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Random Matrices and Non-Exact C^*-algebras

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1 Introduction

In the paper [HT2], we gave new proofs based on random matrix methods of the following two results:

(1) Any unital exact stably finite C^*-algebra has a tracial state.

(2) If $\mathcal{A}$ is a unital exact C^*-algebra, then any state on $K_0(\mathcal{A})$ comes from a tracial state on $\mathcal{A}$.

For each of the results (1) and (2), one may ask whether or not it holds without the assumption that the C^*-algebra be exact. These two problems are still open, and both problems are equivalent to Kaplansky's famous problem, whether all AW^*-factors of type II_1 are von Neumann algebras (cf. [Ha] and [BR]).

In the present note, we provide examples which show that the method used in [HT2] cannot be employed to show that (1) and (2) hold for all C^*-algebras.

As in [HT2], we let $\mathrm{GRM}(m, n, \sigma^2)$ denote the class of complex Gaussian $m \times n$ random matrices of the form

$$B = (b(i,j))_{\substack{1\le i\le m\\ 1\le j\le n}}$$

for which the $2mn$ real random variables $\mathrm{Re}(b(i,j))$, $\mathrm{Im}(b(i,j))$ are independent and Gaussian distributed random variables with mean 0 and variance $\sigma^2/2$, defined on a probability space $(\Omega, \mathcal{F}, P)$. Moreover, for any bounded operator A on a Hilbert space, we denote by $\mathrm{sp}(A)$ the spectrum of A.

The proofs of (1) and (2) above given in [HT2] were both based on the following theorem:

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1.1 Theorem. (cf. [HT2]) *Let $a_1, a_2, \dots, a_r$ be elements of a unital exact C^*-algebra $\mathcal{A}$. Let further $(\Omega, \mathcal{F}, P)$ be a fixed probability space, and let, for each n in $\mathbb{N}$, $Y_1^{(n)}, \dots, Y_r^{(n)}$ be independent Gaussian random matrices defined on Ω and lying in the class $\mathrm{GRM}(n, n, \frac{1}{n})$ defined below. Put*

$$S_n = \sum_{i=1}^{r} a_i \otimes Y_i^{(n)}, \qquad (n \in \mathbb{N}),$$

and let c be a positive real number. We then have

(i) *If $\| \sum_{i=1}^r a_i^* a_i \| \leq c$ and $\| \sum_{i=1}^r a_i a_i^* \| \leq 1$, then for almost all ω in Ω,*

$$\limsup_{n\to\infty} \max \big\{ \mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big) \big\} \leq \big(\sqrt{c}+1\big)^2.$$

(ii) *If $\sum_{i=1}^r a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$, $\| \sum_{i=1}^r a_i a_i^* \| \leq 1$, and $c \geq 1$, then for almost all ω in Ω,*

$$\liminf_{n\to\infty} \min \big\{ \mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big) \big\} \geq \big(\sqrt{c}-1\big)^2. \qquad \square$$

The upper and lower bounds $(\sqrt{c}+1)^2$ and $(\sqrt{c}-1)^2$ in Theorem 1.1 are best possible. This follows from

1.2 Theorem. (cf. [Th]) *Let $\mathcal{B}$ be a unital exact C^*-algebra and let $b_1, b_2, \dots, b_s$ be elements of $\mathcal{B}$ satisfying that*

$$\sum_{i=1}^{s} b_i^* b_i = c\mathbf{1}_{\mathcal{B}} \quad \text{and} \quad \sum_{i=1}^{s} b_i b_i^* = \mathbf{1}_{\mathcal{B}},$$

for some real number c in $[1, \infty[$. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_s^{(n)}$ in $\mathrm{GRM}(n, n, \frac{1}{n})$, and put $T_n = \sum_{i=1}^s b_i \otimes Y_i^{(n)}$. Then for almost all ω in Ω,

$$\max \big\{ \mathrm{sp}(T_n(\omega)^* T_n(\omega)) \big\} \to (\sqrt{c}+1)^2, \quad \text{as } n \to \infty,$$

and

$$\min \big\{ \mathrm{sp}(T_n(\omega)^* T_n(\omega)) \big\} \to (\sqrt{c}-1)^2, \quad \text{as } n \to \infty. \qquad \square$$

Let $C^*(\mathbb{F}_r)$ denote the full C^*-algebra associated with the free group $\mathbb{F}_r$ on r generators, and let $u_1, \dots, u_r$ denote the unitary generators of $C^*(\mathbb{F}_r)$. In [HT2, Proposition 4.9] it was proved, that with $a_i = r^{-1/2} u_i$, $i = 1, \dots, r$, and $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$ as in Theorem 1.1, one has:

$$\liminf_{n\to\infty} \max\{\mathrm{sp}(S_n(\omega)^* S_n(\omega))\} \geq \big(\tfrac{8}{3\pi}\big)^2 r.$$

In particular, for $c \geq 1$ and $r \geq 6c$, the upper bound in Theorem 1.1 is violated because $6c > (\frac{3\pi}{8})^2 4c > (\frac{3\pi}{8})^2 (\sqrt{c}+1)^2$. The upper bound in Theorem 1.2 is also violated in the general non-exact case provided that $c \geq 1$ and $r \geq 8c$ (see Remark 4.5 at the end of this paper). The main result in this note concerns the lower bound in Theorem 1.1 and Theorem 1.2:

1.3 Main Theorem. (cf. Theorem 3.5 and Theorem 4.4)

(a) Let $\mathcal{A}(r,c)$ denote the universal unital C^*-algebra generated by r elements $a_1,\dots,a_r$, satisfying that:

$$\sum_{i=1}^r a_i^*a_i = c\mathbf{1} \quad \text{and} \quad \sum_{i=1}^r a_ia_i^* \le \mathbf{1},$$

where $1 \le c \le r$. Put $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$ as in Theorem 1.1. If $r \ge 13c$, then for almost all ω in Ω, $0 \in \mathrm{sp}(S_n(\omega)^*S_n(\omega))$, eventually as $n \to \infty$.

(b) Let $\mathcal{B}(s,c)$ denote the universal unital C^*-algebra generated by s elements $b_1,\dots,b_s$, satisfying that:

$$\sum_{i=1}^s b_i^*b_i = c\mathbf{1} \quad \text{and} \quad \sum_{i=1}^s b_ib_i^* = \mathbf{1},$$

where $1 \le c \le s-1$. Put $T_n = \sum_{i=1}^s b_i \otimes Y_i^{(n)}$ as in Theorem 1.2. If $s \ge 14c$, then for almost all ω in Ω, $0 \in \mathrm{sp}(T_n(\omega)^*T_n(\omega))$, eventually as $n \to \infty$.

The Main Theorem above clearly shows that the lower bounds in Theorem 1.1 and Theorem 1.2 are violated for general (non-exact) C^*-algebras, when $c > 1$. The proofs in [HT2] of the statements (1) and (2) in the beginning of this introduction did not fully use the exact lower bound $(\sqrt{c}-1)^2$ in Theorem 1.1, but just the fact that in the exact case, we have, for almost all ω, that $0 \notin \mathrm{sp}(S_n(\omega)^*S_n(\omega))$ eventually as $n \to \infty$, when $c > 1$. The Main Theorem above shows that even this fails in the general non-exact case.

Finally, some conventions and notation that are used throughout the paper:

As we have already practiced, in most of this paper we omit mentioning the underlying probability space $(\Omega,\mathcal{F},P)$, and it is understood that all random matrices/variables are defined on this one probability space. By tr_n we denote the normalized trace on $M_n(\mathbb{C})$, and we put $\mathrm{Tr}_n = n\cdot\mathrm{tr}_n$. Furthermore, we denote by $\mathbf{1}_n$ the unit matrix in $M_n(\mathbb{C})$.

2 Some technical Lemmas

The first lemma is elementary and well-known. For completeness we include a proof.

2.1 Lemma. *Let A be a fixed matrix in $M_n(\mathbb{C})$ and consider the two linear mappings $L_A, R_A\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by:*

$$L_A(B) = AB, \quad \text{and} \quad R_A(B) = BA, \quad (B \in M_n(\mathbb{C})).$$

Then

$$\det_{\mathbb{R}}(L_A) = \det_{\mathbb{R}}(R_A) = |\det(A)|^{2n},$$

where $\det_{\mathbb{R}}(L_A)$ (resp. $\det_{\mathbb{R}}(R_A)$) denotes the determinant of the matrix of L_A (resp. R_A) w.r.t. an arbitrary basis for the $2n^2$ dimensional real vector space $M_n(\mathbb{C})$.

