WHEN IS A MOVING AVERAGE A SEMIMARTINGALE?

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Abstract. We consider a moving average process

$$X_t = \int_{\mathbb{R}} (\varphi(u-t) - \varphi(u))\, dB_u, \quad t \geq 0,$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel function that equals zero on $(0, \infty)$ and $(B_u)_{u \in \mathbb{R}}$ is a two-sided Brownian motion. We provide the necessary and sufficient conditions for $(X_t)_{t \geq 0}$ to be a semimartingale on the filtration $\mathcal{F}_t^B = \sigma(B_s;\, s \in (-\infty, t])$.

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1 Introduction

1. Problem setting. Let $(B_u)_{u \in \mathbb{R}}$ be a two-sided Brownian motion, i.e. $(B_u)_{u \geq 0}$ and $(B_{-u})_{u \geq 0}$ are independent Brownian motions started at zero. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel function that is equal to zero on $(0, \infty)$ and satisfies the condition

$$\forall t \geq 0, \quad \int_{\mathbb{R}} (\varphi(u-t) - \varphi(u))^2 du < \infty. \tag{1.1}$$

This property holds, in particular, for any $\varphi \in L^2(\mathbb{R})$. However, there exist functions $\varphi \notin L^2(\mathbb{R})$ for which (1.1) is true (see examples below).

One can define the *moving average* process

$$X_t = \int_{\mathbb{R}} (\varphi(u-t) - \varphi(u))\, dB_u, \quad t \geq 0. \tag{1.2}$$

The process X is adapted to the filtration $\mathcal{F}_t^B = \sigma(B_s;\, s \in (-\infty, t])$, $t \geq 0$. The question arises: for which φ is X a semimartingale with respect to $(\mathcal{F}_t^B)$?

2. Motivation. Recall that a *financial model* is a collection $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathsf{P};\, (S_t)_{t \geq 0}\big)$, where S is a $(\mathcal{F}_t)$-adapted process meaning the (discounted) price of an asset on the securities market. In almost all the models of the modern mathematical finance, S is supposed to be a $(\mathcal{F}_t)$-semimartingale (we cite the definition of a semimartingale in Section 2). There are two basic reasons for such an assumption.

First, the *capital of a self-financing portfolio* in the above model is expressed as a stochastic integral $\int_0^t H_u\,dS_u$, where $(H_t)_{t\geq 0}$ is a $(\mathcal{F}_t)$-predictable process. The famous Bichteler-Dellacherie-Mokobodski theorem (see [1]) asserts that the stochastic integral $\int_0^t H_u\,dS_u$ can be constructed in a "reasonable" way for all the $(\mathcal{F}_t)$-predictable processes if and only if X is a $(\mathcal{F}_t)$-semimartingale.

Second, if S is locally bounded and satisfies the *No Free Lunch with Vanishing Risk* condition for simple integrands, then S is a $(\mathcal{F}_t)$-semimartingale. This was proved by F. Delbaen and W. Schachermayer [3; Theorem 7.2]).

Now, consider a model

$$S_t = e^{X_t}, \quad \mathcal{F}_t = \mathcal{F}_t^B, \quad t \geq 0, \tag{1.3}$$

where X is given by (1.2). Using such a type of models, one can easily simulate the following effects that are observed with the real prices:

- long-range dependence (future increments of S depend on the past);
- negative correlation of the increments of the logarithmic price process $\ln S$ over disjoint intervals.

In view of the reasoning given above, it is important to know for which φ the process S given by (1.3) is a $(\mathcal{F}_t)$-semimartingale. By Itô's formula, S is a $(\mathcal{F}_t)$-semimartingale if and only if X is a $(\mathcal{F}_t)$-semimartingale. This leads us to the problem posed above.

3. Examples. If $\varphi(u) = I(u \leq 0)$, then $X_t = B_t$ $(t \geq 0)$. If $\varphi(u) = c_\alpha |u|^{\alpha-1/2} I(u < 0)$, where $\alpha \in (0,1)$, then X is a fractional Brownian motion of dimension α (see the paper [6] by B. Mandelbrot and M. Van Ness). It is well known (see, for instance, [5; Ch. 4, §9, Example 2]) that, for $\alpha \in (0,1/2)\cup(1/2,1)$, X is not a semimartingale on its own filtration and hence, is not a $(\mathcal{F}_t^B)$-semimartingale.

