LÉVY PROCESSES AND CONVOLUTION SEMIGROUPS WITH PARAMETER IN A CONE AND THEIR SUBORDINATION

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ABSTRACT. Convolution semigroups and Lévy processes with parameter in a cone K are defined. Compared to ordinary convolution semigroups and Lévy processes (corresponding to $K = \mathbb{R}_+$) the case of a general cone K is more complicated in that there is generally not a one-to-one correspondence between semigroups and Lévy processes. Thus, in particular we have to distinguish subordination of cone-parameter convolution semigroups and of cone-parameter Lévy processes.

Several fundamental properties of cone-parameter convolution semigroups and Lévy processes are derived. In the study the distinction between cones with and without a strong basis is important. Conditions that a cone-parameter convolution semigroup is generative (that is, there is a cone-parameter Lévy process in law associated with it) are derived and examples of non-generative semigroups are given.

1. INTRODUCTION

Usual Lévy processes and convolution semigroups have $\mathbb{R}_+ = [0, \infty)$ as domain of the parameter. The basic correspondences among them are formulated as follows, see Sato [22], [24]. (i) The class of convolution semigroups $\{\mu_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ corresponds to the class of infinitely divisible distributions μ through $\mu = \mu_1$. This is due to the fact that the characteristic function $\widehat{\mu}_t(z)$ of μ_t satisfies $\widehat{\mu}_t(z) = \widehat{\mu}_1(z)^t$. (ii) The class of Lévy processes in law $\{X_t \colon t \geqslant 0\}$ on $\mathbb{R}^d$ corresponds to the class of convolution semigroups on $\mathbb{R}^d$ through $\mu_t = \mathcal{L}(X_t)$, the distribution of X_t. This correspondence is one-to-one if processes with the same law are identified. Here we recall that a Lévy process in law is continuous in probability, but, unlike Lévy processes, the sample functions need not be cadlag. (iii) Every Lévy process in law has a modification which is a Lévy process.

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A natural generalization of $\mathbb{R}_+$ is a cone K in a Euclidean space. In this paper we study K-parameter convolution semigroups, Lévy processes and Lévy processes in law, and investigate whether the correspondences above are generalized. Further, we study the generalization of subordination to the cone-parameter case. Thus, we develop the study of the $\mathbb{R}_+^N$-parameter case initiated by Barndorff-Nielsen, Pedersen and Sato [1].

The cone K induces a partial order from which the notions of K-increasingness and K-decreasingness are defined. In the definition of a K-parameter Lévy process $\{X_s\colon s \in K\}$ we require independence of $X_{s^2} - X_{s^1}, \ldots, X_{s^n} - X_{s^{n-1}}$ for every K-increasing sequence $\{s^1, \ldots, s^n\}$ and increment stationarity in the sense that $\mathcal{L}(X_{s^2} - X_{s^1}) = \mathcal{L}(X_{s^4} - X_{s^3})$ for $s^2 - s^1 = s^4 - s^3 \in K$ together with the condition $X_0 = 0$ a. s. The cadlag property of sample functions of a Lévy process is now replaced by the K-cadlag property, the meaning of which will be made precise in Section 2. As in the usual case, we introduce the notion of a K-parameter Lévy process in law, dropping the requirement of the K-cadlag property, but retaining the continuity in probability. A K-parameter convolution semigroup is defined to be a class of probability measures $\{\mu_s\colon s \in K\}$ having the property that $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$ for $s^1, s^2 \in K$ and satisfying the continuity condition that $\mu_{ts} \to \delta_0$ as $t \downarrow 0$ for every $s \in K$.

Any K-parameter Lévy process in law $\{X_s\colon s \in K\}$ induces a K-parameter convolution semigroup $\{\mu_s\colon s \in K\}$ by $\mu_s = \mathcal{L}(X_s)$. We study questions on the converse. Given a K-parameter convolution semigroup, can we find a K-parameter Lévy process in law that induces the semigroup ? Is it unique in law when we can find one ? The situation is radically different according as the cone K has a strong basis or not. We say that $\{e^1, \ldots, e^N\}$ is a strong basis of an N-dimensional cone K if $e^1, \ldots, e^N$ are linearly independent vectors belonging to K and if every $s \in K$ is expressed as $s = s_1 e^1 + \cdots + s_N e^N$ with nonnegative $s_1, \ldots, s_N$. We say that $\{e^1, \ldots, e^N\}$ is a weak basis of K if the former condition is satisfied. When K has a strong basis $\{e^1, \ldots, e^N\}$, we have two important examples of K-parameter Lévy processes. Let $\{V_t^1\}, \ldots, \{V_t^N\}$ be independent $\mathbb{R}_+$-parameter Lévy processes, where $\{V_t^j\}$ is $\mathbb{R}^{d_j}$-valued. One example is

$$X_s = (V_{s_1}^1, \ldots, V_{s_N}^N)^\top \text{ for } s = s_1 e^1 + \cdots + s_N e^N. \tag{1.1}$$

Another is

$$X_s = V_{s_1}^1 + \cdots + V_{s_N}^N \text{ for } s = s_1 e^1 + \cdots + s_N e^N, \tag{1.2}$$

assuming that $d_1 = \cdots = d_N$.

Our main results in Sections 3 and 4 are as follows. Let $\{e^1, \ldots, e^N\}$ be a weak basis of K.

1. If $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$, then μ_s is determined by $\mu_{e^1}, \ldots, \mu_{e^N}$ as $\widehat{\mu}_s(z) = \widehat{\mu}_{e^1}(z)^{s_1} \ldots \widehat{\mu}_{e^N}(z)^{s_N}$ for $s = s_1 e^1 + \cdots + s_N e^N \in K$, where $s_1, \ldots, s_N$ are not necessarily nonnegative.

2. We say that a set of infinitely divisible distributions $\{\rho_1, \ldots, \rho_N\}$ is *admissible* with respect to $\{e^1, \ldots, e^N\}$ if there is a K-parameter convolution semigroup $\{\mu_s\}$ such that $\mu_{e^j} = \rho_j$ for $j = 1, \ldots, N$. If $\{e^1, \ldots, e^N\}$ is a strong basis, then $\{\rho_1, \ldots, \rho_N\}$ is always admissible. If $\{e^1, \ldots, e^N\}$ is not a strong basis, then there exists a set $\{\rho_1, \ldots, \rho_N\}$ which is not admissible, and a necessary and sufficient condition for admissibility is given.

3. Given a K-parameter convolution semigroup $\{\mu_s\}$, we say that it is *generative* if there is a K-parameter Lévy process in law $\{X_s\}$ such that $\mathcal{L}(X_s) = \mu_s$. We say that $\{\mu_s\}$ is *unique-generative* or *multiple-generative* according as such a Lévy process in law is unique in law or not. A remarkable difference from the usual $\mathbb{R}_+$-parameter case is the existence of the non-generative case. For the cone $M^+_{d\times d}$ of $d \times d$ nonnegative-definite symmetric matrices we introduce a natural convolution semigroup $\{\mu_s \colon s \in M^+_{d\times d}\}$ by $\mu_s = N_d(0, s)$, the Gaussian distribution on $\mathbb{R}^d$ with mean 0 and covariance matrix s. We show that $\{\mu_s\}$ is non-generative. The fact that generative semigroups are not always unique-generative is essentially recognized in [1].

4. If K has a strong basis, then every K-parameter convolution semigroup on $\mathbb{R}^d$ is generative; in fact, a K-parameter Lévy process of the form (1.2) is associated. Without the assumption of the existence of a strong basis, any K-parameter purely non-Gaussian convolution semigroup on $\mathbb{R}^d$ is generative, and any K-parameter convolution semigroup on $\mathbb{R}$ (that is, $d = 1$) is generative.

5. When $\{e^1, \ldots, e^N\}$ is a strong basis, a sufficient condition and a necessary condition for unique-generativeness of $\{\mu_s \colon s \in K\}$ are formulated in terms of the supports of μ_{e^j}, $j = 1, \ldots, N$. For example, if $\{\mu_s\}$ is the semigroup induced by a K-parameter Lévy process of the form (1.1), then it is unique-generative. In the Gaussian case, a necessary and sufficient condition is given.

6. If K has a strong basis and if $\{X_s\}$ is a K-parameter Lévy process in law associated with a unique-generative K-parameter convolution semigroup, then $\{X_s\}$ has a modification which is a K-parameter Lévy process.

Unlike the usual $\mathbb{R}_+$-parameter case there is generally not a one-to-one correspondence between subordination of K-parameter convolution semigroups and that of K-parameter Lévy processes. This is due to the existence of multiple-generative and non-generative semigroups. Therefore, in Section 5 we formulate both subordination of K-parameter convolution semigroups and of K-parameter Lévy processes, and study the change of generating triplets under these transformations. Further, we study preservation of selfdecomposability, the L_m property and stability under these operations. This constitutes a partial extension of the results of [1]. As an application of subordination of cone-parameter convolution semigroups, we give a characterization of multivariate type G distributions introduced by Barndorff-Nielsen and Pérez-Abreu [2].

As to works earlier than [1], we mention that Bochner, [4] pp. 106–108, made a heuristic discussion of cone-parameter convolution semigroups, and that there exist several studies of $\mathbb{R}_+^N$-parameter Lévy processes of the form (1.1) or (1.2). Dynkin [8], Evans [10], Fitzsimmons and Salisbury [11] worked on processes which generalize the process $\{X_s\}$ of (1.1). Hirsch [12] and Khoshnevisan, Xiao and Zhong [14] studied the process (1.2).

There are many papers on multiparameter Brownian motions and Lévy processes. Lévy [16] introduced a multiparameter Brownian motion with parameter in $\mathbb{R}^N$ and the papers of Chentsov [5] and McKean [18] followed. Mori [19] characterized similar processes in the purely non-Gaussian setting. Another process called Brownian sheet (with parameter in $\mathbb{R}_+^N$) was studied by Orey and Pruitt [20], Talagrand [27], Khoshnevisan and Shi [13] and others. The multiparameter stable processes of Ehm [9] and the two-parameter Lévy processes of Vares [28] and Lagaize [15] are generalizations of the Brownian sheet. Lévy's multiparameter Brownian motion restricted to a cone K with dimension $\geqslant 2$ does not satisfy the independence of the increments along K-increasing sequences. The Brownian sheet and the processes in Ehm [9], Vares [28] and Lagaize [15] do not have the stationarity of the increments in the $\mathbb{R}_+^N$-increasing direction.

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2. Preliminaries on cones

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{C}$ be the sets of real numbers, rational numbers, positive integers and complex numbers, respectively. Let $\mathbb{R}_+ = [0, \infty)$. Throughout the paper let N, M and d be positive integers. Elements of $\mathbb{R}^d$ are column vectors. We denote the coordinates of $x \in \mathbb{R}^d$ by x_j, and use either the notation $x = (x_j)_{1 \leqslant j \leqslant d}$ or $x = (x_1, \ldots, x_d)^\top$. The inner product on $\mathbb{R}^d$ is $\langle x, y \rangle$ and the norm is $|x|$. When $d_1, \ldots, d_n$ are positive integers and $x^j \in \mathbb{R}^{d_j}$ for $j = 1, \ldots, n$, then $(x^1, \ldots, x^n)^\top$ denotes the stacked vector

$$(x^1, \ldots, x^n)^\top = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \tag{2.1}$$

which is an element of $\mathbb{R}^{d_1 + \cdots + d_n}$.

Let $ID(\mathbb{R}^d)$ be the class of infinitely divisible distributions on $\mathbb{R}^d$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. For $\mu \in ID(\mathbb{R}^d)$ and $t \geqslant 0$, denote $\mu^t = \mu^{t*}$. The characteristic function of μ is $\widehat{\mu}(z) = \int_{\mathbb{R}^d} e^{\mathrm{i}\langle z, x \rangle} \mu(dx)$, $z \in \mathbb{R}^d$. Let $\mathcal{L}(X)$ be the distribution (law) of a random variable X. By $X \stackrel{\mathrm{d}}{=} Y$ we mean $\mathcal{L}(X) = \mathcal{L}(Y)$. For probability measures μ_n $(n = 1, 2, \ldots)$ and μ on $\mathbb{R}^d$, $\mu_n \to \mu$ means weak convergence of μ_n to μ. For a measure μ on $\mathbb{R}^d$, $\mathrm{Supp}(\mu)$ denotes the support of μ as defined e.g. in [22], p. 148. Let δ_c denote a distribution concentrated at a point c. Such a distribution is called trivial. For $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

We use the word cone in the following sense.

Definition 2.1. A subset K of $\mathbb{R}^M$ is a *cone* if it is a non-empty closed convex set closed under multiplication by nonnegative reals ($s \in K$ and $a \geqslant 0$ imply $as \in K$) and containing no straight line through 0 ($s \in K$ and $-s \in K$ imply $s = 0$) and if $K \neq \{0\}$.

Throughout this paper, K is a cone in $\mathbb{R}^M$ unless otherwise stated. Notice that K is closed under addition. Therefore, if $s^1, \ldots, s^n$ are in K, then $t_1 s^1 + \cdots + t_n s^n \in K$ for any nonnegative reals $t_1, \ldots, t_n$.

Let L be the linear subspace generated by K, that is, the smallest linear subspace of $\mathbb{R}^M$ that contains K. If $\dim L = N$, then we say that K is an *N-dimensional cone.* If $\dim L = M$, then K is said to be *nondegenerate.*

If $\{e^1, \ldots, e^N\}$ is a linearly independent system in $\mathbb{R}^M$, then the set of vectors $s = s_1 e^1 + \cdots + s_N e^N$ with nonnegative $s_1, \ldots, s_N$ is the smallest cone that contains $e^1, \ldots, e^N$. It is called the cone generated by $\{e^1, \ldots, e^N\}$.

Definition 2.2. Let K be an N-dimensional cone in $\mathbb{R}^M$. If $\{e^1, \ldots, e^N\}$ is a linearly independent system such that K is the cone generated by it, then $\{e^1, \ldots, e^N\}$ is called a *strong basis* of K. If $\{e^1, \ldots, e^N\}$ is a basis of the linear subspace L generated by K and if $e^1, \ldots, e^N$ are in K, then $\{e^1, \ldots, e^N\}$ is called a *weak basis* of K.

Any cone has a weak basis. A cone in $\mathbb{R}$ is either $[0, \infty)$ or $(-\infty, 0]$, and has a strong basis. Any nondegenerate cone in $\mathbb{R}^2$ is a closed sector with angle $< \pi$ and has a strong basis. A nondegenerate cone in $\mathbb{R}^3$ has a strong basis if and only if it is a triangular cone. For any M, the nonnegative orthant $\mathbb{R}^M_+$ is a cone with a strong basis.

Definition 2.3. Write $s^1 \leqslant_K s^2$ if $s^2 - s^1 \in K$. A sequence $\{s^n\}_{n=1,2,\ldots}$ in $\mathbb{R}^M$ is *K-increasing* if $s^n \leqslant_K s^{n+1}$ for each n; *K-decreasing* if $s^{n+1} \leqslant_K s^n$ for each n. A mapping f from $[0, \infty)$ into $\mathbb{R}^M$ is *K-increasing* if $f(t_1) \leqslant_K f(t_2)$ for $t_1 \leqslant t_2$; *K-decreasing* if $f(t_2) \leqslant_K f(t_1)$ for $t_1 \leqslant t_2$.

More generally, let K_1 and K_2 be cones in $\mathbb{R}^{M_1}$ and $\mathbb{R}^{M_2}$, respectively. A mapping f from K_1 into $\mathbb{R}^{M_2}$ is *(K_1, K_2)-increasing* if $s^1 \leqslant_{K_1} s^2$ implies $f(s^1) \leqslant_{K_2} f(s^2)$; *$(K_1, K_2)$-decreasing* if $s^1 \leqslant_{K_1} s^2$ implies $f(s^2) \leqslant_{K_2} f(s^1)$.

The following facts are basic for cones. The proofs are left to the reader. We call H a *strictly supporting hyperplane* of K, if H is an $(M-1)$-dimensional linear subspace such that $H \cap K = \{0\}$.

Proposition 2.4. *A cone K in $\mathbb{R}^M$ has the following properties.*

(i) *There exists a strictly supporting hyperplane H of K.*

(ii) *Let H be a strictly supporting hyperplane of K and let $s^0 \in K \setminus \{0\}$. Then the hyperplane $s^0 + H$ does not contain 0. Let D be the closed half space containing 0 with boundary $s^0 + H$. Then $K \cap D$ is a bounded set.*

(iii) *If $\{s^n\}_{n=1,2,\ldots}$ is a K-decreasing sequence in K, then it is convergent.*

A weak basis of K is not unique. But, a strong basis of K is essentially unique, if it exists.

Proposition 2.5. *Let $\{e^1, \ldots, e^N\}$ be a strong basis of K with $|e^j| = 1$ for $j = 1, \ldots, N$. Then it is unique up to change of the order.*

Remark 2.6. Given s^1, s^2 in a cone K in $\mathbb{R}^M$, we call $u \in K$ the greatest lower bound of s^1 and s^2 in the partial order $\leqslant_K$ and write $u = s^1 \wedge_K s^2$, if

$$\{v \in K \colon v \leqslant_K s^1\} \cap \{v \in K \colon v \leqslant_K s^2\} = \{v \in K \colon v \leqslant_K u\}. \tag{2.2}$$

If K has a strong basis $\{e^1, \ldots, e^N\}$, then for any $s^1, s^2 \in K$, $s^1 \wedge_K s^2$ exists. Indeed, if $s^j = s^j_1 e^1 + \cdots + s^j_N e^N$ for $j = 1, 2$, then $s^1 \wedge_K s^2 = (s^1_1 \wedge s^2_1)e^1 + \cdots + (s^1_N \wedge s^2_N)e^N$.

Let K be a circular cone in $\mathbb{R}^3$. Then, for $s^1, s^2 \in K$, $s^1 \wedge_K s^2$ does not necessarily exist. This is seen in the following way. Denote $x = (x_j)_{1 \leqslant j \leqslant 3} \in \mathbb{R}^3$ and let K have the x_3-axis as the axis of rotation. We have $\{v \in K : v \leqslant_K s\} = (s - K) \cap K$ for $s \in K$. The section of the left-hand side of (2.2) by a plane $x_3 =$ constant is not a disc if $s^1 - s^2 \notin K \cup (-K)$. Thus, the relation (2.2) is not always possible.

Similarly, one can define the least upper bound. As above, the least upper bound exists when K has a strong basis, but generally not when K is a circular cone.

Definition 2.7. Let K be a cone in $\mathbb{R}^M$. Let $K' = \{u \in \mathbb{R}^M : \langle u, s \rangle \geqslant 0$ for all $s \in K\}$. Then K' is again a cone in $\mathbb{R}^M$. It is called the *dual cone* of K.

We have $(K')' = K$. If $K = \mathbb{R}^M_+$, then $K = K'$. For two cones K_1, K_2 in $\mathbb{R}^M$, we have $K_1 \subseteq K_2$ if and only if $K'_1 \supseteq K'_2$.

