VALUATION OF ASIAN BASKET OPTIONS WITH QUASI-MONTE CARLO TECHNIQUES AND SINGULAR VALUE DECOMPOSITION

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Abstract. We propose pricing methods for European-style Asian arithmetic average basket options in a Black-Scholes framework based on a QMC method. The nature of QMC methods enables us to enhance the accuracy by decomposing the correlation structure of the noise in the problem using singular value decomposition. This leads to optimal utilization of the low discrepancy sequence, and gives several orders of magnitude enhanced performance over conventional QMC and standard MC methods.

1. Introduction

There are no closed form pricing formulas for the European-style Asian arithmetic average options (hereafter Asian options), neither the single asset option nor the basket option. Both problems must be solved by numerical solution methods, and are computer intensive tasks. The option price is given by an expected value, and the pricing is therefore an integration problem. In this paper we formulate the pricing problems explicitly as multi-dimensional integrals, which enables us to use quasi-Monte Carlo (QMC) methods to approximate their values. The main goal is the pricing of the Asian basket option, but the single asset option is also discussed as an introduction to the basket case and for comparative studies between different path discretization schemes.

It is well known that Asian options and other path dependent options hold certain properties that can be exploited to increase the convergence rate when calculating their values with QMC methods. This is done by combining the QMC method with variance reduction techniques. Singular value decomposition (SVD) of the noise term in the problem is suggested. We propose to use a combination of SV-decomposition of the covariance matrix of the Brownian paths, *and*

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a representation of the volatility matrix for the assets in the basket by using SV-decomposition of the covariance matrix of asset returns. We demonstrate that this approach leads to considerably better convergence properties than in the case where conventional discretization is used on the Brownian paths, and Cholesky-decomposition is used to create the volatility matrix from the covariance matrix of the returns. We have also included the Brownian Bridge method for the discretization of the Brownian paths in order to compare with the SVD approach for the single asset case, and we show that the proposed SVD method work better than the Brownian Bridge method as well.

For Asian basket options the number of dimensions in the problem may grow significantly as it is the product of the number of assets in the basket and the number of time discretization points. We show how to avoid performing an SV-decomposition of the full problem by using the direct matrix product to combine the decompositions of each of the covariance structures (the path and the basket) into a matrix that describes the full system. This reengineering of the problem enables us to exploit the QMC method better. This is because low discrepancy sequences, which is the basic part of the QMC approach, often have the property that some elements of the sample vector have better discrepancy characteristics than other, and by reengineering the problem we adapt it to this property.

Section 2 describes the financial market in which we will do our analysis, and section 3 gives some background information on multi-dimensional Brownian motion which is useful in our context. In section 4 we give a short motivation for the use of QMC methods and give arguments for why the methods we use in our approach do work. Section 5 introduces the general expression for the value of the claim. In section 6 we point out the properties of the asset price process we have to use in order to formulate the Asian basket option problem as an integral. The section focuses on the *conventional* QMC approach to outline the general concept and to show how we construct the algorithms we compare the SVD methods with. We describe the Brownian Bridge technique in section 7, and the SVD in section 8. In section 9 we present numerical results comparing the different methods, and finally we conclude in section 10.

2. The market

We operate in the context of a complete, standard financial market $\mathcal{M}$, with constant risk-free rate r and volatility matrix σ. The price processes of the assets in this market are governed by a set of stochastic differential equations (SDEs). There are $N + 1$ assets in the market, one risk free asset and N risky assets. The model for the risky assets is the so called geometric Brownian motion. For a comprehensive survey of the assumptions and properties of the market see [KS98]. The solution to the SDEs is achieved by the development of a risk free measure P_0 and straightforward use of Ito's formula. This leads to the following expressions for the price processes:

$$S_0(t) = e^{rt} \tag{2.1}$$

$$S_n(t) = S_n(0)\exp\Big((r - \frac{1}{2}\sum_{d=1}^{N}\sigma_{nd}^2)t + \sum_{d=1}^{N}\sigma_{nd}W_0^{(d)}(t)\Big), \quad n = 1,\dots,N. \tag{2.2}$$

The volatility matrix $\sigma \in \mathbb{R}^{N\times N}$ is such that $\sigma\sigma^T \equiv \sigma^2$ is the covariance matrix of the returns of the assets. Note that the relation defining σ is not unique, and this is a key feature for the enhancement of the numerical methods. The stochastic process $W_0(t)$ is an N dimensional Brownian motion under the risk free measure P_0. In the following analysis it is convenient to write the price process for the risky assets like

$$S_n(u) = h_n(u-t, S(t), \sigma(W_0(u) - W_0(t))), \quad 0 \le t \le u \le T, \tag{2.3}$$

where $h : [0,\infty) \times \mathbb{R}_+^N \times \mathbb{R}^N \to \mathbb{R}_+^N$ is the function defined by

$$h_n(t,p,w) \equiv p_n \exp\Big((r - \frac{1}{2}\sum_{d=1}^{N}\sigma_{nd}^2)t + w_n\Big), \quad n = 1,\dots,N. \tag{2.4}$$

The process $W_0(t)$ is an essential part of the market $\mathcal{M}$. $\theta \in \mathbb{R}^N$ is given by the relation $b - r\underline{1} = \sigma\theta$, where $b \in \mathbb{R}^N$ is the vector of drift coefficients for the assets of $\mathcal{M}$. θ is called the market price of risk. By using the Girsanov theorem it can be shown that

$$W_0(t) = W(t) + \int_0^t \theta ds, \quad \forall t \in [0,T] \tag{2.5}$$

is an N-dimensional Brownian motion under the risk free measure P_0 relative to the filtration $\{\mathcal{F}(t)\}$ of $W(t)$. The process $W(t)$ is the Brownian motion observed for the assets in the market under the market induced probability measure P. For a more comprehensive survey of these aspects, see [KS98, Ch. 1].

3. Useful properties of the Brownian motion

In this section we present some well-known properties of Gaussian processes which are useful for our approach. The Brownian motion $W_0(t) \in \mathbb{R}^N$ is a Gaussian process, which means that the random variable $Z = (W_0(t_0),\dots,W_0(t_K) \in \mathbb{R}^{N(K+1)}$ has a normal distribution. The covariance matrix of Z is given by

$$C_Z^2 = \begin{bmatrix} t_0 I & t_0 I & \dots & t_0 I \\ t_0 I & t_1 I & \dots & t_1 I \\ \vdots & \vdots & \ddots & \vdots \\ t_0 I & t_1 I & \dots & t_K I \end{bmatrix}, \tag{3.1}$$

where I is the $N \times N$ identity matrix, see e.g. [Øks98]. Let C^2 be the covariance matrix of Z for the case where $N = 1$. If we construct the process