The reason why the fractional Brownian motion is not a semimartingale is that the function $c_\alpha|u|^{\alpha-1/2}I(u<0)$ behaves badly near zero (it vanishes at zero for $\alpha \in (1/2,1)$ and tends to infinity near zero for $\alpha \in (0,1/2)$). P. Cheredito [2] considered a function φ of the following form: it equals $c_\alpha|u|^{\alpha-1/2}$ on $(-\infty,\delta]$ and equals $au+b$ on $[\delta,0]$. Here, $\delta<0$ and a, b are chosen in such a way that φ is continuous at the point δ. It is proved in [2] that the process X constructed by this function is a $(\mathcal{F}_t^B)$-semimartingale.

More generally, it is proved in [2] that if there exist $\alpha \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R})$ such that

$$\varphi(u) = \alpha + \int_u^0 \psi(v)\,dv, \quad u \leq 0 \tag{1.4}$$

(recall that $\varphi = 0$ on $(0,\infty)$), then X is a $(\mathcal{F}_t^B)$-semimartingale.

4. The main result. We prove in this paper the reverse statement: if X is a $(\mathcal{F}_t^B)$-semimartingale, then there exist $\alpha \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R})$ such that (1.4) holds (Theorem 3.1).

One may ask the following question: for which φ is X a semimartingale with respect to its own filtration $(\mathcal{F}_t^X)$? The general theory of semimartingales (see [5; Ch. 4, §6, Theorem 1]) guarantees that if X is a $(\mathcal{F}_t^B)$-semimartingale, then it is also a $(\mathcal{F}_t^X)$-semimartingale. Hence, condition (1.4) is sufficient for X to be a $(\mathcal{F}_t^X)$-semimartingale. However, it turns out that this condition is not necessary (see Example 3.4).

2 Definitions and Known Results

Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P}\big)$ be a filtered probability space. A $(\mathcal{F}_t)$-adapted càdlàg (i.e. right-continuous with left-hand limits) process $(X_t)_{t\geq 0}$ is called a $(\mathcal{F}_t)$*-semimartingale* if there exists a decomposition

$$X_t = X_0 + A_t + M_t \tag{2.1}$$

into a finite-variation process A and a $(\mathcal{F}_t)$-local martingale M with $A_0 = M_0 = 0$.

Proposition 2.1. *Suppose that X is a continuous $(\mathcal{F}_t)$-semimartingale. Then there exists a decomposition of the form* (2.1) *into a continuous finite-variation process and a continuous $(\mathcal{F}_t)$-local martingale M with $A_0 = M_0 = 0$. Moreover, such a decomposition is unique. It is called the canonical decomposition of X.*

For the proof, see [4; Ch. I, (4.24)].

Proposition 2.2. *Let $(X_t)_{t\geq 0}$ be a Gaussian semimartingale on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$. Set*

$$\begin{aligned}
H &= \big\{t \geq 0 : \mathsf{P}\{\Delta X_t \neq 0\} > 0\big\},\\
X'_t &= \int_0^t I(s \notin H)\, dX_s,\\
X''_t &= \int_0^t I(s \in H)\, dX_s.
\end{aligned}$$

Then H is countable and X' is continuous.

For the proof, see [9] or [5; Ch. 4, §9, Theorem 1].

Proposition 2.3. *Let $(X_t)_{t\geq 0}$ be a càdlàg Gaussian process with $\mathsf{E}X_t = 0$ $(t \geq 0)$. Then X is a finite-variation process if and only if there exists an increasing right-continuous function F such that, for any $0 \leq s \leq t$,*

$$\big(\mathsf{E}(X_t - X_s)^2\big)^{1/2} \leq F(t) - F(s).$$

For the proof, see [9] or [5; Ch. 4, §9, Theorem 3].

