CURVATURE AND EINSTEIN EQUATION FOR THE JACOBI GROUP MANIFOLD

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Introduction

In 1956 A. Selberg discovered weakly symmetric Riemannian spaces S as homogeneous spaces for certain locally compact groups G. He then studied the algebra A of all linear invariant operators, applying this finally to his trace formula and to Dirichlet series [5]. The fundamental property of A is that it is commutative. The basic example of S is the group manifold for a reductive Lie group G. The harmonic analysis, discrete groups and Selberg's trace formula on reductive Lie groups was (and is) extensively studied by many mathematicians after Selberg in connection with representation theory with application to number theory.

One of the simplest and most important examples of non-reductive Lie groups is the Jacobi group G^J, on which the classical Jacobi theta series are living. In spite of the very old and glorious history of Jacobi theta series the Jacobi group became known as an important group for

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further study only in recent years in connection with Kac-Moody algebras, see [4] and references therein. The connection with number theory was emphasized in the books of Eichler and Zagier [3] and Berndt and Schmidt [1]. Generally, the Jacobi group is a semidirect product of a symplectic group with a Heisenberg group. From this definition it is clear how to describe irreducible representations of G^J, but it is not a simple problem (cf [1]).

In this paper we consider G^J as a group manifold with a special chart of coordinates, suited for our purpose. We study the differential geometry on this manifold having in mind the development of the spectral theory of invariant differential operators on G^J and certain applications to mathematical physics.

Specifically we study the Jacobi group G^J of degree 1 over the real numbers as the semidirect product of $SL(2,\mathbb{R})$ with the 3-dimensional Heisenberg group. This group can also be considered as a subgroup of the symplectic group $Sp(2,\mathbb{R})$ (cf [1]). On the group G^J we define coordinates (y,x,ϕ,u,v,ψ), where (y,x,ϕ) are the Iwasawa coordinates on $SL(2,\mathbb{R})$ and (u,v,ψ) coordinates on the Heisenberg group. In this way G^J becomes a 6-dimensional C^∞-manifold M, and the group G^J acts on M by left multiplication. The action is described in a simple way in terms of the above coordinates (1.5).

We then study the G^J-invariant differential forms on M, obtaining four independent quadratic differential forms (Lemma 1.1). The first two forms are the well-known differential forms (1.10) related to $SL(2,\mathbb{R})$ and introduced by Selberg, the last two forms (1.12) also involve the Heisenberg group. We then study the 6-dimensional Riemannian manifold with this metric $ds^2(\varepsilon)$ depending on 4 parameters $\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$. The metric is positive definite when all $\varepsilon_i > 0$. (Lemma 2.1). We calculate the Christoffel symbols, the Ricci tensor and the scalar curvature $R(\varepsilon)$. It turns out that $R(\varepsilon)$ for each ε is constant on M (Theorem 2.3). The Einstein equation (2.4) has a non-zero right hand side, which takes a simpler form for certain values of ε. The Einstein equations can be seen as the Euler-Lagrange equations corresponding to an action $S_g(\varepsilon)$ contributed by the field in the absence of matter, taken to be the Hilbert action of the field, given by

$$S_g(\varepsilon) = \int_\Omega R(\varepsilon)d\mu(\varepsilon), d\mu(\varepsilon) = \sqrt{det g_{ij}(\varepsilon)}dydxd\phi dudvd\psi$$

over certain domains Ω in M, cf [2] for the classical theory. We can then introduce Riemannian manifolds obtained as quotients of M by discrete arithmetical subgroups of G^J based on subgroups of the modular group. These manifolds are non-compact of finite volume with a 4-parameter family of Laplacians corresponding to the family of quadratic differential forms described in this paper. (Lemma 1.1). This gives rise to an extension of the spectral theory of automorphic forms to this family of Laplacians on such arithmetical Jacobi manifolds. This theory will be developed in a forthcoming paper.

1. Definition of G^J.

The Jacobi group is a semidirect product of a symplectic group with a Heisenberg group. We will consider here the simplest case of degree 1 over the real numbers

$$G^J = SL(2;\mathbb{R}) \ltimes H(3;\mathbb{R}) \tag{1.1}$$

and the 3-dimensional Heisenberg group $H(3;\mathbb{R})$ is isomorphic to the group of upper triangular unipotent 3 x 3 matrices. From this definition we can see a certain similarity with the Poincaré group G^P which is a semidirect product of 0(3,1) with the group of translations of $\mathbb{R}^4$.

