Return to equilibrium for Pauli-Fierz systems

Jan Dereziński[1] and Vojkan Jakšić[2]

[1]Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warszawa, Poland

[2]Department of Mathematics and Statistics
University of Ottawa, 585 King Edward Avenue
Ottawa, ON, K1N 6N5, Canada

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Contents

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1 Introduction

A quantum system is often described by a W^*-algebra $\mathfrak{M}$ with a σ-weakly continuous group of automorphisms τ^t. Such a pair $(\mathfrak{M}, \tau^t)$ is usually called a W^*-dynamical system.

Definition 1.1 *We say that $(\mathfrak{M}, \tau^t)$ satisfies the return to equilibrium property if*

1. *there exists a unique τ^t-invariant normal state ω on $\mathfrak{M}$;*
2. *if ϕ is any normal state on $\mathfrak{M}$, then*

$$\lim_{t\to\infty} \phi(\tau^t(A)) = \omega(A), \quad A \in \mathfrak{M}.$$

The conventional wisdom (based on our everyday experience) says that the following statement should be true in some sense.

Quasitheorem *Suppose that the W^*-dynamical system $(\mathfrak{M}, \tau^t)$ describes a physical system that is*

1. *infinitely extended;*
2. *a localized perturbation of a a thermal equilibrium system;*
3. *sufficiently regular;*
4. *sufficiently generic.*

Then it should satisfy the return to equilibrum property.

The main goal of our paper is the proof that a certain class of Pauli-Fierz systems satisfies the return to equilibrium property. Pauli-Fierz systems is the name that we give to a class of some W^*-dynamical systems that are often used to describe the interaction of a small quantum system with a "bosonic reservoir". In our paper, the small system will be always described by a finite dimensional Hilbert space $\mathcal{K}$. The bosons will be described by the Fock space over the one particle space $\mathcal{Z}$, denoted $\Gamma_{\rm s}(\mathcal{Z})$. The bosons can be interpreted eg. as photons or phonons. Pauli-Fierz systems usually arise in physics as simplified versions of the non-relativistic QED.

In the case of finite volume Pauli-Fierz systems, the W^*-algebra of observables is simply $B(\mathcal{H})$ for $\mathcal{H} = \mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{Z})$. From the point of view of the classification of W^*-algebras, such algebras are type I factors, so they are rather simple-minded. The dynamics is generated by a certain bounded from below operator H, called the Pauli-Fierz Hamiltonian. The thermal equilibrium state is given by the density matrix $\mathrm{e}^{-\beta H}/\mathrm{Tr}\mathrm{e}^{-\beta H}$. Note however, that one does not expect the return to equilibrium property to be true for finite volume systems.

In the thermodynamical (infinite volume) limit at positive temperature it is not possible to describe a bosonic reservoir by a type I algebra. Instead, one has to use the so-called Araki-Woods representations, which lead to type III algebras $\mathfrak{M}_{\beta,1}$, which we call (left) Araki-Woods algebras.

Using Araki-Woods algebras we can define a one parameter family of W^*-dynamical systems $(B(\mathcal{K}) \otimes \mathfrak{M}_{\beta,1}, \tau^t_\beta)$, which we call thermal Pauli-Fierz systems. Note that they are defined in a canonical way given the inverse temperature $\beta \in [0, \infty]$ and the Pauli-Fierz Hamiltonian H. For the zero temperature, that is for $\beta = \infty$, the system $(B(\mathcal{K}) \otimes \mathfrak{M}_{\infty,1}, \tau^t_\infty)$ is just the type I W^*-algebra $B(\mathcal{H})$ equipped with the dynamics generated by H. For $\beta \in]0, \infty[$, the systems $(\mathfrak{M}_\beta, \tau^t_\beta)$ can be interpreted as the thermodynamical limit of the same underlying physical system. Their algebras are type III and, in general, they are non-equivalent for distinct β. The analysis of the family of W^*-dynamical systems $(\mathfrak{M}_{\beta,1}, \tau^t_\beta)$, for $\beta \in]0, \infty]$, is the main subject of this paper.

The Pauli-Fierz systems that we consider satisfy the two first conditions that we mentioned in the "quasitheorem". They describe an infinitely extended system—which for $\beta < \infty$ mathematically is

expressed in the fact that their W^*-algebra is of type III (a rather nontrivial kind of a W^*-algebra). The choice of the Planck law in the Araki-Woods representation expresses the fact that the system is a localized perturbation of the thermal equuilibrium. Nevertheless, these conditions are not sufficient to guarantee the return to equilibrium property. It is easy to see that this property is violated eg. for free Pauli-Fierz systems.

There are several approaches that can be used to express the "regularity" of interacting Pauli-Fierz systems (the third condition in our "quasitheorem"). The approach that we adopted is based on the idea of Jakšić and Pillet of gluing negative and positive frequences of the bosons [JP1, JP2]. This allows us to define a "conjugate operator"—the generator of translations in the spectral variables. Unlike in the original approach of Jakšić and Pillet, we do not assume the analyticity with respect to this conjugate operator, but only the differentiability (of sufficiently high order). This is a much weaker assumption than the analyticity and allows us to treat Pauli-Fierz systems more efficiently, especially at small temperatures.

To express the genericity of the interaction we use some simple algebraic conditions on the interaction derived from the so-called Fermi Golden Rule. The Fermi Golden Rule describes how to compute the eigenvalues and resonances of a self-adjoint operator to the second order. In particular, it can be used to predict which eigenvalues will disappear when we switch the interaction on for small coupling constant (and turn into resonances, if the system is sufficiently analytic).

It turns out that in the case of thermal Pauli-Fierz systems the computations of the Fermi Golden Rule lead to two distinct conditions that guarantee the return to equilibrium property. The first one applies only to the zero temperature ($\beta = \infty$) and is relatively easy to compute. What is more interesting is what happens at positive temperatures ($\beta < \infty$). For the whole range of positive temperatures $\beta \in]0, \infty[$ the Fermi Golden Rule gives a single condition that guarantees the return to equilibrium for sufficiently small nonzero coupling constants. In order to check this condition, first we need to construct a certain (finite dimensional) $*$-algebra. The definition of this $*$-algebra depends only on the Hamiltonian H and not on the inverse temperature β. If this $*$-algebra has a trivial commutator, then for any positive temperature, for a sufficiently small nonzero coupling constant, the return to equilibrium property holds.

If we assume the two kinds of generic assumptions (the zero-temperature and the positive temperature one) plus a sufficient regularity of the interaction, then we can prove that the return to equlibrium is uniform in temperature (at least for not too large temperatures). In other words, we can show that for any $\beta_0 > 0$ there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$ the return to equilibrium is true. This result would be impossible to obtain using just the translation analyticity. In fact, it was this result, which was the main motivation for us to study the Mourre theory version of the Jakšić-Pillet approach in [DJ].

Before we explain the tools that we use in our paper, let us recall some important elements of the theory of W^*-algebras that we will use [BR, DJP]. One of the most important concepts of the modern theory of W^*-algebras is the so-called standard representation [Ha] (see also [A2, Co, BR]). We say that a quadruple $(\pi, \mathcal{H}, J, \mathcal{H}_+)$ is a standard representation of a W^*-algebra $\mathfrak{M}$, if $\pi : \mathfrak{M} \to B(\mathcal{H})$ is a $*$-representation, J is an antiunitary involution on $\mathcal{H}$ and $\mathcal{H}_+$ is a self-dual cone in $\mathcal{H}$ satisfying the following conditions:
1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$,
2) $JAJ = A^*$ for A in the center of $\mathfrak{M}$,
3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}_+$,
4) $\pi(A)J\pi(A)\mathcal{H}_+ \subset \mathcal{H}_+$ for $A \in \mathfrak{M}$.
It can be shown that every W^*-algebra possesses an essentially unique standard representation.

The standard representation has several important properties. First of all, every normal state ω on $\mathfrak{M}$ has a unique vector representative $\Omega \in \mathcal{H}_+$. Secondly, every W^*-dynamics τ^t on $\mathfrak{M}$ has a canonical implementation in the standard representation by a strongly continuous unitary group, so that

$$\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}.$$

for a certain self-adjoint operator L. The operator L is fixed by the condition $\mathrm{e}^{\mathrm{i}tL}\mathcal{H}_+ \subset \mathcal{H}_+$ and is called the Liouvillean. It encodes all the properties of τ^t in a particularly convenient way. This encoding is

described partly in the following theorem:

Theorem 1.2 *1. The Liouvillean L has no eigenvalues iff the the W^*-dynamics τ^t has no invariant states.*

2. The Liouvillean L has exactly one nondegenerate eigenvalue at zero iff the W^*-dynamics τ^t has a single invariant state.

3. Suppose L has no singular continuous spectrum, has exactly one nondegenerate eigenvalue at zero and the corresponding eigenstate is separating for $\mathfrak{M}$. Then the return to equilibrium property is true for $(\mathfrak{M}, \tau^t)$.

As we see from the point 3) of the above theorem, the return to equilibrium follows from the appropriate spectral properties of the Liouvillean. The main goal of this paper is to prove these spectral properties. This proof consists of two basic steps.

First let us describe the first part. We want to show that the Liouvillean L_β has a zero eigenvalue. In the case of Pauli-Fierz Liouvilleans with $\beta < \infty$, this follows from our forthcoming paper [DJP], where we extend the well known perturbation theory for KMS states due to Araki [A1, BR] to the case of unbounded perturbations. By these results the Pauli-Fierz Lioubilleans L_β always have an eigenvector with the eigenvalue zero. This eigenvector represents a β-KMS state for the dynamics τ^t_β. By the general theory of KMS states, this eigenvector is always separating.

In the case $\beta = \infty$, the Liouvillean also has an eigenvector with the zero eigenvalue. This eigenvector corresponds to the ground state of H. It is not separating for the algebra though. Therefore, even if we prove that it is the unique eigenvector of L_∞ and that L_∞ has no singular continuous spectrum, we cannot conclude that the return to equlibrium holds.

The second part of the argument is based on the paper [DJ], which was devoted to the study of spectral properties of Pauli-Fierz operators. The results of [DJ] imply that under generic assumptions the space of eigenvectors of L_β is at most 1-dimensional. The main idea of the approach of [DJ] is to apply the Mourre theory to the Pauli-Fierz operator restricted to the orthogonal complement of the vacuum sector and then to use the Feshbach method to get spectral information about the full operator. These results are conveniently formalized in the so-called Fermi Golden Rule Operator, that encodes the shift of eigenvalues due to the second order perturbation theory.

Now, combining the two parts, we obtain the main result of our paper saying that Pauli-Fierz Liouvilleans L_β, generically, have no singular spectrum except for a single nondegenerate eigenvalue at zero. Under some additional conditions, our results are true uniformly for $\beta \in [\beta_0, \infty]$

For $\beta < \infty$, β-KMS vectors are separating, as a consequence we obtain the return to equilibrium property.

We argued above that Pauli-Fierz systems at positive temperatures should involve Araki-Woods algebras and the W^*-algebraic formalism. This point of view can be justified by invoking the thermodynamical limit of finite volume systems. This argument can be made easily rigorous, although we are not going to do it in this paper. Instead, in the appendix we included an extensive discussion which shows how Araki-Woods representation and thermal Pauli-Fierz systems arise in the finite volume case. In this case, there is no real need to use the W^*-algebraic language at all. In fact, all the Liouvilleans L_β are unitarily equivalent to the operator $H \otimes -1 \otimes \overline{H}$. Nevertheless, even in this case, using various representations seems to be advantageous from the purely algebraic and computational point of view.

In order to make the problem more accessible to a wider audience, we included a whole section where a simplified version of our main result is described. Note however, that this section is completely independent of the rest of our paper and it can be skipped by a more advanced reader.

1.1 Comparison with the literature

Hamiltonians similar to those considered in our paper can be traced back to a famous paper by Dirac [Di]. Since then, they appear frequently in the physics literature [PF, He, CT]. In the recent years there has been a revival of interest in rigorous results about these operators, starting with such papers as [JP1, Sp2, BFS1, Sk, DG].

From the technical point of view, the results of our paper concern mainly spectral properties of a certain class of Pauli-Fierz operators. A large part of the literature on spectral analysis of Pauli-Fierz operators can be divided into two classes. The first uses the generator of translations as the main tool and the second—the generator of dilations.

The idea of using the generator of translations started in the work of Jakšić and Pillet [JP1, JP2], where it was applied to deform analytically positive temperature Pauli-Fierz Liouvilleans. The infinitesimal version of this method based on the Mourre theory was studied in [DJ]. That paper was a technical preparation for the present paper. In fact, in the introduction to [DJ] we roughly described the applications contained in this paper (without, however, giving exact conditions).

The generator of translations is also the main tool of the interesting paper by Merkli [M], which is devoted to the proof of the return to equilibrium in the mean. That paper, however, does not use the results of [DJ], instead it is based on the technique of a "modified conjugate operator" originally due to Huebner and Spohn, elaborated later in [BFSS]. The results of Merkli have a lot of similarity with the results of this paper. One of the differences is the fact that Merkli is interested in the return to equlibrium in the mean, which means that he does not need to show the absence of singular continuous spectrum. His proof is based on the virial theorem, whereas the method of [DJ], on which the present paper is based, is based on the limiting absorbtion principle. The most important difference between our and Merkli's approach, however, is that whereas in our approach the main role is played by the study of the Pauli-Fierz operator restricted to the orthogonal complement of vacuum states, Merkli works on the whole Hilbert space. We believe that the method applied by Merkli is much more complicated and indirect than ours. Merkli's main result is not uniform in the temperature.

The generators of dilations in the context of massless Pauli-Fierz operators were used first by Bach-Froehlich-Sigal in a series of papers [BFS1, BFS2, BFS3, BFSS]. In [BFS4] they applied this technique to study the return to equilibrium uniformly in temperature. A distinctive feature of their approach is the so-called renormalization group technique, which in this context is meant to describe an iterative procedure based on the Feshbach method, used to control the spectrum of Pauli-Fierz operators. The results of [BFS4] resemble closely ours and Merkli's. Strictly speaking, however, the conditions of the result of Bach-Froehlich-Sigal are not comparable to ours and one can find interactions which can be treated with one method and not by the other.

[BFS4] gives conditions that guarantee the return to equilibrium uniformly in the temperature. These conditions are different than ours. What concerns the infrared decay of the interaction, they are somewhat less restrictive than our conditions. This appears to be due to Bach-Froehlich-Sigal's use of the generator of dilations, and not of translations as in our paper. Note, however, that the methods of our paper seem more transparent than those of [BFS4].

Let us stress that the algebraic conditions on the interaction that guarantee the return to equilibrium for small nonzero coupling constant, which are given in our paper, are very simple and natural. These conditions seem to appear for the first time in the literature. Somewhat similar conditions in the context of the generator of a dynamical semigroup were described by Spohn in [Sp1].

The Fermi Golden Rule Operator, that is one of the main tools of our paper, is closely related to the generator of the so-called dynamical quantum semigroup obtained in the weak coupling limit, sometimes called the Davies generator. The Davies generator was introduced in [Da] and was extensively studied in the literature, eg in [Sp1]. Let us stress that the Davies generator is a different object from the Fermi Golden Rule Operator. It has a more direct physical meaning in terms of the evolution of expectation values. It always has a stationary state. On the other hand, the Fermi Golden Rule Operator may have

no stationary state at all (eg. in the case of two thermal reservoirs at different temperatures). Its main role is to describe the 2nd order correction to the eigenvalues and resonances.

One can distinguish two basic approaches to the algebraic quantum statistical physics. Either one tries to describe a physical system by a C^*-dynamical system (a C^*-algebra with a strongly continuous 1-parameter group of $*$-automorphisms) or a W^*-dynamical system. From the conceptual point of view the first approach has some advantages. In fact, it was more popular in the early years of algebraic quantum statistical physics. Unfortunately, realistic physical systems, especially those involving bosons, often do not fit easily into the C^*-algebraic framework.

The most comprehensive exposition of algebraic quantum statistical physics is [BR]. It describes both the C^* and W^*-algebraic approaches. Nevertheless, it is the C^*-algebraic approach, which is emphasized in this monograph.

The point of view of our paper, which is purely W^*-algebraic, owes a lot to the work of Jakšić-Pillet [JP1, JP2]. They made it clear that the analysis of algebraic properties of quantum system can be based on the study of spectral properties of the Liouvillean (although these ideas can be traced to earlier works, such as [Ja]).

An extensive discussion of the algebraic framework for quantum statistical systems inspired by [JP1, JP2] is contained also in [BFS4].

Our presentation of the algebraic approach to Pauli-Fierz systems has some points that are absent or difficult to find elsewhere. We tried to explain the role of the standard representation and its relationship to what we call the semi-standard representation. Our exposition of the Araki-Woods representation and confined systems should help the reader understand how the W^*-algebraic description of quantum systems arises.

2 Simplified version of the main results

This section contains a self-contained description of a simplified version of the main results of this paper. Readers who prefer a more complete exposition can skip this section altogether.

2.1 Pauli-Fierz system at zero temperature

Suppose we consider a small quantum system interacting with scalar massless bosons. We assume that the small system is described by a finite dimensional Hilbert space $\mathcal{K}$ and a self-adjoint operator K. The bosons are described by the one-particle space $\mathbb{R}^d$, where $\xi \in \mathbb{R}^d$ is their momentum and $|\xi|$ is their energy (dispersion relation). The full Hilbert space of the system at zero temperature is $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$, where $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ denotes the Fock space over the 1-particle space $L^2(\mathbb{R}^d)$, and the free Hamiltonian is

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi,$$

where $a^*(\xi)/a(\xi)$ are the creation/annihillation operators of the boson of momentum ξ. We fix an operator-valued function $\mathbb{R}^d \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$. The interaction is given by the operator

$$V := \int (v(\xi) \otimes a^*(\xi) + v^*(\xi) \otimes a(\xi)) \mathrm{d}\xi.$$

The full Pauli-Fierz Hamiltonian equals

$$H := H_{\mathrm{fr}} + \lambda V,$$

where $\lambda \in \mathbb{R}$. Observables of our system at zero temperature are described by the W^*-algebra

$$B\big(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))\big) \tag{2.1}$$

and the dynamics is given by

$$\tau^t(A) := \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}. \tag{2.2}$$

2.2 Araki-Woods representations

Suppose now that the bosonic field has a nonzero densty given by a function

$$\mathbb{R}^d \ni \xi \mapsto \rho(\xi) \in [0, \infty[.$$

The W^*-algebra $B(\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d)))$ does not describe adequately such bosons any more. It needs to be replaced by the algebra $\mathfrak{M}_{\rho,\mathrm{l}}$, the (left) Araki-Woods algebra at density ρ.

The algebra $\mathfrak{M}_{\rho,\mathrm{l}}$ is constructed as follows. It is represented on the Hilbert space $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$. The creation/annihillation operators corresponding to the first $L^2(\mathbb{R}^d)$ (which describe excitations) are denoted by $a_{\mathrm{l}}^*(\xi)/a_{\mathrm{l}}(\xi)$ and those corresponding to the second $L^2(\mathbb{R}^d)$ (describing holes) are denoted by $a_{\mathrm{r}}^*(\xi)/a_{\mathrm{r}}(\xi)$. (l/r stand for left/right). $\mathfrak{M}_{\rho,\mathrm{l}}$ is generated by the operators of the form

$$\exp \mathrm{i}\big(\int (f(\xi)(1+\rho(\xi))^{\frac{1}{2}} a_{\mathrm{l}}^*(\xi) + \overline{f}(\xi)\rho(\xi)^{\frac{1}{2}} a_{\mathrm{r}}(\xi) + \mathrm{hc})\mathrm{d}\xi\big),$$

where $f \in L^2(\mathbb{R}^d)$ satisfy $\int |f(\xi)|^2 \rho(\xi)\mathrm{d}\xi < \infty$.

2.3 Pauli-Fierz systems at a nonzero density

Suppose now that the small quantum system, described by the Hilbert space $\mathcal{K}$ interacts with the bosons of density ρ. Instead of the algebra (2.1), adequate at temperature zero (equivalently, at $\rho = 0$), they should be described by the W^*-algebra

$$B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}. \tag{2.3}$$

This algebra has an obvious representation in the Hilbert space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$, which we call the semistandard representation.

The dynamics (2.2) should now be replaced by the dynamics

$$\tau_\rho^t(A) := \mathrm{e}^{\mathrm{i}tL_\rho^{\mathrm{semi}}} A \mathrm{e}^{\mathrm{i}tL_\rho^{\mathrm{semi}}}, \tag{2.4}$$

where the self-adjoint operator L_ρ^{semi}, which we call the Pauli-Fierz semi-Liouvillean at density ρ is defined as follows:

$$L_{\mathrm{fr}}^{\mathrm{semi}} := K \otimes 1 + 1 \otimes \int \big(|\xi| a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - |\xi| a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi)\big)\mathrm{d}\xi;$$

$$Q_\rho^{\mathrm{semi}} = \int \big((1+\rho(\xi))^{\frac{1}{2}} v(\xi) \otimes a_{\mathrm{l}}^*(\xi) + \rho(\xi)^{\frac{1}{2}} v^*(\xi) \otimes a_{\mathrm{r}}(\xi)\big)\mathrm{d}\xi + \mathrm{hc},$$

$$L_\rho^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}} + \lambda Q_\rho^{\mathrm{semi}}.$$

Let us stress that the W^*-dynamical system

$$(B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}, \tau_\rho^t), \tag{2.5}$$

which we call the Pauli-Fierz system at density ρ, is canonically defined given H and ρ.

Remark 2.1 *The physical interpretation of a Pauli-Fierz system at density ρ can be described as follows.*

First we consider the finite volume version of the system (2.1) with the W^*-dynamics (2.4). The density matrix

$$\Gamma(\rho(1+\rho)^{-1})/\mathrm{Tr}\Gamma\rho(1+\rho)^{-1},$$

defines a state on the W^*-algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$. It is a quasi-free state of density ρ. One can pass to the thermodynamical limit looking at states that in the bosonic variables are local perturbations of this state. This limit is equal to (2.5).

Thus if the volume was finite, all the systems (2.5) would be unitarily equivalent and completely described by the operator H. The operators $L_\rho^{\rm semi}$ would be unitarily equivalent for different ρ. This is no longer true in the infinite volume case.

2.4 Pauli-Fierz systems in a standard representation

As mentioned in the introduction, every W^*-algebra has a distinguished representation, called the standard representation. In the case of the algebra 2.3, it can be naturally realized in the Hilbert space $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d))$. In this representation, the operator $A\otimes B\in B(\mathcal{K})\otimes\mathfrak{M}_{\rho,{\rm l}}$ is represented as $A\otimes 1_{\overline{\mathcal{K}}}\otimes B$. ($\overline{\mathcal{K}}$ denotes the space complex conjugate to $\mathcal{K}$).

Let us describe the Liouvillean of the dynamics τ_ρ^t:

$$L_{\rm fr}:=K\otimes 1\otimes 1-1\otimes\overline{K}\otimes 1+1\otimes 1\otimes\int\big(|\xi|a_{\rm l}^*(\xi)a_{\rm l}(\xi)-|\xi|a_{\rm r}^*(\xi)a_{\rm r}(\xi)\big){\rm d}\xi;$$

$$Q_\rho=\int\big((1+\rho(\xi))^{\frac12}v(\xi)\otimes 1\otimes a_{\rm l}^*(\xi)+\rho(\xi)^{\frac12}v^*(\xi)\otimes 1\otimes a_{\rm r}(\xi)\big){\rm d}\xi+{\rm hc},$$

$$JQ_\rho J=\int\big((1+\rho(\xi))^{\frac12}1\otimes\overline{v}(\xi)\otimes a_{\rm r}^*(\xi)+\rho(\xi)^{\frac12}1\otimes\overline{v}^*(\xi)\otimes a_{\rm l}(\xi)\big){\rm d}\xi+{\rm hc},$$

$$L_\rho:=L_{\rm fr}+\lambda Q_\rho-\lambda JQ_\rho J.$$

Let us note that from the mathematical point of view Pauli-Fierz Hamiltonians, semi-Liouvilleans and Liouvilleans are quite similar. They belong to the class of operators that we call Pauli-Fierz operators. A detailed study of spectral properties of Pauli-Fierz operators has been done in our previous paper [DJ]. In this present paper we apply these results.

Remark 2.2 *Let us remark that in the case of the finite volume the Liouvilleans L_ρ for distinct densities ρ are unitarily equivalent to one another. This is no longer true in the infinite volume*

2.5 Thermal Pauli-Fierz systems

The most important class of Pauli-Fierz systems corresponds to the family of densities given by the Planck law

$$\rho(\xi)=({\rm e}^{\beta|\xi|}-1)^{-1}.$$

Such systems are called thermal Pauli-Fierz systems at inverse temperature β. Let us restrict our attention to such systems.

Let us change slightly the notation: instead of the subscript ρ_β let us use β, thus we will write L_β, τ_β^t instead of L_{ρ_β} and $\tau_{\rho_\beta}^t$.

Let us describe (a simplified version of) the main result of our paper. We will use the following notation. ${\rm sp}(K)$ will denote the spectrum of K. If $k_1,k_2\in{\rm sp}(K)$ (that is, if k_1,k_2 are eigenvalues of K) we will write $v^{k_1,k_2}(\xi)$ for the matrix elements of $v(\xi)$ between the corresponding eigenspaces of K. If either k_1 or k_2 does not belong to ${\rm sp}(K)$, we simply set $v^{k_1,k_2}(\xi)=0$. We will write k_0 for the infimum of the spectrum of K. The spectral projection of K onto $k\in{\rm sp}(K)$ will be denoted by $1_k(K)$.

Theorem 2.3 *Suppose the following conditions are satisfied:*

1. *$\delta > 0$, and for $n = 0, 1, 2, 3$ we have*

$$\| \tfrac{\xi \mathrm{d}^n}{|\xi| \mathrm{d}\xi^n} |\xi|^{\frac{d-1}{2}} v(\xi) \|^2 \leq c \begin{cases} |\xi|^{3+\delta-n}, & |\xi| < 1, \\ (1+|\xi|)^{-\frac{1}{2}-\delta-n} & |\xi \geq 1. \end{cases}$$

2. *$\dim 1_{k_0} = 1$ (the operator K has a nondegenerate smallest eigenvalue);*

3. *There exists $c > 0$ such that for any $k \in \mathrm{sp}(K)$, $k \neq k_0$,*

$$\sum_{p \in \mathbb{R}} (v^*)^{k,k-p}(p\omega) v^{k-p,k}(p\omega) p^{d-1} \mathrm{d}\omega \geq c 1_k(K);$$

4. *If $B \in B(\mathcal{K})$ and for any $\xi \in \mathbb{R}^d$*

$$B \sum_{k \in \mathrm{sp}(K)} v^{k-|\xi|,k}(\xi) = \sum_{k \in \mathrm{sp}(K)} v^{k-|\xi|,k}(\xi) B,$$

$$B^* \sum_{k \in \mathrm{sp}(K)} v^{k-|\xi|,k}(\xi) = \sum_{k \in \mathrm{sp}(K)} v^{k-|\xi|,k}(\xi) B^*,$$

then B is proportional to the identity operator.

