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Screening and D-brane Dynamics in Finite Temperature Superstring theory

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Abstract

The thermal dynamics of D-branes and of open superstrings in background gauge fields is studied. It is shown that D-brane dynamics forbids constant velocity motion at finite temperature. T-duality is used to interpret this feature as a consequence of the absence of an equilibrium state of charged strings at finite temperature in a constant background electric field, as a result of Debye screening of electric fields. The effective action for the Polyakov loop operator is computed and the corresponding screening solutions are described. The finite temperature theory is also used to illustrate the importance of carefully incorporating Wu–Yang terms into the string path integral for compact target spaces.

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1 Introduction and Summary

One of the fundamental problems in superstring theory is to understand the behaviour of strings in external fields. It is relevant to question about the existence and nature of possible non-perturbative vacuum states. Using duality, it is also important for understanding various aspects of D-brane dynamics. An example of an external field problem which has been studied extensively is that with a constant electromagnetic field [1]–[6]. Gauge fields couple to charges which live at the endpoints of open strings, and the modification of the vacuum energy by a slowly varying gauge field is known to be described by the Born–Infeld action [1]

$$S_{\rm BI} = -\frac{1}{(2\pi)^9(\alpha')^5 g_s}\int d^{10}x\ {\rm Tr}\left\{\sqrt{-\det_{\mu,\nu}\left(\eta_{\mu\nu}+2\pi e\alpha' F_{\mu\nu}\right)}+\ldots\right\} \tag{1.1}$$

where the factor of inverse string coupling comes from the fact that this is a tree level (disc) amplitude. Here and in the following ... will denote higher order terms in the (covariant) derivative expansion of the external fields. There are corrections to the effective action (1.1) at the one-loop (annulus diagram) and higher loop orders. By T-duality, the Born–Infeld action gives the effective dynamics of D-branes [7] and is very useful in determining the couplings of degrees of freedom in the D-brane to string theory as well as supergravity degrees of freedom. In this paper we will study the effective action for gauge fields in a thermal state of superstring theory. We will be particularly interested in how the Born–Infeld action (1.1) is modified by temperature and what this modification implies about electric fields. We shall compute the leading temperature corrections to (1.1) for weakly-coupled strings and slowly-varying external fields.

In the path integral approach to finite temperature superstring theory, the spacetime is taken as ten dimensional Euclidean space with time x^0 compactified on a circle of circumference $\beta = 1/k_{\rm B}T$, where $k_{\rm B}$ is Boltzmann's constant and T is the temperature. Then, Euclidean external gauge fields must be periodic in x^0,

$$A_\mu(x^0,\vec{x}) = A_\mu(x^0+\beta,\vec{x}) \tag{1.2}$$

In this case, it is impossible to fix a gauge where $A_0 = 0$. On the other hand, it is possible to fix a gauge where A_0 is independent of the Euclidean time x^0 and is diagonal, $(A_0)_{bc}(x^0,\vec{x}) = a(\vec{x})^b\,\delta_{bc}$. We will consider the special case where the external gauge field is in an Abelian $U(1)$ subgroup of the full non-Abelian Chan–Paton gauge group, and set all spatial components of the gauge fields to 0. The generator of the Abelian subgroup is characterized by the eigenvalues e_b which are interpreted as $U(1)$ charges and

$$A_0(\vec{x}) = \begin{pmatrix} e_1 & 0 & 0 & \ldots \\ 0 & e_2 & 0 & \ldots \\ 0 & 0 & e_3 & \ldots \\ \ldots & & & \ldots \end{pmatrix} a(\vec{x}) \tag{1.3}$$

The disc amplitude (1.1) is unmodified at finite temperature, because the disc worldsheet cannot wrap the cylindrical target space and thus cannot distinguish between a compactified and an un-compactified spacetime. The first corrections due to temperature appear in the annulus amplitude. We shall find that the free energy is given by the functional

$$F[a] = \int d^9x \,\left\{ \frac{1}{(2\pi)^7(\alpha')^5 g_s} \sum_b \sqrt{1 + \left(2\pi\alpha' e_b \vec{\nabla} a(\vec{x})\right)^2} + \sum_{b,c} V_{\rm eff}\left[(e_b - e_c)a(\vec{x})\right] + \ldots \right\} \tag{1.4}$$

where, for the type I NSR superstring, the effective potential is

$$V_{\rm eff}\,[z] = \frac{32\pi}{(2\pi^2\alpha')^5} \int_0^\infty \frac{dt}{t^6}\, \Theta_2 \left(\frac{\beta}{\pi}\, z \,\middle|\, \frac{i\beta^2}{2\pi^2\alpha' t} \right) \prod_{n=1}^\infty \left(\frac{1 + \,{\rm e}^{\,-2\pi n t}}{1 - \,{\rm e}^{\,-2\pi n t}} \right)^8 \tag{1.5}$$

and Θ_α denote the standard Jacobi theta-functions.

The dependence of the effective potential on the temporal component of the gauge field is familiar from finite temperature gauge theory. In particular, it implies Debye screening of electric fields. Consider the linearized equation for the minima of the free energy, which after rescaling can be written in the form

$$-\,\vec{\nabla}^2 a(\vec{x}) + \mu^2 a(\vec{x}) = 0 \tag{1.6}$$

This equation has exponentially decaying solutions $a(\vec{x}) \sim \,{\rm e}^{\,-\mu|\vec{x}|}$ where, at low temperatures ($\sqrt{\alpha'}k_{\rm B}T \ll 1$), the Debye screening mass μ is given by

$$\mu^2 = 3\pi^2 \cdot 2^{22}\, g_s(\alpha')^3 (k_{\rm B}T)^8 \, \frac{\sum_{b,c}(e_b - e_c)^2}{\sum_b e_b^2} + O\left({\rm e}^{\,-1/\sqrt{\alpha'}k_{\rm B}T} \right) \tag{1.7}$$

It is clear for this reason that constant electric fields cannot be extrema of the effective action. Another way to see that the existence of constant electric fields is incompatible with the existence of a Debye mass is the following. If we try to make a constant Abelian electric field by choosing the background gauge field

$$a(\vec{x}) = \vec{E} \cdot \vec{x} \tag{1.8}$$

then the integral over $\vec{x}$ that one would have to do in computing the temperature corrections to the free energy would vanish because of the property

$$\int d^9x \;\Theta_2 \left(\frac{\beta}{\pi}\, \vec{E} \cdot \vec{x} \,\middle|\, \frac{i\beta^2}{2\pi^2\alpha' t} \right) = 0 \tag{1.9}$$

which is a consequence of supersymmetry. This property can be interpreted as a result of Debye screening, i.e. that the finite temperature string theory forbids constant electric fields. All states except the ground state contain excitations of charged particles and therefore have infinite energy. In fact, because of the Schwinger mechanism [5], even the ground state is unstable.

In the following sections we will demonstrate this result directly using the boundary state formalism. Two different gauges are commonly used to study constant electric fields, the static gauge, $(A_0, \vec{A}) = (\vec{E}\cdot\vec{x}, 0)$, and the temporal gauge, $(A_0, \vec{A}) = (0, -\vec{E}x^0)$. In the temporal gauge, the gauge potential is not periodic in Euclidean time, but it is periodic up to a gauge transformation,

$$\vec{A}(x^0+\beta, \vec{x}) = \vec{A}(x^0, \vec{x}) + \vec{\nabla}\left(-\beta\vec{E}\cdot\vec{x}\right) \tag{1.10}$$

In this case, it is necessary to augment the usual coupling of the edge of a charged open string to the gauge field by adding a Wu–Yang term [8], $\oint A_\mu\, dx^\mu \to \oint A_\mu\, dx^\mu + \chi$, in order to compensate the gauge transformation (1.10). The geometrical reasoning for adding this term is explained in Appendix A. In the next section we shall use the example of a relativistic charged particle to illustrate how the Wu–Yang term is essential if the partition function is to be independent of the gauge choice. One byproduct of the following analysis will therefore be the importance of incorporating such terms into the superstring path integral for generic target space compactifications involving external fields.

Under T-duality, external gauge fields map onto the trajectories of D-branes, whose long wavelength dynamics are described by supergravity. Thermal states of D-branes are of interest as non-BPS states of superstring theory. In the supergravity picture, they have a natural Hawking temperature and radiation which can be interpreted as the emission of closed string modes by a non-extremal D-brane configuration. It has been suggested [9] that the gravitational Hawking temperature and the temperature of a Boltzmann gas of D-branes should be identified. It is therefore natural to study the thermodynamics of D-branes and to understand the special aspects of D-brane dynamics which arise when they are in a thermal state. It has been shown [10] that the effective action for a gas of D0-branes which is obtained by summing over physical (GSO projected) superstring states is

$$S_{\rm eff}[\vec{x}] = \int_0^\beta d\tau\ \left\{\frac{1}{\sqrt{\alpha'}g_s}\sqrt{1+\dot{\vec{x}}(\tau)^2} - \frac{256}{\beta}\ {\rm e}^{\,-2\beta|\vec{x}|/2\pi\alpha'} + \ldots\right\} \tag{1.11}$$

The first term, which is the relativistic free particle Lagrangian for the D0-branes, follows from the Born–Infeld action (1.1) and T-duality. It is correct up to higher derivatives of the relative position vector $\vec{x}(\tau)$ with respect to the Euclidean time τ. The second term, which we have given the low temperature limit of, comes from the annulus amplitude and is valid only for static branes. Corrections to the annulus amplitude at finite temperature which take into account the time dependence of $\vec{x}$ are not known.

Unlike at zero temperature, where the system of static D0-branes is a BPS state and this potential would vanish due to supersymmetry, temperature breaks supersymmetry and leaves the residual short-ranged attractive interaction. The potential in Eq. (1.11) was obtained from the superstring annulus free energy with Dirichlet boundary conditions [11]

$$F[\vec{x},\beta,a_0] = \frac{8}{\pi\sqrt{2\pi\alpha'}}\int_0^\infty \frac{dt}{t^{3/2}}\ {\rm e}^{\,-\vec{x}^{\,2}t/2\pi\alpha'}\ \Theta_2\left(\frac{\beta}{\pi}\,a_0\,\middle|\,\frac{i\beta^2}{2\pi^2\alpha' t}\right)\prod_{n=1}^\infty\left(\frac{1+\ {\rm e}^{\,-2\pi nt}}{1-\ {\rm e}^{\,-2\pi nt}}\right)^8 \tag{1.12}$$

and incorporating the dynamics of the time component of the gauge field $a_0(\tau)$ on the D-particle worldline. On the other hand, at zero temperature the motion of an assembly of D0-branes at constant relative velocity $\vec{v} = \dot{\vec{x}}$ also breaks supersymmetry. The leading velocity dependent gravitational attraction between a pair of D0-branes in ten dimensional supergravity is proportional to $\vec{v}^{\,4}/|\vec{x}|^7$ [7, 12]. The exact one-loop potential at zero temperature is again given by the annulus amplitude which in this case is [7, 13, 14]

$$F[\vec{x},\vec{v}] = \frac{1}{\sqrt{2\pi\alpha'}} \int_0^\infty \frac{dt}{t}\ \mathrm{e}^{-b^2t/2\pi\alpha'}\ \frac{\Theta_1\left(\frac{\epsilon t}{2}\middle|\, it\right)^4}{\Theta_1(\epsilon t|it)} \left[\mathrm{e}^{-\pi t/12} \prod_{n=1}^\infty \left(1-\mathrm{e}^{-2\pi n t}\right)\right]^{-9} \tag{1.13}$$

where b is the impact parameter for the scattering and ϵ is the relative rapidity of the two branes. In the following we will describe the situation when one tries to describe these two vacuum amplitudes collectively.