Example 2.8. Let

$$e^1 = (\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}, 1)^\top, \quad e^2 = (-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}, 1)^\top, \quad e^3 = (0, -1, 1)^\top \tag{2.3}$$

in $\mathbb{R}^3$. These points are on the circle $x_1^2 + x_2^2 = 1$, $x_3 = 1$, and form an equilateral triangle. Let Γ_1 and Γ_2 be the line segments from e^3 to e^1 and from e^2 to e^3, respectively. Let C be the arc from e^1 to e^2 of the circle. Let D be the closed convex set on the plane $x_3 = 1$, surrounded by Γ_1, C and Γ_2. Let $K = \{s = tu \in \mathbb{R}^3 : u \in D$ and $t \geqslant 0\}$. Then $\{e^1, e^2, e^3\}$ is a weak basis of K. Any $s \in \mathbb{R}^3$ is expressed as $s = s_1 e^1 + s_2 e^2 + s_3 e^3$ with $s_j \in \mathbb{R}$. For any $u \in \mathbb{R}^3$ we have

$$\langle u, s \rangle = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 \text{ with } \alpha_j = \langle u, e^j \rangle \text{ for } j = 1, 2, 3. \tag{2.4}$$

Then, $u \in K'$ if and only if $\alpha_j \geqslant 0$ for $j = 1, 2, 3$ and

$$a\alpha_1 + (1 - a)\alpha_2 - a(1 - a)\alpha_3 \geqslant 0 \text{ for } 0 \leqslant a \leqslant 1. \tag{2.5}$$

An alternative characterization is that $u \in K'$ if and only if $\alpha_j \geqslant 0$ for $j = 1, 2, 3$ and

$$\alpha_3^{1/2} \leqslant \alpha_1^{1/2} + \alpha_2^{1/2}. \tag{2.6}$$

Indeed, a few calculations show that $s \in C$ if and only if

$$s = (1 - a(1 - a))^{-1}(ae^1 + (1 - a)e^2 - a(1 - a)e^3) \text{ with } 0 \leqslant a \leqslant 1. \tag{2.7}$$

Using this it follows that $u \in K'$ if and only if (2.5) holds and $\alpha_j \geqslant 0$ for all j. Then notice that nonnegative reals $\alpha_1, \alpha_2, \alpha_3$ satisfy (2.5) if and only if they satisfy (2.6).

Example 2.9. Let K be the least cone in $\mathbb{R}^3$ containing $e^1, \dots, e^4$, where

$$e^1 = (0,0,1)^\top,\ e^2 = (1,1,1)^\top,\ e^3 = (1,0,1)^\top,\ e^4 = (0,1,1)^\top.$$

That is, $K = K_1 \cup K_2$ where K_1 is the cone generated by $\{e^1, e^2, e^3\}$ and K_2 is the cone generated by $\{e^1, e^2, e^4\}$. Note that the section $K \cap \{(x_1, x_2, x_3)^\top \colon x_1, x_2 \in \mathbb{R}\}$ for $x_3 > 0$ is the square with vertices $(0,0,x_3)^\top, (x_3, 0, x_3)^\top, (x_3, x_3, x_3)^\top$ and $(0, x_3, x_3)^\top$.

Let us use $\{e^1, e^2, e^3\}$ as a weak basis of K. For any $u \in \mathbb{R}^3$ $\langle u, s\rangle$ is written as in (2.4). Since $e^4 = e^1 + e^2 - e^3$ it follows that $u \in K'$ if and only if $\alpha_j \geqslant 0$ for $j = 1,2,3$ and $\alpha_3 \leqslant \alpha_1 + \alpha_2$. In particular, there are vectors $u^1, \dots, u^4 \in K'$ such that $\langle u^1, s\rangle = s_1$, $\langle u^2, s\rangle = s_2$, $\langle u^3, s\rangle = s_1 + s_3$, $\langle u^4, s\rangle = s_2 + s_3$. Moreover, it is easily seen that any $u \in K'$ is written as $u = \beta_1 u^1 + \dots + \beta_4 u^4$ where $\beta_1, \dots, \beta_4 \geqslant 0$.

Proposition 2.10. *Let K be an N-dimensional cone in $\mathbb{R}^M$. Let L be the linear subspace generated by K and let T be a linear transformation from L to $\mathbb{R}^{\widetilde{M}}$ such that $\dim(TL) = N$. Denote by T^{-1} the inverse of T defined on TL. Define $\widetilde{K} = TK$, the image of K by T. Then, $\widetilde{K}$ is an N-dimensional cone in $\mathbb{R}^{\widetilde{M}}$. We have $u^1 \leqslant_{\widetilde{K}} u^2$ if and only if $T^{-1}u^1 \leqslant_K T^{-1}u^2$. A system $\{u^1, \dots, u^N\}$ is a strong basis (resp. a weak basis) of $\widetilde{K}$ if and only if $\{T^{-1}u^1, \dots, T^{-1}u^N\}$ is a strong basis (resp. a weak basis) of K.*

The proof is easy and omitted. In the situation above we say that K and $\widetilde{K}$ are *isomorphic* cones and call T an *isomorphism* from K to $\widetilde{K}$.

Example 2.11. Any N-dimensional cone K with a strong basis is isomorphic to $\mathbb{R}^N_+$. The isomorphism is given by a mapping between strong bases.

Example 2.12. Let $d \geqslant 2$ and let $K = M^+_{d\times d}$ be the set of symmetric nonnegative-definite $d \times d$ matrices $s = (s_{jk})_{j,k=1}^d \in K$. The lower triangle, $(s_{jk})_{k\leqslant j}$ with $d(d+1)/2$ entries, determines s. We identify K with a subset of $\mathbb{R}^{d(d+1)/2}$, considering $(s_{jk})_{k\leqslant j}$ as a column vector. Then K is a nondegenerate cone in $\mathbb{R}^{d(d+1)/2}$ and does not have a strong basis, which will follow from Theorems 4.7 and 4.13. For $d = 2$ this is seen also from the following isomorphism.

If $K = M^+_{2\times 2}$ then s is identified with $(x_1, x_2, x_3)^\top$, where $x_1 = s_{11}, x_2 = s_{22}, x_3 = s_{21}$, and hence $K = \{(x_1, x_2, x_3)^\top \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1x_2 - x_3^2 \geqslant 0\}$. In this case K is isomorphic to a circular cone in $\mathbb{R}^3$. Indeed, consider the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3)^\top = (u_1, u_2, u_3)^\top$ with $x_1 = u_1 + u_3, x_2 = -u_1 + u_3, x_3 = u_2$. Then $u \in \widetilde{K} = TK$ is expressed as $u_1 + u_3 \geqslant 0, -u_1 + u_3 \geqslant$

$0, (u_1 + u_3)(-u_1 + u_3) - u_2^2 \geqslant 0$. This is written as $u_3 \geqslant 0, u_3^2 - u_1^2 - u_2^2 \geqslant 0$, which describes a circular cone. For $d \geqslant 3$ it is unlikely that the cone $M_{d\times d}^+$ is isomorphic to a cone expressible by quadratic equations, because the property $\det(s) = 0$ is written as an equation of degree d.

Definition 2.13. Let f be a mapping from a cone K in $\mathbb{R}^M$ into $\mathbb{R}^d$.

(i) We say that f is K-right continuous at $s^0 \in K$, if, for every K-decreasing sequence $\{s^n\}_{n=1,2,\ldots}$ in K with $|s^n - s^0| \to 0$, we have $|f(s^n) - f(s^0)| \to 0$.

(ii) We say that f has K-left limits at $s^0 \in K \setminus \{0\}$, if, for every K-increasing sequence $\{s^n\}_{n=1,2,\ldots}$ in $K \setminus \{s^0\}$ satisfying $|s^n - s^0| \to 0$, $\lim_{n\to\infty} f(s^n)$ exists in $\mathbb{R}^d$.

(iii) We say f is K-cadlag if it is K-right continuous at each $s^0 \in K$ and has K-left limits at each $s^0 \in K \setminus \{0\}$.

When $f\colon K \to \mathbb{R}$ has K-left limits at $s^0 \in K$ then $\lim_{n\to\infty} f(s^n)$ may depend on the choice of the K-increasing sequence $\{s^n\}$. But, we now show that if K is an N-dimensional cone with a strong basis, then any K-left continuous mapping has at most $2^N - 1$ different left limits at each point. Let K be with a strong basis $\{e^1, \ldots, e^N\}$. Let $s^0 \in K$ and $\{s^n\}_{n=1,2,\ldots}$ be a sequence in K. Write s^0 and s^n as $s^0 = s_1^0 e^1 + \cdots + s_N^0 e^N$ and $s^n = s_1^n e^1 + \cdots + s_N^n e^N$. Note that $s^n \leqslant_K s^{n+1}$ if and only if $s_j^n \leqslant s_j^{n+1}$ for all $j = 1, \ldots, N$. Thus, $\{s^n\}_{n=1,2,\ldots}$ is K-increasing with $|s^n - s^0| \to 0$ if and only if $\{s_j^n\}_{n=1,2,\ldots}$ is an increasing sequence in $\mathbb{R}_+$ which tends to s_j^0 for each j. Let a be a nonempty subset of $\{1, \ldots, N\}$. We use the notation $s^n \uparrow_a s^0$ if $\{s^n\}_{n=1,2,\ldots}$ is K-increasing with $|s^n - s^0| \to 0$ such that $s_j^n < s_j^0$ for $j \in a$ and all n, and $s_j^n = s_j^0$ for $j \notin a$ and n sufficiently large. Let $p_{s^0} = \{j\colon s_j^0 > 0\}$.

Lemma 2.14. *Let K have a strong basis $\{e^1, \ldots, e^N\}$.*

(i) *Let $\{s^n\}_{n=1,2,\ldots}$ be K-increasing in $K\setminus\{s^0\}$ with $|s^n - s^0| \to 0$. Then there is a unique nonempty subset a of p_{s^0} such that $s^n \uparrow_a s^0$. This particular a is given by $a = \{j\colon s_j^n < s_j^0$ for all $n\}$.*

(ii) *Let $f\colon K \to \mathbb{R}^d$ have K-left limits at $s^0 \in K\setminus\{0\}$. Then there is a family $\{f^a(s_0)\colon a \subseteq p_{s^0}$, a nonempty$\}$ in $\mathbb{R}^d$ such that if a is a nonempty subset of p_{s^0} and $\{s^n\}_{n=1,2,\ldots}$ is a sequence in K with $s^n \uparrow_a s^0$, then $f(s^n) \to f^a(s^0)$.*

Note that to prove (ii) we must show that if $\{s^n\}$ and $\{r^n\}$ are sequences in K with $s^n, r^n \uparrow_a s^0$, then $\lim f(s^n) = \lim f(r^n)$. Details are left to the reader.

3. Cone-parameter Lévy processes and convolution semigroups

In this section we define cone-parameter Lévy processes and convolution semigroups. Several examples and properties will be discussed as well.

Definition 3.1. Let $\{X_s\colon s\in K\}$ be a collection of random variables on $\mathbb{R}^d$ defined on a probability space $(\Omega,\mathcal{F},P)$. Then, $\{X_s\colon s\in K\}$ is a *K-parameter Lévy process* on $\mathbb{R}^d$ if the following five conditions are satisfied.

(i) If $n\geqslant 3$ and $\{s_j\}_{j=1,\dots,n}$ is K-increasing in K, then $X_{s^{j+1}}-X_{s^j}$, $j=1,\dots,n-1$, are independent.

(ii) If $s^1,\dots,s^4\in K$ and $s^2-s^1=s^4-s^3\in K$, then $X_{s^2}-X_{s^1}\stackrel{\mathrm{d}}{=}X_{s^4}-X_{s^3}$.

(iii) $X_0=0$ a. s.

(iv) $X_s(\omega)$ is K-cadlag in s for almost all $\omega\in\Omega$.

(v) If $s^0\in K$ and $\{s^n\}_{n=1,2,\dots}$ is a sequence in K with $|s^n-s^0|\to 0$, then $X_{s^n}\to X_{s^0}$ in probability.

If $\{X_s\colon s\in K\}$ satisfies (i)–(iii) and (v), then $\{X_s\colon s\in K\}$ is called a *K-parameter Lévy process in law.*

Remark 3.2. (i) Note that with $K=\mathbb{R}_+$ the definition of an $\mathbb{R}_+$-parameter Lévy process reduces to the definition of a Lévy process in [22]. Similarly, an $\mathbb{R}_+$-parameter Lévy process in law is a Lévy process in law, as defined in [22].
(ii) Recall that $\{X_s\colon s\in K\}$ is called measurable if the mapping $X_s(\omega)$ from $(\omega,s)\in\Omega\times K$ into $\mathbb{R}^d$ is measurable with respect to $(\mathcal{F}\times\mathcal{B}(K),\mathcal{B}(\mathbb{R}^d))$. A K-parameter Lévy process is automatically measurable if condition (iv) of Definition 3.1 holds for all ω (not only for almost all ω), or if the underlying probability space is complete. More generally, any K-parameter Lévy process in law has a measurable modification. This follows from the fact that any process which is continuous in probability has a measurable modification; see Cohn [6], Theorem 2.

Proposition 3.3. *A process $\{X_s\colon s\in K\}$ is a K-parameter Lévy process if and only if it satisfies* (i)–(iv) *of Definition* 3.1.

We postpone the proof. Next we define convolution semigroups.

Definition 3.4. A family $\{\mu_s\colon s\in K\}$ of probability measures on $\mathbb{R}^d$ is a *K-parameter convolution semigroup* if

(i) $\mu_{s^1}*\mu_{s^2}=\mu_{s^1+s^2}$ for all $s^1,s^2\in K$,

(ii) $\mu_{ts}\to\delta_0$ for $s\in K$ as $t\downarrow 0$.

The following fact is basic.

Proposition 3.5. *Let $\{X_s \colon s \in K\}$ be a K-parameter Lévy process in law on $\mathbb{R}^d$ and let $\mu_s = \mathcal{L}(X_s)$. Then $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup.*

Proof. We have

$$\mu_{s^1+s^2} = \mathcal{L}(X_{s^1+s^2}) = \mathcal{L}(X_{s^1} + (X_{s^1+s^2} - X_{s^1})) = \mathcal{L}(X_{s^1}) * \mathcal{L}(X_{s^2}) = \mu_{s^1} * \mu_{s^2},$$

that is, (i) in Definition 3.4. Let $s \in K$ and $t_n \in \mathbb{R}_+$ with $t_n \downarrow 0$. Then $t_n s \to 0$ which, by Definition 3.1 (iii) and (v), gives (ii) in Definition 3.4. $\square$

Let us provide some examples of K-parameter Lévy processes (in law) and K-parameter convolution semigroups. The proof of the first lemma is left to the reader.

Lemma 3.6. *Let $\{X_s^1 \colon s \in K\}$, ..., $\{X_s^n \colon s \in K\}$ be independent K-parameter Lévy processes (resp. Lévy processes in law) on $\mathbb{R}^d$. Let $X_s = X_s^1 + \cdots + X_s^n$. Then $\{X_s \colon s \in K\}$ is a K-parameter Lévy process (resp. Lévy process in law) on $\mathbb{R}^d$.*

Example 3.7. Let K be a cone in $\mathbb{R}^M$ and K' be the dual cone of K. Let $u \in K'$. Let $\{V_t \colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$. Then, we get a K-parameter Lévy process $\{X_s \colon s \in K\}$ on $\mathbb{R}^d$ by letting $X_s = V_{\langle u,s \rangle}$.

Example 3.8. Let K have a strong basis $\{e^1, \ldots, e^N\}$. Then, in each of the following three constructions of X_s for $s = s_1 e^1 + \cdots + s_N e^N \in K$, we obtain a K-parameter Lévy process $\{X_s \colon s \in K\}$ on $\mathbb{R}^d$.

(i) Let $\{V_t \colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$. Fix $(c_j)_{1 \leqslant j \leqslant N}$ with $c_j \geqslant 0$ for $1 \leqslant j \leqslant N$. Define $X_s = V_{c_1 s_1 + \cdots + c_N s_N}$.

(ii) Let $\{V_t^j \colon t \geqslant 0\}$, $j = 1, \ldots, N$, be independent Lévy processes on $\mathbb{R}^d$. Define $X_s = V_{s_1}^1 + \cdots + V_{s_N}^N$.

(iii) For each $j = 1, \ldots, N$, let $\{U_t^j \colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^{d_j}$. Assume that they are independent. Let $d = d_1 + \cdots + d_N$. Define $X_s = (U_{s_1}^1, \ldots, U_{s_N}^N)^\top$.

Example 3.9. Let $K = M_{d\times d}^+$ with $d \geqslant 2$. For $s \in K$ let μ_s be the Gaussian measure on $\mathbb{R}^d$, defined as $\mu_s = N_d(0, s)$, the d-dimensional Gaussian distribution with mean zero and covariance matrix s. Then, obviously, $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$. We call it the *canonical $M_{d\times d}^+$-parameter convolution semigroup.*

Remark 3.10. Let K and $\widetilde{K}$ be isomorphic cones as in Proposition 2.10. If $\{\mu_s\colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$, then $\{\widetilde{\mu}_u\colon u \in \widetilde{K}\}$ defined by $\widetilde{\mu}_u = \mu_{T^{-1}u}$ is a $\widetilde{K}$-parameter convolution semigroup. If $\{X_s\colon s \in K\}$ is a K-parameter Lévy process (resp. a K-parameter Lévy process in law) on $\mathbb{R}^d$, then $\{\widetilde{X}_u\colon u \in \widetilde{K}\}$ defined by $\widetilde{X}_u = X_{T^{-1}u}$ is a $\widetilde{K}$-parameter Lévy process (resp. a $\widetilde{K}$-parameter Lévy process in law). The semigroup $\{\widetilde{\mu}_u\}$ and the process $\{\widetilde{X}_u\}$ have the same structures as $\{\mu_s\}$ and $\{X_s\}$, respectively.

Remark 3.11. Let K_1 and K_2 be cones in $\mathbb{R}^M$ such that $K_1 \subseteq K_2$. If $\{X_s\colon s \in K_2\}$ is a K_2-parameter Lévy process (resp. a K_2-parameter Lévy process in law) then its restriction $\{X_s\colon s \in K_1\}$ is a K_1-parameter Lévy process (resp. a K_1-parameter Lévy process in law). If $\{\mu_s\colon s \in K_2\}$ is a K_2-parameter convolution semigroup then its restriction $\{\mu_s\colon s \in K_1\}$ is a K_1-parameter convolution semigroup.

In particular, if $\{\mu_s\colon s \in K\}$ is a K-parameter convolution semigroup then $\{\mu_{ts}\colon t \geqslant 0\}$ is an $\mathbb{R}_+$-parameter convolution semigroup for $s \in K$, and if $\{X_s\colon s \in K\}$ is a K-parameter Lévy process (resp. a K-parameter Lévy process in law), then $\{X_{ts}\colon t \geqslant 0\}$ is a Lévy process (resp. a Lévy process in law).

It follows from the preceding remark that, if $\{\mu_s\colon s \in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$, then $\mu_s \in ID(\mathbb{R}^d)$ for each s. Thus cone-parameter convolution semigroups can be studied within the framework of the theory of infinitely divisible distributions. In the rest of this section we study convolution semigroups from this viewpoint. A deeper study of cone-parameter Lévy processes and convolution semigroups is postponed to Section 4.

For $z, x \in \mathbb{R}^d$ let $g(z,x)$ be the function

$$g(z,x) = e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_{\{|x|\leqslant 1\}}(x). \tag{3.1}$$

For $\mu \in ID(\mathbb{R}^d)$ and $r \in \mathbb{R}$, we define $\widehat{\mu}(z)^r$, $z \in \mathbb{R}^d$, as $\widehat{\mu}(z)^r = e^{r \log \widehat{\mu}(z)}$, where $\log \widehat{\mu}(z)$ is the distinguished logarithm of $\widehat{\mu}(z)$ in [22], p. 33. In other words,

$$\widehat{\mu}(z)^r = \exp\left[r \left(-\tfrac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d} g(z,x)\nu(dx) \right) \right],$$

where (A, ν, γ) is the triplet or the generating triplet of μ in [22], p. 38. The matrix A and the measure ν are respectively the Gaussian covariance matrix and the Lévy measure of μ. The vector γ is a location parameter. If ν satisfies $\int_{|x|\leqslant 1} |x|\nu(dx) < \infty$, let γ^0 be the drift μ, that is $\gamma^0 = \gamma - \int_{|x|\leqslant 1} x\nu(dx)$.