Then for any $\beta_0 > 0$ there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$ and $\beta \in [\beta_0, \infty]$, the Liouvillean L_β has no singular continuous spectrum and a unique eigenvalue, which is at zero and nondegenerate. Consequently, for $0 < |\lambda| \leq \lambda_0$ and $\beta \in [\beta_0, \infty[$, the Pauli-Fierz system $(B(\mathcal{K}) \otimes \mathfrak{M}_{\beta,1}, \tau_\beta^t)$ satisfies the return to equilibrium property.

3 Basic notation and facts

3.1 Miscellanea

We set $\mathbb{C}_+ := \{z \in \mathbb{C} \ : \ \mathrm{Im} z > 0\}$.

B^{cl} denotes the closure of a set B. In particular, $\mathbb{C}_+^{\mathrm{cl}} = \{z \in \mathbb{C} \ : \ \mathrm{Im} z \geq 0\}$.

We will use notation $\delta(p)$ for the Dirac delta at 0, $\mathcal{P}p^{-1}$ for the principal value of p^{-1} and $(p + \mathrm{i}0)^{-1}$ for $\lim_{\downarrow 0}(p + \mathrm{i}\epsilon)^{-1}$. Thus if $\mathbb{R} \ni p \mapsto f(p)$ is a continuous function, then we will write

$$\int f(p)\delta(p)\mathrm{d}p = f(0),$$

$$\int f(p)\mathcal{P}p^{-1}\mathrm{d}p = \lim_{\epsilon \downarrow 0} \big(\textstyle\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \big) f(p) p^{-1} \mathrm{d}p,$$

$$\int f(p)(p + \mathrm{i}0)^{-1}\mathrm{d}p = \lim_{\epsilon \downarrow 0} \int f(p)(p + \mathrm{i}\epsilon)^{-1}\mathrm{d}p,$$

provided the above limits exist.

We will sometimes use the so-called Sochocki formula:

$$(p + \mathrm{i}0)^{-1} = \mathcal{P}p^{-1} - \mathrm{i}\pi\delta(p).$$

3.2 Operators in Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space with the scalar product $(\Psi|\Phi)$, $\Psi, \Phi \in \mathcal{H}$.

$B(\mathcal{H})$, $B_+(\mathcal{H})$ and $U(\mathcal{H})$ denote the set of bounded, bounded positive and unitary operators on $\mathcal{H}$.

$l^2(\mathcal{H})$ will denote the space of Hilbert-Schmidt operators on $\mathcal{H}$ with the scalar product $\mathrm{Tr}(B^*A)$ and $l_+(\mathcal{H})$ the positive Hilbert-Schmidt operators.

If $\Psi \in \mathcal{H}$, then $|\Psi)$ denotes the operator

$$\mathbb{C} \ni \lambda \mapsto \lambda\Psi \in \mathcal{H},$$

and $(\Psi| := |\Psi)^*$ denotes the operator

$$\mathcal{H} \ni \Phi \mapsto (\Psi|\Phi) \in \mathbb{C}.$$

In particular, if $\|\Psi\| = 1$, then $|\Psi)(\Psi|$ is the orthogonal projection onto Ψ.

If A is an operator on $\mathcal{H}$, then $\mathrm{sp}(A)$ denotes its spectrum and $\mathrm{sp}_{\mathrm{p}}(A)$ its point spectrum (the set of eigenvalues of A).

If A is self-adjoint and Θ is a Borel subset of $\mathbb{R}$, then $1_\Theta(A)$ denotes the spectral projection of A onto Θ. $1^{\mathrm{p}}(A)$ denotes the projection onto the subspace spanned by the eigenvectors of A. $1^{\mathrm{ac}}(A)$ denotes the projection onto the absolutely continuous part of the spectrum of A and $1^{\mathrm{sc}}(A) := 1 - 1^{\mathrm{ac}}(A) - 1^{\mathrm{p}}(A)$ denotes the projection onto the singular continuous part of the spectrum.

If A is closed and Θ is an isolated subset of $\mathrm{sp}(A)$ (closed and open in the relative topology of $\mathrm{sp}(A)$), then $1_\Theta(A)$ denotes the spectral (Riesz) projection of A onto Θ.

If $z \in \mathrm{sp}(A)$ is an isolated point of $\mathrm{sp}(A)$, or A is self-adjoint, we will write $1_z(A)$ instead of $1_{\{z\}}(A)$.

3.3 Fermi Golden Rule Operator

In this section we introduce a general framework, which can be used to formalize 2nd order perturbation theory of linear operators, incorporating elements of the Feshbach method.

Suppose that $\mathcal{H}$ is a Hilbert space with a distinguished finite dimensional subspace $\mathcal{H}^{\mathrm{v}}$. We set $\mathcal{H}^{\overline{\mathrm{v}}} := (\mathcal{H}^{\mathrm{v}})^\perp$. We will use the 2×2 matrix notation for operators, that is if $B \in B(\mathcal{H})$ it will be sometimes written as

$$B = \begin{bmatrix} B^{\mathrm{vv}} & B^{\mathrm{v}\overline{\mathrm{v}}} \\ B^{\overline{\mathrm{v}}\mathrm{v}} & B^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{3.6}$$

Suppose that L_{fr} is a self-adjoint operator that leaves $\mathcal{H}^{\mathrm{v}}$ invariant. Thus it can be written in the block matrix form as

$$L_{\mathrm{fr}} = \begin{bmatrix} L_{\mathrm{fr}}^{\mathrm{vv}} & 0 \\ 0 & L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}, \tag{3.7}$$

We will often write E instead of $L_{\mathrm{fr}}^{\mathrm{vv}}$, and $\mathcal{E}$ for $\mathcal{H}^{\mathrm{v}}$. We define the subspaces

$$\mathcal{E}^e := \mathrm{Ran} 1_e(E), \quad e \in \mathrm{sp}(E),$$

We have the decomposition

$$\mathcal{E} = \bigoplus_{e \in \mathrm{sp}(E)} \mathcal{E}^e.$$

We will write

$$B^{e_1,e_2} := 1_{e_1}(E) B 1_{e_2}(E), \qquad e_1, e_2 \in \mathrm{sp}(E), \quad B \in B(\mathcal{E}).$$

Let Q be self-adjoint operator on $\mathcal{H}$ such that $Q^{\mathrm{vv}} = 0$ and $Q^{\mathrm{v}\overline{\mathrm{v}}} = (Q^{\overline{\mathrm{v}}\mathrm{v}})^*$ is bounded.

For $z \notin \mathrm{sp}(L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$ we introduce

$$w(z) \quad := Q^{\mathrm{v}\overline{\mathrm{v}}}(z 1^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}. \tag{3.8}$$

Let us assume that for $e \in \mathrm{sp}(E)$, there exists

$$\lim_{\epsilon \downarrow 0} w(e + \mathrm{i}\epsilon)^{ee} =: w(e + \mathrm{i}0)^{ee}.$$

The Fermi Golden Rule Operator (abbreviated as FGRO) is defined as

$$\Gamma = \sum_{e \in \mathrm{sp}(E)} w(e + \mathrm{i}0)^{ee}$$

We will say that Γ is the FGRO associated with the triple $(\mathcal{H}^{\mathrm{v}}, L_{\mathrm{fr}}, Q)$

In general, Γ is not self-adjoint and it is convenient to use the following notation for its real and imaginary part:

$$\Gamma^{\mathrm{R}} := \mathrm{Re}\Gamma = \tfrac{1}{2}(\Gamma + \Gamma^*), \quad \Gamma^{\mathrm{I}} := \mathrm{Im}\Gamma = \tfrac{1}{2\mathrm{i}}(\Gamma - \Gamma^*).$$

Clearly, Γ is a dissipative operator, this means that

$$\Gamma = \Gamma^{\mathrm{R}} + \mathrm{i}\Gamma^{\mathrm{I}}, \qquad (\Gamma^{\mathrm{R}})^* = \Gamma^{\mathrm{R}}, \qquad \Gamma^{\mathrm{I}} \leq 0.$$

Besides,

$$\Gamma^{ee} = w(e + \mathrm{i}0)^{ee}, \quad \Gamma^{e_1,e_2} = 0, \quad e_1 \neq e_2;$$

$$E\Gamma = \Gamma E, \quad E\Gamma^{\mathrm{R}} = \Gamma^{\mathrm{R}} E, \quad E\Gamma^{\mathrm{I}} = \Gamma^{\mathrm{I}} E.$$

If we assume that $L^{\overline{\mathrm{vv}}} + Q^{\overline{\mathrm{vv}}}$ is essentially self-adjoint on $\mathcal{D}(L^{\overline{\mathrm{vv}}}) \cap \mathcal{D}(Q^{\overline{\mathrm{vv}}})$, then we can define the self-adjoint operator

$$L_\lambda = L_{\mathrm{fr}} + \lambda Q,$$

The main application of the Fermi Golden Rule Operator Γ is to describe in a concise way the second order perturbation predictions for the point spectrum of L_λ.

In our previous paper [DJ] we proved that under some assumptions the operator

$$1_{\mathrm{sp}(\Gamma) \cap \mathbb{R}}(\Gamma)(E + \lambda^2 \Gamma) \tag{3.9}$$

can be used to predict the possible approximate location of eigenvalues of L_λ and to estimate from above their multiplicity. Our present paper will heavily depend on this result.

Note also that

$$E + \lambda^2 \Gamma \tag{3.10}$$

can be used to predict the approximate location and multiplicities of eigenvalues and resonances of L_λ.

3.4 Space $L^2(\mathbb{R})$

In this subsection we describe some operators acting on $L^2(\mathbb{R})$.

Let r denote the self-adjoint operator of multiplication by the variable in $\mathbb{R}$. That means

$$r\Psi(p) := p\Psi(p).$$

Note that throughout the paper in the context of the space $L^2(\mathbb{R})$, the generic name for a variable in $\mathbb{R}$ will be p. On the other hand, the multiplication operator on $L^2(\mathbb{R})$ by its natural variable will be r.

We introduce the self-adjoint operator s on $L^2(\mathbb{R})$:

$$s\Psi(p) := \frac{1}{\mathrm{i}} \nabla_p \Psi(p).$$

Note that $[s, r] = \mathrm{i}$. We will write $\langle s \rangle := (1 + s^2)^{\frac{1}{2}}$.

The following theorem is well known:

Theorem 3.1 *1) Suppose that $n = 1, 2, \ldots$ and $\eta > n - \frac{1}{2}$. Then the function*

$$\mathbb{C}_+ \ni z \mapsto \langle s\rangle^{-\eta}(z-r)^{-n}\langle s\rangle^{-\eta} \in B(L^2(\mathbb{R}))$$

extends from $\mathbb{C}_+$ to a continuous function on $\mathbb{C}_+^{\mathrm{cl}}$.
2) Let $\eta > \frac{1}{2}$. Then for $f \in \mathcal{D}(\langle s\rangle^{\eta})$, there exists a unique continuous function $\mathbb{R} \mapsto f(p) \in \mathbb{C}$ which is a representant of f. For any $p \in \mathbb{R}$ and $\eta > \frac{1}{2}$ define a linear functional $\pi_{p,\eta} : L^2(\mathbb{R}) \to \mathbb{C}$ by

$$\pi_{p,\eta} g := \big(\langle s\rangle^{-\eta} g\big)(p).$$

Then

1. *$\pi_{p,\eta}$ is bounded;*
2. *$\mathbb{R} \ni p \mapsto \pi_{p,\eta} \in B(L^2(\mathbb{R}), \mathbb{C})$ is continuous;*
3. *For $\frac{1}{2} < \eta_1 \leq \eta_2$ we have*
$$\pi_{p,\eta_1}\langle s\rangle^{\eta_1-\eta_2} = \pi_{p,\eta_2}.$$

Remark 3.2 *Define an operator $\pi_p : C(\mathbb{R}) \cap L^2(\mathbb{R}) \to \mathbb{C}$ by $\pi_p f := f(p)$. Then $\pi_{p,\eta} = \pi_p\langle s\rangle^{-\eta}$.*

3.5 Space $L^2(\mathbb{R}, \mathcal{E})$

Let $\mathcal{E}$ be a Hilbert space (not necessarily separable). We say that a function $\mathbb{R} \ni p \mapsto \Psi(p) \in \mathcal{E}$ belongs to $\mathcal{L}^2(\mathbb{R}, \mathcal{E})$ iff

1. There exists a separable subspace $\mathcal{E}_0$ such that for any $p \in \mathbb{R}$, $\Psi(p) \in \mathcal{E}_0$.
2. For any $\Phi \in \mathcal{E}$, $\mathbb{R} \ni p \mapsto (\Phi|\Psi(p)) \in \mathbb{C}$ is measurable.
3. $\int \|\Psi(p)\|^2 \mathrm{d}p < \infty$.

Let $p \mapsto \Psi(p)$ belong to $\mathcal{L}^2(\mathbb{R}, \mathcal{E})$. We say that it belongs to $\mathcal{N}(\mathbb{R}, \mathcal{E})$ iff $\Psi(p) = 0$ for almost all $p \in \mathbb{R}$. We define

$$L^2(\mathbb{R}, \mathcal{E}) := \mathcal{L}^2(\mathbb{R}, \mathcal{E})/\mathcal{N}(\mathbb{R}, \mathcal{E}).$$

We easily see that there exists a unique unitary operator

$$\mathcal{E} \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}, \mathcal{E}),$$

such that $\Psi \otimes f \in \mathcal{E} \otimes L^2(\mathbb{R})$ is mapped onto $p \mapsto f(p)\Psi$.

Suppose now that $q \in B(\mathcal{K}, L^2(\mathbb{R}) \otimes \mathcal{E})$ and $\langle s\rangle^{\eta} \otimes 1_{\mathcal{E}}\, q$ is bounded. Then for $p \in \mathbb{R}$ we can define

$$q(p) := 1_{\mathcal{E}} \otimes \pi_{p,\eta}\langle s\rangle^{\eta}\; q \in B(\mathcal{K}, \mathcal{E}).$$

Clearly, the definition of $q(p)$ does not depend on the choice of $\eta > \frac{1}{2}$ and

$$\mathbb{R} \ni p \mapsto q(p) \in B(\mathcal{K}, \mathcal{E})$$

is a continuous function.

If $f \in L^\infty(\mathbb{R})$, we have the identity

$$q^* f(r) q = \int q^*(p) f(p) q(p) \mathrm{d}p. \tag{3.11}$$

Note the estimate

$$\|q^* q\| \leq \int \|q^*(p) q(p)\| \mathrm{d}p = \int \|q(p)\|^2 \mathrm{d}p. \tag{3.12}$$

3.6 Conjugate Hilbert spaces

If $\mathcal{K}$ is a Hilbert space, then the space $\overline{\mathcal{K}}$ conjugate to $\mathcal{K}$ is any Hilbert space with a distinguished antiunitary map

$$\mathcal{K} \ni \Psi \mapsto \overline{\Psi} \in \overline{\mathcal{K}}. \tag{3.13}$$

The map (3.13) is called the (external) conjugation on $\mathcal{K}$.

Note that, by the Riesz lemma, $\overline{\mathcal{K}}$ is naturally isomorphic to the dual space to $\mathcal{K}$, that is to $B(\mathcal{K}, \mathbb{C})$:

$$\overline{\mathcal{K}} \ni \overline{\Psi} \mapsto (\Psi| \in B(\mathcal{K}, \mathbb{C}).$$

If $A \in B(\mathcal{K})$, then $\overline{A} \in B(\overline{\mathcal{K}})$ is defined as

$$\overline{\mathcal{K}} \ni \overline{\Psi} \mapsto \overline{A}\,\overline{\Psi} := \overline{A\Psi} \in \overline{\mathcal{K}}.$$

Thus we can identify $B(\overline{\mathcal{K}})$ with $\overline{B(\mathcal{K})}$.

Remark 3.3 *For typographical reasons, sometimes it is convenient to use a different notation for conjugation. One can denote (3.13) by*

$$\mathcal{K} \ni \Psi \mapsto \kappa\Psi \in \overline{\mathcal{K}}. \tag{3.14}$$

The inverse of (3.14) will also be denoted by κ. In such a case $\kappa\Psi$ will replace $\overline{\Psi}$ for $\Psi \in \mathcal{K}$; $\kappa A \kappa$ will replace $\overline{A}$ for $A \in B(\mathcal{K})$.

Remark 3.4 *We say that*

$$\mathcal{K} \ni \Psi \mapsto \overline{\Psi} \in \mathcal{K} \tag{3.15}$$

is an (internal) conjugation on $\mathcal{K}$ iff 3.15 is an antilinear map on $\mathcal{K}$ and $\overline{\overline{\Psi}} = \Psi$. If we fix an internal conjugation in $\mathcal{K}$, then we can idenify $\overline{\mathcal{K}}$ with $\mathcal{K}$.

3.7 The identification of Hilbert-Schmidt operators with $\mathcal{K} \otimes \overline{\mathcal{K}}$

We will often use the identification of the set of Hilbert-Schmidt operators, $l^2(\mathcal{K})$ with $\mathcal{K} \otimes \overline{\mathcal{K}}$, so that $|\Phi_1)(\Phi_2| \in l^2(\mathcal{K})$ corresponds to $\Psi_1 \otimes \overline{\Psi_2} \in \mathcal{K} \otimes \overline{\mathcal{K}}$. This identification can be sometimes confusing. To avoid misunderstanding we will try to make clear which convention we use at the moment.

In particular, let us note the following identities valid for $B \in B(\mathcal{K})$ and $C \in \mathcal{K} \otimes \overline{\mathcal{K}} \simeq l^2(\mathcal{K})$:

$$B \otimes 1_{\overline{\mathcal{K}}}\, C = BC, \tag{3.16}$$

$$1_{\mathcal{K}} \otimes \overline{B}\, C = CB^*. \tag{3.17}$$

Note that on the left of these formulas C is interpreted as an element of $\mathcal{K} \otimes \overline{\mathcal{K}}$ whereas on the right, as an element of $l^2(\mathcal{K})$.

3.8 The $\star$ conjugation

In this subsection we introduce a certain antilinear map $\star$ from a dense subspace of $B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ to a dense subspace of $B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$.

Let $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$. We say that v is $\star$-conjugable iff there exists $v^\star \in B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$ such that

$$(\Phi \otimes w | v\Psi) = (v^\star \Phi | \Psi \otimes \overline{w}), \quad \Phi, \Psi \in \mathcal{E}, \quad w \in \mathcal{W}.$$

(If such an operator $v^\star$ exists, then it is unique.)

Remark 3.5 *Given an orthonormal basis $\{w_i \,:\, i \in I\}$ in $\mathcal{W}$, any $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ can be decomposed as*

$$v = \sum_{i \in I} B_i \otimes |w_i), \tag{3.18}$$

where $B_i \in B(\mathcal{K})$ and the sum should be understood in terms of the strong operator convergence. Note that

$$v^* v = \sum_{i \in I} B_i^* B_i. \tag{3.19}$$

v is $\star$-conjugable iff

$$\sum_{i \in I} B_i B_i^* \tag{3.20}$$

is bounded. If this is the case,

$$v^\star := \sum_{i \in I} B_i^* \otimes |\overline{w_i}).$$

and (3.20) equals $v^{\star *} v^\star$.

Theorem 3.6 *Suppose that either $\mathcal{K}$ or $\mathcal{W}$ are finite dimensional. Then all $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ are $\star$-conjugable. If $n := \min\big((\dim \mathcal{K})^2, \dim \mathcal{W}\big)$, then*

$$\|v^\star\| \leq \sqrt{n} \|v\|.$$

Clearly, $\dim B(\mathcal{K}) = (\dim \mathcal{K})^2$. Therefore, we can choose an orthonormal system $\{w_i\}$ in $\mathcal{W}$ with at most n elements such that (3.18) is true. Now

$$\begin{aligned} \|v^\star\|^2 = \Big\| \sum_i B_i B_i^* \Big\| \quad &\leq \sum_i \|B_i B_i^*\| \\ &= \sum_i \|B_i^* B_i\| \\ &\leq n \Big\| \sum_i B_i^* B_i \Big\| = n \|v\|^2. \end{aligned}$$

□

Remark 3.7 *If $\mathcal{W}$ and $\mathcal{K}$ are infinite dimensional, it is easy to find an example of $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ which is not $\star$-conjugable.*

In what follows, if ρ is an operator on $\mathcal{W}$ and $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$, we will write ρv instead of $1_{\mathcal{K}} \otimes \rho \, v$.

Proposition 3.8 *1) If v is $\star$-conjugable, then so is $v^\star$ and*

$$v^{\star\star} = v.$$

2) If $\rho \in B(\mathcal{W})$, then

$$(\rho v)^\star = \overline{\rho} v^\star. \tag{3.21}$$

3) If $B \in B(\mathcal{K})$, then $(vB)^\star = B^* \otimes 1_{\mathcal{W}} \, v^\star$.

3.9 Coupling Hilbert-Schmidt operators

Suppose that $\mathcal{K}$ is a Hilbert space. In this subsection we describe some notation and identities related to the space $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{W}\simeq l^2(\mathcal{K})\otimes\mathcal{W}$.

Note that this subsection, except for the definition of $\check{\otimes}$, can be skipped on the first reading.

It is useful to define the operation of "tensoring in the middle", which will be denoted by $\check{\otimes}$. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be arbitrary Hilbert spaces. If $\overline{B}\in B(\overline{\mathcal{K}})$ and $A\in B(\mathcal{K}\otimes\mathcal{H}_1,\mathcal{K}\otimes\mathcal{H}_2)$, we define

$$\overline{B}\check{\otimes}A := \theta\otimes 1_{\mathcal{H}_1}\ \overline{B}\otimes A\ \theta\otimes 1_{\mathcal{H}_2}\in B(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_1,\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_2), \tag{3.22}$$

where $\theta:\mathcal{K}\otimes\overline{\mathcal{K}}\to\overline{\mathcal{K}}\otimes\mathcal{K}$ is defined as $\theta\,\Psi_1\otimes\overline{\Psi_2}:=\overline{\Psi_2}\otimes\Psi_1$.

In other words, if $C\in B(\mathcal{K})$, $A\in B(\mathcal{H}_2,\mathcal{H}_1)$, we set

$$\overline{B}\,\check{\otimes}\,C\otimes A := C\otimes\overline{B}\otimes A.$$

We use Tr to denote the trace. In the context of coupled systems Tr will be reserved for the trace over the space $\mathcal{K}$. To denote the partial trace over the space $\mathcal{W}$ we will use tr, in particular, if $C\in l^1_+(\mathcal{K}\otimes\mathcal{W})$, then $\mathrm{tr}C$ will be an element of $l^1_+(\mathcal{K})$.

Proposition 3.9 *Let*

$$B\in l^2(\mathcal{K}),\quad D\in l^2(\mathcal{K},\mathcal{K}\otimes\mathcal{W}),$$

$$v_{\mathrm{l}}\in B(\mathcal{K},\mathcal{K}\otimes\mathcal{W}),\quad v_{\mathrm{r}}\in B(\mathcal{K},\mathcal{K}\otimes\overline{\mathcal{W}}).$$

Then the following statements hold:

$$\begin{aligned}
&1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}\ B = v_{\mathrm{l}}B,\\
&1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}^*\ D = v_{\mathrm{l}}^*D;\\
&1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}\ B = B\otimes 1_{\mathcal{W}}\ v_{\mathrm{r}}^\star,\\
&1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}^*\ D = \mathrm{tr}Dv_{\mathrm{r}}^{\star *};
\end{aligned}\tag{3.23}$$

(on the left we use the $\mathcal{K}\otimes\overline{\mathcal{K}}$ notation for B, D and on the right the $l^2(\mathcal{K})$ notation).

Proof. It is sufficient to assume that $v_{\mathrm{l}}=C\otimes|w)$ and $v_{\mathrm{r}}=C\otimes|\overline{w})$ for some $C\in B(\mathcal{K})$, $w\in\mathcal{W}$. Then

$$\begin{aligned}
&1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}} = C\otimes 1_{\overline{\mathcal{K}}}\otimes|w),\\
&1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}^* = C^*\otimes 1_{\overline{\mathcal{K}}}\otimes(w|,\\
&1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}} = 1_{\mathcal{K}}\otimes\overline{C}\otimes|w),\\
&1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}^* = 1_{\mathcal{K}}\otimes\overline{C}^*\otimes(w|.
\end{aligned}\tag{3.24}$$

We may also assume that $D=B\otimes|w_0)$ for some $w_0\in\mathcal{W}$. Using (3.16) and (3.17) we get

$$\begin{aligned}
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}\ B &= CB\otimes|w) &&= C\otimes|w)\ B,\\
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}^*\ D &= C^*B\,(w|w_0) &&= C^*\otimes(w|\ B\otimes|w_0);\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}\ B &= BC^*\otimes|w) &&= B\otimes 1_{\mathcal{W}}\ \big(C\otimes|\overline{w})\big)^\star,\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}^*\ D &= BC(w|w_0) &&= \mathrm{tr}\left(B\otimes|w_0)\ \big(C^*\otimes(\overline{w}|\big)^{\star *}\right).
\end{aligned}\tag{3.25}$$

□

Proposition 3.10 *Let $B \in B(\mathcal{K})$, and $\rho \in B(\mathcal{W})$. For $v_{\rm l} \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$, $v_{\rm r} \in B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$, we have*

$$1_{\mathcal{K}} \otimes \overline{v_{\rm r}}^* \ \rho \ 1_{\overline{\mathcal{K}}} \check{\otimes} v_{\rm l} \ B = {\rm tr} \rho v_{\rm l} B v_{\rm r}^{\star *} = v_{\rm l}^{\star *} \ B \otimes \overline{\rho}^* \ v_{\rm r}. \tag{3.26}$$

$$1_{\overline{\mathcal{K}}} \check{\otimes} v_{\rm l}^* \ \rho \ 1_{\mathcal{K}} \otimes \overline{v_{\rm r}} \ B = {\rm tr} v_{\rm l}^\star \ B \ v_{\rm r}^* \overline{\rho}^* = v_{\rm l}^* \ B \otimes \rho \ v_{\rm r}^\star. \tag{3.27}$$

For $v_{{\rm l},1}, v_{{\rm l},2} \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$, we have

$$1_{\overline{\mathcal{K}}} \check{\otimes} v_{{\rm l},1}^* \ \rho \ 1_{\overline{\mathcal{K}}} \check{\otimes} v_{{\rm l},2} \ B = {\rm tr} v_{{\rm l},1}^\star v_{{\rm l},2}^{\star *} \ B \otimes \overline{\rho}^* = v_{{\rm l},1}^* \rho v_{{\rm l},2} B. \tag{3.28}$$

For $v_{{\rm r},1}, v_{{\rm r},2} \in B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$, we have

$$1_{\mathcal{K}} \otimes \overline{v}_{{\rm r},1}^* \ \rho \ 1_{\mathcal{K}} \otimes \overline{v}_{{\rm r},2} \ B = {\rm tr} B \otimes \rho \ v_{{\rm r},2}^\star v_{{\rm r},1}^{\star *} = B v_{{\rm r},2}^* \overline{\rho}^* v_{{\rm r},1}. \tag{3.29}$$

Proof. We will prove only (3.26). First note that

$$1_{\mathcal{K}} \otimes \overline{v}_{\rm r}^\star \ \rho \ 1_{\overline{\mathcal{K}}} \check{\otimes} v_{\rm l} \ B = 1_{\mathcal{K}} \otimes \overline{v}_{\rm r}^\star \ \rho \ v_{\rm l} B = {\rm tr} \rho v_{\rm l} B v_{\rm r}^{\star *}. \tag{3.30}$$

We take $v_{\rm l} = C_{\rm l} \otimes |w_{\rm l})$ and $v_{\rm r} = C_{\rm r} \otimes |\overline{w}_{\rm r})$ for some $C_{\rm l}, C_{\rm r} \in B(\mathcal{K})$, $w_{\rm l}, w_{\rm r} \in \mathcal{W}$. Then the left hand side of (3.30) equals

$$\begin{aligned}
&{\rm tr}\Big(C_{\rm l} \otimes |\rho w_{\rm l}) \ B \ \big(C_{\rm r} \otimes |\overline{w}_{\rm r})\big)^{\star *}\Big) = {\rm tr}\Big(C_{\rm l} \otimes |\rho w_{\rm l}) \ B \ C_{\rm r} \otimes (w_{\rm r}|\Big) \\
&= C_{\rm l} B C_{\rm r} \, {\rm tr}\big(\rho |w_{\rm l})(w_{\rm r}|\big) = C_{\rm l} B C_{\rm r} (\overline{w}_{\rm l}|\overline{\rho}^* \overline{w}_{\rm r}) \\
&= C_{\rm l} \otimes (\overline{w}_{\rm l}| \ B \otimes \overline{\rho}^* \ C_{\rm r} \otimes |\overline{w}_{\rm r}) = \big(C_{\rm l} \otimes |w_{\rm l})\big)^{\star *} B \otimes \overline{\rho}^* \, C_{\rm r} \otimes |\overline{w}_{\rm r}) = v_{\rm l}^{\star *} B \otimes \overline{\rho}^* v_{\rm r}.
\end{aligned}$$

□

4 Representations of CCR and Fock spaces

In this section we introduce notation related to second quantization and Fock spaces and recall some of properties of second quantization. Most of this material, especially from the first two subsections, is quite standard, see eg. [BR], [BSZ], [DG], although various conventions about notation are used in the literature.