A collection of r.v. $(\xi_i)_{i\in\mathbb{N}}$ is called a *Gaussian system* if for any $n \in \mathbb{N}$, the vector $(\xi_1, \dots, \xi_n)$ is Gaussian.

Let $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \dots$. A $\mathcal{B}(\mathbb{R}^\infty)$-measurable function $N : \mathbb{R}^\infty \to [0, \infty]$ is called a *pseudonorm* if the set $N^{-1}([0, \infty))$ is a linear subspace of $\mathbb{R}^\infty$ and N is a seminorm on this subspace.

Proposition 2.4. *Suppose that $(\xi_i)_{i\in\mathbb{N}}$ is a Gaussian system with $\mathsf{E}\xi_i = 0$ $(i \in \mathbb{N})$. Let N be a pseudonorm on $\mathbb{R}^\infty$ such that $\mathsf{P}\{N(\xi) < \infty\} > 0$. Then there exists $\varepsilon > 0$ such that*

$$\mathsf{E}\exp\{\varepsilon N^2(\xi)\} < \infty.$$

For the proof, see [9] or [5; Ch. 4, §9, Lemma 2].

3 The Main Result

Theorem 3.1. *Let $(B_u)_{u\in\mathbb{R}}$ be a two-sided Brownian motion. Suppose that φ is a Borel function that is equal to zero on $(0,\infty)$ and satisfies condition (1.1). Let $(X_t)_{t\geq 0}$ be the moving average process given by (1.2). Then the following conditions are equivalent:*

(i) *X is a $(\mathcal{F}_t^B)$-semimartingale;*

(ii) *there exist $\alpha \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R})$ such that, for a.e. $u \leq 0$,*

$$\varphi(u) = \alpha + \int_u^0 \psi(v)\,dv. \tag{3.1}$$

Moreover, if (i), (ii) are satisfied, then X is a continuous $(\mathcal{F}_t^B)$-semimartingale with the canonical decomposition

$$X_t = \int_{\mathbb{R}} (\chi(u-t) - \chi(u))\,dB_u + \alpha B_t, \quad t \geq 0, \tag{3.2}$$

where α is given by (3.1) and $\chi(u) = \varphi(u) - \alpha\, I(u \leq 0)$. In particular, both processes A and M in the canonical decomposition are moving averages.

The proof of Theorem 3.1 is based on several auxiliary lemmas.

Lemma 3.2. *Let*

$$F : \mathbb{R}_+ \ni t \longmapsto F_t \in L^2(\mathbb{R}).$$

Suppose that there exists $c \geq 0$ such that, for any $0 \leq s \leq t$,

$$\|F_t - F_s\|_{L^2(\mathbb{R})} \leq c(t-s).$$

Then there exists a $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$-measurable function

$$\Psi : \mathbb{R}_+ \times \mathbb{R} \ni (t,u) \longmapsto \Psi_t(u) \in \mathbb{R}$$

such that, for any $t \geq 0$, $\|\Psi_t\|_{L^2(\mathbb{R})} \leq c$ and, for any $t \geq 0$,

$$F_t(u) - F_0(u) = \int_0^t \Psi_r(u)\,dr \quad \text{for a.e. } u \in \mathbb{R}.$$

Moreover, if $\widetilde{\Psi}$ is another function with the stated properties, then the set $\big\{(t,u) \in \mathbb{R}_+ \times \mathbb{R} : \Psi_t(u) \neq \widetilde{\Psi}_t(u)\big\}$ has zero Lebesgue measure.

`Proof`. *Existence*. Let $(h^i)_{i\in\mathbb{N}}$ be an orthonormal basis in $L^2(\mathbb{R})$. For any $i \in \mathbb{N}$, the function

$$f^i : [0,\infty) \ni t \longmapsto \langle F_t, h^i \rangle_{L^2(\mathbb{R})} \in \mathbb{R}$$

is Lipschitz continuous. Thus, for each $i \in \mathbb{N}$, there exists a Borel function $\psi^i : \mathbb{R}_+ \to \mathbb{R}$ with $|\psi^i| \leq c$ such that, for any $t \geq 0$,

$$f^i(t) - f^i(0) = \int_0^t \psi^i(r)\,dr$$

(see [7]).