Consider now the symplectic group $Sp(2,\mathbb{R})$, which is by definition the group of matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \epsilon GL(4,\mathbb{R})$$

where A, B, C, D are 2 x 2-matrices with the properties

$$A^tD - C^tB = I, A^tC = C^tA, B^tD = D^tB$$

and t means transposition of a matrix, I is the 2 x 2 identity matrix. It turns out (see [1] p1) that G^J can be seen as the subgroup of $Sp(2,\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} & * & & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will consider G^J as a homogeneous space M. The group G^J acts on M by group multiplication from the left. It is clear that M is a real C^∞ manifold. We want to see now the differential one-forms on M invariant under this action. The manifold M can be covered by one chart of coordinates. A convenient system of coordinates is the following. Any element $g\epsilon G^J$ can be written in the form

$$g = \begin{pmatrix} \alpha & 0 & \beta & b \\ \nu & 1 & \mu & c \\ \gamma & 0 & \delta & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mu = \beta a + \delta b, \nu = \alpha a + \gamma b \tag{1.2}$$

where

$$\eta' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \epsilon SL(2,\mathbb{R}); a, b, c\epsilon\mathbb{R}$$

On the group manifold of $SL(2,\mathbb{R})$ we have the system of Iwasawa coordinates given by the Iwasawa decomposition; for $\overset{\circ}{g}\ \epsilon SL(2\mathbb{R})$ we have

$$\overset{\circ}{g}= n(x)a(y)k(\phi) \quad , \quad n(x) = 2\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

$$a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}, k(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & cos\phi \end{pmatrix}, \tag{1.3}$$

$$x \in \mathbb{R}, y > 0, \phi \in \mathbb{R} \mod 2\pi$$

It is convenient also to think of $x+iy = z$ as points of the upper-half plane. The group multiplication by $\overset{\circ}{g}\ \epsilon SL(2,\mathbb{R})$ is equivalent to the following action on the x, y, ϕ coordinates

$$z' = x' + iy' = \frac{\alpha z + \beta}{\gamma z + \delta}, \phi' = \phi + arg(\gamma z + \delta) \tag{1.4}$$

where the argument $argz$ of a complex number z is taken with $-\pi \leq argz < \pi$. Since we have G^J as a semidirect product(1.1), or in the coordinates (1.2)

$$g = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \mu \\ \nu & 1 & \mu & c \\ 0 & 0 & 1 & -\nu \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We will introduce on the group manifold M the system of coordinates of $m\epsilon M, m = (y, x, \phi, u, v, \psi)$, where (y, x, ϕ) are the Iwasawa coordinates on $SL(2,\mathbb{R})$ and u, v, ψ the coordinates on the Heisenberg group manifold $u = a, v = b, \psi = c$. It is not difficult to see that for g from (1.2) and $m = m(y, x, \phi, u, v, \psi)\epsilon M, m' = gm, m' = (y', x', \phi', u', v', \psi')\epsilon M$ we have

$$\begin{cases} z' = \frac{\alpha z+\beta}{\gamma z+\delta}, \phi' = \phi + arg(\gamma z+\delta), u' = \delta u - \gamma v + a, v' = \alpha v - \beta u + b \\ z' = x' + iy', \psi' = \psi + (\alpha v - \beta u)a + (\gamma v - \delta u)b + c \end{cases} \tag{1.5}$$

We introduce now the matrix valued invariant differential one form on M. For $g \epsilon G^J$ we define the differential dg; if $g = (g_{;j})_{i,j=1}^4$, then $dg = (dg_{ij})_{i,j=1}^4$ It is obvious that the matrix $g^{-1}dg$ is G^J-invariant. We have

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -\mu \\ -\nu & 1 & -\mu & -c \\ 0 & 0 & 1 & \nu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & 0 & -\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \delta & 0 & -\beta & -\beta a - \delta b \\ -a & 1 & -b & -c \\ -\gamma & 0 & \alpha & \alpha a + \gamma b \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.6}$$

$$dg = \begin{pmatrix} d\alpha & 0 & d\beta & db \\ \alpha da + a d\alpha + \gamma db + b d\gamma & 0 & a d\beta + \beta da + b d\delta + \delta db & dc \\ d\gamma & 0 & d\delta & -da \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we obtain

$$g^{-1}dg = \begin{pmatrix} \delta da - \beta d\gamma & 0 & \delta d\beta - \beta d\delta & \delta db + \beta da \\ \alpha da + \gamma db & 0 & \beta da + \delta db & -adb + dc + bda \\ -\gamma d\alpha + \alpha d\gamma & 0 & -\gamma d\beta + \alpha d\delta & -\gamma db - \alpha da \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.7}$$

All linear forms from the matrix (1.7) (the nonzero matrix elements) are G^J-invariant 1-forms on the manifold M. Some of them obviously are coming from the group $SL(2,\mathbb{R})$, namely