4.1 Representations of CCR

Suppose that $\mathcal{Z}_0$ is a unitary space, that means, $\mathcal{Z}_0$ is a complex vector space equipped with a positive definite sesquilinear product $(\cdot|\cdot)$. Let $\mathcal{H}$ be a Hilbert space. A map $\mathcal{Z}_0 \ni z \mapsto W_\pi(z) \in U(\mathcal{H})$ is a representation of CCR if

$$W_\pi(z_1) W_\pi(z_2) = {\rm e}^{-\frac{\rm i}{2}{\rm Im}(z_1|z_2)} W(z_1 + z_2), \quad z_1, z_2 \in \mathcal{Z}_0.$$

It is called regular if

$$t \mapsto W_\pi(tz) \ \text{ is strongly continuous for any } \ z \in \mathcal{Z}_0,$$

and

If we have a regular representation of CCR, using Stone's theorem, for $z \in \mathcal{Z}_0$, we can define the field operators

$$\phi_\pi(z) := \frac{\rm d}{{\rm i}{\rm d}t} W_\pi(tz), \tag{4.31}$$

and the creation and annihillation operators

$$z(a_\pi^*) := \frac{1}{\sqrt{2}}(\phi_\pi(z) - \mathrm{i}\phi(\mathrm{i}z)), \quad \overline{z}(a_\pi) := \frac{1}{\sqrt{2}}(\phi_\pi(z) + \mathrm{i}\phi(\mathrm{i}z)). \tag{4.32}$$

Clearly, the field operators are self-adjoint. The creation and annihillation operators can be shown to be closed.

The von Neumann algebra generated by $W_\pi(z)$ for $z \in \mathcal{Z}_0$ is called the von Neumann algebra associated with the representation $z \mapsto W_\pi(z)$.

Suppose κ is a conjugation in $\mathcal{Z}_0$ (an antiunitary map satisfying $\kappa^2 = 1$). Let $\mathcal{Z}_{0,\kappa} := \{z \in \mathcal{Z}_0 \,:\, \kappa z = z\}$. Then the operators $W_\pi(z)$, $z \in \mathcal{Z}_{0,\kappa}$ generate a commutative subalgebra in $B(\mathcal{H})$. The operators in this subalgebra will be called κ-real.

4.2 Fock spaces

Let $\mathcal{Z}$ is a Hilbert space. Set $\Gamma_{\mathrm{s}}^n(\mathcal{Z}) := \otimes_{\mathrm{s}}^n \mathcal{Z}$ and let

$$\Gamma_{\mathrm{s}}(\mathcal{Z}) := \bigoplus_{n=0}^{\infty} \Gamma_{\mathrm{s}}^n(\mathcal{Z})$$

denote the bosonic Fock space. Ω will denote the vacuum. For $z \in \mathcal{Z}$, the creation operator $z(a^*)$ is defined as

$$z(a^*)\Psi := \sqrt{n+1}\, z \otimes_{\mathrm{s}} \Psi, \quad \Psi \in \Gamma_{\mathrm{s}}^n(\mathcal{Z}).$$

$z(a^*)$ extends to a closed operator. Its adjoint

$$\overline{z}(a) := (z(a^*))^*,$$

is called the annihillation operator. The field and Weyl operators are defined as $\phi(z) := \frac{1}{\sqrt{2}}(z(a^*) + \overline{z}(a))$ and $W(z) = \mathrm{e}^{\mathrm{i}\phi(z)}$. It is well known that $\mathcal{Z} \ni z \mapsto W(z) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ is a regular representation of CCR.

In the usual way, we define the operators $\mathrm{d}\Gamma(h) : \Gamma_{\mathrm{s}}(\mathcal{Z}) \to \Gamma_{\mathrm{s}}(\mathcal{Z})$ for an operator h on $\mathcal{Z}$ and $\Gamma(u) : \Gamma_{\mathrm{s}}(\mathcal{Z}_1) \to \Gamma_{\mathrm{s}}(\mathcal{Z}_2)$ for an operator u from $\mathcal{Z}_1$ to $\mathcal{Z}_2$.

If $\mathcal{Z}_1, \mathcal{Z}_2$ are two Hilbert spaces then

$$U\Psi_n \otimes \Psi_m := \sqrt{\frac{(n+m)!}{n!m!}}\Psi_n \otimes_{\mathrm{s}} \Psi_m, \quad \Psi_n \in \Gamma_{\mathrm{s}}^n(\mathcal{Z}_1), \quad \Psi_m \in \Gamma_{\mathrm{s}}^m(\mathcal{Z}_2), \tag{4.33}$$

defines a unitary map

$$U : \Gamma_{\mathrm{s}}(\mathcal{Z}_1) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_2) \to \Gamma_{\mathrm{s}}(\mathcal{Z}_1 \oplus \mathcal{Z}_2).$$

U is sometimes called the exponential map.

4.3 Squeezed states

This subsection will be used only in Appendix B, and can be skipped on the first reading. In the mathematical literature squeezed states can be traced back to the work of Friedrichs [Fr].

Let $c \in \mathcal{Z} \otimes_{\mathrm{s}} \mathcal{Z}$. The vector c can be identified with an operator $c \in l^2(\overline{\mathcal{Z}}, \mathcal{Z})$ such that $c^* = \overline{c}$. We define an unbounded operator a^*ca^* on $\Gamma_{\mathrm{s}}(\mathcal{Z})$ such that for $\Psi_n \in \Gamma_{\mathrm{s}}^n(\mathcal{Z})$

$$a^*ca^*\Psi_n := \sqrt{(n+2)(n+1)}\, c \otimes \Psi_n. \tag{4.34}$$

(Note that on the left of (4.34) we interpret c as an element of $l^2(\overline{\mathcal{Z}}, \mathcal{Z})$ and on the right as an element of $\mathcal{Z} \otimes_{\mathrm{s}} \mathcal{Z}$). The adjoint of a^*ca^* is denoted by ac^*a.

Theorem 4.1 *Assume that $c^*c < 1$.*
1) The formula

$$\Omega_c := \det(1-cc^*)^{\frac{1}{4}} \exp(\tfrac{1}{2}a^*ca^*)\Omega$$

defines a vector in $\Gamma_{\rm s}(\mathcal{Z})$. It is the unique vector satisfying

$$\|\Omega_c\| = 1, \quad (\Omega_c|\Omega) > 0, \qquad (\overline{z}(a) - c\overline{z}(a^*))\Omega_c = 0, \quad z \in \mathcal{Z}.$$

2) The formula

$$R_c := \det(1-cc^*)^{\frac{1}{4}} \exp(-\tfrac{1}{2}a^*ca^*)\Gamma(1-cc^*)^{\frac{1}{2}} \exp(\tfrac{1}{2}ac^*a). \tag{4.35}$$

defines a unitary operator on $\Gamma_{\rm s}(\mathcal{Z})$.
3) $\Omega = R_c\Omega_c$,
4)

$$R_cW(z)R_c^* = W\Big((1-cc^*)^{-\frac{1}{2}}z + (1-cc^*)^{-\frac{1}{2}}c\overline{z}\Big).$$

5) If h, u are operators on $\mathcal{Z}$ and $hc - c\overline{h}^* = 0$, $uc\overline{u}^{*-1} = c$, then

$$R_c{\rm d}\Gamma(h)R_c^* = {\rm d}\Gamma(h), \quad R_c\Gamma(u)R_c^* = \Gamma(u).$$

4.4 Araki-Woods representation

In this subsection we describe the Araki-Woods representation of canonical commutation relations and the corresponding W^*-algebras. These representations were introduced in [AW].

Consider a Hilbert space $\mathcal{Z}$. Consider the Fock space $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. For $(z_1,\overline{z}_2) \in \mathcal{Z}\oplus\overline{\mathcal{Z}}$, $W(z_1,\overline{z}_2)$ will denote the corresponding Weyl perator. Let $\tau : \mathcal{Z}\oplus\overline{\mathcal{Z}} \to \overline{\mathcal{Z}}\oplus\mathcal{Z}$ be the flip operator, that is

$$\mathcal{Z}\oplus\overline{\mathcal{Z}} \ni (z_1,\overline{z_2}) \mapsto \tau(z_1,\overline{z_2}) := (\overline{z_2},z_1) \in \overline{\mathcal{Z}}\oplus\mathcal{Z}. \tag{4.36}$$

Let ϵ be

$$\mathcal{Z}\oplus\overline{\mathcal{Z}} \ni (z_1,\overline{z_2}) \mapsto \epsilon(z_1,\overline{z_2}) := (z_2,\overline{z_1}) \in \mathcal{Z}\oplus\overline{\mathcal{Z}}. \tag{4.37}$$

Note that τ is linear, ϵ antilinear and

$$\epsilon(z_1,\overline{z_2}) = \overline{\tau(z_1,\overline{z_2})}. \tag{4.38}$$

Fix a positive operator ρ on $\mathcal{Z}$. Let $\mathcal{Q}(\rho) \subset \mathcal{Z}$ denote the form domain of ρ. For $z \in \mathcal{Q}(\rho)$, we define two unitary operators acting on $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ as follows:

$$W_{\rho,{\rm l}}(z) := W\big((1+\rho)^{\frac{1}{2}}z, \overline{\rho}^{\frac{1}{2}}\overline{z}\big),$$

$$W_{\rho,{\rm r}}(\overline{z}) := W\big(\rho^{\frac{1}{2}}z, (1+\overline{\rho})^{\frac{1}{2}}\overline{z}\big).$$

Let $\mathfrak{M}_{\rho,{\rm l}}$ and $\mathfrak{M}_{\rho,{\rm r}}$ denote the von Neumann algebras in $B(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})))$ generated by $W_{\rho,{\rm l}}(z)$ and $W_{\rho,{\rm r}}(\overline{z})$. They will be called the left and the right Araki-Woods algebras.

Theorem 4.2 *1a) $\mathcal{Z} \supset \mathcal{Q}(\rho) \ni z \mapsto W_{\rho,{\rm l}}(z) \in U(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ is a regular representation of CCR. This means in particular*

$$W_{\rho,{\rm l}}(z_1)W_{\rho,{\rm l}}(z_2) = {\rm e}^{-\frac{1}{2}{\rm Im}(z_1|z_2)}W_{\rho,{\rm l}}(z_1+z_2).$$

The corresponding field, creation and annihillation operators are affilliated to $\mathfrak{M}_{\rho,\mathrm{l}}$ and are equal

$$\begin{aligned}\phi_{\rho,\mathrm{l}}(z) &= \phi\Big((1+\rho)^{\frac{1}{2}}z, \overline{\rho}^{\frac{1}{2}}\overline{z}\Big),\\ z(a^*_{\rho,\mathrm{l}}) &= \Big((1+\rho)^{\frac{1}{2}}z, 0\Big)(a^*) + \Big(0, \rho^{\frac{1}{2}}z\Big)(a),\\ \overline{z}(a_{\rho,\mathrm{l}}) &= \Big((1+\overline{\rho})^{\frac{1}{2}}\overline{z}, 0\Big)(a) + \Big(0, \overline{\rho}^{\frac{1}{2}}\overline{z}\Big)(a^*).\end{aligned}$$

1b) $\overline{\mathcal{Z}} \supset \overline{\mathcal{Q}(\rho)} \in \overline{z} \mapsto W_{\rho,\mathrm{r}}(\overline{z}) \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ is a regular representation of CCR. This means in particular

$$W_{\rho,\mathrm{r}}(\overline{z_1})W_{\rho,\mathrm{r}}(\overline{z_2}) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Im}(z_1|z_2)}W_{\rho,\mathrm{r}}(\overline{z_1}+\overline{z_2}).$$

The corresponding field, creation and annihillation operators are affilliated to $\mathfrak{M}_{\rho,\mathrm{r}}$ and are equal

$$\begin{aligned}\phi_{\rho,\mathrm{r}}(\overline{z}) &= \phi\Big(\rho^{\frac{1}{2}}z, (1+\overline{\rho})^{\frac{1}{2}}\overline{z}\Big),\\ \overline{z}(a^*_{\rho,\mathrm{r}}) &= \Big(\overline{\rho}^{\frac{1}{2}}\overline{z}, 0\Big)(a) + \Big(0, (1+\overline{\rho})^{\frac{1}{2}}\overline{z}\Big)(a^*),\\ z(a_{\rho,\mathrm{r}}) &= \Big(\rho^{\frac{1}{2}}z, 0\Big)(a^*) + \Big(0, (1+\rho)^{\frac{1}{2}}z\Big)(a).\end{aligned}$$

2) We have the natural unitary identification

$$l^2(\Gamma_{\mathrm{s}}(\mathcal{Z})) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \overline{\Gamma_{\mathrm{s}}(\mathcal{Z})} \simeq \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma(\overline{\mathcal{Z}}) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}). \tag{4.39}$$

Let $U : l^2(\Gamma_{\mathrm{s}}(\mathcal{Z})) \to \Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ denote the identification given by (4.39) and set

$$\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}}) := U^* l^2_+(\Gamma_{\mathrm{s}}(\mathcal{Z})). \tag{4.40}$$

Then $(\mathfrak{M}_{\rho,\mathrm{l}}, \Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})), \Gamma(\epsilon), \Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ is a W^*-algebra in a standard form (see introduction and eg. [DJP]);
3) We have

$$\Gamma(\epsilon)W_{\rho,\mathrm{l}}(z)\Gamma(\epsilon) = W_{\rho,\mathrm{r}}(\overline{z}).$$

Consequently, $\mathfrak{M}_{\rho,\mathrm{l}} = \Gamma(\epsilon)\mathfrak{M}_{\rho,\mathrm{r}}\Gamma(\epsilon)$ and

$$\Gamma(\epsilon)\phi_{\rho,\mathrm{l}}(z)\Gamma(\epsilon) = \phi_{\rho,\mathrm{r}}(\overline{z}),$$

$$\Gamma(\epsilon)z(a^*_{\rho,\mathrm{l}})\Gamma(\epsilon) = \overline{z}(a^*_{\rho,\mathrm{r}}), \quad \Gamma(\epsilon)\overline{z}(a_{\rho,\mathrm{l}})\Gamma(\epsilon) = z(a_{\rho,\mathrm{r}}),$$

4) The vacuum expectation value of the Weyl operators (the "Schwinger function" or "generating function") is equal

$$\big(\Omega|W_{\rho,\mathrm{l}}(z)\Omega\big) = \exp\Big(-\tfrac{1}{4}(z|z) - \tfrac{1}{2}(z|\rho z)\Big).$$

5) Let h be a self-adjoint operator on $\mathcal{Z}$ commuting with ρ. Set

$$L = \mathrm{d}\Gamma(h\oplus(-\overline{h})).$$

Let τ^t acts on the Weyl operators as follows:

$$\tau^t(W_{\rho,\mathrm{l}}(z)) = W_{\rho,\mathrm{l}}(\mathrm{e}^{\mathrm{i}th}z).$$

Then τ^t extends to a W^*-dynamics on $\mathfrak{M}_{\rho,\mathrm{l}}$ and L is the standard Liouvillean of τ^t.
6) Ω is a (β,τ^t)-KMS vector iff h is related to ρ by

$$h = \beta^{-1}(\log(\rho+1) - \log\rho), \tag{4.41}$$

or equivalently by

$$\rho = (\mathrm{e}^{\beta h} - 1)^{-1}.$$

We have then

$$(\Omega|W_{\rho,\mathrm{l}}(z)\Omega) \quad = \exp\Big(-\frac{1}{4}\Big(z|\frac{\mathrm{e}^{\beta h}+1}{\mathrm{e}^{\beta h}-1}z\Big)\Big).$$

It is convenient to introduce the following self-adjoint operator on $\mathcal{Z} \oplus \overline{\mathcal{Z}}$:

$$r := h \oplus (-\overline{h}).$$

We have

$$\tau r \tau = -\overline{r}, \qquad \epsilon r \epsilon = -r, \qquad ULU^* = \mathrm{d}\Gamma(r).$$

If Ω is a (β, τ^t)-KMS vector, then

$$UW_{\rho,\mathrm{l}}(z)U^* = W(|1 - \mathrm{e}^{-\beta r}|^{\frac{1}{2}}(z, \overline{z})),$$

$$U\phi_{\rho,\mathrm{l}}(z)U^* = \phi(|1 - \mathrm{e}^{-\beta r}|^{\frac{1}{2}}(z, \overline{z})),$$

$$UW_{\rho,\mathrm{r}}(z)U^* = W(|1 - \mathrm{e}^{\beta r}|^{\frac{1}{2}}(z, \overline{z})),$$

$$U\phi_{\rho,\mathrm{r}}(z)U^* = W(|1 - \mathrm{e}^{\beta r}|^{\frac{1}{2}}(z, \overline{z})).$$

5 Abstract Pauli-Fierz operators

In this section, first we introduce the notation that we will use to describe the interaction of a second-quantized system with another system. Then we introduce the class of Pauli-Fierz operators. We also describe a number of results about these operators contained in the literature, notably in [DJ], which we will use later on.

In this section we look at Pauli-Fierz operators just as certain abstract self-adjoint operators. Only in the next two sections we will put them in the context of W^*-dynamical systems.

5.1 Creation/annihillation operators in coupled systems

Suppose that $\mathcal{W}$ is a Hilbert space. Consider a bosonic system described by the Fock space $\Gamma_{\mathrm{s}}(\mathcal{W})$ interacting with a "small quantum system" described by a Hilbert space $\mathcal{E}$. Their composite system will be described by the Hilbert space $\mathcal{E} \otimes \Gamma_{\mathrm{s}}(\mathcal{W})$. In this section we will describe a formalism which can be used to describe such coupled systems.

For $q \in B(\mathcal{K}, \mathcal{E} \otimes \mathcal{W})$ we define the creation operator $q(a^*)$ as the (unbounded) quadratic form on $\mathcal{E} \otimes \Gamma_{\mathrm{s}}(\mathcal{W})$ whose only nonzero matrix elements are between $\Psi_{n+1} \in \mathcal{E} \otimes \Gamma_{\mathrm{s}}^{n+1}(\mathcal{W})$ and $\Psi_n \in \mathcal{E} \otimes \Gamma_{\mathrm{s}}^{n}(\mathcal{W})$, for $n = 0, 1, 2, \ldots$, and are equal

$$(\Psi_{n+1}|q(a^*)\Psi_n) := \sqrt{n+1}(\Psi_{n+1}|\, q \otimes 1_{\mathcal{W}}^{\otimes n}\, \Psi_n)$$

The annihillation operator $q^*(a)$ is defined as the quadratic form on $\mathcal{E} \otimes \Gamma_{\mathrm{s}}(\mathcal{W})$ whose only nonzero matrix elements are between $\Psi_{n-1} \in \mathcal{E} \otimes \Gamma_{\mathrm{s}}^{n-1}(\mathcal{W})$ and $\Psi_n \in \mathcal{E} \otimes \Gamma_{\mathrm{s}}^{n}(\mathcal{W})$, for $n = 1, 2, \ldots$, and are equal

$$(\Psi_{n-1}|q^*(a)\Psi_n) := \sqrt{n}(\Psi_{n-1}|\, q^* \otimes 1_{\mathcal{W}}^{\otimes(n-1)}\, \Psi_n).$$

Proposition 5.1 *The operators $q(a^*)$ and $q^*(a)$ extend to closed operators that satisfy*

$$q(a^*)^* = q^*(a).$$

Remark 5.2 *Suppose that $B \in B(\mathcal{E})$, $w \in \mathcal{W}$ and $q = B \otimes |w)$, so that $q^* = B^* \otimes (w|$. Then*

$$q(a^*) = B \otimes w(a^*), \quad q^*(a) = B^* \otimes \overline{w}(a).$$

Let us note the following obvious estimates

Lemma 5.3 *For $q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ and $\Psi_n \in \Gamma_{\rm s}^n(\mathcal{W})$ we have*

$$\|q(a^*)\Psi_n\| \leq \sqrt{n+1}\|q\|\|\Psi\|, \quad \|q^*(a)\Psi_n\| \leq \sqrt{n}\|q\|\|\Psi\|. \tag{5.42}$$

For further reference note that if $\Psi_0 \in \mathcal{E} \simeq \mathcal{E} \otimes \Gamma_{\rm s}^0(\mathcal{W})$, then

$$q(a^*)\Psi_0 = q\Psi_0 \in \mathcal{E} \otimes \mathcal{W} \simeq \mathcal{E} \otimes \Gamma_{\rm s}^1\mathcal{W}), \tag{5.43}$$

and if $\Psi_1 \in \mathcal{E} \otimes \mathcal{W} \simeq \mathcal{E} \otimes \Gamma_{\rm s}^1(\mathcal{W})$, then

$$q^*(a)\Psi_1 = q^*\Psi_1 \in \mathcal{E} \simeq \mathcal{E} \otimes \Gamma_{\rm s}^0(\mathcal{W}). \tag{5.44}$$

5.2 Essential self-adjointness of Pauli-Fierz operators

In this subsection we introduce the class of operators which play the main role in our paper. They are describe the interaction of a small quantum system with free bosons through an interaction linear in the field. They often appear in the physics literature, especially as a simplified version of the non-relativistic QED. From the mathematical point of view they provide one of the simplest nontrivial classes of operators defined in the framework of the second quantization. In the literature they appear under a variety of names. We call this class Pauli-Fierz operators, as in [DG] and [DJ].

Suppose that $\mathcal{E}$, $\mathcal{W}$ are Hilbert spaces. Let $\mathcal{E}$ be finite dimensional. Let E be a self-adjoint operator on $\mathcal{E}$, $q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ and r a self-adjoint operator on $\mathcal{W}$. A self-adjoint operator on $\mathcal{E} \otimes \Gamma_{\rm s}(\mathcal{W})$ of the form

$$L_{\rm fr} := E \otimes 1 + 1 \otimes {\rm d}\Gamma(r)$$

will be called a free Pauli-Fierz operator,

$$Q := q(a^*) + q^*(a)$$

—a Pauli-Fierz interaction and

$$L := L_{\rm fr} + \lambda Q$$

—an interacting Pauli-Fierz operator (λ is a real parameter).

We know two sets of assumptions that guarantee the essential self-adjointness of Pauli-Fierz operators:

Theorem 5.4 *1) If $r \geq 0$ and $r^{-\frac{1}{2}}q$ is bounded, then L is self-adjoint on $\mathcal{D}(L_{\rm fr})$.*
2) If $|r|q$ is bounded, then L is essentially self-adjoint on $\mathcal{D}(L_{\rm fr}) \cap \mathcal{D}(Q)$.

Proof. The proofs of both 1) and 2) can be found in [DJ]. Note, however, that 1) was well known before, see eg. [BFS1]). □

5.3 Fermi Golden Rule Operator for Pauli-Fierz operators

First let us describe a certain condition that is very convenient if we want to study spectral properties of Pauli-Fierz operators. This condition is due to Jakšić-Pillet [JP1, JP2] and was used also in [DJ].

We assume that there exists a Hilbert space $\mathcal{G}$ and a unitary operator $U : \mathcal{W} \to L^2(\mathbb{R}) \otimes \mathcal{G}$ such that the operator UrU^* is the operator of multiplication by the variable in $\mathbb{R}$. We fix such an operator U and identify $\mathcal{W}$ with $L^2(\mathbb{R}) \otimes \mathcal{G}$. As in Subsection 3.4 introduce the self-adjoint operator s on $L^2(\mathbb{R}) \otimes \mathcal{G}$

$$s\Psi(p) := \frac{1}{\mathrm{i}} \nabla_p \Psi(p), \quad p \in \mathbb{R}. \tag{5.45}$$

Let $\mathcal{H}^{\mathrm{v}} := \mathcal{E} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{W})$ be the distinguished subspace of $\mathcal{H} := \mathcal{E} \otimes \Gamma_{\mathrm{s}}(\mathcal{W})$. Note that by using

$$\mathcal{E} \ni \Psi \mapsto \Psi \otimes \Omega \in \mathcal{H}^{\mathrm{v}},$$

we can identify $\mathcal{E}$ with $\mathcal{H}^{\mathrm{v}}$. Likewise, we can identify $L_{\mathrm{fr}}^{\mathrm{vv}}$ with the operator E on $\mathcal{E}$, (which justifies the notation introduced already in Subsection 3.3).

Proposition 5.5 *Suppose* $\langle s \rangle^\eta q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ *with* $\eta > \frac{1}{2}$. *Then the function*

$$\begin{aligned} w(z) &:= Q^{\mathrm{v}\overline{\mathrm{v}}}(z 1^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}} \\ &= q^*(z - E \otimes 1 - 1 \otimes r)^{-1} q, \end{aligned}$$

defined for $z \in \mathbb{C}_+$ *extends by continuity to* $\mathbb{C}_+^{\mathrm{cl}}$. *Moreover* $\mathbb{R} \ni p \mapsto q(p) \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{G})$, *defined as in Subsection 3.5, is a continuous function.*

From now on let us assume $\langle s \rangle^\eta q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ with $\eta > \frac{1}{2}$. Clearly, L_{fr} is a self-adjoint operator preserving $\mathcal{H}^{\mathrm{v}}$, $Q^{\mathrm{vv}} = 0$ and $Q^{\mathrm{v}\overline{\mathrm{v}}}$ is bounded. Moreover, the boundary values of $w(z)$ exist. Therefore, the FGRO Γ for the triple $(\mathcal{E} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{W}), L_{\mathrm{fr}}, Q)$ is well defined.

