RH-05-00
October 2000

Discrete approximations to integrals over unparametrized paths

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Abstract. We discuss measures on spaces of unparametrized paths related to the Wiener measure. These measures arise naturally in the study of one-dimensional gravity coupled to scalar fields. Two kinds of discrete approximations are defined, the piecewise linear and the hypercubic approximations. The convergence of these approximations in the sense of weak convergence of measures is proven. We describe a family of sets of unparametrized paths that are analogous to cylinder sets of parametrized paths. Integrals over some of these sets are evaluated in terms of Dirichlet propagators in bounded regions.

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1 Introduction

In quantum field theory and string theory one frequently encounters the problem of integrating over geometrical objects, e.g., Riemannian manifolds or Riemannian manifolds with some additional structure. One wishes to define a measure on sets of geometrical objects and integrate functions that are independent of the coordinates used to describe the objects. The prime example of a theory where this problem arises is the path integral quantization of general relativity where one attempts to give meaning to expressions of the form

$$\langle F \rangle = \int e^{-S(g)} F([g]) D[g],$$

where g is a Riemannian metric on a manifold M, $[g]$ is the equivalence class of g under diffeomorhisms of M, F is a function and $S(g)$ is a diffeomorphism invariant action functional, e.g., the Einstein–Hilbert action [1, 2]. Giving a mathematical meaning to expressions of this form is largely an unsolved problem but some headway has been made, mainly in two dimensions, see [3] and references therein.

One of the strategies used in physics to deal with functional integrals of this type is to introduce discretizations of the geometrical objects under consideration and try to prove convergence of the discretization as a cutoff parameter, e.g., a lattice spacing, is taken to zero. It inspires confidence in the results obtained when different discretizations lead to identical continuum results. This approach is described in detail in the monograph [3].

For one-dimensional objects, i.e., when the functional integral is over paths, the situation is radically different from the higher dimensional analogues, since we have measures on parametrized paths in $\mathbb{R}^d$ (for example the Wiener measure) that are mathematically well-understood and give rise to measures on unparametrized paths as we shall discuss below. We study two different discretizations of integrals over unparametrized paths and show that the discrete measures converge to the appropriate continuum measure.

In ordinary quantum field theory applications of random paths it is often the convergence of regularized propagators that is of main interest and various results in this vein have been known for a long time. Our main interest is the convergence of the underlying measures on unparametrized paths, whereas convergence of propagators merely means convergence of the total volume of the measures. Corresponding problems in non-relativistic quantum mechanics normally involve only parametrized paths. In this case various aspects of discrete approximations pertaining to the Wiener-measure on paths parametrized by a finite time interval have been discussed by many authors, see, e.g., [4] and references therein.

This paper is organized as follows. In the next section we introduce the models of discretized random paths we wish to study and give a proof of pointwise convergence of the lattice propagator to the continuum propagator, that will be needed later. In section 3 we define the appropriate path spaces, the continuum measures and the discretized measures. In section 4 we use standard tools of probability theory to prove the convergence of the discretized measures. In section 5 we determine a family of sets of unparamterized paths that generates the Borel sets of unparametrized paths and plays a role similar to the one played by cylinder sets for the Wiener-measure. Finally, in section 6 we apply the results of the previous sections to evaluate the measure of some of these sets.

2 Propagators

Let Δ denote the Laplacian in $\mathbb{R}^d$. It is well-known that the Euclidean propagator

$$G(x,y) = 2(-\Delta + m^2)^{-1}(x,y) \tag{1}$$

of a scalar particle of mass $m > 0$ in $\mathbb{R}^d$ has the path integral representation

$$G(x,y) = \int_{\omega : x \to y} e^{-m|\omega|}\, D\omega, \tag{2}$$

where ω is a path from x to y in $\mathbb{R}^d$ and $|\omega|$ denotes its length. The most straightforward interpretation of the formal expression on the right hand side of Eq. (2) is obtained by regarding it as a limit of lattice propagators. We replace $\mathbb{R}^d$ by the hypercubic lattice $a\mathbb{Z}^d$ with lattice spacing a and define a lattice propagator as

$$G^a(x,y) = a^{2-d} \sum_{\omega : x \to y} e^{-m(a)|\omega|} \tag{3}$$

for $x, y \in a\mathbb{Z}^d$ where the sum is over all lattice paths from x to y. The prefactor a^{2-d} is dictated by dimensional considerations and the dependence of the parameter $m(a)$ (lattice mass) on a is determined by the requirement that $G^a(x,y)$ converge to $G(x,y)$ as $a \to 0$.

Using translation invariance we may set $G(x,y) = G(x-y)$ and $G^a(x,y) = G^a(x-y)$. The Fourier transform of the lattice propagator is then

$$\begin{aligned} \widehat{G^a}(k) &= a^d \sum_{x\in a\mathbb{Z}^d} G^a(x)e^{-ik\cdot x} \\ &= e^{am(a)}\left(m^2 + 2a^{-2}\sum_{j=1}^{d}(1-\cos(ak_j))\right)^{-1} \end{aligned} \tag{4}$$

where $k \in [-\pi/a, \pi/a]^d$ and $m(a)$ is given by the equation

$$e^{am(a)} = 2d + m^2a^2. \tag{5}$$

Evidently this implies the desired uniform convergence in momentum space

$$d^{-1}\widehat{G^a}(k) \to 2(k^2+m^2)^{-1} = \widehat{G}(k) \tag{6}$$

as $a \to 0$, for any $k \in \mathbb{R}^d$.

Pointwise convergence in space-time can be obtained as follows. We extend the lattice propagator from $a\mathbb{Z}^d$ to a smooth function on $\mathbb{R}^d$ by setting

$$G^a(x) = \frac{1}{(2\pi)^d}\int_{[-\pi/a,\pi/a]^d} \widehat{G^a}(k)e^{-ik\cdot x}\,dk \tag{7}$$

for any $x \in \mathbb{R}^d$. For $\alpha = (\alpha_1, \ldots, \alpha_d)$, where the α_i's are non-negative integers, let

$$\partial^\alpha = \prod_{i=1}^{d}\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}.$$

Defining $\widehat{G^a}(k) = 0$ outside $[-\pi/a, \pi/a]^d$ it is easily verified that

$$\partial^\alpha\widehat{G^a}(k) \to \partial^\alpha\widehat{G}(k)$$

uniformly on $\mathbb{R}^d$ for any multiindex α. Moreover, there is a constant c_α such that

$$|\partial^\alpha\widehat{G^a}(k)| \leq c_\alpha(k^2+m^2)^{-1-|\alpha|/2} \tag{8}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Thus, choosing $|\alpha| > d$, the right hand side of Eq. (8) is integrable so the dominated convergence theorem together with Fourier inversion implies that

$$d^{-1}x^\alpha G^a(x) \to x^\alpha G(x) \tag{9}$$

as $a \to 0$, where $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$. In particular,

$$d^{-1} G^a(x) \to G(x) \tag{10}$$

for $x \neq 0$.

There is another path integral representation of the propagator $G(x, y)$ introduced in [5] whose analogue for surfaces has played an important role in string theory in recent years [6]. The alternative representation is given by

$$G(x, y) = \int_{\omega : x \to y} \exp\left(-\frac{1}{2} \int \left(|\dot{\omega}|^2 e^{-1} + m^2 e\right) dt\right) D\omega De, \tag{11}$$

where the integration is over paths ω in $\mathbb{R}^d$ from x to y and over intrinsic metrics e on the paths. An intrinsic metric on the path is simply a positive definite function defined on the path. In order to give a meaning to Eq. (11), we note an important common feature of the two action functionals

$$S_1(\omega) = m|\omega| = m \int |\dot{\omega}| \, dt \tag{12}$$

and

$$S_2(\omega, e) = \frac{1}{2} \int \left(|\dot{\omega}|^2 e^{-1} + m^2 e\right) dt \tag{13}$$

which occur in the path integrals (2) and (11). The actions are invariant under reparametrizations

$$\begin{aligned} t' &= \varphi(t) \\ \omega'(t') &= \omega(t) \\ e'(t') &= \frac{e(t)}{\dot{\varphi}(t)}, \end{aligned} \tag{14}$$

where φ is an increasing diffeomorphism between intervals. Thus the path integrations in Eqs. (2) and (11) should be regarded as being taken over diffeomorphism classes of paths in the first case and over diffeomorphism classes of paths and metrics in the second one. The standard method for dealing with functional integration over such orbit spaces is the so called Faddeev-Popov procedure. We discuss the orbit spaces and the appropriate measures on them more thoroughly in Section 3. For the moment we note that any pair (ω, e) can uniquely be reparametrized to (ω', e') such that the parameter interval of the latter is $[0,1]$ and the metric e' is constant on $[0,1]$ and equal to the volume

$$T \equiv \int e(t)dt \tag{15}$$

of e, which is parametrization independent. It follows that the path integral (11) is effectively an integral over T and over paths ω parametrized on $[0,1]$. An interpretation of (11) is then obtained by subdividing $[0,1]$ into N subintervals of length N^{-1} and letting ω be an N-step piecewise linear path $x = x_0 \to x_1 \to \cdots \to x_N = y$ for which

$$\int_0^1 |\dot{\omega}|^2 \, dt = \frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{x_{i+1}-x_i}{N^{-1}}\right)^2 = N\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2 \, .$$

Setting

$$a^2 = \frac{T}{N}$$

we have

$$\begin{aligned}
H^a(x,y) \equiv \; & \frac{a^2}{(2\pi a^2)^{d/2}}\sum_{N=1}^{\infty}\int\prod_{i=1}^{N-1}\frac{dx_i}{(2\pi a^2)^{\frac{d}{2}}}\exp\left(-\frac{1}{2}\sum_{i=0}^{N-1}\frac{|x_{i+1}-x_i|^2}{a^2}-\frac{1}{2}m^2a^2N\right)\\
= \; & a^2\sum_{N=1}^{\infty}(2\pi a^2 N)^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{2a^2N}-\frac{1}{2}m^2a^2N\right)\\
\to \; & \int_0^{\infty}(2\pi T)^{-\frac{d}{2}}\exp\left(-\frac{|x-y|^2}{2T}-\frac{1}{2}m^2T\right)dT\\
= \; & G(x,y)
\end{aligned} \tag{16}$$

for $x \neq y$, as $a \to 0$. Hence, the function $H^a(x,y)$ defined here provides a discrete approximation to $G(x,y)$. In the same way as for the hypercubic lattice approximation we show in the Section 4 that the measures on piecewise linear paths defined by the approximation H^a converge to a continuum path measure which attributes a proper meaning to Eq. (11).