Proof. The usual $n \times n$ matrix units e_{kl}, $1 \le k, l \le n$, form a basis for the *complex* vector space $M_n(\mathbb{C})$. If we list them in reverse lexicographic order, i.e.,

$$e_{11}, e_{21}, \dots, e_{n1},\ e_{12}, e_{22}, \dots, e_{n2},\ \dots, e_{1n}, e_{2n}, \dots, e_{nn}, \tag{2.1}$$

then the matrix for L_A w.r.t. this (ordered) basis is the $n^2 \times n^2$ matrix:

$$S = \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix},$$

where A is repeated n times along the diagonal. A basis for the *real* vector space $M_n(\mathbb{C})$ can be obtained by adding to the list in (2.1) the same elements multiplied by $i = \sqrt{-1}$. The matrix for L_A w.r.t. this basis is the $2n^2 \times 2n^2$ matrix:

$$\begin{pmatrix} \mathrm{Re}(S) & -\mathrm{Im}(S) \\ \mathrm{Im}(S) & \mathrm{Re}(S) \end{pmatrix}.$$

Note next that the matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1}_{n^2} & i\mathbf{1}_{n^2} \\ i\mathbf{1}_{n^2} & \mathbf{1}_{n^2} \end{pmatrix}$ is a unitary in $M_{2n^2}(\mathbb{C})$, and that

$$U \begin{pmatrix} \mathrm{Re}(S) & -\mathrm{Im}(S) \\ \mathrm{Im}(S) & \mathrm{Re}(S) \end{pmatrix} U^* = \begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix},$$

where $\overline{S}$ denotes the complex conjugate of S. Thus,

$$\det_{\mathbb{R}}(L_A) = \det \begin{pmatrix} \mathrm{Re}(S) & -\mathrm{Im}(S) \\ \mathrm{Im}(S) & \mathrm{Re}(S) \end{pmatrix} = \det \begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix} = |\det(S)|^2 = |\det(A)|^{2n},$$

as desired.

To calculate $\det_{\mathbb{R}}(R_A)$, we list instead the matrix units in lexicographic order, i.e.,

$$e_{11}, e_{12}, \dots, e_{1n},\ e_{21}, e_{22}, \dots, e_{2n},\ \dots, e_{n1}, e_{n2}, \dots, e_{nn}.$$

With respect to this (ordered) basis, the matrix for R_A is the $n^2 \times n^2$ matrix:

$$T = \begin{pmatrix} A^t & & & 0 \\ & A^t & & \\ & & \ddots & \\ 0 & & & A^t \end{pmatrix}.$$

By the same arguments as those given above, it follows thus that

$$\det_{\mathbb{R}}(R_A) = |\det(A^t)|^{2n} = |\det(A)|^{2n},$$

as desired. ∎

2.2 Lemma. *Consider the following two diffeomorphisms of the open set* $M_n(\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})$ *in* $M_n(\mathbb{C}) \times M_n(\mathbb{C})$*:*

$$\gamma(y_1,y_2) = \big(y_1y_2^{-1}, \big[(y_1y_2^{-1})^*(y_1y_2^{-1}) + \mathbf{1}_n\big]^{1/2}y_2\big),$$

and

$$\rho(y_1,y_2) = \big(y_2^{-1}y_1, y_2\big[(y_2^{-1}y_1)(y_2^{-1}y_1)^* + \mathbf{1}_n\big]^{1/2}\big).$$

Then the composed map $\varphi = \rho^{-1}\circ\gamma$ *has Jacobi-determinant:*

$$J(\varphi) = \det_{\mathbb{R}}(\varphi') = 1.$$

Moreover, if $(z_1,z_2) = \varphi(y_1,y_2)$*, then*

(i) $y_1y_2^{-1} = z_2^{-1}z_1$.

(ii) $\mathrm{Tr}_n(y_1^*y_1 + y_2^*y_2) = \mathrm{Tr}_n(z_1^*z_1 + z_2^*z_2)$.

Proof. We start by computing the Jacobi-determinant of the mappings γ^{-1} and ρ^{-1}. Note first that

$$\gamma^{-1}(x_1,x_2) = \big(x_1(x_1^*x_1 + \mathbf{1}_n)^{-1/2}x_2, (x_1^*x_1 + \mathbf{1}_n)^{-1/2}x_2\big),$$

for (x_1,x_2) in $M_n(\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$. Note also that $\gamma^{-1} = \sigma_1\circ\sigma_2$, where σ_1,σ_2 are the diffeomorphisms of $M_n(\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ given by:

$$\begin{aligned}\sigma_1(v_1,v_2) &= (v_1v_2, v_2),\\ \sigma_2(x_1,x_2) &= \big(x_1, (x_1^*x_1 + \mathbf{1}_n)^{-1/2}x_2\big).\end{aligned}$$

For fixed (u_1,u_2) in $M_n(\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$, the derivatives $\sigma_j'(u_1,u_2)$, $j\in\{1,2\}$, are (real) linear maps of $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ into itself. Hence, these maps can be written in the form:

$$\sigma_j'(u_1,u_2)\colon \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \mapsto \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where A_j, B_j, C_j, D_j are (real) linear maps on $M_n(\mathbb{C})$. For each j in $\{1,2\}$, we can easily compute the diagonal elements A_j, D_j, and some of the diagonal elements, namely

$$\sigma_1'(v_1,v_2) = \begin{pmatrix} R_{v_2} & L_{v_1} \\ 0 & \mathbf{1}_n \end{pmatrix},$$

and

$$\sigma_2'(x_1,x_2) = \begin{pmatrix} \mathbf{1}_n & 0 \\ * & L_{(x_1^*x_1+\mathbf{1}_n)^{-1/2}} \end{pmatrix},$$

where “$*$” means an undetermined entry. From the equations above, it follows that for each j, the Jacobi-determinant $J(\sigma_j) = \det_{\mathbb{R}}(\sigma_j')$ is just the product of the determinants of the diagonal entries in the corresponding matrix above. Hence, by Lemma 2.1,

$$\begin{aligned} J(\sigma_1)(v_1, v_2) &= |\det(v_2)|^{2n}, \\ J(\sigma_2)(x_1, x_2) &= \big|\det((x_1^* x_1 + \mathbf{1}_n)^{-1/2})\big|^{2n} = (\det(x_1^* x_1 + \mathbf{1}_n))^{-n}. \end{aligned}$$

Thus, for (x_1, x_2) in $M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$, we have

$$\begin{aligned} J(\gamma^{-1})(x_1, x_2) &= J(\sigma_1)(\sigma_2(x_1, x_2)) \cdot J(\sigma_2)(x_1, x_2) \\ &= \big|\det((x_1^* x_1 + \mathbf{1}_n)^{-1/2} x_2)\big|^{2n} (\det(x_1^* x_1 + \mathbf{1}_n))^{-n} \\ &= |\det(x_2)|^{2n} (\det(x_1^* x_1 + \mathbf{1}_n))^{-2n}. \end{aligned} \tag{2.2}$$

Regarding the Jacobi-determinant $J(\rho^{-1})$, note first that

$$\rho^{-1}(x_1, x_2) = \big(x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2} x_1, x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2}\big).$$

As above, we may write ρ^{-1} in the form: $\rho^{-1} = \tau_1 \circ \tau_2$, where τ_1, τ_2 are the diffeomorphisms of $M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ given by:

$$\begin{aligned} \tau_1(w_1, w_2) &= (w_2 w_1, w_2) \\ \tau_2(x_1, x_2) &= \big(x_1, x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2}\big). \end{aligned}$$

The derivatives of τ_1 and τ_2 have the form:

$$\tau_1'(w_1, w_2) = \begin{pmatrix} L_{w_2} & R_{w_1} \\ 0 & \mathbf{1}_n \end{pmatrix},$$

and

$$\tau_2'(x_1, x_2) = \begin{pmatrix} \mathbf{1}_n & 0 \\ * & R_{(x_1 x_1^* + \mathbf{1}_n)^{-1/2}} \end{pmatrix}.$$

Arguing then as above, we get that

$$J(\rho^{-1})(x_1, x_2) = |\det(x_2)|^{2n} \big(\det(x_1 x_1^* + \mathbf{1}_n)\big)^{-2n}. \tag{2.3}$$

We are now ready to calculate the Jacobi-determinant $J(\varphi)$: Let (y_1, y_2) be a pair of matrices in $M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ and put $(x_1, x_2) = \gamma(y_1, y_2)$. Since $\varphi = \rho^{-1} \circ \gamma$ we have

$$J(\varphi)(y_1, y_2) = J(\rho^{-1})(x_1, x_2) \cdot J(\gamma)(y_1, y_2) = \frac{J(\rho^{-1})(x_1, x_2)}{J(\gamma^{-1})(x_1, x_2)}. \tag{2.4}$$

Since $x_1^* x_1$ and $x_1 x_1^*$ have the same eigenvalues (counted with multiplicity), we have $\det(x_1^* x_1 + \mathbf{1}_n) = \det(x_1 x_1^* + \mathbf{1}_n)$, and combining this with (2.2)-(2.4), it follows, finally, that $J(\varphi)(y_1, y_2) = 1$, as desired.