T-duality maps free open strings to open strings whose endpoints are attached to D-branes. It is implemented in a straightforward way when the string action depends only on worldsheet derivatives of a string embedding coordinate. It then proceeds by replacing $\partial_a x^i$ by $i\epsilon_{ab}\partial_b x^i$ and the resulting replacement of Neumann boundary conditions for x^i with Dirichlet conditions. We are interested in examining the T-dual of the string in a constant electric field $\vec{E}$. This should produce an open string whose endpoints are constrained by Dirichlet boundary conditions to end on a D-brane which is moving with a constant velocity $\vec{v} = 2\pi\alpha' e\vec{E}$ [7]. Indeed, in the temporal gauge, the coupling of the constant electric field $\exp ie\oint \vec{E}x^0\cdot\partial_{\rm t}\vec{x}$ is replaced by the vertex operator for a moving D-brane $\exp -e\oint \vec{E}x^0\cdot\partial_{\rm n}\vec{x}$ [15], where $\partial_{\rm t}$ and $\partial_{\rm n}$ are the derivatives tangential and normal to the boundary of the string worldsheet, respectively. However, we shall find that, at finite temperature, both of these couplings must be augmented by a Wu–Yang term which breaks the translation invariance in the direction of the electric field $\vec{E}$. The T-dual of the Wu–Yang term can be obtained in principle, but it is a complicated, non-local expression. We shall find, however, that it has a simple presentation within the boundary state formalism. Once it is taken into account, we can show that, just as Debye screening forbids constant electric fields in open superstring theory, it also forbids the constant motion of D-branes. This implies that there is a damping of their motion analogous to Debye screening.

2 Thermodynamic Partition Function in a Constant Electric Field

2.1 Relativistic Particle

To illustrate the general ideas, it is instructive to begin with the case of a relativistic charged scalar particle in a constant electric field at finite temperature. The extension to strings will be straightforward and the final result can be represented as a sum over particles with masses given by the open string spectrum and appropriate degeneracies. The diagonal elements of the (unnormalized) thermal density matrix are given by the Euclidean path integral

$$\rho(\vec{y},\vec{y};\beta) = \frac{1}{2}\int_0^\infty \frac{ds}{s}\ \mathrm{e}^{-\frac{1}{2}M^2 s} \int Dx^\mu(t)\ \mathrm{e}^{-\frac{1}{2}\int_0^s dt\, \dot{x}_\mu^2(t)}\, \Phi[x^\mu(t)], \tag{2.1}$$

where $\Phi[x^\mu(t)]$ is the Abelian phase factor in the given electromagnetic background. The temperature $T = 1/k_\mathrm{B}\beta$ enters via the boundary conditions

$$\begin{aligned} x^0(s) &= x^0(0) + n\beta \quad (\text{integer } n), \\ \vec{x}(s) &= \vec{x}(0) = \vec{y}, \end{aligned} \tag{2.2}$$

which compactifies the Euclidean time coordinate x^0 on a circle of circumference β. The parameter s plays the role of the proper time during which the particle propagates along the given trajectory $x^\mu(t)$ that can wind n times around the space-time cylinder, while M is the particle mass. The path integral (2.1) is to be summed over all winding numbers $n \in \mathbf{Z}$.

The constant electric field can be described in the static gauge by the vector potential

$$\begin{aligned} A_0(x) &= \frac{2\pi\nu}{\beta} - \vec{F}\cdot\vec{x}, \\ \vec{A}(x) &= \vec{0}\,, \end{aligned} \tag{2.3}$$

where we have denoted the "temporal" component F_{0i} of the Euclidean field strength tensor by $\vec{F} \equiv F_i (i = 1, \ldots, d-1)$. It is related to the electric field $\vec{E}$ in Minkowski space by

$$\vec{F} = i\vec{E}. \tag{2.4}$$

The constant ν in Eq. (2.3) can always be taken to lie in the interval $[0, 1)$ due to the periodicity of x^0. It plays a crucial role at finite temperature as we will discuss, but for now ν is simply associated with choosing a reference point where the potential A_0 vanishes. Since A_μ is single valued for the gauge choice (2.3), the phase factor is simply

$$\Phi[x^\mu(t)] \equiv \exp i\int_0^s dt\ A_\mu\big(x(t)\big)\,\dot{x}^\mu(t) = \exp\left(2\pi i\nu n - i\int_0^s dt\ \vec{F}\cdot\vec{x}(t)\,\dot{x}^0(t)\right). \tag{2.5}$$

Note that one can choose a more general gauge

$$\begin{aligned} A_0(x) &= \frac{2\pi\nu}{\beta} - (1-c)\vec{F}\cdot\vec{x}, \\ \vec{A}(x) &= c\vec{F}x^0, \end{aligned} \tag{2.6}$$

so that (2.3) corresponds to the choice $c = 0$. For this choice, $\vec{A}$ is multivalued due to the boundary condition (2.2) so that the Wu–Yang term [8] has to be included in the phase factor which now takes the form

$$\Phi[x^\mu(t)] \\ = \exp\left(2\pi i\nu n - i(1-c)\int_0^s dt\ \vec{F}\cdot\vec{x}(t)\,\dot{x}^0(t) + ic\int_0^s dt\ x^0(t)\,\vec{F}\cdot\dot{\vec{x}}(t) - icn\beta\vec{F}\cdot\vec{x}(0)\right). \tag{2.7}$$

This formula is derived in Appendix A. Integrating by parts, it is easy to see that (2.5) and (2.7) coincide as they should due to gauge invariance.

Substituting (2.5) in Eq. (2.1), we get a Gaussian path integral of the form of that for the harmonic oscillator [16]. It is convenient to choose coordinates where $\vec{F} = (F, 0, \ldots, 0)$. The result then reads

$$\rho(\vec{y},\vec{y};\beta) = \frac{\beta}{2}\int_0^\infty \frac{ds}{s}\ \mathrm{e}^{-\frac{1}{2}M^2 s}\,\frac{1}{(2\pi s)^{d/2-1}}\,\frac{F}{4\pi\sinh\frac{Fs}{2}}\,\Theta_3\left(\nu - \frac{\beta\vec{F}\cdot\vec{y}}{2\pi}\,\middle|\,\frac{i\beta^2 F}{4\pi\tanh\frac{Fs}{2}}\right) \tag{2.8}$$

where Θ_3 is the usual Jacobi theta function

$$\Theta_3\left(\nu\,|i\tau\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi\tau n^2 + 2\pi i\nu n} \tag{2.9}$$

and d is the dimension of space-time. Eq. (2.8) is similar to the density matrix for (spinor) quantum electrodynamics at finite temperature in the presence of a constant background electric field [17]. The path integral derivation of it is sketched in Appendix B.

The formula (2.8) resembles the density matrix for a harmonic oscillator of frequency F. This feature can be easily understood in second quantization, where the eigenvalues of the operator $i\partial_0$ which enters the Klein–Gordon operator are given by the Matsubara frequencies $2\pi m/\beta$, which results at each level m in the Hamiltonian

$$H_m = -\frac{1}{2}\vec{\nabla}^{\,2} - \frac{1}{2}\Big(\partial_0 - iA_0\Big)^2 = -\frac{1}{2}\vec{\nabla}^{\,2} + \frac{1}{2}\left(\frac{2\pi(m-\nu)}{\beta} + Fx^1\right)^2 \tag{2.10}$$

for the harmonic oscillator of frequency F oscillating along the axis 1 with a shifted position of x^1. The density matrix is then given by

$$\rho(\vec{y},\vec{y};\beta) = \frac{1}{2}\int_0^\infty \frac{ds}{s}\ \mathrm{e}^{-\frac{1}{2}M^2 s}\,\frac{1}{(2\pi s)^{d/2-1}}\sqrt{\frac{F}{2\pi\sinh Fs}}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-F\tanh\frac{Fs}{2}\left(y^1 + \frac{2\pi(m-\nu)}{\beta F}\right)^2}. \tag{2.11}$$

After a Poisson resummation which is the statement that

$$\Theta_3(\nu|i\tau) = \frac{1}{\sqrt{\tau}}\ \mathrm{e}^{-\pi\nu^2/\tau}\,\Theta_3\left(\frac{i\nu}{\tau}\,\middle|\,\frac{i}{\tau}\right), \tag{2.12}$$

we get Eq. (2.8).

The dependence on $\vec{y}$ in Eq. (2.8) implies the violation of translational invariance of the theory. Its appearance is most easily understood in the temporal gauge (given by (2.6) with $c = 1$) where only $\vec{A}$ depends on x. The term $-\beta\vec{F}\cdot\vec{y}/2\pi$ in the first argument of the theta function in (2.8) comes in this gauge entirely from the Wu–Yang term

$$-n\beta\vec{F}\cdot\vec{x}(0) = -n\beta\vec{F}\cdot\vec{y}\,, \tag{2.13}$$

which was added to the exponent of the phase factor Φ to guarantee global gauge invariance.

If we were instead to choose a vector potential which is a periodic function of $t = x^0$, then the gauge field would be single-valued and no Wu–Yang term would be required. The simplest example is a field $\vec{A}(t)$ which is a linear function of t for $0 \leq t < \beta$ and then returns to its initial value $\vec{A}(0)$ at $t = \beta$,

$$\vec{A}_{\rm per}(t) = \vec{F}t\Big(1 - \theta(t-\beta)\Big), \tag{2.14}$$

where θ is the step function. The thermal density matrix is then translationally invariant:

$$\rho_{\rm per}(\vec{y},\vec{y};\beta) = \frac{\beta}{2}\int\limits_0^\infty \frac{ds}{s}\;{\rm e}^{-\frac{1}{2}M^2 s}\,\frac{1}{(2\pi s)^{d/2-1}}\,\frac{F}{4\pi\sinh\frac{Fs}{2}}\,\Theta_3\left(\nu\,\middle|\,\frac{i\beta^2 F}{4\pi\tanh\frac{Fs}{2}}\right)\,. \tag{2.15}$$

The choice of periodic vector potential (2.14) differs from what is considered above for a constant electric field since

$$\dot{\vec{A}}_{\rm per}(t) = \vec{F} - \vec{F}\beta\,\delta(t-\beta). \tag{2.16}$$

The lesson to be learned from this example is that the thermal partition function in a constant electric field is trivial since the integration of Eq. (2.8) over $\vec{y}$ picks up only the term of winding number $n = 0$ and the temperature dependence disappears. There are no excited states in the external field and only the ground state exists as a stable configuration of the charged particle. This feature occurs because there is no globally defined periodic vector potential $\vec{A}(x^0)$ in this case as is required by the standard formulation. Only in time-dependent backgrounds, such as Eq. (2.14) which involves a time-localized point source of electric field, does there exist excited configurations of the system. In this latter case, the imaginary part of Eq. (2.15) at the poles of the contour integration, i.e. at $Fs = 2\pi i\ell$ (integer ℓ), yields the standard (zero temperature) Schwinger probability amplitude for the creation of charged particle pairs in scalar quantum electrodynamics. Note that there is a big difference between constant electric and magnetic fields, because in the latter case the field strength tensor does not have a component in the compactified direction and the magnetic field would enter only in the pre-exponential factors of the thermal density matrix, while the argument of the theta function would be the same as without the field.