For any $t \geq 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \left(k \int_{t}^{t+1/k} \psi^i(r)\, dr \right)^2 = k^2 \sum_{i=1}^{n} (f^i(t+1/k) - f^i(t))^2 \leq c^2.$$

On the other hand, for any $i \in \mathbb{N}$,

$$k \int_{t}^{t+1/k} \psi^i(r)\, dr \xrightarrow[k\to\infty]{} \psi^i(t) \quad \text{for a.e. } t \geq 0.$$

Hence, for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (\psi^i(t))^2 \leq c^2 \quad \text{for a.e. } t \geq 0,$$

and therefore,

$$\sum_{i=1}^{\infty} (\psi^i(t))^2 \leq c^2 \quad \text{for a.e. } t \geq 0. \tag{3.3}$$

Thus, the function

$$\Psi_t(u) := \sum_{i=1}^{\infty} \psi^i(t)\, h^i(u)$$

is well defined for a.e. $(t, u) \in \mathbb{R}_+ \times \mathbb{R}$.

It follows from (3.3) that, for a.e. $t \geq 0$,

$$\int_{\mathbb{R}} (\Psi_t(u))^2 du \leq c^2.$$

We can change Ψ on a set of zero Lebesgue measure in such a way that this inequality will hold for any $t \geq 0$. Hence, for any $t \geq 0$, the function G_t defined as

$$G_t(u) = \int_0^t \Psi_r(u)\, dr, \quad u \in \mathbb{R}$$

belongs to $L^2(\mathbb{R})$. By Fubini's theorem, for any $t \geq 0$, $i \in \mathbb{N}$,

$$\begin{aligned}
\langle G_t, h^i \rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} h^i(u) \int_0^t \Psi_r(u)\, dr du = \int_0^t \int_{\mathbb{R}} h^i(u)\, \Psi_r(u)\, du dr \\
&= \int_0^t \langle h^i, \Psi_r \rangle\, dr = \int_0^t \psi^i(r)\, dr \\
&= f^i(t) - f^i(0) = \langle F_t - F_0, h^i \rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Hence, for any $t \geq 0$, $G_t \overset{\text{a.e.}}{=} F_t - F_0$.

Uniqueness. Let $\widetilde{\Psi}$ be another function with the stated properties. Then, for a.e. $u \in \mathbb{R}$, the function

$$H_u(t) = \int_0^t (\Psi_r(u) - \widetilde{\Psi}_r(u))\, dr, \quad t \geq 0$$

is well defined and absolutely continuous on $\mathbb{R}_+$. Since for any $t \geq 0$,

$$\int_0^t (\Psi_r(u) - \widetilde{\Psi}_r(u))\, du = 0 \quad \text{for a.e. } u \in \mathbb{R},$$

we deduce that, for a.e. $u \in \mathbb{R}$, the function H_u is identically equal to zero. Furthermore, for a.e. $(t,u) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\Psi_t(u) - \widetilde{\Psi}_t(u) = \frac{d}{dt} H_u(t)$$

(see [7]). This completes the proof. $\square$

The following statement is borrowed from [9].

Lemma 3.3. *Let $(X_t)_{t\in[0,1]}$ be a continuous Gaussian semimartingale on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$. Let $X_t = X_0 + A_t + M_t$ be the canonical decomposition of X. Then, for any sequence $T^n = \{t_0^n, \dots, t_n^n\}$ of partitions of $[0,1]$ with $\max_{i=0,\dots,n-1}(t_{i+1}^n - t_i^n) \xrightarrow[n\to\infty]{} 0$, we have*

$$\sum_{i=0}^{n-1} \mathsf{E}\big[X_{t_{i+1}^n} - X_{t_i^n} \big| \mathcal{F}_{t_i^n}\big] \xrightarrow[n\to\infty]{L^2} A_1. \tag{3.4}$$