$X = (\sigma W_0(t_0), \ldots, \sigma W_0(t_K)) \in \mathbb{R}^{N(K+1)}$, where $W_0(t_k) \in \mathbb{R}^N$ and $\sigma^2 \in \mathbb{R}^{N\times N}$ is positive semidefinite, the covariance matrix $C_X^2 \in \mathbb{R}^{N(K+1)\times N(K+1)}$ of the process X is given by

$$C_X^2 = \begin{bmatrix} t_0\sigma^2 & t_0\sigma^2 & \ldots & t_0\sigma^2 \\ t_0\sigma^2 & t_1\sigma^2 & \ldots & t_1\sigma^2 \\ \vdots & \vdots & \ddots & \\ t_0\sigma^2 & t_1\sigma^2 & \ldots & t_K\sigma^2 \end{bmatrix}. \tag{3.2}$$

In the notation of the direct product of matrices we can write $C_X^2 = C^2 \otimes \sigma^2$. To see that C_X^2 is given by (3.2), consider the process $\hat{Z}_0 = \sigma W_0(t_0) \in \mathbb{R}^N$: We know that $\hat{Z}_0 \sim \mathcal{N}_N(0, t_0\sigma\sigma^T)$. By using this for each of the N processes $\hat{Z}_K$ contained in $Z \in R^{N(K+1)}$ we get to the expression (3.2). It can be shown that the eigenvalues of C_X^2 are found directly from the eigenvalues of the matrix σ^2 and the eigenvalues of C^2 by the relation $\Lambda_{C_X^2} = \Lambda_{C^2} \otimes \Lambda_{\sigma^2} \in \mathbb{R}_+^{N(K+1)\times N(K+1)}$. See [Lam93] and [Lan69] for a full treatment of the direct matrix product. The eigenvalue property enables us to find an ordering of the total set of eigenvalues. In section 4 we will look into the QMC method, and reveal the advantage of knowing the eigenvalues and their ordering in finance problems.

4. Quasi-Monte Carlo methods

In this section we give a brief survey of the QMC-technique. The goal is to evaluate an integral of the form

$$\int_{[0,1]^s} f(y)dy \approx \frac{1}{L+1}\sum_{l=0}^{L} f(y^l). \tag{4.1}$$

The sequence $\{y^l\}$ of vectors $y^l = (y_1^l, \ldots, y_s^l) \in [0,1]^s,\ l = 0, \ldots, L$ used for the approximation can be generated by a systematic combinatorial approach, giving a conventional grid-based numerical integration algorithm. The problem with this approach is that the complexity grows exponentially with the dimension s, leading to practically useless algorithms for $s > 5$. If, on the other hand, the sequence is created by letting each vector y^l be independent uniform random variables in $[0,1]^s$, we get the conventional MC approach. This enables us to increase the number of evaluation points in a smooth manner, filling the domain of integration gradually. QMC methods keep this nice feature of the MC approach, but uses number sequences which are not random. These number sequences are constructed with the intention of filling the domain of integration as evenly as possible, resulting in methods where the approximation of the integral can be obtained with even fewer integrand evaluations than in the conventional MC Methods. Conventional MC method only converges at an order of $\mathcal{O}(1/\sqrt{L})$, but QMC methods are able to increase this rate. The employment of QMC methods are closely linked to the formulation of the problem as a multi dimensional integral, and strict control of the use of the number sequence $\{y^l\}$ in the construction of the distributions used. QMC methods are based on the approach of removing randomness from the generation of sampling sequences. The idea is to look for fixed sequences that perform better than random sequences in a well defined sense. The measurement of this behavior is not trivial in general, and these uniform distributed sequences are objects of extensive research, see e.g. [NX96], [Owe99], [Owe98]. The discrepancy of the sequence is used to measure how well distributed the samples are, see e.g. [JBT96], [AP97], [Pas97] for more details. Discrepancy is defined as follows

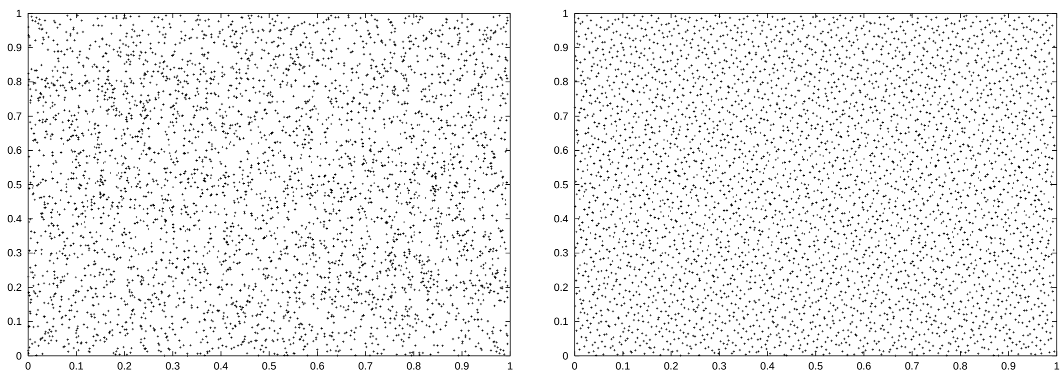

FIGURE 1. An illustration of the ability to fill the domain $[0,1]^2$ uniformly by the use of the conventional pseudo-random numbers (left), and the Halton leaped low discrepancy sequence (right).

Definition 4.1. *Let $\mathcal{B}$ be a family of shapes which are subsets of $[0,1]^s$. Given a sequence $\{y^l\}$ of sample points. The discrepancy of $\{y^l\}$ with respect to $\mathcal{B}$ is*

$$D_L(B,\{y^l\}) = \sup_{B\in\mathcal{B}} \left| \frac{\#\{y^l \in B\}}{L} - \lambda(B) \right|, \tag{4.2}$$

where $\lambda(B)$ is the volume of B and y^l, $l = 1,\dots,L$ are elements of the sequence $\{y^l\}$.

The definition says that we are finding the maximum difference between the fraction of points inside one of the shapes and the volume of the shape. When the set of shapes $\mathcal{B}$ is the set of boxes with a corner at the origin, this is called the star discrepancy $D_L^*(\{y^l\})$.

The Koksma-Hlawka theorem gives an upper bound for the error in QMC methods, see e.g. [KN74], [Nie78], [Nie87]. It is given as the product of the variance of the function that is integrated and the discrepancy.

Theorem 4.1 (Koksma-Hlawka theorem)**.** *Let $\hat{I}$ be the estimator of the integral I over the domain Ω. Then an upper bound for the error is*

$$\left| I - \hat{I} \right| \leq V(f) D_L^*(\{y^l\}), \tag{4.3}$$

where $V(f)$ is the total variation of the function f over Ω in the sense of Hardy and Krause.

See [MC94], [MC95] or [Nie92] for extensive surveys on this subject, and [Woz91] and [MC94] for the alternative approach involving the so called Woźniakowski's identity. The theorem says that QMC methods yields an integration error that is proportional to the discrepancy of the point sequence used. In s dimensions, it is possible to find sequences $\{y^l\}$ such that

$$D_L^*(\{y^l\}) = \mathcal{O}(\frac{(\log L)^{s-1}}{L}). \tag{4.4}$$

The given error bound is thus better than that of the conventional MC method as the number of simulations L grows to infinity, but it is evident that L needs to be very large for reasonable sized s (the dimension of the problem) in order for the benefit to appear. In practice, however, the theoretical bounds for QMC methods are conservative, (see [KW97] for a general survey or [Dah00] for an example involving European basket options). The measure of discrepancy is mainly used as a criteria for constructing good low discrepancy sequences rather than finding error bounds of integration rules.