Turning now to the equation (ii), consider, as above, (y_1, y_2) in $M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$, and put $(x_1, x_2) = \gamma(y_1, y_2)$. Furthermore, define $(z_1, z_2) = \varphi(y_1, y_2) = \rho^{-1}(x_1, x_2)$. Then $(x_1, x_2) = \gamma(y_1, y_2) = \rho(z_1, z_2)$, and in particular $x_1 = y_1 y_2^{-1} = z_2^{-1} z_1$, which proves (i).

Finally, regarding the equation (ii), let (x_1, x_2), (y_1, y_2) and (z_1, z_2) be as above, and note then that

$$\begin{aligned}(y_1, y_2) &= \gamma^{-1}(x_1, x_2) = \big(x_1(x_1^* x_1 + \mathbf{1}_n)^{-1/2} x_2, (x_1^* x_1 + \mathbf{1}_n)^{-1/2} x_2\big),\\ (z_1, z_2) &= \rho^{-1}(x_1, x_2) = \big(x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2} x_1, x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2}\big).\end{aligned}$$

Thus,

$$y_1^* y_1 + y_2^* y_2 = x_2^*(x_1^* x_1 + \mathbf{1}_n)^{-1/2}(x_1^* x_1 + \mathbf{1}_n)(x_1^* x_1 + \mathbf{1}_n)^{-1/2} x_2 = x_2^* x_2,$$

and

$$z_1 z_1^* + z_2 z_2^* = x_2(x_1 x_1^* + \mathbf{1}_n)^{-1/2}(x_1 x_1^* + \mathbf{1}_n)(x_1 x_1^* + \mathbf{1}_n)^{-1/2} x_2^* = x_2 x_2^*.$$

Therefore,

$$\mathrm{Tr}_n(y_1^* y_1 + y_2^* y_2) = \mathrm{Tr}_n(z_1 z_1^* + z_2 z_2^*) = \mathrm{Tr}_n(z_1^* z_1 + z_2^* z_2),$$

which proves (ii). ∎

2.3 Lemma. *Let Y_1, Y_2 be independent random matrices in $\mathrm{GRM}(n, n, \sigma^2)$, and put*

$$N = \{\omega \in \Omega \mid Y_2(\omega) \notin \mathrm{GL}(n, \mathbb{C})\}.$$

Define then the random matrices Z_1, Z_2 by:

$$(Z_1(\omega), Z_2(\omega)) = \begin{cases} \varphi(Y_1(\omega), Y_2(\omega)), & \text{if } \omega \in \Omega \setminus N,\\ (0, 0), & \text{if } \omega \in N,\end{cases}$$

where $\varphi = \rho^{-1} \circ \gamma$ as in Lemma 2.2. Then Z_1, Z_2 are independent random matrices in $\mathrm{GRM}(n, n, \sigma^2)$, and

$$Z_2(\omega) Y_1(\omega) = Z_1(\omega) Y_2(\omega), \quad \text{for all } \omega \text{ in } \Omega.$$

Proof. We note first that N is a null-set in Ω. This follows from the facts that the set $\{A \in M_n(\mathbb{C}) \mid \det(A) = 0\}$ is a null-set w.r.t. Lebesgue measure on $M_n(\mathbb{C})$ $(\simeq \mathbb{R}^{2n^2})$, and that the distribution of (the entries of) Y_2 has density w.r.t. Lebesgue measure.

Note next, that it follows from the definition of the class $\mathrm{GRM}(n, n, \sigma^2)$ given in the introduction, that the joint distribution of the pair (Y_1, Y_2) has the following density w.r.t. Lebesgue measure on $M_n(\mathbb{C}) \times M_n(\mathbb{C})$:

$$f(y_1, y_2) = (\pi\sigma^2)^{-2n^2} \exp\big(-\tfrac{1}{\sigma^2}\mathrm{Tr}_n(y_1^* y_1 + y_2^* y_2)\big), \quad (y_1, y_2 \in M_n(\mathbb{C})).$$

Since φ is a bijection of $M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ onto itself with Jacobi-determinant equal to 1 (cf. Lemma 2.2), the joint density of (Z_1, Z_2) is (except for a Lebesgue null-set) given by:

$$g(z_1, z_2) = f(\varphi^{-1}(z_1, z_2)), \quad ((z_1, z_2) \in M_n(\mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})).$$

If we put $(y_1, y_2) = \varphi^{-1}(z_1, z_2)$, then by Lemma 2.2,

$$\mathrm{Tr}_n(y_1^* y_1 + y_2^* y_2) = \mathrm{Tr}_n(z_1^* z_1 + z_2^* z_2).$$

Thus, the joint density of (Z_1, Z_2) is given by:

$$g(z_1, z_2) = (\pi\sigma^2)^{-2n^2} \exp\big(-\tfrac{1}{\sigma^2}\mathrm{Tr}_n(z_1^* z_1 + z_2^* z_2)\big), \quad (z_1, z_2 \in M_n(\mathbb{C})),$$

and this implies that Z_1, Z_2 are independent random matrices in $\mathrm{GRM}(n, n, \sigma^2)$.

For ω in $\Omega \setminus N$, it follows from Lemma 2.2 that

$$Y_1(\omega) Y_2(\omega)^{-1} = Z_2(\omega)^{-1} Z_1(\omega).$$

Hence we have that

$$Z_2(\omega) Y_1(\omega) = Z_1(\omega) Y_2(\omega), \quad (\omega \in \Omega \setminus N),$$

and the same identity holds trivially for ω in N. ∎

2.4 Corollary. *Let Y_1, Y_2 be independent random matrices in* $\mathrm{GRM}(n, n, \sigma^2)$. *Then there exist random matrices Z_1, Z_2 satisfying the following three conditions:*

- (i) *Z_1, Z_2 are independent random matrices in* $\mathrm{GRM}(n, n, \sigma^2)$.
- (ii) *The entries of Z_1 and Z_2 are Borel functions (in $2n^2$ complex variables) of the entries of Y_1 and Y_2.*
- (iii) $Z_1 Y_1^t + Z_2 Y_2^t = 0$.

Proof. Note first that (Y_1^t, Y_2^t) is also a pair of independent random matrices in the class $\mathrm{GRM}(n, n, \sigma^2)$. Let (Z_1^0, Z_2^0) be the pair of random matrices obtained by application of Lemma 2.3 to (Y_1^t, Y_2^t). Then Z_1^0, Z_2^0 are independent random matrices in $\mathrm{GRM}(n, n, \sigma^2)$, whose entries are Borel functions of the entries of Y_1 and Y_2, and furthermore:

$$Z_2^0 Y_1^t - Z_1^0 Y_2^t = 0.$$

Thus, the pair $(Z_1, Z_2) = (Z_2^0, -Z_1^0)$ satisfies all the requirements. ∎

3 Violation of Lower Bound in $\mathcal{A}(r, c)$

Let n be a positive integer, and consider the standard basis $\{\xi_1^{(n)}, \dots, \xi_n^{(n)}\}$ for $\mathbb{C}^n$. In the following we shall denote by η_n the unit vector in $\mathbb{C}^n \otimes \mathbb{C}^n$ defined as follows:

$$\eta_n = n^{-1/2} \sum_{j=1}^{n} \xi_j^{(n)} \otimes \xi_j^{(n)}.$$

3.1 Lemma. *Let q be a positive integer, and let $a_1, \dots, a_q$, $b_1, \dots, b_q$ be matrices in $M_n(\mathbb{C})$ satisfying that $\sum_{i=1}^q a_i b_i^t = 0 \in M_n(\mathbb{C})$. Then $\sum_{i=1}^q (a_i \otimes b_i)\eta_n = 0$.*

Proof. For any k, l in $\{1, 2, \dots, n\}$, we have

$$\begin{aligned}
\Big\langle \Big(\sum_{i=1}^{q} (a_i \otimes b_i)\eta_n \Big), \xi_k^{(n)} \otimes \xi_l^{(n)} \Big\rangle &= \sum_{i=1}^{q} \big\langle (a_i \otimes b_i)\eta_n, \xi_k^{(n)} \otimes \xi_l^{(n)} \big\rangle \\
&= n^{-1/2} \sum_{i=1}^{q} \sum_{j=1}^{n} \langle a_i \xi_j^{(n)}, \xi_k^{(n)} \rangle \cdot \langle b_i \xi_j^{(n)}, \xi_l^{(n)} \rangle \\
&= n^{-1/2} \sum_{i=1}^{q} \sum_{j=1}^{n} (a_i)_{kj} (b_i)_{lj} = n^{-1/2} \sum_{i=1}^{q} (a_i b_i^t)_{kl} \\
&= n^{-1/2} \Big(\sum_{i=1}^{q} a_i b_i^t \Big)_{kl} = 0.
\end{aligned}$$

Since the set $\{\xi_k^{(n)} \otimes \xi_l^{(n)} \mid k, l = 1, 2, \dots, n\}$ is a basis for $\mathbb{C}^n \otimes \mathbb{C}^n$, the calculation above shows that $\sum_{i=1}^q (a_i \otimes b_i)\eta_n = 0$. ■

In the following we consider for q in $\mathbb{N}$ the Cuntz algebra $\mathcal{O}_q$, i.e., the unital C^*-algebra generated by elements $s_1, \dots, s_q$ satisfying the conditions:

$$s_i^* s_j = \delta_{i,j} \mathbf{1} \ (i, j = 1, 2, \dots, q), \quad \text{and} \quad \sum_{i=1}^{q} s_i s_i^* = \mathbf{1}.$$

We shall consider $\mathcal{O}_q$ as acting on a Hilbert space $\mathcal{H}_q$.