`Proof`. We have

$$Q_n(X) := \sum_{i=0}^{n-1} \big(X_{t_{i+1}^n} - X_{t_i^n}\big)^2 \xrightarrow[n\to\infty]{\mathsf{P}} \langle X\rangle_1. \tag{3.5}$$

Hence, there exists a subsequence $(n(k))$ such that

$$Q_{n(k)}(X) \xrightarrow[k\to\infty]{\text{a.s.}} \langle X\rangle_1.$$

Then

$$\sup_{k\in\mathbb{N}} Q_{n(k)}(X) < \infty \quad \text{a.s.}$$

In view of Proposition 2.4, there exists $\varepsilon > 0$ such that

$$\mathsf{E} \exp\big\{\varepsilon \sup_{k\in\mathbb{N}} Q_{n(k)}(X)\big\} < \infty. \tag{3.6}$$

Consequently,

$$\mathsf{E} \exp\{\varepsilon \langle X\rangle_1\} = \mathsf{E} \exp\{\varepsilon \langle M\rangle_1\} < \infty.$$

It follows now from the Burkholder-Davis-Gundy inequalities (see [8; Ch. IV, (4.1)]) that M is a square-integrable martingale.

Set

$$Q_n(A) = \sum_{i=1}^{n-1} \big(A_{t_{i+1}^n} - A_{t_i^n}\big)^2.$$

As A is a continuous finite-variation process,

$$Q_n(A) \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{3.7}$$

Moreover, $Q_n(A) \le 2Q_n(X) + 2Q_n(M)$, where $Q_n(X)$ is given by (3.5) and

$$Q_n(M) = \sum_{i=0}^{n-1} \big(M_{t_{i+1}^n} - M_{t_i^n}\big)^2.$$

It follows from (3.5) and (3.6) that

$$Q_{n(k)}(X) \xrightarrow[k\to\infty]{L^1} \langle X\rangle_1.$$

Furthermore,

$$Q_n(M) \xrightarrow[n\to\infty]{\mathsf{P}} \langle M\rangle_1.$$

As M is square-integrable, $\mathsf{E}Q_n(M) = \mathsf{E}\langle M\rangle_1$. This leads to

$$Q_n(M) \xrightarrow[n\to\infty]{L^1} \langle M\rangle_1.$$

Hence, the family $(Q_{n(k)})_{k\in\mathbb{N}}$ is uniformly integrable. In view of (3.7),

$$Q_{n(k)}(A) \xrightarrow[k\to\infty]{L^1} 0.$$

As M is a martingale, we have

$$\begin{aligned}
&\mathsf{E}\Bigg(A_1 - \sum_{i=0}^{n(k)-1} \mathsf{E}\big[X_{t_{i+1}^{n(k)}} - X_{t_i^{n(k)}} \big| \mathcal{F}_{t_i^{n(k)}}\big]\Bigg)^2\\
&= \mathsf{E}\Bigg(\sum_{i=0}^{n(k)-1} \Big(A_{t_{i+1}^{n(k)}} - A_{t_i^{n(k)}} - \mathsf{E}\big[A_{t_{i+1}^{n(k)}} - A_{t_i^{n(k)}} \big| \mathcal{F}_{t_i^{n(k)}}\big]\Big)\Bigg)^2\\
&\le \sum_{i=0}^{n(k)-1} \mathsf{E}\big(A_{t_{i+1}^{n(k)}} - A_{t_i^{n(k)}}\big)^2 = \mathsf{E}Q_{n(k)}(A) \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

Thus, we have extracted a subsequence $(T^{n(k)})$ from the sequence (T^n) such that the convergence in (3.4) holds along this subsequence. As this can be done for any sequence (T^n), we deduce that the convergence in (3.4) also holds along (T^n). $\square$

`Proof of Theorem 3.1.` **(ii)⇒(i)** It follows from Fubini's theorem that the function φ given by (3.1) satisfies condition (1.1). Hence, χ also satisfies this condition and the process

$$A_t = \int_{\mathbb{R}} (\chi(u-t) - \chi(u))\, dB_u, \quad t \ge 0$$

is well defined. Obviously, $X_t = A_t + \alpha B_t$.