2.2 Strings

Consider an open superstring with independent $U(1)$ charges e_1 and e_2 at its endpoints in an electromagnetic background. Here, strictly speaking, by an Abelian background field we mean a field in an Abelian subgroup of the Chan–Paton gauge group $O(32)$ of type-I superstring theory. We are interested in the case $e_1 \neq e_2$. However, unitarity requires the existence of neutral strings in the spectrum. Consider a string scattering amplitude with the given charges at the endpoints. An amplitude with an even number of external legs can be sliced in many different ways into intermediate states. Some of these intermediate states will consist of open strings with the charge e_1 or e_2 at both of its endpoints. An amplitude with an odd number of external legs necessarily involves at least one neutral string in the scattering process. Although these neutral string states will not be relevant to Debye screening or the corresponding T-dual D-brane dynamics, they do contribute to the total scattering amplitudes which will involve sums of the form

$$\mathcal{F} = \frac{1}{2} \sum_{e_1, e_2 \in \mathcal{Q}} \mathcal{F}(e_1, e_2) \tag{2.17}$$

where $\mathcal{Q}$ is the set of charges in the decomposition of the fundamental representation of the open superstring Chan–Paton gauge group under the embedding of $U(1)$ induced by the background electromagnetic field.

The open superstring partition function in a constant magnetic field at finite temperature has been calculated in [6]. We will now show that charged superstrings at finite temperature forbid constant electric fields. The bosonic sector of this system can be described in first quantization by the Polyakov path integral

$$\mathcal{F}_b(e_1, e_2) = \int Dg_{ab} \int Dx^\mu \ \mathrm{e}^{\,-S_b[g_{ab}, x^\mu]} \tag{2.18}$$

where the action in the conformal gauge and in Euclidean spacetime is

$$S_b = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z \ \partial_z x_\mu \, \partial_{\bar{z}} x^\mu + i e_1 \oint_{\partial_1 \Sigma} A_\mu(x) \ dx^\mu + i e_2 \oint_{\partial_2 \Sigma} A_\mu(x) \ dx^\mu \tag{2.19}$$

Finite temperature only affects the system when the string worldsheet Σ can wrap the compact time direction, so the leading string diagram of interest is the annulus[1] with local coordinates $z = \rho \, \mathrm{e}^{\,2\pi i \sigma}$ with $0 \leq \sigma \leq 1$ and $a \leq \rho \leq 1$, where $a = \mathrm{e}^{\,-t}$ is the Teichmüller parameter of the annulus. We enforce the periodicity constraint in the Euclidean time by substituting

$$x^0(\rho, \sigma) \ \mapsto \ x^0(\rho, \sigma) + n\beta\sigma \tag{2.20}$$

and then later on summing the path integral over all winding numbers $n \in \mathbf{Z}$.

[1]The Möbius strip has only a single connected boundary, so that only neutral strings contribute to the Möbius amplitude. As mentioned above, neutral superstrings will not play a significant role in the subsequent analysis.

We shall choose the periodic gauge field configuration (2.3). With these choices the action (2.19) becomes

$$\begin{aligned} S_b &= \frac{1}{4\pi\alpha'}\int d^2z\ \partial_z x_\mu\,\partial_{\bar z}x^\mu + 2\pi i\nu n(e_2-e_1) + \frac{n^2\beta^2 t}{8\pi^2\alpha'} \\ &\quad + ie_2 n\beta\int_0^1 d\sigma\ \vec F\cdot\vec x(1,\sigma) - ie_1 n\beta\int_0^1 d\sigma\ \vec F\cdot\vec x(a,\sigma) \\ &\quad + ie_2\int_0^1 d\sigma\ \vec F\cdot\vec x(1,\sigma)\,\partial_\sigma x^0(1,\sigma) - ie_1\int_0^1 d\sigma\ \vec F\cdot\vec x(a,\sigma)\,\partial_\sigma x^0(a,\sigma). \end{aligned} \tag{2.21}$$

There is a zero mode on the annulus given by $x^\mu = y^\mu =$ const. Normally, the action does not depend on it and integrating it out in the path integral produces a factor βV_{d-1}, with V_{d-1} the volume of space. In the present case it contributes to Eq. (2.18) the factor

$$\beta\int d^{d-1}y\ \mathrm{e}^{\,i(e_2-e_1)n\beta\vec F\cdot\vec y} = \beta\,(2\pi)^{d-1}\,\delta^{(d-1)}\Big((e_2-e_1)n\beta\vec F\Big). \tag{2.22}$$

So unless $n = 0$ (the zero temperature condition), or $\beta = 0$ (the dual of the zero temperature condition), or $e_1 = e_2$ (neutral strings), the free energy of the string gas is zero. Thus for charged strings at finite temperature in a constant background electric field, the partition function picks up only the β independent $n = 0$ sector.

This simple calculation exemplifies the fact that the translation non-invariance of the finite temperature string theory produces a non-trivial zero mode integration which localizes the string gas onto its ground state configuration, unless the strings themselves are neutral. This is of course anticipated from the example of the relativistic particle above and the fact that the one-loop string free energy comes from an infinite tower of particle states. In the following we will argue that this ground state localization is evidence for Debye screening of electric fields and the appearance of a Debye mass. We will then use T-duality to translate this into a statement about D-brane dynamics.

3 Debye Screening in Superstring Theory

3.1 Boundary State Formalism

We will now compute the free energy of charged open superstrings in a constant electric field by performing a modular transformation $t = 1/s$ of the annulus amplitude, which interchanges the roles of $\tau = -\ln\rho$ and σ, and working in the closed string channel with a cylindrical worldsheet, i.e. the boundary state formalism. In this representation, $0 \le \tau \le s$ parametrizes the length of the cylinder so that the boundaries are at $\tau = 0, s$, and $0 \le \sigma \le 1$ parametrizes the closed string which propagates through the cylinder with time coordinate τ. The standard field theoretic proper time is then $S = 2\pi\alpha' s$ (the same as for a particle). Boundary states are coherent closed string states which insert a boundary on the worldsheet and enforce on it the appropriate boundary conditions. They are constructed by applying the Boltzmann weight operator $\mathrm{e}^{-S_b}|_{\tau=0}$, constructed from the action (2.19), to the vacuum state of the closed string Hilbert space [2]. Finite temperature only affects the bosonic zero modes, and we therefore consider only these contributions in detail. The bosonic part of the boundary state which corresponds to one end of an open superstring is created in part by the Wilson loop operator

$$\Phi[x^\mu(\tau,\sigma)] = \exp ie\left(\int_0^1 d\sigma\ A_\mu(x)\,\partial_\sigma x^\mu - \chi(x)\right) \tag{3.1}$$

where e is the charge of the open string endpoint and $\chi(x)$ is the appropriate Wu–Yang term described in Appendix A. For a constant electric field $\vec{F}$, we introduce the Euclidean angle $\mathcal{E}$ defined by

$$\cos\mathcal{E} = \frac{1}{\sqrt{1+(2\pi\alpha' e\vec{F})^2}} \tag{3.2}$$

We shall choose coordinates in which $\vec{F} = (F,0,\ldots,0)$ and the gauge (2.6) with $c = 1$. Then the Wu–Yang term is

$$\chi(x) = Fx^1(\sigma=0)\left[x^0(\sigma=1) - x^0(\sigma=0)\right] \tag{3.3}$$

To see what effects finite temperature imposes on the boundary states, we consider (3.1) as an operator on the closed string Hilbert space. The closed string mode expansions are

$$\begin{aligned} x^0(\tau,\sigma) &= y^0 + \frac{2\pi i\alpha' n^0\tau}{s\beta} + \frac{w^0\beta\sigma}{2\pi} + \sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{in}\left(a_n^0\,\mathrm{e}^{-2\pi n(\tau/s+i\sigma)} + \tilde{a}_n^0\,\mathrm{e}^{-2\pi n(\tau/s-i\sigma)}\right) \\ \vec{x}(\tau,\sigma) &= \vec{y} + \frac{i\alpha'\vec{p}\tau}{s} + \sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{in}\left(\vec{a}_n\,\mathrm{e}^{-2\pi n(\tau/s+i\sigma)} + \vec{\tilde{a}}_n\,\mathrm{e}^{-2\pi n(\tau/s-i\sigma)}\right) \end{aligned} \tag{3.4}$$

in the sector of Kaluza–Klein momentum n^0 and winding number w^0 around the compact Euclidean temperature direction. The Wu–Yang term (3.3) may then be written as

$$\chi(x) = \frac{Fw^0\beta}{2\pi}\left[y^1 + \frac{i\alpha' p^1\tau}{s} + \sqrt{\alpha'}\sum_{n\neq 0}\frac{\mathrm{e}^{-2\pi n\tau/s}}{in}\left(a_n^1 + \tilde{a}_n^1\right)\right] \tag{3.5}$$

The total zero mode contribution to the action at $\tau = 0$ is

$$-iS_{w^0} = -iew^0\left(2\pi\nu - \frac{\beta F}{2\pi}\,y^1\right) \tag{3.6}$$

and it is straightforward to see that the oscillator part of (3.5) cancels the extra boundary integration in (3.1) which comes from the winding number term in the mode expansion of x^0 in Eq. (3.4). The total oscillator contribution to the action is therefore the standard one for open strings in constant background electric fields. This simple calculation illustrates the importance of incorporating the Wu–Yang term in compactifications involving external fields.

