that (4.12) reduces to (4.6).

(ii) The Fourier coefficients of $\{X_t\}_{t\in[0,1]}$ are $A_0 = \int_0^1 X_t \mathrm{d}t$ and

$$A_k = \sqrt{2}\int_0^1 X_t \cos(2\pi k t)\mathrm{d}t, \quad B_k = \sqrt{2}\int_0^1 X_t \sin(2\pi k t)\mathrm{d}t, \quad k = 1, 2, \dots . \tag{4.13}$$

Then, $A_k, k = 0, 1, \dots$ and $B_k, k = 1, 2, \dots$ are independent with $\mathcal{L}(A_0) = N(0, \lambda_0)$ and $\mathcal{L}(A_k) = \mathcal{L}(B_k) = N(0, \lambda_k)$. Here, $\lambda_k^{-1} = \alpha + \beta(2\pi k)^2, \quad k = 0, 1, \dots,$ and $\alpha, \beta$ are the parameters given by (4.3). Using a shift of a measure for the Fourier coefficients we get (4.6).

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