In conventional MC methods it is common to use some known algorithm like the Polar-Marsaglia or the Marsaglia-Bray to simulate values from the standard normal distribution, (see [Rip87, Ch. 3] for an overview of such methods). These methods are known as rejection methods, which means that some combinations of the uniform distributed variables used in the algorithms are rejected. When using QMC methods however, we need to ensure that we do not reject any of the uniform distributed numbers. This is because we have to maintain the low discrepancy characteristics of the uniform distributed sequence. Standard rejection methods can therefore not be used directly in QMC methods. Smoothed rejection methods developed in [Caf94] and [Caf98] can however be used with QMC methods, but are harder to implement.

A low discrepancy sequence which is rather simple to implement is the Halton sequence. It was first presented in [Hal60]. In this paper we are going to use an extension of the Halton sequence denoted the Halton leaped sequence. It was presented in [KW97], together with good leap values. We have used the leap value 31 for the numerical experiments in this paper. We are not going to dwell on the choice of sequence here, but mention that other types exist; We have the Sobol sequence first introduced in [Sob67], with an improved implementation presented in [AS79]. We also have the Faure sequence [Fau82] and the Van der

Corput sequence [Pag92]. These named sequences have been shown to belong to a generalized family of (t, s)-sequences for which [Nie92] is a comprehensive reference. An other family of sequences are produced by so called lattice methods for which [SJ94] is the definitive reference. The research in this field is rich and extensive, both in comparative studies between sequences (and their extensions) for a variety of dimension and integrands (e.g. in [KW97], [MC94] and [LL00b]), in connection with variance reduction techniques (e.g. [Owe98] and [LL00a]), and in solution of concrete problems in finance (e.g. [Pas97], [PT95], [MC95], [LL98], [BG96] and [PAG96]).

In many finance problems the so called effective dimension d_s for the problem is actually lower than the real dimension s. (See e.g [CMO97] and [CM96] for finance problems, and [SW98] for a general discussion). This property is present both for path dependent option problems and multi-asset options. The problem of pricing Asian basket options has a mix of both, and some of the challenge is to pinpoint the effective dimensions of the problem. The concept of effective dimension is closely linked to the so called ANOVA decomposition. (See e.g. [Hoe48], [ES81] or [Owe98, Owe99]). It is used to find a representation of the integrand as a sum $\mathcal{F}_{\mathcal{A}}$ of orthogonal functions. If each of these orthogonal functions depends only on a distinct subset of the coordinates, the integrand can be written as a sum of integrals of functions of lower dimension, and the complexity of the problem has been reduced with regards to the integral dimension. Even if we are not able to reduce the dimension of the original integrand by this approach, we can find that some of the orthogonal functions in $\mathcal{F}_{\mathcal{A}}$, say $\mathcal{F}_{\mathcal{A}^C}$, have little effect on the value of the integral. Then if $\mathcal{F}_{\mathcal{A}} - \mathcal{F}_{\mathcal{A}^C}$ have dimension d_s, and d_s is lower than the dimension of original integral, but estimates the true value within acceptable limits (ε), we say that the original problem has effective dimension d_s. In finance problems we can often achieve a representation involving matrices describing the connection between the different variables linearly as arguments to the exponential function, i.e $f(x) = \exp\left(\sum_i c_i x_i\right),\ c_i \leq c_{i+1},\ \forall\ i < s$. This is the case in the problem we are studying, and we achieve this by using the SV-decomposition. If we truncate the sum $\sum_i c_i x_i$ at some point d, where $c_d < \hat{\varepsilon} \ll c_0$ we will get a good approximation of the original problem by evaluating the integral over this lower dimensional integrand. The effective dimension found by the SV-decomposition approach and the effective dimension from the ANOVA approach are compatible, since we can write the exponential function as a sum of polynomials through a series expansion. This means that QMC-methods are well suited for integrals of functions with low effective dimension. Especially if we can find the dimensions having effect, and are able to employ a low discrepancy sequence $\{y^l\}$ for which we know the elements y^l_i having the lowest discrepancy. A numerical test to find the effective dimension of the single asset Asian option problem is performed in subsection 9.1.

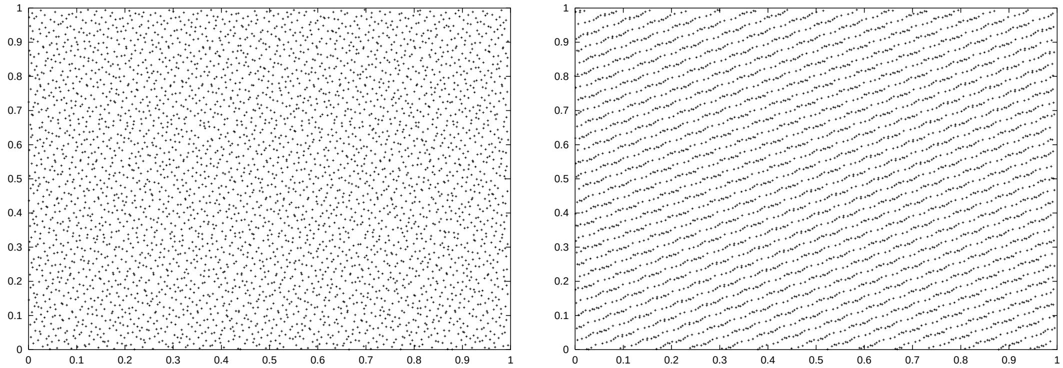

Figure 2. The initial elements have better abilities to fill the domain than the one further back in the low discrepancy vector. Projections onto $[0,1]^2$ of Halton leaped. $(0,1$ and $42,43)$.

5. The value of the claim

An Asian option is actually a special type of European contingent claim, which is defined as a cumulative income process. Without going into details (which can be found in [KS98, Ch. 2.4]), we state that the value at time t of an Asian option is

$$V(t) = e^{-r(T-t)}\mathrm{E}_0[\varphi(\Upsilon(\mathcal{T}))|\mathcal{F}(t)], \tag{5.1}$$

where $\varphi(\cdot)$ is a Borel measurable function. This function can for example be given by

$$\varphi(\Upsilon(\mathcal{T})) = (\Upsilon(\mathcal{T}) - q)^+, \tag{5.2}$$

resulting in the European-style Asian option. A variety of different option contracts fits into this framework by choosing different functions $\Upsilon(\mathcal{T}) : \mathbb{R}^{K+1} \to \mathbb{R}$, $\varphi(\cdot) \in \mathbb{R}$ and $\mathcal{T} \in \mathbb{R}^{K+1}$. All of them, however, are European contingent claims.