In the bosonic sector, the boundary state $|B,e\rangle$ is required to satisfy the rotated boundary conditions which follow from varying the worldsheet action (2.19) [2, 5],

$$\begin{aligned}\left(\partial_\tau x^0 + 2\pi i\alpha' eF\,\partial_\sigma x^1\right)\Big|_{\tau=0}|B,e\rangle &= 0\\ \left(\partial_\tau x^1 - 2\pi i\alpha' eF\,\partial_\sigma x^0\right)\Big|_{\tau=0}|B,e\rangle &= 0\\ \partial_\tau x^j\Big|_{\tau=0}|B,e\rangle &= 0 \quad \forall j>1\end{aligned} \tag{3.7}$$

This gives

$$|B,e\rangle = \frac{1}{\cos\mathcal{E}}\;|B_x,e\rangle^{(0)}\exp\mathcal{O}(\mathcal{E})|0\rangle_a|0\rangle_{\tilde a}\;|B_{\rm gh}\rangle\;|B_\psi,\mathcal{E}\rangle \tag{3.8}$$

where the bosonic zero-mode contributions to the boundary state are

$$\begin{aligned}|B_x,e\rangle^{(0)} &= \sum_{w^0=-\infty}^{\infty}{\rm e}^{\,iS_{w^0}}\left|n^0=0,w^0\right\rangle\prod_{j=1}^{d-1}\left|k^j=0\right\rangle\\ &= \sum_{w^0=-\infty}^{\infty}{\rm e}^{\,2\pi ie\nu w^0}\left|n^0=0,w^0\right\rangle\left|k^1=-\frac{e\beta Fw^0}{2\pi}\right\rangle\prod_{j=2}^{d-1}\left|k^j=0\right\rangle\end{aligned} \tag{3.9}$$

Here $|n^0,w^0\rangle$ is the bosonic vacuum state which is normalized as

$$\langle n',w'|n,w\rangle = \Phi_\beta\,\delta_{nn'}\delta_{ww'} \tag{3.10}$$

where Φ_β is an appropriate volume factor which can be taken to be the "self-dual volume" [18] of the compact direction that is fixed by T-duality invariance to have the asymptotic behaviours

$$\Phi_\beta \simeq \beta \quad \text{for}\ \ \beta\to\infty \quad , \quad \Phi_\beta \simeq \frac{4\pi^2\alpha'}{\beta} \quad \text{for}\ \ \beta\to 0 \tag{3.11}$$

The electric field dependent normalization in Eq. (3.8) is the Born–Infeld Lagrangian for the boundary gauge fields [2], and $|B_{\rm gh}\rangle$ denotes the boundary state for the ghost and superghost degrees of freedom which is unaffected by temperature and the electric field. The $|k^j\rangle$ are the usual continuum momentum eigenstates in the $d-1$ uncompactified directions, while $|0\rangle_a$ and $|0\rangle_{\tilde a}$ are Fock vacua for the bosonic closed string oscillator modes. The Bogoliubov transformation

$$\mathcal{O}(\mathcal{E}) = \sum_{n=1}^{\infty}\left[\sum_{j=2}^{d-1}\frac{1}{n}\,a^j_{-n}\tilde a^j_{-n} + \left(a^0_{-n}\ ,\ a^1_{-n}\right)\begin{pmatrix}\cos 2\mathcal{E} & \sin 2\mathcal{E}\\ -\sin 2\mathcal{E} & \cos 2\mathcal{E}\end{pmatrix}\begin{pmatrix}\tilde a^0_{-n}\\ \tilde a^1_{-n}\end{pmatrix}\right] \tag{3.12}$$

encodes the boundary conditions (3.7) on the bosonic oscillatory modes (see [13, 14, 19], for example). It is unaffected by temperature and so contributes the same quantity as at $T=0$. The fermionic boundary coupling in the worldsheet superstring action is of the form $ie\int_0^1 d\sigma\,\overline{\psi}^\mu F_{\mu\nu}\psi^\nu$. The fermionic boundary state $|B_\psi,\mathcal{E}\rangle$ is therefore also unaffected by finite temperature, and it is given by a similar rotation as in Eq. (3.12) on the fermionic oscillators and on the Ramond zero-mode states (see [2, 14, 20] for the explicit expressions). However, finite temperature breaks supersymmetry and modifies the GSO projection operator due to the winding numbers w^0 around the compact temporal direction [21]. This means that the sum over worldsheet spin structures contains an extra weighting $(-1)^{w^0}$ for the $(-,+)$ spin structure in the Neveu-Schwarz sector.

We shall now evaluate the one-loop superstring vacuum amplitude which is given by the propagator

$$\begin{aligned}\mathcal{F}(e_1,e_2) &= \frac{1}{\pi(2\alpha')^{d/2}}\left\langle B,e_2\left|\frac{1}{L_0+\tilde{L}_0-\eta}\right|B,e_1\right\rangle \\ &= \frac{1}{\pi^2(2\alpha')^{d/2}}\int\limits_0^\infty ds\ \left\langle B,e_2\ \right|\ \mathrm{e}^{\,-\pi s(L_0+\tilde{L}_0-\eta)}\ \left|\ B,e_1\right\rangle \end{aligned} \tag{3.13}$$

where s is the modular parameter of the cylinder and the normal ordering intercept is $\eta=2$ in the bosonic sector, while $\eta=\frac{1}{2}$ in the fermionic Neveu-Schwarz sector and $\eta=0$ in the Ramond sector. The bosonic part of the closed string Hamiltonian is

$$L_0^b+\tilde{L}_0^b-2=\frac{8\pi^2\alpha'(n^0)^2}{\beta^2}+\frac{(w^0)^2\beta^2}{2\pi^2\alpha'}+2\alpha'\vec{p}^{\,2}+N_b+\tilde{N}_b-2 \tag{3.14}$$

where N_b and $\tilde{N}_b$ are the usual number operators for the bosonic oscillatory and ghost modes. After some algebra we find (setting $d=10$)

$$\begin{aligned}\mathcal{F}(e_1,e_2) &= \frac{16\sin\pi\epsilon}{(2\pi\alpha')^5\cos\mathcal{E}_1\cos\mathcal{E}_2}\,\Phi_\beta V_8\sum_{w^0=-\infty}^{\infty}\mathrm{e}^{\,2\pi i\nu w^0(e_1-e_2)}\,\delta\Big((e_1-e_2)w^0\beta F\Big) \\ &\quad\times\int\limits_0^\infty ds\ \ \mathrm{e}^{\,-\frac{s}{2\pi\alpha'}(w^0)^2\beta^2\left[1+(2\pi\alpha' e_1F)^2\right]}\,\frac{\Theta_1'(0|is)^{-3}}{\Theta_1(\epsilon|is)} \\ &\quad\times\Big[\Theta_3(\epsilon|is)\,\Theta_3(0|is)^3-(-1)^{w^0}\,\Theta_4(\epsilon|is)\,\Theta_4(0|is)^3-\ \Theta_2(\epsilon|is)\,\Theta_2(0|is)^3\Big]\end{aligned} \tag{3.15}$$

where Θ_α denote the usual Jacobi theta functions which can be expressed in terms of the triple product formulas

$$\begin{aligned}\Theta_1(\nu|i\tau) &= 2\,\mathrm{e}^{\,-\pi\tau/4}\,\sin\pi\nu\prod_{n=1}^\infty\Big(1-\ \mathrm{e}^{\,-2\pi n\tau}\Big)\Big(1-\ \mathrm{e}^{\,-2\pi n\tau+2\pi i\nu}\Big)\Big(1-\ \mathrm{e}^{\,-2\pi n\tau-2\pi i\nu}\Big) \\ \Theta_2(\nu|i\tau) &= 2\,\mathrm{e}^{\,-\pi\tau/4}\,\cos\pi\nu\prod_{n=1}^\infty\Big(1-\ \mathrm{e}^{\,-2\pi n\tau}\Big)\Big(1+\ \mathrm{e}^{\,-2\pi n\tau+2\pi i\nu}\Big)\Big(1+\ \mathrm{e}^{\,-2\pi n\tau-2\pi i\nu}\Big) \\ \Theta_3(\nu|i\tau) &= \prod_{n=1}^\infty\Big(1-\ \mathrm{e}^{\,-2\pi n\tau}\Big)\Big(1+\ \mathrm{e}^{\,-(2n-1)\pi\tau+2\pi i\nu}\Big)\Big(1+\ \mathrm{e}^{\,-(2n-1)\pi\tau-2\pi i\nu}\Big) \\ \Theta_4(\nu|i\tau) &= \prod_{n=1}^\infty\Big(1-\ \mathrm{e}^{\,-2\pi n\tau}\Big)\Big(1-\ \mathrm{e}^{\,-(2n-1)\pi\tau+2\pi i\nu}\Big)\Big(1-\ \mathrm{e}^{\,-(2n-1)\pi\tau-2\pi i\nu}\Big)\end{aligned} \tag{3.16}$$

and $\Theta_1'(0|i\tau)=\partial_\nu\Theta_1(\nu|i\tau)|_{\nu=0}$. In Eq. (3.15) we have introduced the twist parameter

$$\pi\epsilon=\mathcal{E}_1-\mathcal{E}_2 \tag{3.17}$$

and used the normalization

$$\delta^{(d-2)}(0) = \frac{V_{d-2}}{(2\pi)^{d-2}} \tag{3.18}$$

A factor of $\Theta_1'(0|it)^{-4}$ in Eq. (3.15) comes from the bosonic oscillator modes in the eight directions transverse to the 0–1 plane, and a factor $\Theta_1'(0|it)$ comes from the ghosts while the rotated light-like degrees of freedom contribute $\Theta_1(\epsilon|it)$. These latter two contributions cancel each other in the neutral string limit $e_1 = e_2$ or in the absence of the electric field. The sum of theta functions in the last line of Eq. (3.15) similarly comes from the fermionic oscillators with the first two representing the contributions from the Neveu-Schwarz spin structures and the last one the Ramond sector component. The delta function comes from the overlap $\langle k_2^1|k_1^1\rangle$ after using the orthogonality condition (3.10).

We see once again that the free energy of charged superstrings is independent of temperature. The triviality in the present case arises because the operator (3.6) translates the momentum k^1 in the boundary states (3.9) by the *dual* momentum in the temporal direction. In the case of neutral superstrings, whereby $e_1 = e_2 = e$ and $\epsilon = 0$, we may use the Jacobi abstruse identity

$$\Theta_3(0|is)^4 - \Theta_4(0|is)^4 = \Theta_2(0|is)^4 \tag{3.19}$$

and the series expansion

$$\Theta_2(\nu|i\tau) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{-\pi(2q+1)^2\tau/4 + i\pi(2q+1)\nu} \tag{3.20}$$

to write the amplitude (3.15) in the form

$$\begin{aligned} \mathcal{F}(e,e) &= \frac{32}{(2\pi\alpha')^5}\,\Phi_\beta V_9\left[1+(2\pi\alpha' eF)^2\right] \\ &\times \int_0^\infty ds\ \Theta_2\left(0\left|\frac{i\beta^2 s}{2\pi^2\alpha'}\ \left[1+(2\pi\alpha' eF)^2\right]\right.\right)\left[\frac{\Theta_4(0|is)}{\Theta_1'(0|is)}\right]^4 \end{aligned} \tag{3.21}$$

Eq. (3.21) coincides with the standard expression for the partition function of the neutral superstring in a constant background electric field [4, 6] (after the modular transformation $s = 1/t$ and Wick rotation to the Minkowski electric field (2.4)). Taking into account of the fact that for a given set of charges e_1, e_2 the spectrum must contain neutral strings (c.f. Eq. (2.17)), we arrive at the total one-loop annulus amplitude in the closed string representation,[2]

$$\begin{aligned} \mathcal{F}_{\{e_1,e_2\}} &= \frac{16}{(2\pi\alpha')^5}\,\Phi_\beta V_9 \int_0^\infty ds\ \frac{1}{\Theta_1'(0|is)^4} \\ &\times \left\{\Theta_4(0|is)^4 \sum_{a=1,2}\left[1+(2\pi\alpha' e_a F)^2\right]\ \Theta_2\left(0\left|\frac{i\beta^2 s}{2\pi^2\alpha'}\ \left[1+(2\pi\alpha' e_a F)^2\right]\right.\right)\right. . \end{aligned}$$

[2]To this expression should be added the contribution of neutral string states from the Möbius amplitude. The total result would have the same qualitative properties.

$$+\,2\sin\pi\epsilon\,\sqrt{1+(2\pi\alpha' e_1F)^2}\,\sqrt{1+(2\pi\alpha' e_2F)^2}\;\Theta_1\left(\frac{\epsilon}{2}\Big|\,is\right)^4\,\frac{\Theta_1'(0|is)}{\Theta_1(\epsilon|is)}\Bigg\} \tag{3.22}$$

where we have used the formula

$$\Theta_3\left(\nu\,|i\tau\,\right)\,\Theta_3\left(0\,|i\tau\,\right)^3-\Theta_4\left(\nu\,|i\tau\,\right)\,\Theta_4\left(0\,|i\tau\,\right)^3-\Theta_2\left(\nu\,|i\tau\,\right)\,\Theta_2\left(0\,|i\tau\,\right)^3=2\,\Theta_1\left(\frac{\nu}{2}\Big|\,i\tau\right)^4 \tag{3.23}$$

which is a consequence of the Riemann identity [13]. Upon examining the region of convergence of the $s\to\infty$ modular parameter integration in Eq. (3.22), we arrive at the usual modification [4, 6] of the critical Hagedorn temperature of the free open superstring gas due to the presence of the electric field,

$$T_{\rm H}(F)=\frac{1}{2\pi k_{\rm B}\sqrt{2\alpha'}}\,\sqrt{1+(2\pi\alpha' e_0F)^2} \tag{3.24}$$

where e_0 is the minimum unit of electric charge.