Proposition 3.12. *Let $\{\mu_s\colon s\in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Then, $\mu_0=\delta_0$ and $\mu_s\in ID(\mathbb{R}^d)$ for $s\in K$. We have $\mu_{ts}=\mu_s^t$ for $t\geqslant 0$. Thus, for the triplet (A_s,ν_s,γ_s) of μ_s,*

$$A_{s^1+s^2}=A_{s^1}+A_{s^2},\quad \nu_{s^1+s^2}=\nu_{s^1}+\nu_{s^2},\quad \gamma_{s^1+s^2}=\gamma_{s^1}+\gamma_{s^2},\tag{3.2}$$

$$A_{ts}=tA_s,\quad \nu_{ts}=t\nu_s,\quad \gamma_{ts}=t\gamma_s.\tag{3.3}$$

If, moreover, $\int_{|x|\leqslant 1}|x|\nu_s(dx)<\infty$ for all $s\in K$, then, for the drift γ_s^0 of μ_s, we have

$$\gamma^0_{s^1+s^2}=\gamma^0_{s^1}+\gamma^0_{s^2},\quad \gamma^0_{ts}=t\gamma^0_s.\tag{3.4}$$

Proof. Since $\{\mu_{ts}\colon t\geqslant 0\}$ is an $\mathbb{R}_+$-parameter convolution semigroup as noted in Remark 3.11 we have $\mu_0=\delta_0$, $\mu_s\in ID(\mathbb{R}^d)$ and $\mu_{ts}=\mu_s^t$. Equations (3.2)–(3.4) are obvious consequences. □

Theorem 3.13. *Let $\{\mu_s\colon s\in K\}$ be a K-parameter convolution semigroup with triplets (A_s,ν_s,γ_s). Let $\{e^1,\dots,e^N\}$ be a weak basis on K. Then, for all $s\in K$, μ_s is determined by $\mu_{e^1},\dots,\mu_{e^N}$. More precisely, for $s=s_1e^1+\dots+s_Ne^N\in K$ we have*

$$\widehat{\mu}_s(z)=\widehat{\mu}_{e^1}(z)^{s_1}\dots\widehat{\mu}_{e^N}(z)^{s_N},\quad z\in\mathbb{R}^d,\tag{3.5}$$

$$A_s=s_1A_{e^1}+\dots+s_NA_{e^N},\nu_s=s_1\nu_{e^1}+\dots+s_N\nu_{e^N},\gamma_s=s_1\gamma_{e^1}+\dots+s_N\gamma_{e^N}.\tag{3.6}$$

The second equality is understood to hold on the class of Borel sets B such that $\inf_{x\in B}|x|>0$.

If $\{s^n\}_{n=1,2,\dots}$ is a sequence in K with $|s^n-s^0|\to 0$, then $\mu_{s^n}\to\mu_{s^0}$.

Proof. Any $s\in K$ is represented uniquely as $s=s_1e^1+\dots+s_Ne^N$, with $s_1,\dots,s_N\in\mathbb{R}$. Let $s_j^+=s_j\vee 0$ and $s_j^-=-(s_j\wedge 0)$. Then $s_j=s_j^+-s_j^-$. We have $s=s'-s''$ with $s'=s_1^+e^1+\dots+s_N^+e^N\in K$ and $s''=s_1^-e^1+\dots+s_N^-e^N\in K$. Hence $\mu_s*\mu_{s''}=\mu_{s'}$. Noting that $\widehat{\mu}_{s''}(z)\neq 0$ by infinite divisibility, we have

$$\widehat{\mu}_s(z)=\frac{\widehat{\mu}_{s'}(z)}{\widehat{\mu}_{s''}(z)}=\frac{\widehat{\mu}_{e^1}(z)^{s_1^+}\dots\widehat{\mu}_{e^N}(z)^{s_N^+}}{\widehat{\mu}_{e^1}(z)^{s_1^-}\dots\widehat{\mu}_{e^N}(z)^{s_N^-}},$$

which is (3.5). Now (3.6) is a consequence of (3.5) by the uniqueness of the expression as formulated in [22], E 12.2.

To prove that $\mu_{s^n}\to\mu_{s^0}$ for $s^n\to s^0$, decompose s^n as $s^n=s_1^ne^1+\dots+s_N^ne^N$ for $n=0,1,\dots$. Then $s_j^n\to s_j^0$ for $j=1,\dots,N$ and (3.5) shows that $\widehat{\mu}_{s^n}(z)\to\widehat{\mu}_{s^0}(z)$ for all z. □

Proof of Proposition 3.3. Let $\{X_s \colon s \in K\}$ satisfy (i)–(iv) of Definition 3.1. We show that it is continuous in probability.

For $s \in K$ let $\mu_s = \mathcal{L}(X_s)$. First we show that $\{\mu_s \colon s \in K\}$ is a K-parameter convolution semigroup. By repeating the first part of the proof of Proposition 3.5 it follows that $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$. By K-right continuity of the paths and by $X_0 = 0$ a. s. it follows that $\mu_{t_n s} \to \delta_0$ whenever $t_n \downarrow 0$.

Let $\{s^n\}_{n=1,2,\ldots} \subseteq K$ and $s^0 \in K$ with $|s^n - s^0| \to 0$. Let $\{e^1, \ldots, e^N\}$ be a weak basis of K and decompose s^n and s^0 as $s^n = s_1^n e^1 + \cdots + s_N^n e^N$ and $s^0 = s_1^0 e^1 + \cdots + s_N^0 e^N$ where $s_j^n, s_j^0 \in \mathbb{R}$ for all j and n. Define u^n by $u^n = u_1^n e^1 + \cdots + u_N^n e^N$, where $u_j^n = s_j^n \vee s_j^0$ for $j = 1, \ldots, N$. Since $u_j^n - s_j^n \geqslant 0$ for all j we have $u^n - s^n \in K$, that is $s^n \leqslant_K u^n$ and $u^n \in K$. Similarly, $s^0 \leqslant_K u^n$. Since $X_{s^n} - X_{s^0} = [X_{u^n} - X_{s^0}] - [X_{u^n} - X_{s^n}]$ it suffices to prove that the two terms on the right-hand side converge to zero in probability. As $u^n - s^n, u^n - s^0 \to 0$, the result follows from Definition 3.1 (ii) and the last assertion in Theorem 3.13. $\square$

Definition 3.14. Let $\{e^1, \ldots, e^N\}$ be a weak basis of K and let $\rho_1, \ldots, \rho_N \in ID(\mathbb{R}^d)$. We call $\{\rho_1, \ldots, \rho_N\}$ *admissible with respect to* $\{e^1, \ldots, e^N\}$, if there exists (uniquely, by Theorem 3.13) a K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ such that $\mu_{e^j} = \rho_j$ for $j = 1, \ldots, N$.

Let us consider the problem what condition guarantees that $\{\rho_1, \ldots, \rho_N\}$ is admissible with respect to $\{e^1, \ldots, e^N\}$.

Theorem 3.15. *Let $\{e^1, \ldots, e^N\}$ be a weak basis of K. Let $\rho_1, \ldots, \rho_N \in ID(\mathbb{R}^d)$ and let (A_j, ν_j, γ_j) be the generating triplet of ρ_j. Then the following three statements are equivalent.*

(i) *$\{\rho_1, \ldots, \rho_N\}$ is admissible with respect to $\{e^1, \ldots, e^N\}$.*

(ii) *If $s_1, \ldots, s_N \in \mathbb{R}$ are such that $s_1 e^1 + \cdots + s_N e^N \in K$, then $\widehat{\rho}_1(z)^{s_1} \ldots \widehat{\rho}_N(z)^{s_N}$ is an infinitely divisible characteristic function.*

(iii) *If $s_1, \ldots, s_N \in \mathbb{R}$ are such that $s_1 e^1 + \cdots + s_N e^N \in K$, then $s_1 A_1 + \cdots + s_N A_N$ is nonnegative-definite and $s_1 \nu_1 + \cdots + s_N \nu_N$ is nonnegative.*

Proof. By Theorem 3.13, (i) implies (ii). Conversely, suppose that (ii) is true. For each $s \in K$, define $\mu_s \in ID(\mathbb{R}^d)$ by (3.5) with $\mu_{e^j} = \rho_j$. Since $s_1, \ldots, s_N$ are determined by s, this is well-defined by virtue of (ii). The property $\mu_{s^1+s^2} = \mu_{s^1} * \mu_{s^2}$ is obvious. If $s \to 0$, then $s_j \to 0$ for $1 \leqslant j \leqslant N$, and hence $\mu_s \to \delta_0$. This shows (i). The equivalence of (ii) and (iii) is a consequence of E 12.3 of [22]. $\square$

Corollary 3.16. *Let $\{e^1, \ldots, e^N\}$ be a weak basis of K. Then, every choice of $\{\rho_1, \ldots, \rho_N\}$ in $ID(\mathbb{R}^d)$ is admissible with respect to $\{e^1, \ldots, e^N\}$ if and only if the system $\{e^1, \ldots, e^N\}$ is a strong basis of K.*

Proof. If $\{e^1, \ldots, e^N\}$ is a strong basis, then condition (ii) of the theorem above is automatically satisfied, since $s_j \geqslant 0$ for $j = 1, \ldots, N$. Conversely, suppose that $\{e^1, \ldots, e^N\}$ is not a strong basis. Then, we can choose j_0 such that there exists $s = s_1 e^1 + \cdots + s_N e^N \in K$ with $s_{j_0} < 0$. Let $\rho \in ID(\mathbb{R}^d)$ be nontrivial and $\rho_j = \rho$ for $j \neq j_0$ and $\rho_{j_0} = \rho^c$ with c so large that $(1-c)s_{j_0} > s_1 + \cdots + s_N$. By the theorem above, $\{\rho_1, \ldots, \rho_N\}$ is then not admissible with respect to $\{e^1, \ldots, e^N\}$. $\square$

Example 3.17. Let e^1, e^2, e^3 and K be as in Example 2.8. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if the following condition (3.7) or, equivalently, (3.8) is satisfied:

$$(3.7) \qquad \begin{cases} aA_1 + (1-a)A_2 - a(1-a)A_3 \in M^+_{d\times d} & \text{for } 0 < a < 1, \\ a\nu_1 + (1-a)\nu_2 - a(1-a)\nu_3 \geqslant 0 & \text{for } 0 < a < 1, \end{cases}$$

$$(3.8) \qquad \begin{cases} \langle A_3 z, z\rangle^{1/2} \leqslant \langle A_1 z, z\rangle^{1/2} + \langle A_2 z, z\rangle^{1/2} & \text{for } z \in \mathbb{R}^d, \\ \nu_3(B)^{1/2} \leqslant \nu_1(B)^{1/2} + \nu_2(B)^{1/2} & \text{for } B \in \mathcal{B}(\mathbb{R}^d). \end{cases}$$

Indeed, for $\alpha_1, \alpha_2, \alpha_3 \geqslant 0$, the condition that $\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 \geqslant 0$ for all $s = s_1 e^1 + s_2 e^2 + s_3 e^3 \in K$ is expressed by the condition (2.5) or, equivalently, (2.6). Hence, by Theorem 3.15 we get the result.

For example, if $\rho_1 = \rho_2 = \rho$ with triplet (A, ν, γ), then the admissibility condition for $\{\rho, \rho, \rho_3\}$ is that $4A - A_3 \in M^+_{d\times d}$ and $4\nu - \nu_3 \geqslant 0$.

Example 3.18. Let K be the circular cone in $\mathbb{R}^3$ defined by $x_1^2 + x_2^2 \leqslant x_3^2$ and $x_3 \geqslant 0$. Let e^1, e^2, e^3 be as in (2.3). These form a weak basis of K. Notice that the points e^1, e^2, e^3 are located on the circle C defined by $x_1^2 + x_2^2 = 1, x_3 = 1$ and that the triangle $e^1 e^2 e^3$ is equilateral. Thus K is the union of three cones, each of which is isomorphic to the cone of Example 2.8. Hence we conclude the following from Example 3.17. Let $\rho_j \in ID(\mathbb{R}^d)$ for $j = 1, 2, 3$. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if, for $(k, l, m) = (1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$,

$$(3.9) \qquad \begin{cases} aA_k + (1-a)A_l - a(1-a)A_m \in M^+_{d\times d} & \text{for } 0 < a < 1, \\ a\nu_k + (1-a)\nu_l - a(1-a)\nu_m \geqslant 0 & \text{for } 0 < a < 1 \end{cases}$$

or, equivalently,

$$
\begin{cases} \langle A_m z, z\rangle^{1/2} \leqslant \langle A_k z, z\rangle^{1/2} + \langle A_l z, z\rangle^{1/2} & \text{for } z \in \mathbb{R}^d, \\ \nu_m(B)^{1/2} \leqslant \nu_k(B)^{1/2} + \nu_l(B)^{1/2} & \text{for } B \in \mathcal{B}(\mathbb{R}^d). \end{cases} \tag{3.10}
$$

For example, for any $\rho \in ID(\mathbb{R}^d)$, $\{\rho, \rho, \rho\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ and the associated semigroup $\{\mu_s\colon s \in K\}$ satisfies $\mu_s = \rho$ for any $s \in C$, which is proved from (2.7). As another example, let $\rho_1 = \rho_2 = \rho \in ID(\mathbb{R}^d)$ with triplet (A, ν, γ). Then, like in Example 3.17, $\{\rho, \rho, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if $4A - A_3 \in M^+_{d\times d}$ and $4\nu - \nu_3 \geqslant 0$.

Suppose that $\mathrm{Supp}(\rho_j) \subseteq L_j$ for $j = 1, 2, 3$, where L_j are linear subspaces of $\mathbb{R}^d$ such that any $x \in L_1 + L_2 + L_3$ is uniquely decomposed as $x = x^1 + x^2 + x^3$ with $x^j \in L_j$. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ only if each ρ_j is trivial, as will be seen in Corollary 3.21.

Example 3.19. Let $e^1, \dots, e^4$ and K be as in Example 2.9. Let $\rho_j \in ID(\mathbb{R}^d)$ for $j = 1, 2, 3$. Then, $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if $A_1 + A_2 - A_3$ is nonnegative-definite and $\nu_1 + \nu_2 - \nu_3 \geqslant 0$. This is an immediate consequence of the characterization of the dual cone given in Example 2.9.

Let us give some other applications of Theorem 3.13. For a $d \times d$ matrix A, $A(\mathbb{R}^d) = \{Ax\colon x \in \mathbb{R}^d\}$ denotes the range of A.

Proposition 3.20. *Let $L_1, \dots, L_N$ be linear subspaces of $\mathbb{R}^d$ and set $L = L_1 + \cdots + L_N$. Assume that any $x \in L$ is uniquely decomposed as $x = x^1 + \cdots + x^N$ with $x^j \in L_j$ for $j = 1, \dots, N$. Let $\{e^1, \dots, e^N\}$ be a weak basis of K. Let $\{\mu_s\colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ such that* $\mathrm{Supp}(\mu_{e^j}) \subseteq L_j$ *for $j = 1, \dots, N$. If there is $s \in K$ satisfying $s = s_1 e^1 + \cdots + s_N e^N$ with $s_{j_0} < 0$, then $\mu_{e^{j_0}}$ is trivial.*

Proof. Step 1. Let us prove the assertion under the assumption that L_j, $j = 1, \dots, N$, are orthogonal. Let (A_j, ν_j, γ_j) be the generating triplet of μ_{e^j}. It follows from $\mathrm{Supp}(\mu_{e^j}) \subseteq L_j$ that $A_j(\mathbb{R}^d) \subseteq L_j$, $\mathrm{Supp}(\nu_j) \subseteq L_j$ and $\gamma_j \in L_j$ (cf. Proposition 24.17 of [22]). Now choose s such that $s_{j_0} < 0$. Let $z \in L_{j_0}$. Then, by (3.6) $0 \leqslant \langle z, (s_1 A_1 + \cdots + s_N A_N) z\rangle = s_{j_0}\langle z, A_{j_0} z\rangle$. Hence $\langle z, A_{j_0} z\rangle = 0$. It follows that $A_{j_0} z = 0$. Since $A_j(\mathbb{R}^d) = \{A_j z\colon z \in A_j(\mathbb{R}^d)\}$ and $A_j(\mathbb{R}^d) \subseteq L_j$, we see that $A_j(\mathbb{R}^d) = \{A_j z\colon z \in L_j\}$. Therefore, $A_{j_0}(\mathbb{R}^d) = \{0\}$, that is, $A_{j_0} = 0$. Let B be a Borel set in L_{j_0}. Then $\nu_j(B) \leqslant \nu_j(L_{j_0} \cap L_j) = 0$ for $j \neq j_0$. Hence $s_{j_0}\nu_{j_0}(B) \geqslant 0$. Since $s_{j_0} < 0$, this means that $\nu_{j_0}(B) = 0$. That is, $\nu_{j_0} = 0$. Thus, $\mu_{e^{j_0}}$ is trivial.

Step 2. General case. There exists a linear transformation T from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that the images $L_j^\sharp$ of L_j by T, $j = 1, \dots, N$, are orthogonal. Denote $\mu_s^\sharp(B) = \mu_s(T^{-1}B)$. It is readily seen that $\{\mu_s^\sharp \colon s \in K\}$ is a convolution semigroup. Since $\mu_{e^j}^\sharp(L_j^\sharp) = \mu_{e^j}(T^{-1}L_j^\sharp) = \mu_{e^j}(L_j) = 1$, we have $\mathrm{Supp}(\mu_{e^j}^\sharp) \subseteq L_j^\sharp$. Hence, by Step 1, $\mu_{e^{j_0}}^\sharp$ is trivial, that is, $\mu_{e^{j_0}}$ is trivial. $\square$

Corollary 3.21. *Under the same assumptions as as in Proposition* 3.20*, if, for every* j*, there is an* $s^j = s_1^j e^1 + \cdots + s_N^j e^N \in K$ *satisfying* $s_j^j < 0$*, then* $\{\mu_s \colon s \in K\}$ *is trivial.*

Let K and $\widetilde{K}$ be cones satisfying $K \subseteq \widetilde{K}$. Let $\{\mu_s \colon s \in K\}$ and $\{\widetilde{\mu}_s \colon s \in \widetilde{K}\}$ be, respectively, K- and $\widetilde{K}$-parameter convolution semigroups on $\mathbb{R}^d$. We say that $\{\widetilde{\mu}_s \colon s \in \widetilde{K}\}$ is an extension of $\{\mu_s \colon s \in K\}$ if $\widetilde{\mu}_s = \mu_s$ for all $s \in K$.

Proposition 3.22. *Let* K *be an* N*-dimensional cone with strong basis* $\{e^1, \dots, e^N\}$*. Then there exists a* K*-parameter convolution semigroup* $\{\mu_s \colon s \in K\}$ *on* $\mathbb{R}$ *such that, for any* N*-dimensional cone* $\widetilde{K}$ *satisfying* $\widetilde{K} \supseteq K$ *and* $\widetilde{K} \neq K$*,* $\{\mu_s \colon s \in K\}$ *is not extendable to a* $\widetilde{K}$*-parameter convolution semigroup. In particular if, for the Lévy measures* ν_j *of* μ_{e^j}*, there are* $B_j \in \mathcal{B}(\mathbb{R})$*,* $j = 1, \dots, N$*, such that* $\nu_j(B_j) > 0$ *and* $\nu_k(B_j) = 0$ *for* $k \neq j$*, then* $\{\mu_s \colon s \in K\}$ *is not extendable.*

Proof. Let $\{\mu_s \colon s \in K\}$ be as above and let $\widetilde{K}$ be an N-dimensional cone satisfying $\widetilde{K} \supseteq K$ and $\widetilde{K} \neq K$. Suppose that $\{\mu_s \colon s \in K\}$ is extendable to $\{\widetilde{\mu}_s \colon s \in \widetilde{K}\}$. Since $\{e^1, \dots, e^N\}$ is a weak basis of $\widetilde{K}$ but not a strong basis, there is $s \in \widetilde{K}$ such that $s = s_1 e^1 + \cdots + s_N e^N$ with $s_j < 0$ for some j. The Lévy measure $\widetilde{\nu}_s$ of $\widetilde{\mu}_s$ satisfies $\widetilde{\nu}_s = s_1 \nu_1 + \cdots + s_N \nu_N$ by Theorem 3.13. Hence $\widetilde{\nu}_s(B_j) = s_j \nu_j(B_j) < 0$, which is absurd. $\square$

4. Generative and non-generative convolution semigroups

In this section we make a deeper study of the relations between K-parameter convolution semigroups and K-parameter Lévy processes in law. Proposition 3.5 motivates the following definition.

