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Nonperturbative Dynamics of Noncommutative Gauge Theory

J. Ambjørn[1)*], Y.M. Makeenko[1) 2)†], J. Nishimura[1)‡] and R.J. Szabo[1)§]

[1)] *The Niels Bohr Institute*
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

[2)] *Institute of Theoretical and Experimental Physics*
B. Cheremushkinskaya 25, 117218 Moscow, Russia

Abstract

We present a nonperturbative lattice formulation of noncommutative Yang-Mills theories in arbitrary even dimension. We show that lattice regularization of a noncommutative field theory requires finite lattice volume which automatically provides both an ultraviolet and an infrared cutoff. We demonstrate explicitly Morita equivalence of commutative U(p) gauge theory with $p \cdot n_f$ flavours of fundamental matter fields on a lattice of size L with twisted boundary conditions and noncommutative U(1) gauge theory with n_f species of matter on a lattice of size $p \cdot L$ with single-valued fields. We discuss the relation with twisted large N reduced models and construct observables in noncommutative gauge theory with matter.

*e-mail address : ambjorn@nbi.dk
†e-mail address : makeenko@nbi.dk
‡Permanent address : Department of Physics, Nagoya University, Nagoya 464-8602, Japan, e-mail address : nisimura@nbi.dk
§e-mail address : szabo@nbi.dk

1 Introduction

It has been suggested for some time that noncommutative geometry is a natural framework to describe nonperturbative string theory. This belief has been supported by the fact that Matrix Theory [1] or the IIB matrix model [2], which are conjectured to provide nonperturbative definitions of string theories, give rise to noncommutative Yang-Mills theory on a toroidal compactification [3]. The particular toroidal compactification can be interpreted in terms of the presence of a background B field [4], which can also be understood in the context of open string quantization [5]. Noncommutative gauge theories possess a number of important properties inherent from noncommutative geometry, where there is a remarkable geometric equivalence relation on certain classes of noncommutative spaces known as Morita equivalence. In noncommutative Yang-Mills theory, this implies a duality between gauge theories over different noncommutative tori, for example, which relates a Yang-Mills theory with background magnetic flux to a gauge theory with gauge group of lower rank and no background flux. It allows one to interpolate continuously, through noncommutative Yang-Mills theories, between two ordinary Yang-Mills theories with gauge groups of different rank and appropriate background magnetic fluxes. Furthermore, in certain instances, there is the remarkable fact [6] that the non-abelian nature of a gauge group can be absorbed into the noncommutativity of spacetime by mapping a U(p) gauge theory with multi-valued gauge fields to a U(1) gauge theory with single-valued fields on a dual noncommutative torus.

While most of the results concerning Morita equivalence are obtained at the classical level, there are strong indications that it persists in regularized perturbation theory. In particular, the one-loop divergences coincide [7] for noncommutative gauge theory and ordinary commutative Yang-Mills theory on $\mathbb{R}^4$ after a proper rescaling of the coupling constant. Moreover, it can be shown [8, 9, 10] that noncommutative gauge theory is equivalent to all orders of perturbation theory to a twisted large N reduced model [11, 12]. Following this technique, gauge-invariant observables for noncommutative Yang-Mills theory have been constructed. A surprising new class of observables, which are expressed in terms of open Wilson loops, exists in noncommutative gauge theory [9, 13] in addition to those expressed via a counterpart of the standard closed Wilson loops of the reduced model.

Lattice gauge theories [14] are a standard tool for nonperturbative investigations of non-supersymmetric gauge theories. In this Letter we will apply a lattice formulation of noncommutative gauge theory to study its nonperturbative properties and, in particular, Morita equivalence. The construction is an extension of a previous work [13] in which a unified framework was developed which naturally interpolates between the Matrix theory compactification and the twisted large N reduced model versions of noncommutative Yang-Mills theory. It was shown that a finite N matrix model obtained from a constrained twisted Eguchi-Kawai model is equivalent to a manifestly star-gauge invariant formulation of noncommutative U(1) gauge theory on a lattice of finite extent. In the following we will further investigate the properties of noncommutative Yang-Mills theories in arbitrary even dimension, and in particular reconsider the lattice formulation of noncommutative gauge theory from a general point of view, without specifying a particular representation of the noncommutative algebra. Because the lattice provides a regularization of the theory, all the results obtained are rigorous in this sense. We will show that in noncommutative geometry, when one discretizes

spacetime, consistency of the algebra requires spacetime to be compactified as well, thereby automatically providing an infrared cutoff. This is similar to the correspondence between infrared and ultraviolet in noncommutative field theories discovered [15] in perturbation theory. We will also explicitly construct a map between the fields in commutative U(p) gauge theory on a lattice of size L with twisted boundary conditions (representing the existence of a 't Hooft flux) and noncommutative U(1) gauge theory on a lattice of size $p \cdot L$ with periodic boundary conditions, thereby demonstrating their Morita equivalence.

The technique introduced also allows us to study properties of noncommutative gauge theory with matter. We will consider the coupling to matter fields in the fundamental representation of the gauge group and explicitly demonstrate Morita equivalence between commutative U(p) gauge theory with $p \cdot n_f$ different flavours of matter on a lattice of size L with twisted boundary conditions and noncommutative U(1) gauge theory with n_f flavours of matter on a lattice of size $p \cdot L$ with periodic boundary conditions. In the simplest case of $L = 1$ we recover the reduced model introduced in [16]. We discuss observables represented by the matter fields and rewrite them in terms of Wilson loops by integrating out the matter. This is the analog of the original construction [14] of observables of ordinary lattice gauge theory from matter field averages. We will construct, along these lines, observables which correspond to open Wilson loops and emphasize their role in the nonperturbative dynamics of noncommutative gauge theory.