3.2 Effective Action for the Polyakov Loop

One can readily see from Eq. (3.15) the correlation between the dependence of the correlator of Polyakov loops on the temporal gauge field component A_0 and the vanishing of the electric field. For a charged string at finite temperature, in order for the first exponential in Eq. (3.15) to depend on A_0, the delta function must force $F=0$. Taking this zero field limit, we can use Eq. (3.15) to see the occurrence of Debye screening from the fact that there is a non-trivial effective action for the Polyakov loop operator. Instead of Eq. (3.1), we will consider a more general condensate of photon vertices defined by the path ordered Polyakov loop operator

$$\mathcal{P}[A]=\prod_\alpha\,{\rm Tr}\;{\rm P}\;\exp i\oint_{\partial_\alpha\Sigma}A_\mu(x)\;dx^\mu \tag{3.25}$$

associated with the open superstring Chan–Paton gauge group. The boundary of the string worldsheet is in general a set of disconnected closed loops, $\partial\Sigma=\bigcup_\alpha\partial_\alpha\Sigma$. The screening of electric fields owes to special properties of the finite temperature theory. As all observables must be periodic in x^0, at least up to a gauge transformation, the temporal gauge field $A_0(x)$ has a special status, since not all gauge transformations are allowed. The gauge transformations U which are allowed are those for which U is periodic up to an element of the center of the gauge group,

$$U(x^0+\beta,\vec{x})=\,{\rm e}^{\,i\theta}\,U(x^0,\vec{x}) \tag{3.26}$$

This symmetry is a result of the fact that all string states transform in either the adjoint or other zero N-ality representations of the Chan–Paton gauge group.[3] Related to this

[3]Note that $\theta=\pi$ for type-I superstrings, but later on we shall consider analogous statements for D-branes in which the gauge group will be $U(N)$ for some N and θ is arbitrary.

symmetry is the fact that screening occurs only for fields in the $su(N)$ subalgebra of $u(N)$ where $\sum_b (A_0)_{bb} = 0 \bmod N$. The true gauge group is not $U(N)$ but rather the quotient of $U(N)$ by its center, or $U(N)/U(1) = SU(N)/\mathbf{Z}_N$.

This feature results in two facts. First of all, we can always choose a gauge where A_0 is time-independent and diagonal. However, at finite temperature one cannot completely remove the A_0's by the residual Abelian gauge invariance because of the existence of the non-trivial gauge invariant holonomy (3.25). Secondly, there is a global symmetry under the simultaneous translation of all diagonal elements of A_0,

$$A_0^{aa}(\vec{x}) \to A_0^{aa}(\vec{x}) + \theta \tag{3.27}$$

This global symmetry can be restricted to the $\mathbf{Z}_N$ subgroup of $U(1)$ where $\theta = 2\pi n/\beta$ (integer n). It is a result of the existence of large gauge transformations,

$$A_\mu \to U\left(A_\mu - i\partial_\mu\right)U^{-1} \tag{3.28}$$

where $U = \mathrm{e}^{\,2\pi i n x^0 E^b/\beta}$ and E^b is the matrix whose only non-vanishing entry is $(E^b)_{bb} = 1$. This gauge transform is periodic in x^0 and under which

$$A_0 \to A_0 + \frac{2\pi n}{\beta}\,E^b \tag{3.29}$$

The effective potential should exhibit this invariance. In a static diagonal gauge where the worldsheet boundary wraps the periodic time direction, the Polyakov loop operator is given by $\mathcal{P} = \sum_b \mathrm{e}^{\,i\beta a^b(x)}$. Then $F(x) \equiv -\beta^{-1}\ln\langle\mathcal{P}(x)\rangle$ is the free energy that would be required to introduce a heavy fundamental representation quark into the system and thereby gives information about confinement.

For example, via a gauge transformation we can consider the background field

$$A_0^{bc}(x) = \delta^{bc}\,a_0^b \tag{3.30}$$

where a_0^b are constants. When the worldsheet Σ is an annulus, the effective action is a sum over the differences between all Abelian $U(1)^N$ charges a_0^b located at the two boundaries of Σ. This is easily incorporated into the boundary state formalism described above, and from Eq. (3.15) we find that the effective potential for the Polyakov loop operator is given by

$$,\,[A] = \frac{32}{(2\pi\alpha')^5}\,\Phi_\beta V_9 \int_0^\infty \frac{dt}{t^6}\,\sum_{b,c}\Theta_2\left(\frac{\beta}{\pi}\left(a_0^b - a_0^c\right)\,\middle|\,\frac{i\beta^2}{2\pi^2\alpha' t}\right)\left[\frac{\Theta_2(0|it)}{\Theta_1'(0|it)}\right]^4 \tag{3.31}$$

where we have made a modular transformation $s = 1/t$ and used the Poisson resummation formula

$$\frac{1}{\tau}\,\frac{\Theta_4(\nu|i\tau)}{\Theta_1'(0|i\tau)} = \mathrm{e}^{\,-\pi\nu^2/\tau}\,\frac{\Theta_2\left(\frac{i\nu}{\tau}\,\middle|\,\frac{i}{\tau}\right)}{\Theta_1'\left(0\,\middle|\,\frac{i}{\tau}\right)} \tag{3.32}$$

The case of a single charged superstring is associated with gauge group $U(2)$ and $a_0^b = 2\pi\nu e_b$ in Eq. (3.31).

The effective action can be written more explicitly as a sum over superstring states and temporal winding numbers. The integration over the Teichmüller parameter t of the annulus may be evaluated by expanding the ratio of theta functions in Eq. (3.31) using the formula

$$8\prod_{n=1}^{\infty}\left(\frac{1+\,\mathrm{e}^{\,-2\pi nt}}{1-\,\mathrm{e}^{\,-2\pi nt}}\right)^{8}=\sum_{N=0}^{\infty}d_N\,\mathrm{e}^{\,-2\pi Nt} \tag{3.33}$$

where d_N is the degeneracy of superstring states at level N. Using Eq. (3.20) this gives

$$,\,[A]=2^6(\alpha')^{-5/2}\,\pi^2\,\frac{\Phi_\beta V_9}{(\pi\beta)^5}\sum_{N=0}^{\infty}d_N\,N^{5/2}\sum_{q=-\infty}^{\infty}\frac{K_5\left(\beta|2q+1|\sqrt{\frac{N}{\alpha'}}\right)}{|2q+1|^5}\sum_{b,c}\mathrm{e}^{\,i\beta(2q+1)(a_0^b-a_0^c)} \tag{3.34}$$

where $K_5(z)$ is the irregular modified Bessel function of order 5. Using the asymptotic behaviours

$$K_5(z)\simeq\frac{3\cdot 2^7}{z^5}\quad\text{for }\ |z|\to 0\ \ ,\ \ K_5(z)\simeq\,\mathrm{e}^{\,-z}\sqrt{\frac{\pi}{2z}}\quad\text{for }\ |z|\to\infty \tag{3.35}$$

the low temperature limit of Eq. (3.34) picks up the lowest $|2q+1|=1$ winding modes giving

$$,\,[A]\simeq 2^6\,\Phi_\beta V_9\left(\frac{3\cdot 2^{10}}{\pi^4\beta^{10}}+\frac{\sqrt{2}\pi d_1}{(\alpha')^{9/4}}\,\frac{\mathrm{e}^{\,-\beta/\sqrt{\alpha'}}}{(\pi\beta)^{11/2}}\right)\sum_{b,c}\cos\left(\beta(a_0^b-a_0^c)\right)\qquad\text{for }\ \beta\to\infty \tag{3.36}$$

The first term in Eq. (3.36) comes from the lowest lying $N=0$ states which have degeneracy $d_0=8$, while the second term comes from the first excited $N=1$ levels.

3.3 Some Properties of the Effective Action

The coefficient of the expansion of Eq. (3.34) or Eq. (3.36) to order A_0^2 yields an expression for the Debye screening mass $\mu^2/2$. The leading contribution is given in Eq. (1.7) and it is the same as the Debye mass that one would calculate in ordinary ten-dimensional Yang-Mills theory. This is due to the fact that the dominant term in Eq. (3.36) at low temperatures comes from only the particle-like excitations of the strings. The leading stringy corrections are exponentially suppressed by factors $\mathrm{e}^{\,-\beta/\sqrt{\alpha'}}$. In the low temperature regime these corrections play no role in the Debye screening. This exponential suppression is a result of supersymmetry which leads to cancellation of the leading order stringy corrections to the mass μ. It is interesting that stringy effects only play a role at temperatures near the Hagedorn transition. Notice also that $\mu^2\propto T/T_{\rm H}(0)$, so that well below the critical temperature the Debye mass is small and the electric fields become more and more long-ranged. This suggests that there should exist gauge field configurations with a relatively "mild" time dependence for which there is no screening of the (time dependent) electric fields (such as the example given at the end of section 2.1).

The functional $,\,[A]$ above is the leading term in the derivative expansion of the full effective potential for the gauge field $a(x)^b=e_b\,a(x)$. As discussed in section 1, it should be added to the tree-level Born–Infeld Lagrangian for the field $A_0(\vec{x})$, thereby determining the one-loop, temperature corrections to the gauge field effective action of the superstring theory. The solutions of the resulting equations of motion then determine the allowed, on-shell gauge field configurations which lead to a conformally-invariant theory at finite temperature. At low temperatures, the modified Born–Infeld action is given by the free energy

$$F[a] = \frac{\sum_b e_b^2}{(2\pi)^5(\alpha')^3 g_s} \int d^9x \left\{ \sum_b \frac{\sqrt{1+\left(2\pi\alpha' e_b \vec{\nabla} a(x)\right)^2}}{(2\pi\alpha')^2 \sum_c e_c^2} + \sum_{b,c} \frac{\mu^2}{\beta^2} \cos\Big(\beta(e_b - e_c)a(x)\Big) + \dots \right\} \tag{3.37}$$

We can think of the modified Born–Infeld action (3.37) as a generalization of the sine-Gordon theory representation of the classical Coulomb gas where the standard kinetic term $(\vec{\nabla}a)^2$ is replaced by the Born–Infeld Lagrangian. The sine-Gordon theory has well-known soliton solutions, which correspond to solitary waves of the plasma phase of the Coulomb gas. It is interesting to note that the Born–Infeld generalization of the Coulomb gas (3.37) also has solitons. Consider the ansatz where $a(x)$ depends on only one space coordinate. Then the non-linear equation for the extrema of (3.37) is solved by the function $a(x)$ which is obtained from the integral

$$\int \frac{\pi\beta(2\pi\alpha')^{-1/2}\ da}{2\mu\Big(\sum_b e_b^2\Big)\left(\sum_{b,c}\Big|\sin\Big((e_b-e_c)\beta a/2\Big)\Big|\right)\sqrt{\frac{\mu^2 g_s \sum_b e_b^2}{(2\pi)^5(\alpha')^3\beta^2}\ \sum_{b,c}\sin^2\Big((e_b-e_c)\beta a/2\Big)+1}} = x \tag{3.38}$$

These solitons also exist in gauge field theories as $\mathbf{Z}_N$ domain walls.

