NBI-HE-99-36
September 1999

Branched Polymers Re-Revisited

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MPS-RR 1999-36

ISSN 1398-599X

Abstract

We point out some misconceptions in a recent paper by H. Aoki et al. In particular, the claim that the two-point function of branched polymers behaves as p^{-4} instead of p^{-2} for large p is mistaken and in no way a precondition for the Hausdorff dimension of branched polymers having the well known value four.

1 Introduction

Branched polymers provide us with one of the simplest generalizations of random walk. The theory of branched polymers is well known (see e.g. [1] and references therein).

In a recent paper [2] it is claimed that the standard treatment of the theory of branched polymers, as presented for instance in [1], is not correct. In this note we show that this claim is false.

The problem under discussion concerns the propagation of branched polymers in target space. This is analogous to the propagation of a particle in space-time or the propagation of a bosonic string in 26-dimensional space-time. The grand canonical ensemble provides a convenient framework for developing the theory and this is how it is done in [1] and in many of the papers referred to therein. This approach is claimed to be erroneous in [2], where the discussion is based on the canonical ensemble. As we explain below, the basic requirement of the authors of [2] on the relation between the two approaches, namely that *the correlation functions in the grand canonical ensemble at the critical value of fugacity should reproduce those in the canonical ensemble for large N:*

$$\lim_{N\to\infty} \frac{G_N}{Z_N} = \lim_{\mu\to\mu_c} \frac{G_\mu}{Z_\mu}, \tag{1}$$

is not relevant, and in fact incorrect.

In addition, we explain that the three main conclusions of [2] i.e. (i) that the 2-point function $G_\mu(p)$ differentiated w.r.t. the "chemical" potential μ behaves like $1/p^4$ for large p, (ii) that the 3-point function behaves like

$$G^3_\mu(p,q) \sim G_\mu(p)G_\mu(q)G_\mu(p+q), \tag{2}$$

and (iii) that the 3-point function differentiated w.r.t. μ is given by

$$\frac{dG^3_\mu(p,q)}{d\mu} \sim G'_\mu(p)G_\mu(q)G_\mu(p+q)+G_\mu(p)G'_\mu(q)G_\mu(p+q)+G_\mu(p)G_\mu(q)G'_\mu(p+q), \tag{3}$$

(Eqs. (47-48) in [2]), are all obviously correct, and in fact trivial consequences of what the authors call the "naive" approach. However, their claim that only the differentiated three point function is consistent with a correct "thermodynamical" limit is misleading. Eq. (2) *is* the correct 3-point function, defined in the same way as one would define a 3-point function in string theory, and it does not contain any non-universal part. Eq. (3) is *not* the 3-point function but (by definition) the derivative of the 3-point function with respect to the chemical potential.

Finally, the authors of [2] give some comments concerning the nature of "baby universes" in the theory of branched polymers are which are supposed to support the claims made. We point out that these speculations are mistaken.

2 Explanation

Let us introduce some notation. The random walk representation of the Euclidean propagator is given by

$$G_\mu(x_1, x_2) = \sum_{N=1}^{\infty} e^{-\mu N} \int \prod_{i=1}^{N} dy_i \prod_{i=0}^{N} f(y_{i+1} - y_i), \quad y_{N+1} = x_2,\ y_0 = x_1, \tag{4}$$

where $f(x)$ is a suitable weight function which should fall off sufficiently fast. We call (4) the grand canonical partition function for the random walk, μ the chemical potential and $e^{-\mu}$ the fugacity. There is a critical value, μ_c, for μ above which the sum in (4) is convergent and below which it is divergent. We can write

$$G_\mu(x_1, x_2) = \sum_N e^{-\Delta\mu N} G_N(x_1, x_2), \qquad \Delta\mu = \mu - \mu_c, \tag{5}$$

and $G_N(x_1, x_2)$ is called the canonical partition function for the random walk, since the "internal" volume (number of steps) N is fixed. By Fourier transformation we define $G_\mu(p)$ and $G_N(p)$. Close to the critical point μ_c we have

$$G_\mu(p) \sim \frac{1}{\Delta\mu + p^2}, \qquad G_N(p) \sim e^{-p^2 N}, \tag{6}$$

such that

$$G_\mu(p) \sim \int dN\ e^{-\Delta\mu N} G_N(p). \tag{7}$$

It is clear that the relation (1) is not satisfied for random walks, and there is no reason why it should be satisfied, since the lefthand side is the heat kernel and the righthand side is the propagator.

