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ISSN 1398-599X MPS-RR 1999-3

Spectral theory of Pauli-Fierz Hamiltonians I.

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May 7, 1999

Abstract

We study spectral properties of Pauli-Fierz Hamiltonians which are commonly used to describe the interaction of a small quantum system with a bosonic free field. We give precise estimates on the location and multiplicity of the singular spectrum of such Hamiltonians. Applications of these estimates, which will be discussed elsewhere, concern spectral and ergodic theory of non-relativistic QED. Our proof has two ingredients: the Feshbach method, which is developed in an abstract framework, and the Mourre method applied to the Hamiltonian restricted to the sector orthogonal to the vacuum.

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1 Introduction

In this paper we study spectral properties of a certain class of self-adjoint operators which appear in non-relativistic physics. They are commonly used to describe the interaction

of a small quantum system (an "atom") with a bosonic free field ("radiation" or a "heat bath"). We will refer to them as Pauli-Fierz Hamiltonians (see [PF], [Bl], [BFSS], [DG]).

In a few words, the main result of our paper can be described as follows: the predictions of the second-order perturbation theory for embedded eigenvalues of a large class of Pauli-Fierz Hamiltonians are correct for a sufficiently small coupling constant. A large part of our argument is abstract and uses only certain structural properties of Pauli-Fierz Hamiltonians which are common to many different problems of mathematical physics. Therefore we would like to devote the first part of the introduction to a description of the general structure of our results and arguments. Only afterwards we will explain them in the context of Pauli-Fierz Hamiltonians.

1.1 The conjugate operator method

Let H be a self-adjoint operator and Θ a fixed open subset of the real line. First, we would like to describe two well-known methods used in the study of the spectrum of the operator H inside Θ: the analytic deformation method and the Mourre theory. These two methods have a lot in common and can be viewed as two versions of one method that we will call the conjugate operator method. Although in this paper we will use only the Mourre theory, it is helpful to keep in mind the intuition derived from the analytic deformation method.

In what follows $\sigma(B)$ will denote the spectrum of the operator B and $\sigma_{\rm pp}(B)$ will denote its pure point spectrum.

(1) The analytic deformation approach. One considers a family of operators

$$H(\xi) := \mathrm{e}^{\mathrm{i}\xi S} H \mathrm{e}^{-\mathrm{i}\xi S}, \tag{1.1}$$

where S is an appropriately chosen self-adjoint operator (sometimes called a conjugate operator). The basic assumptions that one imposes on H and S are the following:
(a) The family $H(\xi)$ is analytic in some strip $|\mathrm{Im}\xi| < a$.
(b) For $\mathrm{Im}\xi < 0$, the essential spectrum of $H(\xi)$ "moves down" below Θ, uncovering a region below the real axis, which belongs to the unphysical sheet of the complex plane.

In the uncovered region, $H(\xi)$ may have some discrete eigenvalues. One can show that these eigenvalues do not depend on ξ and that the eigenvalues of $H(\xi)$ contained in $\Theta \subset \mathbf{R}$ coincide with $\sigma_{\rm pp}(H) \cap \Theta$. The non-real eigenvalues of $H(\xi)$ are called resonances. All these eigenvalues can be studied by standard methods of perturbation theory developed for isolated eigenvalues.

The analytic deformation method gives the following practical criterion for the study of the spectral properties of H: If for $\mathrm{Im}\xi < 0$ the deformed Hamiltonian $H(\xi)$ has no eigenvalues in Θ, then H has no pure point spectrum in Θ. Even if $H(\xi)$ has some real eigenvalues in Θ, the method allows one to exclude the singular continuous spectrum

inside this set.

(2) Mourre's theory and Limiting Absorption Principle. This is an infinitesimal version of the analytic deformation approach. Probably the most advanced version of the Mourre theory can be found in [BG] and [BGS]. Below we briefly describe the Mourre theory following essentially [BG].

One again considers a family of operators (1.1), where now ξ is restricted to the real line. Let $n = 0, 1, \ldots$, $0 \leq \theta < 1$ and $\nu = n + \theta$. The basic assumptions of the Mourre theory are:
(a_ν) The nth derivative of $\xi \mapsto (z - H(\xi))^{-1}$ is θ-Hölder continuous.
(b) (The Mourre estimate). For any $x \in \Theta$ there exists an open interval $I \ni x$, a positive number $C_0 > 0$ and a compact operator K such that

$$\mathbf{1}_I(H)\mathrm{i}[S, H]\mathbf{1}_I(H) \geq C_0\mathbf{1}_I(H) + K. \tag{1.2}$$

(Here $\mathbf{1}_I(H)$ denotes the spectral projection of H onto I.)

If (a_ν) holds with $\nu = 1$, (b) and some other technical assumptions hold, then one can show that $\sigma_{\mathrm{pp}}(H) \cap \Theta$ is a discrete set which consists of eigenvalues of finite multiplicity. If in addition $\nu > 1$ and $\mu > \frac{1}{2}$, then for $x \in \Theta \setminus \sigma_{\mathrm{pp}}(H)$ one can establish the existence of

$$\langle S\rangle^{-\mu}(x + \mathrm{i}0 - H)^{-1}\langle S\rangle^{-\mu} := \lim_{y \downarrow 0}\langle S\rangle^{-\mu}(x + \mathrm{i}y - H)^{-1}\langle S\rangle^{-\mu}. \tag{1.3}$$

($\langle S\rangle$ denotes $(1 + S^2)^{\frac{1}{2}}$.) Note that (1.3) implies the absence of singular continuous spectrum in Θ. Moreover, if $\nu \geq \mu + \frac{1}{2}$ then the function (1.3) is $C^{n-1}(\Theta \setminus \sigma_{\mathrm{pp}}(H))$ and its $(n-1)$st derivative is θ-Hölder continuous. Statements similar to the existence of (1.3) usually go under the name of the Limiting Absorption Principle.

The Mourre method in the form described above does not give much information about the location and the multiplicity of $\sigma_{\mathrm{pp}}(H)$. However, if for all $x \in \Theta$ there is no compact operator K in the Mourre estimate (1.2), then $\sigma_{\mathrm{pp}}(H) \cap \Theta$ is empty.

The two methods described above are complementary. The analytic deformation method typically yields stronger results and allows study of resonances, which are of considerable physical interest. This method, however, is usually applicable to a restricted class of Hamiltonians that meet the analyticity condition. The Mourre theory approach is of much wider applicability but it yields weaker results. In particular, resonances cannot be studied with this approach.

The analytic deformation technique was started in [AC], [BC]. For more information about the early literature on this subject see [Si], [RS4].

The Mourre theory originated in [Mo] and was further developed in [PSS], [JMP], [AHS], [FH], [CFKS], [ABG], [BG], [BG].

Both these methods were first applied to Schrödinger operators where S was the generator of dilations. The generator of dilations is often applicable in situations where

the spectrum of the Hamiltonian covers the half-line. The subset of the lower half-plane uncovered by the analytic deformation in this case is the wedge $-a < \arg z \leq 0$.

Another choice of S that can be found in the literature is the generator of translations. With such S, the set uncovered by the analytic deformation is the strip $-a < \mathrm{Im} z \leq 0$. This choice was made in works devoted to the Stark Hamiltonian. A similar choice (the generator of translations in energy) was made in [JP1], [JP2] in the context of Pauli-Fierz Hamiltonians, and we will keep the same S here.

As we have mentioned earlier, in this paper we study the Mourre theory for Pauli-Fierz Hamiltonians. In the continuation of this paper [DJP], we develop the analytic deformation approach and study the resonances following the earlier work [JP1].

Our treatment of the Mourre theory follows [BG]. One of the differences between our approach and [BG] is that we use weighted spaces instead of Besov spaces. This makes our treatment somewhat less general, but also more elementary than that of [BG]. There are some other differences, due to the special properties of Pauli-Fierz Hamiltonians, which we will describe later.

1.2 The Feshbach Method

The structural properties of the Pauli-Fierz Hamiltonians which play an important role in our paper can be described as follows. They are self-adjoint Hamiltonians of the form

$$H = H_{\mathrm{fr}} + \lambda V,$$

on a Hilbert space $\mathcal{H}$. This Hilbert space has a distinguished decomposition

$$\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}. \tag{1.4}$$

With respect to this decomposition, the full Hamiltonian can be written as a 2×2 matrix

$$H = \begin{bmatrix} H^{\mathrm{vv}} & H^{\mathrm{v}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\mathrm{v}} & H^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{1.5}$$

We will use a similar notation for other operators, for instance,

$$\mathbf{1} = \begin{bmatrix} \mathbf{1}^{\mathrm{vv}} & 0 \\ 0 & \mathbf{1}^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{1.6}$$

We assume that the "free Hamiltonian" has the form

$$H_{\mathrm{fr}} = \begin{bmatrix} H_{\mathrm{fr}}^{\mathrm{vv}} & 0 \\ 0 & H_{\mathrm{fr}}^{\overline{\mathrm{vv}}} \end{bmatrix}, \tag{1.7}$$

and that the perturbation has the form

$$V = \begin{bmatrix} 0 & V^{\mathrm{v}\overline{\mathrm{v}}} \\ V^{\overline{\mathrm{v}}\mathrm{v}} & V^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{1.8}$$

To explain the Feshbach formula in its simplest form, we will assume that $z \notin \sigma(H^{\overline{\mathrm{vv}}})$. We remark that this is not the most interesting case in the context of our paper, since in our case $\sigma(H^{\overline{\mathrm{vv}}}) = \mathbf{R}$ and we want to study embedded eigenvalues. However, the assumption $z \notin \sigma(H^{\overline{\mathrm{vv}}})$ allows us to explain the Feshbach method with the least amount of technical assumptions.

For $z \notin \sigma(H^{\overline{\mathrm{vv}}})$ we introduce the following objects:

$$\begin{aligned} W_{\mathrm{v}}(z) &:= H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}, \\ G_{\mathrm{v}}(z) &:= z\mathbf{1}^{\mathrm{vv}} - H^{\mathrm{vv}} - W_{\mathrm{v}}(z). \end{aligned} \tag{1.9}$$

In the physics literature, the operator $W_{\mathrm{v}}(z)$ is sometimes called *the self-energy*. We propose to call $G_{\mathrm{v}}(z)$ *the resonance function*.

One can show that $z \notin \sigma(H)$ iff $0 \notin \sigma(G_{\mathrm{v}}(z))$. Moreover, if $0 \notin \sigma(G_{\mathrm{v}}(z))$, then one can express the resolvent of H in terms of the resolvent of $H^{\overline{\mathrm{vv}}}$ with the help of the following identity:

$$\begin{aligned} (z-H)^{-1} &= \left(\mathbf{1}^{\mathrm{vv}} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}\right) G_{\mathrm{v}}^{-1}(z) \left(\mathbf{1}^{\mathrm{vv}} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right) \\ &\quad +(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}. \end{aligned} \tag{1.10}$$

We call (1.10) *the Feshbach formula*. This formula was discovered independently by many physicists and mathematicians and it is known under a variety of names – the Grushin, Krein, Livshic formula. In the physics literature, where it is especially widely used and known (see for instance [CT]), it is usually called the Feshbach formula, and we keep this name. It was used recently in a context similar to ours in [BFS1, BFS2]. We refer the reader to [BFS1, How, MeMo] for more information on the literature about this formula.

1.3 Combining the Feshbach Method with the Mourre Theory

Next we want to describe how we study $\sigma_{\mathrm{pp}}(H)$ embedded in $\sigma(H)$ with the help of the Feshbach formula. To that end we study the boundary values of $(z-H)^{-1}$ at the real axis using the expression (1.10). We choose an operator S on $\mathcal{H}$ of the form

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{1.11}$$

Let us list the most important additional properties of H and S that we use in our analysis.
(a'_ν) The family $H(\xi)^{\overline{\mathrm{vv}}}$ satisfies an assumption analogous to (a_ν).
(b') The following global Mourre estimate holds:

$$\mathrm{i}[S^{\overline{\mathrm{vv}}}, H^{\overline{\mathrm{vv}}}] \geq C_0 > 0. \tag{1.12}$$

(c') $V^{\mathrm{v}\overline{\mathrm{v}}}(1+|S|)^{\nu-\frac{1}{2}}$ is bounded.

Using (a'_ν) with $\nu > 1$, (b') and some additional technical assumptions, we develop the Mourre theory for $H^{\overline{\mathrm{vv}}}$, which implies that $H^{\overline{\mathrm{vv}}}$ satisfies the Limiting Absorption Principle uniformly on the whole real line. More precisely, for $\mu > \frac{1}{2}$ we prove that the limit

$$\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}((x+\mathrm{i}0)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu} := \lim_{y\downarrow 0}\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}((x+\mathrm{i}y)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu} \quad (1.13)$$

exists and is uniformly bounded in x and λ. Moreover, if $\nu \geq \mu + \frac{1}{2} = n + \theta$ for some $0 < \theta \leq 1$ then the function (1.13) is in $C^{n-1}(\mathbf{R})$ and its $(n-1)$st derivative is θ-Hölder continuous.

As we have mentioned before, our treatment of the Mourre theory follows [BG]. Nevertheless, there are some important differences. First, the spectrum of $H^{\overline{\mathrm{vv}}}$ covers the whole real line while [BG] make the assumption that Hamiltonian has a spectral gap. More importantly, in our case the commutator $\mathrm{i}[H^{\overline{\mathrm{vv}}}, S^{\overline{\mathrm{vv}}}]$ is not bounded relatively to $H^{\overline{\mathrm{vv}}}$. This leads to some difficulties related to the infrared problem of QED which require delicate arguments.

If (a'_ν), (b') and (c') hold one easily shows, using (1.13), that the limit

$$W_{\mathrm{v}}(x+\mathrm{i}0) := \lim_{y\downarrow 0} W_{\mathrm{v}}(x+\mathrm{i}y)$$

exists. Now, it follows from the Feshbach formula that if $x \in \mathbf{R}$ and $0 \notin \sigma(G_{\mathrm{v}}(x+\mathrm{i}0))$, then the Limiting Absorption Principle for H holds. Moreover, we can show that $0 \in \sigma_{\mathrm{disc}}(G_{\mathrm{v}}(x+\mathrm{i}0))$ implies $x \in \sigma_{\mathrm{pp}}(H)$ and that the multiplicity of 0 as the eigenvalue of $G_{\mathrm{v}}(x+\mathrm{i}0)$ is equal to the multiplicity of x as the eigenvalue of H. Thus, the study of $\sigma_{\mathrm{pp}}(H)$ can be reduced to the study of $G_{\mathrm{v}}(x+\mathrm{i}0)$.

An additional property useful in our analysis is the bound

$$\langle S\rangle^{-\mu}(x+\mathrm{i}0 - H^{\overline{\mathrm{vv}}})^{-1}\langle S\rangle^{-\mu} - \langle S\rangle^{-\mu}(x+\mathrm{i}0 - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1}\langle S\rangle^{-\mu} = O(\lambda^\kappa), \quad (1.14)$$

where $\kappa = \frac{\nu-1}{\nu}$. These results together with the Feshbach formula give us a lot of control over the resolvent of the full Hamiltonian H. In particular, we are able to describe the approximate location of the pure point spectrum and to give sharp estimates on its multiplicity.

Our approach is reminiscent of what can be found in the early literature on stationary scattering theory, eq. in [Fr]. It has a lot in common with typical presentations of perturbation of embedded eigenvalues found in physics textbooks.

1.4 Main results

As we have said before, our results concern a certain class of Pauli-Fierz Hamiltonians. While we still postpone the description of these operators, let us mention that for our purposes the most important properties of these Hamiltonians are the following: They are self-adjoint operators of the form described in (1.5), (1.7), (1.8). In addition, they satisfy

a certain hypothesis, called $S(\nu)$, which resembles the assumption (a_ν) of the Mourre theory.

We introduce also the following auxiliary object:

$$w(z) := V^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}V^{\overline{\mathrm{v}}\mathrm{v}}.$$

Note that $\lambda^2 w(z)$ is the second-order approximation to the self-energy $W_{\mathrm{v}}(z)$.

If $\nu > \frac{1}{2}$, it follows from the Mourre theory for $H_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ that $w(z)$ has the boundary values on the real line which we denote by $w(x+\mathrm{i}0)$. It is easy to see that $w(x+\mathrm{i}0)$ is a dissipative operator, that is, $\mathrm{Im}w(x+\mathrm{i}0) \leq 0$.

Let us describe the main results of our paper. Let H be a Pauli-Fierz Hamiltonian satisfying appropriate conditions. We assume $S(\nu)$ with $\nu > 1$ and set $\kappa = \frac{\nu-1}{\nu}$.

(a) Our first result is Theorem 6.2. In this theorem we show that outside of an $O(\lambda^2)$ neighborhood of $\sigma(H^{\mathrm{v}\mathrm{v}})$, the spectrum of H is purely absolutely continuous and that the Limiting Absorption Principle holds.

(b) Let k be an isolated eigenvalue of $H^{\mathrm{v}\mathrm{v}}$. Theorem 6.3 describes the structure of the spectrum of H in an $O(\lambda^2)$ neighborhood of k. Let p_k be the projection of $H_{\mathrm{fr}}^{\mathrm{v}\mathrm{v}}$ onto k. Set

$$w_k := p_k w(k+\mathrm{i}0)p_k.$$

It is easy to see that this operator is dissipative. If

$$\sigma(w_k) \cap \mathbf{R} = \emptyset, \tag{1.15}$$

then we will say that the Fermi Golden Rule assumption for k holds. Under this assumption we can show that the spectrum of H is purely absolutely continuous in a neighborhood of k and that the Limiting Absorption Principle holds.