Definition 4.1. Let $\{\mu_s \colon s \in K\}$ be a convolution semigroup on $\mathbb{R}^d$.
(i) A K-parameter Lévy process in law $\{X_s \colon s \in K\}$ is *associated* with $\{\mu_s \colon s \in K\}$ if $\mu_s = \mathcal{L}(X_s)$ for all $s \in K$;
(ii) $\{\mu_s \colon s \in K\}$ is *generative* if there exists a K-parameter Lévy process in law associated with it. Otherwise it is called *non-generative*;

(iii) $\{\mu_s\colon s \in K\}$ is *unique-generative* if it is generative and any two K-parameter Lévy processes in law, $\{X^1_s\colon s \in K\}$ and $\{X^2_s\colon s \in K\}$, associated with it satisfy $\{X^1_s\colon s \in K\} \overset{\mathrm{d}}{=} \{X^2_s\colon s \in K\}$, which denotes that the two processes have a common system of finite-dimensional marginals; $\{\mu_s\colon s \in K\}$ is *multiple-generative* if it is generative and not unique-generative.

In the case $K = \mathbb{R}_+$ it is well-known that any $\mathbb{R}_+$-parameter convolution semigroup is unique-generative. But in Remark 4.6 we will give an example of a multiple-generative convolution semigroup. In Theorem 4.13 we will construct a class of non-generative convolution semigroups, which includes the canonical $M^+_{d\times d}$-parameter convolution semigroup. On the other hand, we will prove that if $\{\mu_s\colon s \in K\}$ is a convolution semigroup on $\mathbb{R}^d$, then each of the following three conditions is sufficient for $\{\mu_s\}$ to be generative: (i) K has a strong basis, (ii) μ_s is purely non-Gaussian for all s, (iii) $d = 1$. When K has a strong basis and μ_s is Gaussian we give a necessary and sufficient condition that $\{\mu_s\}$ is unique-generative. First we state a few properties of generative convolution semigroups.

Definition 4.2. Let $\{X_s\colon s \in K\}$ be a K-parameter Lévy process in law on $\mathbb{R}^d$. If $\{s^j\}_{1\leqslant j\leqslant n}$ is K-increasing, then let us call $\mathcal{L}((X_{s^j})_{1\leqslant j\leqslant n})$ a *K-increasing marginal distribution* of $\{X_s\colon s \in K\}$.

Theorem 4.3. *Let $\{\mu_s\colon s \in K\}$ denote a generative K-parameter convolution semigroup. Then $\{\mu_s\colon s \in K\}$ determines uniquely all K-increasing marginal distributions of a K-parameter Lévy process in law $\{X_s\colon s \in K\}$ associated with it.*

Proof. Let $\{s^j\}_{1\leqslant j\leqslant n}$ be K-increasing. Let $s^0 = 0$. Then $X_{s^j} - X_{s^{j-1}}$, $j = 1,\dots,n$, are independent and $\mathcal{L}(X_{s^j} - X_{s^{j-1}}) = \mathcal{L}(X_{s^j - s^{j-1}}) = \mu_{s^j - s^{j-1}}$. Hence $\mathcal{L}((X_{s^j} - X_{s^{j-1}})_{1\leqslant j\leqslant n})$ is the direct product of $\mu_{s^j - s^{j-1}}$, $1 \leqslant j \leqslant n$. Since $(X_{s^j})_{1\leqslant j\leqslant n}$ is obtained from $(X_{s^j} - X_{s^{j-1}})_{1\leqslant j\leqslant n}$ by a linear transformation, its distribution is determined. $\square$

Let us give a method of construction of K-parameter Lévy processes in law.

Proposition 4.4. *Let $\{\mu_s\colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ and let $n \geqslant 2$. For each $j = 1,\dots,n$ let $\{X^j_s\colon s \in K\}$ be a K-parameter Lévy process (resp. Lévy process in law) associated with $\{\mu_s\colon s \in K\}$. Let U_j be nonnegative random variables such that $1 = U_1 + \cdots + U_n$ a.s. Suppose that $\{X^1_s\colon s \in K\},\dots,\{X^n_s\colon s \in K\}$ and $(U_1,\dots,U_n)^\top$ are independent. Define $\{X_s\colon s \in K\}$ by*

$X_s = X^1_{U_1 s} + \cdots + X^n_{U_n s}$ *for* $s \in K$*. Then* $\{X_s \colon s \in K\}$ *is a* K*-parameter Lévy process* (*resp. Lévy process in law*) *associated with* $\{\mu_s \colon s \in K\}$*.*

Proof. First assume that $U_1, \ldots, U_n$ are nonrandom. Then it follows from Lemma 3.6 that $\{X_s\}$ is a K-parameter Lévy process in law. Moreover, for $s \in K$ we have

$$\mathcal{L}(X_s) = \mathcal{L}(X^1_s)^{U_1} * \cdots * \mathcal{L}(X^n_s)^{U_n} = \mu_s^{U_1} * \cdots * \mu_s^{U_n} = \mu_s,$$

that is, $\{X_s\}$ is associated with $\{\mu_s\}$.

If $U_1, \ldots, U_n$ are random we hence have that $\{X_s\}$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$ conditional on $(U_1, \ldots, U_n)$. It is easily seen that the same holds in the unconditional distribution.

If the paths of $\{X^j_s\}$ are K-cadlag a. s., then the same holds for $\{X_s\}$. Thus, the property of being a K-parameter Lévy process is inherited from $\{X^j_s\}$ to $\{X_s\}$. $\square$

Let $(\mathbb{R}^d)^K$ be the set of mappings $\omega = (\omega(s))_{s \in K}$ from K into $\mathbb{R}^d$ and let $\mathcal{B}(\mathbb{R}^d)^K$ be the σ-algebra generated by the coordinate mappings $\xi_s(\omega) = \omega(s), s \in K$. If $\{X_s \colon s \in K\}$ is a K-parameter Lévy process in law, then it induces a unique probability measure Q on $((\mathbb{R}^d)^K, \mathcal{B}(\mathbb{R}^d)^K)$ such that $\{X_s \colon s \in K\}$ is identical in law with $\{\xi_s \colon s \in K\}$ under Q. We call Q the distribution (or law) of $\{X_s \colon s \in K\}$ and denote $Q = \mathcal{L}(\{X_s \colon s \in K\})$. The finite-dimensional marginals of $\{\xi_s\}$ under Q are called the marginals of Q. For a K-parameter convolution semigroup $\{\mu_s \colon s \in K\}$ denote the set of distributions of K-parameter Lévy processes in law associated with it by $\mathbb{L}(\{\mu_s \colon s \in K\})$. Then, $\{\mu_s \colon s \in K\}$ is generative (resp. multiple-generative, unique-generative, non-generative) if and only if $\mathbb{L}(\{\mu_s \colon s \in K\})$ is nonempty (resp. has more than one element, is a singleton, is empty).

Theorem 4.5. *Let* $\{\mu_s \colon s \in K\}$ *be a multiple-generative convolution semigroup. Then* $\mathbb{L}(\{\mu_s \colon s \in K\})$ *is a convex set of probability measures.*

Proof. Let $Q^0, Q^1 \in \mathbb{L}(\{\mu_s \colon s \in K\})$ and $p \in [0,1]$. Let $\{X^0_s \colon s \in K\}$ and $\{X^1_s \colon s \in K\}$ be K-parameter Lévy processes in law with $Q^j = \mathcal{L}(\{X^j_s \colon s \in K\})$ for $j = 0, 1$, and U be a random variable such that $\{X^0_s \colon s \in K\}, \{X^1_s \colon s \in K\}$ and U are independent and $p = P(U = 1) = 1 - P(U = 0)$. Define $X_s = X^0_{Us} + X^1_{(1-U)s}$ for $s \in K$. Then from Proposition 4.4 it follows that $\{X_s \colon s \in K\}$ is a K-parameter Lévy process in law associated with $\{\mu_s \colon s \in K\}$. Let $Q = \mathcal{L}(\{X_s \colon s \in K\})$. For $n \geqslant 1$, $s^1, \ldots, s^n \in K$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$, we have

$$Q(\xi_{s^1} \in B_1, \ldots, \xi_{s^n} \in B_n) = P(X_{s^1} \in B_1, \ldots, X_{s^n} \in B_n)$$

$$= \quad pP(X^0_{s^1} \in B_1, \ldots, X^0_{s^n} \in B_n) + (1-p)P(X^1_{s^1} \in B_1, \ldots, X^1_{s^n} \in B_n),$$

that is, $pQ^0 + (1-p)Q^1 = Q \in \mathbb{L}(\{\mu_s \colon s \in K\})$, as desired. $\square$

Remark 4.6. In the setting of Theorem 4.5 let $Q \in \mathbb{L}(\{\mu_s \colon s \in K\})$. Then, all K-increasing marginals of Q are infinitely divisible. But, in general the marginals of Q need not be infinitely divisible. To illustrate, let $K = \mathbb{R}^2_+$. Let $\{\mu_s \colon s \in \mathbb{R}^2_+\}$ be the convolution semigroup on $\mathbb{R}$ given by $\mu_s = N(0, s_1 + s_2)$ for $s = (s_1, s_2)^\top \in \mathbb{R}^2_+$. For $j = 1, 2, 3$, let $\{V^j_t \colon t \geqslant 0\}$ be independent standard Wiener processes on $\mathbb{R}$. Define $\{X^0_s \colon s \in \mathbb{R}^2_+\}$ by $X^0_s = V^1_{s_1} + V^2_{s_2}$, and $\{X^1_s \colon s \in \mathbb{R}^2_+\}$ by $X^1_s = V^3_{s_1+s_2}$. Let Q^0 and Q^1 be the respectively laws. Since $Q^0 \neq Q^1$, $\{\mu_s \colon s \in \mathbb{R}^2_+\}$ is multiple-generative. Let $0 < p < 1$ and let $Q = pQ^0 + (1-p)Q^1$. Then the distribution μ of $(\xi_{e^1}, \xi_{e^2})^\top$ under Q is not infinitely divisible, where $e^1 = (1, 0)^\top$ and $e^2 = (0, 1)^\top$.

The proof is as follows. For any $B \in \mathcal{B}(\mathbb{R}^2)$, $\mu(B) = pN_2(0, \mathrm{diag}(1, 1))(B) + (1-p)\rho(B)$, where ρ is a degenerate Gaussian concentrated on the line $L_1 = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 = x_2\}$. Suppose that μ is infinitely divisible. Then the projection σ of μ onto the line $L_2 = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 = -x_2\}$ has to be infinitely divisible by Proposition 11.10 of [22]. But σ is a mixture of a Gaussian distribution and a point mass at the origin, which is not infinitely divisible by Remark 26.3 of [22].

The next result shows that when K has a strong basis any convolution semigroup is generative, and we give a characterization of the unique-generative convolution semigroups.

Theorem 4.7. *Let K have a strong basis $\{e^1, \ldots, e^N\}$ and let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Let $Y_s = V^1_{s_1} + \cdots + V^N_{s_N}$ for $s = s_1 e^1 + \cdots + s_N e^N \in K$, where $\{V^j_t \colon t \geqslant 0\}$, $j = 1, \ldots, N$, are independent Lévy processes satisfying $\mathcal{L}(V^j_1) = \mu_{e^j}$ for $j = 1, \ldots, N$.*

(i) *The semigroup $\{\mu_s\}$ is generative. In particular, $\{Y_s \colon s \in K\}$ is a K-parameter Lévy process associated with $\{\mu_s\}$.*

(ii) *The following three statements* (a)–(c) *are equivalent:*

(a) *$\{\mu_s\}$ is unique-generative.*

(b) *Any K-parameter Lévy process in law $\{X_s \colon s \in K\}$ associated with $\{\mu_s \colon s \in K\}$ satisfies $\{X_s \colon s \in K\} \stackrel{\mathrm{d}}{=} \{Y_s \colon s \in K\}$.*

(c) *For any K-parameter Lévy process in law $\{X_s \colon s \in K\}$ associated with $\{\mu_s \colon s \in K\}$ and any $s = s_1 e^1 + \cdots + s_N e^N \in K$ we have $X_s = X_{s_1 e^1} + \cdots + X_{s_N e^N}$ a. s.*

(iii) *If* $\{\mu_s\}$ *is unique-generative, then any* K*-parameter Lévy process in law* $\{X_s \colon s \in K\}$ *associated with* $\{\mu_s\}$ *has a* K*-parameter Lévy process modification.*

Remark 4.8. (i) We do not know whether every K-parameter Lévy process in law has a K-parameter Lévy process modification.

(ii) From Theorem 4.7 (ii) it follows that if $\{\mu_s \colon s \in K\}$ is unique-generative, then the N processes $\{X_{te^1} \colon t \geqslant 0\}, \dots, \{X_{te^N} \colon t \geqslant 0\}$ are independent whenever $\{X_s \colon s \in K\}$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$.

Proof of Theorem 4.7. (i) Example 3.8 shows that $\{Y_s\}$ is a K-parameter Lévy process. To see that it is associated with $\{\mu_s\}$, note that for $s = s_1 e^1 + \dots + s_N e^N$ we have $\mathcal{L}(Y_s) = \mathcal{L}(V^1_{s_1}) * \dots * \mathcal{L}(V^N_{s_N}) = \mu^{s_1}_{e^1} * \dots * \mu^{s_N}_{e^N} = \mu_s$.

(ii) It follows directly from (i) that (a) and (b) are equivalent. Assume that $\{\mu_s\}$ is unique-generative. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$ and let $s = s_1 e^1 + \dots + s_N e^N \in K$. Then, from (b),

$$P(X_{s_1 e^1 + \dots + s_N e^N} = X_{s_1 e^1} + \dots + X_{s_N e^N}) = P(Y_{s_1 e^1 + \dots + s_N e^N} = Y_{s_1 e^1} + \dots + Y_{s_N e^N})$$

and since this probability trivially is 1, we get (c).

Conversely, assume that (c) holds. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. Let $n \geqslant 1$ and $0 = s_0 \leqslant s_1 \leqslant \dots \leqslant s_n$. Define random vectors $Z_{i,j}$ for $i = 1, \dots, N, j = 0, \dots, n$ by

$$Z_{i,j} = X_{s_n e^1 + \dots + s_n e^{i-1} + s_j e^i}.$$

Thus, $Z_{i,0} = Z_{i-1,n}$ for $i \geqslant 2$ and $Z_{1,0} = 0$. If follows from (ii) of Definition 3.1 that $Z_{i,j} - Z_{i,j-1}$ with $i = 1, \dots, N$ and $j = 1, \dots, n$ are independent. Since

$$Z_{i,j} = X_{s_n e^1} + \dots + X_{s_n e^{i-1}} + X_{s_j e^i} \quad a.\,s.$$

by (c), we see that $X_{s_j e^i} - X_{s_{j-1} e^i}$ with $i = 1, \dots, N$ and $j = 1, \dots, n$ are independent. Since this holds for arbitrary $n \geqslant 1$ and $0 \leqslant s_1 \leqslant \dots \leqslant s_n$, $\{X_{te^1} \colon t \geqslant 0\}, \dots, \{X_{te^N} \colon t \geqslant 0\}$ are independent Lévy processes in law with $\mathcal{L}(X_{e^j}) = \mu_{e^j}$ for all j. Choosing their modifications which are Lévy processes we now see that (b) holds.

(iii) Let $\{\mu_s\}$ be unique-generative. Let $\{X_s\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. Since $\{X_{te^j} \colon t \geqslant 0\}$ is a Lévy process in law by Remark 3.11, it has a Lévy process modification $\{U^j_t \colon t \geqslant 0\}$. For simplicity let $\{U^j_t \colon t \geqslant 0\}$ be chosen such that all paths are cadlag. For $s = s_1 e^1 + \dots + s_N e^N \in K$ define X'_s as $X'_s = U^1_{s_1} + \dots + U^N_{s_N}$. Then $\{X'_s \colon s \in K\}$ is a modification of $\{X_s \colon s \in K\}$

by (c). We claim that all paths of $\{X'_s\colon s \in K\}$ are K-cadlag. Indeed, K-right continuity follows from right continuity of U^j_t. If $s^n = s^n_1 e^1 + \cdots + s^n_N e^N$ is K-increasing, $s^n \in K \setminus \{s^0\}$ and $s^n \to s^0 = s^0_1 e^1 + \cdots + s^0_N e^N$, then, by Lemma 2.14, there exists a unique nonempty subset a of $\{1, \ldots, N\}$ such that, $s^n \uparrow_a s^0$. Therefore, $\lim_{n\to\infty} X'_{s^n} = \sum_{j\notin a} U^j_{s^0_j} + \sum_{j\in a} \lim_{n\to\infty} U^j_{s^n_j}$ exists. □

Corollary 4.9. *Let K have a strong basis. If $\{\mu_s\colon s \in K\}$ is unique-generative and $\{X_s\colon s \in K\}$ is a K-parameter Lévy process in law associated with it, then any finite-dimensional marginal of $\{X_s\colon s \in K\}$ is infinitely divisible.*

This is a consequence of (ii) of the theorem above. This fact should be compared with Remark 4.6.

Next we give a sufficient condition for $\{\mu_s\colon s \in K\}$ to be unique-generative. Recall that a subset L of $\mathbb{R}^d$ is an additive subgroup if $x - y \in L$ whenever x and y are in L. For instance, a linear subspace is an additive subgroup. As another example note that $\mathbb{Q}$ is an additive subgroup of $\mathbb{R}$; in particular we see that additive subgroups need not be closed.

Theorem 4.10. *Let K have a strong basis $\{e^1, \ldots, e^N\}$ and $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. For $j = 1, \ldots, N$ let L_j be an additive subgroup of $\mathbb{R}^d$ such that $L_j \in \mathcal{B}(\mathbb{R}^d)$. Assume that for all $i \neq j$ we have $L_i \cap L_j = \{0\}$. Let $\mu_{te^j}(L_j) = 1$ for $t \geqslant 0$ and $j = 1, \ldots, N$. Then $\{\mu_s\}$ is unique-generative.*

Proof. We use induction in N. In the case $N = 1$ the theorem is trivially true. Assume that the theorem holds for $N - 1$ in place of N. Let $\{X_s\colon s \in K\}$ be a K-parameter Lévy process in law associated with $\{\mu_s\}$. By Theorem 4.7 it is enough to verify condition (c). Consider the $(N - 1)$-dimensional cones K_1 and K_2 generated by $\{e^2, \ldots, e^N\}$ and by $\{e^1, e^3, \ldots, e^N\}$, respectively. Then, by the induction hypothesis, both $\{\mu_s\colon s \in K_1\}$ and $\{\mu_s\colon s \in K_2\}$ are unique-generative. The restrictions $\{X_s\colon s \in K_1\}$ and $\{X_s\colon s \in K_2\}$ are associated with $\{\mu_s\colon s \in K_1\}$ and $\{\mu_s\colon s \in K_2\}$, respectively. Let $s = s_1 e^1 + \cdots + s_N e^N \in K$ and define $s^1 = s - s_1 e^1 \in K_1$ and $s^2 = s - s_2 e^2 \in K_2$. Using condition (c) for the two restrictions, we decompose X_s as

$$(4.1) \quad X_s = X_{s^1} + (X_s - X_{s^1}) \overset{a.s.}{=} X_{s_2 e^2} + \cdots + X_{s_N e^N} + (X_s - X_{s^1}),$$

$$(4.2) \quad X_s = X_{s^2} + (X_s - X_{s^2}) \overset{a.s.}{=} X_{s_1 e^1} + X_{s_3 e^3} + \cdots + X_{s_N e^N} + (X_s - X_{s^2}).$$

By equating (4.1) and (4.2) it follows that $(X_s - X_{s^1}) - X_{s_1 e^1} \overset{a.s.}{=} (X_s - X_{s^2}) - X_{s_2 e^2}$. The left-hand side is concentrated on L_1 and the right-hand side on L_2. Therefore, $X_s - X_{s^1} = X_{s_1 e^1}$ a.s. Inserting this in (4.1) we get the a.s. identity in (c) for $\{X_s \colon s \in K\}$. $\square$

Example 4.11. (i) In the case $N = 2$ the additive subgroups $L_1 = \mathbb{Q}^d$ and $L_2 = (c\mathbb{Q})^d$ with $c \in \mathbb{R} \setminus \mathbb{Q}$ satisfy the condition $L_1 \cap L_2 = \{0\}$.