Let us extend ψ to the whole real line by putting $\psi = 0$ on $(0, \infty)$. Then, for a.e. $u \in \mathbb{R}$,

$$\chi(u) = \int_u^0 \psi(v)\, dv.$$

Hence, for any $0 \le s \le t$,

$$\mathsf{E}(A_t - A_s)^2 = \int_{\mathbb{R}} (\chi(u-t) - \chi(u-s))^2 du = \int_{\mathbb{R}} \Big(\int_{u-t}^{u-s} \psi(v)\, dv \Big)^2 du$$
$$\le (t-s) \int_{\mathbb{R}} \int_{u-t}^{u-s} \psi^2(v)\, dv du = (t-s)^2 \int_{\mathbb{R}} \psi^2(v)\, dv.$$

By Kolmogorov's criterion, the process $(A_t)_{t \ge 0}$ admits a continuous modification. By Proposition 2.3, A is a finite-variation process. The process αB is a $(\mathcal{F}_t^B)$-martingale. Hence, X is a continuous $(\mathcal{F}_t^B)$-semimartingale with canonical decomposition (3.2).

(i)$\Rightarrow$(ii) Consider the set

$$H = \big\{ t \ge 0 : \mathsf{P}\{\Delta X_t \ne 0\} > 0 \big\}.$$

It follows from the equality $X_{r+t} - X_{r+s} \overset{\text{law}}{=} X_t - X_s$ that either $H = [0, \infty)$ or $H = \emptyset$. As H is countable (see Proposition 2.2), $H = \emptyset$. Applying Proposition 2.2 once more, we conclude that X is continuous.

For any $0 \le s \le t$, the r.v.

$$\int_{\mathbb{R}} (\varphi(u-t) - \varphi(u-s))\, I(u > s)\, dB_u$$

is independent of $\mathcal{F}_s^B$. Hence, for any $0 \le s \le t$,

$$\mathsf{E}[X_t - X_s | \mathcal{F}_s^B] = \int_{\mathbb{R}} (\varphi(u-t) - \varphi(u-s))\, I(u \le s)\, dB_u.$$

Let A be the finite-variation process appearing in the canonical decomposition of X. It follows from Lemma 3.3 that, for any $t \ge 0$ and any sequence $T^n = \{t_0^n, \dots, t_n^n\}$ of partitions of $[0, t]$ with $\max_{i=0,\dots,n-1}(t_{i+1}^n - t_i^n) \xrightarrow[n \to \infty]{} 0$, we have

$$A_t = (L^2) \lim_{n \to \infty} \int_{\mathbb{R}} \Big(\sum_{i=0}^{n-1} (\varphi(u - t_{i+1}^n) - \varphi(u - t_i^n))\, I(u \le t_i^n) \Big) dB_u = \int_{\mathbb{R}} F_t(u)\, dB_u, \quad (3.8)$$

where

$$F_t(\cdot) := (L^2(\mathbb{R})) \lim_{n \to \infty} \sum_{i=0}^{n-1} (\varphi(\cdot - t_{i+1}^n) - \varphi(\cdot - t_i^n))\, I(\cdot \le t_i^n). \quad (3.9)$$

Using (3.8) and (3.9), one can easily verify that $A_{t+r} - A_{s+r} \overset{\text{law}}{=} A_t - A_s$. Combining this with Proposition 2.3, we conclude that there exists $c \ge 0$ such that, for any $0 \le s \le t$,

$$\big(\mathsf{E}(A_t - A_s)^2 \big)^{1/2} \le c(t-s).$$

This means that, for any $0 \le s \le t$,

$$\|F_t - F_s\|_{L^2(\mathbb{R})} \le c(t-s).$$

Lemma 3.2 yields a $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$-measurable function Ψ such that, for any $t \ge 0$, $\|\Psi_t\|_{L^2(\mathbb{R})} \le c$ and, for any $t \ge 0$,

$$F_t(u) = \int_0^t \Psi_r(u)\, dr \quad \text{for a.e. } u \in \mathbb{R} \quad (3.10)$$