3 The continuum measures and discrete approximations

As noted in the previous section the appropriate space to integrate over in Eqs. (2) and (11) consists of equivalence classes of paths under reparametrizations. In this section we define those orbit spaces and the relevant measures.

3.1 Picewise linear paths

It is convenient to start with Eq. (11) and for notational and technical simplicity to consider first paths with only one fixed endpoint x. Let $\Gamma(x)$ be the space consisting of pairs (e, ω) where $e : [0,1] \to \mathbb{R}$ is a positive continuous function and $\omega : [0,1] \to \mathbb{R}^d$ is continuous with $\omega(0) = x$. Let $\mathrm{Diff}_+[0,1]$ denote the set of all increasing diffeomorphisms of the unit interval. As remarked in the previous section there is a unique $\varphi \in \mathrm{Diff}_+[0,1]$ such that the reparametrised pair (e', ω') defined by Eq. (14) has $e' = T$ where T is a constant. Hence we conclude that

$$\tilde{\Gamma}(x) \equiv \Gamma(x)/\,\mathrm{Diff}_+[0,1] = \mathbb{R}_+ \times \Omega(x),$$

where $\Omega(x)$ denotes the set of continuous paths $\omega : [0,1] \to \mathbb{R}^d$ with $\omega(0) = x$.

Let us define a metric $\tilde{d}$ on $\tilde{\Gamma}(x)$ by

$$\tilde{d}((T,\omega),(T',\omega')) = |T - T'| + d(\omega,\omega'),$$

where d is the standard uniform metric on $\Omega(x)$ defined by

$$d(\omega,\omega') = \sup\{|\omega(s) - \omega'(s)| : s \in [0,1]\}.$$

Equipped with $\tilde{d}$ the set $\tilde{\Gamma}(x)$ becomes a separable metric space. The discussion of probability measures and their convergence properties is particularly convenient on complete metric spaces (see, e.g., [8]). Since $\Omega(x)$ with the metric d is complete we can complete $\tilde{\Gamma}(x)$ by adjoining $0 \times \Omega(x)$. This will be assumed in the folowing. All measures on $\tilde{\Gamma}(x)$ that will be considered vanish identically on $0 \times \Omega(x)$.

On $\Omega(x)$ we have the family of Wiener measures W_x^t, $t > 0$, defined on the Borel subsets of $\Omega(x)$. Here t denotes the variance of the measure. We note that W_x^t is uniquely defined by the characteristic functions of its finite dimensional distributions which are given for $0 < t_1 < t_2 < \cdots < t_n \leq 1$ by

$$\begin{aligned} p^t_{t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) &= \int \exp\left(i\xi_1\cdot\omega(t_1) + \cdots + i\xi_n\cdot\omega(t_n)\right)\, dW_x^t(\omega) \\ &= \int \prod_{i=1}^n dx_i (2\pi t(t_i - t_{i-1}))^{-d/2} \exp\left(-\frac{|x_i - x_{i-1}|^2}{2t(t_i - t_{i-1})} + i\xi_i\cdot x_i\right) \\ &= \exp\left(-\frac{t}{2}\sum_{i=1}^n (t_i - t_{i-1})(\xi_i + \cdots + \xi_n)^2 + ix\cdot(\xi_1 + \cdots + \xi_n)\right), \end{aligned} \tag{17}$$

where $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$, $t_0 = 0$ and $x_0 = x$.

For a Borel set $B \subseteq \tilde{\Gamma}(x)$ we let

$$B_t = \{\omega : (t, \omega) \in B\} \quad \text{for} \quad t > 0\,,$$

and define the measure W_x on $\tilde{\Gamma}(x)$ by

$$W_x(B) = \int_0^\infty e^{-\frac{1}{2}m^2 t}\, W_x^t(B_t) dt\,.$$

The above definition requires $t \to W_x^t(B_t)$ to be a measurable function. Rather than proving this directly we show that this must be the case by giving an alternative definition of W_x. First, let $x = 0$ and consider the product M of Lebesgue measure on $\mathbb{R}_+$ and W_0^1 on $\Omega(0)$, i.e.,

$$dM(t, \omega) = dt dW_0^1(\omega)\,.$$

Defining a homeomorphism h of $\mathbb{R}_+ \times \Omega(0)$ onto itself by $h(t, \omega) = (t, t^{-\frac{1}{2}}\omega)$ and observing that

$$W_0^t(A) = W_0^1(t^{-\frac{1}{2}}A)$$

for Borel sets $A \subseteq \Omega(0)$, it follows that we have a measure W_0 on $\tilde{\Gamma}(0)$ given by

$$W_0(B) = \int_B e^{-\frac{1}{2}m^2 t} d(M \circ h)(t, \omega)\,,$$

where the measure $M \circ h$ on $\mathbb{R}_+ \times \Omega(0)$ is defined by $(M \circ h)(B) = M(h(B))$ for Borel sets $B \subseteq \mathbb{R}_+ \times \Omega(0)$. This shows that W_0 is well defined. For arbitrary x we obtain W_x as the translation of W_0 by x.

To set up the discrete approximation to W_x, given by Eq. (16) for the propagator, let $\tilde{\Gamma}_{a,N}(x) \subseteq \tilde{\Gamma}(x)$ be the set of pairs (T, ω), where $T = a^2 N$ and ω is an N-step piecewise linear path $x = x_0 \to x_1 \to \cdots \to x_N$ such that the step $x_{i-1} \to x_i$ is parametrized linearly by the interval $[(i-1)/N, i/N]$. Define the measure $W_{x,a,N}$ on $\tilde{\Gamma}(x)$ supported on $\tilde{\Gamma}_{a,N}(x)$ by

$$dW_{x,a,N}(T, \omega) = \prod_{i=1}^N dx_i (2\pi a^2)^{-\frac{d}{2}} \exp\left(-\frac{1}{2a^2}|x_i - x_{i-1}|^2\right)\,. \tag{18}$$

For $N = 0$ we let $W_{x,a,0}$ be the Dirac measure at the trivial (constant) path. The approximating measure $W_{x,a}$ on $\tilde{\Gamma}(x)$ is supported on the set

$$\tilde{\Gamma}_a(x) \equiv \bigcup_{N=0}^\infty \tilde{\Gamma}_{a,N}(x)$$

and defined by

$$W_{x,a} = (1 - e^{-\frac{1}{2}m^2a^2}) \sum_{N=0}^{\infty} e^{-\frac{1}{2}m^2a^2N} W_{x,a,N} \ . \tag{19}$$

The normalization factor in Eq. (19) has been chosen such that $W_{x,a}$ is a probability measure, whereas the volume of W_x is $W_x(\tilde{\Gamma}(x)) = \frac{2}{m^2}$. We prove the following result in the next section.

Theorem 3.1. $\quad W_{x,a} \to \frac{m^2}{2} W_x \quad \text{as} \quad a \to 0 \ .$

Here and in the following convergence of measures is in the sense of *weak convergence*, i.e.,

$$\int f dW_{x,a} \to \int f dW_x \quad \text{as} \ \ a \to 0 \ ,$$

for all bounded continuous functions f on $\tilde{\Gamma}(x)$.

3.2 Lattice paths

Next let us discuss the measure pertaining to Eq. (2). The relevant orbit space is now

$$\tilde{\Omega}(x) = \Omega(x) / \operatorname{Diff}_+[0,1] = \{[\omega] : \omega \in \Omega(x)\} \, ,$$

where $[\omega] = \{\omega \circ \varphi \mid \varphi \in \operatorname{Diff}_+[0,1]\}$. The quotient space $\tilde{\Omega}(x)$ inherits in a standard fashion a pseudo-metric $\bar{d}$ from the metric d on $\Omega(x)$, given by

$$\bar{d}([\omega],[\omega']) = \inf\{d(\omega, \omega' \circ \varphi) : \varphi \in \operatorname{Diff}_+[0,1]\} \ .$$

Here the term pseudo-metric means that $\bar{d}([\omega],[\omega']) = 0$ may occur even if $[\omega] \neq [\omega']$. For example, we have $\bar{d}([\omega],[\omega \circ f]) = 0$ whenever $f : [0,1] \to [0,1]$ is a uniform limit of increasing diffeomorphisms. This defect is eliminated by taking a further quotient setting

$$\bar{\Omega}(x) = \{\bar{\omega} : \omega \in \Omega(x)\} \ ,$$

where $\bar{\omega} = \{[\omega'] : \bar{d}([\omega],[\omega']) = 0\}$. Then $\bar{d}$ defines a metric on $\bar{\Omega}(x)$, and it is straightforward to verify that $\bar{\Omega}(x)$ is a complete separable metric space.