(ii) Let the setting be as in Example 3.8 (iii). Let $\{\mu_s \colon s \in K\}$ be the K-parameter convolution semigroup defined by $\mu_s = \mathcal{L}(X_s)$ for $s \in K$. Then, by the theorem above, $\{\mu_s \colon s \in K\}$ is unique-generative.

In the following lemma we discuss the consequences of changing the location parameters in the triplets of a K-parameter convolution semigroup. The proof is left to the reader. Then we study the problem of non-generativeness.

Lemma 4.12. *Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Let $\{\gamma_s^\sharp \colon s \in K\}$ be a family of constants in $\mathbb{R}^d$ such that $\gamma^\sharp_{s^1+s^2} = \gamma^\sharp_{s^1} + \gamma^\sharp_{s^2}$ for $s^1, s^2 \in K$ and $\gamma^\sharp_{ts} = t\gamma^\sharp_s$ for $s \in K, t \geqslant 0$. Let $\mu^\sharp_s = \mu_s * \delta_{\gamma^\sharp_s}$. Then $\{\mu^\sharp_s \colon s \in K\}$ is a convolution semigroup. Moreover, $\{\mu_s \colon s \in K\}$ is unique-generative (resp. non-generative, multiple-generative) if and only if $\{\mu^\sharp_s \colon s \in K\}$ is unique-generative (resp. non-generative, multiple-generative).*

Theorem 4.13. *Let $K = M^+_{d\times d}$ with $d \geqslant 2$. Let $\{\mu_s \colon s \in K\}$ be a nontrivial K-parameter convolution semigroup on $\mathbb{R}^d$ such that $\int |x|^2 \mu_s(dx) < \infty$ and the covariance matrix v_s of μ_s satisfies $v_s \leqslant_K s$ for all $s \in K$. Then $\{\mu_s\}$ is non-generative. In particular, the canonical $M^+_{d\times d}$-parameter convolution semigroup defined in Example 3.9 is non-generative.*

Proof. The mean m_s of μ_s satisfies $m_{s^1+s^2} = m_{s^1} + m_{s^2}$ and $m_{ts} = tm_s$. Hence, by Lemma 4.12 we may and do assume that μ_s has mean zero. The covariance matrix satisfies $v_{s^1+s^2} = v_{s^1} + v_{s^2}$ and $v_{ts} = tv_s$.

Step 1. Proof in the case $d = 2$. Suppose there exists a K-parameter Lévy process in law $\{X_s \colon s \in K\}$ on $\mathbb{R}^2$ associated with $\{\mu_s\}$. Let

$$e^1 = \begin{pmatrix} 1 & 2^{1/2} \\ 2^{1/2} & 2 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 2 & 2^{1/2} \\ 2^{1/2} & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 2 & 2^{1/2} \\ 2^{1/2} & 2 \end{pmatrix}.$$

Let K_0 be the cone generated by $\{e^1, e^2\}$. Since e^1 and e^2 have rank one, there are $t_1, t_2 \in [0,1]$ such that $v_{e^1} = t_1 e^1$ and $v_{e^2} = t_2 e^2$. This is easily seen using diagonalization by orthogonal matrices. It follows that for any $t \geqslant 0$, μ_{te^1} and μ_{te^2} are concentrated on L_1 and L_2, respectively, where $L_1 = \{(a, 2^{1/2}a) \colon a \in \mathbb{R}\}$ and $L_2 = \{(2^{1/2}a, a) \colon a \in \mathbb{R}\}$. Hence, by Theorem 4.10, the restriction $\{\mu_s \colon s \in K_0\}$ is unique-generative. Since $\{X_s \colon s \in K_0\}$ is a K_0-parameter Lévy process in law associated with $\{\mu_s \colon s \in K_0\}$, it follows from Remark 4.8 that X_{e^1} and X_{e^2} are independent. Let $(X_s)_j$ denote the jth coordinate of X_s. Since $v_{e^3-e^1} \leqslant_K e^3 - e^1 = \operatorname{diag}(1,0)$ and since $X_{e^3} - X_{e^1} \stackrel{\mathrm{d}}{=} X_{e^3-e^1}$, we have $(X_{e^3} - X_{e^1})_2 = 0$ a.s. Similarly, $(X_{e^3} - X_{e^2})_1 = 0$ a.s. Now, using $X_{e^3} = X_{e^j} + (X_{e^3} - X_{e^j})$ for $j = 1, 2$, we get $(X_{e^3})_1 = (X_{e^2})_1$ and $(X_{e^3})_2 = (X_{e^1})_2$ a.s. Hence $(X_{e^3})_2$ and $(X_{e^3})_1$ are independent. It follows that v_{e^3} is diagonal, say, $v_{e^3} = \operatorname{diag}(a_1, a_2)$ with $a_1, a_2 \geqslant 0$. We have $v_{e^3-e^1} = \operatorname{diag}(t, 0)$ with $t \geqslant 0$ since $v_{e^3-e^1} \leqslant_K e^3 - e^1$. Now, looking at nondiagonal entries of $v_{e^1} = v_{e^3} - v_{e^3-e^1}$ and $v_{e^1} = t_1 e^1$, we conclude that $t_1 = 0$. Thus $v_{e^1} = 0$. Hence $v_{e^3} = v_{e^3-e^1} \leqslant_K e^3 - e^1$, which shows that $a_2 = 0$. The same kind of argument gives $a_1 = 0$ and $v_{e^2} = v_{e^3} = 0$. It follows that $\mu_{e^1} = \mu_{e^2} = \mu_{e^3} = \delta_0$. Since the system $\{e^1, e^2, e^3\}$ is linearly independent, it is a weak basis of K. Hence, by Theorem 3.13, $\mu_s = \delta_0$ for all $s \in K$, contradicting the assumption of nontriviality. Therefore, the associated Lévy process in law does not exist.

Step 2. Proof in the case $d \geqslant 2$. Suppose that we can find a K-parameter Lévy process in law $\{X_s \colon s \in K\}$ on $\mathbb{R}^d$ associated with $\{\mu_s \colon s \in K\}$. Since $\{\mu_s\}$ is nontrivial, there is $s^0 \in K$ such that $v_{s^0} \neq 0$. Let $p = \operatorname{rank}(s^0)$. Then $p \geqslant 1$. Using diagonalization, we can decompose s^0 as $s^0 = s^1 + \cdots + s^p$, where, for each $j \geqslant 1$, $s^j \in K$ and $\operatorname{rank}(s^j) = 1$. Since $v_{s^0} = v_{s^1} + \cdots + v_{s^p}$, we have $v_{s^j} \neq 0$ for some $j \geqslant 1$. Thus we may and do assume that $\operatorname{rank}(s^0) = 1$ and $v_{s^0} \neq 0$. There is a $d \times d$ orthogonal matrix r such that $rs^0r' = \operatorname{diag}(a, 0, \ldots, 0)$ with $a > 0$, where r' is the transpose of r. Define

$$
\begin{aligned}
K_0 &= \{s = (s_{jk})_{j,k=1}^d \in K \colon s_{jk} = 0 \text{ except for } j,k \in \{1,2\}\},\\
K_1 &= \{r'sr \colon s \in K_0\}.
\end{aligned}
$$

Then K_1 is a cone and $s^0 \in K_1$.

Notice that $\operatorname{cov}(rX_s) = rv_sr'$ for $s \in K$, since $\operatorname{cov}(X_s) = v_s$. If $s \in K_1$, then $rv_sr' \leqslant_K rsr' \in K_0$ and hence $rv_sr' \in K_0$. Therefore, if $s \in K_1$, then $(rX_s)_j = 0$ a.s. for $j \neq 1, 2$.

For $u \in M^+_{2\times 2}$ let $T_0u \in K_0$ be the natural extension of u and let $Tu = r'(T_0u)r$. Then T is an isomorphism from $M^+_{2\times 2}$ to K_1. Define $X^0_u = ((rX_{Tu})_1, (rX_{Tu})_2)^\top$ for $u \in M^+_{2\times 2}$. Then $\{X_{Tu} \colon u \in M^+_{2\times 2}\}$ is an $M^+_{2\times 2}$-parameter Lévy process in law on

$\mathbb{R}^d$, and such is $\{rX_{Tu}\colon u \in M^+_{2\times 2}\}$. It follows that $\{X^0_u\colon u \in M^+_{2\times 2}\}$ is an $M^+_{2\times 2}$-parameter Lévy process in law on $\mathbb{R}^2$. Let $\mu^0_u = \mathcal{L}(X^0_u)$. Then $\{\mu^0_u\colon u \in M^+_{2\times 2}\}$ is an $M^+_{2\times 2}$-parameter convolution semigroup on $\mathbb{R}^2$ and $\mathrm{cov}(\mu^0_u)$ equals the restriction of $rv_{Tu}r'$ to the first 2×2 block. Since $rv_{Tu}r' \leqslant_K r(Tu)r' = T_0 u \in K_0$, we see that $\mathrm{cov}(\mu^0_u) \leqslant_{M^+_{2\times 2}} u$. We have $\mathrm{cov}(\mu^0_{u^0}) \neq 0$, where u^0 is chosen so that $Tu^0 = s^0$. But this is impossible in view of Step 1. Hence, $\{X_s\colon s \in K\}$ does not exist. $\square$

Example 4.14. Let $K = M^+_{2\times 2}$ and $\mu_s = N_2(0,s)$. Note that $M^+_{2\times 2}$ has a weak basis $\{e^1, e^2, e^3\}$, where

$$e^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ e^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ e^3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let K_0 be the cone generated by $\{e^1, e^2, e^3\}$. Then, from Theorem 4.10 it follows that $\{\mu_s\colon s \in K_0\}$ is a unique-generative K_0-parameter convolution semigroup. Note also that, by Theorem 4.7 (ii), any K_0-parameter Lévy process in law associated with $\{\mu_s\colon s \in K_0\}$ is identical in law with

$$\left\{(V^1_{s_1}, 0)^\top + (0, V^2_{s_2})^\top + (V^3_{s_3}, V^3_{s_3})^\top\colon s = s_1e^1 + s_2e^2 + s_3e^3 \in K_0\right\},$$

where $\{V^1_t : t \geqslant 0\}, \{V^2_t\colon t \geqslant 0\}$ and $\{V^3_t\colon t \geqslant 0\}$ are independent standard Wiener processes on $\mathbb{R}$. In particular, it follows that any K_0-parameter Lévy process in law associated with $\{\mu_s\colon s \in K_0\}$ has a continuous modification.

Remark 4.15. Let K be a circular cone in $\mathbb{R}^3$. Then there is a Gaussian K-parameter convolution semigroup on $\mathbb{R}^2$ which is non-generative.

Indeed, by Proposition 2.10 and Remark 3.10 we may assume that $K = \{u = (u_1, u_2, u_3)^\top \in \mathbb{R}^3 : u_1^2 + u_2^2 \leqslant u_3^2, u_3 \geqslant 0\}$. Then, by Example 2.12, K is isomorphic to the cone $M^+_{2\times 2}$. Let $T\colon K \to M^+_{2\times 2}$ be an isomorphism. For $u \in K$ let $\mu_u = N_2(0, Tu)$. Since the canonical $M^+_{2\times 2}$-parameter convolution semigroup is non-generative, so is the K-parameter convolution semigroup $\{\mu_u\}$.

In the direction converse to Theorem 4.10 we consider the following question: When is a K-parameter convolution semigroup for K with a strong basis multiple-generative ?

Theorem 4.16. *Let K have a strong basis $\{e^1, \ldots, e^N\}$. Let $\{\mu_s : s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ with triplet (A_s, ν_s, γ_s). Assume that for some i and k with $i \neq k$ we have either* (i) *or* (ii)*, where*

(i) $A_{e^i}(\mathbb{R}^d) \cap A_{e^k}(\mathbb{R}^d) \neq \{0\}$*;*

(ii) ν_{e^i} *and* ν_{e^k} *are not mutually singular.*

Then $\{\mu_s : s \in K\}$ *is multiple-generative.*

To prove this result we need two lemmas, the proof of which are left to the reader.

Lemma 4.17. *Assume* $A_{e^1}(\mathbb{R}^d) \cap A_{e^2}(\mathbb{R}^d) \neq \{0\}$*. Then there exist three symmetric nonnegative-definite matrices* A^0, A^1, A^2 *such that* A^0 *is nonzero,* $A_{e^1} = A^0 + A^1$ *and* $A_{e^2} = A^0 + A^2$*.*

Lemma 4.18. *Assume that the Lévy measures* ν_{e^1} *and* ν_{e^2} *are not mutually singular. Then there exist three Lévy measures* ν^0, ν^1 *and* ν^2 *on* $\mathbb{R}^d$*, such that* ν^0 *is nontrivial,* $\nu_{e^1} = \nu^0 + \nu^1$ *and* $\nu_{e^2} = \nu^0 + \nu^2$*.*

Proof of Theorem 4.16. Let us for simplicity assume that either (i) or (ii) holds with $i = 1$ and $k = 2$. Then, by virtue of the two lemmas above, there exist three generating triplets (A^j, ν^j, γ^j), $j = 0, 1, 2$, such that A^0 or ν^0 is non-zero and such that $(A_{e^j}, \nu_{e^j}, \gamma_{e^j}) = (A^0 + A^j, \nu^0 + \nu^j, \gamma^0 + \gamma^j)$ for $j = 1, 2$. Let $\{V_t^j : t \geqslant 0\}, j = 0, \ldots, N$, be independent Lévy processes on $\mathbb{R}^d$ such that $\mathcal{L}(V_1^j)$ has triplet (A^j, ν^j, γ^j) for $j = 0, 1, 2$ and $\mathcal{L}(V_1^j)$ has triplet $(A_{e^j}, \nu_{e^j}, \gamma_{e^j})$ for $j = 3, \ldots, N$. Define $\{X_s : s \in K\}$ by $X_s = V^0_{s_1+s_2} + V^1_{s_1} + \cdots + V^N_{s_N}$ for $s = s_1 e^1 + \cdots + s_N e^N \in K$. Then $\{X_s\}$ is a K-parameter Lévy process by Lemma 3.6 and Example 3.8, and it is associated with $\{\mu_s\}$. Since $\{V_t^0\}$ is a non-trivial Lévy process, $\{X_{te^1}\}$ and $\{X_{te^2}\}$ are not independent. Thus, by Remark 4.8 (ii), $\{\mu_s \colon s \in K\}$ is multiple-generative. □

We have the following necessary and sufficient condition that a semigroup is unique-generative when K has a strong basis and the semigroup is Gaussian.

Theorem 4.19. *Let* K *have a strong basis* $\{e^1, \ldots, e^N\}$ *and* $\{\mu_s : s \in K\}$ *be a* K*-parameter convolution semigroup on* $\mathbb{R}^d$*. Let* μ_s *be Gaussian, that is* μ_s *has generating triplet* $(A_s, 0, \gamma_s)$*, for* $s \in K$*. Then* $\{\mu_s\}$ *is unique-generative if and only if for all* $i \neq j$ *we have* $A_{e^i}(\mathbb{R}^d) \cap A_{e^j}(\mathbb{R}^d) = \{0\}$*.*

Proof. If for some $i \neq j$ we have $A_{e^i}(\mathbb{R}^d) \cap A_{e^j}(\mathbb{R}^d) \neq \{0\}$ then by Theorem 4.16 $\{\mu_s\}$ is multiple-generative. Conversely assume that $A_{e^i}(\mathbb{R}^d) \cap A_{e^j}(\mathbb{R}^d) = \{0\}$ for all $i \neq j$. Let $L_j = A_{e^j}(\mathbb{R}^d)$ for $j = 1, \ldots, N$. Let $\mu_s^\sharp = \mu_s * \delta_{-\gamma_s}$. Then $\mu^\sharp_{te^j}(L_j) = 1$ for every $t \geqslant 0$ and j. By Theorem 4.10 the convolution semigroup $\{\mu_s^\sharp\}$ is unique-generative, and by Lemma 4.12 the same holds for $\{\mu_s\}$. □

In another direction, we now let K be a general cone and consider the case where μ_s is purely non-Gaussian.

Lemma 4.20. *Let $\{e^1, \dots, e^N\}$ be a weak basis of K and let $\{\mu_s \colon s \in K\}$ be a convolution semigroup such that μ_s has triplet $(0, \nu_s, 0)$ for $s \in K$. Let $\nu = \nu_{e^1} + \cdots + \nu_{e^N}$. Then, for each $s \in K$, ν_s is absolutely continuous with respect to ν. Moreover, the family $\{\phi_s \colon s \in K\}$ of densities ϕ_s of ν_s with respect to ν can be chosen such that*

(i) $\phi_{e^1}(x) + \cdots + \phi_{e^N}(x) \leqslant 1$ *for* $x \in \mathbb{R}^d$,

(ii) $\phi_s(x) = s_1 \phi_{e^1}(x) + \cdots + s_N \phi_{e^N}(x)$ *for* $s \in K$ *and* $x \in \mathbb{R}^d$,

(iii) $s^n \to s$ *implies* $\phi_{s^n}(x) \to \phi_s(x)$ *for* $x \in \mathbb{R}^d$,

(iv) $\phi_s(x) \geqslant 0$ *for* $s \in K$ *and* $x \in \mathbb{R}^d$.

Proof. Let $s = s_1 e^1 + \cdots + s_N e^N$ and let $K_0 = \{s \in K \colon s_1, \dots, s_N \in \mathbb{Q}\}$. Note that $e^1, \dots, e^N \in K_0$. Since $\nu_s = s_1 \nu_{e^1} + \cdots + s_N \nu_{e^N}$ by Theorem 3.13 it follows that ν_s is absolutely continuous with respect to ν. Fix a density ϕ_s^0 of ν_s with respect to ν. Then

(4.3) $\phi^0_{e^1}(x) + \cdots + \phi^0_{e^N}(x) = 1,\ \phi^0_s(x) = s_1 \phi^0_{e^1}(x) + \cdots + s_N \phi^0_{e^N}(x),\ \phi^0_s(x) \geqslant 0,$

each holding for ν-almost every x. Let $B = \{x \in \mathbb{R}^d \colon$ (4.3) holds for all $s \in K_0\}$. Then $\nu(\mathbb{R}^d \setminus B) = 0$. Define

$$
\begin{aligned}
&\phi_s(x) = \phi^0_s(x) \text{ for } s \in K_0 \text{ and } x \in B,\\
&\phi_s(x) = s_1 \phi^0_{e^1}(x) + \cdots + s_N \phi^0_{e^N}(x) \text{ for } s \in K \setminus K_0 \text{ and } x \in B,\\
&\phi_s(x) = 0 \text{ for } s \in K \text{ and } x \in \mathbb{R}^d \setminus B.
\end{aligned}
$$

Then, ϕ_s is a density of ν_s with respect to ν; (i) and (ii) are from the definition of ϕ_s; (iii) is from (ii) since $s^n \to s$ if and only if $s^n_j \to s$ for $j = 1, \dots, N$; (iv) is from the definition for $s \in K_0$ and by approximation using (iii) for s for $s \in K \setminus K_0$. $\square$

Consider the family $\{\phi_s \colon s \in K\}$ of densities of Lemma 4.20 and define, for $s \in K$,

$$
D_s = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \colon 0 \leqslant t \leqslant \phi_s(x)\}. \tag{4.4}
$$

Theorem 4.21. *Let K be an arbitrary cone. Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ such that μ_s is purely non-Gaussian for all s, that is μ_s has triplet $(0, \nu_s, \gamma_s)$. Then $\{\mu_s\}$ is generative.*

To construct an associated K-parameter Lévy process in law, let $\{J(A) \colon A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)\}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, be a Poisson random measure with intensity measure $\lambda(d(t,x)) = dt\,\nu(dx)$, where $\nu = \nu_{e^1} + \cdots + \nu_{e^N}$. For $s \in K$

define

$$(4.5)\quad X_s = \int_{D_s} x1_{\{|x|\leqslant 1\}}(x)(J(d(t,x)) - \lambda(d(t,x))) + \int_{D_s} x1_{\{|x|>1\}}(x)J(d(t,x)) + \gamma_s.$$

Then $\{X_s\colon s \in K\}$ *is a* K*-parameter Lévy process in law associated with* $\{\mu_s\}$*.*

If, in addition, $\int_{\mathbb{R}^d}(1 \wedge |x|)\nu_s(dx) < \infty$ *for all* $s \in K$*, then* $\{X_s\colon s \in K\}$ *is a* K*-parameter Lévy process.*

The first integral on the right-hand side of (4.5) is a stochastic integral only determined up to null sets. Hence, we may change $X_s(\omega)$ on a null set of ω's while (4.5) remains true. Thus, the last statement says that it is possible to choose $X_s(\omega)$ for $\omega \in \Omega$ and $s \in K$ such that all paths are K-cadlag.