If the Fermi Golden Rule assumption fails, we show that outside an $O(\lambda^{2+\kappa})$ neighborhood of $k + \lambda^2\sigma(w_k)$, the spectrum of H is purely absolutely continuous and that the Limiting Absorption Principle holds.

c) If the Fermi Golden Rule assumption fails and $m \in \sigma_{\mathrm{disc}}(w_k) \cap \mathbf{R}$, Theorem 6.4 describes the spectrum of H in a neighborhood of $k + \lambda^2 m$ where, by the second-order perturbation theory, we can expect some eigenvalues of H. Let $p_{k,m}$ be the projection of w_k onto m (we will prove that this projection is orthogonal). We know from (b) that $\sigma_{\mathrm{pp}}(H)$ around $k + \lambda^2 m$ is located in an $O(\lambda^{2+\kappa})$ neighborhood of $k + \lambda^2 m$. In Theorem 6.4 we show that if $S(\nu)$ holds with $\nu > 2$, then the dimension of this point spectrum is not bigger than $\dim p_{k,m}$. Moreover, the Limiting Absorption Principle holds away from $\sigma_{\mathrm{pp}}(H)$.

To summarize, we show that isolated eigenvalues of $H^{\mathrm{v}\mathrm{v}}$, which may give rise to a cluster of eigenvalues of the size $O(\lambda^2)$, split into subclusters of size $O(\lambda^{2+\kappa})$ with the multiplicities estimated from above by the predictions of the second-order perturbation theory. Outside these eigenvalues, the Limiting Absorption Principle holds. Note that as $\nu \to \infty$, $\kappa \to 1$, as expected from the analytic case.

Let us stress that our results in (b) and (c) hold even if k has infinite multiplicity. If $\dim \mathcal{K} = \infty$, this situation is typical for Pauli-Fierz Hamiltonians which arise in quantum statistical mechanics.

Clearly, the notion of Fermi's Golden Rule plays a key role in our argument. Let us remark that the Fermi Golden Rule idea goes back to Dirac [Di] (for the history of the name see [Ha], Section I.1.5). He used it to develop non-relativistic quantum electrodynamics (QED), that is, the theory of emission and absorption of light by matter at low energies. The first triumph of his theory was the derivation of Einstein's A-B law from the first principles of quantum mechanics. In a certain sense, the mathematical foundation of Dirac's theory is the subject of this paper.

1.5 Pauli-Fierz Hamiltonians

We now describe the operators that we call Pauli-Fierz Hamiltonians. These operators describe the dynamics of an (arbitrary) quantum system, which we call $\mathcal{A}$, interacting with a free bosonic field, which we call $\mathcal{R}$. The system $\mathcal{A}$ is described by a Hilbert space $\mathcal{K}$ and a self-adjoint operator K. The one-particle bosonic space is assumed to be $\mathfrak{h} = L^2(\mathbf{R}) \otimes \mathfrak{g}$. The one-particle energy of bosons is the operator of multiplication by $\omega \in \mathbf{R}$ on $L^2(\mathbf{R})$. (The reader may find it surprising that the bosonic energy is unbounded both from below and above. Further on we will explain how the usual physical systems with the energy bounded from below fit in our framework.) After the second quantization, the system $\mathcal{R}$ is described by the Fock space $\Gamma(\mathfrak{h})$ and its Hamiltonian is $\mathrm{d}\Gamma(\omega)$. The composite system is described by the Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$. In the absence of interaction, its Hamiltonian is

$$H_{\mathrm{fr}} = K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega). \tag{1.16}$$

The full Hamiltonian is given by

$$H = H_{\mathrm{fr}} + \lambda V, \tag{1.17}$$

where $V = \varphi(\alpha)$, λ is a real constant and $\varphi(\alpha)$ is the field operator corresponding to $\alpha \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. α describes the coupling of the systems $\mathcal{A}$ and $\mathcal{R}$ and will be called the form-factor, following the terminology used in physics.

We split the Hilbert space into $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$, where $\mathcal{H}^{\mathrm{v}} := \mathcal{K} \otimes \Gamma_0(\mathfrak{h})$ is the vacuum sector. We will call $\mathcal{H}^{\overline{\mathrm{v}}} := (\mathcal{H}^{\mathrm{v}})^{\perp}$ the radiation sector. It is easy to see that the operators H, H_{fr} and V are of the form (1.5), (1.7), (1.8). Note in particular that $H^{\mathrm{vv}} = K$.

The conjugate operator S, which plays the crucial role in the Mourre theory, is given by $S := \mathbf{1} \otimes \mathrm{d}\Gamma(s)$, where $s = -\mathrm{i}\partial_\omega$ acts on $\mathfrak{h}$. Note that in the absence of interaction we have

$$\mathrm{i}[S, H_{\mathrm{fr}}] = N,$$

where N is the number operator. Therefore, on $\mathcal{H}^{\overline{\mathrm{v}}}$ we have a global Mourre estimate:

$$\mathrm{i}[S^{\overline{\mathrm{vv}}}, H_{\mathrm{fr}}^{\overline{\mathrm{vv}}}] = N^{\overline{\mathrm{vv}}} \geq 1.$$

A similar estimate holds in the interacting case: for sufficiently small λ, we can find $C_0 > 0$ such that

$$\mathrm{i}[S^{\overline{\mathrm{v}\mathrm{v}}}, H^{\overline{\mathrm{v}\mathrm{v}}}] \geq C_0 N^{\overline{\mathrm{v}\mathrm{v}}}. \tag{1.18}$$

This relation will be the initial building block in our development of the Mourre theory in Chapter 7.

1.6 Physical systems and Pauli-Fierz Hamiltonians

The best known physical system described by a Hamiltonian (1.17) is the Hamiltonian of non-relativistic QED. This Hamiltonian describes the second-quantized electromagnetic field interacting with a localized piece of matter (see eq. [CT], [BFSS]). We remark that the interaction part of this Hamiltonian has terms both linear and quadratic in the fields, whereas our Hamiltonian has just linear terms. In fact, it is for reasons of space that we have decided to discuss the linear case only – our techniques easily extend to couplings which are quadratic in the field. We have taken care to present our analysis in such a form that various extensions could be easily carried out.

In the case of QED the bosons are called photons. The one-particle space of the photons is $\tilde{\mathfrak{h}} := \mathbf{C}^2 \otimes L^2(\mathbf{R}^3)$ ($\mathbf{C}^2$ accounts for the spin). The energy of a single photon with momentum $\vec{k} \in \mathbf{R}^3$ is given by $|\vec{k}|$. By using the polar coordinates, we can identify $\tilde{\mathfrak{h}}$ with $L^2(\mathbf{R}_+) \otimes \mathfrak{g}$, where $\mathfrak{g} = \mathbf{C}^2 \otimes L^2(S^2)$ (S^2 denotes the unit sphere). After this change of variables, the photon energy operator becomes the operator of multiplication by $\tilde{\omega} \in \mathbf{R}_+$ on $L^2(\mathbf{R}_+) \otimes \mathfrak{g}$. Thus, the photonic energy is bounded from below. In fact, typical Pauli-Fierz Hamiltonians that one finds in physics textbooks are bounded from below. Below we will explain how our results can be applied to such physical systems.

First, let us point out that a Hamiltonian bounded from below is characteristic of systems at zero temperature. If we want to study these systems at a positive temperature, we should use a different Hamiltonian. In the language of algebraic quantum statistical physics, we should consider the KMS state for a given temperature and work in its GNS representation. In the case of the Bose fields this leads to the so-called Araki-Woods representation of CCR. In this representation one has to double the one-particle photon space by adding a space corresponding to the "holes". Next, these two spaces can be "glued" together, as explained in [JP1], [JP2], [JP3], which leads to a space of the form $L^2(\mathbf{R}) \otimes \mathfrak{g}$, with the energy operator given by multiplication by $\omega \in \mathbf{R}$. Thus, even if the Hamiltonian of the zero-temperature systems are bounded from below, their positive temperature analogs are not and they fall into the class considered in our paper. Note that the representation $L^2(\mathbf{R}) \otimes \mathfrak{g}$ for the quantum problem has its analogs in the Lax-Philips theory of the wave equation (see e.g. [RS3]), and in classical models considered in [JP4], [JP5].

The usual zero-temperature systems in the Schrödinger picture clearly do not belong to the class considered in our paper. There is, however, a simple trick which allows us to embed a zero-temperature system into a larger system satisfying the assumptions of our

paper. This is done as follows. We add a non-physical copy of a free bosonic field which is not coupled to the physical system. In this way, one-particle space becomes isomorphic to $L^2(\mathbf{R}) \otimes \mathfrak{g}$. After this modification we obtain an extended Hilbert space and an extended Hamiltonian, which belong to the class considered in this paper. The pure point and singular continuous spectrum of the system do not change after gluing the non-physical bosons. Therefore, all the spectral results we prove for the extended Hamiltonian remain valid for the zero-temperature Hamiltonian.

In the above remarks we have considered a zero-temperature system in the Schrödinger picture, described by a Hamiltonian $\tilde{H}$. One can argue that it is more appropriate from the physical point of view to study the Heisenberg picture, for which the Hamiltonian is given by

$$\tilde{H} \otimes \mathbf{1} - \mathbf{1} \otimes \tilde{H}. \tag{1.19}$$

Even if $\tilde{H}$ is bounded from below, (1.19) is not and belongs to the class considered in our paper.

To summarize, the Mourre theory we develop works both at zero and positive temperature and our paper gives a unified treatment of the spectral theory of Pauli-Fierz Hamiltonians over the complete range of temperatures. We regard this as a substantial advantage of our method.

The conventional wisdom says that an eigenvalue embedded in the continuous spectrum will disappear under the influence of a generic perturbation. In the case of a free positive-temperature Hamiltonian, zero is a degenerate eigenvalue embedded in the absolutely continuous spectrum. It can be shown that for an essentially arbitrary form-factor α (allowed by the principles of quantum statistical physics), the full Pauli-Fierz Hamiltonian will have an eigenstate with a zero eigenvalue corresponding to the KMS state (see [FNV], [JP2], [DJP]). It follows from our theorems that in a generic situation the positive-temperature system will have no other eigenstates besides the KMS state and that the spectrum outside the zero eigenvalue is purely absolutely continuous. This implies that physical systems described by Pauli-Fierz Hamiltonians enjoy strong ergodic properties at positive temperature (see [JP2], [JP3], [DJP] for details).

It was shown in [AH] and [BFSS] (see also [DuSp], [Sp1], [Sp2]) that a large class of zero-temperature Pauli-Fierz Hamiltonians will have a ground state. It follows from our theorems that in a generic situation, this ground state will be non-degenerate and that the rest of the spectrum of H is purely absolutely continuous.

For further discussion, it is convenient to represent the form-factor α as a function

$$\mathbf{R} \ni \omega \mapsto \alpha(\omega) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{g}),$$

with values in the Banach space of all bounded operators from $\mathcal{K}$ to $\mathcal{K} \otimes \mathfrak{g}$. If α has an analytic continuation to a strip along the real axis, instead of the Mourre theory one can use the analytic deformation method developed in [JP1], [JP2], [DJP]. This method yields in many respects stronger results then the Mourre theory. The analyticity

condition, which this method requires, is satisfied in physically interesting examples in the positive-temperature case, where the function α is analytic in the strip $|\mathrm{Im}\omega| < 2\pi T$ (T is the temperature), see [JP1], [JP2]. However, this analyticity condition is never satisfied in the zero-temperature case. In this case, to define α for the extended Hamiltonian, one glues a zero function on the negative half-line to a non-zero function $\tilde{\alpha}$ on the positive half-line. Clearly, the resulting function cannot be analytic. Fortunately, by gluing a zero function to a non-zero function that vanishes at zero, one can obtain a sufficiently smooth function. This is why the Mourre theory we develop in this paper can be applied in the low-temperature regime and in the zero-temperature case – our theorems hold for the values of the coupling constant which are independent of the temperature. Physically, this is perhaps the most important consequence of our results. The analytic deformation method developed in [JP1], [JP2], [DJP] works for $|\lambda| \leq \Lambda(T)$, where $\Lambda(T) \downarrow 0$ as $T \downarrow 0$.

Finally, we would like to briefly discuss the infrared problem which is inherent in the spectral theory of massless Pauli-Fierz Hamiltonians, and which has plagued quantum electrodynamics since its conception. If the quanta of radiation are massless, their energies can be arbitrarily small. The infrared problem arises from the possibility that a finite amount of energy (relative to an equilibrium or a ground state) is shared by infinitely many quanta of the radiation field. The severity of the infrared problem depends on the values of the function α around zero. Let us discuss first the zero-temperature case.

Consider a zero-temperature system where $\alpha(\omega)$ behaves as ω^δ around zero. Then Hypothesis $S(\nu)$ is satisfied with $\nu > \delta + \frac{1}{2}$. Therefore, Theorems 6.2 and 6.3 hold for $\delta > \frac{1}{2}$ and Theorem 6.4 holds for $\delta > \frac{3}{2}$.

Now, let us consider non-relativistic QED at zero-temperature. In this case,

$$\|\tilde{\alpha}(\vec{k})\| \sim |\rho(\vec{k})||\vec{k}|^{-\frac{1}{2}}, \quad \vec{k} \in \mathbf{R}^3, \tag{1.20}$$

where $\rho(\vec{k})$ is the charge distribution in the momentum representation and $\rho(0)$ is the total charge. If one uses the dipole approximation (see [BFS1] for a discussion) then one gets a milder infrared behavior:

$$\|\tilde{\alpha}(\vec{k})\| \sim |\rho(\vec{k})||\vec{k}|^{\frac{1}{2}}, \quad \vec{k} \in \mathbf{R}^3. \tag{1.21}$$

Let us assume for simplicity that ρ is rotationally invariant. After passing to the polar coordinates, we get $\mathfrak{h} = L^2(\mathbf{R}_+) \otimes L^2(S^2)$, where S^2 is the unit sphere in $\mathbf{R}^3$. Then (1.20) becomes

$$\|\tilde{\alpha}(\omega)\| \sim |\rho(\omega)|\omega^{\frac{1}{2}}, \quad \omega > 0, \tag{1.22}$$

and (1.21) becomes

$$\|\tilde{\alpha}(\omega)\| \sim |\rho(\omega)|\omega^{\frac{3}{2}}, \quad \omega > 0. \tag{1.23}$$

Thus, Theorems 6.2 and 6.3 do not cover (1.20) but apply to (1.21), whereas Theorem 6.4 does not cover either (1.20) or (1.21).

At positive temperature, the infrared singularity is smoothed out by the Planck law, and Theorems 6.2, 6.3 and 6.4 apply to typical models of physical interest.

1.7 Comparison with the literature

In the literature one can find other applications of the Mourre theory to zero-temperature Pauli-Fierz Hamiltonians. Probably the earliest is contained in [HuSp2], where the massive radiation field is considered. The conjugate operator that is used there is the second quantization of the generator of translations in the energy variable. However, unlike in our paper, in [HuSp2] the energy variable is restricted to the positive half-line, hence this operator is not self-adjoint.

The massless case, which is physically more important and technically more demanding due to infrared difficulties, was first considered in [BFS1], albeit in energetically non-interesting regions away from $\sigma(K)$. The main point of this paper, however, is a renormalization group analysis of resonances of massless zero-temperature Pauli-Fierz Hamiltonians.

Concerning the Mourre theory in the massless case, the first complete results are presented in [Sk]. This work uses the same non-self-adjoint S as [HuSp2], which is however approximated with a sequence of self-adjoint operators.

Another choice of S, a suitably modified second quantization of the generator of dilations, is used in [BFSS]. For zero-temperature Hamiltonians, the generator of dilations should in principle allow one to treat a more general class of perturbations than the generator of translations.

All of the above papers, including ours, give results which are valid for a small coupling constant. All values of the coupling constant are covered in [DG], where the Mourre theory for a massive radiation field is developed. Unfortunately, the techniques of [DG] do not seem applicable to the massless case considered here.

At positive temperature, none of the various approaches discussed above is applicable, and in this respect our results stand alone.

Finally, let us remark that in all of the above works the Mourre theory is studied on the whole Hilbert space. The distinct feature of our method is that we apply first Mourre theory to $H^{\overline{\mathrm{vv}}}$ and then use the Feshbach method. We believe that this approach is natural and that it gives a more precise information on the location and multiplicity of embedded eigenvalues.

Some of the results of our paper are quite general. These results concern spectral analysis of a relatively arbitrary linear operator and are related to the Feshbach method. Similar ideas can be found throughout the literature, notably in [BFS1], [BFS2] and [GGK]. We believe that these results are of interest outside of the context of Pauli-Fierz Hamiltonians. In particular, the following of our general results appear to be new: Proposition 3.2 about real eigenvalues of a dissipative operator, Proposition 3.7 and Theorem 3.8 about the Feshbach method for embedded eigenvalues, and Theorem 3.12 and Corollary 3.13 about estimating the number of embedded eigenvalues.

1.8 Organization of the paper

The paper is organized as follows.

In Chapter 2 we introduce notation and, for reference purposes, state some general facts about operators in Hilbert spaces.