2 Noncommutative lattice gauge theory

On a noncommutative space of dimension D, the local coordinates x_μ are replaced by hermitian operators $\hat{x}_\mu$ obeying the commutation relations $[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}$, where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are (dimensionful) real-valued c-numbers. To describe a lattice discretization, we restrict the spacetime points to $x_\mu \in \epsilon\mathbb{Z}$, where ϵ is the lattice spacing. The lattice momentum then has the periodicity $k_\mu \to k_\mu + \frac{2\pi}{\epsilon}\delta_{\mu\nu}$, $\nu = 1, \ldots, D$, which implies an identity of operators such as $\mathrm{e}^{i(k_\mu + 2\pi\delta_{\mu\nu}/\epsilon)\hat{x}_\mu} = \mathrm{e}^{ik_\mu\hat{x}_\mu}$. Acting by $\mathrm{e}^{-ik_\mu\hat{x}_\mu}$ on both sides, we conclude that $\mathrm{e}^{2\pi i\hat{x}_\mu/\epsilon} = \mathbb{1}$ and the momentum k_μ should be restricted as

$$\theta_{\mu\nu}k_\nu \in 2\epsilon\mathbb{Z} \tag{2.1}$$

with $\frac{\pi}{\epsilon^2}\,\theta_{\mu\nu}$ a $D \times D$ integer-valued matrix. Note that the restriction (2.1) on the momentum simply disappears in the commutative case $\theta_{\mu\nu} = 0$, and is quite characteristic of the noncommutative geometry.

The discretization (2.1) of the momentum implies that the lattice is periodic, $x_\mu \sim x_\mu + \Sigma_{\mu\nu}$, $\nu = 1, \ldots, D$, with period matrix Σ given by

$$M_{\mu\lambda}\Sigma_{\nu\lambda} = \frac{\pi}{\epsilon}\,\theta_{\mu\nu}\ , \tag{2.2}$$

where $M_{\mu\nu}$ is a $D \times D$ integer-valued matrix. Thus, we have found that lattice regularization of noncommutative field theory inevitably requires the lattice to be compact as well. In the continuum limit $\epsilon \to 0$, the infrared cutoff goes away as $\frac{1}{\epsilon}$. For any given periodicity $\Sigma_{\mu\nu}$ and noncommutativity $\theta_{\mu\nu}$ in the continuum, one can construct a series of lattice theories

satisfying the restriction (2.2) and approaching the target continuum theory in the $\epsilon \to 0$ limit.

Lattice fields $\phi_i(x)$ on the D dimensional noncommutative spacetime are now replaced by finite dimensional operators $\hat{\phi}_i$. Their coordinate-space representation $\phi_i(x)$ can be obtained using a map

$$\hat{\phi}_i = \sum_x \hat{\Delta}(x)\, \phi_i(x) \tag{2.3}$$

with

$$\hat{\Delta}(x) = \frac{1}{|\det \frac{1}{\epsilon}\,\Sigma|} \sum_{\vec{m}} \left(\prod_{\mu=1}^{D} \left(\hat{Z}_\mu \right)^{m_\mu} \right) \ \mathrm{e}^{\pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu} \ \mathrm{e}^{-2\pi i (\Sigma^{-1})_{\mu\nu} m_\mu x_\nu} = \hat{\Delta}(x)^\dagger \ , \tag{2.4}$$

where the sum in (2.3) runs over all lattice points modulo the lattice periodicity, and the sum in (2.4) goes over all integers m_μ modulo the periodicity $\frac{1}{\epsilon}\,\Sigma_{\nu\mu}$. The operators

$$\hat{Z}_\mu = \mathrm{e}^{2\pi i (\Sigma^{-1})_{\mu\nu} \hat{x}_\nu} \tag{2.5}$$

satisfy the commutation relations

$$\hat{Z}_\mu \hat{Z}_\nu = \mathrm{e}^{-2\pi i \Theta_{\mu\nu}} \ \hat{Z}_\nu \hat{Z}_\mu \ , \tag{2.6}$$

where the dimensionless noncommutativity parameter

$$\Theta_{\mu\nu} = 2\pi (\Sigma^{-1})_{\mu\lambda} \theta_{\lambda\rho} (\Sigma^{-1})_{\nu\rho} \tag{2.7}$$

is necessarily rational-valued on the lattice, since the restriction (2.2) implies that

$$M_{\mu\nu} = \frac{1}{2\epsilon} \Sigma_{\mu\lambda} \Theta_{\lambda\nu} \tag{2.8}$$

should be an integer-valued matrix. The normalization of the trace of operators is fixed by $\mathrm{Tr}\,\hat{\Delta}(x) = 1$. The collection of operators $\hat{\Delta}(x)$ form an orthonormal set and the inverse of the map (2.3) is given by $\phi_i(x) = \mathrm{Tr}\,(\hat{\phi}_i\, \hat{\Delta}(x))$. The product $\hat{\phi}_1 \hat{\phi}_2$ of two operators has coordinate space representation given by the lattice star-product

$$\mathrm{Tr}\left(\hat{\phi}_1 \hat{\phi}_2\, \hat{\Delta}(x) \right) = \frac{1}{|\det \frac{1}{\epsilon}\,\Sigma|} \sum_{y,z} \phi_1(y)\, \phi_2(z)\, \mathrm{e}^{-2i(\theta^{-1})_{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)} \stackrel{\mathrm{def}}{=} \phi_1(x) \star \phi_2(x) \tag{2.9}$$

where we have assumed that $(M^{-1})_{\mu\nu}$ is an integer-valued matrix[1].