It is not hard to see that the same space $\bar{\Omega}(x)$ results from the above construction if, e.g., we replace $\mathrm{Diff}_+[0,1]$ by the group $\mathrm{Homeo}_+[0,1]$ of increasing homeomorphisms of the unit interval. Let us also note that evidently the quotient map $\pi : \Omega(x) \to \bar{\Omega}(x)$ is continuous.

The measure W_x on $\tilde{\Gamma}(x) = \mathbb{R}_+ \times \Omega(x)$ constructed in the previous subsection gives rise to a measure V'_x on $\Omega(x)$ by integration over the t-variable,

$$V'_x(A) = W_x(\mathbb{R}_+ \times A) = \int_0^\infty e^{-\frac{1}{2}m^2 t} W_x^t(A) dt$$

for Borel sets $A \subseteq \Omega(x)$. Transporting this measure to $\bar{\Omega}(x)$ by π we obtain a measure V_x given by

$$V_x(\bar{A}) = V'_x(\pi^{-1}(\bar{A})) \ .$$

This measure is defined on those sets $\bar{A}$ for which $\pi^{-1}(\bar{A})$ is a Borel set. This σ-algebra contains the Borel algebra of $\bar{\Omega}(x)$ since π is continuous and we claim that the measure so defined is the appropriate one to associate to Eq. (2).

In order to define the corresponding lattice approximation let $\Omega_{a,N}(x)$ denote the set of parametrized paths in $x + a\mathbb{Z}^d$ with N steps, such that the ith step is linearly parametrized by $[\frac{i-1}{N}, \frac{i}{N}]$. Here x is an arbitrary point in $\mathbb{R}^d$. We let the discrete measure $V'_{x,a,N}$ on $\Omega(x)$, supported on $\Omega_{a,N}(x)$ be defined by

$$V'_{x,a,N}(\omega) = e^{-\beta_0 N} \quad \text{for} \ \ \omega \in \Omega_{a,N}(x) \ , \tag{20}$$

where $\beta_0 = \log 2d$, i.e., $V'_{x,a,N}$ is a normalized counting measure.

Furthermore, in correspondence with Eqs. (3) and (5) we define the measure $V'_{x,a}$ on $\Omega(x)$ supported on

$$\Omega_a(x) \equiv \bigcup_{N=0}^{\infty} \Omega_{a,N}(x)$$

by

$$V'_{x,a} = (1 - e^{-\frac{1}{2d}m^2 a^2}) \sum_{N=0}^{\infty} e^{-\frac{1}{2d}m^2 a^2 N} V'_{x,a,N} \ . \tag{21}$$

Here, $V'_{x,a,0}$ denotes the Dirac measure at the trivial path in $\Omega(x)$, and the normalisation has been chosen such that $V'_{x,a}$ is a probability measure.

Similarly, we define

$$\bar{\Omega}_{a,N}(x) = \pi(\Omega_{a,N}(x))$$

and

$$\bar{\Omega}_a(x) = \pi(\Omega_a(x)) = \bigcup_{N=0}^{\infty} \bar{\Omega}_{a,N}(x)\,.$$

Correspondingly we define the transported measures $V_{x,a,N}$ and $V_{x,a}$ given by

$$V_{x,a,N}(\bar{A}) = V'_{x,a,N}(\pi^{-1}(\bar{A})) \tag{22}$$

and

$$V_{x,a}(\bar{A}) = (1 - e^{-\frac{1}{2d}m^2a^2}) \sum_{N=0}^{\infty} e^{-\frac{1}{2d}m^2a^2N} V_{x,a,N}(\bar{A}) \tag{23}$$

for Borel sets $\bar{A} \subseteq \bar{\Omega}(x)$. With these definitions we then have

Theorem 3.2. $\quad V_{x,a} \to \frac{m^2}{2} V_x \quad$ as $\quad a \to 0$.

This result is proven in the subsequent section as a consequence of the stronger result $V'_{x,a} \to \frac{m^2}{2} V'_x$ as $a \to 0$.

3.3 Paths with two fixed endpoints

Let us briefly discuss paths with both endpoints x, y fixed. It is straightforward to introduce analogues to the spaces defined above for paths with one fixed endpoint. We shall use the same notation except that x is everywhere replaced by x, y. On $\Omega(x, y)$ the family of Wiener measures $W^t_{x,y}$, $t > 0$, is defined by the characteristic functions

$$\begin{aligned} q^t_{t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) &= \int \exp\left(i\xi_1\cdot\omega(t_1) + \cdots + i\xi_n\cdot\omega(t_n)\right)\, dW^t_{x,y}(\omega) \\ &= \int \prod_{i=1}^{n} dx_i (2\pi t(t_i - t_{i-1}))^{-d/2} \exp\left(-\frac{|x_i - x_{i-1}|^2}{2t(t_i - t_{i-1})} + i\xi_i\cdot x_i\right) \\ &\times \; (2\pi t(1-t_n))^{-d/2} \exp\left(-\frac{|y - x_n|^2}{2t(1-t_n)}\right) \end{aligned} \tag{24}$$

$$= Z^t_{x,y} \exp\left(-\frac{t}{2}\sum_{i=1}^{n}(t_i - t_{i-1})(\xi_i + \cdots + \xi_n)^2 - \left(\sum_{i=1}^{n} t_i\xi_i\right)^2 + i\sum_{i=1}^{n}(t_i y + (1-t_i)x)\xi_i\right)$$

where

$$Z^t_{x,y} = (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}}\,,$$

the volume of $W^t_{x,y}$, is simply the heat kernel.

We then define the measure $W_{x,y}$ on $\tilde{\Gamma}(x,y)$ for $x \neq y$ by

$$W_{x,y}(B) = \int_0^\infty e^{-\frac{1}{2}m^2 t} W^t_{x,y}(B_t) dt \; .$$

where $B \subseteq \tilde{\Gamma}(x,y)$ is a Borel set and $B_t \subseteq \Omega(x,y)$ is defined as previously. The fact that this expression is well defined is shown in a similar way as for W_x by first noting that

$$W^t_{0,0}(A) = t^{-\frac{d}{2}} W^1_{0,0}(t^{-\frac{1}{2}} A)$$

for Borel sets $A \subseteq \Omega(0,0)$, and then using

$$W^t_{x,y}(A) = \exp\left(-\frac{|x-y|^2}{2t}\right) W^1_{0,0}(A - \omega_{x,y}) \; ,$$

where $\omega_{x,y}$ is the linear path from x to y and A is a Borel subset of $\Omega(x,y)$. The last relation is a direct consequence of Eq. (24).

Having defined $W_{x,y}$ the measures $V'_{x,y}$ and $V_{x,y}$ are defined in a similar way as V'_x and V_x.

The piecewise linear approximation is defined in analogy with Eq. (19) by

$$W_{x,y,a} = (1 - e^{-\frac{1}{2}m^2 a^2}) \sum_{N=0}^\infty e^{-\frac{1}{2}m^2 a^2 N} W_{x,y,a,N} \; , \tag{25}$$

where

$$dW_{x,y,a,N}(T,\omega) = \prod_{i=1}^N dx_i (2\pi a^2)^{-\frac{d}{2}} \exp\left(-\frac{1}{2a^2}|x_i - x_{i-1}|^2\right) \tag{26}$$

for an N-step piecewise linear path $\omega : x = x_0 \to x_1 \to \cdots \to x_{N-1} \to x_N = y$. Here $W_{x,y,a,1}$ is the Dirac measure $\delta_{(1,\omega_0)}$, where ω_0 is the linear path from x to y, and $T = a^2 N$ as before.

Similarly, the hypercubic approximation is defined for $x \neq y$, $x - y \in a\mathbb{Z}^d$, by

$$V'_{x,y,a} = (1 - e^{-m^2 a^2}) \sum_{N=1}^\infty e^{-m^2 a^2 N} V'_{x,y,a,N} \; , \tag{27}$$

where

$$V'_{x,y,a,N}(\omega) = a^{-d} e^{-\beta_0 N} \quad \text{for} \;\; \omega \in \Omega_{a,N}(x,y) \; , \tag{28}$$

and $V_{x,y,a}$ is obtained by transporting to $\bar{\Omega}(x,y)$ by the quotient map π. Since in all cases we are interested in the limit $a \to 0$ we shall assume $0 < a < 1$.

It should be noted that in contrast to the case of paths with one fixed endpoint, the approximating measures defined here are not probability measures. The volume of $W_{x,y,a,N}$ is obtained by explicit computation and equals

$$Z_{x,y}^{a^2 N} = (2\pi a^2 N)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2a^2 N}} \ , \tag{29}$$

which by insertion into (25) immediately shows that the volume of $W_{x,y,a}$ equals $\left(1 - e^{-\frac{1}{2}m^2 a^2}\right) a^{-2} H^a(x,y)$ and converges to $\frac{m^2}{2} G(x,y)$ as $a \to 0$ according to Eq. (16). Similarly, the volume of $V_{x,y}$ equals $\left(1 - e^{-\frac{1}{2d}m^2 a^2}\right) a^{-2} G^a(x,y)$ and converges to $\frac{m^2}{2} G(x,y)$ as $a \to 0$ according to Eq. (10). On the other hand, the volume of $W_{x,y}$ and of $V_{x,y}$ both equal $G(x,y)$. The convergence of volumes extends to the following result.

Theorem 3.3. $W_{x,y,a} \to \frac{m^2}{2} W_{x,y}$ and $V_{x,y,a} \to \frac{m^2}{2} V_{x,y}$ as $a \to 0$ for $x \neq y$.