In Chapter 3 we describe some properties of self-adjoint operators in a Hilbert space decomposed in a direct sum of two Hilbert spaces. They are centered around the Feshbach formula. This formula leads to certain identities for the projections onto eigenvectors of the operator H, which we found very appealing and useful in our analysis. It also leads to certain precise estimates on the number of eigenvalues of the operator H. In spite of the fact that they are general and simple, some of the results of Chapter 3 appear to be new. Problems involving embedded eigenvalues arise naturally in spectral geometry, number theory and mathematical physics, and we hope that some of the results of this chapter will find applications outside of our work.

We present in a parallel way results concerning the spectrum of H outside of $\sigma(H^{\overline{\mathrm{vv}}})$ and the embedded point spectrum of H inside $\sigma(H^{\overline{\mathrm{vv}}})$. The results about the spectrum outside $\sigma(H^{\overline{\mathrm{vv}}})$ are less technical and can be partly found in the literature, eg. in [BFS1]. They are not used in the remaining part of our paper. On the other hand, the more difficult results concerning the embedded spectrum inside $\sigma(H^{\overline{\mathrm{vv}}})$ belong to the most important tools of our paper. We believe that it is helpful for the reader to compare these two types of results.

In Chapter 4 we review the basic notions of quantum field theory. This chapter makes the paper essentially self-contained.

In Chapter 5 we introduce Pauli-Fierz Hamiltonians and discuss some of their basic properties.

The main results of the paper are stated in Chapter 6.

Chapter 7 describes the proof of the Limiting Absorption Principle for the operator $H^{\overline{\mathrm{vv}}}$. As we have stressed before, our proof follows the arguments of [BG]. In Section 7.1 we derive a bound on the boundary value of the resolvent of $H^{\overline{\mathrm{vv}}}$, which is the basic ingredient of the Limiting Absorption Principle, following essentially the original arguments of [Mo] and [PSS], with modifications due to [BG]. The main additional difficulty is the infrared problem, which is handled similarly as in [JP1]. In Section 7.2 we study the regularity of the boundary value of the resolvent of $H^{\overline{\mathrm{vv}}}$, following [BG]. In Section 7.4 we estimate the difference of the full and the free resolvent.

Chapter 8 completes the proof of our main results. The main tool is the Feshbach formula, which is applied several times to various decompositions of our Hilbert space.

Acknowledgments. We are grateful to A. Jensen, C.-A. Pillet and E. Skibsted for useful discussions and comments on the manuscript. The research of the first author was a part of the project Nr 2 P03A 019 15 financed by a grant of Komitet Badań Naukowych. A part of this work was done during the visit of the first author to University of Ottawa, which was supported by NSERC, and during his visit to the Aarhus University supported

by MaPhySto funded by the Danish National Research Foundation. The research of the second author was partly supported by NSERC. Part of this work was done during the visit of the second author to Caltech. We are grateful to A. Jensen, E. Skibsted, Y. Last and B. Simon for their hospitality.

2 Preliminaries

In this section we set the notation and, for reference purposes, recall some definitions and facts which will be used in the paper.

$\mathbf{N} := \{0, 1, 2, \ldots\}$ denotes the set of natural numbers (including 0). We set $\langle t \rangle := \sqrt{1+t^2}$. We will also use the shorthand

$$\mathbf{C}_\pm := \{z \in \mathbf{C} \ : \ \pm \mathrm{Im} z > 0\},$$

$$\mathbf{R}_\pm := \{x \in \mathbf{R} \ : \ \pm x > 0\}.$$

The closure of a set $\Omega \subset \mathbf{C}$ we denote by $\overline{\Omega}$.

If $\Omega \subset \mathbf{C}$ and $r > 0$, we set

$$B(\Omega, r) := \{z \in \mathbf{C} \ : \ \mathrm{dist}(\Omega, z) < r\},$$

$$\overline{B}(\Omega, r) := \{z \in \mathbf{C} \ : \ \mathrm{dist}(\Omega, z) \leq r\}.$$

In particular, for $k \in \mathbf{C}$,

$$B(k, r) := B(\{k\}, r), \quad \overline{B}(k, r) := \overline{B}(\{k\}, r)$$

denotes the open/closed ball of center k and radius r.

If $\Theta \subset \mathbf{R}$, we set

$$I(\Theta, r) := \{x \in \mathbf{R} \ : \ \mathrm{dist}(\Theta, x) < r\},$$

$$\overline{I}(\Theta, r) := \{x \in \mathbf{R} \ : \ \mathrm{dist}(\Theta, x) \leq r\}.$$

In particular, for $k \in \mathbf{R}$,

$$I(k, r) := I(\{k\}, r), \quad \overline{I}(k, r) := \overline{I}(\{k\}, r)$$

denotes the open/closed interval of center k and radius r.

Let $\mathcal{H}$ be a Hilbert space. The inner product on $\mathcal{H}$ we denote by $(\cdot \,|\, \cdot)$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. If these two spaces are the same and equal to $\mathcal{H}$, we will write simply $\mathcal{B}(\mathcal{H})$.

Let $\Omega \subset \mathbf{C}$. In this paper we will often deal with operator-valued functions

$$\Omega \ni z \mapsto A(z) \in \mathcal{B}(\mathcal{H}). \tag{2.24}$$

Unless otherwise specified, the various limits of such functions are always defined with respect to the norm of the Banach space $\mathcal{B}(\mathcal{H})$.

Definition 2.1 *Assume that z_0 is not an isolated point of Ω. We say that the function (2.24) is differentiable at z_0 with derivative $A'(z_0) \in \mathcal{B}(\mathcal{H})$ if*

$$\lim_{\substack{z \to z_0 \\ z \in \Omega, z \neq z_0}} \|(z - z_0)^{-1}(A(z) - A(z_0)) - A'(z_0)\| = 0.$$

We say that the function $A(z)$ is differentiable on Ω if it is differentiable at every non-isolated point of Ω.

As usual, we denote the n-th derivative by $\partial_z^n A(z)$.

Definition 2.2 *Let $n \in \mathbf{N}$. We say that a function $\Omega \ni z \mapsto A(z)$ is in the class $C_{\mathrm{u}}^n(\Omega)$ if it has a continuous n-th derivative on Ω and satisfies the bound*

$$\|\partial_z^m A(z)\| \leq C_m, \quad z \in \Omega, \quad m = 0, \ldots, n. \tag{2.25}$$

Let $\ell : \overline{\mathbf{R}}_+ \mapsto \overline{\mathbf{R}}_+$ be a positive continuous function with $\ell(0) = 0$. We say that the function $\Omega \ni z \mapsto A(z)$ is in the class $C_{\mathrm{u}}^{n,\ell}(\Omega)$ if $A \in C_{\mathrm{u}}^n(\Omega)$ and there exists C such that

$$\|\partial_z^n A(z_1) - \partial_z^n A(z_2)\| \leq C\ell(|z_1 - z_2|), \qquad z_1, z_2 \in \Omega. \tag{2.26}$$

If we have a family of functions A_λ defined on sets Ω_λ, $\lambda \in \mathcal{I}$, we say that A_λ is of the class $C_{\mathrm{u}}^n(\Omega_\lambda)$ or $C_{\mathrm{u}}^{n,\ell}(\Omega_\lambda)$ uniformly in λ if the constants C in (2.26) and C_m in (2.25) can be chosen independently of $\lambda \in \mathcal{I}$.

For $0 < \theta \leq 1$ we define functions ℓ_θ on $\overline{\mathbf{R}}_+$ by the formula

$$\ell_\theta(\tau) = \begin{cases} \tau^\theta & \text{if } 0 < \theta < 1 \\ \tau\,(1 + \ln(1 + \tau^{-1})) & \text{if } \theta = 1. \end{cases} \tag{2.27}$$

The classes $C_{\mathrm{u}}^{n,\ell_\theta}$ will figure in the Limiting Absorption Principle which we will establish in this paper. In the sequel we use the shorthands $C_{\mathrm{u}}^{n,\theta} = C_{\mathrm{u}}^{n,\ell_\theta}$.

Let $\Omega \subset \mathbf{C}$ and $\Gamma \subset \partial\Omega$. We say that a continuous function $\Omega \ni z \mapsto A(z)$ extends by continuity to $\Omega \cup \Gamma$ if for every $z_0 \in \Gamma$ the limit

$$A(z_0) := \lim_{z \to z_0, z \in \Omega} A(z)$$

exists, and the extended function is continuous on $\Omega \cup \Gamma$. We will denote the functions extended by continuity with the same letter. If $\Omega \subset \mathbf{C}_\pm$ and $A(z)$ extends by continuity to a part of the real axis, we denote by $A(x \pm \mathrm{i}0)$ its values along $\mathbf{R}$.

In the development of the Mourre theory, we will make use of the following two simple facts.

Proposition 2.3 *Let Ω be an open convex set, and let*

$$\overline{\mathbf{R}}_+ \times \Omega \ni (\epsilon, z) \mapsto A(\epsilon, z) \in \mathcal{B}(\mathcal{H})$$

be a function which is continuously differentiable in each variable separately. Assume further that for some constants C and $0 < \theta \leq 1$,

$$\sup_{z\in\Omega} \|\partial_\epsilon^k \partial_z^l A(\epsilon, z)\| < C\epsilon^{-1}\ell_\theta(\epsilon), \qquad k + l \leq 1.$$

Then, the function $\Omega \ni z \mapsto A(0, z)$ is in the class $C_{\mathrm{u}}^{0,\theta}(\Omega)$.

Proof. For $z_1, z_2 \in \Omega$ and $\epsilon > 0$ we have

$$\begin{aligned} \|A(0, z_1) - A(0, z_2)\| &\leq \textstyle\int_0^\epsilon \|\partial_\tau A(\tau, z_1)\|\mathrm{d}\tau + |z_1 - z_2| \int_0^1 \|\partial_z A(\epsilon, z_1 + t(z_2 - z_1))\|\mathrm{d}t \\ &\quad + \textstyle\int_0^\epsilon \|\partial_\tau A(\tau, z_2)\|\mathrm{d}\tau \\ &\leq 2C \textstyle\int_0^\epsilon \tau^{-1}\ell_\theta(\tau)\mathrm{d}\tau + C|z_1 - z_2|\epsilon^{-1}\ell_\theta(\epsilon). \end{aligned} \tag{2.28}$$

Using the form of functions ℓ_θ one easily shows that for some constants C_θ and all $\epsilon \geq 0$,

$$\int_0^\epsilon \tau^{-1}\ell_\theta(\tau)\mathrm{d}\tau \leq C_\theta \ell_\theta(\epsilon).$$

Combining this estimate with (2.28) and setting $\epsilon = |z_1 - z_2|$ we derive that $A(0, z) \in C_{\mathrm{u}}^{0,\theta}(\Omega)$. □

Proposition 2.4 *Let A_λ, $\lambda \in \mathcal{I}$, be a family of functions defined on open convex sets $\Omega_\lambda \subset \mathbf{C}$. If the family A_λ is of the class $C_{\mathrm{u}}^{n,\ell}(\Omega_\lambda)$ uniformly in λ, then the functions A_λ extend by continuity to $\overline{\Omega}_\lambda$ and the family A_λ is of the class $C_{\mathrm{u}}^{n,\ell}(\overline{\Omega}_\lambda)$ uniformly in λ.*

The proof of this proposition is elementary and we will skip it.

Let H be a closed operator on $\mathcal{H}$. We denote the domain of H by $\mathcal{D}(H)$ and the spectrum of H by $\sigma(H)$. The numerical range of H is defined by

$$\mathfrak{n}(H) := \{(\psi|H\psi) \ : \ \psi \in \mathcal{D}(H), \ \|\psi\| = 1\}.$$

The vector space $\mathcal{D}(H)$ equipped with the graph norm

$$\|\psi\|_H = \|\psi\| + \|H\psi\|,$$

is a Banach space. A vector space $\mathcal{C} \subset \mathcal{D}(H)$ is called a core of H if $\mathcal{C}$ is dense in $\mathcal{D}(H)$ in the graph norm. The following useful fact is well known:

Lemma 2.5 *Let A and B be closed operators and let $\mathcal{D}(B)$ be a core of A. Then any core of B is a core of A.*

For any closed operator H, we say that Ω is an isolated subset of $\sigma(H)$ if it is relatively closed and open subset of $\sigma(H)$. If in addition Ω is bounded, then there exists a simple closed path γ which separates Ω and $\sigma(H) \setminus \Omega$, and we can define the spectral projection of H onto Ω by the formula

$$\mathbf{1}_\Omega(H) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z - H)^{-1} \mathrm{d}z.$$

It is easy to show the following fact:

Lemma 2.6 *If Ω is a bounded isolated subset of $\sigma(H)$ and p is a projection such that $\mathrm{Ran}\mathbf{1}_\Omega(H) = \mathrm{Ran}p$ and $[p, B] = 0$, then $p = \mathbf{1}_\Omega(B)$.*

An isolated point of $\sigma(H)$ will be called an isolated eigenvalue of H. An isolated eigenvalue z_0 of H is called semisimple if $(z-H)^{-1}$ has a simple pole at z_0, or equivalently, if

$$\mathrm{Ran}\mathbf{1}_{\{z_0\}}(H) = \mathrm{Ker}(H - z_0).$$

We denote by $\sigma_{\mathrm{disc}}(H)$ the discrete spectrum of H, that is, the set of all isolated eigenvalues e such that $\dim \mathbf{1}_{\{e\}}(H) < \infty$. The essential spectrum of H is defined by $\sigma_{\mathrm{ess}}(H) := \sigma(H) \setminus \sigma_{\mathrm{disc}}(H)$. The resolvent $(z - H)^{-1}$ is a meromorphic function on $\mathbf{C} \setminus \sigma_{\mathrm{ess}}(H)$ with singularities only at the points $z \in \sigma_{\mathrm{disc}}(H)$. The coefficients of the Laurent expansion at $z_0 \in \sigma_{\mathrm{disc}}(H)$ are finite rank operators.

If H is a self-adjoint operator, we denote by $\sigma_{\mathrm{pp}}(H)$, $\sigma_{\mathrm{sc}}(H)$ and $\sigma_{\mathrm{ac}}(H)$ the pure point, singular continuous and absolutely continuous spectrum of H. The singular spectrum is defined by $\sigma_{\mathrm{sing}}(H) = \overline{\sigma_{\mathrm{pp}}(H)} \cup \sigma_{\mathrm{sc}}(H)$. If Θ a Borel subset of $\mathbf{R}$ then $\mathbf{1}_\Theta(H)$ will denote the spectral projection of H onto Θ. We denote by $\mathbf{1}_\Theta^{\mathrm{pp}}(H)$, $\mathbf{1}_\Theta^{\mathrm{sc}}(H)$, $\mathbf{1}_\Theta^{\mathrm{ac}}(H)$ the spectral projections of H onto Θ associated to the pure point, singular continuous and absolutely continuous spectrum.

We now recall some standard results about linear operators that will be used throughout the paper.

Proposition 2.7 *Let H be a self-adjoint operator. If $z \in \mathbf{C} \setminus \sigma(H)$ then*

$$\|(z - H)^{-1}\| = \frac{1}{\mathrm{dist}(z, \sigma(H))}.$$

An immediate consequence of this proposition is

Proposition 2.8 *Let H be a self-adjoint and V a bounded operator. Then, $\sigma(H + V) \subset \overline{B}(\sigma(H), \|V\|)$ and for $z \in \mathbf{C} \setminus \overline{B}(\sigma(H), \|V\|)$ one has the bound*

$$\|(z - (H + V))^{-1}\| \leq \frac{1}{\mathrm{dist}(z, \sigma(H)) - \|V\|}.$$

The concept of the numerical range allows to formulate related results for closed operators.

Proposition 2.9 *Let H be a closed operator such that $\mathcal{D}(H) = \mathcal{D}(H^*)$. Then, $\sigma(H) \subset \overline{\mathfrak{N}(H)}$ and, for $z \in \mathbf{C} \setminus \overline{\mathfrak{N}(H)}$, one has the bound*

$$\|(z - H)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \mathfrak{N}(H))}.$$

Proposition 2.10 *Let H be a closed operator such that $\mathcal{D}(H) = \mathcal{D}(H^*)$, and let V be a bounded operator. Then, $\sigma(H + V) \subset \overline{B}(\mathfrak{N}(H), \|V\|)$, and for $z \in \mathbf{C} \setminus \overline{B}(\mathfrak{N}(H), \|V\|)$ one has the bound*

$$\|(z - (H + V))^{-1}\| \leq \frac{1}{\operatorname{dist}(z, \mathfrak{N}(H)) - \|V\|}.$$

We say that the operator B is A-bounded if $\mathcal{D}(B) \supset \mathcal{D}(A)$ and

$$\|B\phi\| \leq a\|A\phi\| + b\|\phi\|, \quad \phi \in \mathcal{D}(A). \tag{2.29}$$

The infimum of possible values of a in (2.29) is called the A-bound of B. Recall that if A is closed and the A-bound of B is less than 1, then $A + B$ is closed on $\mathcal{D}(A)$. Clearly, if for some $z_0 \in \mathbf{C}$, $\|B_1(z_0 - A)^{-1}\| = a$, where $B = B_1 + B_2$ and B_2 is bounded, then the A-bound of B is less than or equal to a.