4 T-duality and Moving D-branes

The boundary state formalism of the previous section presents the most efficient way to map the results for the electric field problem into statements about D-brane dynamics at finite temperature. String theory in background gauge fields is related by T-duality to open string theory with moving D-branes. The prerequisite for this relation is translation invariance of the external fields in the directions which will be compactified. This means that the gauge fields should be translationally invariant up to a gauge transformation. Generally this is hard to do in first quantization because the required Wu–Yang terms spoil the translation invariance. But, as we now demonstrate, the boundary state formalism gives a precise prescription for obtaining the T-dual D-brane picture.

For this, we compactify the Neumann string coordinates $x^i \equiv x^i_{\rm N}$ for $i = 1, \dots, d-p-1$ on circles of circumferences L_i. This modifies the closed string mode expansions (3.4) along the first $d-p-1$ spatial directions to

$$x^i_{\rm N}(\tau,\sigma) = y^i + \frac{2\pi i\alpha' n^i \tau}{sL_i} + \frac{w^i L_i \sigma}{2\pi} + \sqrt{\alpha'} \sum_{n\neq 0} \frac{1}{in}\left(a^i_n\, {\rm e}^{-2\pi n(\tau/s+i\sigma)} + \tilde{a}^i_n\, {\rm e}^{-2\pi n(\tau/s-i\sigma)}\right) \tag{4.1}$$

where n^1 is the Kaluza–Klein index and w^1 the winding number around the compactified direction along which the electric field lies. The Wu–Yang term in this case is given in Appendix A and, for the gauge choice of section 3.1, it coincides with Eq. (3.5). The zero mode contribution to the Wilson line integral in Eq. (3.1) at $\tau = 0$ is modified to

$$-\, i\tilde{S}_{w^0,w^1} = -2\pi i e\nu w^0 - \frac{ieF}{2\pi}\left(y^0 L_1 w^1 - y^1 \beta w^0\right) \tag{4.2}$$

which now contains the rotation generator on the spacetime torus (again parametrized by *dual* momenta). The oscillator contributions are again unchanged, and in the boundary states (3.8) the zero mode momentum states in Eq. (3.9) are now also discretized in the first $d-p-1$ spatial directions. Upon applying the zero mode operator $\mathrm{e}^{\,i\tilde{S}_{w^0,w^1}}$ to the Fock vacuum, which shifts the total light-like momentum, we find that the only modification to the boundary state is again in the bosonic zero mode part which now becomes

$$\begin{aligned} |B_x, e\rangle^{(0)} \;=\; & \sum_{w^0,w^1=-\infty}^{\infty} \mathrm{e}^{\,2\pi i e\nu w^0} \left| n^0 = \frac{e\beta L_1 F w^1}{2\pi}\,,\; w^0 \right\rangle \left| n^1 = -\frac{e\beta L_1 F w^0}{2\pi}\,,\; w^1 \right\rangle \\ & \times \prod_{j=2}^{d-p-1} \sum_{w^j=-\infty}^{\infty} |n^j = 0, w^j\rangle \prod_{i\geq d-p} |k^i = 0\rangle \end{aligned} \tag{4.3}$$

The shifts in the Kaluza–Klein integers in Eq. (4.3) imply that the electric field must be quantized as

$$F = \frac{2\pi}{e_0 \beta L_1}\, N \tag{4.4}$$

where N is an integer. The constraint (4.4) is derived in Appendix A from a purely mathematical point of view and is shown to be a topological quantization condition associated with the presence of the Wu–Yang term.

Now we apply a T-duality transformation along the compact spatial directions and write down the corresponding boundary state for a moving Dp-brane. This mapping interchanges the Neumann and Dirichlet boundary conditions for the open string along the first $d-p-1$ spatial directions. The new string coordinates $x_{\rm D}^j$ take values on the dual circles of circumferences

$$\tilde{L}_j = \frac{4\pi^2\alpha'}{L_j} \tag{4.5}$$

and they are defined by the equations

$$\partial_\tau x_{\rm D}^j = -i\,\partial_\sigma x_{\rm N}^j \quad , \quad \partial_\sigma x_{\rm D}^j = i\,\partial_\tau x_{\rm N}^j \qquad j = 1, \ldots, d-p-1 \tag{4.6}$$

The boundary state conditions (3.7) now become

$$\begin{aligned} \partial_\tau \left(x^0 - v x_{\rm D}^1\right) \Big|_{\tau=0} |Dp, y_0; v\rangle &= 0 \\ \partial_\sigma \left(x_{\rm D}^1 + v x^0\right) \Big|_{\tau=0} |Dp, y_0; v\rangle &= 0 \\ \partial_\sigma x_{\rm D}^j \Big|_{\tau=0} |Dp, y_0; v\rangle &= 0 \qquad j = 2, \ldots, d-p-1 \\ \partial_\tau x^i \Big|_{\tau=0} |Dp, y_0; v\rangle &= 0 \qquad i \geq d-p \end{aligned} \tag{4.7}$$

where

$$v = 2\pi\alpha' eF = \frac{eN}{e_0}\,\frac{\tilde{L}_1}{\beta} \tag{4.8}$$

is the velocity of the string endpoint in the direction 1 transverse to the D-brane. The constraint (4.8) can be thought of as momentum quantization along the compact boost direction, with the quantum of velocity equal to the speed in going once around the dual circle in a single unit of Matsubara time. The Dirichlet boundary conditions in Eq. (4.7) may be alternatively expressed by setting the operators equal to fixed positions y_0^i at $\tau = 0$ which we interpret as the transverse coordinates of the Dp-brane. Note that the angle $\mathcal{E}$ defined in Eq. (3.2) is interpreted in the dual picture as the Euclidean rapidity of the D-brane motion, so that the twist parameter (3.17) becomes the relativistic composition of the velocities of two branes [7].

The boundary conditions (4.7) are solved by acting on Fock vacua with the dual version of the boundary operator (3.1) obtained by substituting $x_{\rm N}^i \mapsto x_{\rm D}^i$ for $i = 1, \ldots, d-p-1$:

$$\tilde{\Phi}[x^\mu(\tau,\sigma)] = \exp\left(2\pi i\tilde{\nu}w^0 - \frac{1}{2\pi\alpha'}\int_0^1 d\sigma\ y_1(x^0)\,\partial_\tau x_{\rm D}^1 + \frac{1}{2\pi\alpha'}\,\tilde{\chi}(x)\right) \tag{4.9}$$

where $\tilde{\nu} = e\nu$ is now interpreted as a gauge field living in the D-brane worldvolume, and

$$y_1(x^0) = vx^0 \tag{4.10}$$

is the D-brane trajectory induced by the gauge potential transverse to the brane, which produces in Eq. (4.9) the standard boundary vertex operator for a moving D-brane [15, 19]. The dual of the Wu–Yang term (3.3) is given by

$$\tilde{\chi}(x) = vx_{\rm D}^1(\sigma = 0)\left[x^0(\sigma = 1) - x^0(\sigma = 0)\right] \tag{4.11}$$

which in the closed string parametrization at $\tau = 0$ is obtained from Eq. (3.5) by reflecting the left-moving oscillators $\tilde{a}_n^1 \mapsto -\tilde{a}_n^1$. For the case of D-brane motion, the Wu–Yang term (4.11) is associated with maintaining reparametrization invariance of the D-brane worldvolume along the boosted direction at finite temperature. Its effect when the operator (4.9) is written in terms of closed string mode expansions is identical to that of the electric field problem. Note that off-shell the field (4.11) has a very complicated, non-local form. It is this feature which makes a path integral calculation technically problematic, in contrast to the boundary state formalism which uses on-shell string embedding fields.

Upon decompactifying the dual circles in every spatial direction but the boosted one, we arrive at the Dp-brane boundary state [14, 19, 20]

$$|Dp, y_0; v\rangle = \frac{\sqrt{\pi}}{2}\left(2\pi\sqrt{\alpha'}\right)^{3-p}\frac{1}{\cos\mathcal{E}}\ |Dp_x, y_0; v\rangle^{(0)}\exp\tilde{\mathcal{O}}(\mathcal{E})|0\rangle_a|0\rangle_{\tilde{a}}\ |B_{\rm gh}\rangle\ |B_\psi, \mathcal{E}\rangle \tag{4.12}$$

which represents the source for the closed string modes emitted by the brane. The numerical normalization factor in Eq. (4.12) is the Dp-brane tension [22], while the factor of

$\cos\mathcal{E}$ inserts the appropriate Lorentz contraction factor. The operator $\tilde{\mathcal{O}}(\mathcal{E})$ is obtained from Eq. (3.12) by changing the sign of $\tilde{a}_n^i$ for $i=1,\ldots,d-p-1$. The ghost and fermionic boundary states are identical to those used in section 3.1, while the bosonic zero mode part of the boundary state is now

$$|Dp_x,y_0;v\rangle^{(0)}=\sum_{w^0,\tilde{w}^1=-\infty}^{\infty}\mathrm{e}^{\,i\tilde{S}_{w^0,\tilde{n}^1}}\prod_{j=1}^{d-p-1}\delta\left(y^j-y_0^j\right)\,|n^0=0,w^0\rangle|\tilde{n}^1=0,\tilde{w}^1\rangle\prod_{i=2}^{d-1}|k^i=0\rangle \tag{4.13}$$

where $\tilde{n}^1=w^1$ is the dual momentum and $\tilde{w}^1=n^1$ the dual winding number around the compact boost direction. As before the closed string zero-mode operator (4.2) accounts for the boosted boundary conditions in the light-like plane, while the transverse Dirichlet boundary conditions in Eq. (4.7) are enforced by the delta-function operators in Eq. (4.13) which are defined by the Fourier expansions

$$\begin{aligned}\delta\left(y^1-y_0^1\right)&=\sum_{\tilde{n}^1=-\infty}^{\infty}\mathrm{e}^{\,2\pi i\tilde{n}^1\left(y^1-y_0^1\right)/\tilde{L}_1}\\ \delta\left(y^j-y_0^j\right)&=\int_{-\infty}^{\infty}\frac{dq_j}{2\pi}\,\mathrm{e}^{\,iq_j\left(y^j-y_0^j\right)}\qquad j=2,\ldots,d-p-1\end{aligned} \tag{4.14}$$