The proof is given in the next section.

4 Convergence of the approximations

In a complete separable metric space M there is a standard two-step procedure for proving convergence of a family m_a, $a > 0$, of Borel probability measures to a measure m. The first step is to verify that m_a, $a > 0$, is a *tight* (or *precompact*) family. This means that for every $\eta > 0$ there exists a compact set $K \subseteq M$ such that $m_a(K) \geq 1 - \eta$ for all $a > 0$. The second step is to show that

$$\int_M f \, dm_a \to \int_M f \, dm \tag{30}$$

as $a \to 0$ for a collection of functions that determine the measure in the sense that if the integrals of these functions coincide for two measures then the measures coincide. Of course, the first step is superfluous if one can establish the convergence (30) for all bounded continuous functions f. But this only happens rarely. Generally, the first step ensures that every sequence m_{a_n} from the given family has a convergent subsequence, and the second step then implies that its limit is independent of the chosen sequence or subsequence. For the spaces $\Omega(x)$ and $\Omega(x,y)$ the second step can be accomplished by proving convergence of the characteristic functions of the finite dimensional distributions. For the spaces $\tilde{\Gamma}(x)$ and $\tilde{\Gamma}(x,y)$ a little more is required as we discuss below.

In the following four lemmas we show that the approximations introduced in the previous section form tight families.

Lemma 4.1. $W_{x,a},\ 0 < a < 1$, is a tight family of measures on $\tilde{\Gamma}(x)$.

Proof. The following is an adaptation of the corresponding argument for the piecewise linear approximations to the measure W_x^t (see [8]). According to the Arzela-Ascoli theorem the sets of compact closure in $\Omega(x)$ are the equicontinuous ones. Defining the modulus of continuity

$$m(\omega,\delta) = \sup\{|\omega(s)-\omega(t)| : |s-t| < \delta\} \quad \text{for} \quad \delta > 0,\ \omega \in \Omega(x)\ ,$$

it follows that complements to sets of the form

$$C = \bigcup_{n=1}^{\infty} \left\{ \omega : m(\omega,\delta_n) > \frac{1}{n} \right\} \tag{31}$$

have compact closures in $\Omega(x)$ for an arbitrary sequence $\{\delta_n\}$ of positive numbers. We observe that by Eq. (19)

$$W_{x,a}\left([t_0,+\infty)\times\Omega(x)\right) < \eta \quad \text{if} \quad t_0 > -m^{-2}\log\eta \tag{32}$$

for any $\eta > 0$ and all $a > 0$. In order to prove the lemma it therefore suffices to show that for any $\eta,\ \varepsilon,\ t_0 > 0$ there exists a $\delta > 0$ such that

$$W_{x,a}([0,t_0]\times\{\omega\in\Omega(x) : m(\omega,\delta) > \varepsilon\}) < \eta \tag{33}$$

for all $a > 0$.

By Eq. (19) it follows that Eq. (33) holds if

$$W_{x,a,N}\left(\{(a^2N,\omega)\in\tilde{\Gamma}(x) : m(\omega,\delta) > \varepsilon\}\right) < \eta \quad \text{for} \quad a^2N \le t_0\ . \tag{34}$$

But for a, N as in Eq. (34) we have

$$\begin{aligned} W_{x,a,N}(\{(a^2N,\omega) : m(\omega,\delta) > \varepsilon\}) &= W_{x,1,N}(\{(N,\omega) : m(\omega,\delta) > \frac{\varepsilon}{a}\}) \\ &\le W_{x,1,N}(\{(N,\omega) : m(\omega,\delta) > \frac{\varepsilon\sqrt{N}}{\sqrt{t_0}}\})\ . \end{aligned} \tag{35}$$

Hence, it suffices to show, for given $\eta, \varepsilon > 0$, that

$$W_{x,1,N}(\{(N,\omega) : m(\omega,\delta) > \varepsilon\sqrt{N}\}) < \eta\ , \tag{36}$$

if δ is small enough. This is a well known result (see, e.g., [8] pp. 62-63). For later refrence we briefly recall the argument.

First, note that since paths contributing to (36) are linear on each interval $[\frac{i-1}{N}, \frac{i}{N}]$ we have

$$m_N(\omega,\delta) \equiv \max\left\{|\omega(\frac{i}{N}) - \omega(\frac{j}{N})| : 0 \le i,j \le N,\ |\frac{i}{N} - \frac{j}{N}| < \delta\right\} \ge \frac{1}{3} m(\omega,\delta)$$

for $N \ge \delta^{-1}$. Note also that by uniform continuity of $\omega \in \Omega(x)$ the inequality (36) is fulfilled for sufficiently small δ for each individual N, so we need not worry about small N. Hence we may replace $m(\omega,\delta)$ in (36) by $m_N(\omega,\delta)$, and we may assume $\delta = M^{-1}$, where $M \in \mathbb{N}$ and $N \ge M$.

Next, given $N \ge M$, we choose integers $0 = k_0 < k_1 < \cdots < k_M = N$ such that any subinterval $[\frac{i}{N}, \frac{j}{N}]$ of $[0,1]$ of length $\le \delta$ is contained in one of the intervals $[\frac{k_l}{N}, \frac{k_{l+2}}{N}]$ and such that the latter intervals are all of length $\le 3\delta$. It follows that

$$\begin{aligned}
&W_{x,1,N}\left(\{(N,\omega) : m(\omega,\delta) > \varepsilon\sqrt{N}\}\right)\\
&\le \sum_{l=0}^{M-2} W_{x,1,N}\left(\left\{(N,\omega) : \max_{k_l \le k \le k_{l+2}} |\omega(\frac{k_l}{N}) - \omega(\frac{k}{N})| > \frac{\varepsilon}{6}\sqrt{N}\right\}\right)\\
&\le \sum_{l=0}^{M-2} W_{x,1,N}\left(\left\{(N,\omega) : \max_{k_l \le k \le k_{l+2}} |\omega(\frac{k_l}{N}) - \omega(\frac{k}{N})| > \frac{\delta^{-\frac{1}{2}}\varepsilon}{6\sqrt{3}}\sqrt{k_{l+2}-k_l}\right\}\right) .
\end{aligned}$$

Due to statistical independence of the steps in ω and translation invariance, we have

$$\begin{aligned}
&W_{x,1,N}\left(\left\{(N,\omega) \in \tilde{\Gamma}(x) : \max_{k_l \le k \le k_{l+2}} |\omega(\frac{k_l}{N}) - \omega(\frac{k}{N})| > \alpha\right\}\right)\\
&= W_{0,1,k_{l+2}-k_l}\left(\left\{(k_{l+2}-k_l,\omega) \in \tilde{\Gamma}(0) : \max_{k_l \le k \le k_{l+2}} |\omega(\frac{k}{k_{l+2}-k_l})| > \alpha\right\}\right)
\end{aligned}$$

for $\alpha > 0$. Combining this with the previous inequality we conclude that it is sufficient to show for given $\eta > 0$ the existence of $\delta > 0$ such that for all $N \in \mathbb{N}$

$$\delta^{-1} W_{0,1,N}\left(\left\{(N,\omega) : \max\{|\omega(\frac{i}{N})|\ |\ 0 \le i \le N\} > \delta^{-\frac{1}{2}}\sqrt{N}\right\}\right) < \eta\ .$$

This inequality is a consequence of the Chebychev inequality and the uniform boundedness in N of the moments of $N^{-\frac{1}{2}}|\omega(1)|$ with respect to the measure $W_{0,1,N}$. The details may be found in [8]. $\square$

Lemma 4.2. $V'_{x,a}$, $0 < a < 1$, is a tight family of measures on $\Omega(x)$.

Proof. This is obtained by an argument similar to the one given above. First, applying the Arzela-Ascoli theorem one concludes that it is sufficient to prove for given $\eta, \varepsilon > 0$ that

$$V'_{x,a}(\{\omega \in \Omega(x) : m(\omega, \delta) > \varepsilon\}) < \eta \ . \tag{37}$$

for small enough δ. Second, since the contribution of terms with $a^2 N \geq t_0$ in (21) is less than or equal to $\frac{\eta}{2}$ if $t_0 \geq -m^2 \log \frac{\eta}{2}$ we conclude as above that it is sufficient to show the existence of a $\delta > 0$ such that for all $N \in \mathbb{N}$

$$V'_{x,1,N}(\{\omega \in \Omega(x) : m(\omega, \delta) > \varepsilon\sqrt{N}\}) < \eta \ . \tag{38}$$

The proof of this fact parallels the one for piecewise linear paths referred to above, and uses only the statistical independence of the steps in a path together with the uniform boundedness in N of the moments of $N^{-\frac{1}{2}}|\omega(1)|$ with respect to the measure $V'_{x,1,N}$. We omit the details of the argument. □

Lemma 4.3. $W_{x,y,a}$, $0 < a < 1$, is a tight family of measures on $\tilde{\Gamma}(x, y)$ for $x \neq y$.

Proof. : By definition tightness of the family $W_{x,y,a}$, $a > 0$, means tightness of the corresponding famify of normalized measures. Since, however, the volume of $W_{x,y,a}$ converges to the volume of $\frac{1}{2}m^2 W_{x,y}$ as $a \to 0$, as noted previously, we need not worry about normalisation.