Proposition 2.11 *Suppose that A, B are operators such that A is closed, $\mathcal{D}(B) \supset \mathcal{D}(A)$ and, for some $z_0 \in \mathbf{C}$, we have*

$$\|B_1(z_0 - A)^{-1}\| < 1, \qquad \|(z_0 - A)^{-1}B_1\| < 1,$$

where $B = B_1 + B_2$ and B_2 is bounded. Then

$$(A + B)^* = A^* + B^*. \tag{2.30}$$

Proof. Replacing A with $A - z_0$, where $z_0 \notin \sigma(A)$, we can assume that $z_0 = 0$. We can also subtract the bounded operator B_2 from B without affecting (2.30).

Clearly, $\mathcal{D}(A^* + B^*) \supset \mathcal{D}(A^*)$ and

$$(A + B)^*\big|_{\mathcal{D}(A^*)} = A^* + B^*.$$

We want to show that

$$\mathcal{D}(A^*) \supset \mathcal{D}((A + B)^*). \tag{2.31}$$

Let $\psi \in \mathcal{D}(A)$. Since $\|A^{-1}B\| < 1$, we know that $\mathbf{1} + A^{-1}B$ has a bounded inverse. Hence $\psi_1 := (1 + A^{-1}B)^{-1}\psi$ satisfies

$$\|\psi_1\| \leq C_1\|\psi\|. \tag{2.32}$$

Since $\|BA^{-1}\| < 1$, the operator $\mathbf{1} + BA^{-1}$ has also a bounded inverse. But, on $\mathcal{D}(A)$, $(\mathbf{1} + A^{-1}B)^{-1} = A^{-1}(\mathbf{1} + BA^{-1})^{-1}A$. Therefore, $\mathbf{1} + BA^{-1}$ is bounded as an operator on $\mathcal{D}(A)$. Thus $\psi_1 \in \mathcal{D}(A)$.

Now let $\phi \in \mathcal{D}((A + B)^*)$. Using that $\psi_1 \in \mathcal{D}(A) = \mathcal{D}(A + B)$, we have

$$|(\phi|(A + B)\psi_1)| \leq C\|\psi_1\|. \tag{2.33}$$

Clearly,

$$\begin{aligned} |(\phi|A\psi)| &= |(\phi|A(1 + A^{-1}B)\psi_1) \\ &= (\phi|(A + B)\psi_1) \\ &\leq C\|\psi_1\| \leq C_1\|\psi\|, \end{aligned}$$

where in the last steps we used (2.33) and (2.32). This shows that $\phi \in \mathcal{D}(A^*)$, and ends the proof of the inclusion (2.31). □

3 General theory

3.1 Dissipative operators

We begin with

Definition 3.1 *A closed operator B is called dissipative if $\mathfrak{N}(B) \subset \overline{\mathbf{C}}_-$.*

On $\mathcal{D}(B) \cap \mathcal{D}(B^*)$ we can define

$$\mathrm{Re}B := \frac{1}{2}(B + B^*), \qquad \mathrm{Im}B := \frac{1}{2\mathrm{i}}(B - B^*).$$

Clearly, $\mathrm{Re}B$ and $\mathrm{Im}B$ are symmetric operators.

In all our applications the following condition will be satisfied:

$$\mathcal{D}(B) = \mathcal{D}(B^*), \;\; \mathrm{Im}B \text{ is bounded and } \mathrm{Re}B \text{ is self-adjoint on } \mathcal{D}(B). \tag{3.34}$$

Clearly, under this condition B is dissipative iff $\mathrm{Im}B \leq 0$.

Proposition 3.2 *Let B be a dissipative operator satisfying (3.34) and $e \in \mathbf{R}$. Then,*
(i) $\mathrm{Ker}(B - e) = \mathrm{Ran}\mathbf{1}_{\{\mathrm{e}\}}(\mathrm{Re}B) \cap \mathrm{Ran}\mathbf{1}_{\{0\}}(\mathrm{Im}B)$.
(ii) *Let p be the orthogonal projection onto* $\mathrm{Ker}(B - e)$. *Then* $0 = [p, B]$.
(iii) *If in addition $e \in \sigma_{\mathrm{disc}}(B)$, then the eigenvalue e is semisimple and $p = \mathbf{1}_{\{\mathrm{e}\}}(B)$.*

Proof. Let $\psi \in \mathrm{Ker}(B-e)$. Then $\psi \in \mathcal{D}(B) = \mathcal{D}(B^*)$ and

$$0 = (\psi|(B-e)\psi) = (\psi|(B^*-e)\psi).$$

Hence $0 = (\psi|\mathrm{Im}B\psi)$. Since $\mathrm{Im}B \leq 0$, we derive that $0 = \mathrm{Im}B\psi$. This and $0 = (B-e)\psi$ yield that $e\psi = \mathrm{Re}B\psi$. Hence

$$\mathrm{Ran}\mathbf{1}_{\{e\}}(\mathrm{Re}B) \cap \mathrm{Ran}\mathbf{1}_{\{0\}}(\mathrm{Im}B) \supset \mathrm{Ran}\mathbf{1}_{\{e\}}(B).$$

The inclusion $\subset$ is obvious, and Part (i) follows.

By Part (i) we have

$$p \leq \mathbf{1}_{\{e\}}(\mathrm{Re}(B)), \quad p \leq \mathbf{1}_{\{e\}}(\mathrm{Im}(B)).$$

Hence

$$0 = [p, \mathrm{Re}B] = [p, \mathrm{Im}B].$$

This implies (ii).

To establish Part (iii), we note that if $e \in \sigma_{\mathrm{disc}}(B)$ then e is a pole of $(z-B)^{-1}$. By Proposition 2.9 this pole is simple. Hence e is a semisimple eigenvalue of B. An application of Lemma 2.6 completes the proof of (iii). □

If e is an isolated real eigenvalue of B with $\dim \mathbf{1}_{\{e\}}(B) = \infty$, then $\mathrm{Ker}(B-e)$ may be strictly smaller than $\mathrm{Ran}\mathbf{1}_{\{e\}}(B)$. (We thank E. Skibsted for pointing this out to us).

Proposition 3.3 *Let B be a bounded dissipative operator on a Hilbert space $\mathcal{H}$ such that $\sigma(B) \cap \mathbf{R} \subset \sigma_{\mathrm{disc}}(B)$. Then*

$$c := \sup_{z \in \mathbf{C}_+} \left\| (z-B)^{-1} \mathbf{1}_{\sigma(B)\setminus\mathbf{R}}(B) \right\| < \infty, \tag{3.35}$$

and, for $z \in \overline{\mathbf{C}}_+ \setminus \sigma(B)$,

$$\|(z-B)^{-1}\| \leq \max\Big(\frac{1}{\mathrm{dist}(z, \sigma(B)\cap\mathbf{R})}, c\Big). \tag{3.36}$$

Proof. By Proposition 3.2

$$\mathbf{1}_{\sigma(B)\cap\mathbf{R}} = \sum_{e \in \sigma(B)\cap\mathbf{R}} \mathbf{1}_{\{e\}}(B),$$

is an orthogonal projection. Therefore

$$\|(z-B)^{-1}\| = \max\Big(\|(z-B)^{-1}\mathbf{1}_{\sigma(B)\cap\mathbf{R}}(B)\|, \|(z-B)^{-1}\mathbf{1}_{\sigma(B)\setminus\mathbf{R}}(B)\|\Big). \tag{3.37}$$

Also by Proposition 3.2, $B\mathbf{1}_{\sigma(B)\cap\mathbf{R}}(B)$ is self-adjoint. Hence

$$\|(z-B)^{-1}\mathbf{1}_{\sigma(B)\cap\mathbf{R}}(B)\| = \frac{1}{\mathrm{dist}(z,\sigma(B)\cap\mathbf{R})}. \tag{3.38}$$

Since $B\mathbf{1}_{\sigma(B)\setminus\mathbf{R}}(B)$ is a bounded operator with spectrum contained in $\mathbf{C}_-$, we clearly have (3.35). Now (3.37), (3.35) and (3.38) imply (3.36). □

The following result is an immediate consequence of the previous proposition.

Proposition 3.4 *Let B be a bounded dissipative operator such that $\sigma(B)\cap\mathbf{R}\subset\sigma_{\mathrm{disc}}(B)$, and let c be the constant defined in (3.35). If V is a bounded operator such that $c\|V\|<1$, then*

$$\sigma(B+V)\cap\overline{\mathbf{C}}_+ \subset \overline{B}(\sigma(B)\cap\mathbf{R},\|V\|).$$

Furthermore, for $z\in\overline{\mathbf{C}}_+\setminus\overline{B}(\sigma(B)\cap\mathbf{R},\|V\|)$ one has the bound

$$\|(z-B-V)^{-1}\| \leq \frac{1}{\mathrm{dist}(z,\sigma(B)\cap\mathbf{R})-\|V\|}. \tag{3.39}$$

3.2 The Feshbach formula

Let $\mathcal{H}$ be a Hilbert space decomposed into a direct sum $\mathcal{H}=\mathcal{H}^{\mathrm{v}}\oplus\mathcal{H}^{\overline{\mathrm{v}}}$. The projections onto $\mathcal{H}^{\mathrm{v}}$ and $\mathcal{H}^{\overline{\mathrm{v}}}$ we denote by $\mathbf{1}^{\mathrm{vv}}$ and $\mathbf{1}^{\overline{\mathrm{vv}}}$. In this section we study operators of the form

$$H=\begin{bmatrix} H^{\mathrm{vv}} & H^{\mathrm{v}\overline{\mathrm{v}}} \\ H^{\overline{\mathrm{v}}\mathrm{v}} & H^{\overline{\mathrm{vv}}} \end{bmatrix}, \tag{3.40}$$

We assume that H^{vv} and $H^{\overline{\mathrm{vv}}}$ are closed operators on $\mathcal{H}^{\mathrm{v}}$ and $\mathcal{H}^{\overline{\mathrm{v}}}$; moreover we suppose that $H^{\overline{\mathrm{v}}\mathrm{v}}:\mathcal{H}^{\mathrm{v}}\to\mathcal{H}^{\overline{\mathrm{v}}}$ and $H^{\mathrm{v}\overline{\mathrm{v}}}:\mathcal{H}^{\overline{\mathrm{v}}}\to\mathcal{H}^{\mathrm{v}}$ are bounded operators. Clearly H is closed and $\mathcal{D}(H)=\mathcal{D}(H^{\mathrm{vv}})\oplus\mathcal{D}(H^{\overline{\mathrm{vv}}})$.

For any $z\notin\sigma(H^{\overline{\mathrm{vv}}})$ we define

$$\begin{aligned} W_{\mathrm{v}}(z) &:= H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}}-H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}, \\ G_{\mathrm{v}}(z) &:= z\mathbf{1}^{\mathrm{vv}}-H^{\mathrm{vv}}-W_{\mathrm{v}}(z). \end{aligned} \tag{3.41}$$

In the physics literature, the operator $W_{\mathrm{v}}(z)$ is sometimes called *the self-energy*. For $G_{\mathrm{v}}(z)$ we propose the name *the resonance function*. Note that $W_{\mathrm{v}}(z)$ is an analytic operator-valued function on $\mathbf{C}\setminus\sigma(H^{\overline{\mathrm{vv}}})$, and that $W_{\mathrm{v}}(z)^*=W_{\mathrm{v}}(\overline{z})$. It follows that $G_{\mathrm{v}}(z)$ is an analytic family of type A on $\mathbf{C}\setminus\sigma(H^{\overline{\mathrm{vv}}})$.

The following proposition is well known:

Proposition 3.5 *Assume that $z \notin \sigma(H^{\overline{\mathrm{vv}}})$. Then,*
(i) *$z \in \sigma(H)$ iff $0 \in \sigma(G_{\mathrm{v}}(z))$;*
(ii) *If $0 \notin \sigma(G_{\mathrm{v}}(z))$ then*

$$\begin{aligned}(z-H)^{-1} &= \left(\mathbf{1}^{\mathrm{vv}} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}\right) G_{\mathrm{v}}^{-1}(z) \left(\mathbf{1}^{\mathrm{vv}} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right) \\ &\quad + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}.\end{aligned} \tag{3.42}$$

Proof. Our proof is inspired by [GGK]. Set

$$A(z) := \mathbf{1} + H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}, \quad B(z) := \mathbf{1} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Both $A(z)$ and $B(z)$ have bounded inverses:

$$A^{-1}(z) = \mathbf{1} - H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}, \quad B^{-1}(z) = \mathbf{1} - (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Moreover, both $B(z)$ and $B^{-1}(z)$ are bounded operators on $\mathcal{D}(H)$.

The following identity holds in the sense of operators from $\mathcal{D}(H)$ to $\mathcal{H}$:

$$A(z)(z-H)B(z) = G_{\mathrm{v}}(z) + z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}} \tag{3.43}$$

If $z \notin \sigma(H)$, then the left hand side of (3.43) is invertible. Hence so is the right hand side. This implies that $G_{\mathrm{v}}(z)$ is invertible.

Next we will use the identity (also understood in the sense of operators from $\mathcal{D}(H)$ to $\mathcal{H}$):

$$(z-H) = A^{-1}(z)(G_{\mathrm{v}}(z) + z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})B^{-1}(z). \tag{3.44}$$

If $0 \notin \sigma(G_{\mathrm{v}}(z))$, then the right hand side of (3.44) is invertible. Hence so is $z - H$. This completes the proof of (i).

To show (ii) we note that if $z \notin \sigma(H)$, (3.43) or (3.44) implies

$$(z-H)^{-1} = B(z)(G_{\mathrm{v}}^{-1}(z) + (z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1})A(z),$$

which, after substituting the expressions defining $A(z)$ and $B(z)$, yields (3.42). □

¿From now on we assume in addition that H^{vv} and $H^{\overline{\mathrm{vv}}}$ are self-adjoint and $H^{\mathrm{v}\overline{\mathrm{v}}} = (H^{\overline{\mathrm{v}}\mathrm{v}})^*$. This clearly implies that H is self-adjoint. Moreover, $W_{\mathrm{v}}(z)^* = W_{\mathrm{v}}(\overline{z})$, the operator $W_{\mathrm{v}}(z)$ is dissipative if $\mathrm{Im}z \geq 0$. If $x \in \mathbf{R} \setminus \sigma(H^{\overline{\mathrm{vv}}})$, then $W_{\mathrm{v}}(x)$ is self-adjoint and

$$\frac{\mathrm{d}}{\mathrm{d}x}W_{\mathrm{v}}(x) = -H^{\mathrm{v}\overline{\mathrm{v}}}(x\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-2}H^{\overline{\mathrm{v}}\mathrm{v}} \leq 0. \tag{3.45}$$

The rest of this section is devoted to various refinements of Proposition 3.5. The first result in this direction is

Theorem 3.6 *Assume that $e \in \mathbf{R} \setminus \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$. Then,*
(i) *e is an eigenvalue of H iff 0 is an eigenvalue of $G_{\mathrm{v}}(e)$.*
(ii) $\dim \mathbf{1}_{\{e\}}(H) = \dim \mathbf{1}_{\{0\}}(G_{\mathrm{v}}(e))$.
(iii) *Set $p = \mathbf{1}_{\{0\}}(G_{\mathrm{v}}(e))$. Then $pW'_{\mathrm{v}}(e)p$ is a negative operator and*

$$\begin{aligned} \mathbf{1}_{\{e\}}(H) &= \left(p + (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}} p\right) \\ &\quad \times (p - pW'_{\mathrm{v}}(e)p)^{-1} \left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right). \end{aligned} \tag{3.46}$$

(iv) *If*

$$u := (p - pW'_{\mathrm{v}}(e)p)^{-\frac{1}{2}} \left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right),$$

then u is a partial isometry and

$$uu^* = p, \qquad u^* u = \mathbf{1}_{\{e\}}(H).$$

Proof. Let $H\psi = e\psi$. Then

$$\begin{aligned} H^{\mathrm{v}\mathrm{v}}\psi^{\mathrm{v}} + H^{\mathrm{v}\overline{\mathrm{v}}}\psi^{\overline{\mathrm{v}}} &= e\psi^{\mathrm{v}}, \\ H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}} + H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\psi^{\overline{\mathrm{v}}} &= e\psi^{\overline{\mathrm{v}}}. \end{aligned} \tag{3.47}$$

The second equation gives (recall that $e \notin \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$)

$$\psi^{\overline{\mathrm{v}}} = (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}}. \tag{3.48}$$

Inserting this identity into the first equation of (3.47) we get

$$G_{\mathrm{v}}(e)\psi^{\mathrm{v}} = 0. \tag{3.49}$$

Now, if $\psi^{\mathrm{v}} \in \mathcal{D}(H^{\mathrm{v}\mathrm{v}})$ satisfies (3.49) and $\psi^{\overline{\mathrm{v}}}$ is given in terms of ψ^{v} by (3.48), then $\psi = \psi^{\mathrm{v}} \oplus \psi^{\overline{\mathrm{v}}}$ satisfies $H\psi = e\psi$, which follows by a simple computation. Thus, the projection

$$\psi \mapsto \psi^{\mathrm{v}} \tag{3.50}$$

restricted to $\mathrm{Ran}\mathbf{1}_{\{e\}}(H)$ is a bijection onto $\mathrm{Ran}p$. This yields (i) and (ii).