These delta-functions have the effect of introducing extra terms, proportional to $|\vec{y}|^2$, into the energy levels coming from the masses of the open string excitations which stretch between a pair of branes at separation $|\vec{y}|$. The compactness of the boost direction is required since the operator (4.9) shifts the temporal Kaluza–Klein momentum n^0 by $vq^1/2\pi$, which is only an integer when the momentum q^1 is discretized and the velocity is quantized according to Eq. (4.8). Likewise, the operator $\mathrm{e}^{\,i\tilde{S}_{w^0,\tilde{n}^1}}$ shifts the Kaluza–Klein momentum $\tilde{n}^1$ along the boost direction by $\tilde{L}_1vw^0/\tilde{\beta}$, which with the topological quantization condition (4.8) is an integer only for $w^0=0$. The Dirichlet zero mode boundary states are thus

$$\begin{aligned}|Dp_x,y_0;v\rangle^{(0)}\;&=\;\sum_{\tilde{w}^1=-\infty}^{\infty}\sum_{\tilde{n}^1=-\infty}^{\infty}\mathrm{e}^{\,-2\pi i\tilde{n}^1y_0^1/\tilde{L}_1}\left|n^0=\frac{\beta v\tilde{n}^1}{\tilde{L}_1}\;,\;w^0=0\right\rangle\left|\tilde{n}^1,\tilde{w}^1\right\rangle\\ &\quad\times\prod_{j=2}^{d-p-1}\int_{-\infty}^{\infty}\frac{dq_j}{2\pi}\,\mathrm{e}^{\,-iq_jy_0^j}\,|q_j\rangle\prod_{i\geq d-p}|k^i=0\rangle\end{aligned} \tag{4.15}$$

It is important to realize that the form (4.15) for the boundary state is only valid at finite temperature and velocity. For a static Dp-brane the boost direction can be decompactified and the boundary state is instead given from Eq. (4.13) as

$$|Dp_x,y_0;v=0\rangle^{(0)}=\sum_{w^0=-\infty}^{\infty}\mathrm{e}^{\,2\pi i\tilde{v}w^0}\,|n^0=0,w^0\rangle\prod_{j=1}^{d-p-1}\int_{-\infty}^{\infty}\frac{dq_j}{2\pi}\,\mathrm{e}^{\,-iq_jy_0^j}\,|q_j\rangle\prod_{i\geq d-p}|k^i=0\rangle \tag{4.16}$$

which produces a non-trivial winding mode dependence in the temperature direction. If we had used instead of Eq. (4.8) the dual quantization condition $\tilde{v}=\tilde{N}\tilde{\beta}/\tilde{L}_1,\ \tilde{N}\in\mathbf{Z}$, in Eq. (4.13), then the Kaluza–Klein momenta of the boundary states would be $n^0 = 0$ and $\tilde{n}^1 = -w^0 \tilde{v} \tilde{L}_1/\tilde{\beta}$ while their winding numbers would be arbitrary. This state would then coincide with Eq. (4.16) in the zero velocity limit. However, as shown in Appendix A, Eq. (4.8) has a deep mathematical origin and is the correct one to use for the present problem. It arises from the mathematical property that the light-like torus has non-trivial Čech cohomology which makes it impossible to define a globally non-singular vector potential on it with non-vanishing flux. We see then that the zero mode boundary state (4.15) is the appropriate T-dual of Eq. (3.9) which contains as well a sort of temperature duality transformation, in the sense that the boosted boundary conditions forbid closed string windings around the temporal direction but now the closed string energies are non-vanishing.

We see that from the onset the Dp-brane boundary state represents only the trivial $w^0 = 0$ state at finite temperature. This property may be attributed to the fact that the closed string operator which is used to boost a static D-brane boundary state [14] doesn't commute with the compactification of the boost plane and thereby produces a non-trivial projective phase. Indeed, the free energy

$$\tilde{\mathcal{F}}(v_1, v_2) = \frac{2}{\pi (2\alpha')^{\frac{d}{2}+3-p}} \left\langle Dp, y_0^{(2)}; v_2 \left| \frac{1}{L_0 + \tilde{L}_0 - \eta} \right| Dp, y_0^{(1)}; v_1 \right\rangle \tag{4.17}$$

may be computed using Eq. (4.15) and the results of section 3.1. Here an extra factor of 2 has been inserted to take into account of the fact that one can interchange the roles of the two endpoints of an oriented string [23]. After computing the discrete overlaps of states using the orthogonality conditions, one can safely take the decompactification limit $\tilde{L}_1 \to \infty$ with $\tilde{w}^1 = 0$ and $q_1 = 2\pi \tilde{n}^1/\tilde{L}_1$ a fixed continuum momentum variable. Integrating over the momenta q_j for $2 \leq j \leq d-p-1$, we arrive at

$$\begin{aligned} \tilde{\mathcal{F}}(v_1, v_2) &= 2^{p/2} (2\pi)^{4-2p} \left(4\pi^2 \alpha'\right)^{-p/2} \sqrt{1+v_1^2} \sqrt{1+v_2^2}\, \Phi_\beta V_p \\ &\quad \times \int_0^\infty \frac{ds}{s^{4-p/2}}\ \mathrm{e}^{-b^2/2\pi\alpha' s} \frac{\Theta_1'(0|is)^{-3}}{\Theta_1(\epsilon|is)} \Theta_1\left(\frac{\epsilon}{2}\middle| is\right)^4 \\ &\quad \times \int_{-\infty}^\infty \frac{dq_1}{2\pi}\ \mathrm{e}^{-2\pi\alpha' s q_1^2 \left(1+v_1^2\right) - i q_1 \left(y_0^{(1)1} - y_0^{(2)1}\right)} \delta\Big((v_1 - v_2)\beta q_1\Big) \end{aligned} \tag{4.18}$$

where the impact parameter b of the Dp-brane scattering is defined through

$$b^2 = \sum_{j=2}^{9-p} \left(y_0^{(1)j} - y_0^{(2)j}\right)^2 \tag{4.19}$$

In both cases where either $v_1 \neq v_2$ or $v_1 = v_2$, one reproduces only the zero temperature results for the scattering amplitude of a pair of moving Dp-branes [7, 14] or the cancellation of the gravitational attraction with the Ramond-Ramond repulsion between static branes [23].

In contrast to that of the charged string, the moving D-brane boundary state (4.15) represents only the ground state configuration. This difference is easily understood when one considers the Lorentz invariance of the brane dynamics, i.e. that one can always boost into the rest frame of a single moving D-brane. A pair of static Dp-branes corresponds to a neutral superstring in the T-dual picture, so that it is only possible to write down a non-trivial temperature dependent interaction between a pair of static branes [10, 11], as given in section 1. This triviality comes from the same zero mode operators, associated with the presence of the Wu–Yang term, as in the electric field problem. It is therefore attributed to the Debye screening of electric fields that we described in the previous section. In the present case, the screening comes from the dependence of the static interaction potential on the gauge field a_0 that lives on the brane worldvolume (see Eq. (1.12)) and it results in a damping of the D-brane motion at finite temperature. It would be interesting to understand more precisely how the features of the superstring effective action that we discussed in section 3 affect the non-extremal thermal states of D-branes. In particular, it would be interesting to understand how the information about a possible deconfinement phase transition stored in the effective potential for the Polyakov loop operator is relevant to the corresponding dimensionally-reduced supersymmetric Yang-Mills theory description at finite temperature. This may be useful in determining precisely how to unify the two non-extremal deformations of D-brane configurations (by temperature and velocity) and thereby describe the thermodynamics of their gravitational interactions. It may also prove useful to a better understanding of the statistical mechanics of D-branes.

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Appendix A The Wu–Yang Term

In this appendix we will describe the formalism for adding Wu–Yang terms to the action in the presence of topologically non-trivial gauge fields. Such terms are generally required whenever a gauge connection on a compact space has a non-trivial flux. In that case, it is not a globally defined differential form on the configuration manifold and can only be defined locally with respect to an open covering of the space. Demanding that the action be independent of the choice of covering used (or equivalently of the gauge choice) requires the addition of (generalized) Wu–Yang terms. We shall first describe the formalism generally in the case of an arbitrary compact manifold $\mathcal{M}$, and then afterwards discuss the specific cases of interest in this paper. More details can be found in [24], for example.

Let $\{U_a\}$ be a finite open cover of the manifold $\mathcal{M}$, and assume that the cover is "good", i.e. each U_a and each non-empty intersection of the U_a's is a contractible open set which is diffeomorphic to an open ball in $\mathbf{R}^d$. A gauge field A on a non-trivial line bundle over $\mathcal{M}$ is given by the specification of a one-form $A^{(a)}$ defined everywhere on U_a. On each non-empty intersection of two open sets U_a and U_b of the cover, the corresponding gauge fields $A^{(a)}$ and $A^{(b)}$ are related by a gauge transformation:

$$A^{(a)} - A^{(b)} = d\chi^{(ab)} \qquad \text{on} \;\; U_a \cap U_b \tag{A.1}$$

By definition, the 0-form $\chi^{(ab)}$ satisfies $\chi^{(ab)} = -\chi^{(ba)}$. Now consider the situation on a non-empty triple overlap of sets U_a, U_b and U_c. Summing up the three equations of the form (A.1) which come from the distinct pairwise intersections of the three open sets, we arrive at the equation

$$d\left(\chi^{(ab)} + \chi^{(bc)} + \chi^{(ca)}\right) = 0 \tag{A.2}$$

which by Poincaré's lemma implies that

$$\chi^{(ab)} + \chi^{(bc)} + \chi^{(ca)} = c^{(abc)} = \text{const.} \qquad \text{on} \;\; U_a \cap U_b \cap U_c \tag{A.3}$$

The locally constant functions $c^{(abc)}$ satisfy the cocycle equations

$$c^{(abc)} - c^{(bcd)} + c^{(cda)} - c^{(dab)} = 0 \tag{A.4}$$

and they encode deep topological information about the line bundle over $\mathcal{M}$. They define a two-cocycle of the Čech cohomology group $H^2_{\text{C}}(\mathcal{M}, \mathbf{R})$ of the manifold $\mathcal{M}$ with coefficients in the constant sheaf $\mathbf{R}$. This group measures the obstructions to passing from local to global data on $\mathcal{M}$. There is a natural isomorphism between the Čech cohomology group and the ordinary deRham cohomology group $H^2_{\text{DR}}(\mathcal{M})$.

We now consider the appropriate modification of the Wilson loop operator

$$W[A] = \exp i \oint_{\Gamma} A \tag{A.5}$$

integrated over a cycle , of $\mathcal{M}$. Since A is only locally defined on $\mathcal{M}$, this integral needs to be carefully defined using an open cover of $\mathcal{M}$. Consider a triangulation of $\mathcal{M}$ such that the simplices induce a one-dimensional simplicial decomposition of the cycle , $= \bigcup_a L_a$. One each line L_a there is a gauge field one-form $A^{(a)}$ as above, and so naively the appropriate definition of (A.5) should be

$$W[A] = \exp i \sum_a \int_{L_a} A^{(a)} \tag{A.6}$$

However, the operator (A.6) transforms non-trivially under deformations of the simplicial decomposition. It is straightforward to see that the induced change in integrand is a sum of terms of the form $d\chi^{(ab)}$ (according to (A.1)), so that by Stokes' theorem we should add the term $-\sum_{a,b}\chi^{(ab)}(p_{ab})$ to (A.6) in order to cancel this variation. Here p_{ab} is the intersection point of the lines L_a and L_b. We must then cancel out the dependence on the choice of locations of points p_{ab} in the double overlaps, which according to (A.3) requires the addition of the term $\sum_{a,b,c} c^{(abc)}$.. In this way we arrive at the consistent topological extension of the Wilson loop integral:

$$W[A] = \exp i\left(\sum_a \int_{L_a} A^{(a)} - \sum_{a,b}\chi^{(ab)}(p_{ab}) + \sum_{a,b,c} c^{(abc)}\right) \tag{A.7}$$

The operator (A.7) is independent of the choice of triangulation of the manifold $\mathcal{M}$.