We note first that the volume of $W_{x,y,a,N}$ given by (29) is uniformly bounded in a and N. Hence, in (25) the sum over $N \leq s_0 a^{-2}$ or over $N \geq t_0 a^{-2}$ can be made arbitrarily small for sufficiently small s_0 or sufficiently large t_0, respectively. By the same arguments as in the first part of the proof of Lemma 4.1 it is sufficient to demonstrate the existence of a $\delta > 0$ such that

$$W_{x,y,a,N}(\{(a^2 N, \omega) \in \tilde{\Gamma}(x, y) : m(\omega, \delta) > \varepsilon\}) < \eta \quad \text{for} \quad s_0 \leq a^2 N \leq t_0 \ , \tag{39}$$

for given $\eta, \epsilon, s_0, t_0 > 0$.

We may as before replace $m(\omega, \delta)$ by $m_N(\omega, \delta)$. Assuming $\delta < \frac{1}{3}$ and setting $N_1 = [\frac{2}{3}N] + 1$, $N_2 = [\frac{1}{3}N]$ (where $[\alpha]$ denotes the integer part of α), any subinterval of $[0, 1]$ of lenghth δ is contained in either $[0, N_1/N]$ or in $[N_2/N, 1]$. Hence, we have

$$\begin{aligned} &W_{x,y,a,N}(\{(a^2 N, \omega) : m_N(\omega, \delta) > \varepsilon\}) \\ &\leq W_{x,y,a,N}(\{(a^2 N, \omega) : m^1_N(\omega, \delta) > \varepsilon\}) + W_{x,y,a,N}(\{(a^2 N, \omega) : m^2_N(\omega, \delta) > \varepsilon\}) \ , \end{aligned} \tag{40}$$

where we have set

$$m_N^1(\omega,\delta) = \max\{|\omega(\frac{i}{N}) - \omega(\frac{j}{N})| : 0 \le i,j \le N_1,\ |\frac{i}{N} - \frac{j}{N}| < \delta\}$$

and

$$m_N^2(\omega,\delta) = \max\{|\omega(\frac{i}{N}) - \omega(\frac{j}{N})| : N_2 \le i,j \le N,\ |\frac{i}{N} - \frac{j}{N}| < \delta\}\ .$$

By definition of $W_{x,y,a,N}$ we have

$$\begin{aligned} & W_{x,y,a,N}(\{(a^2N,\omega) : m_N^1(\omega,\delta) > \varepsilon\}) \\ &= \int_{\mathbb{R}^d} du\, Z_{u,y}^{a^2(N-N_1)} W_{x,u,a,N_1}(\{(a^2N_1,\omega) : m_{N_1}(\omega,\delta) > \varepsilon\})\ . \end{aligned} \tag{41}$$

Here $a^2(N-N_1) \ge a^2(\frac{1}{3}N - 1) \ge \frac{1}{6}s_0$ (assuming $N \ge 6$) so

$$Z_{u,y}^{a^2(N-N_1)} \le (\frac{1}{3}\pi s_0)^{-\frac{d}{2}}.$$

Using this estimate together with

$$dW_{x,a,N}(a^2N,\omega) = d\omega(1) dW_{x,\omega(1),a,N}(\omega) \quad \text{for} \quad \omega \in \Omega(x)$$

in Eq. (41) we obtain

$$\begin{aligned} & W_{x,y,a,N}(\{(a^2N,\omega) : m_N^1(\omega,\delta) > \varepsilon\}) \\ &\le (\frac{1}{3}\pi s_0)^{-\frac{d}{2}} W_{x,a,N_1}(\{(a^2N_1,\omega) : m_{N_1}(\omega,\delta) > \varepsilon\})\ . \end{aligned}$$

Finally, using $a^2N_1 \le \frac{2}{3}t_0$ we conclude from the proof of Lemma 4.1 that $W_{x,a,N_1}(\{(a^2N_1,\omega) : m_{N_1}(\omega,\delta) > \varepsilon\})$ can be made arbitrarily small for $a^2N \le t_0$ if δ is chosen small enough.

The second term in (40) can be treated similarly, and the lemma is proven. □

Lemma 4.4. $V'_{x,y,a}$, $0 < a < 1$, is a tight family of measures on $\Omega(x,y)$ for $x \neq y$.

Proof. Only a few modifications of the previous proof are needed.

For the volume $Z_{x,y,1,N}$ of $W_{x,y,1,N}$ we have the following result, which is rather easily derived from its Fourier representation (see, e.g., [9] pp. 76-77]):

$$\lim_{N\to\infty} \left((2\pi N/d)^{\frac{d}{2}} Z_{x,y,1,N} - e^{-\frac{|x-y|^2}{2N/d}} \right) = 0$$

uniformly in $x-y \in \mathbb{Z}^d$. For the volume $Z_{x,y,a,N}$ of $W_{x,y,a,N}$ this means

$$\lim_{N\to\infty}\left((2\pi a^2 N/d)^{\frac{d}{2}} Z_{x,y,a,N} - e^{-\frac{|x-y|^2}{2Na^2/d}}\right) = 0 \tag{42}$$

uniformly in $x-y \in a\mathbb{Z}^d$ and $0 < a < 1$.

As a first consequence of (42) we note that $Z_{x,y,a,N}$ is uniformly bounded in a, N for $a^2 N \geq t_0$ for any given $t_0 > 0$ large enough. It follows that in Eq. (25) the sum over $N \geq t_0 a^{-2}$ can be made arbitrarily small (when applied to any Borel set in $\Omega(x,y)$) by choosing t_0 large enough.

A second consequence is that for any $s_0 > 0$

$$\begin{aligned}
&(1-e^{-\frac{1}{2d}m^2a^2}) \sum_{a^2N\geq s_0}^{\infty} e^{-\frac{1}{2d}m^2a^2N} Z_{x,y,a,N} \\
\to\ &\frac{m^2}{2d}\int_{s_0}^{\infty} Z_{x,y}^{t/d}\, e^{-\frac{1}{2d}m^2 t} dt = \frac{m^2}{2}\int_{s_0/d}^{\infty} Z_{x,y}^{t} e^{-\frac{1}{2}m^2 t} dt
\end{aligned}$$

as $a \to 0$. On the other hand, as we know from Section 2,

$$(1-e^{-\frac{1}{2d}m^2a^2}) \sum_{N=0}^{\infty} e^{-\frac{1}{2d}m^2a^2N} Z_{x,y,a,N} \ \to\ \frac{m^2}{2} G(x,y) = \frac{m^2}{2}\int_0^{\infty} Z_{x,y}^{t} e^{-\frac{1}{2}m^2 t} dt$$

as $a \to 0$. Hence we conclude that the sum in Eq. (25) over $a^2 N \leq s_0$ can be made arbitrarily small for all $a < a_0$ for some $a_0 > 0$. Replacing s_0 by $\min\{s_0, a_0^2\}$ we can arrange that $a_0 = 1$.

It now follows as in the previous proof that it suffices to show for given $\eta, \varepsilon, s_0, t_0 > 0$ that there exists $\delta > 0$ such that

$$V'_{x,y,a,N}(\{\omega : m_N(\omega,\delta) > \varepsilon\}) < \eta \quad \text{for} \quad s_0 < a^2 N < t_0 \ .$$

Following the proof of Lemma 4.3 we have the estimate

$$\begin{aligned}
&V'_{x,y,a,N}(\{(a^2N,\omega) : m_N(\omega,\delta) > \varepsilon\}) \\
&\leq V'_{x,y,a,N}(\{(\omega) : m_N^1(\omega,\delta) > \varepsilon\}) + V'_{x,y,a,N}(\{(\omega) : m_N^2(\omega,\delta) > \varepsilon\}) \ .
\end{aligned} \tag{43}$$

By definition of $V'_{x,y,a,N}$ we can write

$$\begin{aligned}
&V'_{x,y,a,N}(\{(\omega) : m_N^1(\omega,\delta) > \varepsilon\} \\
&= \sum_{u\in a\mathbb{Z}^d} Z_{u,y,a,N-N_1} V'_{x,u,a,N_1}(\{(\omega) : m_{N_1}(\omega,\delta) > \varepsilon\} \ .
\end{aligned}$$

From Eq. (42) and $a^2(N-N_1) \geq \frac{1}{6}s_0$ it follows that $Z_{u,y,a,N-N_1}$ is uniformly bounded in x, y, a, N for $a^2N \geq s_0$ and $n \geq N_0$ for some $N_0 \in \mathbb{N}$. Letting C denote such an upper bound, we have

$$\begin{aligned}
&V'_{x,y,a,N}(\{(\omega) : m^1_N(\omega, \delta) > \varepsilon\} \\
&\leq C \sum_{u \in a\mathbb{Z}^d} V'_{u,x,a,N_1}((\{(\omega)\, m_{N_1}(\omega, \delta) > \varepsilon\} \\
&= CV'_{x,a,N_1}((\{(\omega) : m_{N_1}(\omega, \delta) > \varepsilon\}\ . \qquad (44)
\end{aligned}$$

From the proof of Lemma 4.2 it now follows as above that the right hand side of (44) can be made arbitrarily small for $a^2N \leq t_0$ if δ is chosen sufficiently small. Estimating the second term in (43) similarly and noting again that the case $N < N_0$ can be taken care of separately, the proof of the lemma is complete. $\square$

We are now ready to give proofs of the convergence theorems stated in Section 3. In view of the preceding lemmas and the remarks at the beginning of this section it is sufficient in each case to prove convergence on a measure determining class of functions.