Define $w : \mathcal{H}^{\mathrm{v}} \mapsto \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$ by setting

$$w := p + (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}} p.$$

Then w is the inverse of the map (3.50). Besides,

$$w^* w = p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2} H^{\overline{\mathrm{v}}\mathrm{v}} p$$

restricted to Ran$p$ is a positive, invertible operator. With a slight abuse of the notation we denote its inverse by $(w^*w)^{-1}$. One easily checks that $w(w^*w)^{-1}w^*$ is an orthogonal projection on $\mathrm{Ran}w = \mathrm{Ran}\mathbf{1}_{\{e\}}(H)$. Hence

$$\mathbf{1}_{\{e\}}(H) = w(w^*w)^{-1}w^*.$$

This shows (iii). Part (iv) follows from the identity $u = (w^*w)^{-\frac{1}{2}}w^*$. □

Due to the assumption $e \notin \sigma(H^{\overline{\mathrm{vv}}})$, the proof of Theorem 3.6 was relatively simple. Related results are subtler if $e \in \sigma(H^{\overline{\mathrm{vv}}})$. As a warm up, we prove

Proposition 3.7 *Let $e \in \mathbf{R}$. Suppose that the limit*

$$\lim_{y\downarrow 0} W_{\mathrm{v}}(e + \mathrm{i}y) =: W_{\mathrm{v}}(e + \mathrm{i}0) \tag{3.51}$$

exists. Then $W_{\mathrm{v}}(e + \mathrm{i}0)$ is dissipative.

Assume moreover that e is not an eigenvalue of $H^{\overline{\mathrm{vv}}}$. Then

$$\dim \mathrm{Ker}(H - e) \leq \dim \mathrm{Ker} G_{\mathrm{v}}(e + \mathrm{i}0). \tag{3.52}$$

In particular, if e is an eigenvalue of H, then 0 is an eigenvalue of $G_{\mathrm{v}}(e + \mathrm{i}0)$.

Proof. Assume that $\psi \in \mathcal{D}(H)$ and $H\psi = e\psi$. Since e is not an eigenvalue of $H^{\overline{\mathrm{vv}}}$, $e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}}$ is injective and

$$\psi^{\overline{\mathrm{v}}} = (e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}}. \tag{3.53}$$

Moreover,

$$0 = \mathbf{1}_{\{e\}}(H^{\overline{\mathrm{vv}}}) = -\mathrm{s} - \lim_{y\downarrow 0} \mathrm{i}y((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}. \tag{3.54}$$

Therefore,

$$\begin{aligned} &\mathrm{s} - \lim_{y\downarrow 0} \left(((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} - (e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\right) H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}} \\ &= -\mathrm{s} - \lim_{y\downarrow 0} \mathrm{i}y((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}(e\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}} \quad = 0. \end{aligned} \tag{3.55}$$

Combining (3.53) and (3.55) we get

$$\psi^{\overline{\mathrm{v}}} = \mathrm{s} - \lim_{y\downarrow 0}((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}}\psi^{\mathrm{v}}.$$

Substituting this identity into the first equation in (3.47) we derive that

$$G_{\mathrm{v}}(e + \mathrm{i}0)\psi^{\mathrm{v}} = 0.$$

Now assume that $\psi^{\mathrm{v}} = 0$. Then $\psi \in \mathcal{H}^{\overline{\mathrm{v}}}$ and therefore

$$e\psi = H\psi = H^{\overline{\mathrm{vv}}}\psi.$$

Since $e \notin \sigma_{\mathrm{pp}}(H^{\overline{\mathrm{vv}}})$, $\psi = 0$. Thus, the projection $\psi \mapsto \psi^{\mathrm{v}}$ restricted to $\mathrm{Ran}\mathbf{1}_{\{e\}}(H)$ is an injective map into $\mathrm{Ker} G_{\mathrm{v}}(x + \mathrm{i}0)$. □

The following theorem is the principal result of this section.

Theorem 3.8 *Assume that e is not an eigenvalue of $H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$, that the limit (3.51) exists, and that the function*

$$\mathbf{C}_+ \cup \{e\} \ni z \mapsto W_{\mathrm{v}}(z)$$

is in the class $C^1(\mathbf{C}_+ \cup \{e\})$ (its derivative at $z = e$ we denote by $W'_{\mathrm{v}}(e + \mathrm{i}0)$). Further, assume that $0 \in \sigma_{\mathrm{disc}}(G_{\mathrm{v}}(e + \mathrm{i}0))$. Then the following holds:
(i) *0 is an eigenvalue of H.*
(ii) $\dim \mathbf{1}_{\{e\}}(H) = \dim \mathrm{Ker} G_{\mathrm{v}}(e + \mathrm{i}0)$.
(iii) *Set $p = \mathbf{1}_{\{0\}}(G_{\mathrm{v}}(e + \mathrm{i}0))$. Then, p is the orthogonal projection onto* $\mathrm{Ker}(G_{\mathrm{v}}(e + \mathrm{i}0)$, *the operator*

$$pW_{\mathrm{v}}(e + \mathrm{i}0)p =: pW_{\mathrm{v}}(e)p$$

is self-adjoint, and the operator

$$pW'_{\mathrm{v}}(e + \mathrm{i}0)p =: pW'_{\mathrm{v}}(e)p$$

is negative.
(iv) *The limits*

$$\begin{aligned} \lim_{y\to 0}((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p &=: (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p \\ \lim_{y\to 0} pH^{\mathrm{v}\overline{\mathrm{v}}}((e + \mathrm{i}y)\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} &=: pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}, \end{aligned} \tag{3.56}$$

exist and

$$\begin{aligned} \mathbf{1}_{\{e\}}(H) &= \left(p + (e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}p\right) \\ &\quad \times (p - pW'_{\mathrm{v}}(e)p)^{-1}\left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right). \end{aligned} \tag{3.57}$$

(v) *If*

$$u := (p - pW'_{\mathrm{v}}(e)p)^{-\frac{1}{2}}\left(p + pH^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\right),$$

then u is a partial isometry and

$$uu^* = p, \qquad u^*u = \mathbf{1}_{\{e\}}(H).$$

Proof. We begin with proofs of Parts (iii) and (iv).

We first observe that the relation $W^*_{\mathrm{v}}(z) = W_{\mathrm{v}}(\overline{z})$ yields that the functions $W_{\mathrm{v}}(z)$ and $G_{\mathrm{v}}(z)$ are also of the class $C^1(\mathbf{C}_- \cup \{e\})$.

Since the operator $-G_{\mathrm{v}}(e+\mathrm{i}0)$ is dissipative and $0 \in \sigma_{\mathrm{disc}}(G_{\mathrm{v}}(e+\mathrm{i}0))$, Proposition 3.2 yields that p is an orthogonal projection. It follows that

$$\mathbf{1}_{\{0\}}(G^*_{\mathrm{v}}(e + \mathrm{i}0)) = p.$$

Since $G_{\rm v}^*(e+{\rm i}0)=G_{\rm v}(e-{\rm i}0)$, the identities $pG_{\rm v}(e\pm{\rm i}0)p=0$ can be rewritten as

$$p(e\mathbf{1}^{\rm vv}-H^{\rm vv})p=pW_{\rm v}(e\pm{\rm i}0)p.$$

This yields the relation

$$pW_{\rm v}(e+{\rm i}0)p=pW_{\rm v}(e-{\rm i}0)p=:pW_{\rm v}(e)p. \tag{3.58}$$

Clearly, the operator $pW_{\rm v}(e)p$ is self-adjoint.

Let $\tau\mapsto\gamma(\tau)\in\mathbf{C}_+\cup\{e\}$, $\gamma(0)=e$, be a smooth curve such that $\gamma(0)=e$ and γ is tangent to $\mathbf{R}$ at $\tau=0$. We may assume that $\gamma'(0)=1$. Then,

$$\tau\mapsto{\rm Im}pW_{\rm v}(\gamma(\tau))p,$$

is a function with values in dissipative operators such that

$${\rm Im}pW_{\rm v}(\gamma(0))p={\rm Im}pW_{\rm v}(e)p=0.$$

It follows that

$$0=\frac{\rm d}{{\rm d}\tau}{\rm Im}pW_{\rm v}(\gamma(\tau))p|_{\tau=0}={\rm Im}pW_{\rm v}'(e+{\rm i}0)p.$$

This shows that

$$pW_{\rm v}'(e+{\rm i}0)p=:pW_{\rm v}'(e)p \tag{3.59}$$

is a self-adjoint operator. This proves (iii) except for the part asserting that $pW_{\rm v}'(e)p$ is a negative operator. Using $p=p^*$, $W_{\rm v}'(e-{\rm i}0)=W_{\rm v}'(e+{\rm i}0)^*$ and (3.59), we also have that

$$pW_{\rm v}'(e-{\rm i}0)p=pW_{\rm v}'(e+{\rm i}0)p. \tag{3.60}$$

We now show that the limit

$$(e\mathbf{1}^{\overline{\rm vv}}-H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p:=\lim_{y\to0,y\neq0}((e+{\rm i}y)\mathbf{1}^{\overline{\rm vv}}-H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p, \tag{3.61}$$

exists. Denote the expression inside the limit by $L(y)$. Then, the resolvent identity yields

$$\begin{aligned}(L^*(y_1)-L^*(y_2))(L(y_1)-L(y_2)) &= \tfrac{1}{2{\rm i}y_1}\left(pW_{\rm v}(e-{\rm i}y_1)p-pW_{\rm v}(e+{\rm i}y_1)p\right)\\ &+\tfrac{1}{2{\rm i}y_2}\left(pW_{\rm v}(e-{\rm i}y_2)p-pW_{\rm v}(e+{\rm i}y_2)p\right)\\ &-\tfrac{1}{{\rm i}(y_2+y_1)}\left(pW_{\rm v}(e-{\rm i}y_1)p-pW_{\rm v}(e+{\rm i}y_2)p\right)\\ &-\tfrac{1}{{\rm i}(y_1+y_2)}\left(pW_{\rm v}(e-{\rm i}y_2)p-pW_{\rm v}(e+{\rm i}y_1)p\right).\end{aligned} \tag{3.62}$$

Note that it follows from (3.58) and (3.60) that the function

$$\mathbf{C}_+\cup\mathbf{C}_-\cup\{e\}\ni z\mapsto pW_{\rm v}(z)p, \tag{3.63}$$

is continuously differentiable. This observation and identity (3.62) yield that the sequence $L(y_n)$ is Cauchy whenever $y_n \to 0$. Thus, the limit (3.61) exists and this implies the existence of the limits (3.56) (for the second limit take the adjoint of (3.61)). Since

$$pW'_{\rm v}(e)p = -((e\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p)^*(e\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p,$$

it follows that the operator $pW'_{\rm v}(e)p$ is negative. This completes the proof of Part (iii).

Set $w := p + (e\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}$ and $w(z) := p + (z\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}$. By (3.56) we have

$$\lim_{y\downarrow 0} w(e + {\rm i}y) = w. \tag{3.64}$$

We easily compute that

$$Hw(z) = zw(z) - G_{\rm v}(z)p.$$

Hence

$$\lim_{y\downarrow 0} Hw(e + {\rm i}y) = ew. \tag{3.65}$$

(3.64) and (3.65) imply that ${\rm Ran}w \subset {\rm Ker}(H - e)$. Thus w maps ${\rm Ker}G_{\rm v}(e + {\rm i}0)$ into ${\rm Ker}(H - e)$. Clearly, the inverse of w is $\mathbf{1}^{\rm vv}$ restricted to ${\rm Ker}(H - e)$. Therefore, w is a bijective map from ${\rm Ker}G_{\rm v}(e + {\rm i}0)$ to ${\rm Ker}(H - e)$. Similarly as at the end of the proof of Theorem 3.6, we note that

$$\begin{aligned} w^*w &= p + ((e\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p)^*(e\mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1}H^{\overline{\rm v}{\rm v}}p \\ &= p - pW'_{\rm v}(e)p, \end{aligned}$$

and

$$\mathbf{1}_{\{e\}}(H) = w(w^*w)^{-1}w^*.$$

This proves Relation (3.57) and completes the proof of Part (iv). Part (v) follows from the identity

$$u = (w^*w)^{-\frac{1}{2}}w^*.$$

□

3.3 Counting the eigenvalues

Let $[e_-, e_+] \ni x \mapsto G(x)$ be a function with values in bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that the function G is strictly increasing if the following holds: If $x > y$, then there is $\epsilon > 0$ such that, for all $\psi \in \mathcal{H}$,

$$(\psi|G(x)\psi) > (\psi|G(y)\psi) + \epsilon\|\psi\|^2.$$

Proposition 3.9 *Let $[e_-, e_+] \ni x \mapsto G(x) \in \mathcal{B}(\mathcal{H})$ be a function such that:*
(a) *For all $x \in [e_-, e_+]$, $G(x)$ is self-adjoint.*
(b) *$[e_-, e_+] \ni x \mapsto G(x)$ is continuous and strictly increasing.*
(c) $\dim \mathbf{1}_{[0,\infty[}(G(e_+)) < \infty$.
Then:
(i) The set $\{x \in [e_-, e_+] \ : \ 0 \in \sigma(G(x))\}$ is a finite subset of $[e_-, e_+]$ and, for $x \in [e_-, e_+]$, we have $0 \in \sigma(G(x))$ iff $0 \in \sigma_{\rm disc}(G(x))$.
(ii)

$$\sum_{x \in [e_-, e_+]} \dim \mathrm{Ker}\, G(x) = \dim \mathbf{1}_{[0,\infty[}(G(e_+)) - \dim \mathbf{1}_{]0,\infty[}(G(e_-)).$$

Proof. We will use the Courant–Weyl (min-max) principle ([RS4], Theorem XIII.1, [We]). Let

$$f_n(x) := \inf_{\psi_1, \ldots, \psi_{n-1}} \sup_{\substack{\|\psi\|=1 \\ \psi \in \{\psi_1, \ldots, \psi_{n-1}\}^\perp}} (\psi | G(x) \psi).$$

We set $\Sigma(x) := -\infty$ if $\mathcal{H}$ is finite dimensional, $\Sigma(x) := \inf_n f_n(x)$ otherwise. It follows from the min-max principle that $\Sigma(x) = \sup \sigma_{\rm ess}(G(x))$ (note that $\sup \emptyset = -\infty$). Furthermore, $f_n(x)$ is a non-increasing sequence such that if $\Sigma(x) < f_n(x)$ then $f_n(x)$ is the n-th eigenvalue of $G(x)$ (in the non-increasing order) counted with multiplicites.

Clearly, since $G(x)$ is a strictly increasing continuous function, the functions $f_n(x)$ are also strictly increasing and continuous in x. In particular, it follows that $\Sigma(x)$ is an increasing function. By (c), $\Sigma(e_+) < 0$, hence $\Sigma(x) < 0$ for all $x \in [e_-, e_+]$. Therefore,

$$\dim \mathbf{1}_{[0,\infty[}(G(x)) = \#\{n \ : \ f_n(x) \geq 0\},$$
$$\dim \mathrm{Ker}\, G(x) = \#\{n \ : \ f_n(x) = 0\},$$

and the result follows from the Darboux principle. □

The following two theorems follow by combining the last proposition with Theorems 3.6 and 3.8. We consider a self-adjoint operator H which has the same form as in Section 3.2. We will assume in addition that $H^{\rm vv}$ is a bounded operator. The first theorem describes how to count eigenvalues outside $\sigma(H^{\overline{\rm vv}})$.

Theorem 3.10 *Assume that $[e_-, e_+] \cap \sigma(H^{\overline{\rm vv}}) = \emptyset$ and that $\dim \mathbf{1}_{[0,\infty[}(G_{\rm v}(e_+)) < \infty$. Then,*

$$\dim \mathbf{1}^{\rm pp}_{[e_-, e_+]}(H) = \dim \mathbf{1}_{[e_-, e_+]}(H) = \dim \mathbf{1}_{[0,\infty[}(G_{\rm v}(e_+)) - \dim \mathbf{1}_{]0,\infty[}(G_{\rm v}(e_-)).$$

Proof. We know by Proposition 3.5 (i) that

$$[e_-, e_+] \cap \sigma(H) \subset \{x \in [e_-, e_+] \ : \ 0 \in \sigma(G_{\rm v}(x))\}. \tag{3.66}$$

Since $G'_{\rm v}(x) \geq \mathbf{1}^{\rm vv}$ we see that $G_{\rm v}(x)$ is strictly increasing. Thus we easily see that the function $G(x)$ satisfies the assumptions of Proposition 3.9, which implies that the set (3.66) is finite. Hence

$$\mathbf{1}_{[e_-,e_+]}(H) = \mathbf{1}^{\rm pp}_{[e_-,e_+]}(H) = \sum_{x\in[e_-,e_+]} \mathbf{1}_{\{x\}}(H).$$

It follows from Theorem 3.6 that for all $x \in [e_-, e_+]$,

$$\dim \mathbf{1}_{\{x\}}(H) = \dim \mathrm{Ker}\, G_{\rm v}(x),$$

and the result follows from Proposition 3.9. □

Corollary 3.11 *Assume that* $[e_-, e_+] \cap \sigma(H^{\overline{\rm vv}}) = \emptyset$. *Then*

$$\dim \mathbf{1}_{[e_-,e_+]}(H) \leq \dim \mathcal{H}^{\rm v}. \tag{3.67}$$

If the self-energy is differentiable, one can also count eigenvalues of H inside $\sigma(H^{\overline{\rm vv}})$.