3.2 Lemma. *Let r be an even positive integer and put $q = \frac{r}{2}$. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)}, \dots, Y_r^{(n)}$ in $\mathrm{GRM}(n, n, \frac{1}{n})$. Then, for each n, there exist random operators $b_1^{(n)}, \dots, b_r^{(n)} \colon \Omega \to \mathcal{O}_q \otimes M_n(\mathbb{C})$, such that the following conditions hold for almost all ω in Ω:*

(i) *For any vector ζ in $\mathcal{H}_q$, $\Big(\sum_{i=1}^r b_i^{(n)}(\omega) \otimes Y_i^{(n)}(\omega) \Big)(\zeta \otimes \eta_n) = 0$, for all n in $\mathbb{N}$.*

(ii) *For any positive ϵ, $\sum_{i=1}^r b_i^{(n)}(\omega)^* b_i^{(n)}(\omega) \geq ((\sqrt{r} - 1)^2 - \epsilon)\mathbf{1}_{\mathcal{O}_q \otimes M_n(\mathbb{C})}$, for n sufficiently large.*

(iii) *For any positive* ϵ, $\sum_{i=1}^r b_i^{(n)}(\omega)b_i^{(n)}(\omega)^* \leq ((\sqrt{2}+1)^2+\epsilon)\mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})}$, *for* n *sufficiently large.*

Proof. For n in $\mathbb{N}$ and j in $\{1,2,\dots,q\}$, let $Z_{2j-1}^{(n)}, Z_{2j}^{(n)}$ be the independent random matrices obtained by application of Corollary 2.4 to the random matrices $Y_{2j-1}^{(n)}, Y_{2j}^{(2n)}$. Since the entries of $Z_{2j-1}^{(n)}, Z_{2j}^{(n)}$ are Borel functions of the entries of $Y_{2j-1}^{(n)}, Y_{2j}^{(n)}$, it follows that $Z_1^{(n)}, Z_2^{(n)}, \dots, Z_r^{(n)}$ are r independent random matrices in $\mathrm{GRM}(n,n,\frac{1}{n})$. Moreover, for each j,

$$Z_{2j-1}^{(n)}\big(Y_{2j-1}^{(n)}\big)^t + Z_{2j}^{(n)}\big(Y_{2j}^{(n)}\big)^t = 0. \tag{3.1}$$

Consider next the Cuntz algebra $\mathcal{O}_q$, and let $s_1, s_2, \dots, s_q$ denote the canonical generators of $\mathcal{O}_q$. Then consider the random operators $b_1^{(n)}, \dots, b_r^{(n)} : \Omega \to \mathcal{O}_q \otimes M_n(\mathbb{C})$ defined by:

$$\begin{aligned} b_{2j-1}^{(n)} &= s_j \otimes Z_{2j-1}^{(n)}, \quad (j \in \{1,2,\dots,q\}),\\ b_{2j}^{(n)} &= s_j \otimes Z_{2j}^{(n)}, \qquad (j \in \{1,2,\dots,q\}), \end{aligned}$$

or equivalently,

$$b_i^{(n)} = s_{[\frac{i+1}{2}]} \otimes Z_i^{(n)}, \quad (i \in \{1,2,\dots,r\}).$$

We show that these random operators satisfy the conditions (i)-(iii).

Regarding (i), note that for any n in $\mathbb{N}$ and any vector ζ in the Hilbert space $\mathcal{H}_q$, we have

$$\begin{aligned} \Big(\sum_{i=1}^r b_i^{(n)} \otimes Y_i^{(n)}\Big)(\zeta\otimes\eta_n) &= \Big(\sum_{j=1}^q s_j \otimes \big(Z_{2j-1}^{(n)} \otimes Y_{2j-1}^{(n)} + Z_{2j}^{(n)} \otimes Y_{2j}^{(n)}\big)\Big)(\zeta\otimes\eta_n)\\ &= \sum_{j=1}^q s_j\zeta \otimes \big(Z_{2j-1}^{(n)} \otimes Y_{2j-1}^{(n)} + Z_{2j}^{(n)} \otimes Y_{2j}^{(n)}\big)\eta_n. \end{aligned}$$

Note here that by (3.1) and Lemma 3.1, $(Z_{2j-1}^{(n)} \otimes Y_{2j-1}^{(n)} + Z_{2j}^{(n)} \otimes Y_{2j}^{(n)})\eta_n = 0$, for each j, and hence by the above calculation it follows that (i) holds.

Regarding (ii), we have

$$\begin{aligned} \sum_{i=1}^r \big(b_i^{(n)}\big)^* b_i^{(n)} &= \sum_{i=1}^r s_{[\frac{i+1}{2}]}^* s_{[\frac{i+1}{2}]} \otimes \big(Z_i^{(n)}\big)^* Z_i^{(n)} = \sum_{i=1}^r \mathbf{1}_{\mathcal{O}_q} \otimes \big(Z_i^{(n)}\big)^* Z_i^{(n)}\\ &= \mathbf{1}_{\mathcal{O}_q} \otimes \sum_{i=1}^r \big(Z_i^{(n)}\big)^* Z_i^{(n)} = \mathbf{1}_{\mathcal{O}_q} \otimes T_n^* T_n, \end{aligned} \tag{3.2}$$

where T_n is the random $rn \times n$ matrix given by:

$$T_n = \begin{pmatrix} Z_1^{(n)} \\ Z_2^{(n)} \\ \vdots \\ Z_r^{(n)} \end{pmatrix}.$$

Note that actually $T_n \in \mathrm{GRM}(rn, n, \frac{1}{n})$, and hence it follows from the complex version of Silverstein's Theorem (cf. [Si] and [HT1, Theorem 7.1(ii)]) that

$$\lim_{n\to\infty} \lambda_{\min}(T_n^*T_n) = (\sqrt{r}-1)^2, \quad \text{almost surely,}$$

where $\lambda_{\min}(T_n^*T_n)$ denotes the smallest eigenvalue of $T_n^*T_n$. Hence, for almost all ω in Ω and any ϵ in $]0,\infty[$, there exists n_ω in $\mathbb{N}$, such that $T_n(\omega)^*T_n(\omega) \geq ((\sqrt{r}-1)^2-\epsilon)\mathbf{1}_n$, whenever $n \geq n_\omega$. Combining this with (3.2), it follows that (ii) holds.

Turning then to (iii), note that

$$\begin{aligned}\sum_{i=1}^{r} b_i^{(n)}\big(b_i^{(n)}\big)^* &= \sum_{j=1}^{q} s_js_j^* \otimes \big(Z_{2j-1}^{(n)}\big(Z_{2j-1}^{(n)}\big)^* + Z_{2j}^{(n)}\big(Z_{2j}^{(n)}\big)^*\big)\\ &= \sum_{j=1}^{q} s_js_j^* \otimes \big(R_j^{(n)}\big)^*R_j^{(n)},\end{aligned} \tag{3.3}$$

where, for each j,

$$R_j^{(n)} = \begin{pmatrix}\big(Z_{2j-1}^{(n)}\big)^*\\ \big(Z_{2j}^{(n)}\big)^*\end{pmatrix} \in \mathrm{GRM}(2n, n, \tfrac{1}{n}).$$

By the complex version of Geman's Theorem (cf. [Gem] and [HT1, Theorem 7.1(i)]), it follows that for each j,

$$\lim_{n\to\infty} \big\|\big(R_j^{(n)}\big)^*R_j^{(n)}\big\| = (\sqrt{2}+1)^2, \quad \text{almost surely.}$$