Proof of Theorem 3.1: For $0 < t_1 < \cdots < t_n \leq 1$ and $s \in \mathbb{R}$, $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$ we define the characteristic function $p_{a;t_1,\ldots,t_n}$ of $W_{x,a}$ by

$$p_{a;t_1,\ldots,t_n}(s, \xi_1, \ldots, \xi_n) = \int_{\tilde{\Gamma}(x)} e^{i(st+\xi_1\cdot\omega(t_1)+\cdots+\xi_n\cdot\omega(t_n))} dW_{x,a}(t, \omega)\ , \qquad (45)$$

and similarly the characteristic function $p_{t_1,\ldots,t_n}$ of W_x. We claim it is sufficient to show that $p_{a;t_1,\ldots,t_n} \to \frac{m^2}{2} p_{t_1,\ldots,t_n}$ pointwise as $a \to 0$, for arbitrary $0 < t_1 < \cdots < t_n \leq 1$. In order to see this, it is enough to verify that the measure W_x on $\tilde{\Gamma}(x)$ is determined by its characteristic functions. Let f be a smooth function on $\mathbb{R}_+ \times R^{nd}$ with compact support. Multiplying $p_{t_1,\ldots,t_n}$ by the Fourier transform of f at $(s, \xi_1, \ldots, \xi_n)$ and integrating over $(s, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{nd+1}$ gives by Fubini's theorem $\int_{\tilde{\Gamma}(x)} f(t, \omega(t_1), \ldots, \omega(t_n)) dW_{x,a}(t, \omega)$. A simple limiting argument then shows that measures of sets of the form $\{(t, \omega) : (t, \omega(t_1), \ldots, \omega(t_n)) \in C\}$, where $C \subseteq \mathbb{R}_+ \times \mathbb{R}^{nd}$ is closed, are determined by the characteristic functions. Since sets of this form generate the Borel algebra in $\tilde{\Gamma}(x)$ the claim follows.

Given N let $1 \leq N_1 \leq \cdots \leq N_n \leq N$ be such that $t_i \in]\frac{N_i-1}{N}, \frac{N_i}{N}]$ and set $t'_i = \frac{N_i}{N}$. By an explicit computation, replacing the intermediate times t_i by t'_i in the piecewise linear paths, one finds

$$\int_{\tilde\Gamma(x)} e^{i(st+\xi_1\cdot\omega(t_1)+\cdots+\xi_n\cdot\omega(t_n))} dW_{x,a,N}(t,\omega) = C_{a,N}\, p^{a^2N}_{t'_1,\ldots,t'_n}(s,\xi_1,\ldots,\xi_n).$$

The quantity $C_{a,N}$ which depends on the time differences $t_i - t'_i$ tends to 1 uniformly in N as $a \to 0$. Using the expression (17) for $p^t_{t_1,\ldots,t_n}(s,\xi_1,\ldots,\xi_n)$ it follows easily that

$$\begin{aligned} p_{a;t_1,\ldots,t_n}(s,\xi_1,\ldots,\xi_n) &= (1-e^{-\frac{1}{2}m^2a^2})\sum_{N=0}^{\infty} C_{a,N} e^{-\frac{1}{2}m^2a^2N} e^{isa^2N} p^{a^2N}_{t'_1,\ldots,t'_n}(s,\xi_1,\ldots,\xi_n) \\ &\to \frac{m^2}{2} p_{t_1,\ldots,t_n}(s,\xi_1,\ldots,\xi_n) \end{aligned}$$

as $a \to 0$. □

Proof of Theorem 3.2: We do this by proving the stronger result that $V'_{x,a} \to \frac{m^2}{2} V'_x$ as $a \to 0$. It follows by the same argument as given in the beginning of the previous proof that it is enough to prove convergence of the characteristic function

$$\begin{aligned} p'_{a;t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) &= \int_{\Omega(x)} e^{i(\xi_1\cdot\omega(t_1)+\cdots+\xi_n\cdot\omega(t_n))} dV'_{x,a}(t,\omega) \\ &= (1-e^{-\frac{1}{2}m^2a^2})\sum_{N=0}^{\infty} e^{-\frac{1}{2}m^2a^2N} \sum_{\omega\in\Omega_{a,N}(x)} e^{-\beta_0 N} e^{i(\xi_1\cdot\omega(t_1)+\cdots+\xi_n\cdot\omega(t_n))} \end{aligned}$$

to $\frac{m^2}{2} p_{t_1,\ldots,t_n}$ as $a \to 0$ for arbitrary $0 < t_1 < \cdots < t_n \le 1$. Furthermore, we can assume $x = 0$, since translation by x only gives rise to a factor $e^{ix\cdot(\xi_1+\cdots+\xi_n)}$ in the characteristic functions. Defining N_i and t'_i as in the preceding proof we have

$$\begin{aligned} & p'_{a;t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) \\ &= (1-e^{-\frac{1}{2d}m^2a^2})\sum_{N=0}^{\infty} e^{-\frac{1}{2d}m^2a^2N} \prod_{i=1}^{n}\left(\frac{1}{d}\sum_{\nu=1}^{d}\cos a(\xi_i+\cdots+\xi_n)_\nu\right)^{N_i-N_{i-1}} \\ &= (1-e^{-\frac{1}{2d}m^2a^2})\sum_{N=0}^{\infty} e^{-\frac{1}{2d}m^2a^2N} \prod_{i=1}^{n}\left(\frac{1}{d}\sum_{\nu=1}^{d}\cos a(\xi_i+\cdots+\xi_n)_\nu\right)^{(t'_i-t'_{i-1})N}, \end{aligned}$$

where ν labels the components of the ξ-variables and $N_0 = 0$. Finally, using

$$\left(\frac{1}{d}\sum_{\nu=1}^{d}\cos a(\xi_i+\cdots+\xi_n)_\nu\right)^{\frac{s}{a^2}} \to e^{-\frac{s}{2d}(\xi_i+\cdots+\xi_n)^2}$$

as $a \to 0$, an application of the dominated convergence theorem shows that

$$p'_{a;t_1,\ldots,t_n}(\xi_1,\ldots,\xi_n) \to \frac{m^2}{2}\int_0^\infty dt\, e^{-\frac{1}{2}m^2t} \exp\left(-\frac{t}{2}\sum_{i=1}^{n}(t_i-t_{i-1})(\xi_i+\cdots+\xi_n)^2\right)$$

as $a \to 0$, which is the desired result. $\square$

Proof of Theorem 3.3: The convergence $W_{x,y,a} \to W_{x,y}$ follows by essentially the same proof as of Theorem 3.1. Similarly the convergence $V_{x,y,a} \to V_{x,y}$ is obtained by trivial modifications of the proof of Theorem 3.2. Details are left to the reader.

$\square$

5 Cylinder sets

In this section we define a class of sets of geometric paths which generate the Borel algebra and play a role similar to the one played by cylinder sets in the theory of parametrized paths. We will see in the next section that the measure of these sets can be calculated in a particularly simple way.

A natural condition to put on a parametrized path ω is that the path be located in a particular subset A of $\mathbb{R}^d$ at a given time t, i.e., $\omega(t) \in A$. For geometric paths a condition of this type is meningless but a similar one which has a well defined meaning is the condition that a geometric path $\bar{\omega}$ hit a set A. This means that $\bar{\omega} \cap A \neq \emptyset$, i.e., if ω is a parametrization of $\bar{\omega}$ then there is a time t such that $\omega(t) \in A$. More generally, we can require that a geometric path hit a number of sets in a particular order and/or stay away from other sets. Below we define a certain class of sets defined by such conditions. Other definitions are possible but we find this class simple to work with.

We consider paths with two fixed endpoints x and y. Let $A_1, \ldots, A_n$ be subsets of $\mathbb{R}^d$ and let $\bar{\omega} \in \bar{\Omega}(x,y)$ with parametrization $\omega : [0,1] \to \mathbb{R}^d$. Define

$$\begin{aligned}
\tau_1 &= \sup\{t \geq 0 : \omega([0,t]) \subseteq A_1\} \\
\tau_2 &= \sup\{t \geq \tau_1 : \omega([\tau_1,t]) \subseteq A_2\} \\
&\vdots \\
\tau_n &= \sup\{t \geq \tau_{n-1} : \omega([\tau_{n-1},t]) \subseteq A_n\},
\end{aligned}$$

where by convention $\sup\emptyset = 1$. We then define $Z(A_1, \ldots, A_n)$ as the set of all geometric paths $\bar{\omega} \in \bar{\Omega}(x,y)$ such that

$$\tau_1 < \tau_2 < \ldots < \tau_{n-1} < \tau_n = 1.$$

This defining property is easily seen to be independent of the parametrization ω chosen for $\bar{\omega}$. In fact, $\bar{\omega} \in Z(A_1, \ldots, A_n)$ exactly if it starts at $x \in A_1$, stays inside

A_1 until it leaves A_1 at a point $x_1 = \omega(\tau_1) \in A_2$, then stays in A_2 until it leaves at a point $x_2 = \omega(\tau_2) \in A_3$ and so on until it leaves A_{n-1} at a point $x_{n-1} = \omega(\tau_{n-1}) \in A_n$ and then finally stays in A_n until it ends at $y \in \bar{A}_n$. The values of the escape times τ_i depend of course on the parametrization but their ordering and the points x_i are independent of parametrization.

Proposition 5.1. Let $A_1, \dots, A_n \subseteq \mathbb{R}^d$ be open sets such that $x \in A_1$ and $y \in A_n \setminus \bar{A}_{n-1}$. Furthermore, assume

$$A_{i-1} \cap \partial A_i \cap A_{i+1} = \emptyset \tag{46}$$

for $i = 2, \dots, n-1$. Then $Z(A_1, \dots, A_n)$ is an open subset of $\bar{\Omega}(x, y)$.