Let us introduce the notation

$$q^{e_1, e_2} := 1_{e_1}(E) \otimes 1_{\mathcal{W}}\, q\, 1_{e_2}(E).$$

Then the FGRO Γ equals

$$\Gamma = \sum_{e_1, e_2 \in \mathrm{sp}(E)} (q^*)^{e_1, e_2} (e_1 - e_2 + \mathrm{i}0 - r)^{-1} q^{e_2, e_1}. \tag{5.46}$$

5.4 Spectral theory of Pauli-Fierz operators

The following theorem is a consequence of the main results of [DJ].

Theorem 5.6 *Suppose that* $\eta > 2$, $\epsilon > 0$ *and* c *are fixed. Then there exists* $\lambda_0 > 0$ *such that for any* λ *and* $q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ *satisfying the following conditions:*
1) $0 < |\lambda| < \lambda_0$*;*
2) L *is essentially self-adjoint on* $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q)$*;*
3) $\|\langle s \rangle^\eta q\| \leq c$*;*
4) $\Gamma^{\mathrm{I}} < -\epsilon(1 - 1_0(\Gamma^{\mathrm{I}}))$*;*
we have $\mathrm{sp}_{\mathrm{sc}}(L) = \emptyset$ *and* $\dim 1_{\mathrm{p}}(L) \leq \dim 1_0(\Gamma^{\mathrm{I}})$.

Proof. Fix for the moment $q \in B(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ such that $\|\langle s \rangle^\eta q\| < \infty$ for $\eta > 2$. By Theorems 6.2, 6.3 and 6.4 of [DJ] there exists $\lambda_0 > 0$ such that, if $0 < |\lambda| \leq \lambda_0$ and if L is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q)$, then $\mathrm{sp}_{\mathrm{sc}}(L) = \emptyset$ and $\dim 1_{\mathrm{p}}(L) \leq \dim 1_{\mathbb{R}}(\Gamma)$. But by the general properties of dissipative operators (see Proposition 3.2 (i) of [DJ]), we have $1_{\mathbb{R}}(\Gamma) \leq 1_0(\Gamma^{\mathrm{I}})$. Hence $\dim 1_{\mathrm{p}}(L) \leq \dim 1_0(\Gamma^{\mathrm{I}})$.

If we check the proof of Theorems 6.2, 6.3 and 6.4, we see that the constant λ_0 is uniform if q is subject to the conditions $\|\langle s \rangle^\eta q\| \leq c$ and $\Gamma^{\mathrm{I}} < -\epsilon(1 - 1_0(\Gamma^{\mathrm{I}}))$. □

Remark 5.7 *If in the above theorem we replace 4) with*
4') $\Gamma^{\mathrm{I}} < -\epsilon$;
then we can weaken its assumption and demand only that $\eta > 1$. (Note that in this case $\dim 1_0(\Gamma^{\mathrm{I}}) = 0$, and thus we can conclude that L has no point spectrum).

Let us note that in [DJ] we actually prove much more than what we stated above. The following theorem, adapted from [DJ], expresses in precise terms the intuition about the operator $1_{\mathrm{sp}(\Gamma)\cap\mathbb{R}}(\Gamma)(E + \lambda^2\Gamma)$ mentioned in (3.9). In this paper, however, we will not use this result.

(Below, for $x_0 \in \mathbb{R}$ and $\epsilon > 0$, we use the notation $I(x_0, \epsilon) := [x_0 - \epsilon, x_0 + \epsilon]$).

Theorem 5.8 *Suppose that $\|\langle s\rangle^{\eta} q\| < \infty$ with $\eta > 2$ hold. Suppose that for sufficiently small λ, L is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q)$. Let $\kappa = 1 - \eta^{-1}$. Then there exists $\lambda_0 > 0$ and $\alpha > 0$ such that for $0 < |\lambda| < \lambda_0$, the following is true:*
1) If $e \in \mathrm{sp}(E)$, $m \in \mathrm{sp}(\Gamma^{ee}) \cap \mathbb{R}$, then

$$\dim 1^{\mathrm{p}}_{I(e+\lambda^2 m, \alpha\lambda^{2+\kappa})}(L) \leq \dim 1_m(\Gamma^{ee}).$$

2)

$$\mathrm{sp}_{\mathrm{p}}(L) \subset \bigcup_{e\in\mathrm{sp}(E)} \bigcup_{m\in\mathbb{R}\cap\mathrm{sp}(\Gamma^{ee})} I(e + \lambda^2 m, \alpha\lambda^{2+\kappa}).$$

6 Pauli-Fierz systems

The starting point of this section is a certain Pauli-Fierz operator, denoted H and called a Pauli-Fierz Hamiltonian. The basic feature of H is the positivity of its bosonic energy. In the context of our paper, this implies that H is bounded from below. Physical systems at zero temperature are described by Hamiltonians that are bounded from below. Therefore, we reserve the name *a Pauli-Fierz Hamiltonian* to Pauli-Fierz operators bounded from below.

We present some results concerning the ground state of H and the corresponding FGRO, denoted Γ. Some of these results will be needed later in our study of positive temperature Pauli-Fierz systems.

Subsections 6.5-6.8 are devoted to Pauli-Fierz W^*-dynamical systems at density ρ. Here ρ is a certain positive operator commuting with the 1-particle energy h. These systems are defined in a canonical way from the Pauli-Fierz Hamiltonian H and the density ρ. There are two natural representations of these systems. The representation that we call the semi-standard representation used to be more common in the literature. The standard representation acts on a larger space and looks more complicated. Nevertheless, it is the standard representation which is more useful in the study of Pauli-Fierz systems.

In both semi-standard and standard representation the dynamics can be implemented in by a strongly continuous unitary group. Its generators are called the semi-Liouvillean L^{semi}_{ρ} and the Liouvillean L_{ρ} respectively. Both are examples of Pauli-Fierz operators.

Our main goal is the study of spectral properties of the Liouvilleans. We will describe some of these results that follow from the literature and the previous section. In particular, we will compute the FGRO for L_{ρ}, denoted by Γ_{ρ}.

6.1 Pauli-Fierz Hamiltonians

Throughout this section we assume that K is a self-adjoint operator on a finite dimensional Hilbert space $\mathcal{K}$, h is a positive operator on a Hilbert space $\mathcal{Z}$ and $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$. The self-adjoint operator on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h)$$

will be called a free Pauli-Fierz Hamiltonian and

$$H := H_{\rm fr} + \lambda V,$$

where $\lambda \in \mathbb{R}$ and

$$V := v(a^*) + v^*(a),$$

will be called an interacting Pauli-Fierz Hamiltonian.

Let us list basic assumptions.

Assumption 6.A $h^{-\frac{1}{2}} v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$.

Sometimes we will need a stronger assumption.

Assumption 6.B $h^{-1} v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$.

Theorem 6.1 *1) Suppose Assumption 6.A holds. Then the operator H is self-adjoint on $\mathcal{D}(H_{\rm fr})$ and bounded from below.*
2) Suppose Assumption 6.B is true. Then H has a ground state, that means

$$\dim 1_{\inf {\rm sp}(H)} \geq 1.$$

Proof. In 1) we just repeat the statement of Theorem 5.4 1).

2) was proven in [Ge] (see also [AH] and [BFS1]). □

6.2 Gluing of reservoir 1-particle spaces

Recall that the reservoir 1-particle space is denoted by $\mathcal{Z}$. It is useful to consider the Hilbert space $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ with the self-adjoint operator $r := h \oplus (-\overline{h})$,

The most important assumption that we need is the Jakšć-Pillet gluing condition.

Assumption 6.C *There exists a Hilbert space $\mathcal{G}$ and unitary operator $U : \mathcal{Z} \oplus \overline{\mathcal{Z}} \to L^2(\mathbb{R}) \otimes \mathcal{G}$ such that $U^* r U$ is the operator of multiplication by the variable in $\mathbb{R}$.*

We fix the operator U as in the above assumption, and we introduce the self-adjoint operator s as in (5.45) In what follows we will always suppose Assumption 6.C is true.

Let us note that if $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ is identified with $L^2(\mathbb{R}) \otimes \mathcal{G}$, then $\mathcal{Z}$ is identified with $L^2(\mathbb{R}_+) \otimes \mathcal{G}$, since $h = 1_{[0,\infty[}(r) r$. Likewise, $\overline{\mathcal{Z}}$ is identified with $L^2(\mathbb{R}_-) \otimes \mathcal{G}$. Note also that

$$\overline{\Psi(p)} := (\epsilon \Psi)(-p) \quad \Psi \in L^2(\mathbb{R}, \mathcal{G}). \tag{6.47}$$

defines a internal conjugation in $\mathcal{G}$.

In the expression $(v, 0)$ below we interpret 0 as an operator from $\mathcal{K}$ to $\mathcal{K} \otimes \overline{\mathcal{Z}}$. Thus the operator $(v, 0)$ has the meaning of an operator from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{Z} \oplus \mathcal{K} \otimes \overline{\mathcal{Z}} \simeq \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathcal{K} \otimes L^2(\mathbb{R}) \otimes \mathcal{G}$. It can be written as a function defined for almost all $p \in \mathbb{R}$ with values in $B(\mathcal{K}, \mathcal{K} \otimes \mathcal{G})$:

$$(v, 0)(p) = \begin{cases} v(p), & p > 0 \\ 0, & p \leq 0. \end{cases} \tag{6.48}$$

In the following assumption we have $\eta \geq 0$.

Assumption 6.D$(\eta)_0$ $\langle s \rangle^\eta (v, 0) \in B(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}))$.

6.3 Fermi Golden Rule Operator for Pauli-Fierz Hamiltonians

In this subsection we will calculate the Fermi Golden Rule Operator for the triple $(\mathcal{K}\otimes\Gamma^0_{\rm s}(\mathcal{Z}), H_{\rm fr}, V)$.

First let us fix some notation. If $\mathcal{K}^k = {\rm Ran} 1_{\{k\}}(K)$, then we have a direct sum decomposition

$$\mathcal{K} = \mathop{\oplus}_{k\in{\rm sp}(K)} \mathcal{K}^k. \tag{6.49}$$

We will write

$$B^{k_1,k_2} := 1_{k_1}(K)B1_{k_2}(K), \quad k_1,k_2\in{\rm sp}(K), \quad B\in B(\mathcal{K}),$$

and also

$$v^{k_1,k_2} := 1_{k_1}(K)\otimes 1_{\mathcal{Z}}\, v\, 1_{k_2}(K).$$

If $k\in{\rm sp}(K)$, we define the set of allowed transition energies

$$\mathcal{F}_k := \{k_1 - k \ : \ k_1\in{\rm sp}(K)\}, \qquad \mathcal{F} := \bigcup_{k\in{\rm sp}(K)} \mathcal{F}_k,$$

and the set of allowed positive transition energies

$$\mathcal{F}^+_k := \mathcal{F}_k\cap[0,\infty[, \qquad \mathcal{F}^+ := \bigcup_{k\in{\rm sp}(K)} \mathcal{F}^+_k.$$

Let k_0 denote the ground state energy of K, that is

$$k_0 := \inf{\rm sp}(K).$$

Let $\mathcal{H}^{\rm v} := \mathcal{K}\otimes\Gamma^0_{\rm s}(\mathcal{Z})$.

Proposition 6.2 *Suppose Assumption 6.D$(\eta)_0$ with $\eta>\frac{1}{2}$ is true. Then the function*

$$\begin{aligned} w(z) \ &:= V^{{\rm v}\overline{\rm v}}(z1^{\overline{\rm v}\overline{\rm v}} - H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})^{-1}V^{\overline{\rm v}{\rm v}} \\ &= v^*(z - K\otimes 1 - 1\otimes h)v, \end{aligned}$$

defined for $z\in\mathbb{C}_+$ extends by continuity to $\mathbb{C}^{\rm cl}_+$. Moreover, the function (6.48) is continuous in $p\in\mathbb{R}$.

Proof. We apply the trick of "gluing non-physical free bosons" [DJ]. Consider the extended 1-boson space $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and define the operators $r = h\oplus(-\overline{h})$ and $q = (v,0)$. Note that for $z\in\mathbb{C}_+$,

$$v^*(z-h)^{-1}v = q^*(z-r)^{-1}q.$$

Now the proposition follows by the same arguments as Proposition 5.5.

Exactly as in (5.46), we can compute the FGRO Γ for the triple $(\mathcal{K}\otimes\Gamma^0_{\rm s}(\mathcal{Z}), H_{\rm fr}, V)$, which is equal

$$\Gamma = \sum_{k\in{\rm sp}(K)} \Gamma^{kk},$$

$$\Gamma^{kk} = \sum_{p\in\mathcal{F}_k} (v^*)^{k,k-p}(p+{\rm i}0-h)^{-1}v^{k-p,k}.$$

Moreover, we have

$$\begin{aligned}(\Gamma^{\mathrm{R}})^{kk} &= \sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}\mathcal{P}(p-h)^{-1}v^{k-p,k}\\ &= \sum_{p\in\mathcal{F}_k}\int(v^*)^{k,k-p}(p_1)\mathcal{P}(p-p_1)^{-1}v^{k-p,k}(p_1)\mathrm{d}p_1,\\ (\Gamma^{\mathrm{I}})^{kk} &= -\pi\sum_{p\in\mathcal{F}_k^+}(v^*)^{k,k-p}\delta(p-h)v^{k-p,k}\\ &= -\pi\sum_{p\in\mathcal{F}_k^+}(v^*)^{k,k-p}(p)v^{k-p,k}(p).\end{aligned} \tag{6.50}$$

Above we wrote the formula for Γ^{I} and Γ^{R} in two equivalent forms. In the first form we use the self-adjoint operator h on $\mathcal{Z}$ and the real number $p\in\mathbb{R}$. Strictly speaking, neither the proncipal value $\mathcal{P}(p-h)$ nor the deltafunction $\delta(p-h)$ are well defined as self-adjoint operators. But within the context of (6.50), these formulas are well defined by the integral expressions using the representation of v into a direct integral with the fibers $v(p_1)$.

Let us note that ground states of K belong to the kernel of Γ^{I}:

Proposition 6.3

$$(\Gamma^{\mathrm{I}})^{k_0k_0}=0.$$

In particular, $\Gamma^{k_0k_0}$ *is self-adjoint and*

$$\mathcal{K}^{k_0}\subset\mathrm{Ker}(\Gamma^{\mathrm{I}}). \tag{6.51}$$

Proof. Note that $q(p)=0$ for $p\leq 0$. Therefore, all the terms with $p\leq 0$ drop out from the formulas for Γ^{I}. □

It is easy to see that for a generic interaction v, the kernel of Γ^{I} should coincide with the subspace of ground states of K. Let us formulate this condition as the first generic assumption that we will use in our paper.

Assumption 6.E $\dim\mathrm{Ker}\Gamma^{\mathrm{I}}=\mathcal{K}^{k_0}$.

Our second generic assumption says that the ground state of K is nondegenerate:

Assumption 6.F $\mathcal{K}^{k_0}=1$.

Obviously, we have:

Theorem 6.4 *Suppose that Assumption 6.D*$(\eta)_0$ *with* $\eta>\frac{1}{2}$*, 6.E and 6.F hold. Then* $\dim\mathrm{Ker}\Gamma^{\mathrm{I}}=1$.

6.4 Nondegeneracy of ground states of Pauli-Fierz Hamiltonians

In this subsection we formulate the main result of this section concerning Pauli-Fierz Hamiltonians. It will say that if the interaction v is sufficiently regular and the generic assumptions 6.E and 6.F hold, then the Pauli-Fierz Hamiltonian H for small nonzero coupling constant has a unique ground state.

Before we do this, we make an observation that the appropriate regularity of the gluing implies the assumptions of Theorem 6.1 and hence implies the self-adjointness and the existence of a ground state for H.

Theorem 6.5 *1) Suppose Assumption 6.D$(\eta)_0$ with $\eta > \frac{1}{2}$ holds. Then Assumption 6.A holds.*
2) Suppose Assumption 6.D$(\eta)_0$ with $\eta > \frac{3}{2}$ holds. Then Assumption 6.B is true.

Proof. We use the trick described in the proof of Proposition 6.2. In particular, we use the operators q and r introduced in the proof of this proposition.

To see 1), we note that by Assumption 6.D$(\eta)_0$ with $\eta > \frac{1}{2}$, and Theorem 3.1, for any $p \in \mathbb{R}$, the operator $q^*(p + \mathrm{i}0 - r)^{-1}q$ exists and is bounded. Setting $p = 0$ gives

$$q^*(\mathrm{i}0 - r)^{-1}q = v^*h^{-1}v.$$

Hence, $h^{-\frac{1}{2}}v$ is bounded.

Similarly, to see 2), we note that by Assumption 6.D$(\eta)_0$ with $\eta > \frac{3}{2}$ the operator $q^*(p + \mathrm{i}0 - r)^{-2}q$ exists and is bounded. Clearly,

$$q^*(\mathrm{i}0 - r)^{-2}q = v^*h^{-2}v.$$

Hence $h^{-1}v$ is bounded. □

Now we deduce spectral information on H.

Theorem 6.6 *Suppose Assumptions 6.D$(\eta)_0$ with $\eta > 2$, 6.E and 6.F hold. Then there exists $\lambda_0 > 0$ and such that for $0 < |\lambda| < \lambda_0$, the following is true:*
1) $\dim 1^{\mathrm{p}}(H) = 1$
2) $\mathrm{sp}_{\mathrm{sc}}(H) = \emptyset$.
3) $\mathrm{sp}_{\mathrm{p}}(H) = \inf \mathrm{sp}(H)$.

Proof. We extend the space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ to the space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\overline{\mathcal{Z}}) \simeq \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. Note that $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ can be identified with $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma^0_{\mathrm{s}}(\overline{\mathcal{Z}})$, which is a subspace of $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma^0_{\mathrm{s}}(\overline{\mathcal{Z}})$.

We consider the extended operators

$$\begin{aligned} L_{\mathrm{fr}} &:= H_{\mathrm{fr}} \otimes 1 - 1 \otimes \mathrm{d}\Gamma(h) &&\simeq K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(r) \\ Q &:= V \otimes 1 &&\simeq q(a^*) + q^*(a). \end{aligned}$$

We set

$$L := L_{\mathrm{fr}} + \lambda Q \simeq H \otimes 1 - 1 \otimes \mathrm{d}\Gamma(h).$$

By Theorem 6.5, H is self-adjoint on $\mathcal{D}(H_{\mathrm{fr}})$. Therefore, L is self-adjoint on $\mathcal{D}(L_{\mathrm{fr}})$. Note also that

$$\mathrm{sp}_{\mathrm{p}}(H) = \mathrm{sp}_{\mathrm{p}}(L), \quad \mathrm{sp}_{\mathrm{sc}}(H) = \mathrm{sp}_{\mathrm{sc}}(L). \tag{6.52}$$

Clearly, L is a Pauli-Fierz operator such that $\|\langle s\rangle^\eta q\| < \infty$ for $\eta > 2$. The FGRO for the triple $(\mathcal{K} \otimes \Gamma^0_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), L_{\mathrm{fr}}, Q)$ is equal (after the obvious identification of Hilbert spaces) to the FGRO for the triple $(\mathcal{K} \otimes \Gamma^0_{\mathrm{s}}(\mathcal{Z}), H_{\mathrm{fr}}, V)$, which we studied in the last subsection. By Theorem 6.4, we know that $\dim \mathrm{Ker}\Gamma^{\mathrm{I}} = 1$. Therefore, Theorem 5.6 implies that there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$ we have

$$\dim 1^{\mathrm{p}}(L) \leq 1, \quad \mathrm{sp}_{\mathrm{sc}}(L) = \emptyset.$$

By (6.52), this implies

$$\dim 1^{\mathrm{p}}(H) \leq 1, \quad \mathrm{sp}_{\mathrm{sc}}(H) = \emptyset. \tag{6.53}$$

By Theorem 6.4 2) we have $\dim 1_{\inf \mathrm{sp}(H)}(H) \geq 1$. Combining this with (6.53) we get $\dim 1^{\mathrm{p}}(H) = 1$ and $\mathrm{sp}_{\mathrm{p}}(H) = \inf \mathrm{sp}(H)$. □

6.5 Semi-standard representations of Pauli-Fierz systems

In this subsection we introduce the notion of a Pauli-Fierz W^*-dynamical system. This will be the main subject of the remaining part of this section.

Let ρ be a positive operator on $\mathcal{Z}$. The composite system consisting of $B(\mathcal{K})$ and of a reservoir with density ρ is described by the W^*-algebra

$$B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}, \tag{6.54}$$

where $\mathfrak{M}_{\rho,\mathrm{l}} \subset B(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ is the Araki-Woods W^*-algebra introduced in Subsection 4.4.

This algebra (6.54) acts in the obvious way on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \otimes \overline{\mathcal{Z}})$, in fact we have

$$B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}} \subset B(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

In this subsection we will use this representation, which we will call the semi-standard representation. In the next subsection the same algebra will be considered in the standard representation.

Proposition 6.7 *Suppose that* $(1+\rho)^{\frac{1}{2}}v \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ *and* $\overline{\rho}^{\frac{1}{2}}v^\star \in B(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{Z}})$. *Then following operators on* $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$*:*

$$\begin{aligned} &\Big((1+\rho)^{\frac{1}{2}}v, 0\Big)(a^*) + \Big(0, v^{\star *}\overline{\rho}^{\frac{1}{2}}\Big)(a), \\ &\Big(v^*(1+\rho)^{\frac{1}{2}}, 0\Big)(a) + \Big(0, \overline{\rho}^{\frac{1}{2}}v^\star\Big)(a^*) \end{aligned} \tag{6.55}$$

are affilliated to $B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}$.

Set

$$q_\rho := ((1+\rho)^{\frac{1}{2}}v, \overline{\rho}^{\frac{1}{2}}v^\star) \in B(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

We can also write q_ρ in terms of a direct integral of operators in $B(\mathcal{K}, \mathcal{K} \otimes \mathcal{G})$:

$$q_\rho(p) \quad = \begin{cases} (1+\rho)^{\frac{1}{2}}v(p), & p > 0, \\ \overline{\rho}^{\frac{1}{2}}v^\star(-p), & p < 0; \end{cases}$$

Let Q_ρ^{semi} be the sum of (6.55), that is

$$\begin{aligned} Q_\rho^{\mathrm{semi}} \quad &:= \Big((1+\rho)^{\frac{1}{2}}v, \overline{\rho}^{\frac{1}{2}}v^\star\Big)(a^*) + \Big(v^*(1+\rho)^{\frac{1}{2}}, v^{\star *}\overline{\rho}^{\frac{1}{2}}\Big)(a) \\ &= q_\rho(a^*) + q_\rho^*(a). \end{aligned}$$

The free Pauli-Fierz semi-Liouvillean is the self-adjoint operator on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ defined as

$$\begin{aligned} L_{\mathrm{fr}}^{\mathrm{semi}} \quad &:= K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h \oplus -\overline{h}) \\ &= K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(r). \end{aligned}$$

The full Pauli-Fierz semi-Liouvillean of density ρ is

$$L_\rho^{\mathrm{semi}} := L_{\mathrm{fr}}^{\mathrm{semi}} + Q_\rho^{\mathrm{semi}}. \tag{6.56}$$

Assumption 6.G $_\rho$

$$(1+|r|)q_\rho \in B(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}})). \tag{6.57}$$

Theorem 6.8 *1)*

$$\tau_{\rm fr}^t(A) := {\rm e}^{{\rm i}tL_{\rm fr}} A {\rm e}^{-{\rm i}tL_{\rm fr}}, \qquad A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}$$

defines a W^*-dynamics on $B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}$.
2) Suppose Assumption 6.G$_\rho$ holds. Then $L_\rho^{\rm semi}$ is essentially self-adjoint on $\mathcal{D}(L_{\rm fr}^{\rm semi}) \cap \mathcal{D}(Q_\rho^{\rm semi})$ and

$$\tau_\rho^t(A) := {\rm e}^{{\rm i}tL_\rho^{\rm semi}} A {\rm e}^{-{\rm i}tL_\rho^{\rm semi}}, \qquad A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}},$$

is a W^*-dynamics on $B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}$.

Proof. 1) is obvious. To prove 2) we note that by Assumption 6.G$_\rho$ and [DJP], we obtain the essential self-adjointness of $L_\rho^{\rm semi}$. Now [DJP] implies that τ_ρ^t is a W^*-dynamics. □

Definition 6.9 *The pair $\big(B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}, \tau_\rho^t\big)$ will be called the Pauli-Fierz W^*-dynamical system at density ρ associated to the Hamiltonian H.*

6.6 Standard representation of Pauli-Fierz systems

Consider the representation

$$\pi : B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}} \to B(\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

defined by

$$\pi(A) := 1_{\overline{\mathcal{K}}} \check{\otimes} A, \quad A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}},$$

where $\check{\otimes}$ was introduced in 3.22. Clearly,

$$\pi(B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}) = B(\mathcal{K}) \otimes 1_{\overline{\mathcal{K}}} \otimes \mathfrak{M}_{\rho,{\rm l}}.$$

Set $J := J_{\mathcal{K}} \otimes \Gamma(\epsilon)$, where

$$J_{\mathcal{K}} \Psi_1 \otimes \overline{\Psi_2} := \Psi_2 \otimes \overline{\Psi_1}, \quad \Psi_1, \Psi_2 \in \mathcal{K}, \tag{6.58}$$

and ϵ was introduced in 4.37. Note that

$$J\, B(\mathcal{K}) \otimes 1_{\overline{\mathcal{K}}} \otimes \mathfrak{M}_{\rho,{\rm l}}\, J = 1_{\mathcal{K}} \otimes B(\overline{\mathcal{K}}) \otimes \mathfrak{M}_{\rho,{\rm r}},$$

and if $A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}$, then

$$J\pi(A)J = 1_{\mathcal{K}} \otimes \Big(1_{\overline{\mathcal{K}}} \otimes \Gamma(\tau)\ \overline{A}\ 1_{\overline{\mathcal{K}}} \otimes \Gamma(\tau)\Big),$$

where τ was introduced in 4.36.

Theorem 6.10

$$\Big(\pi,\ \mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}}),\ J,\ (\mathcal{K} \otimes \overline{\mathcal{K}})_+ \otimes \Gamma_{{\rm s},+}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\Big)$$

is a standard representation of $B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}$.