In order to construct a lattice formulation of noncommutative Yang-Mills theory, we need to maintain star-gauge invariance on the lattice. As in the case of ordinary lattice gauge theory [14], this is achieved by putting the $\mathrm{U}(n)$ gauge fields on the links of the lattice [13]. This determines a unitary operator

$$\hat{U}_\mu = \sum_x \hat{\Delta}(x) \otimes U_\mu(x) \ , \tag{2.10}$$

[1] This formula is similar to Ref. [17]. See also [18] for earlier works in this regard.

where $U_\mu(x)$ is an $n \times n$ matrix field on the lattice which is star-unitary, $U_\mu(x) \star U_\mu(x)^\dagger = 1\!\!1_n$. The lattice action is

$$S = -\frac{1}{g^2} \sum_x \sum_{\mu \neq \nu} \mathrm{tr}_{(n)} \Big[U_\mu(x) \star U_\nu(x + \epsilon\hat{\mu}) \star U_\mu(x + \epsilon\hat{\nu})^\dagger \star U_\nu(x)^\dagger \Big] \ , \tag{2.11}$$

and it is invariant under the lattice star-gauge transformation

$$U_\mu(x) \mapsto g(x) \star U_\mu(x) \star g(x + \epsilon\hat{\mu})^\dagger \ , \tag{2.12}$$

where the gauge function $g(x)$ is star-unitary, $g(x) \star g(x)^\dagger = 1\!\!1_n$. The lattice has finite extent, as is already discussed, and the fields $U_\mu(x)$ are single-valued, $U_\mu(x + \Sigma_{\alpha\nu}\hat{\alpha}) = U_\mu(x)$.

3 Lattice Morita equivalence

We will now demonstrate that the noncommutative Yang-Mills theory with action (2.11) is Morita equivalent to a commutative Yang-Mills theory on a lattice of different size and with a different gauge group, thereby generalizing previous results [8, 13] for the twisted Eguchi-Kawai model. We start with a *commutative* U(p) lattice gauge theory with 't Hooft fluxes. The action is

$$S = -\frac{1}{\tilde{g}^2} \sum_x \sum_{\mu \neq \nu} \mathrm{tr}_{(p)} \Big[U_\mu(x)\, U_\nu(x + \epsilon\hat{\mu})\, U_\mu(x + \epsilon\hat{\nu})^\dagger\, U_\nu(x)^\dagger \Big] \ , \tag{3.1}$$

where $U_\mu(x)$ are U(p) gauge fields satisfying the twisted boundary conditions

$$U_\mu(x + \Sigma_{\alpha\nu}\hat{\alpha}) = \Omega_\nu(x)\, U_\mu(x)\, \Omega_\nu(x + \epsilon\hat{\mu})^\dagger \ , \tag{3.2}$$

with period matrix Σ. The transition functions $\Omega_\mu(x)$ are SU(p) matrices obeying the consistency condition [19]

$$\Omega_\mu(x + \Sigma_{\alpha\nu}\hat{\alpha})\ \Omega_\nu(x) = \mathcal{Z}_{\mu\nu}\ \Omega_\nu(x + \Sigma_{\alpha\mu}\hat{\alpha})\ \Omega_\mu(x) \, , \tag{3.3}$$

where $\mathcal{Z}_{\mu\nu} = \mathrm{e}^{2\pi i Q_{\mu\nu}/p} \in \mathbb{Z}_p$. The antisymmetric matrix Q has elements $Q_{\mu\nu} \in \mathbb{Z}$ representing the 't Hooft fluxes. The reason why we have chosen the $\Omega_\mu(x)$ as SU(p), rather than the usual U(p), matrices is that the abelian magnetic flux of a U(p) gauge field, which plays a very important role in Morita equivalences of noncommutative gauge theories [6], arises in the commutative case only via the corresponding 't Hooft flux [20]. A similar consideration with U(p) transition matrices would lead us to the same results.

We take the gauge choice $\Omega_\mu(x) = \Gamma_\mu$ with constant Γ_μ which are called twist eaters. Eq. (3.3) implies the Weyl-'t Hooft commutation relations

$$\Gamma_\mu \Gamma_\nu = \mathrm{e}^{2\pi i Q_{\mu\nu}/p}\, \Gamma_\nu \Gamma_\mu \, . \tag{3.4}$$