Proof. Let $\bar{\omega} \in Z(A_1, \dots, A_n)$. Choose a parametrization ω for $\bar{\omega}$. Since the sets A_i are open we can choose $s_i < \tau_i$ such that $\omega([s_i, \tau_i]) \subseteq A_{i+1}$, see Fig. 1. By the definition of the τ_i's it follows that $\omega([s_i, s_{i+1}]) \subseteq A_{i+1}$ for $i = 0, 1, \dots, n-1$, setting $s_0 = 0$.

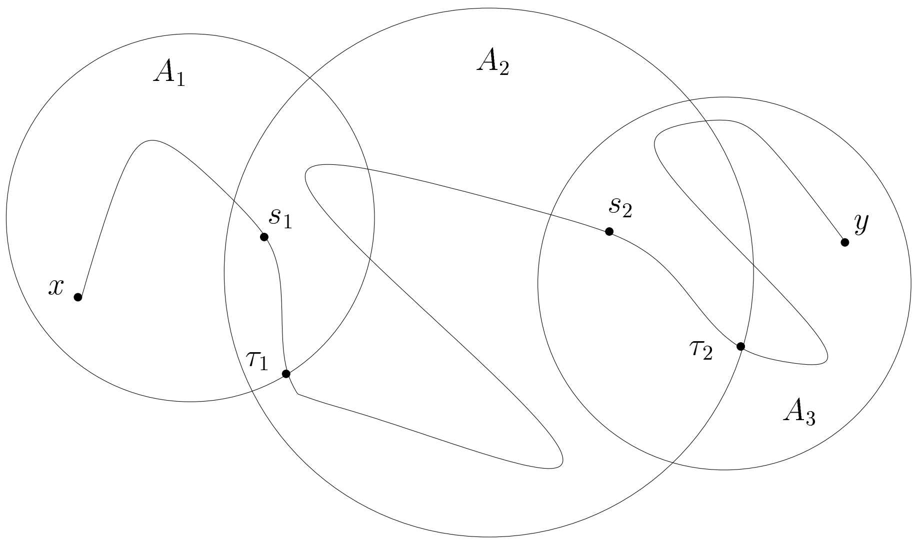

Figure 1: An illustration of the times s_i and τ_i in the case of $n = 3$.

Let $r_i > 0$ be the distance from the compact set $\omega([s_i, s_{i+1}])$ to the boundary of A_{i+1}, $i = 0, 1, \dots, n-1$, and set $r = \min_i r_i$. Now take a geometric path $\bar{\omega}'$ at a distance smaller than r from $\bar{\omega}$. Then there exists a parametrization $\omega' : [0,1] \to \mathbb{R}^d$ of $\bar{\omega}'$ such that

$$\sup_{t \in [0,1]} |\omega(t) - \omega'(t)| < r.$$

In particular it follows that

$$\omega'([s_i, s_{i+1}]) \subseteq A_{i+1}.$$

By the assumption (46) we may from the outset choose the s_i's such that $\omega(s_{i+1}) \notin A_i$. Choosing r smaller, if necessary, we can also assume that r is smaller than the smallest of the distances from $\omega(s_{i+1})$ to A_i, $i = 1, 2, \ldots, n$. Hence, $\omega'(s_{i+1}) \notin A_i$. On the other hand $\omega'(s_i) \in A_i$ so ω' leaves the set A_i at a time $\tau_i' \in [s_i, s_{i+1}]$. It follows that $\tau_1' < \tau_2' < \ldots \tau_n' = 1$ so $\bar{\omega}' \in Z(A_1, \ldots, A_n)$. □

The condition (46) was essential in the above argument because otherwise the paths might never enter the interior of $A_i \setminus A_{i-1}$ for some i. But (46) can be replaced by a weaker condition as we now explain.

Let A be an open set in $\mathbb{R}^d$. We say that a geometric path $\bar{\omega}$ is *tangent to the boundary of* A at $x \in \partial A$ if there is a parametrization ω of $\bar{\omega}$ such that $\omega(t_0) = x$ and there is an $\varepsilon > 0$ such that $\omega(t) \in \bar{A}$ for $0 < |t - t_0| < \varepsilon$. We claim that any path in $Z(A_1, \ldots, A_n)$ which is nowhere tangent to any of the boundaries ∂A_i is an interior point of the set $Z(A_1, \ldots, A_n)$. This can be seen as follows: In addition to the s_i's, choose numbers $s_i' \in [0, 1]$ such that $\tau_i < s_i' < s_i$ and $\omega(s_i') \notin \bar{A}_i$. Now choose $r > 0$ smaller than each of the distances from $\omega(s_i')$ to $\bar{A}_i$. It then follows that a path $\bar{\omega}'$ within a distance r from $\bar{\omega}$ leaves A_i somewhere between s_i and s_i' and hence $\bar{\omega}' \in Z(A_1, \ldots, A_n)$ as before.

It is not hard to see that if $\bar{\omega} \in Z(A_1, \ldots, A_n)$ is tangent to one of the ∂A_i's then $\bar{\omega} \in \partial Z(A_1, \ldots, A_n)$, i.e., there are paths arbitrarily close to $\bar{\omega}$ that are not in $Z(A_1, \ldots, A_n)$, see Fig. 2.

We do not have a proof that the sets $Z(A_1, \ldots, A_n)$ are measurable for general open sets $A_1, \ldots, A_n$. We avoid this problem simply by taking the closures of these sets. We denote the closures by $\bar{Z}(A_1, \ldots A_n)$.

Proposition 5.2. The sets $\bar{Z}(A_1, \ldots, A_n)$ where the A_i's are open balls generate the Borel algebra of geometric paths.

Proof. We will show that any open set in $\bar{\Omega}(x, y)$ can be written as a countable union of $\bar{Z}$-sets. Given $\bar{\omega} \in \bar{\Omega}(x, y)$ and $\varepsilon > 0$ we show that there are open balls $A_1, \ldots, A_n$ such that $\bar{\omega} \in \bar{Z}(A_1, \ldots, A_n)$ and $\bar{Z}(A_1, \ldots, A_n)$ is contained in a ball in $\bar{\Omega}(x, y)$ of radius ε centered on $\bar{\omega}$. Moreover, the A_i's can be taken to have rational centers

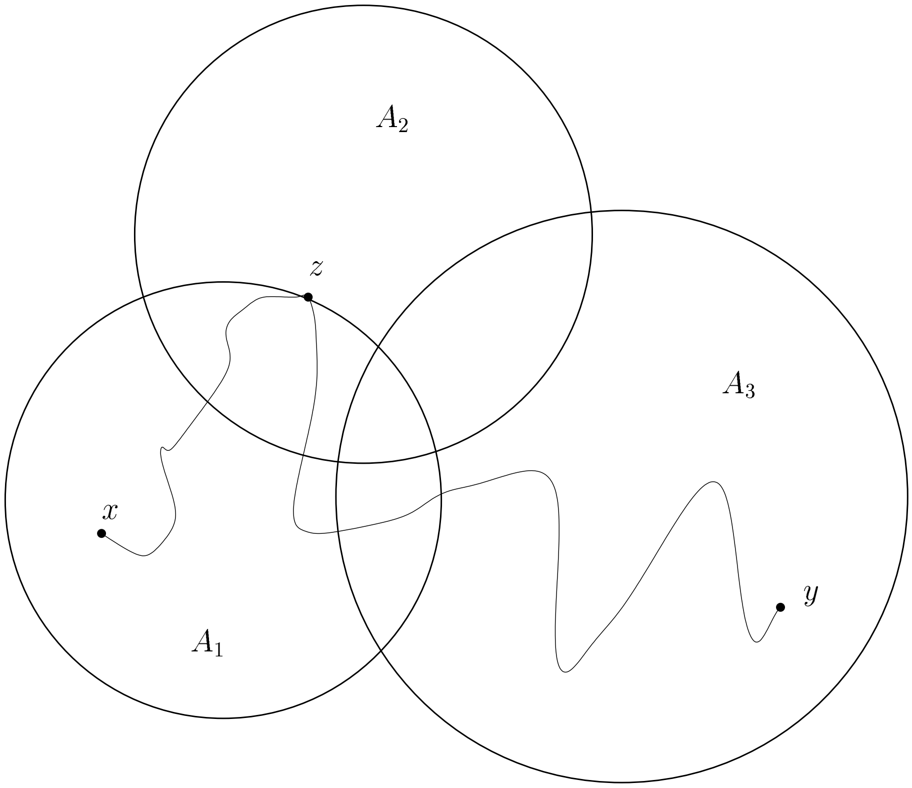

Figure 2: A path from x to y which is tangent to A_2 at the point z. There are paths arbitrarily close to this path which are not in $Z(A_1, A_2, A_3)$.

and radii. It follows then by a standard argument that the $\bar{Z}$-sets generate the Borel algebra.

Let $\bar{\omega} \in U$ where $U \subseteq \bar{\Omega}(x, y)$ is open. Choose a rational number ε so that $\varepsilon < \frac{1}{2}\bar{d}(\bar{\omega}, \partial U)$. Let A_1 be an open ball of radius ε centered at x. If $\bar{\omega}$ is not contained in A_1 let $x_1 \in \mathbb{R}^d$ be the point where $\bar{\omega}$ leaves A_1 for the first time, i.e., if $\omega : [0, 1] \to \mathbb{R}^d$ is a parametrization of $\bar{\omega}$, then $x_1 = \omega(\tau_1)$, where

$$\tau_1 = \sup\{t \in [0, 1] : \omega([0, t]) \subseteq A_1\}$$

as before. Take a point y_1 with rational coordinates such that $|x_1 - y_1| < \varepsilon/3$. Let A_2 be a ball of radius ε cenetred at y_1. If $\bar{\omega}$ stays inside A_2 after it leaves A_1 at x_1 the construction is finished; otherwise let x_2 be the point where $\bar{\omega}$ leaves A_2 for the first time after it left A_1 at x_1 and define y_2 and A_3 in a way analogous to the one used to define y_1 and A_2. The construction continues in this way until we obtain a set A_n inside which $\bar{\omega}$ stays after it leaves A_{n-1}. The construction has to end after a finite number of steps since any paramterization ω of $\bar{\omega}$ is a uniformly continuous map.