Set

$$\begin{aligned} Q_\rho &:= 1_{\overline{\mathcal{K}}} \check{\otimes} Q_\rho^{\rm semi} \\ &= 1_{\overline{\mathcal{K}}} \check{\otimes} \Big((1+\rho)^{\frac{1}{2}} v, \overline{\rho}^{\frac{1}{2}} v^\star\Big)(a^*) + 1_{\overline{\mathcal{K}}} \check{\otimes} \Big(v^*(1+\rho)^{\frac{1}{2}}, v^{\star *} \overline{\rho}^{\frac{1}{2}}\Big)(a) \\ &= \Big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho\Big)(a^*) + \Big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho^*\Big)(a). \end{aligned}$$

Then

$$\begin{aligned} JQ_\rho J &:= 1_{\mathcal{K}} \otimes \Big(1_{\overline{\mathcal{K}}} \otimes \Gamma(\tau)\ \overline{Q_\rho^{\mathrm{semi}}}\ 1_{\overline{\mathcal{K}}} \otimes \Gamma(\tau)\Big) \\ &= 1_{\mathcal{K}} \otimes \Big(\rho^{\frac{1}{2}} \overline{v}^\star, (1+\overline{\rho})^{\frac{1}{2}} \overline{v}\Big)(a^*) + 1_{\mathcal{K}} \otimes \Big(\overline{v}^{\star *} \rho^{\frac{1}{2}}, \overline{v}^* (1+\overline{\rho})^{\frac{1}{2}}\Big)(a) \\ &= \Big(1_{\mathcal{K}} \otimes \overline{\tau q_\rho}\Big)(a^*) + \Big(1_{\mathcal{K}} \otimes \overline{q_\rho^* \tau}\Big)(a). \end{aligned}$$

Clearly Q_ρ is affilliated to $B(\mathcal{K}) \otimes 1_{\overline{\mathcal{K}}} \otimes \mathfrak{M}_{\rho,\mathrm{l}}$ and $JQ_\rho J$ is affilliated to $1_{\mathcal{K}} \otimes B(\overline{\mathcal{K}}) \otimes \mathfrak{M}_{\rho,\mathrm{r}}$.

Set

$$L_{\mathrm{fr}} := K \otimes 1 \otimes 1 - 1 \otimes \overline{K} \otimes 1 + 1 \otimes 1 \otimes \mathrm{d}\Gamma(r),$$

and

$$L_\rho := L_{\mathrm{fr}} + \lambda Q_\rho - \lambda J Q_\rho J, \tag{6.59}$$

Theorem 6.11 *Suppose Assumption 6.G_ρ is true. Then L_ρ is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q_\rho) \cap \mathcal{D}(JQ_\rho J)$. Moreover, L_{fr} is the standard Liouvillean for τ_{fr}^t and L_ρ is the standard Liouvillean for τ_ρ^t. In particular,*

$$\pi(\tau_\rho^t(A)) = \mathrm{e}^{\mathrm{i}tL_\rho} \pi(A) \mathrm{e}^{-\mathrm{i}tL_\rho}, \qquad A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}. \tag{6.60}$$

Proof. (6.57) implies that $(1+|r|)\tau\overline{q}_\rho \in B(\overline{\mathcal{K}}, \overline{\mathcal{K}} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}))$. Hence

$$(1+|r|)(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho - 1_{\mathcal{K}} \otimes \tau\overline{q}_\rho) \in B(\mathcal{K} \otimes \overline{\mathcal{K}}, \mathcal{K} \otimes \overline{\mathcal{K}} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

Hence the essential self-adjointness of L_ρ follows from Theorem 5.4 2).

Clearly,

$$\pi(\tau_\rho^t(A)) = \mathrm{e}^{\mathrm{i}t(L_{\mathrm{fr}} + \lambda Q_\rho)} \pi(A) \mathrm{e}^{-\mathrm{i}t(L_{\mathrm{fr}} + \lambda Q_\rho)}, \qquad A \in B(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}.$$

Now the essential self-adjointness of L_ρ and [DJP] imply that L_ρ is the Liouvillean of τ_ρ^t. □

6.7 Fermi Golden Rule Operator for Pauli-Fierz Liouvilleans

The main objective of this subsection is to compute and study the FGRO for Pauli-Fierz Liouvilleans. As we will see, they enjoy some special algebraic properties.

Let us formulate the following assumption:

Assumption 6.D$(\eta)_\rho$ $\langle s \rangle^\eta q_\rho \in B(\mathcal{E}, \mathcal{E} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}))$.

Let us remark in parenthesis that Assumption 6.D$(\eta)_0$ introduced in Subsection 6.2 is a special case of the Assumption 6.D$(\eta)_\rho$ for $\rho = 0$.

In this subsection we suppose that Assumption 6.D$(\eta)_\rho$ with $\eta > \frac{1}{2}$. Then we can apply the formalism of FGRO's to the triple $\big(\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{Z} \oplus \overline{\mathcal{Z}}), L_{\mathrm{fr}}, Q_\rho - JQ_\rho J\big)$. The FGRO obtained this way will be denoted by Γ_ρ and will be used to study the operator L_ρ.

Define the following self-adjoint operators on $\mathcal{K}$

$$\begin{aligned}\Delta_\rho^{\mathrm{R}} &:= \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(q_\rho^*)^{k,k-p}\mathcal{P}(p-r)^{-1}q_\rho^{k-p,k}\\ &= \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}(1+\rho)\mathcal{P}(p-h)^{-1}v^{k-p,k}\\ &- \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^{\star*})^{k,k+p}\overline{\rho}\mathcal{P}(p-\overline{h})^{-1}(v^\star)^{k+p,k},\end{aligned}$$

$$\begin{aligned}\Delta_\rho^{\mathrm{I}}: &= -\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(q_\rho^*)^{k,k-p}\delta(p-r)q_\rho^{k-p,k}\\ &= -\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k^+}(v^*)^{k,k-p}(1+\rho)\delta(p-h)v^{k-p,k}\\ &-\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k^+}(v^{\star*})^{k,k+p}\overline{\rho}\delta(p-\overline{h})(v^\star)^{k+p,k}\\ &-\pi\sum_{k\in\mathrm{sp}(K)}(v^*)^{k,k}(1+\rho)\delta(h)v^{k,k}.\end{aligned}$$

Set

$$\begin{aligned}\Delta_\rho := \Delta_\rho^{\mathrm{R}} + \mathrm{i}\Delta_\rho^{\mathrm{I}} &= \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(q_\rho^*)^{k,k-p}(p+\mathrm{i}0-r)^{-1}q_\rho^{k-p,k}\\ &= \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}(1+\rho)(p+\mathrm{i}0-h)^{-1}v^{k-p,k}\\ &- \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^{\star*})^{k,k+p}\overline{\rho}(p-\mathrm{i}0-\overline{h})^{-1}(v^\star)^{k+p,k}.\end{aligned}$$

For $B\in l^2(\mathcal{K})$ set also

$$\begin{aligned}\Xi_\rho(B) &:= 2\pi\sum_{k_1,k_2\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_{k_1}\cap\mathcal{F}_{k_2}}(q_\rho^*)^{k_1,k_1-p}\Big(B\otimes\delta(p-r)\Big)(\tau q_\rho^\star)^{k_2-p,k_2}\\ &= 2\pi\sum_{k_1,k_2\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_{k_1}^+\cap\mathcal{F}_{k_2}^+}(v^*)^{k_1,k_1-p}\Big(B\otimes\delta(p-h)(1+\rho)^{\frac12}\rho^{\frac12}\Big)v^{k_2-p,k_2}\\ &+2\pi\sum_{k_1,k_2\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_{k_1}^+\cap\mathcal{F}_{k_2}^+}(v^{\star*})^{k_1,k_1+p}\Big(B\otimes\delta(p-\overline{h})(1+\overline{\rho})^{\frac12}\overline{\rho}^{\frac12}\Big)(v^\star)^{k_2+p,k_2}\\ &+2\pi\sum_{k_1,k_2\in\mathrm{sp}(K)}(v^*)^{k_1,k_1}\Big(B\otimes\delta(h)(1+\rho)^{\frac12}\rho^{\frac12}\Big)v^{k_2,k_2}.\end{aligned}$$

All the formulas for Δ_ρ^{R}, Δ_ρ^{I}, Δ_ρ and Ξ_ρ are written in two equivalent forms. The first forms involve the operators q_ρ. They are more compact than the second forms

The second forms involve the operators v and ρ. They are more directly related to the basic physical quantities of the system. They are however less convenient. Note in particular, that in the formulas for Δ_ρ^{R} and Δ_ρ, the terms with $p=0$ need to be carefully interpreted. (The singularity of $\mathcal{P}(-r)^{-1}$ and $(\mathrm{i}0-r)^{-1}$ is "cut" into two parts in these expressions. This problem is absent in the formulas involving q_ρ).

In the formulas for Δ_ρ^{I} and Ξ_ρ, we singled out the terms with $p=0$ ("the infrared terms"). These terms are due to that part of interacion where the Jakšić-Pillet gluing occurs. They disappear if the infrared behavior of the system is sufficiently mild.

In the expression for Ξ_ρ we use $\tau q_\rho^\star\in B\big(\mathcal{K},\mathcal{K}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big)$. Let us note the following identities concerning this operator:

$$\tau q_\rho^\star = \big(\rho^{\frac12}v,(1+\overline{\rho})^{\frac12}v^\star\big) = \big(\rho^{\frac12}(1+\rho)^{-\frac12},\rho^{-\frac12}(1+\rho)^{\frac12}\big)q_\rho. \tag{6.61}$$

We can write $\tau q_\rho^\star$ fiberwise:

$$\tau q_\rho^\star(p) = \begin{cases} \rho^{\frac{1}{2}} v(p), & p \geq 0, \\ (1+\overline{\rho})^{\frac{1}{2}} v^\star(-p), & p \leq 0; \end{cases}$$

Theorem 6.12 *If $B \in l^2(\mathcal{K})$, we have*

$$\Gamma_\rho(B) = \Delta_\rho B - B\Delta_\rho^* + \mathrm{i}\Xi_\rho(B).$$

The real and the imaginary part of $\Gamma_\rho \in B(l^2(\mathcal{K}))$ are given by

$$\begin{aligned} \Gamma_\rho^{\mathrm{R}}(B) &= \Delta_\rho^{\mathrm{R}} B - B\Delta_\rho^{\mathrm{R}}, \\ \Gamma_\rho^{\mathrm{I}}(B) &= \Delta_\rho^{\mathrm{I}} B + B\Delta_\rho^{\mathrm{I}} + \Xi_\rho(B). \end{aligned}$$

Proof. Using (5.46) we see that

$$\begin{aligned} \Gamma_\rho &= \sum_{e_1,e_2 \in \mathrm{sp}(K\otimes 1 - 1\otimes\overline{K})} \\ &\quad \big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho^* - 1_{\mathcal{K}} \otimes \overline{q_\rho^* \tau}\big)^{e_1,e_2} (e_1 - e_2 + \mathrm{i}0 - r)^{-1} \big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho - 1_{\mathcal{K}} \otimes \overline{\tau q_\rho}\big)^{e_2,e_1}, \end{aligned} \tag{6.62}$$

where the superscripts e_1, e_2 correspond to the decomposition of $\mathcal{K}\otimes\overline{\mathcal{K}}$ into the eigenspaces of $K\otimes 1 - 1\otimes\overline{K}$. Next note that if we now use the superscripts in $\mathrm{sp}(K)$ and use the decomposition of $\mathcal{K}$ into spectral subspaces of K, as described in (6.49), then (6.62) can be rewritten as

$$\begin{aligned} \Gamma_\rho &= \sum_{k_1,k_2 \in \mathrm{sp}(K)} \sum_{p \in \mathcal{F}_{k_1} \cap \mathcal{F}_{k_2}} \\ &\quad \big(1_{\overline{\mathcal{K}}} \check{\otimes} (q_\rho^*)^{k_1,k_1-p} - 1_{\mathcal{K}} \otimes \overline{q_\rho^* \tau}^{k_1,k_1+p}\big)(p + \mathrm{i}0 - r)^{-1} \big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho^{k_2-p,k_2} - 1_{\mathcal{K}} \otimes \overline{\tau q_\rho}^{k_2+p,k_2}\big). \end{aligned} \tag{6.63}$$

Now let $B \in l^2(\mathcal{K})$. We see that $\Gamma_\rho(B)$ consists of 4 types of terms:
Type I Using (3.28), we obtain

$$\begin{aligned} &\big(1_{\overline{\mathcal{K}}} \check{\otimes} (q_\rho^*)^{k,k-p}\big)(p + \mathrm{i}0 - r)^{-1} \big(1_{\overline{\mathcal{K}}} \check{\otimes} q_\rho^{k-p,k}\big) B \\ &= (q_\rho^*)^{k,k-p} (p + \mathrm{i}0 - r)^{-1} q_\rho^{k-p,k} B. \end{aligned}$$

Summing up the above terms over $k \in \mathrm{sp}(K)$, $p \in \mathcal{F}_k$ we obtain $\Delta_\rho B$.
Type II. Using first (3.29) and then $\tau\overline{r}\tau = -r$, we get

$$\begin{aligned} &\big(1_{\mathcal{K}} \otimes (\overline{q_\rho^* \tau})^{k,k-p}\big)(-p + \mathrm{i}0 - r)^{-1} \big(1_{\mathcal{K}} \otimes (\overline{\tau q_\rho})^{k-p,k}\big) B \\ &= B(q_\rho^* \tau)^{k,k-p} (-p + \mathrm{i}0 - \overline{r})^{-1} \tau q_\rho^{k-p,k} \\ &= -B(q_\rho^*)^{k,k-p} (p - \mathrm{i}0 - r)^{-1} q_\rho^{k-p,k}. \end{aligned}$$

Summing up the above terms over $k \in \mathrm{sp}(K)$, $p \in \mathcal{F}_k$ we obtain $-B\Delta_\rho^*$.
Type III. We use (3.27) to obtain

$$\begin{aligned} &(1_{\overline{\mathcal{K}}} \check{\otimes} (q_\rho^*)^{k_2,k_2-p})(p + \mathrm{i}0 - r)^{-1} (1_{\mathcal{K}} \otimes (\overline{\tau q_\rho})^{k_1,k_1-p}) B \\ &= (q_\rho^*)^{k_2,k_2-p} B \otimes (p + \mathrm{i}0 - r)^{-1} B (\tau q_\rho^\star)^{k_1-p,k_1}. \end{aligned}$$

Type IV. We use (3.26) and then $\tau\overline{r}\tau = -r$:

$$\begin{aligned}
&(1_{\mathcal{K}}\otimes(\overline{q}^*_\rho\tau)^{k_1,k_1-p}(-p+\mathrm{i}0-r)^{-1}(1_{\overline{\mathcal{K}}}\check{\otimes}q_\rho^{k_2,k_2-p})B\\
&=(q_\rho^{\star*})^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0-\overline{r})^{-1}\tau q_\rho^{k_1-p,k_1}\\
&=(q_\rho^{\star*}\tau)^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0+r)^{-1}q_\rho^{k_1-p,k_1}\\
&=(q_\rho^{*})^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0+r)^{-1}(\tau q_\rho^{\star})^{k_1-p,k_1}.
\end{aligned}$$

In the last step we used (6.61) and the fact that ρ commutes with h.

The sum of type III and IV terms over $k_1, k_2 \in \mathrm{sp}(K)$, $p \in \mathcal{F}_{k_1} \cap \mathcal{F}_{k_2}$ equals $-\mathrm{i}\Xi_\rho(B)$. □

Set

$$\widetilde{q_\rho}^{\,p} := \sum_{k\in\mathrm{sp}(K)} q_\rho^{k-p,k}, \qquad \widetilde{\tau q_\rho^\star}^{\,p} := \sum_{k\in\mathrm{sp}(K)} (\tau q_\rho^\star)^{k-p,k},$$
$$\widetilde{v}^{\,p} := \sum_{k\in\mathrm{sp}(K)} v^{k-p,k}, \qquad \widetilde{v^\star}^{\,p} := \sum_{k\in\mathrm{sp}(K)} (v^\star)^{k-p,k}.$$

Here is another useful expression for Γ^{I}_ρ:

Theorem 6.13 *Let $B_1, B_2 \in l^2(\mathcal{K})$. Then*

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma^{\mathrm{I}}_\rho(B_2) &= \pi\sum_{p\in\mathcal{F}}\mathrm{Tr}\Big(\widetilde{q_\rho}^{\,p}B_1 - B_1\otimes 1\,\widetilde{\tau q_\rho^\star}^{\,p}\Big)^*\delta(p-r)\Big(\widetilde{q_\rho}^{\,p}B_2 - B_2\otimes 1\,\widetilde{\tau q_\rho^\star}^{\,p}\Big)\\
&= \pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^{\,p}B_1 - B_1\otimes 1\,\rho^{\frac12}\widetilde{v}^{\,p}\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^{\,p}B_2 - B_2\otimes 1\,\rho^{\frac12}\widetilde{v}^{\,p}\Big)\\
&+\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^{\,p}B_2^* - B_2^*\otimes 1\,\rho^{\frac12}\widetilde{v}^{\,p}\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^{\,p}B_1^* - B_1^*\otimes 1\,\rho^{\frac12}\widetilde{v}^{\,p}\Big)\\
&+\pi\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^{0}B_1 - B_1\otimes 1\,\rho^{\frac12}\widetilde{v}^{0}\Big)^*\delta(h)\Big((1+\rho)^{\frac12}\widetilde{v}^{0}B_2 - B_2\otimes 1\,\rho^{\frac12}\widetilde{v}^{0}\Big).
\end{aligned} \tag{6.64}$$

Proof. Recall that

$$-\Delta^{\mathrm{I}}_\rho = \pi\sum_{k,p}(q_\rho^*)^{k,k-p}\delta(p-r)q_\rho^{k-p,k}.$$

Hence

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Delta^{\mathrm{I}}_\rho B_2 &= \pi\sum_{k,p}\mathrm{Tr}\big(q_\rho^{k-p,k}B_1\big)^*\delta(p-r)q_\rho^{k-p,k}B_2\\
&= \pi\sum_{k_1,k_2,p}\mathrm{Tr}\big(q_\rho^{k_1-p,k_1}B_1\big)^*\delta(p-r)(q_\rho^{k_2-p,k_2}B_2).
\end{aligned}$$

There is an alternative formula for $-\Delta^{\mathrm{I}}_\rho$, which follows from (3.26) and $\tau\overline{r}\tau = -r$:

$$\begin{aligned}
-\Delta^{\mathrm{I}}_\rho &= \pi\sum_{k,p}\mathrm{tr}(q_\rho^\star)^{k,k-p}(q_\rho^{\star*})^{k-p,k}\delta(p-\overline{r})\\
&= \pi\sum_{k,p}\mathrm{tr}(\tau q_\rho^\star)^{k,k-p}(q_\rho^{\star*}\tau)^{k-p,k}\delta(p+r)\\
&= \pi\sum_{k,p}\mathrm{tr}(\tau q_\rho^\star)^{k-p,k}(q_\rho^{\star*}\tau)^{k,k-p}\delta(p-r).
\end{aligned}$$

Hence

$$\begin{aligned}
-\mathrm{Tr}B_1^*B_2\Delta^{\mathrm{I}}_\rho &= \pi\sum_{k,p}\mathrm{Tr}\big(B_1(\tau q_\rho^\star)^{k-p,k})^*\delta(p-r)B_2(\tau q_\rho^\star)^{k-p,k}\\
&= \pi\sum_{k_1,k_2,p}\mathrm{Tr}\big(B_1(\tau q_\rho^\star)^{k_1-p,k_1}\big)^*\delta(p-r)B_2(\tau q_\rho^\star)^{k_2-p,k_2} = 0.
\end{aligned}$$

Recall that

$$\tfrac{1}{2}\Xi_\rho(B_2) = \pi \sum_{k_1,k_2,p} (q_\rho^*)^{k_1,k_1-p}\, B_2 \otimes \delta(p-r)\, (\tau q_\rho^\star)^{k_2-p,k_2}. \tag{6.65}$$

Terms coming from Ξ_ρ we split as

$$\mathrm{Tr} B_1^* \Xi_\rho(B_2) = \frac{1}{2}\mathrm{Tr} B_1^* \Xi_\rho(B_2) + \frac{1}{2}\mathrm{Tr} B_1^* \Xi_\rho(B_2). \tag{6.66}$$

The first term on the right of (6.66) we treat as follows:

$$\tfrac{1}{2}\mathrm{Tr} B_1^* \Xi_\rho(B_2) = \sum_{k_1,k_2,p} \mathrm{Tr}\big(q_\rho^{k_1-p,k_1} B_1\big)^*\, \delta(p-r)\, B_2 (\tau q_\rho^\star)^{k_2-p,k_2}.$$

Then we transform the formula (6.65), using (6.61), (3.28) and then $\tau\overline{r}\tau = -r$:

$$\begin{aligned}
\tfrac{1}{2}\Xi_\rho(B_2) &= \pi \sum_{k_1,k_2,p} (q_\rho^{\star *}\tau)^{k_1,k_1-p}\, B_2 \otimes \delta(p-r)\, q_\rho^{k_2-p,k_2} \\
&= \pi \sum_{k_1,k_2,p} \mathrm{tr}(\tau q_\rho)^{k_1,k_1-p}\, B_2\, (q_\rho^{\star *})^{k_2-p,k_2} \delta(p-\overline{r}) \\
&= \pi \sum_{k_1,k_2,p} \mathrm{tr} q_\rho^{k_1,k_1-p}\, B_2\, (q_\rho^{\star *}\tau)^{k_2-p,k_2} \delta(p+r) \\
&= \pi \sum_{k_1,k_2,p} \mathrm{tr} q_\rho^{k_1-p,k_1}\, B_2\, (q_\rho^{\star *}\tau)^{k_2-p,k_2} \delta(p-r).
\end{aligned}$$

Hence the second term in (6.66) can have the form

$$\tfrac{1}{2}\mathrm{Tr} B_1^* \Xi_\rho(B_2) = \pi \sum_{k_1,k_2,p} \mathrm{Tr}\big(B_1 (\tau q_\rho^\star)^{k_2-p,k_2}\big)^*\, \delta(p-r)\, B_2 q_\rho^{k_1-p,k_1}.$$

This ends the proof of the first identity of (6.64).

Let us prove the second identity. We have

$$\begin{aligned}
&\pi \sum_{p\in\mathcal{F}} \mathrm{Tr}\Big(\widetilde{q_\rho}^{\,p} B_1 - B_1 \otimes 1\, \widetilde{\tau q_\rho^\star}^{\,p}\Big)^*\, \delta(p-r) \Big(\widetilde{q_\rho}^{\,p} B_2 - B_2 \otimes 1\, \widetilde{\tau q_\rho^\star}^{\,p}\Big) \\
&= \pi \sum_{p\in\mathcal{F}^+\cup\{0\}} \mathrm{Tr}\Big((1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_1 - B_1 \otimes 1\, \rho^{\frac{1}{2}} \widetilde{v}^p\Big)^*\, \delta(p-h) \Big((1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_2 - B_2 \otimes 1\, \rho^{\frac{1}{2}} \widetilde{v}^p\Big) \\
&+ \pi \sum_{-p\in\mathcal{F}^+} \mathrm{Tr}\Big(\overline{\rho}^{\frac{1}{2}} \widetilde{v^\star}^{\,p} B_1 - B_1 \otimes 1\, (1+\overline{\rho})^{\frac{1}{2}} \widetilde{v^\star}^{\,p}\Big)^*\, \delta(-p-\overline{h}) \Big(\overline{\rho}^{\frac{1}{2}} \widetilde{v^\star}^{\,p} B_2 - B_2 \otimes 1\, (1+\overline{\rho})^{\frac{1}{2}} \widetilde{v^\star}^{\,p}\Big).
\end{aligned} \tag{6.67}$$

The second term on the right side of (6.67) can be transformed into

$$\begin{aligned}
&\pi \sum_{p\in\mathcal{F}^+} \mathrm{Tr}\Big(B_1^* \otimes 1 \rho^{\frac{1}{2}} \widetilde{v}^p B_1^* - (1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_1^*\Big)^{**}\, \delta(p-\overline{h}) \Big(B_2^* \otimes 1 \rho^{\frac{1}{2}} \widetilde{v}^p - (1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_2^*\Big)^\star \\
&= \pi \sum_{p\in\mathcal{F}^+} \mathrm{Tr}\Big(B_1^* \otimes 1 \rho^{\frac{1}{2}} \widetilde{v}^p B_1^* - (1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_1^*\Big) \Big(B_2^* \otimes 1 \rho^{\frac{1}{2}} \widetilde{v}^p - (1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_2^*\Big)^*\, \delta(p-h) \\
&= \pi \sum_{p\in\mathcal{F}^+} \mathrm{Tr}\Big((1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_2^* - B_2^* \otimes 1\, \rho^{\frac{1}{2}} \widetilde{v}^p\Big)^*\, \delta(p-h) \Big((1+\rho)^{\frac{1}{2}} \widetilde{v}^p B_1^* - B_1^* \otimes 1\, \rho^{\frac{1}{2}} \widetilde{v}^p\Big).
\end{aligned}$$

In the first step we used $(\widetilde{v}^p)^\star = \widetilde{v^\star}^{\,-p}$, then we used (3.28) and in the last step we used the cyclicity of trace. □

The operators $\widetilde{q_\rho}^{\,p}$ and $\widetilde{\tau q_\rho^\star}^{\,p}$ can be decomposed along the fibers:

$$\widetilde{q_\rho}^{\,p}(p) = \begin{cases} (1+\rho)^{\frac{1}{2}} \widetilde{v}^p(p), & p \geq 0 \\ \overline{\rho}^{\frac{1}{2}} \widetilde{v^\star}^{\,p}(-p), & p \leq 0; \end{cases}$$

$$\widetilde{\tau q_\rho^\star}^{\,p}(p) = \begin{cases} \rho^{\frac{1}{2}} \widetilde{v}^p(p), & p \geq 0 \\ (1+\overline{\rho})^{\frac{1}{2}} \widetilde{v^\star}^{\,p}(-p), & p \leq 0; \end{cases}$$

Thus the identity of Theorem 6.13 can be rewritten as

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma_\rho^{\mathrm{I}}(B_2) &= \pi \sum_{p\in\mathcal{F}} \mathrm{Tr}\Big(\widetilde{q_\rho}^{\,p}(p)\, B_1 - B_1\otimes 1\, \widetilde{\tau q_\rho^\star}^{\,p}(p)\Big)^*\Big(\widetilde{q_\rho}^{\,p}(p)\, B_2 - B_2\otimes 1\, \widetilde{\tau q_\rho^\star}^{\,p}(p)\Big)\\
&= \pi \sum_{p\in\mathcal{F}^+\cup\{0\}} \mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\, B_1 - B_1\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big((1+\rho)^{\frac12}\widetilde{v}^p(p) B_2 - B_2\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p)\Big)\\
&\quad+\pi \sum_{p\in\mathcal{F}^+} \mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\, B_2^* - B_2^*\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\, B_1^* - B_1^*\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p)\Big)\\
&\quad+\pi\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^0(0)\, B_1 - B_1\otimes 1\, \rho^{\frac12}\widetilde{v}^0(0)\Big)^*\\
&\quad\times\Big((1+\rho)^{\frac12}\widetilde{v}^0(0)\, B_2 - B_2\otimes 1\, \rho^{\frac12}\widetilde{v}^0(0)\Big).
\end{aligned} \tag{6.68}$$

Let us summarize the properties of Γ_ρ.