The theoretical definition of the Asian option is

$$\text{(For single asset option)} \quad \Upsilon_1(t_0, T) = \int_{t_0}^{T} S(u)\mu(du) \tag{5.3}$$

$$\text{(For basket option)} \quad \Upsilon_N(t_0, T) = \int_{t_0}^{T} \sum_{n=1}^{N} S_n(u)\mu(du), \tag{5.4}$$

for some Borel measure μ on $[t_0, T]$. Our formulation is rather general, but the measure μ is usually given by $\mu(du) = (T - t_0)^{-1}du$. Other candidates can, however, easily be handled by our setup. If we for example choose $\mu(du) = \delta_T(du)$, where δ_T is the Dirac point mass at T, we get a European call option. Other examples are given in, e.g. [RS95]. Note also that contracts often are specified with $t_0 = 0$, but in our discussion we only need $t_0 < T$.

In real applications the integrals in the formulations of $\Upsilon(t_0, T)$ and $\Upsilon_B(t_0, T)$ must be approximated, and often these approximations are specified in the contracts by specifying the number of sampling points along the path. For this purpose let $\mu(du) = (T - t_0)^{-1} du$ and $\mathcal{T} = (t_0, t_1, \ldots, t_K)$, $t_K = T$, and specify the number $K + 1$ of sampling points. The length of the intervals $t_k - t_{k-1}$ need not be equal, but we shall assume this here for simplicity. Approximations of (5.3) and (5.4) can then be carried out by using the expressions

$$\text{(For single asset option)} \quad \hat{\Upsilon}_1(\mathcal{T}) = \frac{1}{K+1} \sum_{k=0}^{K} S(t_k) \tag{5.5}$$

$$\text{(For basket option)} \quad \hat{\Upsilon}_N(\mathcal{T}) = \frac{1}{K+1} \sum_{k=0}^{K} \sum_{n=1}^{N} S_n(t_k). \tag{5.6}$$

Note that by choosing $N = 1$ in (5.6), the basket option is a single asset option. These types of approximations are also necessary in order to apply the MC and QMC framework. We will briefly discuss the convergence of (5.3) to (5.5) and (5.4) to (5.6) in section 9. For simplicity we prefer to consider $V(t)$ at $t = 0$, which is the value at the time the option is bought. Note that this does *not* imply $t_0 = 0$. The expression we are going to use throughout the rest of the paper for the value of the Asian basket option thus becomes

$$V_N(0) = e^{-rT} \mathrm{E}_0[\varphi(\hat{\Upsilon}_N(\mathcal{T}))], \tag{5.7}$$

In section 6 we will show the *conventional* way of how the price processes for the risky assets can be expressed as models where the the noise is implemented as independent stochastic processes. This independence formulation enables us to express the expected value in expression (5.7) as an integral over $\mathbb{R}^{N(K+1)}$, see for instance [Øks98, Ch. 2.1]. Furthermore we will show that in the particular case of integrals involving distribution functions, we can convert the integral from $\mathbb{R}^{N(K+1)}$ to $[0,1]^{N(K+1)}$.

6. The conventional way of formulating E_0 as an integral

In the standard formulation of the asset price processes given by (2.3) and (2.4), the Brownian motion used as the driving noise has a built-in correlation structure. In order to formulate the Asian basket option pricing problem as an integral we have to model the price processes in terms of independent stochastic variables. This can be done in several ways, resulting in algorithms with different properties when used together with QMC methods. In this section we outline the *conventional* way of doing this. This approach exploits that the increments $\Delta W_0(t_k)$

are independent, and results in the following expressions:

$$\sum_{k=0}^{K} S_n(t_k) = \sum_{k=0}^{K} h_n(t_k, S(0), \sigma W_0(t_k)) \tag{6.1}$$

$$= S_n(0) h_n(t_0, 1, \bar\epsilon_0)\big(1 + h_n(\Delta t_1, 1, \bar\epsilon_1)\big(1 + h_n(\Delta t_2, 1, \bar\epsilon_2)\big(1 + \cdots\big)\big)\big) \tag{6.2}$$

$$= S_n(t_0) \delta^n_{K,1}(\bar\epsilon_1, \ldots, \bar\epsilon_K), \tag{6.3}$$

where $\Delta t_k = t_k - t_{k-1},\ \bar\epsilon_k \sim \mathcal{N}_N(0, \Delta t_k \sigma^2),\ 0 < k \le K$ are independent of $\mathcal{F}_{t_{k-1}}$, while $\bar\epsilon_0 \sim \mathcal{N}_N(0, t_0\sigma^2)$. The notation including the δ function is achieved by letting

$$\delta^n_{K,K}(\bar\epsilon_K) = 1 + h_n(\Delta t_K, 1, \bar\epsilon_K) \tag{6.4}$$

$$\delta^n_{K,k}(\bar\epsilon_k, \ldots, \bar\epsilon_K) = 1 + h_n(\Delta t_k, 1, \bar\epsilon_k)\delta^n_{K,k+1}(\bar\epsilon_{k+1}, \ldots, \bar\epsilon_K),\ k = K-1, \ldots, 1. \tag{6.5}$$

Note furthermore that the most common method of finding σ is by use of the Cholesky decomposition of σ^2, even if other types like the SV-decomposition exists, and are more suited in finance problems. In section 9, where we compare different approaches, we have used the Cholesky decomposition when calculating values of the Asian basket option by the *conventional* approach, while we have used the SV-decomposition in the SVD approach described in section 8.

With the expression (6.3) we can write the value of the Asian basket option as

$$V_N(0) = e^{-rT} \mathrm{E}_0[\varphi(\hat\Upsilon_N(\mathcal{T}))]$$

$$= e^{-rT} \int_{\mathbb{R}^{N(K+1)}} \varphi\Big(\frac{1}{K+1} \sum_{n=1}^{N} S_n(t_0)\delta^n_{K,1}(x)\Big)\psi(x)dx \tag{6.6}$$

$$= e^{-rT} \int_{[0,1]^{N(K+1)}} \varphi\Big(\frac{1}{K+1} \sum_{n=1}^{N} S_n(t_0)\delta^n_{K,1}(\Psi^{-1}(y))\Big)dy, \tag{6.7}$$

where $\psi : \mathbb{R}^{N(K+1)} \to \mathbb{R}^{N(K+1)}$ is the density of an $N(K+1)$ - dimensional centered Gaussian random variable with covariance matrix equal to the identity, and $\Psi^{-1} : [0,1]^{N(K+1)} \to \mathbb{R}^{N(K+1)}$ is a vector of inverse cumulative normal distribution functions with mean 0 and variance 1: $\Psi^{-1}(y) = (\Psi_1^{-1}(y_1), \ldots, \Psi^{-1}_{N(K+1)}(y_{N(K+1)}))$. In the following it is convenient to introduce the notation $\hat\varphi(Y) \equiv \varphi(\frac{Y}{K+1})$ in order to simplify the expressions. The transformation of the integral over $\mathbb{R}^{N(K+1)}$ in (6.6) to $[0,1]^{N(K+1)}$ in (6.7) is due to the mapping performed by the function $\Psi^{-1}(\cdot)$, and is valid for any inverse cumulative distribution function. In order to find the inverse of $\Psi(\cdot)$ we use that $\Psi_n^{-1}(y_n) = \mathrm{erf}^{-1}(2y_n - 1)$. We do the evaluation by a rational approximation suggested in [Mor98]. Other types of methods for calculating $\Psi^{-1}(\cdot)$ could be employed, but caution must be taken when used together with QMC. (See comments in section 4 on this.) It is evident at this stage that we can approximate the option price by making use of a convenient set of points $\{y\} \subseteq [0,1]^{N(K+1)}$. In the *conventional* QMC approach this is done by using samples y^l from the Halton leaped sequence as input to $\Psi^{-1}(\cdot)$, and calculating the mean value of the integrand in the domain of integration. The value for an Asian basket option for this approach is thus approximated by