In order to find a general solution to (3.2), we need an explicit form of the twist eaters Γ_μ satisfying (3.4) for SU(p) and even spacetime dimension $D = 2d$. For generic rank p and

flux matrix Q, these matrices have been constructed in Ref. [21] and have the dimension of the irreducible representation of (3.4) equal to $p/\tilde{p}_0$ with integer $\tilde{p}_0$. The most general gauge field configuration $U_\mu(x)$ satisfying the constraints (3.2) is determined by two integral matrices $\tilde{P}$ and B, which are constructed in the Appendix, and can be represented in terms of the operator (2.10) as

$$\hat{U}_\mu = \sum_{\vec{m}} \left(\prod_{\nu=1}^{D} \left(\hat{Z}'_\nu \right)^{m_\nu} \right) \, \mathrm{e}^{\pi i \sum_{\nu<\lambda} \Theta'_{\nu\lambda} \, m_\nu m_\lambda} \otimes u_\mu(\vec{m}) \ , \tag{3.5}$$

where $u_\mu(\vec{m})$ is a $\tilde{p}_0 \times \tilde{p}_0$ matrix and $\hat{Z}'_\mu$ is given by

$$\hat{Z}'_\mu = \mathrm{e}^{2\pi i (\Sigma'^{-1})_{\mu\nu} \, \hat{x}_\nu} \otimes \prod_{\rho=1}^{D} (\Gamma_\rho)^{B_{\mu\rho}} \ , \tag{3.6}$$

where $\Sigma' = \Sigma\tilde{P}$ and $\Theta' = -\tilde{P}^{-1}B^\top$. Because of their dependence on the twist eating solutions, the operators (3.6) obey the commutation relations

$$\hat{Z}'_\mu \hat{Z}'_\nu = \mathrm{e}^{-2\pi i \Theta'_{\mu\nu}} \, \hat{Z}'_\nu \hat{Z}'_\mu \ . \tag{3.7}$$

The two matrices Σ' and Θ' must satisfy the general constraint (2.8), which implies that $M = -\frac{1}{2\,\epsilon} \Sigma \, B^\top$ must be an integer-valued matrix. The sum over $\vec{m}$ in (3.5) can then be taken over $\mathbb{Z}^D$ modulo the periodicity $m_\mu \sim m_\mu + \frac{1}{\epsilon} \Sigma'_{\nu\mu}$.

We now introduce the map

$$\hat{\Delta}'(x') = \frac{1}{|\det \frac{1}{\epsilon} \Sigma'|} \sum_{\vec{m}} \left(\prod_{\mu=1}^{D} \left(\hat{Z}'_\mu \right)^{m_\mu} \right) \, \mathrm{e}^{\pi i \sum_{\mu<\nu} \Theta'_{\mu\nu} \, m_\mu m_\nu} \, \mathrm{e}^{-2\pi i \left(\Sigma'^{-1} \right)_{\mu\nu} \, m_\mu \, x'_\nu} \tag{3.8}$$

and decompose the operator $\hat{U}_\mu$ as

$$\hat{U}_\mu = \sum_{x'} \hat{\Delta}'(x') \otimes U'_\mu(x') \ . \tag{3.9}$$

It follows that $U'_\mu(x')$ is a single-valued $\tilde{p}_0 \times \tilde{p}_0$ matrix field on a periodic lattice with period matrix $\Sigma' = \Sigma\tilde{P}$ and the dimensionless noncommutativity matrix Θ'. Unitarity of the operator $\hat{U}_\mu$ further requires that $U'_\mu(x')$ be star-unitary. Finally, we arrive at the action (2.11) with reduced rank $n = \tilde{p}_0$ and coupling constant $g^2 = \tilde{g}^2(p/\tilde{p}_0)$. This shows that an ordinary U(p) lattice gauge theory with twisted gauge fields is equivalent to noncommutative U($\tilde{p}_0$) lattice gauge theory with periodic gauge fields. What we have arrived at is the lattice analog of the well-known fact that, for $\tilde{p}_0 = 1$, noncommutative U(1) Yang-Mills theory with rational-valued deformation parameters $\Theta_{\mu\nu}$ is equivalent to ordinary Yang-Mills theory with gauge group U(p) and non-vanishing 't Hooft flux. From the present point of view, Morita equivalence is regarded as a change of basis $\hat{\Delta}(x) \leftrightarrow \hat{\Delta}'(x')$ for the mapping between operators and fields.

We will now discuss the case $\Sigma = \epsilon \mathbb{1}_D$ in the present construction. A one-site U(p) lattice gauge theory is just the Eguchi-Kawai model [11], and the fact that the boundary condition

is twisted as in (3.2) means that it is actually the twisted Eguchi-Kawai model [12]. To see this explicitly, we reduce the action (3.1) to a single point $x = 0$ by using the constraints (3.2) and arrive at the action

$$S = -\frac{1}{\tilde{g}^2} \sum_{\mu \neq \nu} \mathcal{Z}_{\mu\nu} \ \mathrm{tr}_{(p)} \Big(V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \Big) \tag{3.10}$$

where $V_\mu = U_\mu(0)\,\Gamma_\mu$, $\mu = 1, \ldots, D$, are $p \times p$ unitary matrices. This is the action of the twisted Eguchi-Kawai model, where the phase factor $\mathcal{Z}_{\mu\nu}$ is called the "twist". Thus, the recent proposal [8] that the twisted large N reduced model serves as a concrete definition of noncommutative Yang-Mills theory can be interpreted as the simplest example of Morita equivalence. The possibility of such an interpretation has also been suggested in Ref. [22].