Hence, for almost all ω in Ω and any ϵ in $]0,\infty[$, there exists n_ω in $\mathbb{N}$, such that

$$R_j^{(n)}(\omega)^*R_j^{(n)}(\omega) \leq ((\sqrt{2}+1)^2+\epsilon)\mathbf{1}_n, \quad \text{whenever } n \geq n_\omega.$$

Combining this with (3.3), it follows that for almost all ω, we have

$$\begin{aligned}\sum_{i=1}^{r} b_i^{(n)}(\omega)b_i^{(n)}(\omega)^* &\leq \sum_{j=1}^{q} s_js_j^* \otimes ((\sqrt{2}+1)^2+\epsilon)\mathbf{1}_n = ((\sqrt{2}+1)^2+\epsilon)\cdot\Big(\sum_{j=1}^{q} s_js_j^*\Big)\otimes \mathbf{1}_n\\ &= ((\sqrt{2}+1)^2+\epsilon)\cdot \mathbf{1}_{\mathcal{O}_q}\otimes \mathbf{1}_n,\end{aligned}$$

whenever $n \geq n_\omega$. This verifies (iii). ■

3.3 Proposition. *Let c be a number in $[1,\infty[$, let r be an even positive integer such that $r \geq 12c$, and put $q = \frac{r}{2}$. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)}, \dots, Y_r^{(n)}$ in $\mathrm{GRM}(n, n, \frac{1}{n})$. Then, for each n, there exist random operators $a_1^{(n)}, \dots, a_r^{(n)}\colon \Omega \to \mathcal{O}_q \otimes M_n(\mathbb{C})$, such that the following conditions hold for almost all ω:*

(i) $\sum_{i=1}^{r} a_i^{(n)}(\omega)^*a_i^{(n)}(\omega) = c\mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})}$, *for n sufficiently large.*

(ii) $\sum_{i=1}^r a_i^{(n)}(\omega)a_i^{(n)}(\omega)^* \leq \mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})}$, *for n sufficiently large.*

(iii) *With $V_n = \sum_{i=1}^r a_i^{(n)} \otimes Y_i^{(n)}$, we have $0 \in \mathrm{sp}(V_n(\omega)^*V_n(\omega))$, for n sufficiently large.*

Proof. For each n, let $b_1^{(n)}, \dots, b_r^{(n)} : \Omega \to \mathcal{O}_q \otimes M_n(\mathbb{C})$ be the random operators described in Lemma 3.2, and let $\mathcal{S}$ denote the sure event in Ω, consisting of those ω for which all three conditions (i)-(iii) in Lemma 3.2 are satisfied. We note next that

$$0 < \frac{c((\sqrt{2}+1)^2+\epsilon)}{(\sqrt{r}-1)^2-\epsilon} < 1, \tag{3.4}$$

for ϵ sufficiently small in $]0,\infty[$. Indeed, since $r \geq 2$ the first inequality is obviously fulfilled for sufficiently small ϵ. Moreover,

$$(\sqrt{r}-1)^2 - \epsilon \geq (\sqrt{12c}-\sqrt{c})^2 - \epsilon \geq c\big((\sqrt{12}-1)^2-\epsilon\big) > 0,$$

for ϵ sufficiently small, so that

$$\frac{c((\sqrt{2}+1)^2+\epsilon)}{(\sqrt{r}-1)^2-\epsilon} \leq \frac{(\sqrt{2}+1)^2+\epsilon}{(\sqrt{12}-1)^2-\epsilon}.$$

Since $\frac{(\sqrt{2}+1)^2}{(\sqrt{12}-1)^2} < 1$, it follows that the second inequality in (3.4) holds for small enough ϵ.

Now, fix ϵ in $]0,\infty[$ such that (3.4) holds. Then, for each ω in $\mathcal{S}$, choose n_ω in $\mathbb{N}$ such that each of the conditions in (i)-(iii) of Lemma 3.2 is satisfied whenever $n \geq n_\omega$. It is not hard to see, that n_ω can be chosen in such a way that the mapping $\omega \mapsto n_\omega$ is measurable on $\mathcal{S}$. Then, for each n, we define $a_1^{(n)}, \dots, a_r^{(n)}$ as follows:

$$a_i^{(n)}(\omega) = \begin{cases} \mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})}, & \text{if } \omega \notin \mathcal{S}, \\ \mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})}, & \text{if } \omega \in \mathcal{S} \text{ and } n < n_\omega, \\ c^{1/2} b_i^{(n)}(\omega)\big(\sum_{k=1}^r b_k^{(n)}(\omega)^* b_k^{(n)}(\omega)\big)^{-1/2}, & \text{if } \omega \in \mathcal{S} \text{ and } n \geq n_\omega. \end{cases}$$

Note that by Lemma 3.2(ii) and the choice of n_ω and ϵ, $a_1^{(n)}, \dots, a_r^{(n)}$ are well-defined. Moreover, since the mapping $\omega \mapsto n_\omega$ is measurable, $a_1^{(n)}, \dots, a_r^{(n)}$ are random operators.

Consider now a fixed ω in $\mathcal{S}$. Then, whenever $n \geq n_\omega$, we have:

$$\sum_{i=1}^r a_i^{(n)}(\omega)^* a_i^{(n)}(\omega) = c\mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})},$$

and, by (ii) and (iii) in Lemma 3.2,

$$\begin{aligned}\sum_{i=1}^r a_i^{(n)}(\omega)a_i^{(n)}(\omega)^* &= c\sum_{i=1}^r b_i^{(n)}(\omega)\Big(\sum_{k=1}^r b_k^{(n)}(\omega)^* b_k^{(n)}(\omega)\Big)^{-1} b_i^{(n)}(\omega)^* \\ &\leq \Big(\frac{c}{(\sqrt{r}-1)^2-\epsilon}\Big)\cdot\sum_{i=1}^r b_i^{(n)}(\omega)b_i^{(n)}(\omega)^* \\ &\leq \Big(\frac{c((\sqrt{2}+1)^2+\epsilon)}{(\sqrt{r}-1)^2-\epsilon}\Big)\cdot\mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})} \\ &\leq \mathbf{1}_{\mathcal{O}_q\otimes M_n(\mathbb{C})},\end{aligned}$$

where the last inequality follows from (3.4) and the choice of ϵ. Thus, the random operators $a_1^{(n)}, \dots, a_r^{(n)}$ satisfy conditions (i) and (ii) in the proposition. Regarding condition (iii), let ζ be an arbitrary non-zero vector in $\mathcal{H}_q$ and consider the vector $\zeta \otimes \eta_n$ in $\mathcal{H}_q \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. Then, whenever $\omega \in \mathcal{S}$ and $n \geq n_\omega$, the vector

$$\xi = \Big[\Big(\sum_{k=1}^{r} b_k^{(n)}(\omega)^* b_k^{(n)}(\omega)\Big)^{1/2} \otimes \mathbf{1}_n\Big](\zeta \otimes \eta_n),$$

is non-zero too, and at the same time,

$$\big(V_n(\omega)\big)\xi = \Big(\sum_{i=1}^{r} a_i^{(n)}(\omega) \otimes Y_i^{(n)}(\omega)\Big)\xi = c^{1/2}\Big(\sum_{i=1}^{r} b_i^{(n)}(\omega) \otimes Y_i^{(n)}(\omega)\Big)(\zeta \otimes \eta_n) = 0,$$

by Lemma 3.2(i). This implies that $0 \in \mathrm{sp}(V_n(\omega)^* V_n(\omega))$, whenever $\omega \in \mathcal{S}$ and $n \geq n_\omega$, and thus $a_1^{(n)}, \dots, a_r^{(n)}$ satisfy (iii) too. ■

3.4 Definition. Assume that $r \in \mathbb{N}$ and $c \in [1, \infty[$, such that $r \geq c$. Then by $\mathcal{A}(r,c)$ we denote the universal unital C^*-algebra generated by r elements $a_1, a_2, \dots, a_r$ satisfying the relations:

$$\sum_{i=1}^{r} a_i^* a_i = c\mathbf{1} \quad \text{and} \quad \sum_{i=1}^{r} a_i a_i^* \leq \mathbf{1}. \qquad \square \tag{3.5}$$

Note that the condition $r \geq c$ is necessary and sufficient for the existence of $\mathcal{A}(r,c)$. Indeed, if $a_1, \dots, a_r$ are bounded operators on a Hilbert space $\mathcal{H}$, such that $\sum_{i=1}^r a_i a_i^* \leq \mathbf{1}$, then $\|a_i\|^2 = \|a_i a_i^*\| \leq 1$ for all i, and hence $\|\sum_{i=1}^r a_i^* a_i\| \leq \sum_{i=1}^r \|a_i\|^2 \leq r$. Conversely, let $s_1, \dots, s_r$ denote the generators of the Cuntz algebra $\mathcal{O}_r$, and put $a_i = \sqrt{\frac{c}{r}} s_i$, $i = 1, 2, \dots, r$. Then, if $r \geq c$, the operators $a_1, \dots, a_r$ satisfy condition (3.5).