$$V_N(0) \approx \frac{e^{-rT}}{L} \sum_{l=1}^{L} \hat{\varphi}\Big(\sum_{n=1}^{N} S_n(t_0)\delta_{K,1}^n(\Psi^{-1}(y^l))\Big) \tag{6.8}$$

where $y^l \in [0,1]^{N(K+1)}$ is the l'th vector in the low discrepancy sequence. As L is increased in (6.8) we get better approximations. The approach in this section, however, does not take into account any special structures of the integrand. This must be done in order to exploit that some of the elements of the low discrepancy vector in the QMC method are more evenly distributed than others. In section 4 we have discussed methods of reengineering the integrand to reduce the complexity and to improve convergence rates. The next sections are devoted to the specification of two types of decompositions, the Brownian Bridge approach and the SVD approach.

7. The Brownian Bridge approach

The Brownian Bridge approach is presented for the pricing of Asian options on a single underlying asset only. This is because the coupling between the asset dependency and the time dependency in the basket case is hard to define for the Brownian Bridge approach. We will show in the next section that this coupling can be handled rather easily in the SVD approach. The inclusion of the Brownian Bridge approach is solely for reason of comparison. In section 9 we present results showing that the SVD approach turns out to be better among the two.

When evaluating path dependent options we have to simulate the path – one way or the other. In the formulation leading to the conventional approach, involving the $\delta(\cdot)$ function, this is done by using the independent increment property of Brownian motion. We can, however, achieve a representation of the path by using another approach – the so called Brownian Bridge approach:

(1) Before entering the simulation loop: Choose T, and the number of equal time steps $K = 2^p$. Set $\Delta t = \frac{T}{K}$ and $t_k = k\Delta t,\ k = 1, \ldots, K$.
(2) Inside the simulation loop: Generate Gaussian independent variables ϵ_j^l, for each of the L turns in the simulation loop, distributed according to

$$\epsilon_j^l = \sqrt{\hat{t}_j}\Psi^{-1}(y_j^l) \sim \mathcal{N}(0, \hat{t}_j),\ j = 0, \ldots, K-1 \tag{7.1}$$

$$\text{where } \hat{t}_0 = T \text{ and } \hat{t}_j = \frac{T}{2^{2+\lfloor \log_2 j \rfloor}},\ j = 1, \ldots, K-1 \tag{7.2}$$

(3) The Wiener path $w_l(t)$ is sampled at each t_k as

$$w_l(t_0) = 0 \tag{7.3}$$

$$w_l(t_K) = \sigma\epsilon_0^l \qquad \big(\sim \mathcal{N}(0, \sigma^2 T)\big) \tag{7.4}$$

$$w_l(t_{K/2}) = \frac{1}{2}(w_l(t_0) + w_l(t_K)) + \sigma\epsilon_1^l \qquad \big(\sim \mathcal{N}(0, \sigma^2\frac{T}{2})\big) \tag{7.5}$$

$$w_l(t_{K/4}) = \frac{1}{2}(w_l(t_0) + w_l(t_{K/2})) + \sigma\epsilon_2^l \qquad \big(\sim \mathcal{N}(0, \sigma^2\frac{T}{4})\big) \tag{7.6}$$

$$w_l(t_{3K/4}) = \frac{1}{2}(w_l(t_{K/2}) + w_l(t_K)) + \sigma\epsilon_3^l \qquad \big(\sim \mathcal{N}(0, \sigma^2\frac{T}{4})\big) \tag{7.7}$$

$$\vdots$$

(4) The price path is then calculated by using

$$S^l(t_k) = S(t_0)\exp\left((r - \frac{1}{2}\sigma^2)t_k + w_l(t_k)\right) \tag{7.8}$$

(5) The price of the option is found by averaging:

$$V_1(0) \approx \frac{\mathrm{e}^{-rT}}{L}\sum_{l=1}^{L}\hat{\varphi}\Big(\sum_{k=0}^{K} S^l(t_k)\Big) \tag{7.9}$$

Although the total variance in this representation is the same as in the standard discretization, much more of the variance is contained in the first few steps of the Brownian Bridge formula. This reduces the effective dimension of the simulation and increases the effect of the low discrepancy sequence used. It turns out that this decomposition is not optimal, and the optimal decomposition is given by the SVD method presented next.

8. The SVD approach

A random variable $Y \sim \mathcal{N}_N(0, \Sigma\Sigma^T)$ can be written $Y = \Sigma X$ where $X \sim \mathcal{N}_N(0, I)$, and I is the $N \times N$ identity matrix. In section 3 we discussed properties of multidimensional Brownian motion, and concluded in expression (3.2) with the covariance matrix of the process. Given a covariance matrix $\Sigma^2 = \Sigma\Sigma^T$ there are several alternatives of finding the matrix Σ. The Cholesky decomposition produces a Σ matrix which is triangular, while the Σ matrix from the SV-decomposition can be written as $E\sqrt{\Lambda}$, where E contains the eigenvectors of Σ^2 and Λ is a diagonal matrix with the corresponding eigenvalues in decreasing order on the diagonal. We will use the SV-decomposition *both* for the Wiener path along the time dimension *and* to find a volatility matrix σ used in the modeling of the price process of the underlying assets. The properties of Brownian motion enables us to perform two separate SV-decompositions instead of one large: One for the covariance matrix $C^2 \in \mathbb{R}^{(K+1)\times(K+1)}$ given in section 3, describing the path-dependencies, and one for the covariance matrix $\sigma^2 \in \mathbb{R}^{N\times N}$ for the underlying assets. The eigenvalues of C^2 and σ^2 can then be combined by the direct matrix product into an ordering $O : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ of the total set of eigenvalues for the full problem to give us $\lambda_{O(\cdot,\cdot)}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N(K+1)}$. This method enables us to allocate specific elements y_i^l from the low discrepancy vector y^l to the different orthogonal noise generators (represented by the eigenvalues λ) of the full problem. If y^l is a vector from the Halton leaped sequence, the noise term with the biggest eigenvalue is mapped to y_1^l the next biggest to y_2^l and so forth. In order for this approach to be effective, the discrepancy of y_1^l should be lower than the discrepancy of y_2^l etc. This is a property of many low discrepancy sequences, and the Halton leaped sequence seems to have this characteristic. (See e.g. [KW97]).