Proof of the theorem. According to Lemma 4.12 we may and do assume $\gamma_s = 0$ for all s. Let $D_s^1 = D_s \cap \{(t,x)\colon |x| \leqslant 1\}$, $D_s^2 = D_s \cap \{(t,x)\colon |x| > 1\}$, $f_s^1(t,x) = x1_{D_s^1}(t,x)$ and $f_s^2(t,x) = x1_{D_s^2}(t,x)$. Let $U_s^1 = \int f_s^1(t,x)(J(d(t,x)) - \lambda(d(t,x)))$ and $U_s^2 = \int f_s^2(t,x)J(d(t,x))$. That is, U_s^j is the jth term on the right-hand side of (4.5) for $j = 1, 2$. Using $d\nu_s = \phi_s d\nu$ it follows that

$$\begin{aligned} \lambda(D_s^2) &= \nu_s(\{x\colon |x| > 1\}) < \infty, \\ \int |f_s^1|^2(t,x)\lambda(d(t,x)) &= \int_{|x|\leqslant 1} |x|^2\nu_s(dx) < \infty. \end{aligned}$$

Hence, U_s^2 exists as Lebesgue-Stieltjes integral with respect to $J(d(t,x))$ while U_s^1 exists as stochastic integral with respect to the compensated measure $J(d(t,x)) - \lambda(d(t,x))$. Moreover, it is well-known that for $s^1, s^2 \in K$ and $z \in \mathbb{R}^d$ we have

$$\begin{aligned} (4.6)\quad Ee^{\mathrm{i}\langle z, U_{s^2}^1 - U_{s^1}^1\rangle} &= \exp\int \left(e^{\mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t,x)\rangle} - 1 - \mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t,x)\rangle\right)\lambda(d(t,x)), \\ (4.7)\quad Ee^{\mathrm{i}\langle z, U_{s^2}^2 - U_{s^1}^2\rangle} &= \exp\int \left(e^{\mathrm{i}\langle z, (f_{s^2}^2 - f_{s^1}^2)(t,x)\rangle} - 1\right)\lambda(d(t,x)). \end{aligned}$$

Step 1. Let $s^1, s^2 \in K$ with $s^1 \leqslant_K s^2$. Then, $D_{s^1} \subseteq D_{s^2}$ and

$$(4.8)\quad (f_{s^2}^1 - f_{s^1}^1)(t,x) = x1_{D_{s^2}^1 \setminus D_{s^1}^1}(t,x) = \begin{cases} x1_{\{|x|\leqslant 1\}}(x) & \text{if } \phi_{s^1}(x) < t \leqslant \phi_{s^2}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, using $\phi_{s^2} - \phi_{s^1} = \phi_{s^2 - s^1}$ and $\nu_{s^2 - s^1}(dx) = \phi_{s^2 - s^1}(x)\nu(dx)$, we find that

$$\begin{aligned} &\int \left(e^{\mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t,x)\rangle} - 1 - \mathrm{i}\langle z, (f_{s^2}^1 - f_{s^1}^1)(t,x)\rangle\right)\lambda(d(t,x)) \\ = &\int_{\mathbb{R}^d} x1_{\{|x|\leqslant 1\}}(x)\left(e^{\mathrm{i}\langle z, x\rangle} - 1 - \mathrm{i}\langle z, x\rangle\right)\nu_{s^2 - s^1}(dx). \end{aligned}$$

Inserting this in (4.6) we find that $\mathcal{L}(U^1_{s^2} - U^1_{s^1})$ has triplet $(0, 1_{\{|x|\leqslant 1\}}(x)\nu_{s^2-s^1}(dx), 0)$. Similar arguments show that $\mathcal{L}(U^2_{s^2} - U^2_{s^1})$ has triplet $(0, 1_{\{|x|>1\}}(x)\nu_{s^2-s^1}(dx), 0)$.

Step 2. Let $n \geqslant 2$ and $\{s^j\}_{j=1,\dots,n}$ be K-increasing. Then $D^j_{s^{k-1}} \subseteq D^j_{s^k}$ and $(f^j_{s^k} - f^j_{s^{k-1}})(t,x) = x1_{D^j_{s^k}\setminus D^j_{s^{k-1}}}(t,x)$ for $j = 1,2$ and $k = 2,\dots,n$. Hence, since the sets $D^1_{s^2}\setminus D^1_{s^1},\dots,D^1_{s^n}\setminus D^1_{s^{n-1}}, D^2_{s^2}\setminus D^2_{s^1},\dots,D^2_{s^n}\setminus D^2_{s^{n-1}}$ are disjoint, $U^j_{s^k} - U^j_{s^{k-1}}$, $j = 1,2, k = 2,\dots,n$, are independent; consequently also $X_{s^k} - X_{s^{k-1}} = (U^1_{s^k} - U^1_{s^{k-1}}) + (U^2_{s^k} - U^2_{s^{k-1}})$, $k = 2,\dots,n$, are independent. Moreover, by Step 1, $\mathcal{L}(X_{s^k} - X_{s^{k-1}}) = \mu_{s^k - s^{k-1}}$.

Step 3. Let $s^n, s \in K$ with $s^n \to s$. By Lemma 4.20 (iii) we have $\phi_{s^n}(x) \to \phi_s(x)$ for all $x \in \mathbb{R}^d$. Hence, $1_{D_{s^n}}(t,x) \to 1_{D_s}(t,x)$ for λ-a.e. (t,x). Moreover, by Lemma 4.20 (i),(ii),(iv) it follows that

$$0 \leqslant \phi_r(x) \leqslant |r_1| + \cdots + |r_N| \text{ for } r = r_1e^1 + \ldots + r_Ne^n \in K. \tag{4.9}$$

Decompose s^n and s as $s^n = s^n_1e^1 + \cdots + s^n_Ne^N$ and $s = s_1e^1 + \cdots + s_Ne^N$. Since $s^n_j \to s_j$ for all $j = 1,\dots,N$, (4.9) shows that there exists a constant $c > 0$ such that $1_{D_s}(t,x)$, $1_{D_{s^n}}(t,x) \leqslant 1_{[0,c]}(t)$. Since

$$\begin{aligned}
|e^{\mathrm{i}\langle z,(f^1_{s^n}-f^1_s)(t,x)\rangle} - 1 - \mathrm{i}\langle z,(f^1_{s^n} - f^1_s)(t,x)\rangle| &\leqslant \tfrac{1}{2}|\langle z,(f^1_{s^n} - f^1_s)(t,x)\rangle|^2 \\
&\leqslant \tfrac{1}{2}|z|^2|(f^1_{s^n} - f^1_s)(t,x)|^2 \leqslant \tfrac{1}{2}|z|^2(2|x|)^2 1_{\{|x|\leqslant 1\}}(x)1_{[0,c]}(t), \\
|e^{\mathrm{i}\langle z,(f_{s^n}-f_s)(t,x)\rangle} - 1| &\leqslant 2\,1_{\{|x|>1\}}(x)1_{[0,c]}(t),
\end{aligned}$$

it follows from (4.6)–(4.7) that $\mathcal{L}(U^j_{s^n} - U^j_s) \to \delta_0$ for $j = 1,2$.

Step 4. Note that by Step 2 $\{X_s\colon s \in K\}$ satisfies (i)–(ii) of Definition 3.1. It is immediate that $X_0 = 0$ a.s. Since $X_s = U^1_s + U^2_s$ it follows from Step 3 that $\{X_s\colon s \in K\}$ is continuous in probability. Thus, we have shown that $\{X_s\colon s \in K\}$ is a K-parameter Lévy process in law. Moreover, it is associated with $\{\mu_s\colon s \in K\}$ since we have $\mathcal{L}(X_s) = \mu_s$ for $s \in K$ by Step 2.

Step 5. Now assume in addition that $\int_{\mathbb{R}^d}(1 \wedge |x|)\nu_s(dx) < \infty$ for all s. Let $(T_1,Y_1),(T_2,Y_2),\dots,$ be a random sequence such that $J(d(t,x)) = \sum_m \delta_{(T_m,Y_m)}(d(t,x))$ a.s. Then, using (4.5) we have that

$$X_s = \sum_{m:T_m\leqslant\phi_s(Y_m)} Y_m - \int_{|x|\leqslant 1} x\phi_s(x)\nu(dx) \qquad \text{a.s.} \tag{4.10}$$

where $\sum_{m:T_m\leqslant\phi_s(Y_m)}|Y_m| < \infty$ a.s. We stress that X_s is only determined up to null sets by (4.10). Let us define $X_s(\omega)$ such that all paths are K-cadlag. Let $p \in \mathbb{N}$ and define $u^p = p(e^1 + \cdots + e^N) \in K$. Choose a null set $N \in \mathcal{F}$ such that $\sum_{m:T_m(\omega)\leqslant\phi_{u^p}(Y_m(\omega))} Y_m(\omega)$ is absolutely convergent for all $p \in \mathbb{N}$ and $\omega \in N^c$.

Note that if $s \in K$ then there is some $p \in \mathbb{N}$ such that $s \leqslant_K u^p$. Hence, since $\phi_s(x) \leqslant \phi_{u^p}(x)$ by Lemma 4.20 (ii) and (iv), the series $\sum_{m: T_m(\omega) \leqslant \phi_s(T_m(\omega))} Y_m(\omega)$ is absolutely convergent for all $s \in K$ and all $\omega \in N^c$. For $s \in K$ let

$$X_s(\omega) = \begin{cases} \sum_{m: T_m(\omega) \leqslant \phi_s(Y_m(\omega))} Y_m(\omega) - \int_{|x| \leqslant 1} x\phi_s(x)\nu(dx) & \text{if } \omega \in N^c \\ 0 & \text{if } \omega \in N. \end{cases}$$

Note that $s^1 \leqslant_K s^2$ implies $\phi_{s^1} \leqslant \phi_{s^2}$. Using this it follows that all paths of $\{X_s \colon s \in K\}$ are K-cadlag. In fact, the K-left limits can be calculated as follows. Let $\{s^n\}$ in $K \setminus \{s\}$ be K-increasing with $s^n \to s$. Then

$$X_{s^n} \to \sum_{m: T_m \leqslant \phi_{s^n}(Y_m) \text{ for some } n} Y_m - \int_{|x| \leqslant 1} x\phi_s(x)\nu(dx)$$

pointwise on N^c. Thus, $\{X_s \colon s \in K\}$ is a K-parameter Lévy process. $\square$

In the next result we specialize to the case $d = 1$.

Theorem 4.22. *Let K be an arbitrary cone. Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}$. Then $\{\mu_s\}$ is generative.*

Proof. Let (A_s, ν_s, γ_s) be the triplet of μ_s. Here A_s is a nonnegative number. By the previous theorem there exists a K-parameter Lévy process in law $\{X_s^1\}$ associated with the convolution semigroup $\{\widetilde{\mu}_s\}$, where $\widetilde{\mu}_s$ is the distribution with triplet $(0, \nu_s, \gamma_s)$. Let $\{V_t \colon t \geqslant 0\}$ be a standard Wiener process, independent of $\{X_s^1 \colon s \in K\}$. If $s^1 \leqslant_K s^2$, then $A_{s^1} \leqslant A_{s^2}$. Hence, $\{X_s^2 \colon s \in K\}$ defined by $X_s^2 = V_{A_s}$ is a K-parameter Lévy process in law such that $\mathcal{L}(X_s^2)$ has triplet $(A_s, 0, 0)$. Hence, $\{X_s\}$ defined by $X_s = X_s^1 + X_s^2$ is a K-parameter Lévy process in law associated with $\{\mu_s\}$. $\square$

The following fact on $M^+_{d \times d}$-parameter convolution semigroups is a consequence of Theorem 4.21 combined with Theorem 4.13.

Proposition 4.23. *Let $K = M^+_{d \times d}$ with $d \geqslant 2$. Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ such that $\int |x|^2 \mu_s(dx) < \infty$ and $v_s \leqslant_K s$ for all $s \in K$, where v_s is the covariance matrix of μ_s. Then μ_s is Gaussian, that is, the Lévy measure ν_s of μ_s is zero.*

Proof. Let (A_s, ν_s, γ_s) be the triplet of μ_s. Decompose μ_s as $\mu_s = \mu_s' * \mu_s''$, where μ_s' and μ_s'' are infinitely divisible with triplets $(0, \nu_s, \gamma_s)$ and $(A_s, 0, 0)$, respectively. Then μ_s' and μ_s'' have finite second moments and the covariance matrices v_s' and v_s'' of μ_s', μ_s'' satisfy $v_s = v_s' + v_s''$. Hence, $v_s', v_s'' \leqslant_K s$. Since $\{\mu_s'\}$ is a K-parameter convolution semigroup there is a K-parameter Lévy process associated with it by Theorem 4.21. But, Theorem 4.13 says that this is impossible if $\{\mu'_s\}$ is nontrivial. It follows that $\nu_s = 0$. $\square$

Remark 4.24. Let $d \geqslant 1$ and consider the problem of constructing a family of probability measures $\{\mu_s \colon s \in M^+_{d\times d}\}$ on $\mathbb{R}^d$ which is closed under convolution and satisfies that s is the covariance matrix of μ_s. When $d = 1$ let $M^+_{d\times d} = \mathbb{R}_+$. Then the latter condition is that $s \in \mathbb{R}_+$ is the variance of μ_s. In this case there are many such families. In fact, any infinitely divisible distribution on $\mathbb{R}$ with unit variance corresponds to a family with the desired properties.

Let $d \geqslant 2$. It is remarkable that, up to a change of drift, the canonical $M^+_{d\times d}$-parameter convolution semigroup is the only family with the desired properties. Precisely, if $\{\mu_s \colon s \in M^+_{d\times d}\}$ satisfies the conditions stated above, then $\mu_s = \mu^\sharp_s * \delta_{m_s}$, where m_s is the mean of μ_s and $\{\mu^\sharp_s \colon s \in M^+_{d\times d}\}$ is the canonical $M^+_{d\times d}$-parameter convolution semigroup. This follows since $\{\mu_s * \delta_{-m_s} \colon s \in M^+_{d\times d}\}$ is a convolution semigroup on $\mathbb{R}^d$ satisfying the assumptions of the preceding proposition.

5. Subordination of cone-parameter Lévy processes and convolution semigroups

In this section we extend the concept of subordination to the case where subordinators and subordinands have parameters in K_1 and K_2, respectively. Here K_1 is an N_1-dimensional cone in $\mathbb{R}^{M_1}$ and K_2 is an N_2-dimensional cone in $\mathbb{R}^{M_2}$. Then we discuss inheritance of selfdecomposability, the L_m property and stability from subordinator to subordinated. As the subordinators have to be supported on K_2, we begin with the following lemma.

Lemma 5.1. *Let K_2 be a cone in $\mathbb{R}^{M_2}$. Let $\rho \in ID(\mathbb{R}^{M_2})$ with triplet (A, ν, γ). Then* $\mathrm{Supp}(\rho) \subseteq K_2$ *if and only if*

$$A = 0, \quad \nu(\mathbb{R}^{M_2} \setminus K_2) = 0, \quad \int_{K_2 \cap \{|s| \leqslant 1\}} |s| \nu(ds) < \infty, \quad \gamma^0 \in K_2. \tag{5.1}$$

Here we recall that $\gamma^0 = \gamma - \int_{K_2 \cap \{|s| \leqslant 1\}} s \nu(ds)$, the drift of ρ. The lemma follows either by using Skorohod [25], Chapter 3, Theorem 21 or by using Proposition 2.4 and extending the proof of Theorem 21.5 of [22].

Theorem 5.2. *Let $\{e^1, \dots, e^{N_1}\}$ be a weak basis of K_1. Let $\{\rho_s \colon s \in K_1\}$ be a K_1-parameter convolution semigroup on $\mathbb{R}^{M_2}$. Let (A_s, ν_s, γ_s) be the triplet of ρ_s. Then*

$\operatorname{Supp}(\rho_s) \subseteq K_2$ *for all* $s \in K_1$ *if and only if the following conditions* (5.2) *and* (5.3) *are satisfied:*

$$A_{e^j} = 0,\ \nu_{e^j}(\mathbb{R}^{M_2} \setminus K_2) = 0,\ \text{and} \int_{K_2 \cap \{|s| \leqslant 1\}} |s| \nu_{e^j}(ds) < \infty \quad \text{for } j = 1, \ldots, N_1, \tag{5.2}$$

$$\text{if } s_1, \ldots, s_{N_1} \in \mathbb{R} \text{ are such that } s_1 e^1 + \cdots + s_{N_1} e^{N_1} \in K_1, \text{ then } s_1 \gamma^0_{e^1} + \cdots + s_{N_1} \gamma^0_{e^{N_1}} \in K_2, \text{ where } \gamma^0_{e^j} \text{ is the drift of } \rho_{e^j}. \tag{5.3}$$

If $\{e^1, \ldots, e^{N_1}\}$ is a strong basis, then condition (5.3) is simply written as $\gamma^0_{e^j} \in K_2$ for $j = 1, \ldots, N_1$. If $\{\rho_s \colon s \in K_1\}$ satisfies $\operatorname{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$ then we call it a K_1-parameter convolution semigroup supported on K_2.

Proof of the theorem. Suppose that $\operatorname{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$. Then the triplet (A_s, ν_s, γ_s) satisfies (5.1). By Theorem 3.13 we see that $\gamma^0_s = s_1 \gamma^0_{e^1} + \cdots + s_{N_1} \gamma^0_{N_1}$ for $s = s_1 e^1 + \cdots + s_{N_1} e^{N_1} \in K_1$. Hence (5.2) and (5.3) hold. The converse is similarly proved. $\square$

Corollary 5.3. *Let* $\{\rho_s \colon s \in K_1\}$ *be a* K_1*-parameter convolution semigroup supported on* K_2*. Then it is generative.*

The proof is given by Theorem 4.21 combined with Theorem 5.2.

If there is a K_1-parameter Lévy process associated, the property $\operatorname{Supp}(\rho_s) \subseteq K_2$ is expressed as a path property.

Proposition 5.4. *If* $\{Z_s \colon s \in K_1\}$ *is a* K_1*-parameter Lévy process on* $\mathbb{R}^{M_2}$*, then the following are equivalent.*

(i) $Z_s \in K_2$ *a. s. for each* $s \in K_1$.

(ii) *Almost surely,* Z_s *is* (K_1, K_2)*-increasing as a function of* s.