However, the operator (A.7) is ambiguous up to the choice of constants $c^{(abc)}$ (for example, as we will see below, there are choices of covers which have no triple intersections). Although these constants do not alter the classical theory, in a path integral approach to the quantum theory we must demand that the Wilson loop operator (A.7) be independent of the $c^{(abc)}$'s. This imposes the quantization condition

$$c^{(abc)} = 2\pi\, n^{(abc)} \qquad , \qquad n^{(abc)} \in \mathbf{Z} \tag{A.8}$$

Mathematically, this simply means that $c^{(abc)}/2\pi$ defines a two-cocycle of the *integer* Čech cohomology $H^2_{\rm C}(\mathcal{M},\mathbf{Z})$. In fact, this constraint imposes a quantization condition on the flux of the gauge field through any two-cycle Σ of $\mathcal{M}$. To see this we consider the induced covering of Σ by two-simplices Δ_a, such that the intersection of any two simplices Δ_a and Δ_b is a line L_{ab} while the intersection of any three lines L_{ab}, L_{bc} and L_{ca} is a point p_{abc}. We may then compute

$$\begin{aligned}
F \;\equiv\; \frac{1}{\mathrm{vol}(\Sigma)}\oint_\Sigma dA \;&=\; \frac{1}{\mathrm{vol}(\Sigma)}\sum_a\int_{\Delta_a} dA^{(a)} \\
&= \frac{1}{\mathrm{vol}(\Sigma)}\sum_{a,b}\int_{L_{ab}}\left(A^{(a)} - A^{(b)}\right) \qquad \text{by Stokes' theorem} \\
&= \frac{1}{\mathrm{vol}(\Sigma)}\sum_{a,b}\int_{L_{ab}} d\chi^{(ab)} \qquad \text{by (A.1)} \\
&= \frac{1}{\mathrm{vol}(\Sigma)}\sum_{a,b,c}\left(\chi^{(ab)}(p_{abc}) + \chi^{(bc)}(p_{abc}) + \chi^{(ca)}(p_{abc})\right) \qquad \text{by Stokes' theorem} \\
&= \frac{1}{\mathrm{vol}(\Sigma)}\sum_{a,b,c} c^{(abc)} \qquad \text{by (A.3)}
\end{aligned} \tag{A.9}$$

which using (A.8) gives the flux quantization

$$F = \frac{2\pi N}{\mathrm{vol}(\Sigma)} \tag{A.10}$$

where $N = \sum_{a,b,c} n^{(abc)} \in \mathbf{Z}$.

Let us now turn to the specific examples discussed in the text. For the cases discussed in sections 2 and 3, we take $\mathcal{M} = \mathbf{S}^1 \times \mathbf{R}^{d-1}$, and the Wilson loop integral over the circle $\mathbf{S}^1$. The minimal good covering of $\mathbf{S}^1$ consists of three open sets U_a which respectively overlie the line segments $[0, \frac{\beta}{3}]$, $[\frac{\beta}{3}, \frac{2\beta}{3}]$ and $[\frac{2\beta}{3}, \beta]$. The covering may then be extended trivially through the $\mathbf{R}^{d-1}$ directions to give a good cover of the entire manifold $\mathcal{M}$. For the gauge choice (2.6), the transition functions $\chi^{(12)}$ and $\chi^{(23)}$ may be taken to vanish, while the third one satisfies (A.1) which gives

$$\begin{aligned} \partial_0 \chi^{(13)} &= 0 \\ \vec{\partial} \chi^{(13)} &= n\beta c\vec{F} \end{aligned} \tag{A.11}$$

at $t = 0$, where we have used (2.2). Integrating (A.11) gives

$$\chi^{(13)}(0) = n\beta c\vec{F} \cdot \vec{x}(0) \tag{A.12}$$

and, since the minimal covering has no triple intersections, the Wilson loop operator (A.7) yields the phase factor (2.7). Note that in this case there is no analog of the flux quantization condition (A.10) owing to the absence of non-trivial two-cycles in the present manifold $\mathcal{M}$, or equivalently that $H^2_{\mathrm{C}}(\mathcal{M}, \mathbf{Z}) = 0$ in this case.

Next we consider the case of the manifold $\mathcal{M} = \mathbf{S}^1_\beta \times \mathbf{S}^1_L \times \mathbf{R}^{d-2}$ which is relevant to the analysis in section 4. The minimal good cover of the torus $\mathbf{S}^1_\beta \times \mathbf{S}^1_L$ consists of nine open sets $U^{(\beta)}_a \times U^{(L)}_b$ which are obtained from the product of the minimal good coverings of the circle described above. Again the covering is trivially extended to the whole of $\mathcal{M}$. For the gauge choice (2.6) it follows from the above example that the only non-vanishing transition function is $\chi^{(13)}_{\beta L}$ which is induced by those of $\mathbf{S}^1_\beta$ and $\mathbf{S}^1_L$. Now, however, the condition (A.1) reads

$$\begin{aligned} \partial_0 \chi^{(13)}_{\beta L} &= -mL(1-c)F \\ \vec{\partial} \chi^{(13)}_{\beta L} &= n\beta c\vec{F} \end{aligned} \tag{A.13}$$

at $t = 0$ (with m the winding number around $\mathbf{S}^1_L$), so that we may take

$$\chi^{(13)}_{\beta L}(0) = -mL(1-c)Fx^0(0) + n\beta c\vec{F} \cdot \vec{x}(0) \tag{A.14}$$

The discreteness of the electric field (4.4) along the compactified direction can now be seen as a consequence of the cohomological quantization condition (A.10). Taking $\Sigma = \mathbf{S}^1_\beta \times \mathbf{S}^1_L$, we have $\mathrm{vol}(\Sigma) = \beta L$ and thus

$$F = \frac{2\pi N}{\beta L} \qquad , \qquad N \in \mathbf{Z} \tag{A.15}$$

This constraint comes from the mathematical property $H^2_{\mathrm{C}}(\mathcal{M}, \mathbf{Z}) = \mathbf{Z}$ of the present manifold $\mathcal{M}$.

Appendix B Path Integral Evaluation of the Thermal Density Matrix

To derive Eq. (2.8) from the path integral (2.1), (2.5), we use the mode expansion

$$x^\mu(t) = x^\mu_{\rm cl}(t) + \sum_{k=1}^{\infty} \frac{a^\mu_k}{\pi\sqrt{k}} \sin\frac{2\pi k t}{s} + \sum_{k=1}^{\infty} \frac{b^\mu_k}{\pi\sqrt{k}} \left(\cos\frac{2\pi k t}{s} - 1\right) \tag{B.1}$$

where $x^\mu_{\rm cl}(t)$ obeys the classical equations of motion

$$\begin{aligned} \ddot{x}^0_{\rm cl} + i\vec{F}\cdot\dot{\vec{x}}_{\rm cl} &= 0, \\ \ddot{\vec{x}}_{\rm cl} - i\vec{F}\dot{x}^0_{\rm cl} &= 0 \end{aligned} \tag{B.2}$$

and the boundary conditions (2.2). Then the zero mode $x^\mu_{\rm cl}$ is orthogonal as usual to the modes with $k \geq 1$ with respect to the scalar product of two functions:

$$(x,y) = \int_0^s dt\ \left(\dot{x}_\mu \dot{y}^\mu + iF(x^1\dot{y}^0 + y^1\dot{x}^0)\right), \tag{B.3}$$

since the contributions from the non-zero modes vanish at $t=0$ and $t=s$. Solving Eq. (B.2) we find

$$\begin{aligned} x^0_{\rm cl} &= y^0 + \frac{n\beta}{2}\left(1 - \cosh Ft + \coth\frac{Fs}{2}\sinh Ft\right) \quad 0 \leq y^0 < \beta, \\ x^1_{\rm cl} &= y^1 + \frac{in\beta}{2}\left(\sinh Ft + \coth\frac{Fs}{2}(1-\cosh Ft)\right), \\ x^i_{\rm cl} &= y^i \quad \text{for} \quad i = 2,\ldots,d-1\,. \end{aligned} \tag{B.4}$$

The factor of i in the second line of Eq. (B.4) disappears after the substitution (2.4).

Substituting (B.1), (B.4) into the action yields

$$S = -2\pi\nu n + S_{\rm cl} + \sum_{k=1}^{\infty}\sum_{\mu=0}^{d-1} \frac{k}{s}\left(a^\mu_k a^\mu_k + b^\mu_k b^\mu_k\right) + \sum_{k=1}^{\infty} \frac{iF}{\pi}\left(a^0_k b^1_k - a^1_k b^0_k\right) \tag{B.5}$$

with

$$S_{\rm cl} = \int_0^s dt\ \left(\tfrac{1}{2}(\dot{x}^0_{\rm cl})^2 + \tfrac{1}{2}(\dot{x}^1_{\rm cl})^2 + iFx^1_{\rm cl}\dot{x}^0_{\rm cl}\right). \tag{B.6}$$

The cross term does not appear since the zero modes are orthogonal to the non-zero ones. To calculate $S_{\rm cl}$ it is convenient to make use of Eq. (B.2) and an integration by parts in (B.6) which gives

$$S_{\rm cl} = \frac{1}{2}x^0_{\rm cl}\dot{x}^0_{\rm cl}\Big|_0^s + \frac{1}{2}x^1_{\rm cl}\dot{x}^1_{\rm cl}\Big|_0^s + \frac{iF}{2}x^1_{\rm cl}\dot{x}^0_{\rm cl}\Big|_0^s = \frac{n^2\beta^2 F}{4\tanh\frac{Fs}{2}} + in\beta F y^1. \tag{B.7}$$

This results in the argument of the theta function in Eq. (2.8). The Gaussian integral over the modes a_k^0, b_k^0, a_k^1, and b_k^1 with $k \geq 1$ produces the fluctuation determinant

$$\prod_{k=1}^{\infty} \det \begin{pmatrix} \frac{k}{s} & 0 & 0 & -\frac{iF}{2\pi} \\ 0 & \frac{k}{s} & \frac{iF}{2\pi} & 0 \\ 0 & \frac{iF}{2\pi} & \frac{k}{s} & 0 \\ -\frac{iF}{2\pi} & 0 & 0 & \frac{k}{s} \end{pmatrix}^{-1/2} = \prod_{k=1}^{\infty} \left(\frac{k^2}{s^2} + \frac{F^2}{4\pi^2} \right)^{-1} = \frac{F}{4\pi \sinh \frac{Fs}{2}}, \tag{B.8}$$

where we have used zeta-function regularization in the last equality. This contributes to the pre-exponential factor. The rest of the derivation is standard.

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