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \overset{\circ}{g}{}^{-1} d \overset{\circ}{g} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} d\alpha & d\beta \\ d\gamma & d\delta \end{pmatrix}$$

$$= \begin{pmatrix} \delta d\alpha - \beta d\gamma & \delta d\beta - \beta d\delta \\ -\gamma d\alpha + \alpha d\gamma & -\gamma d\beta + \alpha d\delta \end{pmatrix} \tag{1.8}$$

Applying the Iwasawa coordinates (1.3) to (1.8) we obtain

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\cos 2\phi}{2y}dy + \frac{sin2\phi}{2y}dx & , & -d\phi + \frac{1+\cos 2\phi}{2y}dx - \frac{\sin 2\phi}{2y}dy \\ d\phi + \frac{\cos 2\phi - 1}{2y}dx - \frac{\sin 2\phi}{2y}dy & , & -\frac{\cos 2\phi}{2y}dy - \frac{\sin 2\phi}{2y}dx \end{pmatrix} \quad (1.9)$$

We can see that $a_{11} = -a_{22}$. From (1.9) we obtain two important invariant differential 2-forms.

$$\begin{cases} ds_0^2 = (2a_{11})^2 + (a_{21} + a_{12})^2 = \frac{dx^2+dy^2}{y^2} \\ ds_1^2 = (\frac{a_{21}-a_{12}}{2})^2 = (d\phi - \frac{dx}{2y})^2 \end{cases} \quad (1.10)$$

It is clear that ds_0^2, ds_1^2 are not only $SL(2,\mathbb{R})$ but G^J invariant forms. From (1.7) we have also the following three forms invariant under the G^J action

$$\omega_1 = \alpha da + \gamma db, \omega_2 = \beta da + \delta db, \omega_3 = dc + bda - adb \quad (1.11)$$

Applying the system of coordinates (1.5), we obtain two other G^J invariant 2-forms,

$$\begin{cases} ds_2^2 = \omega_1^2 + \omega_2^2 = (y + \frac{x^2}{y})du^2 + \frac{1}{y}dv^2 + \frac{2x}{y}dudv \\ ds_3^2 = \omega_3^2 = v^2du^2 + u^2dv^2 + d\psi^2 - 2uvdudv + 2vdud\psi - 2udvd\psi \end{cases} \quad (1.12)$$
$$= (vdu - udv + d\psi)^2$$

$$ds^2(\varepsilon) = \varepsilon_0 ds_0^2 + \varepsilon_1 ds_1^2 + \varepsilon_2 ds_2^2 + \varepsilon_3 ds_3^2 \quad (1.13)$$

2. The Ricci tensor and the Einstein equation

We let $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $i, j, k = 1, 2, \cdots, 6$.

The metric tensor $g_{ij}(\varepsilon)$ of $ds^2(\varepsilon)$ is given by

$$g_{ij}(\varepsilon) = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} \frac{\varepsilon_0}{y^2} & 0 & 0 & \\ 0 & \frac{\varepsilon_0}{y^2} + \frac{\varepsilon_1}{4y^2} & -\frac{\varepsilon_1}{2y} & 0 \\ 0 & -\frac{\varepsilon_1}{2y} & \varepsilon_1 & 0 \end{pmatrix}$$

$$B_1 = C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} \varepsilon_2(y+\frac{x^2}{y}) + \varepsilon_3 v^2 & \varepsilon_2\frac{x}{y} - \varepsilon_3 uv & \varepsilon_3 v \\ \varepsilon_2\frac{x}{y} - \varepsilon_3 uv & \varepsilon_2\frac{1}{y} + \varepsilon_3 u^2 & -\varepsilon_3 u \\ \varepsilon_3 v & -\varepsilon_3 u & \varepsilon_3 \end{pmatrix}$$

$$\det g_{ij}(\varepsilon) = \frac{\varepsilon_0^2\varepsilon_1\varepsilon_2^2\varepsilon_3}{y^4}$$

$$g^{ij}(\varepsilon) = g_{ij}^{-1}(\varepsilon) = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{y^2}{\varepsilon_0} & 0 & 0 \\ 0 & \frac{y^2}{\varepsilon_0} & \frac{y}{2\varepsilon_0} \\ 0 & \frac{y}{2\varepsilon_0} & \frac{1}{4\varepsilon_0} + \frac{1}{\varepsilon_1} \end{pmatrix}$$

$$B_2 = C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} \frac{1}{\varepsilon_2 y} & -\frac{x}{\varepsilon_2 y} & -\frac{ux+v}{\varepsilon_2 y} \\ -\frac{x}{\varepsilon_2 y} & \frac{1}{\varepsilon_2}(y+\frac{x^2}{y}) & \frac{1}{\varepsilon_2}(yu+\frac{x(ux+v)}{y}) \\ \frac{1}{\varepsilon_2}(yu+\frac{x(ux+v)}{y}) & \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2}(yu^2+\frac{(v+ux)^2}{y}) & \end{pmatrix}$$