Theorem 3.12 *Assume that* $\sigma_{\rm pp}(H^{\overline{\rm vv}}) \cap [e_-, e_+] = \emptyset$, *and that the following holds:*
(a) *The limit*

$$W_{\rm v}(x + {\rm i}0) := \lim_{y\downarrow 0} W_{\rm v}(x + {\rm i}y) \tag{3.68}$$

exists for all $x \in [e_-, e_+]$.
(b) *The function* $\mathrm{Re}W_{\rm v}(x + {\rm i}0)$ *is differentiable and for some* $\epsilon > 0$ *and all* $x \in [e_-, e_+]$ *it satisfies*

$$\mathrm{Re}G'_{\rm v}(x + {\rm i}0) = \mathbf{1}^{\rm vv} - \mathrm{Re}W'_{\rm v}(x) \geq \epsilon \mathbf{1}^{\rm vv}.$$

(c) $\dim \mathbf{1}_{[0,\infty[}(\mathrm{Re}\, G_{\rm v}(e_+ + {\rm i}0)) < \infty$.
Then,

$$\dim \mathbf{1}^{\rm pp}_{[e_-,e_+]}(H) \leq \dim \mathbf{1}_{[0,\infty[}(\mathrm{Re}G_{\rm v}(e_+ + {\rm i}0)) - \dim \mathbf{1}_{]0,\infty[}(\mathrm{Re}G_{\rm v}(e_- + {\rm i}0)).$$

Proof. Assumptions (a) and Theorem 3.8, and then Proposition 3.2, yield that for $x \in [e_-, e_+]$,

$$\dim \mathbf{1}_{\{x\}}(H) \leq \dim \mathrm{Ker}\, G_{\rm v}(x + {\rm i}0) \leq \dim \mathrm{Ker}\mathrm{Re}G_{\rm v}(x + {\rm i}0). \tag{3.69}$$

Assumption (b) yields that the function $[e_-, e_+] \ni x \mapsto \mathrm{Re}G_{\rm v}(x + {\rm i}0)$ is continuous and strictly increasing. Using also (c) we see that the function $\mathrm{Re}G_{\rm v}(x + {\rm i}0)$ satisfies the conditions of Proposition 3.9. This proposition and Relation (3.69) yield the statement of the theorem. □

Corollary 3.13 *Assume that* $\sigma_{\rm pp}(H^{\overline{\rm vv}}) \cap [e_-, e_+] = \emptyset$*, and that the following holds:*
(a) *The limit*

$$W_{\rm v}(x + {\rm i}0) := \lim_{y \downarrow 0} W_{\rm v}(x + {\rm i}y), \tag{3.70}$$

exists for all $x \in [e_-, e_+]$.
(b) *The function* $[e_-, e_+] \ni x \mapsto {\rm Re} W_{\rm v}(x + {\rm i}0)$ *is differentiable, and for some* $\epsilon > 0$ *and all* $x \in [e_-, e_+]$,

$${\rm Re} G'_{\rm v}(x + {\rm i}0) = \mathbf{1}^{\rm vv} - {\rm Re} W'_{\rm v}(x + {\rm i}0) \geq \epsilon \mathbf{1}^{\rm vv}.$$

Then

$$\dim \mathbf{1}^{\rm pp}_{[e_-, e_+]}(H) \leq \dim \mathcal{H}^{\rm v}. \tag{3.71}$$

4 Fock spaces and all that

In this section we describe in an abstract setting some Hilbert spaces and operators of the quantum field theory. We have attempted to give an essentially self-contained presentation of the topics we will need. For additional information, the reader may consult [RS2], [BSZ], [BR], [GJ], [DG], [De].

Let $\mathfrak{h}$ be a Hilbert space. We set $\mathfrak{h}^{0\otimes} := \mathbf{C}$ and $\mathfrak{h}^{n\otimes} := \mathfrak{h} \otimes \ldots \otimes \mathfrak{h}$. If A is a closed operator on $\mathfrak{h}$, we denote by $A^{\otimes n}$ the closed operator on $\mathfrak{h}^{n\otimes}$ defined by $A \otimes \ldots \otimes A$ (if $n = 0$, $A^{\otimes 0} = 1$). Let S_n be the group of permutations of n elements. For each $\sigma \in S_n$ we define an operator (which we also denote by σ) on the basis elements of $\mathfrak{h}^{n\otimes}$ by

$$\sigma(f_{i_1} \otimes \ldots \otimes f_{i_n}) = f_{\sigma(i_1)} \otimes \ldots \otimes f_{\sigma(i_n)},$$

where $\{f_k\}$ is a basis of $\mathfrak{h}$. σ extends by linearity to a unitary operator on $\mathfrak{h}^{\otimes n}$, which does not depend on a basis. We set

$$P_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.$$

The operator P_n is an orthogonal projection. Let

$$\Gamma_n(\mathfrak{h}) := {\rm Ran} P_n.$$

This Hilbert space is commonly called the n-particle bosonic space.

The symmetric (or boson) Fock space over $\mathfrak{h}$ is defined by

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Gamma_n(\mathfrak{h}).$$

The vector $\Omega = (1, 0, 0, \ldots)$ plays a special role and is called *the vacuum*. A vector $\Psi = (\psi_0, \psi_1, \ldots)$ is called a finite particle vector if $\psi_n = 0$ for all but finitely many n. The set of all finite particle vectors we denote by $\Gamma_{\rm fin}(\mathfrak{h})$.

If A is an operator on $\mathfrak{h}$, we define $\Gamma(A)$ to be the operator which is equal to $A^{\otimes n}$ on $\Gamma_n(\mathfrak{h})$. If ω is a self-adjoint operator on $\mathcal{H}$ then $\Gamma(\exp(\mathrm{i}t\omega))$ is a strongly continuous unitary group on $\Gamma(\mathfrak{h})$, and we denote its generator by $\mathrm{d}\Gamma(\omega)$. Note that

$$\Gamma(\exp(\mathrm{i}t\omega)) = \exp(\mathrm{i}t\mathrm{d}\Gamma(\omega)),$$

and $\mathrm{d}\Gamma(\omega)\Omega = 0$. $\mathrm{d}\Gamma(\omega)$ preserves the n-particle subspaces, and on $\mathcal{D}(\mathrm{d}\Gamma(\omega))) \cap \Gamma_n(\mathfrak{h})$ it acts as

$$\omega \otimes \mathbf{1} \ldots \otimes \mathbf{1} + \mathbf{1} \otimes \omega \ldots \otimes \mathbf{1} + \ldots \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \omega.$$

The number operator is defined by $N = \mathrm{d}\Gamma(\mathbf{1})$.

In the models we will study, the boson Fock space will represent only a part of the system, usually referred to as a "radiation field" or a "heat bath". The other part is an "atom" or a "small system", to which we associate a Hilbert space $\mathcal{K}$. The interaction of these two sub-systems is described by a suitable self-adjoint operator on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$. To describe these interaction operators in a sufficient generality, it is convenient to extend the notion of the usual creation and annihilation operators on the Fock space. For the conventional definitions of these operators we refer the reader to [RS2], Section X.7. The definitions we will use are also discussed in [DG].

Notation. In the rest of the paper, whenever the meaning is clear within the context, we will write A for the operators of the form $\mathbf{1} \otimes A$ and $A \otimes \mathbf{1}$.

Let

$$v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}).$$

Such operators will be called form-factors. We define a linear operator $b(v)$ on $\mathcal{K} \otimes (\oplus_{n=0}^{\infty} \mathfrak{h}^{n\otimes})$ as follows:

$$b(v) : \mathcal{K} \otimes \mathfrak{h}^{0\otimes} \mapsto 0,$$

$$b(v) : \mathcal{K} \otimes \mathfrak{h}^{n\otimes} \mapsto \mathcal{K} \otimes \mathfrak{h}^{(n-1)\otimes},$$

$$b(v)(\psi \otimes \phi_1 \otimes \ldots \otimes \phi_n) := (v^*\psi \otimes \phi_1) \otimes \phi_2 \otimes \ldots \otimes \phi_n.$$

It is not difficult to show that $b(v)$ is a bounded operator which takes $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ into itself. Note that $\|b(v)\| = \|v\|_{\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})}$.

We define the annihilation operator $a(v)$ on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ with domain $\mathcal{K} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ by

$$a(v) = (N+1)^{\frac{1}{2}} b(v).$$

The operator $a(v)$ is closable, and we denote its closure with the same letter.

The adjoint of $a(v)$ we denote by $a^*(v)$ and call it the creation operator. To describe this operator, note that on $\mathcal{K} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h})$ we have

$$a^*(v) = Pb^*(v)(N+1)^{\frac{1}{2}}, \tag{4.72}$$

where $P = \sum_{n=0}^{\infty} P_n$. Moreover, $b^*(v)$ acts on $\mathcal{K} \otimes (\oplus_{n=0}^{\infty} \mathfrak{h}^{n\otimes})$ as follows:

$$b^*(v) : \mathcal{K} \otimes \mathfrak{h}^{n\otimes} \mapsto \mathcal{K} \otimes \mathfrak{h}^{(n+1)\otimes},$$

$$b^*(v)(\psi \otimes \phi_1 \otimes \dots \phi_n) = (v\psi) \otimes \phi_1 \otimes \dots \otimes \phi_n.$$

Here $\psi \in \mathcal{K}$, $\phi_k \in \mathfrak{h}$.

We remark that the map $v \mapsto a(v)$ is anti-linear while the map $v \mapsto a^*(v)$ is linear. In the sequel $a^{\#}(v)$ stands either for $a(v)$ or $a^*(v)$.

The (Segal) field operator is defined by

$$\varphi(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v)).$$

The operator $\varphi(v)$ is essentially self-adjoint on $\mathcal{K} \otimes \mathcal{D}(N^{\frac{1}{2}})$. The following two elementary estimates will be often used in the rest of the paper:

$$\|(N+1)^{-\frac{1}{2}} a^{\#}(v)\| \leq \|v\|, \qquad \|(N+1)^{-\frac{1}{2}} \varphi(v)\| \leq \sqrt{2}\|v\|. \tag{4.73}$$

Note that if v acts as $v\psi = (q\psi) \otimes f$, where f is a fixed vector in $\mathfrak{h}$ and $q \in \mathcal{B}(\mathcal{K})$, then

$$a^*(v) = q \otimes a^*(f), \qquad a(v) = q^* \otimes a(f).$$

Here $a^{\#}(f)$ are the usual creation and annihilation operators on $\Gamma(\mathfrak{h})$. Such form-factors we will call simple.

More generally, if $\{f_n\}$ is an orthogonal basis of $\mathfrak{h}$, for any $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, there are operators $v_n \in \mathcal{B}(\mathcal{K})$ such that

$$v\psi = \sum_n (v_n\psi) \otimes f_n, \qquad \|v\psi\|^2 = \sum_n \|v_n\psi\|^2, \quad \psi \in \mathcal{K}.$$

Thus, every form-factor can be decomposed into a sum of simple form-factors. One can alternatively use this fact to define the operators $a^{\#}(v)$, etc.

If U is a unitary operator on $\mathfrak{h}$, then

$$\Gamma(U)\varphi(v)\Gamma(U^{-1}) = \varphi(Uv). \tag{4.74}$$

¿From this relation it follows that if ω is a self-adjoint operator on $\mathfrak{h}$ such that $\omega v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, then

$$\mathrm{i}[\mathrm{d}\Gamma(\omega), \varphi(v)] = \varphi(\mathrm{i}\omega v). \tag{4.75}$$

We now describe several technical results which will be used in the sequel.

Proposition 4.1 *Let $v \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ and ω an operator on $\mathfrak{h}$. Assume that $\omega \geq 0$ and that ω is invertible on the range of v. Set $K_0 := v^*v$. Then the following estimates hold for any Ψ in the quadratic form domain of $\mathrm{d}\Gamma(\omega)$:*

$$\begin{aligned} \|a(v)\Psi\|^2 &\leq \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi), \\ \|a^*(v)\Psi\|^2 &\leq (\Psi|K_0\Psi) + \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi), \\ \|\varphi(v)\Psi\|^2 &\leq (\Psi|K_0\Psi) + 2\|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi). \end{aligned} \tag{4.76}$$

Remark. Results of this genre go back to the N_τ-estimates of Glimm and Jaffe [GJ], see also [Ar], [BFS1].

Proof. Before we start, we remark that $v^*\omega^{-1}v \in \mathcal{K}$ iff $v^*\omega^{-\frac{1}{2}} \in \mathcal{B}(\mathcal{K} \otimes \mathfrak{h}, \mathfrak{h})$, and that

$$\|v^*\omega^{-1}v\| = \|v^*\omega^{-\frac{1}{2}}\|^2 = \|\omega^{-\frac{1}{2}}v\|^2.$$

Let $\mathcal{Q}$ denote the quadratic form domain of $\mathrm{d}\Gamma(\omega)$. Since

$$[N, a^*(v)a(v)] = [N, a(v)a^*(v)] = 0,$$

it suffices to establish the first two relations for

$$\Psi \in \mathcal{Q} \cap (\mathcal{K} \otimes \Gamma_n(\mathfrak{h})). \tag{4.77}$$

Note also that if $\Psi_i \in \mathcal{K} \otimes \Gamma_{n_i}(\mathfrak{h})$, $i = 1, 2$, then

$$(\Psi_2|a^*(v)\Psi_1) = \begin{cases} 0 & \text{if } n_2 - n_1 \neq 1, \\ \sqrt{n_1 + 1}(\Psi_2|v \otimes \mathbf{1}^{\otimes n_1}\Psi_1) & \text{if } n_2 - n_1 = 1. \end{cases} \tag{4.78}$$

Using Relation (4.78) twice, we derive

$$\begin{aligned} (\Psi|a^*(v)a(v)\Psi) &= n(\Psi|vv^* \otimes \mathbf{1}^{\otimes(n-1)}\Psi) \\ &= n(\mathbf{1}_\mathcal{K} \otimes \omega^{\frac{1}{2}} \otimes \mathbf{1}^{\otimes(n-1)}\Psi|(v^*(\mathbf{1}_\mathcal{K} \otimes \omega^{-\frac{1}{2}}))^* \\ &\qquad \times (v^*(\mathbf{1}_\mathcal{K} \otimes \omega^{-\frac{1}{2}}))(\mathbf{1}_\mathcal{K} \otimes \omega^{\frac{1}{2}} \otimes \mathbf{1}^{\otimes(n-1)}\Psi) \\ &\leq \|v^*(\mathbf{1}_\mathcal{K} \otimes \omega^{-\frac{1}{2}})\|^2\, n(\Psi|\mathbf{1}_\mathcal{K} \otimes \omega \otimes \mathbf{1}^{\otimes(n-1)}\Psi) \\ &= \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi). \end{aligned}$$

This proves the first relation in (4.76). Before we prove the second, it is convenient to introduce some additional notation.

For $i = 1, \ldots, n$, let $\tau_i^{(n)}$ denote the transposition of 1 and i. Recall that $\tau_i^{(n)}$ defines a unitary operator (which we denote by the same letter) on $\mathfrak{h}^{\otimes n}$. Clearly, $(\tau_i^{(n)})^2 = \mathbf{1}$. For any $h \in \mathcal{B}(\mathfrak{h})$ we define

$$h_i^{(n)} := \mathbf{1}^{\otimes(i-1)} \otimes h \otimes \mathbf{1}^{\otimes(n-i)} = \tau_i^{(n)}(h \otimes \mathbf{1}^{\otimes(n-1)})(\tau_i^{(n)})^{-1}.$$

Assume that Ψ satisfies (4.77). Then, using (4.72) and (4.78), we derive that

$$\begin{aligned}(a^*(v)\Psi|a^*(v)\Psi) &= (n+1)(\Psi|(v^*\otimes \mathbf{1}^{\otimes n})P_{n+1}(v\otimes \mathbf{1}^{\otimes n})\Psi)\\ &= \textstyle\sum_{i=1}^{n+1}(\Psi|(v^*\otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(v\otimes \mathbf{1}^{\otimes n})\Psi) \qquad (4.79)\\ &= (\Psi|K_0\Psi) + \textstyle\sum_{i=2}^{n}(\Psi|(v^*\otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(v\otimes \mathbf{1}^{\otimes n})\Psi).\end{aligned}$$

Note that for $i>1$,

$$\begin{aligned}&(\Psi|(v^*\otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(v\otimes \mathbf{1}^{\otimes n})\Psi) =\\ &= ((\omega^{\frac12})_{i-1}^{(n)}\Psi|(v^*\otimes \mathbf{1}^{\otimes n})(\omega^{-\frac12})_i^{(n+1)}(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(\omega^{-\frac12})_i^{(n+1)})(v\otimes \mathbf{1}^{\otimes n})(\omega^{\frac12})_{i-1}^{(n)}\Psi)\\ &= ((\omega^{\frac12})_{i-1}^{(n)}\Psi|(v^*(\mathbf{1}_{\mathcal{K}}\otimes \omega^{-\frac12})\otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(\mathbf{1}_{\mathcal{K}}\otimes \omega^{-\frac12})v\otimes \mathbf{1}^{\otimes n})(\omega^{\frac12})_{i-1}^{(n)}\Psi)\\ &\le \|v^*(\mathbf{1}_{\mathcal{K}}\otimes \omega^{-\frac12})\|\,\|(\mathbf{1}_{\mathcal{K}}\otimes \omega^{-\frac12})v\|\,\|(\omega^{\frac12})_{i-1}^{(n)}\Psi\|^2\\ &= \|v^*\omega^{-1}v\|(\Psi|\omega\otimes \mathbf{1}^{\otimes(n-1)}\Psi).\end{aligned}$$

Thus,

$$\begin{aligned}\textstyle\sum_{i=2}^{n+1}(\Psi|(v^*\otimes \mathbf{1}^{\otimes n})(\mathbf{1}_{\mathcal{K}}\otimes \tau_i^{(n+1)})(v\otimes \mathbf{1}^{\otimes n})\Psi) &\le n\|v^*\omega^{-1}v\|(\Psi|\omega\otimes \mathbf{1}^{\otimes(n-1)}\Psi)\\ &= \|v^*\omega^{-1}v\|(\Psi|\mathrm{d}\Gamma(\omega)\Psi).\end{aligned}$$

Combining this inequality with the identity (4.79), we derive the second relation in (4.76).