The principles of the SVD method for the basket option problem given as a list of tasks are as follows:

(1) Before entering the simulation loop: Find $\sigma \in \mathbb{R}^{N\times N}$ by performing an SV-decomposition of the covariance matrix σ^2, and $C \in \mathbb{R}^{(K+1)\times(K+1)}$ by an SV-decomposition of the covariance matrix C^2. Find a relation $O(\cdot,\cdot)$ between the time discretization point k, the asset n and the ordering of the eigenvalues λ by sorting the output from the direct matrix product in reverse order:

$$\lambda = \lambda_{\sigma^2} \otimes \lambda_{C^2} \tag{8.1}$$

(2) Inside the simulation loop: Create a low discrepancy vector $y^l \in [0,1]^{N(K+1)}$ for each of the L turns in the simulation loop.
(3) Find the corresponding inverse cumulative normal values

$$\epsilon^l_{O(n,k)} = \Psi^{-1}(y^l_{n,k}) \; n = 1,\dots,N, \quad k = 0,\dots,K. \tag{8.2}$$

(4) Find the asset price for each of the N assets in each of the $K+1$ points along the time line. This is done by

$$S_n^l(t_k) = S_n(t_0)\exp\big((r - \frac{1}{2}\sum_{d=1}^{N}\sigma_{nd}^2)t_k + \sum_{d=1}^{N}\sigma_{nd}\sum_{j=0}^{K}C_{kj}\epsilon^l_{O(d,j)}\big), \tag{8.3}$$

(5) Find the average of all the asset prices computed in (8.3) and evaluate $\hat{\varphi}(\hat{\Upsilon}_N)$.
(6) The option price is approximated by performing the described loop L times, averaging the L results of $\hat{\varphi}(\cdot)$, and discounting by e^{-rT}.

The full expression for the approximate value of the Asian basket option by the SVD approach is therefore given by

$$V_N(0) \approx \frac{e^{-rT}}{L} \sum_{l=1}^{L} \hat{\varphi}\Big(\sum_{n=1}^{N} \sum_{k=0}^{K} S_n(t_0) \\ \exp\big((r - \frac{1}{2} \sum_{d=1}^{N} \sigma_{nd}^2) t_k + \sum_{d=1}^{N} \sigma_{nd} \sum_{j=0}^{K} C_{kj} \epsilon^l_{O(d,j)} \big) \Big) \tag{8.4}$$

where σ_{nd} and C_{kj} are elements of the matrices resulting from the SV-decompositions. The matrices σ and C together with the function $O(\cdot, \cdot)$ are the essential parts of this approach.

9. Numerical results

We present the numerical results from simulations of prices of both single asset Asian call options and Asian basket call options. The simulations for single asset options is performed to show the difference between the convergence of the conventional recursive approach, the Brownian Bridge and the SVD method, while the basket option simulations only compare the conventional recursive approach with the SVD approach. We will also look briefly into the convergence of the sum in (5.5) to the value of the integral (5.3) as the number of evaluation points K along the path increases. For the numerical calculations we specify $\varphi(\cdot)$ to be the payoff function of a call option. This specification also effects $\hat{\varphi}(\cdot)$, giving:

$$\varphi(Y) \equiv \big(Y - q\big)^+ \tag{9.1}$$

$$\hat{\varphi}(Y) \equiv \big(\frac{Y}{K+1} - q\big)^+ = \frac{1}{K+1}\big(Y - \hat{q}\big)^+, \tag{9.2}$$

where $\hat{q} \equiv q(K+1)$. In addition we let $t_0 = 0$ in the numerical examples.

9.1. Convergence of $\hat{\Upsilon}$.

The integrand in the expression for $\hat{\Upsilon}_1(T)$ must be approximated when calculating the value of the theoretical expression for the Asian option, and therefore it is interesting to investigate how fast the convergence of the sum in (5.5) to the value of the integral (5.3) is achieved. We do this numerically by looking at the expressions:

$$e^{-rT} E_0[\varphi\big(\int_0^T S(u) T^{-1} du\big)] = \lim_{K\to\infty} e^{-rT} E_0[\hat{\varphi}\big(\sum_{k=0}^{K} S(t_k)\big)] \tag{9.3}$$

$$= \lim_{\substack{K\to\infty \\ L\to\infty}} \frac{e^{-rT}}{L} \sum_{l=1}^{L} \hat{\varphi}\big(\sum_{k=0}^{K} S^l(t_k)\big). \tag{9.4}$$

When performing the calculation we use the SVD method since this have the lowest variance, i.e we can keep L smaller than for the other methods. Figure 3 gives the level of accuracy for a given K compared to the true value of the integral. We see that the convergence is rather fast, and that for $K = 10$, we are well within 0.1% of the value of the integral. This indicates that an effective dimension of $d_s = 10$ for this problem is a conservative estimate. Notice that we do not know the real value of the integral, and therefore these convergence results are purely indicative.

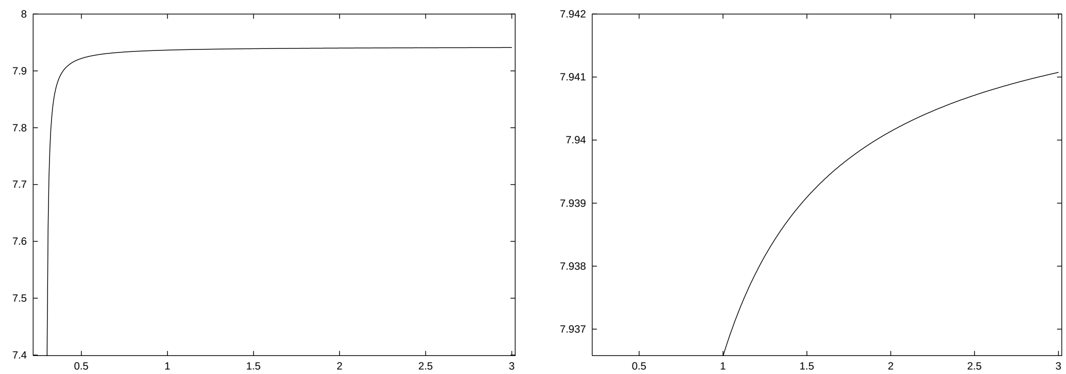

FIGURE 3. The convergence of the option price as $K \to \infty$

9.2. **Single asset Asian options.** We will simulate prices for options where time to maturity is one year $T = 1.0$, initial price $S(0) = 100$ and strike $q = 100$. Furthermore the risk free rate in the market is $r = 0.05$, the volatility is constant $\sigma = 0.3$, and the assets pay no dividends. With this setup we calculate the price for $K = 2^p,\ p \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, by using L number of simulations in the range $L \in [10^3, 10^5]$. The standard MC method (STA), the conventional QMC method by the use of the recursive $\delta(\cdot)$ function (REC), the Brownian Bridge method (BB) and the SVD method (SVD) are compared by use of a set of graphs. The variance of the resulting series are also given as a measure of convergence speed. Notice that each new point in each of the graphs are calculated by using non-overlapping sequences of low discrepancy vectors. The results is shown as graphs where the price of the option is on the Y-axis and the X-axis show the number of simulations on $\log_{10}$ scale. See Figure 4 and Figure 5.