It is important to find conditions on $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$, under which the metric $g_{ij}dx^i dx^j$ has various signatures. We consider the interesting case, where the signature is $(-,-,+,-,-,+)$. This is determined by the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_6$ of the matrix $(g_{ij})_{ij=1}^6$. For the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ we obtain $\lambda_1 = \frac{\varepsilon_0}{y^2} < 0$ if and only if $\varepsilon_0 < 0$, while λ_2 and λ_3 satisfy the equation

$$\lambda^2 - (\frac{1}{y^2}(\varepsilon_0 + \frac{\varepsilon_1}{4}) + \varepsilon_1)\lambda + \frac{\varepsilon_0\varepsilon_1}{y^2} = 0$$

Hence

$$\lambda_2 + \lambda_3 = \frac{1}{y^2}(\varepsilon_0 + \frac{\varepsilon_1}{4}) + \varepsilon_1$$

and

$$\lambda_2\lambda_3 = \frac{\varepsilon_0\varepsilon_1}{y^2}$$

It follows that the form $\varepsilon_0 ds_0^2 + \varepsilon_1 ds_1^2$ has signature $(-,-,+)$ if and only if $\varepsilon_0 < 0, \varepsilon_1 > 0$. Consider now the eigenvalues $\lambda_4, \lambda_5, \lambda_6$. They satisfy the equation

$$-\lambda^3 + (\varepsilon_2(y + \frac{1+x^2}{y}) + \varepsilon_3(1+u^2+v^2))\lambda^2 - (\varepsilon_2\varepsilon_3(y + \frac{1+x^2+(ux+v)^2}{y}) + \varepsilon_2^2)\lambda + \varepsilon_2^2\varepsilon_3 = 0$$

This gives the equations

$$\lambda_4\lambda_5\lambda_6 = \varepsilon_2^2\varepsilon_3 \tag{2.1}$$

$$\lambda_4 + \lambda_5 + \lambda_6 = \varepsilon_2(y + \frac{1+x^2}{y}) + \varepsilon_3(1+u^2+v^2) \tag{2.2}$$

$$\lambda_4\lambda_5 + \lambda_5\lambda_6 + \lambda_6\lambda_4 = \varepsilon_2\varepsilon_3(y + \frac{1+x^2+(ux+v)^2}{y}) + \varepsilon_2^2$$

where $\lambda_4, \lambda_5, \lambda_6$ are functions of x and y, x $\epsilon\mathbb{R}$, $y > 0$ and $u, v\epsilon\mathbb{R}$ Dividing the first equation into the last equation, we get

$$\frac{1}{\lambda_4} + \frac{1}{\lambda_5} + \frac{1}{\lambda_6} = \frac{1}{\varepsilon_2}(y + \frac{1+x^2+(ux+v)^2}{y}) + \frac{1}{\varepsilon_3} \tag{2.3}$$

We shall prove that the form $\varepsilon_2 ds_2^2 + \varepsilon_3 ds_3^2$ has signature $(-,-,+)$ if and only if $\varepsilon_2 < 0, \varepsilon_3 > 0$. Assume first that $\varepsilon_2 < 0, \varepsilon_3 > 0$. Then by (2.1) and a suitable ordering of $\lambda_4, \lambda_5, \lambda_6$ we have $\lambda_4, \lambda_5 < 0, \lambda_6 > 0$. Suppose on the other hand that $\lambda_4, \lambda_5 < 0, \lambda_6 > 0$ for all x, y, u, v. From (2.1) follows that $\varepsilon_3 > 0$ and $\varepsilon_2 \neq 0$. Assume that $\varepsilon_2 > 0$. Then the r.h.s. of (2.2) is positive for all x, y, u, v.