4 Wilson loops on the lattice

We now turn to a description of star-gauge invariant observables on the lattice [9, 13]. We first define the lattice parallel transport operator $\mathcal{U}(x;C)$ which is specified by a contour $C = \{\mu_1, \mu_2, \ldots, \mu_n\}$, where $\mu_j = \pm 1, \pm 2, \cdots \pm D$ and $U_{-\mu}(x) = U_\mu(x - \epsilon\hat{\mu})^\dagger$. It is defined by

$$\mathcal{U}(x;C) = U_{\mu_1}(x) \star U_{\mu_2}(x + \epsilon\hat{\mu}_1) \star \cdots \star U_{\mu_n}\left(x + \epsilon \sum_{j=1}^{n-1} \hat{\mu}_j \right) \tag{4.1}$$

and it transforms under the star-gauge transformation (2.12) as

$$\mathcal{U}(x;C) \mapsto g(x) \star \mathcal{U}(x;C) \star g(x+v)^\dagger \ , \tag{4.2}$$

where $v = \epsilon \sum_{j=1}^{n} \hat{\mu}_j$.

Star-gauge invariant observables are constructed out of the parallel transport operator $\mathcal{U}(x;C)$ using a star-unitary function $S_v(x)$ with the property

$$S_v(x) \star g(x) \star S_v(x)^\dagger = g(x+v) \tag{4.3}$$

for arbitrary functions $g(x)$ on the periodic lattice. The solution to this equation is given by

$$S_v(x) = \mathrm{e}^{ik\cdot x} \tag{4.4}$$

provided that

$$v_\mu = \theta_{\mu\nu} k_\nu + \Sigma_{\mu\nu} n_\nu \tag{4.5}$$

with some integer-valued vector n_μ. Since both k and v are discrete and periodic, there are the same finite number of values that k and v can take, by modding out the periodicity. Therefore, it makes sense to ask whether (4.5) gives a one-to-one correspondence between k and v. The answer is affirmative if and only if there exist $D \times D$ integer-valued matrices J and K which satisfy

$$\frac{1}{\epsilon}\,\Sigma J - 2MK = \mathbb{1}_D \ , \tag{4.6}$$

where M is the integer-valued matrix introduced in (2.2). If this condition is not met, then there exists v for which there is no momentum k satisfying (4.5), and for the other v there are more than one k satisfying (4.5). For example, let us consider the case with $\frac{1}{\epsilon}\Sigma = L\mathbb{1}_D$. If L is even, then (4.6) cannot be satisfied. If L is odd and M^{-1} is an integer-valued matrix, then one can satisfy (4.6) by taking $J = \mathbb{1}_D$ and $K = \frac{(L-1)}{2}M^{-1}$.

By using the function (4.4), with the property (4.3), star-gauge invariant observables can be constructed out of (4.1) as

$$\mathcal{O}(C) = \sum_x \mathrm{tr}_{(n)}\Big(\mathcal{U}(x;C)\Big) \star S_v(x) \ . \tag{4.7}$$

Its star-gauge invariance stems from (4.2) and (4.3). In Eq. (4.7), the parameter k_μ in (4.4) can be interpreted as the total momentum of the contour C. Note that in the commutative case, $\theta_{\mu\nu} = 0$, Eq. (4.5) reproduces the fact that gauge invariant quantities are given either by closed Wilson loops ($n_\nu = 0$) or by Polyakov loops ($n_\nu \neq 0$), i.e. loops winding around the torus. Note also that in the commutative case, the total momentum k_μ is unrestricted. In the noncommutative case, on the other hand, one can construct gauge invariant quantities associated with open loops. In particular, when the condition (4.6) is met, one can construct a gauge invariant quantity for each contour C on the lattice, and the total momentum should be specified uniquely (up to periodicity) through (4.5) depending on the separation vector v_μ of the contour. We will see in the next Section that star-gauge invariant observables constructed in noncommutative Yang-Mills theory reduce smoothly to ordinary Wilson loops in the commutative limit.

5 Coupling to fundamental matter fields

We will now consider noncommutative gauge theories coupled to matter fields in the fundamental representation of the gauge group. One advantage of the lattice formulation is that it allows a hopping parameter expansion (or large mass expansion) of this theory which will enable us to clarify various aspects of the star-gauge invariant observables constructed out of noncommutative gauge fields. For simplicity, we consider a complex scalar field $\phi(x)$ as the matter field. Fermions can be treated in an analogous way. The action for the matter field is

$$S_{\mathrm{matter}} = -\kappa \left\{ \sum_{x,\mu} \phi^\dagger(x) \star U_\mu(x) \star \phi(x+\epsilon\hat{\mu}) + \mathrm{c.c.} \right\} + \sum_x \phi^\dagger(x)\,\phi(x) \ , \tag{5.1}$$

and it is invariant under the star-gauge transformation

$$\phi(x) \mapsto g(x) \star \phi(x) \quad ; \quad \phi^\dagger(x) \mapsto \phi^\dagger(x) \star g^\dagger(x) \quad ; \quad U_\mu(x) \mapsto g(x) \star U_\mu(x) \star g^\dagger(x+\epsilon\hat{\mu}) \ . \tag{5.2}$$

By integrating over the field $\phi(x)$, we perform an expansion in the hopping parameter κ.