Finally, the third relation follows from the first two and the simple estimate

$$\|\varphi(v)\Psi\|^2 \le \|a(v)\Psi\|^2 + \|a^*(v)\Psi\|^2.$$

□

We will also make use of the following estimate.

Lemma 4.2 *Let $v\in B(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})$, $1\ge\delta\ge 0$ and $N_\delta := 1+\delta N$. Then, for any β,*

$$\|(\varphi(v) - N_\delta^{-\beta}\varphi(v)N_\delta^{\beta})\| \le C_\beta\sqrt{\delta}\|v\|.$$

Proof. It suffices to consider the case $\beta\ge 0$. For $\Psi\in\mathcal{K}\otimes\Gamma_n(\mathfrak{h})$ we have

$$\left(a^*(v) - N_\delta^{-\beta}a^*(v)N_\delta^{\beta}\right)\Psi = \sqrt{n+1}\left(1-(\tfrac{1+\delta n}{1+\delta(n+1)})^{\beta}\right)P_{n+1}v\otimes \mathbf{1}^{\otimes n}\Psi.$$

For $0<x\le 1$ we have $|1-(1-x)^\beta|\le C_\beta x$. Hence

$$\begin{aligned}\left\|a^*(v) - N_\delta^{-\beta}a^*(v)N_\delta^{\beta}\right\| &\le \sup_{n\ge0}\sqrt{n+1}\left|\left(1-(\tfrac{1+\delta n}{1+\delta(n+1)})^{\beta}\right)\right|\|v\|\\ &\le \sup_{n\ge0} C_\beta\sqrt{n+1}\tfrac{\delta}{1+\delta(n+1)}\|v\| \qquad (4.80)\\ &\le C_\beta\sqrt{\delta}\|v\|.\end{aligned}$$

After taking adjoints, (4.80) yields

$$\left\|a(v) - N_\delta^{-\beta} a(v) N_\delta^\beta\right\| \leq C_\beta \sqrt{\delta} \|v\|.$$

Clearly, the above two estimates yield the statement. □

The final results which we need is the exponential law for bosonic systems, see e.g. [BSZ], Section 3.2.

Theorem 4.3 *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be Hilbert spaces. There exist a unitary mapping*

$$U : \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \mapsto \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2),$$

with the following properties:
(i) *If A_1 and A_2 are operators on $\mathfrak{h}_1$ and $\mathfrak{h}_2$ then*

$$U(\Gamma(A_1) \otimes \Gamma(A_2))U^{-1} = \Gamma(A_1 \oplus A_2).$$

(ii) *If Ω denotes the vacuum on $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ and Ω_1, Ω_2 the vacua on $\Gamma(\mathfrak{h}_1)$, $\Gamma(\mathfrak{h}_2)$, then*

$$U(\Omega_1 \otimes \Omega_2) = \Omega.$$

(iii) *If $f_1 \in \mathfrak{h}_1$, $f_2 \in \mathfrak{h}_2$, then*

$$U \exp(\mathrm{i}\varphi(f_1) \otimes \exp(\mathrm{i}\varphi(f_2))U^{-1} = \exp(\mathrm{i}\varphi(f_1 \oplus f_2)).$$

(iv) *Let $\mathcal{K}$ be a Hilbert space. Assume that $v_1 \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}_1)$, $f_2 \in \mathfrak{h}_2$ and $v_2 = \mathbf{1}_\mathcal{K} \otimes f_2$. Then $v_1 \oplus v_2$ can be viewed as an element of $B(\mathcal{K}, \mathcal{K} \otimes (\mathfrak{h}_1 \oplus \mathfrak{h}_2))$ and*

$$(\mathbf{1}_\mathcal{K} \otimes U) \exp(\mathrm{i}\varphi(v_1)) \otimes \exp(\mathrm{i}\varphi(v_2))(\mathbf{1}_\mathcal{K} \otimes U)^{-1} = \exp(\mathrm{i}\varphi(v_1 \oplus v_2)).$$

Remark. The properties (ii), (iii) specify U uniquely.

5 Pauli-Fierz Hamiltonians

In this section we define Hamiltonians which we will study.

In quantum physics such Hamiltonians are used to describe systems which consist of two parts: the "small system" $\mathcal{A}$ and the "radiation field" $\mathcal{R}$. The system $\mathcal{A}$ is described by a Hilbert space $\mathcal{K}$ and a self-adjoint operator K on $\mathcal{K}$. The "radiation field" $\mathcal{R}$ is described by a bosonic Fock space. Its 1-particle space is $\mathfrak{h} := L^2(\mathbf{R}) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is an auxiliary Hilbert space. We denote by ω the operator of multiplication by $\omega \in \mathbf{R}$. The Hilbert space of the combined system is $\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$ and its free Hamiltonian is

$$H_{\mathrm{fr}} := K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega).$$

Let $\alpha \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ be a given form factor. The Hamiltonian of the coupled system is formally given by

$$H := H_{\rm fr} + \lambda\varphi(\alpha). \tag{5.81}$$

We make the following hypothesis:

Hypothesis A. H is essentially self-adjoint on $\mathcal{D} := \mathcal{D}(H_{\rm fr}) \cap \mathcal{D}(\varphi(\alpha))$.

If we equip $\mathcal{D}$ with the norm

$$\|\Phi\|_{\mathcal{D}} := \|\Phi\| + \|H_{\rm fr}\Phi\| + \|\varphi(\alpha)\Phi\|,$$

then $\mathcal{D}$ becomes a Banach space. It follows from Hypothesis A and an easy abstract argument (the same that is needed to show Lemma 2.5) that any vector space dense in $\mathcal{D}$ is a core for H. Thus, for instance, $\mathcal{D}_{\rm fin} := \mathcal{D}(K) \otimes (\Gamma_{\rm fin}(\mathfrak{h}) \cap \mathcal{D}(\mathrm{d}\Gamma(|\omega|)))$ is a core of H.

Below we give two explicit conditions that imply Hypothesis A.

5.1 "Positive-temperature systems"

Proposition 5.1 *Assume that the operators $|\omega|\alpha$ and $(|K| \otimes \mathbf{1}_{\mathfrak{h}})\alpha - \alpha|K|$ are bounded. Let*

$$\hat{N} := |K| + \mathrm{d}\Gamma(|\omega| + 1).$$

Then, for any $\lambda \in \mathbf{R}$, H is essentially self-adjoint on any core of $\hat{N}$. Moreover, Hypothesis A *is satisfied.*

Proof. Clearly, H is a well defined symmetric operator on $\mathcal{D}(\hat{N})$. We will prove the proposition by invoking Nelson's commutator theorem ([RS2], Theorem X.37). We must show that there is a constant $d > 0$ such that the following estimates hold for any $\Psi \in \mathcal{D}(\hat{N})$:

$$\begin{aligned} \|H\Psi\| &\leq d\|\hat{N}\Psi\|, \\ |(H\Psi|\hat{N}\Psi) - (\hat{N}\Psi|H\Psi)| &\leq d\|\hat{N}^{\frac{1}{2}}\Psi\|^2. \end{aligned} \tag{5.82}$$

By (4.73), $\lambda\varphi(\alpha)$ is an infinitesimal perturbation of N and therefore of $\hat{N}$. Obviously, $\mathrm{d}\Gamma(\omega)$ is bounded with respect to $\mathrm{d}\Gamma(|\omega|)$. These observation yield the first relation.

Since

$$|(H\Psi|\hat{N}\Psi) - (\hat{N}\Psi|H\Psi)| = |\lambda||(\Psi|[\hat{N}, \varphi(\alpha)]\Psi)|,$$

and

$$\mathrm{i}[\hat{N}, \varphi(\alpha)] = \varphi(\mathrm{i}(|\omega| + 1)\alpha) + \varphi(\beta),$$

where $\beta = \mathrm{i}(|K| \otimes \mathbf{1}_{\mathfrak{h}})\alpha - \mathrm{i}\alpha|K|$, the second relation in (5.82) follows from the estimates (4.73).

Finally, since $\mathcal{D}_{\rm fin}$ is a core for $\hat{N}$, Hypothesis A is satisfied. □

5.2 "Zero-temperature systems"

Let $\tilde{\mathfrak{h}} := L^2(\mathbf{R}_+) \otimes \mathfrak{g}$, where $\mathfrak{g}$ is an auxiliary Hilbert space. We denote by $\tilde{\omega}$ the operator of multiplication by $\omega \in \mathbf{R}_+$. Consider the Hilbert space $\tilde{\mathcal{H}} := \mathcal{K} \otimes \Gamma(\tilde{\mathfrak{h}})$ and the free Hamiltonian

$$\tilde{H}_{\rm fr} := K \otimes \mathbf{1} + \mathbf{1} \otimes {\rm d}\Gamma(\tilde{\omega}). \tag{5.83}$$

Let $\tilde{\alpha} \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \tilde{\mathfrak{h}})$. The Hamiltonian of the coupled system is formally given by

$$\tilde{H} := \tilde{H}_{\rm fr} + \lambda\varphi(\tilde{\alpha}), \tag{5.84}$$

where λ is a real constant.

Proposition 5.2 *Assume that the operator K is bounded from below and that*

$$\tilde{\alpha}^* \tilde{\omega}^{-1} \tilde{\alpha} \in \mathcal{B}(\mathcal{K}). \tag{5.85}$$

Then $\varphi(\tilde{\alpha})$ is infinitesimally small with respect to $\tilde{H}_{\rm fr}$. In particular, for any $\lambda \in \mathbf{R}$, the operator $\tilde{H}$ is self-adjoint on $\mathcal{D}(\tilde{H}_{\rm fr})$.

Proof. Without loss of generality we may assume that K is strictly positive. By (4.76) there is a constant $c > 0$ such that for any $\Psi \in \mathcal{D}(H_{\rm fr})$ and any $\epsilon > 0$,

$$\|\varphi(\tilde{\alpha})\Psi\|^2 \leq c\|\Psi\|^2 + c(\Psi|\tilde{H}_0\Psi)) \leq c(1+\epsilon^{-1})\|\Psi\|^2 + c\epsilon\|\tilde{H}_{\rm fr}\Psi\|^2.$$

It follows that $\varphi(\tilde{\alpha})$ is an infinitesimal perturbation of $\tilde{H}_{\rm fr}$. The other conclusions of the proposition follow from the Kato-Rellich theorem. □

The operator $\tilde{H}$ has a different form than the operator (5.81). Nevertheless, we will show below that by studying operators of the form (5.81) one can obtain information on the "zero-temperature" Hamiltonian $\tilde{H}$.

Consider a Hamiltonian of the form (5.84) and assume that (5.85) holds. This Hamiltonian can be extended to act on the Hilbert space $\tilde{\mathcal{H}} \otimes \Gamma(\tilde{\mathfrak{h}})$ as

$$\tilde{H}^{\rm ext}_{\rm fr} := \tilde{H}_{\rm fr} \otimes \mathbf{1} - \mathbf{1} \otimes {\rm d}\Gamma(\tilde{\omega}),$$
$$\tilde{H}^{\rm ext} := \tilde{H} \otimes \mathbf{1} - \mathbf{1} \otimes {\rm d}\Gamma(\tilde{\omega}).$$

Since $\tilde{\mathcal{H}} \otimes \Gamma_0(\tilde{\mathfrak{h}})$ is an invariant subspace of $\tilde{H}^{\rm ext}$ and

$$\tilde{H} = \tilde{H}^{\rm ext}\Big|_{\tilde{\mathcal{H}} \otimes \Gamma_0(\tilde{\mathfrak{h}})},$$

the spectral properties of $\tilde{H}$ can be inferred from the spectral properties of $\tilde{H}^{\rm ext}$ (note in particular that $\sigma_{\rm pp}(\tilde{H}) = \sigma_{\rm pp}(\tilde{H}^{\rm ext})$ and $\sigma_{\rm sc}(\tilde{H}) = \sigma_{\rm sc}(\tilde{H}^{\rm ext})$). Let us show that $\tilde{H}^{\rm ext}$ is unitarily equivalent to a Hamiltonian of the form (5.81) satisfying Hypothesis A.

Let U be the map from $\Gamma(\tilde{\mathfrak{h}}) \otimes \Gamma(\tilde{\mathfrak{h}})$ to $\Gamma(\tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}})$ defined in Theorem 4.3. Clearly,

$$\mathbf{1}_{\mathcal{K}} \otimes U : \tilde{\mathcal{H}} \otimes \Gamma(\mathfrak{h}) \to \mathcal{K} \otimes \Gamma(\tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}})$$

is a unitary map. Next, we have the unitary map

$$L^2(\mathbf{R}_+) \oplus L^2(\mathbf{R}_+) \ni (f_1, f_2) \mapsto f \in L^2(\mathbf{R}), \tag{5.86}$$

where

$$f(\omega) = \begin{cases} f_1(\omega) & \text{if } \omega \geq 0 \\ f_2(\omega) & \text{if } \omega < 0, \end{cases}$$

which induces the unitary map

$$w : \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{h}} = \Big(L^2(\mathbf{R}_+) \otimes \mathfrak{g}\Big) \oplus \Big(L^2(\mathbf{R}_+) \otimes \mathfrak{g}\Big) \mapsto \mathfrak{h} = L^2(\mathbf{R}) \otimes \mathfrak{g}.$$

Set

$$W := \mathbf{1}_{\mathcal{K}} \otimes (\Gamma(w)U).$$

Clearly,

$$W : \tilde{\mathcal{H}} \otimes \Gamma(\tilde{\mathfrak{h}}) \mapsto \mathcal{H},$$

is a unitary map. Let $\alpha \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ be given by

$$\alpha := \tilde{\alpha} \oplus 0.$$

We have

$$\begin{aligned} w(\tilde{\omega}, -\tilde{\omega})w^* &= \omega, \\ W(\mathrm{d}\Gamma(\tilde{\omega})) \otimes \mathbf{1} - \mathbf{1} \otimes \mathrm{d}\Gamma(\tilde{\omega}))W^* &= \mathrm{d}\Gamma(\omega), \\ W\varphi(\tilde{\alpha}) \otimes \mathbf{1}W^* &= \varphi(\alpha). \end{aligned}$$

Thus

$$\begin{aligned} W\tilde{H}_{\mathrm{fr}}^{\mathrm{ext}}W^* &= K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega), \\ W\tilde{H}^{\mathrm{ext}}W^* &= K \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}\Gamma(\omega) + \lambda\varphi(\alpha). \end{aligned}$$

Hence $W\tilde{H}_{\mathrm{fr}}^{\mathrm{ext}}W^*$ and $W\tilde{H}^{\mathrm{ext}}W^*$ have the form of the operators H_{fr}, H (recall (5.83), (5.84)). Furthermore, the operator $W\tilde{H}^{\mathrm{ext}}W^*$ is self-adjoint on

$$\mathcal{D}(W\tilde{H}_{\mathrm{fr}}^{\mathrm{ext}}W^*) = \mathcal{D}(W\tilde{H}_{\mathrm{fr}}^{\mathrm{ext}}W^*) \cap \mathcal{D}(\varphi(\alpha)).$$

Thus the Hamiltonian $W\tilde{H}^{\mathrm{ext}}W^*$ satisfies Hypothesis A. □

6 Main results

The Hilbert space $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$ has a natural decomposition $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$, where

$$\begin{aligned} \mathcal{H}^{\mathrm{v}} &:= \mathcal{K} \otimes \Gamma_0(\mathfrak{h}), \\ \mathcal{H}^{\overline{\mathrm{v}}} &:= \bigoplus_{n=1}^{\infty} \mathcal{K} \otimes \Gamma_n(\mathfrak{h}). \end{aligned}$$

(v stands for the vacuum). Note that $H_{\mathrm{fr}}^{\mathrm{vv}} = K$, $\mathcal{D}(H_{\mathrm{fr}}) = \mathcal{D}(K) \oplus \mathcal{D}(H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})$ and $\mathcal{D}(\varphi(\alpha)) = \mathcal{K} \oplus \mathcal{D}(\varphi(\alpha)^{\overline{\mathrm{vv}}})$. Hence, if $\mathcal{D}$ is as in Hypothesis A, then $\mathcal{D} = \mathcal{D}(K) \oplus \mathcal{D}(H_{\mathrm{fr}}^{\overline{\mathrm{vv}}}) \cap \mathcal{D}(\varphi(\alpha)^{\overline{\mathrm{vv}}})$. Using Hypothesis A, the fact that $H^{\overline{\mathrm{vv}}} = \lambda\varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}}$ is a bounded operator and the Kato–Rellich theorem we see that $H^{\mathrm{vv}} + H^{\overline{\mathrm{vv}}}$ is essentially self-adjoint on $\mathcal{D}$. Therefore, $H^{\overline{\mathrm{vv}}}$ is essentially self-adjoint on $\mathcal{D}(H_{\mathrm{fr}}^{\overline{\mathrm{vv}}}) \cap \mathcal{D}(\varphi(\alpha)^{\overline{\mathrm{vv}}})$. Thus the formalism and results of Chapter 3 can be applied to the operator H. We will use the notation introduced in Section 3.2. In particular, we recall that the self-energy is defined by

$$\begin{aligned} W_{\mathrm{v}}(z) &= H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} H^{\overline{\mathrm{v}}\mathrm{v}} \\ &= \tfrac{1}{2}\lambda^2 a(\alpha)^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} a^*(\alpha)^{\overline{\mathrm{v}}\mathrm{v}}. \end{aligned}$$

We define a self-adjoint operator s on the Hilbert space $\mathfrak{h}$ by $s := -\mathrm{i}\partial_\omega \otimes \mathbf{1}_{\mathfrak{g}}$, so that $[s, \omega] = -\mathrm{i}$. The conjugate operator is defined by

$$S := \mathbf{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(s). \tag{6.87}$$

For any $\nu \geq 0$ we introduce the following hypothesis:

$$\text{Hypothesis } \ S(\nu). \qquad\qquad \langle s \rangle^{\nu} \alpha \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}).$$

We will compare $W_{\mathrm{v}}(z)$ with its second-order approximation $\lambda^2 w(z)$, where

$$\begin{aligned} w(z) &:= \varphi(\alpha)^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1} \varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}} \\ &= \tfrac{1}{2}\Big(a(\alpha)(z - H_{\mathrm{fr}})^{-1} a^*(\alpha)\Big)^{\mathrm{vv}} \\ &= \tfrac{1}{2}\alpha^*(z - K \otimes \mathbf{1} - \mathbf{1} \otimes \omega)^{-1}\alpha. \end{aligned}$$

We now state an auxiliary result on the regularity properties of the function $w(z)$. Some of these properties will be used in the statement of our main theorems, but we remark that they are of independent interest.