By (2.2)

$$\lambda_6(x,y,u,v) > \varepsilon_2(y + \frac{1+x^2}{y}) + \varepsilon_3(1+u^2+v^2) \tag{2.4}$$

and by (2.3)

$$\lambda_6^{-1}(x,y,u,v) > \frac{1}{\varepsilon_2}(y + \frac{1+x^2+(ux+v)^2}{y}) + \frac{1}{\varepsilon_3} \tag{2.5}$$

Multiplying (2.4) and (2.5), we get

$$1 > y^2 + \frac{1}{y^2} \quad \text{for} \quad y > 0,$$

a contradiction. It follows, that $\varepsilon_2 < 0$, and we have proved

Lemma 2.1. *The metric $(g_{ij})_{ij=1}^6$ has signature (-, -, +, -, -, +) if and only if $\varepsilon_0 < 0, \varepsilon_1 > 0, \varepsilon_2 < 0, \varepsilon_3 > 0$.*

Remark 2.2. Writing $\varepsilon_0 ds_0^2+\varepsilon_1 ds_1^2 = \varepsilon_0\frac{dy^2}{y^2}+(\varepsilon_0+\frac{1}{4}\varepsilon_1)\frac{dx^2}{y^2}-\frac{d\phi dx}{2y}+d\phi^2$ we observe, that the ϕ-independent part of this metric is negative definite if and only if $\varepsilon_0<0, 0<\varepsilon_1<-4\varepsilon_0$

We calculate the Christoffel symbols $\Gamma^k_{ij}(\varepsilon)$, given by

$$\Gamma^k_{ij}(\varepsilon)=\frac{1}{2}g^{kl}(\varepsilon)(\frac{\partial g_{lj}(\varepsilon)}{\partial x^i}+\frac{\partial g_{il}(\varepsilon)}{\partial x^j}-\frac{\partial g_{ij(\varepsilon)}}{\partial x^l}) \tag{2.6}$$

where we sum as usual over the repeated index l from 1 to 6. We obtain

$$\Gamma^1_{ij}(\varepsilon)=\begin{pmatrix} -\frac{1}{y} & 0 & 0 & 0 & 0 & 0\\ 0 & (1+\frac{\varepsilon_1}{4\varepsilon_0})\frac{1}{y} & -\frac{\varepsilon_1}{4\varepsilon_0} & 0 & 0 & 0\\ 0 & -\frac{\varepsilon_1}{4\varepsilon_0} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{\varepsilon_2}{2\varepsilon_0}(x^2-y^2) & \frac{\varepsilon_2 x}{2\varepsilon_0} & 0\\ 0 & 0 & 0 & \frac{\varepsilon_2 x}{2\varepsilon_0} & \frac{\varepsilon_2}{2\varepsilon_0} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^2_{ij}(\varepsilon)=\begin{pmatrix} 0 & -\frac{1}{y}(1+\frac{\varepsilon_1}{8\varepsilon_0}) & \frac{\varepsilon_1}{4\varepsilon_0} & 0 & 0 & 0\\ -\frac{1}{y}(1+\frac{\varepsilon_1}{8\varepsilon_0}) & 0 & 0 & 0 & 0 & 0\\ \frac{\varepsilon_1}{4\varepsilon_0} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{\varepsilon_2}{\varepsilon_0}xy & -\frac{\varepsilon_2}{2\varepsilon_0}y & 0\\ 0 & 0 & 0 & -\frac{\varepsilon_2}{2\varepsilon_0}y & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^3_{ij}(\varepsilon)=\begin{pmatrix} 0 & -(\frac{1}{4}+\frac{\varepsilon_1}{16\varepsilon_0})\frac{1}{y^2} & \frac{\varepsilon_1}{8\varepsilon_0} & 0 & 0 & 0\\ -(\frac{1}{4}+\frac{\varepsilon_1}{16\varepsilon_0})\frac{1}{y^2} & 0 & 0 & 0 & 0 & 0\\ \frac{\varepsilon_1}{8\varepsilon_0} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{\varepsilon_2}{2\varepsilon_0}x & -\frac{\varepsilon_2}{4\varepsilon_0} & 0\\ 0 & 0 & 0 & -\frac{\varepsilon_2}{4\varepsilon_0} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^4_{ij}(\varepsilon)=\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2y} & 0 & 0\\ 0 & 0 & 0 & \frac{x}{2y^2} & \frac{1}{2y^2} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2y} & \frac{x}{2y^2} & 0 & \frac{2\varepsilon_3}{\varepsilon_2}\frac{xv}{y} & \frac{\varepsilon_3}{\varepsilon_2}\frac{v-xu}{y} & \frac{\varepsilon_3}{\varepsilon_2}\frac{x}{y}\\ 0 & \frac{1}{2y^2} & 0 & \frac{\varepsilon_3}{\varepsilon_2}\frac{v-xu}{y} & -2\frac{\varepsilon_3}{\varepsilon_2}\frac{u}{y} & \frac{\varepsilon_3}{\varepsilon_2}\frac{1}{y}\\ 0 & 0 & 0 & \frac{\varepsilon_3}{\varepsilon_2}\frac{x}{y} & \frac{\varepsilon_3}{\varepsilon_2}\frac{1}{y} & 0 \end{pmatrix}$$