Proof. If (ii) holds, then we clearly have (i), since $Z_0 = 0$ a. s. Suppose that (i) holds. If $s^1, s^2 \in K_1$ satisfy $s^1 \leqslant_{K_1} s^2$, then $Z_{s^1} \leqslant_{K_2} Z_{s^2}$ a. s., since $Z_{s^2} - Z_{s^1} \stackrel{\mathrm{d}}{=} Z_{s^2 - s^1} \in K_2$ a. s. Let $K_{1,0}$ be the set of $s \in K_1$ with rational coordinates. Almost surely, for any choice of $s^1, s^2 \in K_{1,0}$ satisfying $s^1 \leqslant_{K_1} s^2$, $Z_{s^1} \leqslant_{K_2} Z_{s^2}$. Approximating s^1 and s^2 by K_1-decreasing sequences in $K_{1,0}$ and using the K_1-right continuity of sample functions, we see that, almost surely, for any choice of $s^1, s^2 \in K_1$ satisfying $s^1 \leqslant_{K_1} s^2$, $Z_{s^1} \leqslant_{K_2} Z_{s^2}$. That is, (ii) holds. $\square$

If $\{Z_s\colon s \in K_1\}$ is a K_1-parameter Lévy process (resp. Lévy process in law) on $\mathbb{R}^{M_2}$ satisfying (i) of Proposition 5.4 then we call it a K_2-valued K_1-parameter Lévy process (resp. Lévy process in law). Note that the preceding proposition is stated for K_1-parameter Lévy processes only; there is no analogous characterization of the sample paths of a K_2-valued K_1-parameter Lévy process in law.

Now we introduce subordination of convolution semigroups. For any measure μ and μ-integrable function f, we write $\mu(f) = \int f(x)\mu(dx)$.

Theorem 5.5. *Let $\{\mu_u\colon u \in K_2\}$ be a K_2-parameter convolution semigroup on $\mathbb{R}^d$ and $\{\rho_s\colon s \in K_1\}$ a K_1-parameter convolution semigroup supported on K_2. Define a probability measure σ_s on $\mathbb{R}^d$ by*

$$\sigma_s(f) = \int_{K_2} \mu_u(f)\rho_s(du) \tag{5.4}$$

for bounded continuous functions f on $\mathbb{R}^d$. Then $\{\sigma_s\colon s \in K_1\}$ is a K_1-parameter convolution semigroup on $\mathbb{R}^d$.

We call this procedure to get $\{\sigma_s\colon s \in K_1\}$ *subordination of* $\{\mu_u\colon u \in K_2\}$ *by* $\{\rho_s\colon s \in K_1\}$. The new convolution semigroup is said to be *subordinate to* $\{\mu_u\colon u \in K_2\}$ *by* $\{\rho_s\colon s \in K_1\}$. Sometimes $\{\mu_u\colon u \in K_2\}$, $\{\rho_s\colon s \in K_1\}$ and $\{\sigma_s\colon s \in K_1\}$ are respectively called subordinand, subordinator and subordinated.

Proof of the theorem. If f is bounded and continuous, then $\mu_u(f)$ is continuous in u by Theorem 3.13, and hence the integral in (5.4) exists. It is linear in f, nonnegative for $f \geqslant 0$, and 1 for $f = 1$. It decreases to 0 whenever $f = f_n(x)$ decreases to 0 on $\mathbb{R}^d$ as $n \to \infty$. Thus there is a unique probability measure σ_s satisfying (5.4) (Dudley [7], Theorem 4.5.2). Moreover, $\{\sigma_s\colon s \in K_1\}$ is a convolution semigroup. Indeed, we have

$$\widehat{\sigma}_s(z) = \int_{K_2} \widehat{\mu}_u(z)\rho_s(du), \qquad z \in \mathbb{R}^d, \tag{5.5}$$

from which the the property $\sigma_{s^1+s^2} = \sigma_{s^1} * \sigma_{s^2}$ is easily verified. As $t \downarrow 0$, ρ_{ts} tends to δ_0, and hence $\widehat{\sigma}_{ts}(z) \to 1$, that is, $\sigma_{ts} \to \delta_0$. □

Next we consider subordination of cone-parameter Lévy processes in law. We have to impose the regularity condition that the processes involved (the subordinator and the subordinand) are measurable processes. But recall from Remark 3.2 (ii) that this is essentially no restriction since any K-parameter Lévy process in law has a measurable modification. Thus, we introduce *subordination of a measurable K_2-parameter Lévy process in law by a measurable K_2-valued K_1-parameter Lévy process in law.* This is an extension of the multivariate subordination introduced in [1].

Theorem 5.6. *Let $\{Z_s \colon s \in K_1\}$ be a measurable K_2-valued K_1-parameter Lévy process in law and $\{X_u \colon u \in K_2\}$ a measurable K_2-parameter Lévy process in law on $\mathbb{R}^d$. Suppose that they are independent. Define $Y_s = X_{Z'_s}$, where $Z'_s = Z_s 1_{K_2}(Z_s)$. Then $\{Y_s \colon s \in K_1\}$ is a measurable K_1-parameter Lévy process in law on $\mathbb{R}^d$.*

If in addition $\{Z_s \colon s \in K_1\}$ is a measurable K_2-valued K_1-parameter Lévy process on K_2 and $\{X_u \colon u \in K_2\}$ a measurable K_2-parameter Lévy process on $\mathbb{R}^d$, then $\{Y_s \colon s \in K_1\}$ is a measurable K_1-parameter Lévy process on $\mathbb{R}^d$.

The processes $\{X_u \colon u \in K_2\}$, $\{Z_s \colon s \in K_1\}$ and $\{Y_s \colon s \in K_1\}$ are subordinand, subordinator and subordinated, respectively. In this case, if we denote $\mathcal{L}(X_u) = \mu_u$, $\mathcal{L}(Z_s) = \rho_s$ and $\mathcal{L}(Y_s) = \sigma_s$, then $\{\sigma_s \colon s \in K_1\}$ is exactly the convolution semigroup obtained by subordination of $\{\mu_u \colon u \in K_2\}$ by $\{\rho_s \colon s \in K_1\}$. However, we cannot proceed in the converse direction, as some cone-parameter convolution semigroups are non-generative.

Proof of the theorem. Since $\{Y_s \colon s \in K_1\}$ appears by composition of two measurable mappings, it is itself measurable. The other properties defining a cone-parameter Lévy process in law are essentially verified as in the first part of the proof of Theorem 3.3 of [1].

Assume that $\{Z_s \colon s \in K_1\}$ is a K_2-valued K_1-parameter Lévy process and $\{X_u \colon u \in K_2\}$ a K_2-parameter Lévy process on $\mathbb{R}^d$. Then, almost surely, $\{Y_s \colon s \in K_1\}$ is K_1-cadlag and is hence a measurable K_1-parameter Lévy process on $\mathbb{R}^d$. $\square$

Let us give the characteristic functions and the triplets of subordinated semigroups. For $v = (v_1, \ldots, v_{N_2})^\top$ and $w = (w_1, \ldots, w_{N_2})^\top$ in $\mathbb{C}^{N_2}$, we write $\langle v, w \rangle = \sum_{k=1}^{N_2} v_k w_k$. In the case of ordinary subordination (that is, $K_1 = K_2 = \mathbb{R}_+$) the following theorem reduces to a well-known result (see [22], Theorem 30.1). In the case where $K_1 = \mathbb{R}_+$ and $K_2 = \mathbb{R}_+^{N_2}$, it is in Theorems 3.3 and 4.7 of [1].

Theorem 5.7. *Let $\{\mu_u \colon u \in K_2\}$, $\{\rho_s \colon s \in K_1\}$ and $\{\sigma_s \colon s \in K_1\}$ be the subordinand, subordinator and subordinated convolution semigroups in Theorem 5.5. Let $\{h^1, \ldots, h^{N_2}\}$ be a weak basis of K_2. Let $(A_k^\mu, \nu_k^\mu, \gamma_k^\mu)$ be the triplet of μ_{h^k} for $k = 1, \ldots, N_2$. Let ν_s^ρ and $\gamma_s^{0\rho}$ be the Lévy measure and the drift of ρ_s for $s \in K_1$ and* *decompose* $\gamma_s^{0\rho}$ *as*

$$\gamma_s^{0\rho} = (\gamma_s^{0\rho})_1 h^1 + \cdots + (\gamma_s^{0\rho})_{N_2} h^{N_2}. \tag{5.6}$$

Let T *be the linear transformation from* $\mathbb{R}^{M_2}$ *onto* $\mathbb{R}^{N_2}$ *defined by*

$$Tu = (u_1, \ldots, u_{N_2})^\top \quad \text{whenever } Ru = u_1 h^1 + \cdots + u_{N_2} h^{N_2},$$

where R *is the orthogonal projection from* $\mathbb{R}^{M_2}$ *to the linear subspace* L_2 *generated by* K_2*. Then we have the following.*

(i) *For any* $s \in K_1$,

$$\widehat{\sigma}_s(z) = \exp \Psi_s^\rho(w), \qquad z \in \mathbb{R}^d, \tag{5.7}$$

where

$$\Psi_s^\rho(w) = \int_{K_2} (e^{\langle w, Tu\rangle} - 1)\nu_s^\rho(du) + \langle T\gamma_s^{0\rho}, w\rangle \tag{5.8}$$

with $w = (w_1, \ldots, w_{N_2})^\top$ *given by*

$$w_k = -\tfrac{1}{2}\langle z, A_k^\mu z\rangle + \int_{\mathbb{R}^d} g(z,x)\nu_k^\mu(dx) + \mathrm{i}\langle \gamma_k^\mu, z\rangle. \tag{5.9}$$

Here $g(z,x)$ *is the function in* (3.1).

(ii) *For any* $s \in K_1$ *the triplet* $(A_s^\sigma, \nu_s^\sigma, \gamma_s^\sigma)$ *of* σ_s *is represented as follows:*

$$A_s^\sigma = \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \, A_k^\mu, \tag{5.10}$$

$$\nu_s^\sigma(B) = \int_{K_2} \mu_u(B)\nu_s^\rho(du) + \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \, \nu_k^\mu(B), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{5.11}$$

$$\gamma_s^\sigma = \int_{K_2} \nu_s^\rho(du) \int_{|x|\leqslant 1} x\mu_u(dx) + \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \, \gamma_k^\mu. \tag{5.12}$$

(iii) *Fix* $s \in K_1$. *If* $\int_{K_2\cap\{|u|\leqslant 1\}} |u|^{1/2}\nu_s^\rho(du) < \infty$ *and* $\gamma_s^{0\rho} = 0$, *then* $A_s^\sigma = 0$, $\int_{|x|\leqslant 1} |x|\nu_s^\sigma(dx) < \infty$, *and the drift* $\gamma_s^{0\sigma}$ *is zero.*

(iv) *Let* K_3 *be a cone in* $\mathbb{R}^d$. *If* $\operatorname{Supp}(\mu_u) \subseteq K_3$ *for all* $u \in K_2$, *then* $\operatorname{Supp}(\sigma_s) \subseteq K_3$ *for all* $s \in K_1$ *and*

$$\gamma_s^{0\sigma} = \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k \, \gamma_k^{0\mu}. \tag{5.13}$$

Proof of Theorem 5.7 (i). We start from the identity (5.5). For $u = u_1 h^1 + \cdots + u_{N_2} h^{N_2} \in K_2$ we have

$$\widehat{\mu}_u(z) = \widehat{\mu}_{h^1}(z)^{u_1} \ldots \widehat{\mu}_{h^{N_2}}(z)^{u_{N_2}} \tag{5.14}$$

$$= \exp\left[\sum_{k=1}^{N_2} u_k \left(-\tfrac{1}{2}\langle z, A_k^\mu z\rangle + \int_{\mathbb{R}^d} g(z,x)\nu_k^\mu(dx) + \mathrm{i}\langle \gamma_k^\mu, z\rangle\right)\right]$$

by Theorem 3.13. Define $T\rho_s$ as $(T\rho_s)(B) = \rho_s(T^{-1}(B))$ for $B \in \mathcal{B}(\mathbb{R}^{N_2})$. Let $K_2^\sharp$ be the set of $w = (w_1, \dots, w_{N_2})^\top \in \mathbb{C}^{N_2}$ such that $\mathrm{Re}\,(u_1 w_1 + \cdots + u_{N_2} w_{N_2}) \leqslant 0$ for all $u_1, \dots, u_{N_2} \in \mathbb{R}$ satisfying $u_1 h^1 + \cdots + u_{N_2} h^{N_2} \in K_2$. We claim that

$$\int_{\mathbb{R}^{N_2}} e^{\langle w, \widetilde{u}\rangle}(T\rho_s)(d\widetilde{u}) = \int_{K_2} e^{\langle w, Tu\rangle}\rho_s(du) = \exp\Psi_s^\rho(w) \quad \text{for } w \in K_2^\sharp. \tag{5.15}$$

By [22], Proposition 11.10, the triplet $(A_s^{T\rho}, \nu_s^{T\rho}, \gamma_s^{T\rho})$ of $T\rho_s$ is given by the triplet $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ of ρ_s as

$$A_s^{T\rho} = TA_s^\rho T', \qquad \nu_s^{T\rho} = [\nu_s^\rho T^{-1}]_{\mathbb{R}^{N_2}\setminus\{0\}},$$
$$\gamma_s^{T\rho} = T\gamma_s^\rho + \int Tu(1_{\{|\widetilde{u}|\leqslant 1\}}(Tu) - 1_{\{|u|\leqslant 1\}}(u))\nu_s^\rho(du),$$

where T' is the transpose of T. Hence, $A_s^{T\rho} = 0$ and

$$\int_{|\widetilde{u}|\leqslant 1} |\widetilde{u}|\nu_s^{T\rho}(d\widetilde{u}) = \int_{|Tu|\leqslant 1} |Tu|\nu_s^\rho(du) \leqslant \text{const}\int_{|u|\leqslant 1}|u|\nu_s^\rho(du) + \int_{|u|>1}\nu_s^\rho(du) < \infty.$$

The drift $\gamma_s^{0T\rho}$ of $T\rho_s$ is represented as $\gamma_s^{0T\rho} = T\gamma_s^{0\rho}$, since

$$\begin{aligned}\gamma_s^{0T\rho} &= \gamma_s^{T\rho} - \int_{|\widetilde{u}|\leqslant 1}\widetilde{u}\nu_s^{T\rho}(d\widetilde{u}) \\ &= T\gamma_s^\rho + \int Tu(1_{\{|\widetilde{u}|\leqslant 1\}}(Tu) - 1_{\{|u|\leqslant 1\}}(u))\nu_s^\rho(du) - \int_{|Tu|\leqslant 1} Tu\nu_s^\rho(du) \\ &= T\gamma_s^\rho - \int_{|u|\leqslant 1} Tu\nu_s^\rho(du) = T\gamma_s^{0\rho}.\end{aligned}$$

Hence, by (5.8), $\int e^{\mathrm{i}\langle z, Tu\rangle}\rho_s(du) = \exp\Psi_s^\rho(\mathrm{i}z)$ for $z \in \mathbb{R}^{N_2}$. If $w \in K_2^\sharp$, then $\mathrm{Re}\,\langle w, Tu\rangle \leqslant 0$ for ρ_s-almost every u and hence $\int e^{\langle w, Tu\rangle}\rho_s(du)$ is finite. Now we can apply Theorem 25.17 of [22]. Thus, if $w \in K_2^\sharp$, then (5.8) is definable and (5.15) holds.

Now (5.7) follows from (5.5), (5.14), and (5.15), because w of (5.9) belongs to $K_2^\sharp$ by Theorem 3.15. This proves (i). □

We prepare lemmas to prove (ii)–(iv). We say a subclass Λ of $ID(\mathbb{R}^d)$ is bounded if $\sup_{|z|\leqslant 1}\langle z, A_\mu z\rangle$, $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu_\mu(dx)$, and $|\gamma_\mu|$ are bounded with respect to $\mu \in \Lambda$. Here $(A_\mu, \nu_\mu, \gamma_\mu)$ is the triplet of μ. The boundedness of Λ in this sense is equivalent to precompactness (see [22], E 12.5).

Lemma 5.8. *Let Λ be a bounded subclass of $ID(\mathbb{R}^d)$. Then there are constants $C(\varepsilon)$, C_1, C_2, C_3 such that, for all $t \geqslant 0$,*

$$\sup_{\mu\in\Lambda}\int_{|x|>\varepsilon}\mu^t(dx) \leqslant C(\varepsilon)t \quad \textit{for } \varepsilon > 0, \tag{5.16}$$

$$\sup_{\mu\in\Lambda}\int_{|x|\leqslant 1}|x|^2\mu^t(dx) \leqslant C_1 t, \tag{5.17}$$

$$\sup_{\mu\in\Lambda}\left|\int_{|x|\leqslant 1}x\mu^t(dx)\right| \leqslant C_2 t, \tag{5.18}$$

$$\sup_{\mu\in\Lambda}\int_{|x|\leqslant 1}|x|\mu^t(dx) \leqslant C_3 t^{1/2}. \tag{5.19}$$

Proof. These follow from [22], Lemma 30.3, its proof, and Example 25.12. □

Lemma 5.9. *Let $\{\mu_s \colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. Then there are constants $C(\varepsilon)$, C_1, C_2, C_3 such that, for all $s \in K$,*

$$\int_{|x|>\varepsilon}\mu_s(dx) \leqslant C(\varepsilon)|s| \quad \textit{for } \varepsilon > 0, \tag{5.20}$$

$$\int_{|x|\leqslant 1}|x|^2\mu_s(dx) \leqslant C_1|s|, \tag{5.21}$$

$$\left|\int_{|x|\leqslant 1}x\mu_s(dx)\right| \leqslant C_2|s|, \tag{5.22}$$

$$\int_{|x|\leqslant 1}|x|\mu_s(dx) \leqslant C_3|s|^{1/2}. \tag{5.23}$$

Proof. Fix a strictly supporting hyperplane H of K and $s^0 \in K \setminus \{0\}$. Let $K_0 = K \cap (s^0 + H)$. Then, by Proposition 2.4 (ii), K_0 is a compact set. Now $\{\mu_s \colon s \in K_0\}$ is a bounded subclass of $ID(\mathbb{R}^d)$. Indeed, let $\{e^1, \dots, e^N\}$ be a weak basis of K. Then $s \in K$ is uniquely expressed as $s = s_1e^1 + \dots + s_Ne^N$, and $s_1, \dots, s_N$ are continuous functions of s. Hence $\sup_{s\in K_0}(|s_1| + \dots + |s_N|) < \infty$. This shows boundedness of $\{\mu_s \colon s \in K_0\}$, in view of (3.6) of Theorem 3.13. Since every $s \in K$ is written as $s = tr$ with some $t \geqslant 0$ and $r \in K_0$, Lemma 5.8 shows that there is $C(\varepsilon)$ such that

$$\int_{|x|>\varepsilon}\mu_s(dx) = \int_{|x|>\varepsilon}{\mu_r}^t(dx) \leqslant C(\varepsilon)t.$$

Let $c = \inf_{r\in K_0}|r|$. We have $c > 0$, since $0 \notin K_0$. Hence $t \leqslant c^{-1}|s|$, and we get (5.20) by changing a constant. The other assertions are proved similarly. □

Proof of Theorem 5.7 (ii)–(iv). First let us prove (ii). We rewrite (5.7). For $w = (w_1, \ldots, w_{N_2})^\top$ of (5.9),

$$\begin{aligned}\langle T\gamma_s^{0\rho}, w\rangle = &-\tfrac{1}{2}\left\langle z, \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k\, A_k^\mu z\right\rangle \\ &+ \int_{\mathbb{R}^d} g(z,x)\left(\sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k\, \nu_k^\mu\right)(dx) + \mathrm{i}\left\langle \sum_{k=1}^{N_2} (\gamma_s^{0\rho})_k\, \gamma_k^\mu, z\right\rangle.\end{aligned}$$

This gives the summation terms in (5.10)–(5.12). Further, for w of (5.9),

$$\begin{aligned}\int_{K_2} (e^{\langle w, Tu\rangle} - 1)\nu_s^\rho(du) &= \int_{K_2} \left(\prod_{k=1}^{N_2} \widehat{\mu}_{h^k}(z)^{u_k} - 1\right) \nu_s^\rho(du) \\ &= \int_{K_2} (\widehat{\mu}_u(z) - 1)\nu_s^\rho(du) = \int_{K_2} \nu_s^\rho(du) \int_{\mathbb{R}^d} (e^{\mathrm{i}\langle z,x\rangle} - 1)\mu_u(dx) \\ &= \int_{K_2} \nu_s^\rho(du) \int_{\mathbb{R}^d} g(z,x)\mu_u(dx) + \mathrm{i}\int_{K_2} \nu_s^\rho(du) \left\langle z, \int_{|x|\leqslant 1} x\mu_u(dx)\right\rangle.\end{aligned}$$

Here the last equality is valid by Lemma 5.9. Define τ_s by $\tau_s(B) = \int_{K_2} \mu_u(B)\nu_s^\rho(du)$ for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Then, using Lemma 5.9, we can prove that $\int_{\mathbb{R}^d} (1 \wedge |x|^2)\tau_s(dx) < \infty$. Thus we get (5.10)–(5.12), where τ_s gives the first term in the expression (5.11).