It is important to quantify the performance of the different approaches, and we have done this by simply calculating the variances of the graphs. Although the prices have been calculated by a deterministic approach, and we therefore can not truly trust statistical measures on the behavior, we believe that the used measure will give some insights. The result of this measurement for an Asian option on a single underlying asset is given for $K = 2^p,\ p = 1, \ldots, 9$ in Table 1. We have done two sets of simulations, one for $L \in [10^3, 10^4]$ and one for $L \in [10^4, 10^5]$.

When we plot these results (for $p \leq 9$) we get the rather illustrative picture in Figure 6, showing that the SVD method has close to constant variance as the

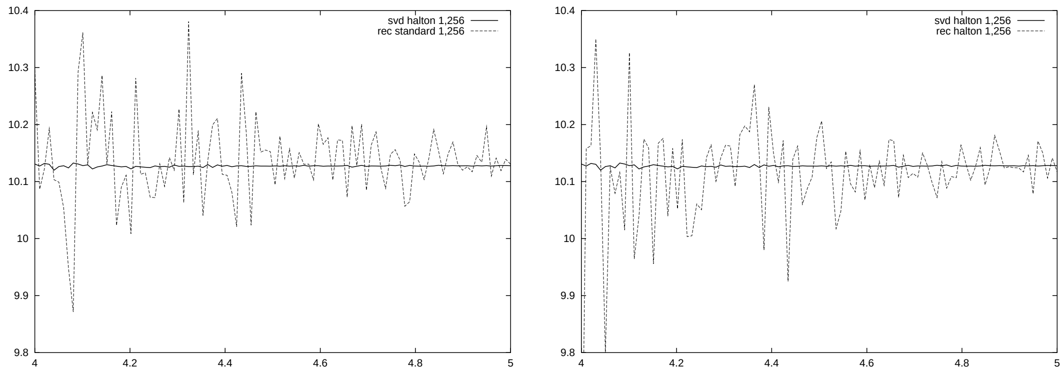

FIGURE 4. Comparing *SVD* method with *STA* and *REC*. For these simulations, $N = 1$ and $K = 256$, and *REC* does not perform better than *STA* for dimensions this high.

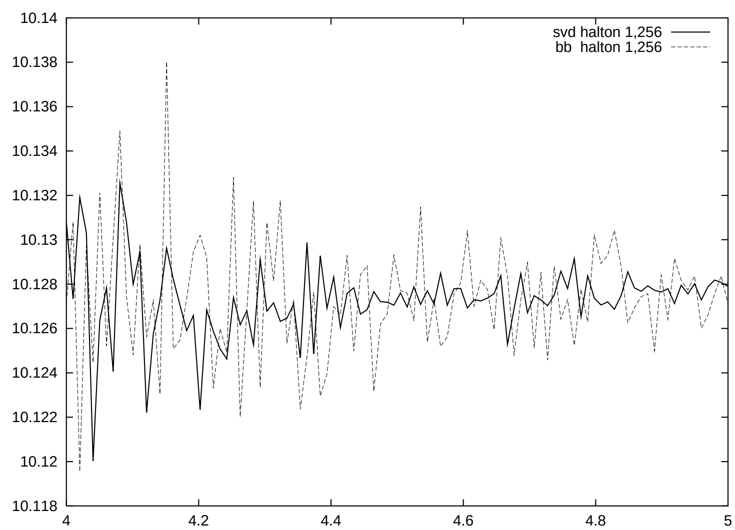

FIGURE 5. Comparing *SVD* with *BB*. We see that *SVD* is slightly better than *BB*. $N = 1$ and $K = 256$.

number of dimensions 2^p are increased, while the variance of the conventional method increases linearly (note that the scale on both axis are logarithmic). The standard (non QMC-approach) has constant high variance, and the Brownian Bridge approach increases a bit in the start but stabilizes on a lower level than the SVD method.

9.3. **Basket Asian options.** When we calculate the value of the basket option, an additional element concerning the N assets in the basket comes into consideration. In section 3 we described how to find the eigenvalues of the full system,

L	p	STANDARD	REC	BB	SVD
$10^3 \to 10^4$	1	0.0750028	0.000164428	0.000234812	0.000349644
$10^3 \to 10^4$	2	0.0880283	0.000484648	0.000313312	0.000217205
$10^3 \to 10^4$	3	0.0943216	0.00216916	0.00030832	0.000221485
$10^3 \to 10^4$	4	0.074528	0.00749465	0.000311825	0.000220451
$10^3 \to 10^4$	5	0.0789701	0.021403	0.000303524	0.000221139
$10^3 \to 10^4$	6	0.0750914	0.01854	0.000307054	0.000221403
$10^3 \to 10^4$	7	0.0805887	0.0351491	0.000305817	0.000221582
$10^3 \to 10^4$	8	0.0490915	0.258421	0.000309338	0.00022173
$10^3 \to 10^4$	9	0.053857	0.407136	0.00031079	0.000221813

L	p	STANDARD	REC	BB	SVD
$10^4 \to 10^5$	1	0.00611862	2.32071e-006	2.16537e-006	3.09931e-006
$10^4 \to 10^5$	2	0.00542252	6.30376e-006	3.75607e-006	2.78211e-006
$10^4 \to 10^5$	3	0.00637498	3.73341e-005	5.16935e-006	2.79055e-006
$10^4 \to 10^5$	4	0.00827774	0.000298189	6.74235e-006	2.77861e-006
$10^4 \to 10^5$	5	0.00709202	0.000459984	7.33869e-006	2.78717e-006
$10^4 \to 10^5$	6	0.00766036	0.000888891	7.40184e-006	2.80059e-006
$10^4 \to 10^5$	7	0.00605674	0.00134237	7.23032e-006	2.8036e-006
$10^4 \to 10^5$	8	0.00528101	0.00532122	7.38271e-006	2.80411e-006
$10^4 \to 10^5$	9	0.0059849	0.0312871	7.45808e-006	2.80269e-006

TABLE 1. The variances are calculated for each series of 100 prices calculated by simulating for L in the range $[10^3, 10^4]$ and $[10^4, 10^5]$. The mapping onto these ranges are logarithmic, i.e there are fewer samples from the end of the interval than the beginning. $N = 1$ and $K = 2^p$.

and in section 8 we showed how to utilize this to optimize the use of the low discrepancy sequence. In this section we will use a setup of the simulation similar to the one used for the single asset option, but in addition we will let the number of assets vary: $N \in \{2, 4, 8, 16, 32, 64\}$.