Theorem 6.14 *1) Γ_ρ commutes with $[K,\cdot]$.*
2) Γ_ρ is dissipative, that means $\Gamma_\rho^{\mathrm{I}} \leq 0$; consequently, $\mathrm{e}^{-\mathrm{i}t\Gamma}$ is a contraction.
3) $\mathrm{e}^{-\Gamma_\rho}$ is completely positive (see eg. [BR]).
4) $B \in \mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$ iff the following two commutation relations hold

$$\begin{aligned}
&(1+\rho)^{\frac12}\widetilde{v}^p(p)B = B\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p), \quad p\in\mathcal{F}^+\cup\{0\},\\
&(1+\rho)^{\frac12}\widetilde{v}^p(p)B^* = B^*\otimes 1\, \rho^{\frac12}\widetilde{v}^p(p), \quad p\in\mathcal{F}^+.
\end{aligned} \tag{6.69}$$

Proof. 1) and 2) are obvious. 3) follows from the well known form of the generator of completely positive semigroups [BR].

To see 4) note that $B \in \mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$ iff $\mathrm{Tr}B^*\Gamma_\rho^{\mathrm{I}}(B) = 0$. This means that all the terms of (6.68) with $B_1 = B_2 = B$ are zero. But this is precisely the condition (6.69). □

6.8 Pauli-Fierz systems with several reservoirs

Suppose that $\mathcal{Z}_i$, $i = 1,\dots,n$ are Hilbert spaces. Suppose that h_i, ρ_i are positive commuting self-adjoint operators on $\mathcal{Z}_i$ and $v_i \in B(\mathcal{K}, \mathcal{K}\otimes\mathcal{Z}_i)$, for $i = 1,\dots,n$. We impose Assumption 6.D$(\eta)_{\rho_i}$ with $\eta > \frac12$ on v_i, for $i = 1,\dots,n$. Then we can define the FGRO for the individual systems, denoted Γ_{i,ρ_i}.

We can consider the composite system given by $\mathcal{Z} := \overset{n}{\underset{i=1}{\oplus}} \mathcal{Z}_i$, $h := \overset{n}{\underset{i=1}{\oplus}} h_i$, $\rho := \overset{n}{\underset{i=1}{\oplus}} \rho_i$ and $v = \sum_{i=1}^n v_i$. We immediately see that the following is true:

Theorem 6.15 *The FGRO for the composite system equals*

$$\Gamma_\rho = \sum_{i=1}^n \Gamma_{i,\rho_i}.$$

7 Thermal Pauli-Fierz systems

7.1 Thermal Pauli-Fierz Liouvilleans

In this section we will consider Pauli-Fierz Liouvilleans at the inverse temperature $\beta \in]0, \infty]$. The setup of this section is very similar to the setup of the previous section. In particular, the operators K, h, v and H, as well as the spaces $\mathcal{K}$ and $\mathcal{Z}$ are such as those introduced in Subsection 6.1.

Let $0 < \beta \leq \infty$. In this section we consider the family of densities

$$\rho_\beta := (e^{\beta h} - 1)^{-1}, \quad \rho_\infty = 0.$$

Note that

$$1 + \rho_\beta = (1 - \mathrm{e}^{-\beta h})^{-1} = \mathrm{e}^{\beta h}\rho_\beta, \quad 1 + \rho_\infty = 1. \tag{7.70}$$

We change slightly the notation for various objects, replacing the subscripts ρ_β by β. For instance we will write q_β, L_β, L_β^{semi}, $\mathfrak{M}_{\beta,\mathrm{l}}$ and τ_β^t instead of q_{ρ_β}, L_{ρ_β}, $L_{\rho_\beta}^{\mathrm{semi}}$, $\mathfrak{M}_{\rho_\beta,\mathrm{l}}$ and $\tau_{\rho_\beta}^t$. We hope that this change of notation will not lead to a confusion. We warn however about one confusing point: the density $\rho = 0$ corresponds now to inverse temperature $\beta = \infty$.

Note that

$$\begin{aligned} q_\beta &= |1 - \mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v, v^\star), \\ \overline{\tau q_\beta} &= |1 - \mathrm{e}^{\beta r}|^{-\frac{1}{2}}(\overline{v^\star}, \overline{v}), \\ \tau q_\beta^\star &= |1 - \mathrm{e}^{\beta r}|^{-\frac{1}{2}}(v, v^\star) = \mathrm{e}^{\beta r/2} q_\beta. \end{aligned}$$

The following hypothesis is just assumption 6.G$_\rho$ for $\rho = \rho_\beta$.

Assumption 7.A$_\beta$

$$(1 + |r|)q_\beta \in B(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

Proposition 7.1 *Let $0 < \beta_0 \leq \beta < \infty$. Then Assumption 7.A$_{\beta_0}$ implies Assumption 7.A$_\beta$.*

Proof. It is sufficient to note that the operator

$$|1 - \mathrm{e}^{-\beta r}|^{-\frac{1}{2}}|1 - \mathrm{e}^{-\beta_0 r}|^{\frac{1}{2}} \tag{7.71}$$

is bounded. In fact, this follows from the fact that the function

$$\mathbb{R} \ni p \mapsto |1 - \mathrm{e}^{-\beta p}|^{-\frac{1}{2}}|1 - \mathrm{e}^{-\beta_0 p}|^{\frac{1}{2}} \in \mathbb{R} \tag{7.72}$$

goes to 1 for $p \to \infty$, to 0 as $p \to -\infty$, and is continuous. □

The following theorem follows immediately from Proposition 7.1 and Theorem 6.11.

Theorem 7.2 *Suppose Assumption 7.A$_{\beta_0}$ is true. Then for any $\beta_0 \leq \beta < \infty$, L_β is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q_\beta) \cap (JQ_\beta J)$.*

Our aim in this section is to study the one-parameter family of W^*-dynamical systems $\big(B(\mathcal{K}) \otimes \mathfrak{M}_{\beta,\mathrm{l}}, \tau_\beta^t\big)$, called thermal Pauli-Fierz W^*-dynamical systems.

7.2 Existence of KMS states for thermal Pauli-Fierz systems

The dynamics $\tau_{\rm fr}^t$ and τ_β^t possess β-KMS states. For the free dynamics it is straightforward, for the interacting dynamics it follows from a version of Araki's theory from [DJP].

On the level of the Liouvilleans this is expressed by the fact that both $L_{\rm fr}$ and L_β possess certain distinguished eigenstates called β-KMS states.

For $0 < \beta < \infty$, set $\gamma_\beta := {\rm e}^{-\beta K/2}/\sqrt{{\rm Tr}{\rm e}^{-\beta K}}$. For $\beta = \infty$ we set

$$\gamma_\infty := 1_{k_0}(K)/\sqrt{{\rm Tr}1_{k_0}(K)},$$

where we recall that $k_0 := \inf {\rm sp}(K)$. Note that

$$[0,\infty] \ni \beta \mapsto \gamma_\beta \in l^1(\mathcal{K})$$

is a continuous function.

Theorem 7.3 *1) For any $\beta \in [0,\infty]$, the vector $\gamma_\beta \otimes \Omega$ is a β-KMS vector for $\tau_{\rm fr}^t$. Clearly, it is an eigenvector of $L_{\rm fr}$ with the eigenvalue 0.*
2) Let $0 < \beta_0 < \infty$. Suppose Assumption 7.A_{β_0} is true. Then for any $\beta_0 \leq \beta < \infty$, $\gamma_\beta \otimes \Omega \in \mathcal{D}({\rm e}^{-\beta(L+\lambda Q_\beta)/2})$ and the vector

$$ {\rm e}^{-\beta(L+\lambda Q_\beta)/2}\gamma_\beta \otimes \Omega. \tag{7.73}$$

is a β-KMS vector for τ_β^t. Consequently, it is an eigenvector of L_β with the eigenvalue 0.

Proof. Let us prove 2). By [DJP], we need to check that

$$\|{\rm e}^{-\lambda\beta Q_\beta/2}\gamma_\beta \otimes \Omega\| < \infty.$$

The next lemma verifies this hypothesis. □

Lemma 7.4 *There exists a constant c such that for any integer n,*

$$\|Q_\beta^n\, \gamma_\beta \otimes \Omega\| \leq c^n \sqrt{(n+1)!}.$$

Proof. Using the formulas

$$Q_\beta := 1_{\overline{\mathcal{K}}} \check{\otimes} q_\beta(a^*) + 1_{\overline{\mathcal{K}}} \check{\otimes} q_\beta^*(a).$$

we decompose Q_β into the sum of a creation and annihilation operator. Then $Q_\beta^n\, \gamma_\beta \otimes \Omega$ splits into the sum of 2^n terms. Applying the estimates (5.42) to each term we derive the estimate

$$\|Q_\beta^n\, \gamma_\beta \otimes \Omega\| \leq 2^n \sqrt{(n+1)!}\, \|q_\beta\|^n. \tag{7.74}$$

□

7.3 FGRO's for thermal Pauli-Fierz Liouvilleans

Let us now consider the formalism of FGRO in the context of thermal Liouvilleans.

Let $\eta \geq 0$ and $0 < \beta \leq \infty$. The following assumption is just Assumption 6.D$(\eta)_\rho$ for $\rho = \rho_\beta$:

Assumption 7.B$(\eta)_\beta$ $\langle s \rangle^\eta q_\beta \in B(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}))$.

Proposition 7.5 *Let $0 < \beta_0 \leq \beta < \infty$. Then Assumption 7.$B(\eta)_{\beta_0}$ implies Assumption 7.$B(\eta)_\beta$.*

Proof. We just note that the function (7.72) which we already studied in the proof of Proposition 7.1, has all bounded derivatives. Therefore, the operator

$$\langle s\rangle^{\eta}|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}|1-\mathrm{e}^{-\beta_0 r}|^{\frac{1}{2}}\langle s\rangle^{-\eta}$$

is bounded for any $\eta\in\mathbb{R}$. □

Throughout this subsection we assume that Assumption 7.B$(\eta)_{\beta_0}$ with $\eta>\frac{1}{2}$ holds. We will describe the FGRO Γ_ρ in the case $\rho=\rho_\beta$, which, consistently with our notation will, be denoted Γ_β. Note that a special attention needs to be devoted to the infrared term in Γ_β.

Proposition 7.6 *There exists*

$$v_{\mathrm{ir}}:=\lim_{p\downarrow 0}\frac{v(p)}{p^{\frac{1}{2}}}=\lim_{p\downarrow 0}\frac{v^\star(-p)}{p^{\frac{1}{2}}}. \tag{7.75}$$

Set

$$\widetilde{v_{\mathrm{ir}}}^0=\sum_p v_{\mathrm{ir}}^{pp}.$$

Noting that the above operators belong to $B(\mathcal{K},\mathcal{K}\otimes\mathcal{G})$ and $\mathcal{G}$ is equipped with an internal conjugation (see (6.47), we have

$$v_{\mathrm{ir}}=(v_{\mathrm{ir}})^\star,\quad (\widetilde{v_{\mathrm{ir}}}^0)^\star=\widetilde{v_{\mathrm{ir}}}^0. \tag{7.76}$$

Moreover, $\Gamma^{\mathrm{I}}_\beta$ can be written in the following two ways:

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma^{\mathrm{I}}_\beta(B_2)\quad &=\pi\sum_{p\in\mathcal{F}^+}|\mathrm{e}^{\beta p}-1|^{-1}\mathrm{Tr}\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_1-B_1\otimes 1\ \widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_2-B_2\otimes 1\ \widetilde{v}^p(p)\Big)\\
&\quad+\pi\sum_{p\in\mathcal{F}^+}|\mathrm{e}^{\beta p}-1|^{-1}\mathrm{Tr}\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_2^*-B_2^*\otimes 1\ \widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_1^*-B_1^*\otimes 1\ \widetilde{v}^p(p)\Big)\\
&\quad+\tfrac{\pi}{\beta}\mathrm{Tr}\Big(\widetilde{v_{\mathrm{ir}}}^0B_2^*-B_2^*\otimes 1\ \widetilde{v_{\mathrm{ir}}}^0\Big)^*\\
&\quad\times\Big(\widetilde{v_{\mathrm{ir}}}^0B_1^*-B_1^*\otimes 1\ \widetilde{v_{\mathrm{ir}}}^0\Big).
\end{aligned}\tag{7.77}$$

Proof. For $p>0$,

$$q_\beta(p)=|1-\mathrm{e}^{-\beta p}|^{-\frac{1}{2}}v(p),$$

$$q_\beta(-p)=|1-\mathrm{e}^{\beta p}|^{-\frac{1}{2}}v^\star(-p).$$

But $\mathbb{R}\ni p\mapsto q_\beta(p)$ is continuous. Hence

$$\begin{aligned}
q_\beta(0)&=\lim_{p\downarrow 0}q_\beta(p)=\beta^{-\frac{1}{2}}\lim_{p\downarrow 0}p^{-\frac{1}{2}}v(p),\\
q_\beta(0)&=\lim_{p\downarrow 0}q_\beta(-p)=\beta^{-\frac{1}{2}}\lim_{p\downarrow 0}p^{-\frac{1}{2}}v^\star(-p).
\end{aligned}$$

This implies the existence of the limits in (7.75) and the identities of (7.76).

The identity (7.77) is a modification of the identity (6.68), where we take into account the identities (7.70) and (7.75). □

Let us define

$$\mathfrak{N} := \Big\{ B \in B(\mathcal{K}) \ : \quad B\otimes 1\, \widetilde{v}^p(p) = \widetilde{v}^p(p) B, \quad p \in \mathcal{F}^+,$$
$$B^*\otimes 1\, \widetilde{v}^p(p) = \widetilde{v}^p(p) B^*, \quad p \in \mathcal{F}^+,$$
$$B\otimes 1\, \widetilde{v_{\rm ir}}^0 = \widetilde{v_{\rm ir}}^0 B \Big\}.$$

Proposition 7.7 *$\mathfrak{N}$ is a $*$-subalgebra of $B(\mathcal{K})$ containing $\mathbb{C}1_{\mathcal{K}}$. Moreover, for any $t \in \mathbb{C}$, $B \in \mathfrak{N}$ we have $\mathrm{e}^{\mathrm{i}tK} B \mathrm{e}^{-\mathrm{i}tK} \in \mathfrak{N}$.*

Proof. It is easy to check that $\mathfrak{N}$ is an algebra. To see that it is preserved by $*$ we note that the first two conditions are manifestly symmetric wrt $*$. Besides, note that $(\widetilde{v_{\rm ir}}^0)^\star = \widetilde{v_{\rm ir}}^0$ implies that

$$B\otimes 1\, \widetilde{v_{\rm ir}}^0 = \widetilde{v_{\rm ir}}^0 B \quad \Rightarrow \quad B^*\otimes 1\, \widetilde{v_{\rm ir}}^0 = \widetilde{v_{\rm ir}}^0 B^*.$$

The obvious identities

$$1\otimes 1\, \widetilde{v}^p(p) = \widetilde{v}^p(p) 1, \quad 1\otimes 1\, \widetilde{v_{\rm ir}}^0 = \widetilde{v_{\rm ir}}^0 1$$

imply that $1 \in \mathfrak{N}$.

Note that

$$\mathrm{e}^{\mathrm{i}tK}\otimes 1\, \widetilde{v}^p(p) \mathrm{e}^{-\mathrm{i}tK} = \mathrm{e}^{\mathrm{i}tp} \widetilde{v}^p(p), \quad p \in \mathcal{F}^+, \quad \mathrm{e}^{\mathrm{i}tK}\otimes 1\, \widetilde{v_{\rm ir}}^0 \mathrm{e}^{-\mathrm{i}tK} = \widetilde{v_{\rm ir}}^0.$$

This implies that $\mathfrak{N}$ is invariant wrt $\mathrm{e}^{\mathrm{i}tK} \cdot \mathrm{e}^{-\mathrm{i}tK}$. □

Theorem 7.8 $\mathrm{Ker}\Gamma_\beta^{\rm I}$ *consists of operators of the form $\mathrm{e}^{-\beta K/2} C$ with $C \in \mathfrak{N}$.*

It is easy to see that the following assumption is satisfied for a generic intreaction v.

Assumption 7.C $\mathfrak{N} = \mathbb{C}1_{\mathcal{K}}$.

Theorem 7.9 *If Assumption 7.C is satisfied, then $\mathrm{sp}(\Gamma_\beta) \cap \mathbb{R} = \{0\}$ and $\mathrm{Ran}1_0(\Gamma_\beta) = \mathrm{Ker}\Gamma_\beta^{\rm I}$ is spanned by γ_β.*

Proof. (7.77) can be rewritten as

$$\begin{aligned}
\Gamma_\beta^{\rm I} \ &= \pi \sum_{p\in\mathcal{F}^+} |\mathrm{e}^{\beta p} - 1|^{-1} \mathrm{Tr} \Big(\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B_1 - \mathrm{e}^{\beta K/2} B_1 \otimes 1\, \widetilde{v}^p(p) \Big)^* \\
&\quad \times \mathrm{e}^{-\beta K} \otimes 1 \Big(\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B_2 - \mathrm{e}^{\beta K/2} B_2 \otimes 1\, \widetilde{v}^p(p) \Big) \\
&+ \pi \sum_{p\in\mathcal{F}^+} |\mathrm{e}^{\beta p} - 1|^{-1} \mathrm{Tr} \Big(\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B_2^* - \mathrm{e}^{\beta K/2} B_2^* \otimes 1\, \widetilde{v}^p(p) \Big)^* \\
&\quad \times \mathrm{e}^{-\beta K} \otimes 1 \Big(\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B_1^* - \mathrm{e}^{\beta K/2} B_1^* \otimes 1\, \widetilde{v}^p(p) \Big) \\
&+ \tfrac{\pi}{\beta} \mathrm{Tr} \Big(\widetilde{v_{\rm ir}}^0 \mathrm{e}^{\beta K/2} B_2^* - \mathrm{e}^{\beta K/2} B_2^* \otimes 1\, \widetilde{v_{\rm ir}}^0 \Big)^* \\
&\quad \times \mathrm{e}^{-\beta K} \otimes 1 \Big(\widetilde{v_{\rm ir}}^0 \mathrm{e}^{\beta K/2} B_1^* - \mathrm{e}^{\beta K/2} B_1^* \otimes 1\, \widetilde{v_{\rm ir}}^0 \Big).
\end{aligned}$$

Hence, $B \in \mathrm{Ker}\Gamma_\beta^{\rm I}$ iff

$$\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B - \mathrm{e}^{\beta K/2} B \otimes 1\, \widetilde{v}^p(p) = 0,$$
$$\widetilde{v}^p(p) \mathrm{e}^{\beta K/2} B^* - \mathrm{e}^{\beta K/2} B^* \otimes 1\, \widetilde{v}^p(p) = 0,$$
$$\widetilde{v}^0_{\rm ir}(0) \mathrm{e}^{\beta K/2} B - \mathrm{e}^{\beta K/2} B \otimes 1\, \widetilde{v}^0_{\rm ir}(0) = 0.$$

This clearly implies 1).

1) together with the assumption 7.C implies that $\mathrm{Ker}\Gamma_\beta^{\rm I}$ is spanned by γ_β. Clearly, $[\Delta_\beta, \gamma_\beta] = 0$. Hence $\gamma_\beta \in \mathrm{Ker}\Gamma_\beta^{\rm R}$. Therefore, $\mathrm{Ran}1_0(\Gamma_\beta)$ is spanned by γ_β. This ends the proof of 2). □

7.4 Return to equilibrium for a fixed positive temperature

In this subsection we describe conditions that for any fixed positive temperature guarantee the return to equilibrium property. The result will be not uniform in the temperature.

Theorem 7.10 *Let $0 < \beta_0 < \infty$, $\eta > 2$. Suppose that Assumptions 7.A_{β_0}, 7.$B(\eta)_{\beta_0}$ and 7.C are satisfied. Let $\beta \in [\beta_0, \infty[$. Then there exists $\lambda_0(\beta) > 0$ such that for $0 < |\lambda| < \lambda_0(\beta)$ we have*

$$\mathrm{sp}_{\mathrm{p}}(L_\beta) = \{0\}, \qquad \dim 1_0(L_\beta) = 1, \qquad \mathrm{sp}_{\mathrm{sc}}(L_\beta) = \emptyset.$$

Proof. By Theorem 7.9 we know that for $\beta \in [\beta_0, \infty[$, we have $\dim \mathrm{Ker}(\Gamma^{\mathrm{I}}_\beta) = 1$. Therefore, by Theorem 5.6, there exists $\lambda_0(\beta)$ such that for $0 < |\lambda| < \lambda_0(\beta)$ we have

$$\dim 1_0(L_\beta) \leq 1, \qquad \mathrm{sp}_{\mathrm{sc}}(L_\beta) = \emptyset.$$

But by Theorem 7.3, $\dim 1_0(L_\beta) \geq 1$. □

7.5 0-temperature Pauli-Fierz Liouvilleans

At the zero temperature, all the properties of the Liouvillean, denoted L_∞, follow easily from the properties of the Hamiltonian. They are described in this subsection.

If we consider $\beta = \infty$, it is convenient to use the identification of the space $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ with the space

$$\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \overline{\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})}.$$

Under this identification, the 0-temperature Liouvillean becomes

$$L_\infty = H \otimes 1 - 1 \otimes \overline{H}.$$

The FGRO for L_∞, denoted Γ_∞ can be expressed in terms of the FGRO for H, denoted Γ as follows. If Γ_∞ is given by Δ_∞ and Ξ_∞ as in Theorem 6.12 then

$$\Delta_\infty = \Gamma, \qquad \Xi_\infty = 0.$$

The following theorem follows immediately from Theorem 6.6:

Theorem 7.11 *Under assumptions of Theorem 6.6, there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ we have*

$$\mathrm{sp}_{\mathrm{p}}(L_\infty) = \{0\}, \qquad \dim 1_0(L_\infty) = 1, \qquad \mathrm{sp}_{\mathrm{sc}}(L_\infty) = \emptyset.$$

7.6 Uniform in temperature estimate on the FGRO

In this subsection we study the FGRO Γ_β uniformly in the temperature. We give the conditions that guarantee that the kernel of Γ_β is spanned by the Gibbs state γ_β and that on the orthogonal complement to γ_β is uniformly dissipative. The key assumptions are the two generic assumptions that were used at the zero temperature, the is 6.E and 6.F, and the generic assumption for the positive temperature, that is 7.C.

Theorem 7.12 *Let $0 < \beta_0 < \infty$, $\eta > \frac{1}{2}$. Suppose that Assumption 7.B$(\eta)_{\beta_0}$ is satisfied. Then 1)*

$$]0, \infty] \ni \beta \mapsto \Gamma_\beta \in B(l^2(\mathcal{K}))$$

is a continuous function.
2) Assume in addition that 6.E, 6.F and 7.C are satisfied. Let $0 < \beta_1$. Then there exists $\epsilon > 0$ such that for $\beta \in [\beta_1, \infty]$

$$\mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\beta(B) \leq -\epsilon(\mathrm{Tr} B^* B - |\mathrm{Tr} B \gamma_\beta|^2). \tag{7.78}$$

Proof. The continuity of Γ_β in $\beta \in]0, \infty]$ is obvious from (7.77)

Let us consider first $\beta = \infty$. Assumptions 6.E implies that there exists $\epsilon(\infty) > 0$ such that

$$\Delta^{\mathrm{I}}_\infty \leq -\epsilon(\infty)(1 - 1_{k_0}(K)). \tag{7.79}$$

Recall that $\Gamma^{\mathrm{I}}_\infty(B) = \Delta^{\mathrm{I}}_\infty B + B\Delta^{\mathrm{I}}_\infty$. Hence, using (7.79) and at the last step using Assumption 6.F we obtain

$$\begin{aligned} \mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\infty(B) &\leq -\epsilon(\infty)\big(\mathrm{Tr} B^*(1 - 1_{k_0}(K))B + \mathrm{Tr} B^* B(1 - 1_{k_0}(K))\big) \\ &\leq -\epsilon(\infty)\big(\mathrm{Tr} B^* B - \mathrm{Tr} B^* 1_{k_0}(K) B 1_{k_0}(K)\big) \\ &= -\epsilon(\infty)(\mathrm{Tr} B^* B - |\mathrm{Tr} B \gamma_\infty|^2). \end{aligned}$$

Next, let us consider $\beta < \infty$. It follows from Theorem 7.9 2) that for any $\beta \in [\beta_0, \infty[$, there exists $\epsilon(\beta) > 0$ such that

$$\mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\beta(B) \leq -\epsilon(\beta)(\mathrm{Tr} B^* B - |\mathrm{Tr} B \gamma_\beta|^2).$$

The compactness of $[\beta_1, \infty]$, the continuity of $[\beta_1, \infty] \ni \beta \mapsto \Gamma^{\mathrm{I}}_\beta$ and of $[\beta_1, \infty] \ni \beta \mapsto \gamma_\beta$ imply that one can chose $\epsilon > 0$ such that (7.78) is true. □

7.7 Uniform in temperature return to equilibrium

In this subsection we describe the main result of this paper. We give conditions that imply that, uniformly in the temperature for small nonzero coupling constant the Liouvillean L_β has no singular continuous spectrum and only one nondegenerate eigenvalue. For positive temperatures this implies the return to equilibrium property.

One of the main ingredients of the proof is the uniform assumption on the FGRO Γ_β obtained in the previous subsection. The second ingredient is a uniform assumption on the regularity of the interaction, which we will formulate below.

For any η and β_0 let us make the following assumption

Assumption 7.D$(\eta)_{\beta_0}$ $\sup_{\beta_0 \leq \beta \leq \infty} \|\langle s \rangle^\eta q_\beta\| < \infty$.

Theorem 7.13 *Let $0 < \beta_0 < \infty$. Suppose Assumptions 7.A$_{\beta_0}$, 7.D$(\eta)_{\beta_0}$ with $\eta > 2$, 6.E 6.F and 7.C are satisfied. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and $\beta \in [\beta_0, \infty]$ we have*

$$\mathrm{sp}_{\mathrm{p}}(L_\beta) = \{0\}, \qquad \dim 1_0(L_\beta) = 1, \qquad \mathrm{sp}_{\mathrm{sc}}(L_\beta) = \emptyset.$$

Proof. By Theorem 7.2 we know that L_β is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q - JQJ)$ for $\beta \in [\beta_0, \infty[$ and any λ. By Theorem 6.5, we know that L_∞ is self-adjoint on $\mathcal{D}(L_{\mathrm{fr}})$ and any λ.