3.5 Theorem. *Let c be a positive number in $[1, \infty[$, and let r be positive integer such that $r \geq 13c$. Consider the universal C^*-algebra $\mathcal{A}(r,c)$, and let $a_1, a_2, \dots, a_r$ be the canonical generators of $\mathcal{A}(r,c)$. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)}, \dots, Y_r^{(n)}$ in $\mathrm{GRM}(n, n, \frac{1}{n})$, and put $S_n = \sum_{i=1}^r a_i \otimes Y_i^{(n)}$. Then for almost all ω in Ω, we have*

$$0 \in \mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big), \quad \text{for } n \text{ sufficiently large.} \tag{3.6}$$

Proof. The proof is divided into two cases:

(i) In this case we assume that r is even. Then, since $r \geq 12c$, we may, for each n in $\mathbb{N}$, consider the random operators $a_1^{(n)}, \dots, a_r^{(n)} \colon \Omega \to \mathcal{O}_q \otimes M_n(\mathbb{C})$ described in Proposition 3.3 (recall that $q = \frac{r}{2}$). Let $\mathcal{S}$ be the sure event in Ω, consisting of those ω for which all three statements (i)-(iii) in Proposition 3.3 are satisfied. We show that (3.6) is satisfied for all ω in $\mathcal{S}$: Consider a fixed ω in $\mathcal{S}$, and then choose n_ω in Ω such that each of the conditions in (i)-(iii) of Proposition 3.3 is satisfied whenever $n \geq n_\omega$. Then, let n be a fixed positive integer, such that $n \geq n_\omega$, and consider the operators

$a_1^{(n)}(\omega), \dots, a_r^{(n)}(\omega)$ in $\mathcal{O}_q \otimes M_n(\mathbb{C})$. Since these operators satisfy the conditions in (i) and (ii) of Proposition 3.3, it follows by the universal property of $\mathcal{A}(r,c)$, that there exists a $*$-homomorphism $\Phi_\omega^{(n)} \colon \mathcal{A}(r,c) \to \mathcal{O}_q \otimes M_n(\mathbb{C})$, such that $\Phi_\omega^{(n)}(a_i) = a_i^{(n)}(\omega)$, for each i in $\{1,2,\dots,r\}$. Consider then the $*$-homomorphism

$$\Phi_\omega^{(n)} \otimes \mathrm{id}_n \colon \mathcal{A}(r,c) \otimes M_n(\mathbb{C}) \to \mathcal{O}_q \otimes M_n(\mathbb{C}) \otimes M_n(\mathbb{C}),$$

where id_n is the identity mapping on $M_n(\mathbb{C})$. Note that

$$\Phi_\omega^{(n)} \otimes \mathrm{id}_n(S_n(\omega)^* S_n(\omega)) = V_n(\omega)^* V_n(\omega),$$

where $V_n = \sum_{i=1}^r a_i^{(n)} \otimes Y_i^{(n)}$. This implies, in particular, that $\mathrm{sp}(S_n(\omega)^* S_n(\omega)) \supseteq \mathrm{sp}(V_n(\omega)^* V_n(\omega))$, and since $0 \in \mathrm{sp}(V_n(\omega)^* V_n(\omega))$ (cf. Proposition 3.3(iii)), we have verified that $0 \in \mathrm{sp}(S_n(\omega)^* S_n(\omega))$, whenever $n \geq n_\omega$. This concludes the proof of case (i).

(ii) In this case we assume that r is odd. Consider then, in addition, the C^*-algebra $\mathcal{A}(r-1,c)$, and let, at this point, $g_1, \dots, g_{r-1}$ denote the canonical generators of $\mathcal{A}(r-1,c)$. Then consider the operators $f_1, \dots, f_r$ in $\mathcal{A}(r-1,c)$ defined by

$$f_i = \begin{cases} g_i, & \text{if } i \in \{1,2,\dots,r-1\}, \\ 0, & \text{if } i = r. \end{cases}$$

Note that $\sum_{i=1}^r f_i^* f_i = c\mathbf{1}_{\mathcal{A}(r-1,c)}$ and $\sum_{i=1}^r f_i f_i^* \leq \mathbf{1}_{\mathcal{A}(r-1,c)}$, and hence, by the universal property of $\mathcal{A}(r,c)$, there exists a $*$-homomorphism $\Phi \colon \mathcal{A}(r,c) \to \mathcal{A}(r-1,c)$, such that $\Phi(a_i) = f_i$ for all i. Consider then, for each n, the $*$-homomorphism

$$\Phi \otimes \mathrm{id}_n \colon \mathcal{A}(r,c) \otimes M_n(\mathbb{C}) \to \mathcal{A}(r-1,c) \otimes M_n(\mathbb{C}),$$

and note that for all ω in Ω,

$$\Phi \otimes \mathrm{id}_n(S_n(\omega)^* S_n(\omega)) = W_n(\omega)^* W_n(\omega),$$

where $W_n = \sum_{i=1}^r f_i \otimes Y_i^{(n)}$. This implies, in particular, that

$$\mathrm{sp}(S_n(\omega)^* S_n(\omega)) \supseteq \mathrm{sp}(W_n(\omega)^* W_n(\omega)), \quad \text{for all } \omega \text{ in } \Omega \text{ and all } n \text{ in } \mathbb{N}. \tag{3.7}$$

Note here that

$$W_n = \sum_{i=1}^r f_i \otimes Y_i^{(n)} = \sum_{i=1}^{r-1} g_i \otimes Y_i^{(n)}.$$

Since $g_1, \dots, g_{r-1}$ are the canonical generators of the C^*-algebra $\mathcal{A}(r-1,c)$, and since $r-1 \geq 13c-1 \geq 12c$, it follows thus from case (i) proved above, that for almost all ω in Ω, $0 \in \mathrm{sp}(W_n(\omega)^* W_n(\omega))$ for n sufficiently large. Combining this with (3.7), we get the desired conclusion in case (ii) too. ■

4 Violation of lower bound in $\mathcal{B}(r,c)$

4.1 Definition. Assume that $r \in \mathbb{N}$ and $c \in [1,\infty[$, such that $r \geq c$. Then by $\mathcal{B}(r,c)$ we denote the universal unital C^*-algebra generated by r elements $b_1, b_2, \dots, b_r$ satisfying the relations

$$\sum_{i=1}^{r} b_i^* b_i = c\mathbf{1} \quad \text{and} \quad \sum_{i=1}^{r} b_i b_i^* = \mathbf{1}, \tag{4.1}$$

provided that these relations can be fulfilled in a C^*-algebra. $\square$

Regarding the existence of $\mathcal{B}(r,c)$, the condition $r \geq c$ is necessary by the same argument we gave when considering the question of existence for $\mathcal{A}(r,c)$. However, this condition is not sufficient to ensure that $\mathcal{B}(r,c)$ exists. In fact, for given r, there are values of c in $]r-1,r[$ for which $\mathcal{B}(r,c)$ does not exist! We shall not prove that assertion here, but merely verify that $\mathcal{B}(r,c)$ is well-defined whenever $c \in [1, r-1] \cup \{r\}$.

If $c = r$, then the canonical generators $s_1, \dots, s_r$ of the Cuntz algebra $\mathcal{O}_r$ satisfy (4.1). If $c = r-1$ then the canonical generators of the Cuntz algebra $\mathcal{O}_{r-1}$ together with 0 form r operators satisfying (4.1). Finally, if $c < r-1$, we have $r \geq [c]+2$. From [HT2, Lemma 8.3] it follows that there exist elements $x_1, x_2, \dots, x_{[c]+2}$ of the Cuntz algebra $\mathcal{O}_2$, satisfying that

$$\sum_{i=1}^{[c]+2} x_i^* x_i = c\mathbf{1}_{\mathcal{O}_2} \quad \text{and} \quad \sum_{i=1}^{[c]+2} x_i x_i^* = \mathbf{1}_{\mathcal{O}_2}.$$

Extending then the set $\{x_1, x_2, \dots, x_{[c]+2}\}$ by $r-[c]-2$ copies of 0, we obtain r operators satisfying (4.1).

We shall need the following lemma.

4.2 Lemma. *Let $\mathcal{H}$ be a Hilbert space, and let $a_1, \dots, a_r$ be elements of $\mathcal{B}(\mathcal{H})$, such that $\sum_{i=1}^{r} a_i^* a_i = c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ and $\sum_{i=1}^{r} a_i a_i^* \leq \mathbf{1}_{\mathcal{B}(\mathcal{H})}$ for some constant c, such that $1 \leq c \leq r$.*

Then there exist a Hilbert space $\tilde{\mathcal{H}}$, and elements $\tilde{a}_1, \dots, \tilde{a}_{r+1}$ of $\mathcal{B}(\tilde{\mathcal{H}})$, such that the following conditions hold:

(i) $\tilde{\mathcal{H}} \supseteq \mathcal{H}$.