The effective action $\Gamma_{\mathrm{eff}}[U]$ for the gauge field $U_\mu(x)$ induced by the integration over $\phi(x)$ is given by

$$\Gamma_{\mathrm{eff}}[U] = -\ln \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} \left\langle \left(\sum_{x,\mu} \phi^\dagger(x) \star U_\mu(x) \star \phi(x+\epsilon\hat{\mu}) + \mathrm{c.c.} \right)^n \right\rangle_{\kappa=0} \ . \tag{5.3}$$

Throughout this section, $\langle \cdots \rangle$ refers to the vacuum expectation value for fixed gauge background, namely we integrate over the matter fields only. Using Wick's theorem, we obtain[2]

$$\Gamma_{\rm eff}[U] = \sum_C \frac{\kappa^{L(C)}}{L(C)} \sum_x {\rm tr}_{(n)} \Big(\mathcal{U}(x;C) \Big) \ , \tag{5.4}$$

where $\sum_C$ denotes the sum over all closed loops on the lattice and $L(C)$ denotes the length of the loop C in lattice units. In this way, we encounter observables associated with closed loops[3].

Let us now consider star-gauge invariant observables involving matter fields such as

$$G[f] = \left\langle \sum_x \phi^\dagger(x) \star \phi(x) \star f(x) \right\rangle \ , \tag{5.5}$$

which is star-gauge invariant for arbitrary functions $f(x)$ on the lattice which can be regarded as the wavefunction of the composite operator $\phi^\dagger(x) \star \phi(x)$. We first make a Fourier transform of $f(x)$ and express $G[f]$ as

$$G[f] = \sum_{\vec{k}} \tilde{f}(k) \left\langle \sum_x \phi^\dagger(x) \star {\rm e}^{ik\cdot x} \star \phi(x+v) \right\rangle \ , \tag{5.6}$$

where $v_\mu = \theta_{\mu\nu} k_\nu$. We have used the fact that ${\rm e}^{ik\cdot x}$ acts as a translation operator in the noncommutative field theory as follows from Eqs. (4.3) and (4.4). Integrating over the matter fields using the hopping parameter expansion, we obtain

$$G[f] = \sum_{\vec{k}} \tilde{f}(k) \sum_C \kappa^{L(C)} \sum_x {\rm tr}_{(n)} \Big(\mathcal{U}(x;C) \star {\rm e}^{ik\cdot x} \Big) \tag{5.7}$$

where $\sum_C$ now denotes the sum over all lattice loops beginning at the origin and ending at v_μ. Thus we find the observables encountered in the previous Section which are associated with open loops. In the commutative case, the separation vector v_μ of the loops should vanish independently of the momentum k_μ. Thus the summation over the loop C contains only closed loops. The summation over $\vec{k}$ can then be done explicitly reproducing the wavefunction $f(x)$. In the noncommutative case, the summation over the loop C depends on k_μ and we are not allowed to reverse the order of the summations. The larger k_μ is, the larger becomes the separation vector v_μ of the two ends of the loop C. This is a characteristic phenomenon in noncommutative field theories. If one would like to have a higher resolution in one direction, say in the $\mu = 1$ direction, by increasing k_1, then the object will extend in the other directions proportionally to $\theta_{\mu 1} k_1$. On the other hand, if one considers the case

[2] A calculation of the hopping parameter expansion with fundamental fermions has been performed by A. Zubkov.

[3] We can also introduce n_f flavours of the matter field and take the limit $n_f \to \infty$ with $\kappa \sim n_f^{-1/4}$. In that case, only the simplest closed loops of lengths 4, i.e. the boundaries of lattice plaquettes, remain in (5.4). The single-plaquette action (2.11) can thus be induced by fundamental matter fields, similarly to the commutative case [23].

when $\tilde{f}(k)$ has support only for finite momentum k_μ, then one can take the $\theta_{\mu\nu} \to 0$ limit smoothly, reproducing the commutative case.

The considerations in this Section show that the star-gauge invariant observables indeed play the same fundamental role as ordinary Wilson loops do in commutative gauge theories. As the knowledge of all the Wilson loop correlators would give all the information of the gauge theory including matter fields, so do the star-gauge invariant observables in the noncommutative case. We have also seen explicitly how these observables reduce smoothly to ordinary Wilson loops in the commutative limit.

6 Morita equivalence with fundamental matter fields

In this final Section, we discuss Morita equivalence of noncommutative Yang-Mills theories coupled to fundamental matter fields. We show, in particular, that the twisted Eguchi-Kawai model with fundamental matter as constructed in Ref. [16] is equivalent to noncommutative Yang-Mills theory with fundamental matter fields. We use the setup of Section 3. We consider *commutative* $\mathrm{U}(p)$ gauge theory and introduce N_f flavours of matter in the fundamental representation. We represent it as $\Phi(x)_{ij}$, where $i = 1, \ldots, p$ represents the color index and $j = 1, \ldots, N_f$ represents the flavour index. The spacetime is discretized as $x_\mu \in \epsilon\mathbb{Z}$. For the present purpose, it is essential to take N_f to be an integral multiple of p. In what follows, we assume that $N_f = p$ for simplicity. Then, $\Phi(x)_{ij}$ is a $p \times p$ general complex matrix.