To show (iii), let $\int_{K_2 \cap \{|u|\leqslant 1\}} |u|^{1/2}\nu_s^\rho(du) < \infty$ and $\gamma_s^{0\rho} = 0$. Then $A_s^\sigma = 0$ by (5.10). Use (5.11), (5.12) and (5.23) and notice that

$$\begin{aligned}\int_{|x|\leqslant 1} |x|\nu_s^\sigma(dx) &= \int_{K_2} \nu_s^\rho(du) \int_{|x|\leqslant 1} |x|\mu_u(dx) \\ &\leqslant C_3 \int_{|u|\leqslant 1} |u|^{1/2}\nu_s^\rho(du) + \int_{|u|>1} \nu_s^\rho(du) < \infty\end{aligned}$$

and that

$$\gamma_s^{0\sigma} = \gamma_s^\sigma - \int_{|x|\leqslant 1} x\nu_s^\sigma(dx) = \gamma_s^\sigma - \int_{K_2} \nu_s^\rho(du) \int_{|x|\leqslant 1} x\mu_u(dx) = 0.$$

Thus (iii) is true.

Let us show (iv). Assume that $\mathrm{Supp}(\mu_u) \subseteq K_3$ for $u \in K_2$. Since $\mathrm{Supp}(\rho_s) \subseteq K_2$ for all $s \in K_1$, we have $\mathrm{Supp}(\sigma_s) \subseteq K_3$ for all $s \in K_1$. Hence, by Lemma 5.1, $\int_{|x|\leqslant 1} |x|\nu_s^\sigma(dx) < \infty$. Thus the drift $\gamma_s^{0\sigma}$ of σ_s exists and $\gamma_s^{0\sigma} = \gamma_s^\sigma - \int_{|x|\leqslant 1} x\nu_s^\sigma(dx)$. The drift $\gamma_u^{0\mu}$ of μ_u also exists and has a similar expression. Now using (5.11) and (5.12), we get (5.13). □

A random variable Y on $\mathbb{R}$ (or its distribution) is said to be of type G if $Y \overset{\mathrm{d}}{=} Z^{1/2}X$ where X is a standard Gaussian, Z is nonnegative and infinitely divisible, and

X and Z are independent (see [21]). Equivalently, Y is of type G if its distribution is the same as the distribution at a fixed time of a Lévy process on $\mathbb{R}$ subordinate to Brownian motion. Barndorff-Nielsen and Pérez-Abreu [2] say that an $\mathbb{R}^d$-valued random variable Y (or its distribution) is of type ext$G$ if, for any $c \in \mathbb{R}^d$, $\langle c, Y\rangle$ is of type G. They say that an $\mathbb{R}^d$-valued random variable Y (or its distribution) is of type mult$G$ if

$$Y \stackrel{\mathrm{d}}{=} Z^{1/2} X, \tag{5.24}$$

where X is standard Gaussian on $\mathbb{R}^d$, Z is an $M^+_{d\times d}$-valued infinitely divisible random variable, $Z^{1/2}$ is the nonnegative-definite symmetric square root of Z, and X and Z are independent. If Y is of type mult$G$, then Y is of type ext$G$. Maejima and Rosiński [17] say that a probability measure μ on $\mathbb{R}^d$ (or a random vector with distribution μ) is of type G (we call it type G in the MR sense) if μ is symmetric, infinitely divisible with Gaussian covariance matrix arbitrary and Lévy measure ν represented as $\nu(B) = E[\nu_0(X^{-1}B)]$ for $B \in \mathcal{B}(\mathbb{R}^d)$ where ν_0 is a measure on $\mathbb{R}^d$ and X is standard Gaussian on $\mathbb{R}$. They show that μ is of type mult$G$ if it is of type G in the MR sense, and that type ext$G$ distributions are not always of type G in the MR sense. Type mult$G$ is related to subordination of cone-parameter convolution semigroups.

Theorem 5.10. *If $\{\sigma_t\colon t \geqslant 0\}$ is an $\mathbb{R}_+$-parameter convolution semigroup on $\mathbb{R}^d$ subordinate to the canonical $M^+_{d\times d}$-parameter convolution semigroup $\{\mu_u\colon u \in M^+_{d\times d}\}$ by an $\mathbb{R}_+$-parameter convolution semigroup $\{\rho_t\colon t \geqslant 0\}$ supported on $M^+_{d\times d}$, then, for any $t \geqslant 0$, σ_t is of type* mult$G$. *Conversely, any distribution on $\mathbb{R}^d$ of type* mult$G$ *is expressible as σ_1 of such an $\mathbb{R}_+$-parameter convolution semigroup $\{\sigma_t\colon t \geqslant 0\}$.*

Proof. Let $\{\sigma_t\colon t \geqslant 0\}$ be as stated above. Then, by (5.5) and by the definition of the canonical $M^+_{d\times d}$-parameter convolution semigroup,

$$\widehat{\sigma}_t(z) = \int_{M^+_{d\times d}} e^{-\langle z,uz\rangle/2}\rho_t(du), \quad z \in \mathbb{R}^d. \tag{5.25}$$

Let Z_t be a random variable on $M^+_{d\times d}$ with distribution ρ_t, X a standard Gaussian on $\mathbb{R}^d$, where X and Z_t are independent. Then

$$Ee^{\mathrm{i}\langle z, Z_t^{1/2}X\rangle} = Ee^{-\langle z, Z_t z\rangle/2} = \int_{M^+_{d\times d}} e^{-\langle z,uz\rangle/2}\rho_t(du).$$

Therefore $\sigma_t = \mathcal{L}({Z_t}^{1/2}X)$, that is, σ_t is of type mult$G$.

The converse is obvious, since we can construct a convolution semigroup $\{\rho_t\colon t \geqslant 0\}$ supported on $M^+_{d\times d}$ with $\rho_1 = \mathcal{L}(Z)$ from a given $M^+_{d\times d}$-valued infinitely divisible random variable Z. $\square$

Remark 5.11. Let $\sigma = \mathcal{L}(Y)$ be a distribution on $\mathbb{R}^d$ of type mult$G$ which satisfies (5.24) using Z and X and let ν^ρ and $\gamma^{0\rho}$ be the Lévy measure and the drift of $\rho = \mathcal{L}(Z)$. Note that ν^ρ is a measure on $M^+_{d\times d}$ and $\gamma^{0\rho} \in M^+_{d\times d}$. Then, by Theorem 5.10, σ is infinitely divisible and we can apply Theorem 5.7 to find the triplet $(A^\sigma, \nu^\sigma, \gamma^\sigma)$ of σ. Thus, we obtain that

$$\widehat{\sigma}(z) = \exp\left[\int_{M^+_{d\times d}} (e^{-\langle z,uz\rangle/2} - 1)\nu^\rho(du) - \tfrac{1}{2}\langle z, \gamma^{0\rho} z\rangle\right],$$

and $A^\sigma = \gamma^{0\rho}$, $\gamma^\sigma = 0$ and $\nu^\sigma(B) = \int_{M^+_{d\times d}} \mu_u(B)\nu^\rho(du)$ with $\mu_u = N_d(0,u)$. These results are noticed in [2] without using subordination.

Inheritance of selfdecomposability and the L_m-property from subordinator to subordinated in subordination of an $\mathbb{R}^{N_2}_+$-parameter Lévy process was studied in [1]. In the rest of this section we extend their results to the cone-parameter case. Our method of proof is simpler than that of [1]. However, since we do not consider operator selfdecomposability and operator stability, the results here do not cover those in [1].

A distribution μ on $\mathbb{R}^d$ is said to be *selfdecomposable* if, for every $b > 1$, there is a distribution μ' on $\mathbb{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu'}(z), \qquad z \in \mathbb{R}^d. \tag{5.26}$$

The class of selfdecomposable distributions on $\mathbb{R}^d$ is denoted by $L_0 = L_0(\mathbb{R}^d)$. Thus we also call them *of class* L_0. If $\mu \in L_0$, then μ is infinitely divisible, μ' is uniquely determined by μ and b, and μ' is also infinitely divisible.

For $m = 1, 2, \ldots$, $L_m = L_m(\mathbb{R}^d)$ is inductively defined as follows: $\mu \in L_m(\mathbb{R}^d)$ if and only if $\mu \in L_0(\mathbb{R}^d)$ and, for every $b > 1$, $\mu' \in L_{m-1}(\mathbb{R}^d)$. The class $L_\infty = L_\infty(\mathbb{R}^d)$ is defined to be the intersection of $L_m(\mathbb{R}^d)$ for $m = 0, 1, 2, \ldots$. We have

$$ID \supset L_0 \supset L_1 \supset \cdots \supset L_\infty \supset \mathfrak{S}, \tag{5.27}$$

where $\mathfrak{S} = \mathfrak{S}(\mathbb{R}^d)$ is the class of stable distributions on $\mathbb{R}^d$.

Definition 5.12. Let K be a cone in $\mathbb{R}^M$. Let $\{\mu_s\colon s \in K\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$. It is called *of class* L_m if, for every $s \in K$, $\mu_s \in L_m(\mathbb{R}^d)$.

Here $m \in \{0, 1, \dots, \infty\}$. Let $0 < \alpha \leqslant 2$. We call $\{\mu_s \colon s \in K\}$ *strictly α-stable* if, for every $s \in K$,

$$\mu_{as}(B) = \mu_s(a^{-1/\alpha}B) \quad \text{for all } a > 0 \text{ and } B \in \mathcal{B}(\mathbb{R}^d). \tag{5.28}$$

If $\mu_{as} = \delta_0$ for all $a > 0$, then it satisfies (5.28) for every α. Our terminology is different from [22] in this respect. In [22] this case is excluded from the definition of strict α-stability. If $\{\mu_s\}$ is supported on a cone and $\mu_s \neq \delta_0$ for some s, then it cannot be strictly α-stable for $\alpha \in (1, 2]$. If $\{\mu_s\}$ is supported on a cone and strictly 1-stable, then μ_s is trivial for all s. These follow from Lemma 5.1.

Theorem 5.13. *Let $\{\sigma_s \colon s \in K_1\}$ be a K_1-parameter convolution semigroup on $\mathbb{R}^d$ subordinate to a K_2-parameter convolution semigroup $\{\mu_u \colon u \in K_2\}$ by a K_1-parameter convolution semigroup $\{\rho_s \colon s \in K_1\}$ supported on K_2. Let $0 < \alpha \leqslant 2$. Suppose that $\{\mu_u \colon u \in K_2\}$ is strictly α-stable. Then the following are true.*

(i) *Let $m \in \{0, 1, \dots, \infty\}$. If $\{\rho_s \colon s \in K_1\}$ is of class L_m, then $\{\sigma_s \colon s \in K_1\}$ is of class L_m.*

(ii) *Let $0 < \alpha' \leqslant 1$. If $\{\rho_s \colon s \in K_1\}$ is strictly α'-stable, then $\{\sigma_s \colon s \in K_1\}$ is strictly $\alpha\alpha'$-stable.*

We need two lemmas.

Lemma 5.14. *Let K be a cone in $\mathbb{R}^M$. Let $\mu \in L_0(\mathbb{R}^M)$ satisfying* $\mathrm{Supp}(\mu) \subseteq K$. *Then, for any $b > 1$, the probability measure μ' defined by* (5.26) *satisfies* $\mathrm{Supp}(\mu') \subseteq K$.

Proof. We fix $b > 1$ and denote by μ'' the probability measure defined by $\widehat{\mu''}(z) = \widehat{\mu}(b^{-1}z)$. Thus (5.26) means that $\mu = \mu' * \mu''$. Let (A, ν, γ), (A', ν', γ') and (A'', ν'', γ'') be the triplets of μ, μ', and μ'', respectively. Then, $A = A' + A''$, $\nu = \nu' + \nu''$ and $\gamma = \gamma' + \gamma''$. Applying Lemma 5.1, we have

$$A = 0, \quad \nu(\mathbb{R}^M \setminus K) = 0, \quad \int_{|s| \leqslant 1} |s| \nu(ds) < \infty, \quad \gamma^0 \in K,$$

where γ^0 is the drift of μ. Therefore, we have $A' = 0$, $\nu'(\mathbb{R}^M \setminus K) = 0$, $\int_{|s| \leqslant 1} |s| \nu'(ds) < \infty$, and similarly for A'' and ν''. Thus μ' and μ'' have drifts $\gamma^{0\prime}$ and $\gamma^{0\prime\prime}$, and $\gamma^0 = \gamma^{0\prime} + \gamma^{0\prime\prime}$. Since $\gamma^{0\prime\prime} = b^{-1}\gamma^0$, we have $\gamma^{0\prime} = (1 - b^{-1})\gamma^0 \in K$. Now we can conclude that μ' is supported on K, using Lemma 5.1 again. $\square$

Lemma 5.15. *Let K be a cone in $\mathbb{R}^M$. Let $\{\mu_s\colon s \in K\}$ be a K-parameter convolution semigroup of class L_0 on $\mathbb{R}^d$. Fix $b > 1$ and define μ'_s by*

$$\widehat{\mu}_s(z) = \widehat{\mu}_s(b^{-1}z)\widehat{\mu}'_s(z). \tag{5.29}$$

Then $\{\mu'_s\colon s \in K\}$ is a K-parameter convolution semigroup.

Proof. We have $\widehat{\mu}_{s^1+s^2}(z) = \widehat{\mu}_{s^1}(z)\widehat{\mu}_{s^2}(z) = \widehat{\mu}_{s^1+s^2}(b^{-1}z)\widehat{\mu'}_{s^1}(z)\widehat{\mu'}_{s^2}(z)$. On the other hand, $\widehat{\mu}_{s^1+s^2}(z) = \widehat{\mu}_{s^1+s^2}(b^{-1}z)\widehat{\mu'}_{s^1+s^2}(z)$. Since $\widehat{\mu}_s(z) \neq 0$, we have $\widehat{\mu'}_{s^1+s^2}(z) = \widehat{\mu'}_{s^1}(z)\widehat{\mu'}_{s^2}(z)$. As $t \downarrow 0$, $\widehat{\mu}_{ts}(z) \to 1$ and hence, by (5.29), $\widehat{\mu'}_{ts}(z) \to 1$. Therefore, $\{\mu'_s\colon s \in K\}$ is a K-parameter convolution semigroup. $\square$

Proof of Theorem 5.13. (i) Suppose that $\{\rho_s\colon s \in K\}$ is of class L_0. Fix $b > 1$. There are ρ'_s and ρ''_s such that $\rho_s = \rho'_s * \rho''_s$ and $\widehat{\rho''}_s(z) = \widehat{\rho}_s(b^{-1}z)$. Since $\mathrm{Supp}(\rho_s) \subseteq K_2$, we have $\mathrm{Supp}(\rho'_s) \subseteq K_2$ by Lemma 5.14. It is evident that $\mathrm{Supp}(\rho''_s) \subseteq K_2$. Therefore, by (5.5),

$$\begin{aligned}
\widehat{\sigma}_s(z) &= \int_{K_2} \widehat{\mu}_u(z)\rho_s(du) = \iint_{K_2\times K_2} \widehat{\mu}_{u^1+u^2}(z)\rho'_s(du^1)\rho''_s(du^2) \\
&= \iint_{K_2\times K_2} \widehat{\mu}_{u^1}(z)\widehat{\mu}_{u^2}(z)\rho'_s(du^1)\rho''_s(du^2) \\
&= \int_{K_2} \widehat{\mu}_{u^1}(z)\rho'_s(du^1)\int_{K_2} \widehat{\mu}_{b^{-1}u^2}(z)\rho_s(du^2).
\end{aligned}$$

Now we utilize the assumption that $\widehat{\mu}_{au}(z) = \widehat{\mu}_u(a^{1/\alpha}z)$ for $a > 0$. Then

$$\widehat{\sigma}_s(z) = \widehat{\sigma}_s(b^{-1/\alpha}z)\int_{K_2} \widehat{\mu}_{u^1}(z)\rho'_s(du^1). \tag{5.30}$$

By Lemma 5.15, $\int_{K_2} \widehat{\mu}_{u^1}(z)\rho'_s(du^1)$ is the characteristic function of a subordinated convolution semigroup. Since $b^{1/\alpha}$ can be an arbitrary real larger than 1, (5.30) shows that $\sigma_s \in L_0$, that is, $\{\sigma_s\colon s \in K_1\}$ is of class L_0.

If $\{\rho_s\colon s \in K_1\}$ is of class L_1, then $\{\rho'_s\colon s \in K_1\}$ is of class L_0 by the definition of the class L_1 and $\int_{K_2} \widehat{\mu}_{u^1}(z)\rho'_s(du^1)$ is the characteristic function of a convolution semigroup of class L_0, which, combined with (5.30), shows that $\{\sigma_s\colon s \in K_1\}$ is of class L_1. Repeating this argument, we see that, if $\{\rho_s\colon s \in K_1\}$ is of class L_m for some $m < \infty$, then $\{\sigma_s\colon s \in K_1\}$ is of class L_m. Finally, if $\{\rho_s\colon s \in K_1\}$ is of class L_∞, then $\{\sigma_s\colon s \in K_1\}$ is of class L_m for all $m < \infty$, that is, it is of class L_∞.

(ii) Assume that $\{\rho_s\colon s \in K_1\}$ is strictly α'-stable. Then

$$\widehat{\sigma}_{as}(z) = \int_{K_2} \widehat{\mu}_u(z)\rho_{as}(du) = \int_{K_2} \widehat{\mu}_{a^{1/\alpha'}u}(z)\rho_s(du)$$

$$= \int_{K_2} \widehat{\mu}_u(a^{1/(\alpha\alpha')} z)\rho_s(du) = \widehat{\sigma}_s(a^{1/(\alpha\alpha')} z).$$

This shows that $\{\sigma_s \colon s \in K_1\}$ is strictly $\alpha\alpha'$-stable. $\square$

Remark 5.16. Let Y be a random variable of type mult$G$ on $\mathbb{R}^d$. Then $\mathcal{L}(Y)$ can be embedded into an $\mathbb{R}_+$-parameter convolution semigroup subordinate to the canonical $M^+_{d\times d}$-parameter convolution semigroup, which is strictly 2-stable. Hence we can apply Theorem 5.13. Thus, if the $M^+_{d\times d}$-valued random variable Z in (5.24) is of class L_m, then Y is of class L_m.

Remark 5.17. The problem how much we can weaken the assumption of strict α-stability of $\{\mu_u \colon u \in K_2\}$ in Theorem 5.13 is open even in the case of the ordinary subordination. In the subordination of Brownian motion with drift on $\mathbb{R}^d$ (2-stable but not strictly 2-stable), the selfdecomposability is inherited from subordinator to subordinated if $d = 1$ (Sato [23]), but it is not always inherited if $d \geqslant 2$ (Takano [26]).

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