(ii) $\tilde{a}_{i|\mathcal{H}} = \begin{cases} a_i, & \text{if } 1 \leq i \leq r, \\ 0, & \text{if } i = r+1. \end{cases}$

(iii) $\sum_{i=1}^{r+1} \tilde{a}_i^* \tilde{a}_i = c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}$ *and* $\sum_{i=1}^{r+1} \tilde{a}_i \tilde{a}_i^* = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}$.

Proof. The lemma, as well as its proof, is a slight modification of [HT2, Lemma 8.4]. Let $s_1, \dots, s_r$ be the canonical generators of the Cuntz algebra $\mathcal{O}_r$, acting on the Hilbert space $\mathcal{H}_r$. Put $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_r \otimes l_2(\mathbb{N}) \simeq l_2(\mathbb{N}, \mathcal{H} \otimes \mathcal{H}_r)$. Then an operator a in $\mathcal{B}(\tilde{\mathcal{H}})$ can be realized as a an infinite matrix $(a_{ij})_{i,j\in\mathbb{N}}$ with entries a_{ij} from $\mathcal{B}(\mathcal{H}\otimes\mathcal{H}_r)$. Next, we define a sequence of selfadjoint operators $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathcal{H})$ by:

$$h_1=\sum_{i=1}^r a_ia_i^* \quad\text{and}\quad h_{n+1}=\tfrac{1}{r}\big((c-1)\mathbf{1}_{\mathcal{B}(\mathcal{H})}+h_n\big),\ (n\in\mathbb{N}).$$

By the assumptions, $0\le h_1\le \mathbf{1}_{\mathcal{B}(\mathcal{H})}$. Moreover, since $1\le c\le r$, we have $\varphi([0,1])\subseteq[0,1]$, where φ is the map: $\varphi(t)=r^{-1}((c-1)+t)$, $t\in\mathbb{R}$. Hence, by induction, $0\le h_n\le\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ for all n in $\mathbb{N}$. Define now operators $\tilde{a}_1,\dots,\tilde{a}_r$ in $\mathcal{B}(\tilde{\mathcal{H}})$ by the diagonal matrices

$$\tilde{a}_i=\begin{pmatrix} a_i\otimes\mathbf{1}_{\mathcal{B}(\mathcal{H}_r)} & & & \mathrm{O}\\ & \sqrt{h_2}\otimes s_i & & \\ & & \sqrt{h_3}\otimes s_i & \\ \mathrm{O} & & & \ddots\end{pmatrix},\quad i=1,2,\dots,r,$$

and put

$$\tilde{a}_{r+1}=\begin{pmatrix} 0 & \sqrt{\mathbf{1}_{\mathcal{B}(\mathcal{H})}-h_1}\otimes\mathbf{1}_{\mathcal{B}(\mathcal{H}_r)} & & \mathrm{O}\\ & 0 & \sqrt{\mathbf{1}_{\mathcal{B}(\mathcal{H})}-h_2}\otimes\mathbf{1}_{\mathcal{B}(\mathcal{H}_r)} & \\ & & 0 & \ddots\\ \mathrm{O} & & & \ddots\end{pmatrix},$$

so that $(\tilde{a}_{r+1})_{ij}=0$ whenever $j\ne i+1$. Since $rh_{n+1}+(\mathbf{1}_{\mathcal{B}(\mathcal{H})}-h_n)=c\mathbf{1}_{\mathcal{B}(\mathcal{H})}$ for all n, it follows by standard calculations that

$$\sum_{i=1}^r\tilde{a}_i^*\tilde{a}_i=c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}\quad\text{and}\quad\sum_{i=1}^r\tilde{a}_i\tilde{a}_i{}^*=\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}.$$

Let η_1 be a unit vector in $\mathcal{H}_r$ and let $(\varepsilon_1,\varepsilon_2,\dots)$ be the standard basis for $l_2(\mathbb{N})$. Put

$$\iota_{\mathcal{H}}(\xi)=\xi\otimes\eta_1\otimes\varepsilon_1,\quad(\xi\in\mathcal{H}).$$

Then $\iota_{\mathcal{H}}\colon\mathcal{H}\to\tilde{\mathcal{H}}$ is an isometry, and

$$\tilde{a}_i\iota_{\mathcal{H}}=\begin{cases}\iota_{\mathcal{H}}a_i, & \text{if}\quad i\in\{1,2,\dots,r\},\\ 0, & \text{if}\quad i=r+1.\end{cases}$$

Thus, if we identify $\mathcal{H}$ by $\iota_{\mathcal{H}}(\mathcal{H})\subseteq\tilde{\mathcal{H}}$, via the isometry $\iota_{\mathcal{H}}$, the conditions (i), (ii) and (iii) are satisfied. ■

4.3 Lemma. *Let c be a real number in $[1,\infty[$ and let r be a positive integer such that $r\ge c$. Consider the universal C^*-algebra $\mathcal{A}(r,c)$, and assume that $\mathcal{A}(r,c)$ acts on the Hilbert space $\mathcal{H}$.*

Then, there exists a completely positive map $\Psi\colon\mathcal{B}(r+1,c)\to\mathcal{B}(\mathcal{H})$, satisfying that

$$\Psi(b_i^*b_j)=\begin{cases}a_i^*a_j, & \text{if } \max\{i,j\}\le r,\\ 0, & \text{if } \max\{i,j\}=r+1,\end{cases}\tag{4.2}$$

where $b_1,\dots,b_{r+1}$ are the canonical generators of $\mathcal{B}(r+1,c)$, and $a_1,\dots,a_r$ are the canonical generators of $\mathcal{A}(r,c)$.

Note before the proof, that $c \leq (r+1)-1$, and hence it follows from the discussion proceeding Definition 4.1 that $\mathcal{B}(r+1,c)$ is well-defined.

Proof of Lemma 4.3. By application of Lemma 4.2 to the operators $a_1,\dots,a_r \in \mathcal{A}(r,c) \subseteq \mathcal{B}(\mathcal{H})$, it follows that there exist a Hilbert space $\tilde{\mathcal{H}}$ and operators $\tilde{a}_1,\dots,\tilde{a}_{r+1}$ in $\mathcal{B}(\tilde{\mathcal{H}})$, such that the conditions (i)-(iii) in Lemma 4.2 are satisfied. In particular,

$$\sum_{i=1}^{r+1} \tilde{a}_i^*\tilde{a}_i = c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})} \quad \text{and} \quad \sum_{i=1}^{r+1} \tilde{a}_i\tilde{a}_i^* = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})},$$

and hence by the universal property of $\mathcal{B}(r+1,c)$, there exists a $*$-homomorphism $\Phi\colon \mathcal{B}(r+1,c) \to \mathcal{B}(\tilde{\mathcal{H}})$, such that $\Phi(b_i) = \tilde{a}_i$ for all i in $\{1,2,\dots,r+1\}$.

Next, let $P_{\mathcal{H}}\colon \tilde{\mathcal{H}} \to \mathcal{H}$ denote the orthogonal projection of $\tilde{\mathcal{H}}$ onto $\mathcal{H}$, and define the mapping $\Psi\colon \mathcal{B}(r+1,c) \to \mathcal{B}(\mathcal{H})$ by:

$$\Psi(b) = P_{\mathcal{H}}\Phi(b)_{|\mathcal{H}}, \quad (b \in \mathcal{B}(r+1,c)).$$

Then Ψ is a unital completely positive mapping, and for any i,j in $\{1,2,\dots,r+1\}$,

$$\Psi(b_i^*b_j) = P_{\mathcal{H}}\Phi(b_i)^*\Phi(b_j)_{|\mathcal{H}} = P_{\mathcal{H}}\tilde{a}_i^*\tilde{a}_{j|\mathcal{H}}.$$

Now, if $i,j \in \{1,2,\dots,r\}$, then by (ii) in Lemma 4.2, $\Psi(b_i^*b_j) = P_{\mathcal{H}}\tilde{a}_i^*\tilde{a}_{j|\mathcal{H}} = a_i^*a_j$. If $j = r+1$, we get similarly that $\Psi(b_i^*b_j) = 0$ by (ii) in Lemma 4.2. Finally, if $i = r+1$ then $P_{\mathcal{H}}\tilde{a}_i{}^* = 0$ by (ii) in Lemma 4.2, and hence also $\Psi(b_i^*b_j) = P_{\mathcal{H}}\tilde{a}_i^*\tilde{a}_{j|\mathcal{H}} = 0$.