(note that $F_0 = 0$). In what follows, we will abbreviate (3.10) to

$$F_t \overset{\text{a.e.}}{=} \int_0^t \Psi_r \, dr.$$

Let us introduce the family $(\Theta_x)_{x \in \mathbb{R}}$ of shunt operators on $L^2(\mathbb{R})$ defined by

$$\Theta_x : L^2(\mathbb{R}) \ni h(\cdot) \longmapsto h(\cdot - x) \in L^2(\mathbb{R}).$$

Obviously, for any $s \geq 0$, $t \geq 0$,

$$\Theta_s F_t \overset{\text{a.e.}}{=} \int_0^t (\Theta_s \Psi_r) \, dr.$$

On the other hand, it follows from (3.9) that, for any $s \geq 0$, $t \geq 0$,

$$\Theta_s F_t \overset{\text{a.e.}}{=} F_{t+s} - F_s \overset{\text{a.e.}}{=} \int_s^{t+s} \Psi_r \, dr \overset{\text{a.e.}}{=} \int_0^t \Psi_{r+s} \, dr.$$

Comparing these two representations of $\Theta_s F_t$ and using the uniqueness result in Lemma 3.2, we get, for any $s \geq 0$,

$$\Theta_s \Psi_r \overset{\text{a.e.}}{=} \Psi_{r+s} \quad \text{for a.e. } r \geq 0.$$

Thus, the complement to the set $\left\{(s,t) \in \mathbb{R}_+^2 : \Psi_t \overset{\text{a.e.}}{=} \Theta_{t-s} \Psi_s\right\}$ has zero Lebesgue measure. Therefore, there exists $s_0 \geq 0$ such that, for a.e. $t \geq 0$,

$$\Psi_t \overset{\text{a.e.}}{=} \Theta_{t-s_0} \Psi_{s_0}. \tag{3.11}$$

By changing Ψ on a set of zero Lebesgue measure, we get a function $\widetilde{\Psi}$ for which (3.11) is true with any $t \geq 0$. Then, for any $t \geq 0$,

$$\widetilde{\Psi}_t \overset{\text{a.e.}}{=} \Theta_t \widetilde{\Psi}_0. \tag{3.12}$$

Set $\psi = \widetilde{\Psi}_0$. Since $F_t(u) = 0$ for $u \geq t$ (this follows from (3.9)), we deduce that $\psi(v) = 0$ for a.e. $v \geq 0$. Set

$$\chi(u) = \int_u^0 \psi(v) \, dv, \quad u \in \mathbb{R}.$$

Then, for any $t \geq 0$, $u \in \mathbb{R}$,

$$\chi(u-t) - \chi(u) = \int_{u-t}^u \psi(v) \, dv.$$

On the other hand, (3.10) and (3.12) yield, for any $t \geq 0$,

$$F_t(u) = \int_0^t \Psi_r(u) \, dr = \int_0^t \widetilde{\Psi}_r(u) \, dr = \int_0^t \psi(u-r) \, dr = \int_{u-t}^u \psi(v) \, dv \quad \text{for a.e. } u \in \mathbb{R}.$$

Consequently, for any $t \geq 0$,

$$F_t(u) = \chi(u-t) - \chi(u) \quad \text{for a.e. } u \in \mathbb{R}.$$

Recalling now (3.8), we get, for any $t \geq 0$,

$$A_t = \int_{\mathbb{R}} (\chi(u-t) - \chi(u))\, dB_u.$$

Let M be the local martingale appearing in the canonical decomposition of X. The above reasoning shows that, for any $t \geq 0$,

$$M_t = \int_{\mathbb{R}} \big((\varphi - \chi)(u-t) - (\varphi - \chi)(u)\big)\, dB_u.$$

It is easy to verify that, for any $t \geq 0$,

$$0 = \mathsf{E}[M_t | \mathcal{F}_0^B] = \int_{\mathbb{R}} \big((\varphi - \chi)(u-t) - (\varphi - \chi)(u)\big)\, I(u \leq 0)\, dB_u.$$