From the above construction it is clear that $\bar{\omega} \in Z(A_1, \dots, A_n)$. Moreover, if $\bar{\omega}'$ is another path in $Z(A_1, \dots, A_n)$ then $\bar{d}(\bar{\omega}, \bar{\omega}') \leq 2\varepsilon$ because we can choose a parametrization ω' of $\bar{\omega}'$ such that the τ_i's coincide for ω and ω' and hence, for any $t \in [0,1]$, $\omega(t)$ and $\omega'(t)$ both belong to the same A_j, $j = 1, \dots, n$. We conclude that $Z(A_1, \dots, A_n)$ and hence $\bar{Z}(A_1, \dots, A_n)$ is contained in a closed ball in $\bar{\Omega}(x,y)$ of radius 2ε centered on $\bar{\omega}$. This ball is contained in U and the proof is complete. $\square$

We remark that the proof of the above result can of course be adapted to the case where the sets A_i are boxes in $\mathbb{R}^d$ rather than balls.

6 Integrating over cylinder sets

In this section we show that the lattice approximation to the measure of the $\bar{Z}$-sets converges and we derive some formulae for the measure of these sets in terms of Dirichlet propagators.

Let A be a bounded set in $\mathbb{R}^d$ with a smooth boundary. Let x and y be two different points in the interior of A. We recall that the Dirichlet Green function for $\frac{1}{2}(-\Delta + m^2)$ with data on ∂A, denoted $G_A^D(x,y)$, is given by the Wiener integral over all paths from x to y that avoid ∂A. This fact is established in, e.g., [7] for the corresponding heat kernel and hence follows for the propagator by integrating over time.

In the following discussion the endpoints x and y will be kept fixed and for simplicity we denote the measure $V_{x,y}$ by μ. Accordingly we can write

$$G_A^D(x,y) = \int_{\bar{Z}(A)} d\mu = \mu(\bar{Z}(A)). \tag{47}$$

We are interested in generalizing this formula to the case of $\bar{Z}(A_1, \dots, A_n)$ with $n > 1$ and showing that

$$\lim_{a \to 0} V_{x,y,a}(\bar{Z}(A_1, \dots, A_n)) = \mu(\bar{Z}(A_1, \dots, A_n)). \tag{48}$$

In order to minimize technical complications let us assume that the sets A_i are boxes so their boundaries are contained in hyperplanes.

Let us consider a family of boxes $A_1, \dots, A_n$ in $\mathbb{R}^d$ with the propery that the intersection of any two different boundaries ∂A_i and ∂A_j has codimension 2 or greater, i.e., the boundaries never overlap. Let us define O_1 as the collection of all paths in $\bar{\Omega}(x, y)$ that are somewhere tangent to one of the hyperplanes that make up the boundaries of the A_i's. Let O_2 be the collection of all paths in $\bar{\Omega}(x, y)$ that meet one or more of the intersections $\partial A_i \cap \partial A_j$, $i \neq j$. Put $O = O_1 \cup O_2$. It can be checked that the set of paths that are somewhere tangent to a given hyperplane is a measurable set with measure zero. It has measure zero since the probability that a Wiener path intersects a hyperplane exactly once in a time interval is zero, see, e.g., [10] Chapter 12. The set O_2 is easily seen to be closed and hence measurable. Its measure is zero since the codimension of the intersections $\partial A_i \cap \partial A_j$ is greater than 1. Thus, O is a measurable set with measure 0. The boundary of $Z(A_1, \dots, A_n)$ consists of paths for which either two of the τ_i's coincide or the path is tangent to one of the boundaries ∂A_i. Hence, $\partial \bar{Z}(A_1, \dots, A_n) \subseteq \partial Z(A_1, \dots, A_n) \subseteq O$. *We can therefore conclude from [8] Theorem 2.1 that the convergence (48) takes place for boxes* and the argument can be extended to the case of A_i's with piecewise smooth boundaries.

Let us now turn to the calculation of the measure of the $\bar{Z}$-sets. Let A be as before. Since $G_A^D(x, z) = 0$ for $z \in \partial A$ we have

$$\int_{\partial A} \frac{\partial}{\partial n} \left(G_A^D(x, z) G(z, y) \right) \, dS = \int_{\partial A} \frac{\partial G_A^D}{\partial n}(x, z) G(z, y) \, dS \tag{49}$$

where $\frac{\partial}{\partial n}$ is the normal derivative to ∂A with respect to z. Let Y_A be the collection of all paths from x to y which hit the boundary ∂A. An application of the divergence theorem and Eq. (47) lead to

$$\mu(Y_A) = \int_{\partial A} \frac{\partial G_A^D}{\partial n}(x, z) G(z, y) \, dS. \tag{50}$$

More generally, it can be argued that

$$P_A(z) = \frac{\partial G_A^D}{\partial n}(x, z) G(z, y) \tag{51}$$

is (up to the constant factor $\mu(Y_A)$) the *conditional probability density* that a path from x to y which hits the boundary ∂A hits it for the first time at the point $z \in \partial A$, and $P_A(z)$ is given by an integral over all paths from x to y which hit the boundary of A and hit it for the first time in z.

It is convenient to extend the Dirichlet Green functions G_A^D to all of $\mathbb{R}^d$ such that they are 0 outside A. Let us now consider the case $x \in A_1$, $y \in A_2 \setminus A_1$, $A_1 \cap A_2 \neq \emptyset$.

Then the measure of $\bar{Z}(A_1, A_2)$ is the integral over all paths from x to y which leave A_1 for the first time at a point $z \in \partial A_1 \cap A_2$ and stay in A_2 after they leave A_1. The integral over these paths is obtained by analogy with Eq. (50) as

$$\mu(\bar{Z}(A_1, A_2)) = \int_{\partial A_1} \frac{\partial G^D_{A_1}}{\partial n}(x,z) G^D_{A_2}(z,y)\, dS \tag{52}$$

and

$$\frac{\partial G^D_{A_1}}{\partial n}(x,z) \frac{G^D_{A_2}(z,y)}{\mu(\bar{Z}(A_1, A_2)} \tag{53}$$

is the conditional probability density that a path in $\bar{Z}(A_1, A_2)$ leaves A_1 for the first time in the point z.

It is straightforward to generalize the above considerations to the case of arbitrary n, i.e., $\bar{Z}(A_1, \ldots, A_n)$. By the Markov property of the Brownian paths we have

$$\mu(\bar{Z}(A_1, \ldots, A_n)) = \int_{\partial A_1} \cdots \int_{\partial A_{n-1}} \prod_{i=1}^{n-1} \frac{\partial G^D_{A_i}}{\partial n_i}(z_{i-1}, z_i) G^D_{A_n}(z_{n-1}, y)\, dS_1 \ldots dS_{n-1}, \tag{54}$$

where we have set $z_0 = x$ and $\frac{\partial}{\partial n_i}$ denotes the normal derivative to ∂A_i with respect to z_i. In this integral formula z_i is the point where the path first leaves A_i after hitting $A_1, \ldots, A_{i-1}$ in that order.

We note that for $n = 1$ the convergence (48), i.e., the convergence of the lattice approximations to the Dirichlet propagators is well known for sufficiently nice sets A. This convergence can also be proved directly without the use of measure theory. We also note that all the integration formulae above have clear lattice analogues for arbitrary n.

7 Conclusion

We have in this paper defined integration over geometric paths and studied natural discretized measures on spaces of such paths. Two different discretizations were discussed, one with a metric degree of freedom and one without. We have proven the convergence of the discretized measures and thereby in particular established the convergence of the discrete approximations to the integrals over paths that one is normally interested in for physics applications. We furthermore introduced, in the case without a metric degree of freedom, a natural class of sets of geometric paths which play the role of cylinder sets and generate the Borel algebra and we have shown how to calculate the measure of these sets in terms of Dirichlet propagators.

One, perhaps disappointing but not entirely unexpected, outcome of our analysis is that no technical simplifications are obtained by considering only parametrization independent quantities, i.e., by restricting to inherently physical degrees of freedom. In particular, it is hard to get a technical handle on geometric paths without introducing parametrizations to calculate with as is usually done in theories with a local gauge invariance.

One of the main motivations for this study was to obtain some insight into the corresponding problem for random surfaces. The random surface case is far more difficult than the one considered in this paper since the measures on parametrized surfaces which correspond to Wiener measure on paths are not well understood. Some of the ideas we have discussed here can be carried over to embedded surfaces but modifications would be needed since points on a geometric surface cannot be ordered like the points on a geometric path. For nonimbedded surfaces a new approach is required. In the absence of imbedding degrees of freedom, points on the surface have to be identified in terms of intrinsic geometric degrees of freedom like curvature. How to do this in a systematic fashion is far from obvious.

Acknowledgement. We are indebted to Institut Mittag Leffler for hospitality in the spring of 1999 when this work was begun. T.J. is also indebted to the Niels Bohr Institute and the CERN Theory Division for hospitality. The work of B.D. is supported in part by MatPhySto funded by the Danish National Research Foundation. This research was partly supported by TMR grant no. HPRN-CT-1999-00161.

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