The action for the matter part of the $\mathrm{U}(p)$ gauge theory is given by

$$S_{\text{matter}} = -\kappa \left\{ \sum_{x,\mu} \mathrm{tr}_{(p)} \Big(\Phi^\dagger(x) U_\mu(x) \Phi(x + \epsilon\hat{\mu}) \Big) + \text{c.c.} \right\} + \sum_x \mathrm{tr}_{(p)} \Big(\Phi^\dagger(x) \Phi(x) \Big) \ , \qquad (6.1)$$

which is invariant under the gauge transformation $U_\mu(x) \mapsto g(x) U_\mu(x) g(x + \epsilon\hat{\mu})$ and $\Phi(x) \mapsto g(x)\Phi(x)$. The action (6.1) has also a global $\mathrm{U}(N_f)$ flavour symmetry $\Phi(x) \mapsto \Phi(x)\, g'$, where $g' \in \mathrm{U}(N_f)$. We impose a constraint on $\Phi(x)$ and $U_\mu(x)$ given by the twisted boundary conditions

$$U_\mu(x + \Sigma_{\alpha\nu}\hat{\alpha}) = \Gamma_\nu\, U_\mu(x)\, \Gamma_\nu^\dagger \ , \qquad \Phi(x + \Sigma_{\alpha\nu}\hat{\alpha}) = \Gamma_\nu\, \Phi(x)\, \Gamma_\nu^\dagger \ , \qquad (6.2)$$

where Σ is the period matrix. Since the transition matrix lives in $\mathrm{SU}(p)$, this implies a nonvanishing 't Hooft flux of the gauge field, as we discussed in Section 3. Notice also that in (6.2), the Γ_ν on the left of $\Phi(x)$ represents a (global) gauge transformation, whereas the $\Gamma_\nu^\dagger$ on the right of $\Phi(x)$ represents a rotation in the flavour space. This is the trick to introducing the fundamental representation in Morita equivalence. We have made use of the global flavour symmetry $\mathrm{SU}(N_f)$ to mimic the boundary condition for the adjoint representation. Such an idea first appeared in Ref. [24] in the context of supersymmetric theories.

Solving the constraint as we did in Section 3, we find that the resulting theory is *noncommutative* $\mathrm{U}(\tilde{p}_0)$ lattice gauge theory coupled to $\tilde{p}_0$ flavours of matter fields in the fundamental representation, with periodic boundary conditions. We can set $\Sigma = \epsilon \mathbb{1}_D$ to obtain the twisted Eguchi-Kawai model with matter fields. This is the model introduced in Ref. [16] as a model which reproduces large N gauge theory with N_f flavours in the Veneziano limit

$N_f \sim N \to \infty$. Now, we have found that the same model with $N_f = n_f N$ can be interpreted as noncommutative $\mathrm{U}(\tilde{p}_0)$ gauge theory with periodic boundary conditions including $n_f \tilde{p}_0$ flavours of matter fields in the fundamental representation. The one-loop beta-function for $\tilde{p}_0 = 1$ and fermionic matter has been calculated in Ref. [25]. The result agrees with $\mathrm{SU}(N)$ commutative Yang-Mills theory with $N_f = n_f N$ flavours as it should due to Morita equivalence. In the present case, there is no extra infrared singularity associated with non-planar diagrams [15] since one-loop diagrams with fundamental matter are always planar.

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Appendix A General representation of twist eaters

The explicit form of the twist eaters Γ_μ satisfying (3.4) for $\mathrm{SU}(p)$ with generic rank p, flux matrix Q and even dimension $D = 2d$ can be constructed as follows [21]. Using the discrete symmetry of the D-dimensional torus under the $\mathrm{SL}(D, \mathbb{Z})$ geometrical automorphism group, Q can be represented in a canonical skew-diagonal form

$$Q = \begin{pmatrix} 0 & -q_1 & & & \\ q_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -q_d \\ & & & q_d & 0 \end{pmatrix} . \tag{A.1}$$

Given the d independent fluxes $q_i \in \mathbb{Z}$, we introduce the integers $p_i = \gcd(q_i, p)$, $\tilde{p}_i = p/p_i$, and $\tilde{q}_i = q_i/p_i$. By construction, $\tilde{p}_i$ and $\tilde{q}_i$ are co-prime. A necessary and sufficient condition for the existence of solutions to (3.4) is that the integer $\tilde{p}_1 \cdots \tilde{p}_d$, the dimension of irreducible representation of the Weyl-'t Hooft algebra, divides the rank p [21]. In that case we write

$$p = \tilde{p}_0 \prod_{i=1}^{d} \tilde{p}_i \tag{A.2}$$

and the twist eating solutions may then be given on the subgroup $\mathrm{SU}(\tilde{p}_1) \otimes \cdots \otimes \mathrm{SU}(\tilde{p}_d) \otimes \mathrm{SU}(\tilde{p}_0)$ of $\mathrm{SU}(p)$ such that Γ_{i-1}, Γ_i are constructed from the Weyl-'t Hooft matrices on $\mathrm{SU}(\tilde{p}_i)$. The subgroup of $\mathrm{GL}(p, \mathbb{C})$ consisting of matrices which commute with the twist eaters Γ_μ is then $\mathrm{GL}(\tilde{p}_0, \mathbb{C})$.