By Theorem 7.12, we know that there exists $\epsilon > 0$ such that for $\beta \in [\beta_0, \infty]$ we have

$$\Gamma^{\mathrm{I}}_\beta < -\epsilon(1 - 1_0(\Gamma^{\mathrm{I}})).$$

By Assumption 7.D$(\eta)_{\beta_0}$ with $\eta > 2$ we know that

$$\|\langle s\rangle^{\eta}\big(1_{\overline{\mathcal{K}}}\check{\otimes}q_\beta - 1_{\mathcal{K}}\otimes\overline{\tau q_\beta}\big)\| \leq 2c.$$

Hence by Theorem 5.6, there exists $\lambda_0 > 0$ such that for $0 < |\lambda| \leq \lambda_0$ we have

$$\dim 1_{\mathrm{p}}(L_\beta) \leq \dim \Gamma_\beta^{\mathrm{I}} = 1, \qquad \mathrm{sp}_{\mathrm{sc}}(L_\beta) = \emptyset.$$

But by Theorem 7.11 and 7.3, $\dim 1_0(L_\beta) \geq 1$. □

7.8 An infrared condition for a uniform return to equilibrium

The Assumption 7.D$(\eta)_{\beta_0}$ seems somewhat difficult to check in practice. Therefore, in this subsection we will describe some other conditions, which are easier to verify in applications and imply the Assumption 7.D$(\eta)_{\beta_0}$.

First of all, we would like to have a condition that involves directly v, and not the whole family q_β.

Secondly, in realistic physical systems the function $[0,\infty[\ni p\mapsto v(p)$ is usually smooth (even analytic) outside $p = 0$. Therefore, our main concern is the point $p = 0$, where we can expect that $v(p)$ go to zero as p^α for some $\alpha > 0$. The assumptions stated below will try to optimize this α with our method.

Theorem 7.14 *Suppose that $3 > \eta > 2$, $\eta' > \eta$, $\beta_0 > 0$. Assume that for $n = 0, 1, 2, 3$ we have*

$$\big\|(1+|r|)^{-\frac{3}{2}+\eta'}|r|^{n-\frac{3}{2}-\eta'}s^n(v,v^\star)\big\| < \infty. \tag{7.80}$$

Then Assumption 7.D$(\eta)_{\beta_0}$ holds, that means

$$\sup_{\beta\in[\beta_0,\infty]}\|\langle s\rangle^{\eta}|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v,v^\star)\| < \infty.$$

The following corollaries present slightly weaker but maybe more explicit conditions that guarantee the uniform regularity of the interaction. In both corollaries $\eta, \eta', \beta_0, \beta_0'$. are as in Theorem 7.14

Corollary 7.15 *Suppose that*

$$\begin{aligned}&\textstyle\int_0^\infty (1+p)^{-3+2\eta'}p^{-3-2\eta'+2n}\|\frac{\mathrm{d}^n}{\mathrm{d}p^n}v(p)\|^2\mathrm{d}p < \infty, \quad n = 0,1,2,3,\\ &\textstyle\frac{\mathrm{d}^n}{\mathrm{d}p^n}v(0) = 0, \quad n = 0,1,2.\end{aligned} \tag{7.81}$$

Then Assumption 7.D$(\eta)_{\beta_0}$ holds.

Proof. Since $\mathcal{K}$ is finite dimensional, the $\star$ conjugation is az bounded linear map. Therefore, (7.81) implies

$$\begin{aligned}&\textstyle\int_0^\infty (1+p)^{-3+2\eta'}p^{-3-2\eta'+2n}\|\frac{\mathrm{d}^n}{\mathrm{d}p^n}v^\star(p)\|^2\mathrm{d}p < \infty, \quad n = 0,1,2,3,\\ &\textstyle\frac{\mathrm{d}^n}{\mathrm{d}p^n}v^\star(0) = 0, \quad n = 0,1,2.\end{aligned} \tag{7.82}$$

Because of the first derivatives of $v(p)$ and $v^\star(p)$ vanish at zero, for $n = 0, 1, 2, 3$ we have

$$\big(s^n(v,v^\star)\big)(p) = \begin{cases} \frac{\mathrm{d}^n}{\mathrm{d}p^n}v(p), & p \geq 0;\\ \frac{\mathrm{d}^n}{\mathrm{d}p^n}v^\star(-p), & p \leq 0.\end{cases}$$

Hence

$$\begin{aligned}
&\|(1+|r|)^{-\frac{3}{2}+\eta'}|r|^{n-\frac{3}{2}-\eta'}s^n(v,v^\star)\|^2\\
&=\|(v,v^\star)^*s^n(1+|r|)^{-3+2\eta'}|r|^{2n-3-2\eta'}s^n(v,v^\star)\|\\
&=\Big\|\int_0^\infty p^{2n-3-2\eta'}(1+p)^{-3+2\eta'}\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v^*(p)\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v(p)\mathrm{d}p+\int_0^\infty p^{2n-3-2\eta'}(1+p)^{-3+2\eta'}\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v^{\star*}(p)\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v^\star(p)\mathrm{d}p\Big\|\\
&\leq\int_0^\infty p^{2n-3-2\eta'}(1+p)^{-3+2\eta'}\|\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v(p)\|^2\mathrm{d}p+\int_0^\infty p^{2n-3-2\eta'}(1+p)^{-3+2\eta'}\|\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v^\star(p)\|^2\mathrm{d}p.
\end{aligned}$$

□

The following corollary follows easily from the previous corollary:

Corollary 7.16 *Suppose that for $p\in[0,\infty[$, $\delta>0$ and for $n=0,1,2,3$ we have*

$$\|\tfrac{\mathrm{d}^n}{\mathrm{d}p^n}v(p)\|\leq c\begin{cases}p^{\eta'-n+1} & 0\leq p<1;\\ (1+p)^{-\frac{1}{2}-\delta-n} & p\geq 1\end{cases}$$

Then Assumption 7.D(η)$_{\beta_0}$ holds.

The proof of Theorem 7.14 is divided into a number of steps.

Lemma 7.17 *Let $0\leq\delta\leq 1$ and $\xi_\delta(p):=|p|^\delta(1+|p|)^{-\delta}$. Then there exists c_δ such that*

$$|\xi_\delta(p)-\xi_\delta(p+t)|\leq c_\delta|t|^\delta$$

($\xi_\delta(p)$ is globally δ-Hölder).

Proof. The product of globally δ-Hölder functions is globally δ-Hölder. Clearly, $(1+|p|)^{-\delta}$ is globally 1-Hölder. Hence it is sufficient to prove the global δ-Hölder property for p^δ, that is

$$\big||p|^\delta-|p+t|^\delta\big|\leq c_\delta|t|^\delta. \tag{7.83}$$

It is also easy to see that it suffices to prove (7.83) for $p,p+t\geq 0$.

By the concavity of $p\mapsto p^\delta$, we have

$$\left|\frac{p}{p+t}\right|^\delta+\left|\frac{t}{p+t}\right|^\delta\leq\left|\frac{p}{2(p+t)}+\frac{t}{2(p+t)}\right|^\delta=\frac{1}{2^\delta}.$$

Hence

$$2^{-\delta}|p|^\delta+|t|^\delta\leq|p|^\delta+|t|^\delta\leq 2^{-\delta}|p+t|^\delta,$$

which implies (7.83) wih $c_\delta=2^\delta$. □

Lemma 7.18 *Let r, s be the operators on $L^2(\mathbb{R})$ introduced in Subsection 3.4. Let $0<\delta<1$, $\delta<\delta'$. Then*

$$\||s|^\delta\Psi\|\leq c\|(1+|r|)^{\delta'}|r|^{-\delta'-1}\Psi\|+c\|(1+|r|)^{\delta'}|r|^{-\delta'}s\Psi\|.$$

Proof. Set $\Phi := (1+|r|)^{\delta'}|r|^{-\delta'}\Psi$. Now, using $\mathrm{e}^{\mathrm{i}ts}r = (r+t)\mathrm{e}^{\mathrm{i}ts}$, we obtain

$$\begin{aligned}\|\mathrm{e}^{\mathrm{i}ts}\Psi - \Psi\| &\leq \big\|\big((1+|r+t|)^{-\delta'}|r+t|^{\delta'} - (1+|r|)^{-\delta'}|r|^{\delta'}\big)\mathrm{e}^{\mathrm{i}ts}\Phi\big\| \\ &\quad + \|(1+|r|)^{-\delta'}|r|^{\delta'}(\mathrm{e}^{\mathrm{i}ts}\Phi - \Phi)\|^2.\end{aligned} \tag{7.84}$$

By Lemma 7.17, the first term on the right of (7.84) is estimated from above by

$$c\min(t^{\delta'}, 1)\|\Phi\|.$$

The second term on the right of (7.84) is less than

$$c\min(t\|s\Phi\|, \|\Phi\|).$$

Note that for any $0 < \delta < 1$ there exists c_δ such that for any $p \in \mathbb{R}$

$$|p|^{2\delta} = c_\delta \int_0^\infty |\mathrm{e}^{\mathrm{i}tp} - 1|^2 t^{-1-2\delta}\mathrm{d}t.$$

Using this identity, we obtain

$$\begin{aligned}\||s|^\delta\Psi\|^2 &= c\textstyle\int_0^\infty \|\mathrm{e}^{\mathrm{i}ts}\Psi - \Psi\|^2 t^{-1-2\delta}\mathrm{d}t \\ &\leq c(\|\Phi\| + \|s\Phi\|)^2.\end{aligned}$$

Next we note that

$$s\Phi = (1+|r|)^{\delta'}|r|^{-\delta'}s\Psi + \delta'\mathrm{sgn}r\big(-(1+|r|)^{\delta'}|r|^{-\delta'-1} + (1+|r|)^{\delta'-1}|r|^{\delta'}\big)\Psi.$$

Thus

$$\|s\Phi\| \leq c\|(1+|r|)^{\delta'}|r|^{-\delta'}s\Psi\| + c\|(1+|r|)^{\delta'-1}|r|^{-\delta'}\Psi\|.$$

□

Proof of Theorem 7.14 Set $\theta_j(p) := \frac{\mathrm{d}^j}{\mathrm{d}p^j}|1-\mathrm{e}^{-p}|^{-\frac{1}{2}}$. We use Lemma 7.18 with $\delta = \eta - 2$ and $\delta' = \eta' - 2$ and Ψ replaced with $s^2|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v, v^\star)$. Then

$$\begin{aligned}\||s|^\eta|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v,v^\star)\| &\leq c\|(1+|r|)^{\eta'-2}|r|^{-\eta'+1}s^2|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v,v^\star)\| \\ &\quad + c\|(1+|r|)^{\eta'-2}|r|^{-\eta'+2}s^3|1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v,v^\star)\| \\ &\leq c\sum_{j=0}^{2}\|(1+|r|)^{\eta'-2}|r|^{-\eta'+1}\beta^j\theta_j(\beta r)s^{2-j}(v,v^\star)\| \\ &\quad + c\sum_{j=0}^{3}\|(1+|r|)^{\eta'-2}|r|^{-\eta'+2}\beta^j\theta_j(\beta r)s^{3-j}(v,v^\star)\|.\end{aligned} \tag{7.85}$$

We have

$$|\theta_0(p)| \leq c\begin{cases}|p|^{-\frac{1}{2}}(1+|p|)^{\frac{1}{2}}, & p > 0 \\ |p|^{-\frac{1}{2}}(1+|p|)^{\frac{1}{2}}\mathrm{e}^{-|p|/2}, & p < 0.\end{cases}$$

Therefore, for $\beta \in [\beta_0, \infty]$, we have

$$|\theta_0(\beta p)| \leq c\begin{cases}|p|^{-\frac{1}{2}}(1+|p|)^{\frac{1}{2}}, & p > 0 \\ |p|^{-\frac{1}{2}}(1+|p|)^{\frac{1}{2}}\mathrm{e}^{-\beta_0|p|/2}, & p < 0.\end{cases}$$

Hence the two terms on the right hand side of (7.85) with θ_0 can be estimated as

$$\sum_{n=2}^{3}\|(1+|r|)^{\eta'-2}|r|^{-\eta'+n-1}\theta_0(\beta r)s^n(v,v^\star)\| \quad \leq c\sum_{n=2}^{3}\|(1,\mathrm{e}^{-\beta_0\overline{h}/2})(1+|r|)^{\eta'-\frac{3}{2}}|r|^{-\eta'+n-\frac{3}{2}}s^n(v,v^\star)\|$$

Next we note that

$$|\theta_j(p)|\leq c|p|^{-\frac{1}{2}-j}(1+|p|)^{\frac{1}{2}+j}\mathrm{e}^{-|p|/2},\quad j=1,2,\ldots.$$

Hence for $\beta\in[\beta_0,\infty]$

$$|\theta_j(\beta p)|\leq c\beta^{-\frac{1}{2}-j}|p|^{-\frac{1}{2}-j}\mathrm{e}^{-\beta_0'|p|/2},\quad j=1,2,\ldots.$$

Therefore, the terms on the right hand side of (7.85) can be estimated as

$$\begin{aligned}&\sum_{j=1}^{2}\|(1+|r|)^{\eta'-2}|r|^{-\eta'+1}\beta^j\theta_j(\beta r)s^{2-j}(v,v^\star)\|\\ &+\sum_{j=1}^{3}\|(1+|r|)^{\eta'-2}|r|^{-\eta'+2}\beta^j\theta_j(\beta r)s^{3-j}(v,v^\star)\| \quad \leq \beta^{-\frac{1}{2}}\sum_{n=0}^{2}\|\mathrm{e}^{-\beta_0'|r|/2}|r|^{-\eta'-\frac{3}{2}+n}s^n(v,v^\star)\|.\end{aligned}$$

□

7.9 Pauli-Fierz systems with two thermal reservoirs

Suppose that $\mathcal{Z}_i$, are Hilbert spaces for $i=1,2$. Suppose that h_i are positive self-adjoint operators on $\mathcal{Z}_i$ and $v_i\in B(\mathcal{K},\mathcal{K}\oplus\mathcal{Z}_i)$, for $i=1,2$.

Assume that $\beta_i\in]0,\infty]$, $i=1,2$. Consider the systems described by h_i, $\rho_{\beta_i}=(\mathrm{e}^{\beta_i h_i}-1)^{-1}$. Then we can define the FGRO for the individual systems, denoted Γ_{i,β_i}.

We can consider the composite system given by $\mathcal{Z}:=\mathcal{Z}_1\oplus\mathcal{Z}_2$, $h:=h_i\oplus h_2$, $\rho:=\rho_{\beta_1}\oplus\rho_{\beta_2}$ and $v=v_1+v_2$. The FGRO for the composite system will be denoted by Γ_{β_1,β_2}. By Theorem 6.15, we clearly have

$$\Gamma_{\beta_1,\beta_2}=\sum_{i=1}^{n}\Gamma_{i,\beta_i}. \tag{7.86}$$

Theorem 7.19 *Let $\beta_1,\beta_2\in]0,\infty[$ and $\beta_1\neq\beta_2$. Let $\eta>\frac{1}{2}$. Let Assumptions 7.B(η)$_{\beta_i}$ and 7.C be satisfied for the systems $i=1,2$. Then* $\mathrm{Ker}\Gamma_{(\beta_1,\beta_2)}=\{0\}$.

Proof. By Theorem 7.9, $\mathrm{Ker}\Gamma^{\mathrm{I}}_{i,\beta_i}$ is spanned by γ_{β_i}. But by (7.86),

$$\mathrm{Ker}\Gamma^{\mathrm{I}}_{\beta_1,\beta_2}=\mathrm{Ker}\Gamma^{\mathrm{I}}_{1,\beta_1}\cap\mathrm{Ker}\Gamma^{\mathrm{I}}_{2,\beta_2}.$$

□

The Liouvillean of the composite system will be denoted L_{β_1,β_2}.

Theorem 7.20 *Let $\beta_1,\beta_2\in]0,\infty[$ and $\beta_1\neq\beta_2$. Let $\eta>1$. Let Assumptions 7.A$_{\beta_i}$, 7.B(η)$_{\beta_i}$ and 7.C be satisfied for the systems $i=1,2$. Then there exists $\lambda_0>0$ such that for $0<|\lambda|<\lambda_0$ we have*

$$\mathrm{sp}_{\mathrm{p}}(L_{\beta_1,\beta_2})=\emptyset,\qquad\qquad \mathrm{sp}_{\mathrm{sc}}(L_{\beta_1,\beta_2})=\emptyset.$$

Proof. We apply Thorem 5.7 and 7.19. □

8 Examples of gluing

In the whole paper the conditions on Pauli-Fierz systems were quite abstract in order to keep the notation as simple as possible and the models as general as possible. Nevertheless, it is not easy to see whether concrete physical systems satisfy the assumptions of these results. In particular, it is difficult to see when the gluing technique of Jakšić-Pillet is applicable.

In this section we will show two examples of systems where one can introduce the gluing in a natural way. Both examples involve massless particles in a Euclidean space. In the absence of interaction such systems "glue well". If the interaction is present and the temperature is positive, the main problem is the infra-red behavior of the interaction. We will see for instance, that in dimension 3, if the interaction behaves at small momenta as $|\xi|^{-\frac{1}{2}}$ and satisfies the appropriate reality conditions (if it involves fields and no conjugate fields), then the positive and negative frequencies glue well. But this is exactly the infrared behavior of QED, as noted in [JP1].

8.1 Massless scalar particles

Let $\Xi = \mathbb{R}^d$ be a Euclidean space (where $\xi \in \Xi$ denotes the momentum). Massless spin 0 particles are described by the 1-particle space $L^2(\Xi),)$ with the 1-particle energy $|\xi|$.

The gluing map is defined as

$$L^2(\Xi) \oplus \overline{L^2(\Xi)} \ni (\Psi_+, \overline{\Psi}_-) \mapsto \Psi \in L^2(\mathbb{R}) \otimes L^2(S^{d-1}), \tag{8.87}$$

$$\Psi(p,\omega) := \begin{cases} p^{\frac{d-1}{2}} \Psi_+(p\omega), & p > 0, \\ (-p)^{\frac{d-1}{2}} \overline{\Psi}_-(-p\omega), & p < 0. \end{cases}$$

(Above, $(p,\omega) \in \mathbb{R} \times S^{d-1}$).

Fix the interaction $\Xi \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$. Consider the Pauli-Fierz Hamiltonian

$$\begin{aligned} H &:= K \otimes 1 + 1 \otimes \int a^*(\xi) a(\xi) |\xi| \, \mathrm{d}\xi \\ &\quad + \lambda \int \big(v(\xi) \otimes a^*(\xi) + v^*(\xi) \otimes a(\xi) \big) \mathrm{d}\xi. \end{aligned}$$

Fix the density $\Xi \ni \xi \mapsto \rho(\xi) \in \mathbb{R}_+$. The semi-Liouvillean at density ρ is given by

$$\begin{aligned} L_\rho^{\mathrm{semi}} &:= K \otimes 1 + 1 \otimes \int \big(|\xi| a_{\mathrm{l}}^*(\xi) a_{\mathrm{l}}(\xi) - \int |\xi| a_{\mathrm{r}}^*(\xi) a_{\mathrm{r}}(\xi) \big) \, \mathrm{d}\xi \\ &\quad + \lambda \int \big((1+\rho(\xi))^{\frac{1}{2}} v(\xi) \otimes a_{\mathrm{l}}^*(\xi) + \rho(\xi)^{\frac{1}{2}} v^*(\xi) \otimes a_{\mathrm{r}}(\xi) \big) \mathrm{d}\xi \\ &\quad + \lambda \int \big((1+\rho(\xi))^{\frac{1}{2}} v^*(\xi) \otimes a_{\mathrm{l}}(\xi) + \rho(\xi)^{\frac{1}{2}} v(\xi) \otimes a_{\mathrm{r}}^*(\xi) \big) \mathrm{d}\xi \\ &= K \otimes 1 + 1 \otimes \int p a^*(p,\omega) a(p,\omega) \, \mathrm{d}p \mathrm{d}\omega \\ &\quad + \lambda \int \big(q_\rho(p,\omega) \otimes a^*(p,\omega) + q_\rho(p,\omega) \otimes a(p,\omega) \big) \, \mathrm{d}p \mathrm{d}\omega, \end{aligned}$$

where

$$q_\rho(p,\omega) := \begin{cases} p^{\frac{d-1}{2}} (1+\rho(p\omega))^{\frac{1}{2}} v(p\omega), & p > 0 \\ (-p)^{\frac{d-1}{2}} \rho(-p\omega)^{\frac{1}{2}} v^*(-p\omega), & p < 0. \end{cases}$$

The space $L^2(\Xi)$ is equipped with a physically motivated conjugation

$$\kappa \Psi(\xi) := \overline{\Psi}(-\xi).$$

Assume now that the interaction is given by κ-real fields (see the end of Subsection 3.4). This means

$$v^*(\xi) = v(-\xi). \tag{8.88}$$

Assume also that the density ρ commutes with the conjugation κ, or in other words

$$\rho(\xi) = \rho(-\xi).$$

Then

$$q_\rho(p,\omega) = \begin{cases} p^{\frac{d-1}{2}}(1+\rho(p\omega))^{\frac{1}{2}}v(p\omega), & p>0 \\ (-p)^{\frac{d-1}{2}}\rho(p\omega)^{\frac{1}{2}}v(p\omega), & p<0. \end{cases}$$

In particular, assume that $v(\xi) = |\xi|^{1-\frac{d}{2}}\tilde{v}(\xi)$ and $\rho(\xi) = (\mathrm{e}^{\beta|\xi|}-1)^{-1}$. Then

$$q_\rho(p,\omega) := \left(\frac{1-\mathrm{e}^{-\beta p}}{p}\right)^{\frac{1}{2}}\tilde{v}(p\omega).$$

Therefore, if $\tilde{v}(\xi)$ is analytic, then $q_\rho(p,\omega)$ is analytic in p.

Remark 8.1 *In relativistic quantum theory κ-real fields are usually called just "fields" and κ-imaginary fields are called "conjugate fields". Only "fields" appear in the interaction of Hamiltonians derived from local Lagrangians. Therefore, (8.88) is satisfied in realistic Hamiltonians derived from models of relativistic quantum field theory, such as the usual Hamiltonian of nonrelativistic QED (see eg [BFS1] or the introduction to [DJ]).*

8.2 Massless vector particles

The case of vector particles is very similar to that of scalar particles.

Suppose that $L^2(\Xi,\Xi)$ denotes the Hilbert space of square integrable vector fields on Ξ. Massless vector particles are described by the 1-particle space $L^2_{\mathrm{tr}}(\Xi,\Xi)$, consisting of transversal vector fields on Ξ, that is

$$L^2_{\mathrm{tr}}(\Xi,\Xi) = \{\Psi \in L^2(\Xi,\Xi) \ : \ \xi\cdot\Psi(\xi) = 0, \ \xi\in\Xi\}.$$

($\xi\cdot\Psi(\xi)$ denotes the scalar product of ξ and $\Psi(\xi)$).

The 1-particle energy is $|\xi|$. The distinguished conjugation can be chosen to be

$$\kappa\Psi(\xi) := \overline{\Psi}(-\xi).$$

Remark 8.2 *Note that $L^2_{\mathrm{tr}}(\mathbb{R}^1,\mathbb{R}^1) = \{0\}$ and $L^2_{\mathrm{tr}}(\mathbb{R}^2,\mathbb{R}^2)$ is naturally isomorphic to $L^2(\mathbb{R}^2)$ by the map*

$$L^2(\mathbb{R}^2) \ni \Psi \mapsto S\Psi \in L^2_{\mathrm{tr}}(\mathbb{R}^2,\mathbb{R}^2)$$

with

$$(S\Psi)_i(\xi) = \sum_j \Psi(\xi)\frac{\xi_j}{|\xi|}\epsilon_{ij}.$$

where $\epsilon_{i,j}$ is the completely antisymmetic tensor such that $\epsilon_{1,2}=1$.

The gluing map is defined as the unitary map

$$L^2_{\mathrm{tr}}(\Xi,\Xi)\oplus\overline{L^2_{\mathrm{tr}}(\Xi,\Xi)} \ni (\Psi_+,\overline{\Psi}_-) \mapsto \Psi \in L^2(\mathbb{R})\otimes L^2(TS^{d-1}),$$

$$\Psi(p,\omega) := \begin{cases} p^{\frac{d-1}{2}}\Psi_+(p\omega), & p>0, \\ (-p)^{\frac{d-1}{2}}\overline{\Psi}_-(-p\omega), & p<0. \end{cases}$$

Here TS^{d-1} denotes the tangent bundle of the $d-1$-dimensional sphere and $L^2(TS^{d-1})$ is the Hilbert space of square integrable vector fields on S^{d-1}.

Then we repeat verbatim the constructions of the spin 0 case.

A Abstract type I factors

According to the W^*-algebraic approach to quantum statistical physics, a quantum system is described by an (abstract) W^*-algebra together with a W^*-dynamics and normal states. These data alone can be used to compute physical quantities. In particular, we need not assume that the W^*-algebra acts on a Hilbert space.

Of course, in practice it is convenient to assume that the W^*-algebra is represented on a Hilbert space and the dynamics is unitarily implemented. In some cases there are several natural representations having their own advantages. One of them will be the standard representation. Often there are other representations that are simpler.

If we consider quantum systems in a finite volume, which we will call confined quantum systems, then it is usually sufficient to describe them by the algebra of all bounded operators on a certain Hilbert space. In the language of the classification of W^*-algebras they are called type I factors. Of course, such algebras are simple-minded and no sophisticated W^*-algebraic approach is needed to study them.

In the appendix we describe how various constructions that we used in our paper look in the case of confined systems. In particular, we will explain the connection between the Pauli-Fierz Hamiltonian H and the Pauli-Fierz Liouvilleans and semi-Liouvilleans.

In this section we describe a type I factor first in its minimal representation and then in its standard representation. There are two different equivalent realizations of the standard representation with a different notation.

A.1 Type I factors—irreducible representation

The space of normal functional on $B(\mathcal{H})$ can be identified with $l^1(\mathcal{H})$ (trace class operators) by the formula

$$\psi(A) = \mathrm{Tr}\rho A, \quad \rho \in l^1(\mathcal{H}), \quad A \in B(\mathcal{H}). \tag{A.89}$$

In particular, states are determined by positive trace one operators, called density matrices. A state given by a density matrix ρ is faithful iff $\mathrm{Ker}\rho = \{0\}$.

If $\tau \in \mathrm{Aut}(B(\mathcal{H}))$, then there exists $W \in U(\mathcal{H})$ such that

$$\tau(A) = WAW^*, \quad A \in B(\mathcal{H}). \tag{A.90}$$

If $\mathbb{R} \ni t \mapsto \tau^t \in \mathrm{Aut}(B(\mathcal{H}))$ is a W^*-dynamics, then there exists a self-adjoint operator H on $\mathcal{H}$ such that

$$\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}, \quad A \in B(\mathcal{H}).$$

A state given by (A.89) is invariant wrt the W^*-dynamics (A.90) iff H commutes with ρ. There exists a (β, τ^t)–KMS state iff $\mathrm{Tr}\mathrm{e}^{-\beta H} < \infty$ and then it has the density matrix $\mathrm{e}^{-\beta H}/\mathrm{Tr}\mathrm{e}^{-\beta H}$.