Altogether, we have verified that Ψ has the desired properties. ∎

4.4 Theorem. *Let c be a real number in $[1,\infty[$ and let s be a positive integer such that $s \geq 14c$. Consider the universal C^*-algebra $\mathcal{B}(s,c)$, and let $b_1,b_2,\dots,b_s$ be the canonical generators. Consider further, for each n in $\mathbb{N}$, independent random matrices $Y_1^{(n)},\dots,Y_s^{(n)}$ in $\mathrm{GRM}(n,n,\frac{1}{n})$, and define: $T_n = \sum_{i=1}^{s} b_i \otimes Y_i^{(n)}$. Then for almost all ω in Ω, we have*

$$0 \in \mathrm{sp}\big(T_n(\omega)^*T_n(\omega)\big), \quad \textit{for } n \textit{ sufficiently large.} \tag{4.3}$$

Proof. Put $r = s-1$, and note that $r \geq 14c-1 \geq 13c$. Consider then the universal C^*-algebra $\mathcal{A}(r,c)$, and assume that $\mathcal{A}(r,c)$ acts on the Hilbert space $\mathcal{H}$. Let, as usual, $a_1,a_2,\dots,a_r$, denote the canonical generators of $\mathcal{A}(r,c)$, and define, for each n, $S_n = \sum_{i=1}^{r} a_i \otimes Y_i^{(n)}$, where $Y_1^{(n)},\dots,Y_r^{(n)}$ are the first r of the random matrices $Y_1^{(n)},\dots,Y_s^{(n)}$ set out in the theorem. It follows then from Theorem 3.5 that for almost all ω in Ω,

$$0 \in \mathrm{sp}\big(S_n(\omega)^*S_n(\omega)\big), \quad \text{for } n \text{ sufficiently large.} \tag{4.4}$$

By Lemma 4.3, there exists a unital completely positive mapping $\Psi\colon \mathcal{B}(s,c) \to \mathcal{B}(\mathcal{H})$, such that

$$\Psi(b_i^*b_j) = \begin{cases} a_i^*a_j, & \text{if } \max\{i,j\} \leq r, \\ 0, & \text{if } \max\{i,j\} = r+1. \end{cases}$$

Consider then, for each n in $\mathbb{N}$, the unital positive linear mapping

$$\Psi \otimes \mathrm{id}_n \colon \mathcal{B}(s,c) \otimes M_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}),$$

and note that

$$\begin{aligned}\Psi \otimes \mathrm{id}_n(T_n^* T_n) &= \Psi \otimes \mathrm{id}_n\Big(\sum_{i,j=1}^{s} b_i^* b_j \otimes (Y_i^{(n)})^* Y_j^{(n)}\Big) \\ &= \sum_{i,j=1}^{r} a_i^* a_j \otimes (Y_i^{(n)})^* Y_j^{(n)} = S_n^* S_n.\end{aligned} \tag{4.5}$$

This implies, that for any n in $\mathbb{N}$ and any ω in Ω, we have:

$$0 \in \mathrm{sp}\big(S_n(\omega)^* S_n(\omega)\big) \implies 0 \in \mathrm{sp}\big(T_n(\omega)^* T_n(\omega)\big). \tag{4.6}$$

Indeed, if $\omega \in \Omega$ and $n \in \mathbb{N}$ such that $0 \notin \mathrm{sp}(T_n(\omega)^* T_n(\omega))$, then $T_n(\omega)^* T_n(\omega) \geq \epsilon \mathbf{1}_{\mathcal{B}(s,c)\otimes M_n(\mathbb{C})}$ for some strictly positive number ϵ. By (4.5), and since $\Psi \otimes \mathrm{id}_n$ is unital and positive, this implies that $S_n(\omega)^* S_n(\omega) \geq \epsilon \mathbf{1}_{\mathcal{B}(\mathcal{H})\otimes M_n(\mathbb{C})}$, which, in turn, implies that $0 \notin \mathrm{sp}(S_n(\omega)^* S_n(\omega))$.

Combining then (4.6) with (4.4), it follows immediately that (4.3) holds for almost all ω in Ω. ■

4.5 Remark. The method of proof used above can also be used to show, that the upper bound in Theorem 1.2 is violated for the generators $b_1, \dots, b_s$ of $\mathcal{B}(s,c)$, when $s \geq 8c$:

Assume that $c \geq 1$ and $s \geq c+1$. Let q be the unique integer for which $c \leq q < c+1$, and put $r = s - q \geq 1$. Consider then the full C^*-algebra $C^*(\mathbb{F}_r)$ of the free group $\mathbb{F}_r$ on r generators, and assume that $C^*(\mathbb{F}_r)$ acts on the Hilbert space $\mathcal{H}$. Put $a_i = r^{-1/2} u_i$, $i = 1, \dots, r$, where $u_1, \dots, u_r$ are the unitary generators of $C^*(\mathbb{F}_r)$. Moreover, let $s_1, \dots, s_q$ be the generators of the Cuntz algebra $\mathcal{O}_q$ acting on the Hilbert space $\mathcal{H}_q$. Define then a sequence of real numbers $(\gamma_i)_{i\in\mathbb{N}}$ by the equations:

$$\gamma_1 = 1 \quad \text{and} \quad \gamma_{i+1} = \tfrac{1}{q}(q - c + \gamma_i), \ (i \in \mathbb{N}).$$

Since $1 \leq c \leq q$, we get by induction, that $\gamma_i \in [0,1]$ for all i in $\mathbb{N}$. Put $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_q \otimes l_2(\mathbb{N})$, and consider the operators $\tilde{a}_1, \dots, \tilde{a}_s$ in $\mathcal{B}(\tilde{\mathcal{H}})$ defined by:

$$\tilde{a}_i = \begin{pmatrix} a_i \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}_q)} & & & O \\ & \sqrt{\gamma_2} a_i \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}_q)} & & \\ & & \sqrt{\gamma_3} a_i \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}_q)} & \\ O & & & \ddots \end{pmatrix},$$

for $i = 1, 2, \dots, r$ and

$$\tilde{a}_{r+j} = \begin{pmatrix} 0 & & & & O \\ \sqrt{1-\gamma_2}\mathbf{1}_{\mathcal{B}(\mathcal{H})} \otimes s_j & 0 & & & \\ & \sqrt{1-\gamma_3}\mathbf{1}_{\mathcal{B}(\mathcal{H})} \otimes s_j & 0 & & \\ & & \sqrt{1-\gamma_4}\mathbf{1}_{\mathcal{B}(\mathcal{H})} \otimes s_j & \ddots & \\ O & & & \ddots & \end{pmatrix},$$

for $j = 1, 2, \ldots, q$. Then it is elementary to check that

$$\sum_{i=1}^{s} \tilde{a}_i^* \tilde{a}_i = c\mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})} \quad \text{and} \quad \sum_{i=1}^{s} \tilde{a}_i \tilde{a}_i^* = \mathbf{1}_{\mathcal{B}(\tilde{\mathcal{H}})}.$$

Moreover, with the embedding of $\mathcal{H}$ in $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_q \otimes l_2(\mathbb{N})$ defined in the proof of Lemma 4.2, one has

$$\tilde{a_i}^*{}_{|\mathcal{H}} = \begin{cases} a_i^*, & \text{if } 1 \le i \le r, \\ 0, & \text{if } r+1 \le i \le s. \end{cases}$$

Thus, as in the proof of Lemma 4.3, there exists a completely positive map $\Phi \colon \mathcal{B}(s, c) \to \mathcal{B}(\mathcal{H})$, such that

$$\Phi(b_i b_j^*) = \begin{cases} a_i a_j^*, & \text{if } \max\{i, j\} \le r, \\ 0, & \text{if } \max\{i, j\} > r. \end{cases}$$

This, together with the identity $\|tt^*\| = \|t\|^2$ for operators t on a Hilbert space, gives:

$$\Big\| \sum_{i=1}^{s} b_i \otimes Y_i^{(n)} \Big\|^2 \ge \Big\| \sum_{i=1}^{r} a_i \otimes Y_i^{(n)} \Big\|^2,$$

for all n in $\mathbb{N}$. Hence, it follows from [HT2, Proposition 4.9] that

$$\liminf_{n\to\infty} \Big\| \sum_{i=1}^{s} b_i \otimes Y_i^{(n)} \Big\|^2 \ge \left(\tfrac{8}{3\pi}\right)^2 r \ge \left(\tfrac{8}{3\pi}\right)^2 (s - c - 1),$$

almost surely. Thus, whenever $s \ge 8c$, we have

$$\liminf_{n\to\infty} \Big\| \sum_{i=1}^{s} b_i \otimes Y_i^{(n)} \Big\|^2 \ge \left(\tfrac{8}{3\pi}\right)^2 6c > 4c \ge (\sqrt{c} + 1)^2,$$

almost surely, which proves the assertion. □

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