The general solution to Eq. (3.2) for $U_\mu(x)$ is determined by two $D \times D$ integral matrices

which, in the basis (A.1) where Q is skew-diagonal, read

$$\tilde{P} = \begin{pmatrix} \tilde{p}_1 & 0 & & & \\ 0 & \tilde{p}_1 & & & \\ & & \ddots & & \\ & & & \tilde{p}_d & 0 \\ & & & 0 & \tilde{p}_d \end{pmatrix} \quad , \qquad B = \begin{pmatrix} 0 & -b_1 & & & \\ b_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -b_d \\ & & & b_d & 0 \end{pmatrix} \tag{A.3}$$

with integral b_i obeying $a_i\tilde{p}_i + b_i\tilde{q}_i = 1$, $i = 1, \ldots, d$, for some integer a_i. Defining matrices A and $\tilde{Q}$ using a_i and $\tilde{q}_i$ as with $\tilde{P}$ and B in (A.3), respectively, the previous condition can be written as $A\tilde{P} + B\tilde{Q} = \mathbb{1}_D$, which is invariant under the $\mathrm{SL}(D,\mathbb{Z})$ transformation

$$\begin{aligned} A &\mapsto \Lambda^\top A\,\Lambda \qquad\quad ; \qquad \tilde{P} \mapsto \Lambda^{-1}\,\tilde{P}\,(\Lambda^{-1})^\top \\ B &\mapsto \Lambda^\top B\,\left(\Lambda^{-1}\right)^\top \quad ; \qquad \tilde{Q} \mapsto \Lambda^\top\,\tilde{Q}\,(\Lambda^{-1})^\top\ , \end{aligned} \tag{A.4}$$

where $\Lambda \in \mathrm{SL}\,(D,\mathbb{Z})$. The flux matrix Q can be written in terms of $\tilde{P}$ and $\tilde{Q}$ in an $\mathrm{SL}(D,\mathbb{Z})$ covariant way as $Q = p\,\tilde{Q}\,\tilde{P}^{-1}$, and it transforms as $Q \mapsto \Lambda^\top Q\,\Lambda$. One can use this symmetry to rotate Q back to general form and thus find the corresponding $\tilde{P}$ and B which are needed to construct the general solution to Eq. (3.2).

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112, `hep-th/9610043`.

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. **B498** (1997) 467, `hep-th/9612115`.

[3] A. Connes, M.R. Douglas and A. Schwarz, J. High Energy Phys. **9802** (1998) 003, `hep-th/9711162`.

[4] D. Bigatti, Phys. Lett. **B451** (1999) 324, `hep-th/9804120`; D. Bigatti and L. Susskind, `hep-th/9908056`.

[5] N. Seiberg and E. Witten, J. High Energy Phys. **9909** (1999) 032, `hep-th/9908142`.

[6] A. Schwarz, Nucl. Phys. **B534** (1998) 720, `hep-th/9805034`; G. Landi, F. Lizzi and R.J. Szabo, Commun. Math. Phys. **206** (1999) 603, `hep-th/9806099`; D. Brace, B. Morariu and B. Zumino, Nucl. Phys. **B545** (1999) 192, `hep-th/9810099`; B. Pioline and A. Schwarz, J. High. Energy Phys. **9908** (1999) 021, `hep-th/9908019`.

[7] J.C. Várilly and J.M. Gracia-Bondía, Int. J. Mod. Phys. **A14** (1999) 1305, `hep-th/9804001`; M. Chaichian, A. Demichev and P. Presnajder, `hep-th/9812180`; C.P. Martín and D. Sanchez-Ruiz, Phys. Rev. Lett. **83** (1999) 476, `hep-th/9903077`; M.M. Sheikh-Jabbari, J. High Energy Phys. **9906** (1999) 015, `hep-th/9903107`; T. Krajewski and R. Wulkenhaar, `hep-th/9903187`.

[8] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, `hep-th/9908141`.

[9] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, `hep-th/9910004`.

[10] S. Iso, H. Kawai and Y. Kitazawa, hep-th/0001027.

[11] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48** (1982) 1063.

[12] A. Gonzalez-Arroyo and M. Okawa, Phys. Lett. **120B** (1983) 174; Phys. Rev. **D27** (1983) 2397; T. Eguchi and R. Nakayama, Phys. Lett. **122B** (1983) 59.

[13] J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, J. High Energy Phys. **9911** (1999) 029, hep-th/9911041.

[14] K.G. Wilson, Phys. Rev. **D10** (1974) 2445.

[15] S. Minwalla, M. Van Raamsdonk and N. Seiberg, J. High Energy Phys. **0002** (2000) 020, hep-th/9912072.

[16] S.R. Das, Phys. Lett. **132B** (1983) 155.

[17] I. Bars and D. Minic, hep-th/9910091.

[18] D.B. Fairlie and C.K. Zachos, Phys. Lett. **B224** (1989) 101; M. Li, Nucl. Phys. **B499** (1997) 149, hep-th/9612222; T. Curtright, D.B. Fairlie and C.K. Zachos, Phys. Lett. **B405** (1997) 37, hep-th/9704037; D.B. Fairlie, Mod. Phys. Lett. **A13** (1998) 263, hep-th/9707190.

[19] G. 't Hooft, Nucl. Phys. **B153** (1979) 141; Commun. Math. Phys. **81** (1981) 267.

[20] D.R. Lebedev, M.I. Polikarpov and A.A. Rosly, Nucl. Phys. **B325** (1989) 138.

[21] P. van Baal and B. van Geemen, J. Math. Phys. **27** (1986) 455; D.R. Lebedev and M.I. Polikarpov, Nucl. Phys. **B269** (1986) 285.

[22] A. Hashimoto and N. Itzhaki, J. High Energy Phys. **9912** (1999) 007, hep-th/9911057.

[23] M. Bander, Phys. Lett. **126B** (1983) 463; H.W. Hamber, Phys. Lett. **126B** (1983) 471.

[24] E. Cohen and C. Gomez, Nucl. Phys. **B223** (1983) 183.

[25] M. Hayakawa, hep-th/9912167.