A.2 Type I factor—standard representation in Hilbert-Schmidt operators

Clearly, the representation of $B(\mathcal{H})$ in $\mathcal{H}$ is not in a standard form. To construct a standard form of $B(\mathcal{H})$, consider the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$, denoted $l^2(\mathcal{H})$, and two injective representations:

$$\begin{aligned} &B(\mathcal{H}) \ni A \mapsto \pi_{\mathrm{l}}(A) \in B(l^2(\mathcal{H})), \quad \pi_{\mathrm{l}}(A)B := AB, \quad B \in l^2(\mathcal{H}); \\ &B(\overline{\mathcal{H}}) \ni \overline{A} \mapsto \pi_{\mathrm{r}}(\overline{A}) \in B(l^2(\mathcal{H})), \quad \pi_{\mathrm{r}}(\overline{A})B := BA^* \quad B \in l^2(\mathcal{H}). \end{aligned} \tag{A.91}$$

Set $J_{\mathcal{H}}B := B^*$, $B \in l^2(\mathcal{H})$. Then

$$J_{\mathcal{H}}\pi_{\mathrm{l}}(A)J_{\mathcal{H}} = \pi_{\mathrm{r}}(\overline{A})$$

and

$$(\pi_l, l^2(\mathcal{H}), J_{\mathcal{H}}, l^2_+(\mathcal{H})))$$

is a standard representation of $B(\mathcal{H})$.

If a state on $B(\mathcal{H})$ is given by a density matrix $\rho \in l^1_+(\mathcal{H})$ then its standard vector representative is $\rho^{\frac{1}{2}} \in l^2_+(\mathcal{H})$. If $\tau \in \mathrm{Aut}(B(\mathcal{H}))$ is determined by $W \in U(\mathcal{H})$, then its standard implementation is $\pi_{\rm l}(W)\pi_r(\overline{W})$. If the W^*-dynamics τ^t is given by a self-adjoint operator H, then its standard Liouvillean is $\pi_{\rm l}(H) - \pi_{\rm r}(\overline{H})$.

A.3 Type I factors—standard representation in $\mathcal{H} \otimes \overline{\mathcal{H}}$

An alternative formalism, which can be used to describe a standard form of type I factors, uses the notion of a conjugate Hilbert space,

Under the identification described in Subsection 3.7 the representations (A.91) become

$$\begin{aligned} B(\mathcal{H}) \ni A &\mapsto A \otimes 1_{\overline{\mathcal{H}}} \in B(\mathcal{H} \otimes \overline{\mathcal{H}}); \\ B(\overline{\mathcal{H}}) \ni \overline{A} &\mapsto 1_{\mathcal{H}} \otimes \overline{A} \in B(\mathcal{H} \otimes \overline{\mathcal{H}}). \end{aligned} \tag{A.92}$$

Note that the standard unitary implementation of the automorphism $\tau(A) = WAW^*$ is then equal $W \otimes \overline{W}$. The standard Liouvillean for $\tau^t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}$ equals $L = H \otimes 1 - 1 \otimes \overline{H}$. The modular conjugation is $J_{\mathcal{H}}$ defined by

$$J_{\mathcal{H}} \Psi_1 \otimes \overline{\Psi_2} := \Psi_2 \otimes \overline{\Psi_1}. \tag{A.93}$$

The positive cone is then equal to

$$(\mathcal{H} \otimes \overline{\mathcal{H}})_+ := \mathrm{Cone}\{\Psi \otimes \overline{\Psi} \ : \ \Psi \in \mathcal{H}\}^{\rm cl}.$$

Remark A.1 *Instead of using the conjugate space $\overline{\mathcal{H}}$, one can fix a conjugation c in $\mathcal{H}$, as in Remark 3.4 and then define a standard representation of $B(\mathcal{H})$ in $\mathcal{H} \otimes \mathcal{H}$.*

B Confined Bose gas

In this section we consider the W^*-algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$, where $\mathcal{Z}$ is a certain Hilbert space. As we will see, under certain conditions on the density ρ, in this case the Araki-Woods representation is simply a special form of the standard representation.

B.1 Confined Bose gas—irreducible representation

In this subsection we consider the algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$ acting simply on $\Gamma_{\rm s}(\mathcal{Z})$. Recall that the W^*-algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$ is generated by the CCR representation

$$\mathcal{Z} \ni z \mapsto W(z) \in U(\Gamma_{\rm s}(\mathcal{Z})).$$

Suppose that ρ is a positive operator on a Hilbert space $\mathcal{Z}$. Set $\gamma := \rho(1+\rho)^{-1}$.

Theorem B.1

$$\mathrm{Tr}\Gamma(\gamma) = \det(1-\gamma)^{-1} \tag{B.94}$$

Therefore (B.94) is finite iff

$$\mathrm{Tr}\gamma < \infty, \ \textit{or equivalently,} \ \mathrm{Tr}\rho < \infty. \tag{B.95}$$

Throughout this section we will suppose that (B.95) is true. Define the state ω_ρ on the W^*-algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$ given by the density matrix

$$\Gamma(\gamma)/{\rm Tr}\Gamma(\gamma).$$

Let h be another self-adjoint operator on $\mathcal{Z}$. Define the dynamics on $B(\Gamma_{\rm s}(\mathcal{Z}))$:

$$\tau^t(A) := {\rm e}^{{\rm i}t{\rm d}\Gamma(h)} A {\rm e}^{-{\rm i}t{\rm d}\Gamma(h)} \qquad A \in B(\Gamma_{\rm s}(\mathcal{Z})).$$

Clearly, ω_ρ is τ^t-invariant iff h commutes with ρ.

The state ω_ρ is (β, τ^t)-KMS iff

$$\gamma = {\rm e}^{-\beta h}.$$

B.2 Confined Bose Gas—standard representation

Using at the last step the exponential map, we have the identification

$$l^2(\Gamma_{\rm s}(\mathcal{Z})) \simeq \Gamma_{\rm s}(\mathcal{Z}) \otimes \overline{\Gamma_{\rm s}(\mathcal{Z})} \simeq \Gamma_{\rm s}(\mathcal{Z}) \otimes \Gamma(\overline{\mathcal{Z}}) \simeq \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \tag{B.96}$$

The W^*-algebra $B(\Gamma_{\rm s}(\mathcal{Z}))$ has a standard representation in in the Hilbert space $l^2(\Gamma_{\rm s}(\mathcal{Z}))$, as described in Subsection A.2. It has also a standard representation in the Hilbert space $\Gamma_{\rm s}(\mathcal{Z}) \otimes \overline{\Gamma_{\rm s}(\mathcal{Z})}$, as described in Subsection A.3. Using the identification (B.96) we can use the space $\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ to get a standard representation. (Of course, all these representations are naturally unitarily equivalent).

Let us describe the last representation in detail. Let

$$U : \Gamma_{\rm s}(\mathcal{Z}) \otimes \Gamma_{\rm s}(\overline{\mathcal{Z}}) \to \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \tag{B.97}$$

be the unitary map defined as in (4.33). Define the representations of W^*-algebras

$$\begin{aligned} B(\Gamma_{\rm s}(\mathcal{Z})) \ni A \quad &\mapsto \pi_{\rm l}(A) := U\, A \otimes 1_{\Gamma_{\rm s}(\overline{\mathcal{Z}})}\, U^* \in B(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})), \\ \overline{B(\Gamma_{\rm s}(\mathcal{Z}))} \simeq B(\overline{\Gamma_{\rm s}(\mathcal{Z})}) \simeq B(\Gamma_{\rm s}(\overline{\mathcal{Z}})) \ni \overline{A} \quad &\mapsto \pi_{\rm r}(\overline{A}) := U\, 1_{\Gamma_{\rm s}(\mathcal{Z})} \otimes \overline{A}\, U^* \in B(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})). \end{aligned} \tag{B.98}$$

Consider the representation of CCR

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z_2}) \mapsto W(z_1, \overline{z_2}) \in U(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})). \tag{B.99}$$

It is convenient to split (B.99) into 2 commuting representations of CCR

$$\mathcal{Z} \ni z \mapsto W_{\rm l}(z) := W(z, 0) = \pi_{\rm l}(W(z)) \in U(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})), \tag{B.100}$$

$$\overline{\mathcal{Z}} \ni \overline{z} \mapsto W_{\rm r}(\overline{z}) := W(0, \overline{z}) = \pi_{\rm r}(\overline{W(z)}) \in U(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})). \tag{B.101}$$

The creation/annihillation operators corresponding to (B.100) will be denoted $z(a_{\rm l}^*)$, $\overline{z}(a_{\rm l})$; The creation/annihillation operators corresponding to (B.101) will be denoted $\overline{z}(a_{\rm r}^*)$, $z(a_{\rm r})$.

The algebra $\pi_{\rm l}(B(\Gamma_{\rm s}(\mathcal{Z})))$ is generated by $W_{\rm l}(z)$, $z \in \mathcal{Z}$ and the algebra $\pi_{\rm r}(B(\Gamma_{\rm s}(\overline{\mathcal{Z}})))$ is generated by $W_{\rm r}(\overline{z})$, $z \in \mathcal{Z}$.

Let ϵ be defined as in (4.37) and $\Gamma_{{\rm s},+}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ in 4.40.

Theorem B.2 *1)* $\Big(\pi_{\rm l}, \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), \Gamma(\epsilon), \Gamma_{{\rm s},+}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\Big)$ *is a standard representation of* $B(\Gamma_{\rm s}(\mathcal{Z}))$. *Moreover, if* $1_{\mathcal{Z}}$ *denotes the projection onto* $\mathcal{Z}$, *then*

$$\pi_{\rm l}(B(\Gamma_{\rm s}(\mathcal{Z}))) = \{A \in B(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})) \ : \ A = \Gamma(1_{\mathcal{Z}}) A \Gamma(1_{\mathcal{Z}})\},$$

$$\Gamma(\epsilon)\pi_{\rm l}(A)\Gamma(\epsilon) = \pi_{\rm r}(\overline{A}).$$

2) Let $r := h \oplus (-\overline{h})$. *Then* ${\rm d}\Gamma(r)$ *is the standard Liouvillean of* τ^t.

3) Assume (B.95). Then the standard vector representative of ω_ρ *is*

$$\Omega_\rho := \det(1-\gamma)^{\frac{1}{2}} \exp(\tfrac{1}{2} a_{\rm l}^* \overline{\gamma}^{\frac{1}{2}} a_{\rm r}^* + \tfrac{1}{2} a_{\rm r}^* \gamma^{\frac{1}{2}} a_{\rm l}^*)\Omega. \tag{B.102}$$

Proof. 1) and 2) are straightforward. Let us prove (B.102).

In the representation of Subsection A.2, the standard vector representative of ω_ρ equals

$$(\mathrm{Tr}\Gamma(\gamma))^{-\frac{1}{2}}\Gamma(\gamma^{\frac{1}{2}}).$$

Recall that $(\mathrm{Tr}\Gamma(\gamma))^{-\frac{1}{2}} = \det(1-\gamma)^{\frac{1}{2}}$.

We have

$$\Gamma(\gamma^{\frac{1}{2}}) = \sum_{n=0}^{\infty}(\gamma^{\frac{1}{2}})^{\otimes_{\mathrm{s}}n}.$$

Now in the representation that we use in this subsection $(\gamma^{\frac{1}{2}})^{\otimes_{\mathrm{s}}n}$ corresponds to

$$(\sqrt{n!})^{-2}(a_{\mathrm{r}}^*\gamma^{\frac{1}{2}}a_{\mathrm{l}}^*)^n\Omega = (n!)^{-1}(\tfrac{1}{2}a_{\mathrm{l}}^*\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{r}}^* + \tfrac{1}{2}a_{\mathrm{r}}^*\gamma^{\frac{1}{2}}a_{\mathrm{l}}^*)^n\Omega.$$

Hence $\Gamma(\gamma^{\frac{1}{2}})$ corresponds to

$$\exp(\tfrac{1}{2}a_{\mathrm{l}}^*\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{r}}^* + \tfrac{1}{2}a_{\mathrm{r}}^*\gamma^{\frac{1}{2}}a_{\mathrm{l}}^*)\Omega.$$

Remark B.3 *In (B.102), the notation of (4.35) is used. In fact,*

$$c = \begin{bmatrix} 0 & \gamma^{\frac{1}{2}} \\ \overline{\gamma}^{\frac{1}{2}} & 0 \end{bmatrix} \tag{B.103}$$

is a map $c : \overline{\mathcal{Z}} \oplus \mathcal{Z} \to \mathcal{Z} \oplus \overline{\mathcal{Z}}$ satisfying $\overline{c}^* = c$ and

$$cc^* = \begin{bmatrix} 0 & \gamma^{\frac{1}{2}} \\ \overline{\gamma}^{\frac{1}{2}} & 0 \end{bmatrix}\begin{bmatrix} 0 & \overline{\gamma}^{\frac{1}{2}} \\ \gamma^{\frac{1}{2}} & 0 \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{bmatrix}.$$

Therefore,

$$\det(1-cc^*)^{\frac{1}{2}} = \det(1-\gamma).$$

Thus the vector Ω_ρ is an example of a squeezed state considered in (4.35), that is

$$\Omega_\rho = \det(1-cc^*)^{\frac{1}{4}}\exp(\tfrac{1}{2}(a_{\mathrm{l}}^*, a_{\mathrm{r}}^*)c(a_{\mathrm{l}}^*, a_{\mathrm{r}}^*))\Omega.$$

B.3 Confined Bose gas—standard representation in the Araki-Woods form

Define the following transformation on $\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$:

$$R_\rho := \exp(-\tfrac{1}{2}a_{\mathrm{l}}^*\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{r}}^* - \tfrac{1}{2}a_{\mathrm{r}}^*\gamma^{\frac{1}{2}}a_{\mathrm{l}}^*)\det(1-\gamma)^{\frac{1}{2}}\Gamma(1-\gamma\oplus 1-\overline{\gamma})^{\frac{1}{2}}\exp(\tfrac{1}{2}a_{\mathrm{l}}\gamma^{\frac{1}{2}}a_{\mathrm{r}} + \tfrac{1}{2}a_{\mathrm{r}}\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{l}}).$$

Theorem B.4 *R_ρ is a unitary operator satisfying*

$$R_\rho\Omega_\rho = \Omega, \tag{B.104}$$

$$R_\rho \mathrm{e}^{\mathrm{i}d\Gamma(r)}R_\rho^* = \mathrm{e}^{\mathrm{i}d\Gamma(r)}, \tag{B.105}$$

$$R_\rho\Gamma(\epsilon)R_\rho^* = \Gamma(\epsilon), \tag{B.106}$$

$$R_\rho W(z_1, \overline{z_2})R_\rho^* \quad = W((1+\rho)^{\frac{1}{2}}z_1 + \rho^{\frac{1}{2}}z_2, \overline{\rho}^{\frac{1}{2}}\overline{z_1} + (1+\overline{\rho})^{\frac{1}{2}}\overline{z_2}). \tag{B.107}$$

$$R_\rho\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}}) = \Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}}), \tag{B.108}$$

Proof. Let c be defined as in B.103. Using Remark B.3 and

$$\Gamma(1-cc^*)=\Gamma(1-\gamma\oplus 1-\overline{\gamma}),$$

we see that

$$R_\rho:=\det(1-cc^*)^{\frac{1}{4}}\exp(-\tfrac{1}{2}(a_{\mathrm{l}}^*,a_{\mathrm{r}}^*)c(a_{\mathrm{l}}^*,a_{\mathrm{r}}^*))\Gamma(1-cc^*)^{\frac{1}{2}}\exp(\tfrac{1}{2}ac^*a).$$

Thus R_ρ is in fact the transformation R_c considered in (4.35). Therefore, (B.104) follows from Theorem 4.1 2). (B.105) and (B.106) follow from Theorem 4.1 5). (B.107) follows from Theorem 4.1 4) using

$$R_\rho W(z_1,\overline{z_2})R_\rho^* \;=W\big((1-\gamma)^{-\frac{1}{2}}z_1+(1-\gamma)^{-\frac{1}{2}}\gamma^{\frac{1}{2}}z_2,(1-\overline{\gamma})^{-\frac{1}{2}}\overline{\gamma}^{\frac{1}{2}}\overline{z}_1,(1-\overline{\gamma})^{-\frac{1}{2}}\overline{z}_2\big).$$

Let us prove (B.108). We check that

$$\tfrac{1}{2}a_{\mathrm{l}}^*\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{r}}^*-\tfrac{1}{2}a_{\mathrm{r}}^*\gamma^{\frac{1}{2}}a_{\mathrm{l}}^*,$$

$$\Gamma(1-\gamma\oplus 1-\overline{\gamma})^{\frac{1}{2}},$$

$$\tfrac{1}{2}a_{\mathrm{l}}\gamma^{\frac{1}{2}}a_{\mathrm{r}}+\tfrac{1}{2}a_{\mathrm{r}}\overline{\gamma}^{\frac{1}{2}}a_{\mathrm{l}}$$

map $\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\cap\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ into $\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. Therefore, R_ρ maps $\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ into itself.

A similar argument shows that R_ρ^* maps $\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ into itself. This proves (B.108).□

For $A\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ set

$$\begin{aligned}\theta_{\rho,\mathrm{l}}(A)&:=R_\rho\pi_{\mathrm{l}}(A)R_\rho^*=R_\rho UA\otimes 1_{\Gamma_{\mathrm{s}}(\overline{\mathcal{Z}})}U^*R_\rho^*\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})),\\ \theta_{\rho,\mathrm{r}}(\overline{A})&:=R_\rho\pi_{\mathrm{r}}(\overline{A})R_\rho^*=R_\rho U1_{\Gamma_{\mathrm{s}}(\mathcal{Z})}\otimes\overline{A}U^*R_\rho^*\in B(\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}})),\end{aligned}\tag{B.109}$$

where π_{l}, π_{r} were defined in (B.98) and U in (B.97).

Theorem B.5 *1) R_ρ intertwines between the W_{l}, W_{r} and the Araki-Woods representations:*

$$\theta_{\rho,\mathrm{l}}(W(z))=R_\rho W_{\mathrm{l}}(z)R_\rho^*=W_{\rho,\mathrm{l}}(z)=W((1+\rho)^{\frac{1}{2}}z,\overline{\rho}^{\frac{1}{2}}\overline{z}),$$

$$\theta_{\rho,\mathrm{r}}(W(\overline{z}))=R_\rho W_{\mathrm{r}}(\overline{z})R_\rho^*=W_{\rho,\mathrm{r}}(\overline{z})=W(\rho^{\frac{1}{2}}z,(1+\overline{\rho})^{\frac{1}{2}}\overline{z}).$$

2) $\Big(\theta_{\rho,\mathrm{l}},\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}),\Gamma(\epsilon),\Gamma_{\mathrm{s},+}(\mathcal{Z}\oplus\overline{\mathcal{Z}})\Big)$ is a standard representation of $B(\Gamma_{\mathrm{s}}(\mathcal{Z}))$ and

$$\theta_{\rho,\mathrm{l}}(B(\Gamma_{\mathrm{s}}(\mathcal{Z})))=\mathfrak{M}_{\rho,\mathrm{l}},$$

$$\Gamma(\epsilon)\theta_{\rho,\mathrm{l}}(A)\Gamma(\epsilon)=\theta_{\rho,\mathrm{r}}(\overline{A}).$$

3) $\mathrm{d}\Gamma(r)$ is the standard Liouvillean of τ^t.
4) Ω is the standard vector representative of ω_ρ.

C Confined Pauli-Fierz systems

C.1 Confined Pauli-Fierz systems—irreducible representation

We use the assumptions and notation of the Subsection 5.2. We treat $B(\mathcal{K}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}))$ as the W^*-algebra describing our system. We suppose Assumption 6.A.

Recall that the Pauli-Fierz Hamiltonian is defined as the operator on $\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})$

$$H := K\otimes 1 + 1\otimes \mathrm{d}\Gamma(h) + \lambda\big(v(a^*)+v^*(a)\big),$$

for K a self-adjoint operator on $\mathcal{K}$, $v\in B(\mathcal{K},\mathcal{K}\otimes\mathcal{Z})$ and h an positive operator on $\mathcal{Z}$. Recall that then H is self-adjoint and generates a W^*-dynamics on $B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))$:

$$\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH},\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})). \tag{C.110}$$

Definition C.1 $\big(B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})),\, \mathrm{e}^{\mathrm{i}tH}\cdot\mathrm{e}^{-\mathrm{i}tH}\big)$ *will be called a confined Pauli-Fierz W^*-dynamical system.*

Let us fix a positive operator ρ on $\mathcal{Z}$ commuting with h and satisfying the condition (B.95). We are going to see how the confined Pauli-Fierz W^*-dynamical system looks in 2 different representations of $B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))$, which describe the bosons at the density ρ, and use the Araki-Woods representation $\theta_{\rho,\mathrm{l}} : B(\Gamma_{\rm s}(\mathcal{Z}))\to B(\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ introduced in B.109.

C.2 Confined Pauli-Fierz system—semi-standard representation

We can use the identity representation for $B(\mathcal{K})$ and the Araki-Woods representation for $B(\Gamma_{\rm s}(\mathcal{Z}))$. Thus we obtain the representation

$$\theta^{\rm semi}_{\rho,\mathrm{l}} := 1_{B(\mathcal{K})}\otimes\theta_{\rho,\mathrm{l}} : B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))\to B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})),$$

which will be called the semistandard representation of $B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))$. In other words, $\theta^{\rm semi}_{\rho,\mathrm{l}}$ is defined by

$$\theta^{\rm semi}_{\rho,\mathrm{l}}(A) = 1_{\mathcal{K}}\otimes R_\rho U\ A\otimes 1_{\Gamma_{\rm s}(\overline{\mathcal{Z}})}\ 1_{\mathcal{K}}\otimes U^*R_\rho^*,\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})),$$

where U was defined in (B.97).

Clearly

$$\theta^{\rm semi}_{\rho,\mathrm{l}}\big(B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})\big) = B(\mathcal{K})\otimes\mathfrak{M}_{\rho,\mathrm{l}}.$$

Let $L^{\rm semi}_\rho$ be defined as in (6.56), that is

$$\begin{aligned} L^{\rm semi}_\rho \quad &:= K\otimes 1 + 1\otimes\mathrm{d}\Gamma(r)\\ &\quad +\lambda\Big((1+\rho)^{\frac12}v, \overline{\rho}^{\frac12}v^\star\Big)(a^*) + \lambda\Big(v^*(1+\rho)^{\frac12}, v^{\star *}\overline{\rho}^{\frac12}\Big)(a).\end{aligned}$$

The following idenities show the relationship between the Pauli-Fierz Hamiltonian and the Pauli-Fierz semi-Liouvillean in the confined case. Let us stress that the semi-Liouvilleans $L^{\rm semi}_\rho$ are unitarily equivalent for different ρ.

Theorem C.2

$$\theta^{\rm semi}_{\rho,\mathrm{l}}\Big(\mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}\Big) = \mathrm{e}^{\mathrm{i}tL^{\rm semi}_\rho}\theta^{\rm semi}_{\rho,\mathrm{l}}(A)\mathrm{e}^{-\mathrm{i}tL^{\rm semi}_\rho},\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})),$$

$$L^{\rm semi}_\rho = 1_{\mathcal{K}}\otimes R_\rho U\,\big(H\otimes 1_{\Gamma_{\rm s}(\overline{\mathcal{Z}})} - 1_{\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})}\otimes\mathrm{d}\Gamma(h)\big)\,1_{\mathcal{K}}\otimes U^*R_\rho^*.$$

C.3 Confined Pauli-Fierz system—standard representation

We can use the standard representation for $B(\mathcal{K})$ in $B(\mathcal{K}\otimes\overline{\mathcal{K}})$ and the Araki-Woods representation for $B(\Gamma_{\rm s}(\mathcal{Z}))$. Thus we obtain the representation

$$\theta_{\rho,{\rm l}} : B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))\to B(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})).$$

defined by

$$\theta_{\rho,{\rm l}}(A) := 1_{\overline{\mathcal{K}}}\check{\otimes}\theta^{\rm semi}_{\rho,{\rm l}}(A),\qquad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})).$$

In other words, $\theta_{\rho,{\rm l}}$ is defined by

$$\theta_{\rho,{\rm l}}(A_1\otimes A_2) = A_1\otimes 1_{\overline{\mathcal{K}}}\otimes\theta_{\rho,{\rm l}}(A_2),\quad A_1\in B(\mathcal{K}),\quad A_2\in B(\Gamma_{\rm s}(\mathcal{Z})).$$

One can put it in a yet different way. Introduce the obvious unitary identification

$$\tilde{U} : \mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\overline{\mathcal{Z}})\to\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma(\mathcal{Z}\oplus\overline{\mathcal{Z}}).$$

Then

$$\theta_{\rho,{\rm l}}(A) = 1_{\mathcal{K}\otimes\overline{\mathcal{K}}}\otimes R_\rho\ \tilde{U}\ A\otimes 1_{\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\overline{\mathcal{Z}})}\ \tilde{U}^*\ 1_{\mathcal{K}\otimes\overline{\mathcal{K}}}\otimes R_\rho^*,\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})).$$

Clearly

$$\theta_{\rho,{\rm l}}(B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z}))) = B(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\rho,{\rm l}}.$$

Let L_ρ be defined as in (6.59), that is

$$\begin{aligned} L_\rho\ &:= K\otimes 1_{\overline{\mathcal{K}}}\otimes 1 - 1_{\mathcal{K}}\otimes\overline{K}\otimes 1 + 1_{\mathcal{K}}\otimes 1_{\overline{\mathcal{K}}}\otimes{\rm d}\Gamma(r)\\ &+\lambda 1_{\overline{\mathcal{K}}}\check{\otimes}\Big((1+\rho)^{\frac12}v,\overline{\rho}^{\frac12}v^\star\Big)(a^*) + \lambda 1_{\overline{\mathcal{K}}}\check{\otimes}\Big(v^*(1+\rho)^{\frac12},v^{\star *}\overline{\rho}^{\frac12}\Big)(a),\\ &-\lambda 1_{\mathcal{K}}\otimes\Big(\rho^{\frac12}\overline{v}^\star,(1+\overline{\rho})^{\frac12}\overline{v}\Big)(a^*) - \lambda 1_{\mathcal{K}}\otimes\Big(\overline{v}^{\star *}\rho^{\frac12},\overline{v}^*(1+\overline{\rho})^{\frac12}\Big)(a). \end{aligned}$$

The following identities give the relationship between the Pauli-Fierz Hamiltonian and the Pauli-Fierz Liouvillean in the confined case. Let us stress that the Liouvilleans L_ρ are unitarily equivalent for different ρ.

Theorem C.3

$$\theta_{\rho,{\rm l}}\Big({\rm e}^{{\rm i}tH}A{\rm e}^{-{\rm i}tH}\Big) = {\rm e}^{{\rm i}tL_\rho}\theta_{\rho,{\rm l}}(A){\rm e}^{-{\rm i}tL_\rho},\quad A\in B(\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})).$$

$$L_\rho = 1_{\mathcal{K}\otimes\overline{\mathcal{K}}}\otimes R_\rho\ \tilde{U}\ \Big(H\otimes 1_{\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(\overline{\mathcal{Z}})} - 1_{\mathcal{K}\otimes\Gamma_{\rm s}(\mathcal{Z})}\otimes\overline{H}\Big)\ \tilde{U}^*\ 1_{\mathcal{K}\otimes\overline{\mathcal{K}}}\otimes R_\rho^*.$$

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