$$\Gamma^5_{ij}(\varepsilon) = \begin{pmatrix} A_5 & B_5 \\ C_5 & D_5 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} -\frac{x}{y} & -\frac{1}{2y} & 0 \\ \frac{1}{2}(1-\frac{x^2}{y^2}) & -\frac{x}{2y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} -\frac{x}{y} & \frac{1}{2}(1-\frac{x^2}{y^2}) & 0 \\ -\frac{1}{2y} & -\frac{x}{2y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_5 = \begin{pmatrix} -2\frac{\varepsilon_3}{\varepsilon_2}v(y+\frac{x^2}{y}) & \frac{\varepsilon_3}{\varepsilon_2}(-\frac{xv}{y}+\frac{x^2u}{y}+yu) & -\frac{\varepsilon_3}{\varepsilon_2}(y+\frac{x^2}{y}) \\ \frac{\varepsilon_3}{\varepsilon_2}(-\frac{xv}{y}+\frac{x^2u}{y}+yu) & 2\frac{\varepsilon_3}{\varepsilon_2}\frac{ux}{y} & -\frac{\varepsilon_3}{\varepsilon_2}\frac{x}{y} \\ -\frac{\varepsilon_3}{\varepsilon_2}(y+\frac{x^2}{y}) & -\frac{\varepsilon_3}{\varepsilon_2}(y+\frac{x^2}{y}) & 0 \end{pmatrix}$$

$$\Gamma^6_{ij}(\varepsilon) = \begin{pmatrix} A_6 & B_6 \\ C_6 & D_6 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_6 = \begin{pmatrix} -\frac{1}{2}\frac{v+2xu}{y} & -\frac{u}{2y} & 0 \\ \frac{1}{2}(u-\frac{x(ux+v)}{y^2}) & -\frac{xu+v}{2y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_6 = \begin{pmatrix} -\frac{1}{2}\frac{v+2xu}{y} & \frac{1}{2}(u-\frac{(v+xu)x}{y^2}) & 0 \\ -\frac{u}{2y} & -\frac{xu+v}{2y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D_6 = \begin{pmatrix} -\frac{2\varepsilon_3}{\varepsilon_2}v(yu+\frac{x(ux+v)}{y}) & \frac{\varepsilon_3}{\varepsilon_2}(yu^2+\frac{u^2x^2-v^2}{y}) & -\frac{\varepsilon_3}{\varepsilon_2}(yu+\frac{x(ux+v)}{y}) \\ \frac{\varepsilon_3}{\varepsilon_2}(yu^2+\frac{u^2x^2-v^2}{y}) & 2\frac{\varepsilon_3}{\varepsilon_2}\frac{u}{y}(xu+v) & -\frac{\varepsilon_3}{\varepsilon_2}\frac{xu+v}{y} \\ -\frac{\varepsilon_3}{\varepsilon_2}(yu+\frac{x(ux+v)}{y}) & -\frac{\varepsilon_3}{\varepsilon_2}\frac{xu+v}{y} & 0 \end{pmatrix}$$

The Ricci tensor is given by

$$R_{ik}(\varepsilon) = \frac{\partial\Gamma^l_{ik}(\varepsilon)}{\partial x^l} - \frac{\partial\Gamma^l_{il}(\varepsilon)}{\partial x^k} + (\Gamma^l_{ik}(\varepsilon)\Gamma^m_{lm}(\varepsilon) - \Gamma^m_{il}(\varepsilon)\Gamma^l_{km}(\varepsilon)) \tag{2.7}$$

We obtain

$$R_{ik}(\varepsilon) = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$$

$$S = \begin{pmatrix} -\frac{1}{2y^2}(3+\frac{\varepsilon_1}{4\varepsilon_0}) & 0 & 0 \\ 0 & -\frac{1}{2y^2}(3+\frac{\varepsilon_1}{4\varepsilon_0}-\frac{1}{16}(\frac{\varepsilon_1}{\varepsilon_0})^2) & -(\frac{\varepsilon_1}{\varepsilon_0})^2\frac{1}{16y} \\ 0 & -(\frac{\varepsilon_1}{\varepsilon_0})^2\frac{1}{16y} & \frac{1}{8}(\frac{\varepsilon_1}{\varepsilon_0})^2 \end{pmatrix}$$