Hence, for any $t \geq 0$,

$$\int_{-\infty}^{0} \big((\varphi - \chi)(u-t) - (\varphi - \chi)(u)\big)^2 du = 0.$$

Thus, the set $\big\{(u,v) \in \mathbb{R}_-^2 : (\varphi - \chi)(u) \neq (\varphi - \chi)(v)\big\}$ has zero Lebesgue measure. Consequently, there exists $v_0 \leq 0$ such that, for a.e. $u \leq 0$,

$$(\varphi - \chi)(u) = (\varphi - \chi)(v_0).$$

Denoting $(\varphi - \chi)(v_0)$ by α, we get, for a.e. $u \leq 0$,

$$\varphi(u) = \alpha + \chi(u) = \alpha + \int_u^0 \psi(v)\, dv.$$

This completes the proof. □

In the following example, we construct a function φ such that φ satisfies condition (1.1) and the process X given by (1.2) is a $(\mathcal{F}_t^X)$-semimartingale while φ is not of the form (3.1).

Example 3.4. Consider the complex function

$$f : \mathbb{C} \ni z \longmapsto \frac{z - 1/2}{1 - z/2} \in \mathbb{C}.$$

Let $(a_k)_{k=0}^{\infty}$ be the coefficients of its Taylor decomposition at zero, i.e.

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 2.$$

Since $f(\overline{z}) = \overline{f(z)}$, each a_k is a real number.

Let g denote the restriction of f to $K = \{z : |z| = 1\}$. Let μ be the normalized Lebesgue measure on K. Then the decomposition

$$g(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in K$$

is the Fourier decomposition of g in $L^2(\mu, K)$ (note that $g \in L^2(K, \mu)$ since g is bounded). Hence,

$$\sum_{k=0}^{\infty} a_k^2 < \infty. \tag{3.13}$$

In view of the equality $|g(z)| = 1$ $(z \in K)$, we have, for any $l \in \mathbb{N}$,

$$\begin{aligned} 0 &= \int_K z^l \, d\mu = \int_K z^l g(z)\, \overline{g(z)}\, d\mu \\ &= \int_K \Big(\sum_{k=0}^{\infty} a_k z^{k+l}\Big)\Big(\sum_{m=0}^{\infty} a_m z^{-m}\Big) d\mu \\ &= \sum_{k=0}^{\infty} a_k a_{k+l}. \end{aligned} \tag{3.14}$$

Let us now consider the function

$$\varphi(u) = \sum_{k=0}^{\infty} a_k \, I(u \le -k), \quad u \in \mathbb{R}.$$

Property (3.13) guarantees that φ satisfies condition (1.1). Let $[s,t]$ and $[s',t']$ be two disjoint intervals on $[0,\infty)$ with $t-s \le 1$, $t'-s' \le 1$. The function

$$\Delta_{st}(u) = \varphi(u-t) - \varphi(u-s), \quad u \in \mathbb{R}$$

is a step function that equals a_k on $(s-k, t-k]$ and equals zero on $\mathbb{R} \setminus \bigcup_{k=0}^{\infty} (s-k, t-k]$. The same is true for $\Delta_{s't'}$. Hence, there exist $l \in \mathbb{N}$, $c \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} \Delta_{st}(u)\, \Delta_{s't'}(u)\, du = c \sum_{k=0}^{\infty} a_k a_{k+l} = 0.$$

Using the equality $\Delta_{t_1 t_3} = \Delta_{t_1 t_2} + \Delta_{t_2 t_3}$, one can easily show that, for disjoint intervals $[s,t]$ and $[s',t']$ of arbitrary length,

$$\int_{\mathbb{R}} \Delta_{st}(u)\, \Delta_{s't'}(u)\, du = 0. \tag{3.15}$$

Now, let X be the process given by (1.2). Equality (3.15) shows that the increments $X_t - X_s$ and $X_{t'} - X_{s'}$ over disjoint intervals $[s,t]$ and $[s',t']$ are independent. Hence, X is a martingale on its own filtration. On the other hand, φ does not have the form (3.1) (otherwise all a_k with $k \ge 1$ would be equal to zero). $\square$

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