The results are given in Table 2 and illustrated in Figure 8. In the illustrations we have kept the number of sampling points K constant and increased the number of assets N in the basket. The different methods are labeled *REC* for the conventional QMC method, *SVD1* for the full SVD method including an ordering of the total noise in the problem by the use of the $O(\cdot,\cdot)$ function, and *SVD2* for an SVD method where we have decomposed both time and asset dimensions, but not combined them into an overall ordering. In *SVD2* the N first elements of the low discrepancy vector are used for the noise in the problem stemming from the time discretization point giving the biggest contribution, and these N elements are used in an ordering according to the contributions from the different assets. This should theoretically give the *SVD1*-method best performance,

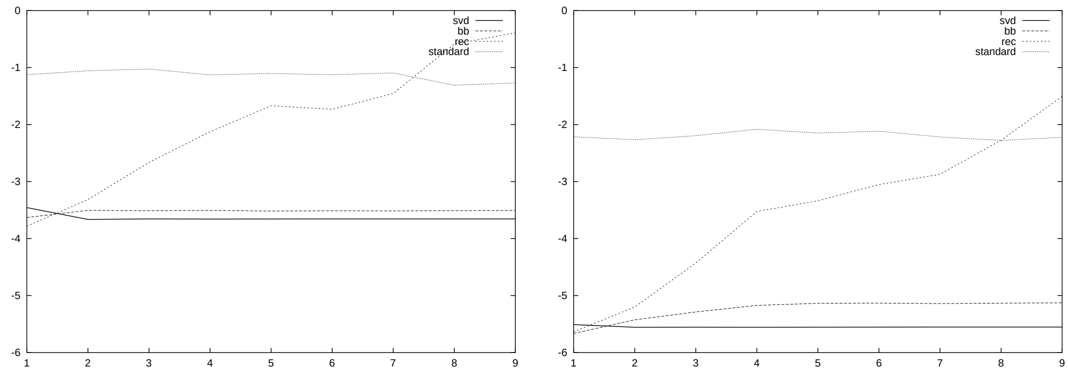

FIGURE 6. The plots have p on the X-axis, and $\log_{10}$ of the variance on the Y-axis. There are 9 estimated variance values for each approach, one for each p. Each series of prices contains 100 values, and are created by simulating prices for L in the range $[10^3, 10^4]$ (left) and $[10^4, 10^5]$ (right). The mapping onto these ranges are logarithmic, i.e there are fewer samples from the end of the interval than the beginning. $N = 1$ and $K = 2^p$.

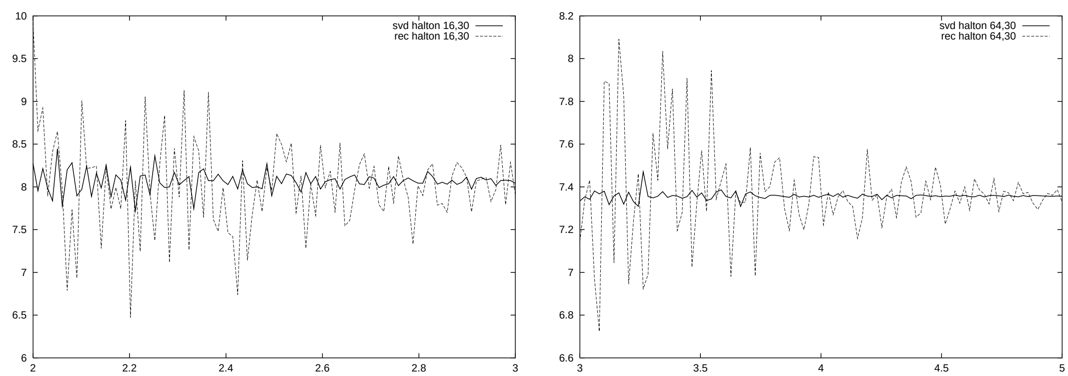

FIGURE 7. Comparison of the methods for different baskets. N and K given in the labels of the plots.

but for the example we have tested, this conclusion can not be drawn. In the numerical studies of the Asian basket option, we have estimated the matrix σ^2 by using asset return time-series from the Oslo Stock Exchange in Norway to get a realistic case.

10. CONCLUSIONS

The use of QMC methods gives faster convergence than conventional MC Methods for both single asset and basket Asian options. By using the low discrepancy

L	i	REC	SVD2	SVD1
$10^3 \to 10^4$	1	0.00727686	0.000156036	0.000141992
$10^3 \to 10^4$	2	0.0108797	0.000231383	0.000234575
$10^3 \to 10^4$	3	0.00988478	0.000276212	0.00019724
$10^3 \to 10^4$	4	0.0240513	0.000312554	0.000224188
$10^3 \to 10^4$	5	0.0374931	0.000709355	0.00073982
$10^3 \to 10^4$	6	0.0486269	0.00028011	0.000391925

TABLE 2. The variance of the option price as the number of assets $N = 2^i$ in the basket is increased. Here $K = 2^5 = 32$.

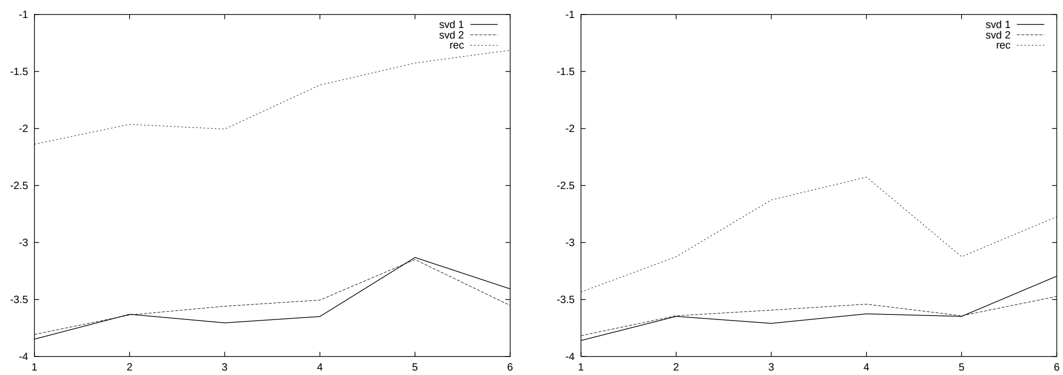

FIGURE 8. The variance of the option price as the number of assets in the basket is increased. $N = 2^i$, i on the X-axis. Time discretization is $K = 2^5 = 32$ (left) and $K = 2^2 = 4$ (right).

sequence more effectively, the examples we have simulated show a large performance gain compared to the conventional QMC method. Furthermore we get better results when using the SVD approach than the Brownian Bridge approach for single asset Asian options. The benefit of the SVD approach increases as the number of sampling points in the time interval increase due to the fact that the conventional QMC method becomes less effective, while the SVD method maintains its efficiency. The conventional QMC method is actually outperformed even by the conventional MC method for very high dimensions ($K \geq 256$), while the QMC method based on SVD remains very good also for high dimensions. The problem clearly has low effective dimension, but while we can estimate this to about 10 for the single asset Asian option, the effective dimension of the basket Asian option will depend on the covariance structure of the assets in the basket, and can therefore vary among different baskets. The numerical tests show that as we increase the number of assets in the basket, the difference between the conventional QMC method and the SVD method is seemingly constant, or slightly increasing as N becomes large.

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