$$T = \begin{pmatrix} -2\frac{\varepsilon_3}{\varepsilon_2}(y+\frac{x^2}{y})+2(\frac{\varepsilon_3}{\varepsilon_2})^2v^2 & -2\frac{\varepsilon_3}{\varepsilon_2}\frac{x}{y}-(\frac{\varepsilon_3}{\varepsilon_2})^2 2uv & (\frac{\varepsilon_3}{\varepsilon_2})^2 2v \\ -2\frac{\varepsilon_3}{\varepsilon_2}\frac{x}{y}-(\frac{\varepsilon_3}{\varepsilon_2})^2 2uv & -2\frac{\varepsilon_3}{\varepsilon_2}\frac{1}{y}+(\frac{\varepsilon_3}{\varepsilon_2})^2 2u^2 & -(\frac{\varepsilon_3}{\varepsilon_2})^2 2u \\ (\frac{\varepsilon_3}{\varepsilon_2})^2 2v & -(\frac{\varepsilon_3}{\varepsilon_2})^2 2u & 2(\frac{\varepsilon_3}{\varepsilon_2})^2 \end{pmatrix}$$

We obtain the following expression for the scalar curvature $R(\varepsilon) = g^{ik}(\varepsilon)R_{ik}(\varepsilon)$

$$R(\varepsilon) = -\frac{3}{\varepsilon_0} - \frac{1}{8}\frac{\varepsilon_1}{\varepsilon_0^2} - 2\frac{\varepsilon_3}{\varepsilon_2^2} \tag{2.8}$$

Thus, for fixed $\varepsilon, R(\varepsilon)$ is constant, and the metric is an Einstein metric. The Einstein equation is

$$R_{ik}(\varepsilon) - CR(\varepsilon)g_{ik}(\varepsilon) = F_{ik}(\varepsilon) \tag{2.9}$$

where C is a universal constant, depending only on the dimension, which here is 6, and where $F_{ik}(\varepsilon)$ is the external gravitational field. Setting $K(\varepsilon) = -CR(\varepsilon)$, we obtain

$$F_{ik}(\varepsilon) = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix} \tag{2.10}$$

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & L_{23} \\ 0 & L_{32} & L_{33} \end{pmatrix}$$

$$L_{11} = \frac{1}{y^2}(-\frac{3}{2} - \frac{1}{8}\frac{\varepsilon_1}{\varepsilon_0} + K\varepsilon_0)$$

$$L_{22} = \frac{1}{y^2}(-\frac{3}{2} - \frac{1}{8}\frac{\varepsilon_1}{\varepsilon_0} + \frac{1}{32}(\frac{\varepsilon_1}{\varepsilon_0})^2 + K(\varepsilon_0 + \frac{\varepsilon_1}{4})$$

$$L_{23} = \frac{1}{y}(-\frac{1}{16}(\frac{\varepsilon_1}{\varepsilon_0})^2 + K(-\frac{\varepsilon_1}{2}))$$

$$L_{32} = \frac{1}{y}(-\frac{1}{16}(\frac{\varepsilon_1}{\varepsilon_0})^2 + K(-\frac{\varepsilon_1}{2}))$$

$$L_{33} = \frac{1}{8}(\frac{\varepsilon_1}{\varepsilon_0})^2 + K\varepsilon_1$$

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

$$M_{11} = (y + \frac{x}{y}^2)(-2\frac{\varepsilon_3}{\varepsilon_2} + K\varepsilon_2) + v^2(2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3)$$

$$M_{12} = \frac{x}{y}(-2\frac{\varepsilon_3}{\varepsilon_2} + K\varepsilon_2) + uv(-2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K(-\varepsilon_3))$$

$$M_{13} = v(2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3)$$

$$M_{21} = \frac{x}{y}(-2(\frac{\varepsilon_3}{\varepsilon_2}) + K\varepsilon_2) + uv(-2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K(-\varepsilon_3))$$

$$M_{22} = \frac{1}{y}(-2\frac{\varepsilon_3}{\varepsilon_2} + K\varepsilon_2) + u^2(2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3)$$

$$M_{23} = u(-2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K(-\varepsilon_3))$$

$$M_{31} = v(2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3)$$

$$M_{32} = u(-2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K(-\varepsilon_3))$$

$$M_{33} = 2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3$$

The two block matrices $F_{ik}(\varepsilon)_{ik=1}^3 = G_{ik}(\varepsilon)$ and $F_{ik}(\varepsilon)_{ik=4}^6 = H_{ik}(\varepsilon)$ comprising $F_{ik(\varepsilon)}$ can be written in the form

$$G_{ik}(\varepsilon) = (-\frac{3}{2} - \frac{1}{8}\frac{\varepsilon_1}{\varepsilon_0} + K\varepsilon_0) \begin{pmatrix} \frac{1}{y^2} & 0 & 0 \\ 0 & \frac{1}{y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + (\frac{1}{16}(\frac{\varepsilon_1}{\varepsilon_0})^2 + K\frac{\varepsilon_1}{2}) \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2y^2} & -\frac{1}{y} \\ 0 & -\frac{1}{y} & 2 \end{pmatrix} \quad (2.11)$$

$$H_{ik}(\varepsilon) = (-2\frac{\varepsilon_3}{\varepsilon_2} + K\varepsilon_2)\begin{pmatrix} y+\frac{x^2}{y} & \frac{x}{y} & 0 \\ \frac{x}{y} & \frac{1}{y} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ (2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3)\begin{pmatrix} v^2 & -uv & v \\ -uv & u^2 & -u \\ v & -u & 1 \end{pmatrix} \quad (2.12)$$