The value of the Hausdorff dimension of the random walk follows from (7). Without going into a detailed derivation (which e.g. can be found in [1]), we simply note that N and $\Delta\mu$ as well as p and x are conjugate variables. In the scaling limit this leads to

$$\langle N \rangle_\mu \sim \frac{1}{\Delta\mu}, \quad |x| \sim \frac{1}{(\Delta\mu)^{\frac{1}{2}}} \qquad \text{i.e.} \quad \langle N \rangle_\mu \sim x^2, \tag{8}$$

where $\langle \cdot \rangle_\mu$ is the expectation in an ensemble of walks whose endpoints are separated by a distance x in imbedding space.

In the case of branched polymers one obtains similarly [1] for μ close to μ_c:

$$G_\mu(p) \sim \frac{1}{\sqrt{\Delta\mu} + p^2}. \tag{9}$$

This result is universal (as for the random walk) and contains no non-scaling part. Standard arguments [1], identical to the ones leading to the Eqs. (8) for the random walk, imply (in the scaling limit):

$$\langle N \rangle_\mu \sim \frac{1}{\Delta\mu}, \quad |x| \sim \frac{1}{(\Delta\mu)^{\frac{1}{4}}}, \qquad \text{i.e.} \quad \langle N \rangle_\mu \sim x^4, \tag{10}$$

which shows that the Hausdorff dimension of branched polymers is four.

As for the random walk the 2-point function G_μ for branched polymers does not satisfy (1), and for the same reason as before this fact does not disqualify it as the correct propagator. Of course the large-p behaviour of derivatives of $G_\mu(p)$ w.r.t. μ is

$$\frac{d^l G_\mu(p)}{d\mu^l} \sim \frac{1}{(\Delta\mu)^{l-\frac{1}{2}} p^4} + O(p^{-6}), \tag{11}$$

for $l > 0$. The fact that this large-p behavior of (11) agrees with the large-p behavior of $G_N(p)$ derived from (9) does not make it the correct propagator for branched polymers. In fact, it is simply the propagator for branched polymers with l marked vertices since each differentiation brings down a factor of N which is the number of ways we can choose a vertex to mark.

Next let us comment on Eqs. (2) and (3). The derivation of (2), as presented in [1], is acknowledged in [2], but the authors object that it does not satisfy (1). We repeat that it is not a relevant objection as we have explained, and contrary to their statements there are no non-universal terms contributing to this equation.

Eq. (3) is an immediate consequence of eq.(2) by differentiating w.r.t. μ on both sides. Indeed, it, and more generally eq. (52) of [2], can also be obtained before taking the scaling limit by writing the factor N coming from the differentiation as a sum of the number of vertices associated with each propagator in the appropriate ϕ^3-graph, taking care of the endpoint contributions for the propagators. In particular, (3) is by definition not the 3-point function for branched polymers, but the 3-point function for branched polymers with one additional marked point.

Let us finally comment on the remarks made in [2] about baby universes. It is claimed that one has a situation similar to two-dimensional gravity where a typical surface will consist of a "parent" universe dressed with small baby universes that are connected to the parent by a bottleneck. The argument is based on a relation for the 1-point function[1]:

$$G^1_N > \int dN' \; N' G_{N'} G^1_{N-N'}. \tag{12}$$

However, this relation is only (approximately) valid when the entropy exponent $\gamma < 0$. In fact, it is usually used to "derive" that $\gamma < 0$ in 2d gravity. For branched polymers $\gamma = 1/2$ and (12) is not valid (inserting $G^1_N \sim N^{\gamma-2} = N^{-3/2}$ clearly shows that (12) is violated for branched polymers (and for all $\gamma > 0$)). The reason is that *if* there are numerous "baby universes" of all sizes, then (12) cannot hold, because the decomposition of a graph with one marked point into a graph with two marked points of size N' and a graph with one marked point of size $N - N'$ is not unique. A more refined treatment [1] leads to the conclusion that $\gamma > 0$ implies $\gamma = 1/2$ under very general conditions. The scenario with baby universes of all sizes when $\gamma > 0$ has been verified in numerous computer simulations (using, by the way, the canonical ensemble!).

[1]For some reason Eq. (55) in [2] misses a combinatorial factor N', leading to erroneous conclusions.

Acknowledgement

J.A. and B.D. thank MaPhySTo – Centre for Mathematical Physics and Stochastic, funded by a grant from The Danish National Research Foundation – for support.

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