From the expressions for $G_{ik}(\varepsilon)$ we see that the condition for $G_{ik}(\varepsilon)$ to be 0 is, setting $\frac{\varepsilon_1}{\varepsilon_0} = \alpha$,

$$-\frac{3}{2} - \frac{1}{8}\alpha + K\varepsilon_0 = 0 \quad \text{and} \quad \frac{1}{8}\alpha + K\varepsilon_0 = 0 \quad \text{or} \quad \alpha = 0$$

This leads to $\varepsilon_1 = -6\varepsilon_0, K\varepsilon_0 = \frac{3}{4}$ and $\varepsilon_1 = 0, K\varepsilon_0 = \frac{3}{2}$. In the first case the operator is hyperbolic. In the second case it is elliptic, but the term $(\frac{dx-d\phi}{2y})^2$ is missing.

From the expressions for the $H_{ik}(\varepsilon)$ we obtain the following conditions for the two terms to be 0

$$-2\frac{\varepsilon_3}{\varepsilon_2} + K\varepsilon_2 = 0 \quad \text{and} \quad 2(\frac{\varepsilon_3}{\varepsilon_2})^2 + K\varepsilon_3 = 0$$

or

$$\frac{\varepsilon_3}{\varepsilon_2^2} = \frac{K}{2} \quad \text{and} \quad \frac{\varepsilon_3}{\varepsilon_2^2} = -\frac{K}{2}$$

Thus, the two terms cannot vanish simultanously, but there are two interesting special cases. In the first, where $\frac{\varepsilon_3}{\varepsilon_2^2} = -\frac{K}{2}$, all $F_{ik}(\varepsilon)$ are functions of x and y. In the other case, where $\frac{\varepsilon_3}{\varepsilon_2^2} = \frac{K}{2}$, $G_{ik}(\varepsilon)$ contains only functions of x and y and is connected with dx, dy, dϕ, while $H_{ik}(\varepsilon)$ contains only functions of u and v and is connected with du, dv, ψ. Inserting the expression (2.8) for $R(\varepsilon)$ in the above conditions with $K = RC$, we get

$$\pm 2\frac{\varepsilon_3}{\varepsilon_2^2} = C(\frac{3}{\varepsilon_0} + 18\frac{\varepsilon_1}{\varepsilon_0^2} + 2\frac{\varepsilon_3}{\varepsilon_2^2})$$

$$(\pm\frac{1}{c} - 1)2\frac{\varepsilon_3}{\varepsilon_2^2} = \frac{3}{\varepsilon_0} + \frac{1}{8}\frac{\varepsilon_1}{\varepsilon_0^2}$$

If in the first case $C = 1$, this gives $\varepsilon_1 = -24\varepsilon_0$, and the equation is hyperbolic, but the part of the metric depending on only x and y is not positive or negative definite. If $C \neq 1$, there are solutions for all

ε_1. If in the second case $C = -1$, we get $\varepsilon_1 = -24\varepsilon_0$. If $C \neq -1$, there are solutions for all ε_1.

We summarize the main result as follows.

Theorem 2.3. *The scalar curvature $R(\varepsilon)$ is a constant given by (2.8) for each ε. The right hand side $F_{ik}(\varepsilon)$ of the Einstein equation (2.9) is given by(2.10) or by the 3X3 block matrices $G_{ik}(\varepsilon)$ and $H_{ik}(\varepsilon)$ of (2.11),(2.12).*

$G_{ik}(\varepsilon) = 0$ if either $\varepsilon_1 = -6\varepsilon_0, K\varepsilon_0 = \frac{3}{4}$
or $\varepsilon_1 = 0, K\varepsilon_0 = \frac{3}{2}$.

$H_{ik}(\varepsilon)$ is a function of (x,y) if $\frac{\varepsilon_3}{\varepsilon_2^2} = -\frac{K}{2}$ and a function of (u,v) if $\frac{\varepsilon_3}{\varepsilon_2^2} = \frac{K}{2}$.

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