Theorem 6.1 *Assume that Hypothesis $S(\nu)$ holds with $\nu > \frac{1}{2}$. Let $n \in \mathbf{N}$ and $0 < \theta \leq 1$ be such that $\nu = n + \frac{1}{2} + \theta$. Then, the function $\mathbf{C}_+ \ni z \mapsto w(z)$ extends by continuity to $\overline{\mathbf{C}}_+$ and is in the class $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$.*

The next three theorems describe our main results. In our model, the spectrum of K plays a role of the threshold set. The first theorem asserts that, away from an $O(\lambda^2)$ neighborhood of $\sigma(K)$, the Limiting Absorption Principle holds.

Theorem 6.2 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu > \frac{1}{2}$ *and* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then, there exists a constant* $\beta_1 > 0$ *such that for* $|\lambda| \leq \Lambda_1$ *the following holds:*
(i) *Set for shortness*

$$\Theta := \mathbf{R} \setminus I(\sigma(K), \lambda^2\beta_1).$$

Then, the function

$$z \mapsto \langle S\rangle^{-\mu}(z - H)^{-1}\langle S\rangle^{-\mu} \tag{6.88}$$

extends by continuity to a function on $\mathbf{C}_+ \cup \Theta$. *In particular, the spectrum of* H *on the set* Θ *is purely absolutely continuous.*
(ii) *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$, *and* μ *be such that* $\nu \geq \mu + \frac{1}{2} = 1 + n + \theta$. *Then, the function (6.88) is of the class* $C^{n,\theta}_{\rm u}$ *of the set*

$$\overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2\beta_1).$$

Our last two theorems describe the structure of the spectrum near an isolated eigenvalue k of K. They incorporate the notion of Fermi's Golden Rule. We remark that in our approach the eigenvalue k may have an infinite multiplicity. Let $p_k = \mathbf{1}_{\{k\}}(K)$. If $S(\nu)$ holds for some $\nu > \frac{1}{2}$ then it follows from Theorem 6.1 that

$$w_k := p_k w(k + \mathrm{i}0) p_k \tag{6.89}$$

is a bounded dissipative operator. We will always consider w_k as an operator on the Hilbert space $\mathrm{Ran} p_k$. In the standard description of atomic radiation, the spectrum of $\mathrm{Im} w_k$ captures the emission and absorption processes and radiative life-time of energy level k (of order λ^2). The spectrum of $\mathrm{Re} w_k$ captures the line shift of this energy level (of order λ^2). If $\sigma(w_k) \cap \mathbf{R} = \emptyset$, that is, if $\mathrm{Im} w_k < 0$, one expects that the energy level k has dissolved into the continuum, and that the spectrum of H in a neighborhood of k is purely absolutely continuous. Among other things, our next theorem justifies rigorously this heuristic expectation.

Theorem 6.3 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$ *and let* $\mu > \frac{1}{2}$. *Let* k *be an isolated eigenvalue of* K. *Assume that*

$$\mathcal{T}_k := \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\rm disc}(w_k).$$

Let β_1 *be the constant from the previous theorem and* $\kappa := 1 - \nu^{-1}$. *Then there exist constants* $\Lambda_2 > 0$ *and* $\beta_2 > 0$ *such that for* $|\lambda| \leq \Lambda_2$ *the following holds:*
(i) *Set for shortness*

$$\Theta(k) := \overline{I}(k, \lambda^2\beta_1) \setminus I(\{k\} + \lambda^2\mathcal{T}_k, |\lambda|^{2+\kappa}\beta_2).$$

Then, the function

$$z \mapsto \langle S\rangle^{-\mu}(z-H)^{-1}\langle S\rangle^{-\mu} \tag{6.90}$$

extends by continuity to a function on $\mathbf{C}_+ \cup \Theta(k)$. *In particular, the spectrum of* H *on the set* $\Theta(k)$ *is purely absolutely continuous.*
(ii) *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$ *and* μ *be such that* $\nu \geq \mu + \frac{1}{2} = n + 1 + \theta$. *Then, the function (6.90) is in the class* $C^{n,\theta}_{\mathrm{u}}$ *of the set*

$$\overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2\beta_1) \setminus B(\{k\} + \lambda^2\mathcal{T}_k, |\lambda|^{2+\kappa}\beta_2\right).$$

The next theorem is perhaps our deepest result. It concerns the situation where $\mathcal{T}_k \neq \emptyset$, and describes the structure of the spectrum of H around a point $m \in \sigma_{\mathrm{disc}}(w_k) \cap \mathbf{R}$. We set $p_{k,m} = \mathbf{1}_{\{m\}}(w_k)$. It follows from Proposition 3.2 that $p_{k,m}$ is an orthogonal projection. We emphasize that in the following theorem we need a stronger assumption on the interaction, namely we need $S(\nu)$ with $\nu > 2$.

Theorem 6.4 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 2$. *Let* k *be an isolated eigenvalue of* K,

$$\mathcal{T}_k := \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\mathrm{disc}}(w_k).$$

and let $m \in \mathcal{T}_k$. *Let* β_1, β_2 *and* κ *be as in the previous theorems. Then there exists a constant* $\Lambda_3 > 0$ *such that for* $|\lambda| \leq \Lambda_3$ *the following holds:*
(i) *Set for shortness*

$$\Theta(k,m) := \overline{I}(k + \lambda^2 m, |\lambda|^{2+\kappa}\beta_2) \cap \overline{I}(k, \lambda^2\beta_1).$$

Then $\dim \mathbf{1}^{\mathrm{pp}}_{\Theta(k,m)} \leq \dim p_{k,m}$. *In particular,* $\sigma_{\mathrm{pp}}(H) \cap \Theta(k,m)$ *is a finite set consisting of eigenvalues of finite multiplicity.*
(ii) *If* $\mu > \frac{1}{2}$, *then the function*

$$z \mapsto \langle S\rangle^{-\mu}(z-H)^{-1}\langle S\rangle^{-\mu} \tag{6.91}$$

extends by continuity to a function on $\mathbf{C}_+ \cup (\Theta(k,m) \setminus \sigma_{\mathrm{pp}}(H))$. *In particular, the spectrum of* H *on* $\Theta(k,m) \setminus \sigma_{\mathrm{pp}}(H)$ *is purely absolutely continuous.*
(iii) *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$ *and* $\nu \geq \mu + \frac{1}{2} = n + 1 + \theta$. *Then, for any* $\epsilon > 0$, *the function (6.91) is in the class* $C^{n,\theta}_{\mathrm{u}}$ *of the set*

$$\overline{\mathbf{C}}_+ \cap \left(\overline{B}(k + \lambda^2 m, |\lambda|^{2+\kappa}\beta_2) \cap \overline{B}(k, \lambda^2\beta_1) \setminus B(\sigma_{\mathrm{pp}}(H), \epsilon)\right).$$

Remark. Theorems 6.2, 6.3 and 6.4 combined together give global spectral information on the part of the real axis where the spectrum of K is discrete. The related results are easy to derive and we will not discuss them in this paper.

7 The Mourre theory on the radiation sector

In this chapter we prove the Limiting Absorption Principle for the Hamiltonian H reduced to the radiation sector. In Section 7.1 we prove the basic form and in Section 7.2 more refined versions of the Limiting Absorption Principle. As we have remarked in the introduction, the Limiting Absorption Principle for $H^{\overline{\mathrm{v}\mathrm{v}}}$ will hold uniformly on $\mathbf{R}$. In Section 7.3 we prove Theorem 6.1. In Section 7.4 we prove an estimate on the difference $(z\mathbf{1}^{\overline{\mathrm{v}\mathrm{v}}} - H^{\overline{\mathrm{v}\mathrm{v}}})^{-1} - (z\mathbf{1}^{\overline{\mathrm{v}\mathrm{v}}} - H_{\mathrm{fr}}^{\overline{\mathrm{v}\mathrm{v}}})^{-1}$.

7.1 Limiting Absorption Principle

This section is devoted to the proof of the following theorem:

Theorem 7.1 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu > \frac{1}{2}$ *and* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then,*

$$\sup_{(\lambda,z)\in[-\Lambda_1,\Lambda_1]\times\mathbf{C}_+} \left\| \langle S^{\overline{\mathrm{v}\mathrm{v}}}\rangle^{-\mu} (z\mathbf{1}^{\overline{\mathrm{v}\mathrm{v}}} - H^{\overline{\mathrm{v}\mathrm{v}}})^{-1} \langle S^{\overline{\mathrm{v}\mathrm{v}}}\rangle^{-\mu} \right\| < \infty.$$

Notation. In this section we will always work in the space $\mathcal{H}^{\overline{\mathrm{v}}}$. Henceforth, until the end of the section we will drop the superscripts $\overline{\mathrm{v}\mathrm{v}}$. Thus, we write H, S, N for the operators $H^{\overline{\mathrm{v}\mathrm{v}}}$, $S^{\overline{\mathrm{v}\mathrm{v}}}$, $N^{\overline{\mathrm{v}\mathrm{v}}}$, etc.

We start by assembling some preliminary definitions and facts. Let

$$\mathbf{S}A := [S, A],$$

$$\mathrm{e}^{\mathrm{i}\tau\mathbf{S}}A := \mathrm{e}^{\mathrm{i}\tau S} A \mathrm{e}^{-\mathrm{i}\tau S}.$$

Note that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}\mathrm{d}\Gamma(\omega) &= \mathrm{d}\Gamma(\omega) + \tau N, \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}N &= N, \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}(a(\alpha)) &= a(\mathrm{e}^{\mathrm{i}\tau s}\alpha), \\ \mathrm{e}^{\mathrm{i}\tau\mathbf{S}}(a^*(\alpha)) &= a^*(\mathrm{e}^{\mathrm{i}\tau s}\alpha). \end{aligned}$$

We choose a real function $\zeta \in C_0^\infty(\mathbf{R})$ such that $\zeta(t) = 1$ in a neighborhood of 0 and set

$$\xi(t) := \mathrm{e}^t \zeta(t). \tag{7.92}$$

Let

$$\mathfrak{D} := \mathcal{D}(H_{\mathrm{fr}}) \cap \mathcal{D}(N). \tag{7.93}$$

For $\epsilon \in \mathbf{R}$ we define

$$\begin{aligned} H_{\epsilon,\mathrm{fr}} &:= H_{\mathrm{fr}} - \mathrm{i}\epsilon N, \\ V_\epsilon &:= \tfrac{1}{\sqrt{2}}(a(\xi(-\epsilon s)\alpha) + a^*(\xi(\epsilon s)\alpha)), \\ H_\epsilon &:= H_{\epsilon,\mathrm{fr}} + \lambda V_\epsilon. \end{aligned} \tag{7.94}$$

where for $\epsilon = 0$ we have $H_{0,\mathrm{fr}} = H_{\mathrm{fr}}$, $H_0 = H$ and for $\epsilon \neq 0$ the domain is chosen as $\mathcal{D}(H_{\epsilon,\mathrm{fr}}) = \mathcal{D}(H_\epsilon) = \mathfrak{D}$.

Remark. We remark that the following identity holds on $\mathfrak{D}$:

$$H_\epsilon = \frac{1}{2\pi} \int \hat{\xi}(\epsilon\tau) \mathrm{e}^{\mathrm{i}\tau\mathbf{S}} H \mathrm{d}\tau. \tag{7.95}$$

Thus, formally, we could write $H_\epsilon = \xi(\epsilon\mathbf{S})H$, following the notation of [BG] and [BGS]. In [BG], a functional calculus was developed for expressions similar to (7.95). In our case, strictly speaking, this calculus does not apply, because H is an unbounded operator. Nevertheless, this calculus certainly motivates the definition of H_ϵ and the algebraic computations of this section.

The basic properties of the operators $H_{\epsilon,\mathrm{fr}}$ are summarized in the following lemma:

Lemma 7.2 *For any $\epsilon \in \mathbf{R}$, $H_{\epsilon,\mathrm{fr}}$ is a normal operator such that $H^*_{\epsilon,\mathrm{fr}} = H_{-\epsilon,\mathrm{fr}}$,*

$$\|H_{\epsilon,\mathrm{fr}}\Psi\|^2 = \|H_{\mathrm{fr}}\Psi\|^2 + \epsilon^2\|N\Psi\|^2, \quad \Psi \in \mathcal{D}(H_{\epsilon,\mathrm{fr}}), \tag{7.96}$$

$$\sigma(H_{\epsilon,\mathrm{fr}}) = \{-\mathrm{i}n\epsilon + \mathbf{R} \,:\, n = 1, 2, \ldots\}.$$

The proof of this lemma is the same as of Proposition 4.2 in [JP1] and we will skip it.

The next lemma gives the basic properties of the operator H_ϵ.

Lemma 7.3 *Assume that Hypothesis $S(0)$ holds. Then the following is true:*
(i) *For any $\epsilon \neq 0$, V_ϵ is an infinitesimal perturbation of $H_{\epsilon,\mathrm{fr}}$. In particular, H_ϵ is a closed operator with domain $\mathfrak{D}$.*
(ii) *For any ϵ we have $H^*_\epsilon = H_{-\epsilon}$.*

Proof. It follows from estimate (4.73) that

$$\|N^{-\frac{1}{2}} V_\epsilon\| \leq 2\|\xi\|_\infty \|\alpha\|, \tag{7.97}$$

and thus V_ϵ is an infinitesimal perturbation of N. This observation and (7.96) yield that V_ϵ is also an infinitesimal perturbation of $H_{\epsilon,\mathrm{fr}}$. This proves (i).

Let us show (ii). Clearly, we can assume that $\epsilon \neq 0$. It is easy to see that we can split V_ϵ as $V_\epsilon = V_{\epsilon,1} + V_{\epsilon,2}$ where $V_{\epsilon,2}$ is bounded and $\|N^{-1}V_{\epsilon,1}\| < 1$, $\|V_{\epsilon,1}N^{-1}\| < 1$. Therefore, we can apply Proposition 2.11. □

For $z \notin \sigma(H_\epsilon)$ we set

$$G_\epsilon(z) := (z - H_\epsilon)^{-1}.$$

In the next 3 lemmas we describe some properties of $G_\epsilon(z)$ that hold under the assumption S(0) and $\epsilon \neq 0$. Although they are formally obvious, they require a proof due to the unboundedness of some of the operators.

For $n = 1, 2, \ldots$ we define closed operators $H_\epsilon^{(n)}$ by the formula

$$H_\epsilon^{(n)} = -\mathrm{i}\delta_{1n}N + \frac{\lambda}{\sqrt{2}}\left((-1)^k a(s^k\xi^{(n)}(-\epsilon s)\alpha) + a^*(s^k\xi^{(n)}(\epsilon s)\alpha)\right), \tag{7.98}$$

where $\delta_{1n} = 1$ if $n = 1$ and 0 otherwise. Here, of course, $\xi^{(n)}$ is the n-th derivative of the function ξ. Note that for any $\Psi \in \mathfrak{D}$

$$\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}H_\epsilon\Psi = H_\epsilon^{(n)}\Psi.$$

Lemma 7.4 *Assume that Hypothesis S(0) holds and let $z \notin \sigma(H_\epsilon)$ be given. Then, the function*

$$\mathbf{R}_+ \ni \epsilon \mapsto G_\epsilon(z), \tag{7.99}$$

is infinitely differentiable and

$$\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}G_\epsilon(z) = \sum_{n_1+n_2+\ldots n_l=n} G_\epsilon(z)H_\epsilon^{(n_1)}G_\epsilon(z)\ldots G_\epsilon(z)H_\epsilon^{(n_l)}G_\epsilon(z). \tag{7.100}$$

Remark. In this section we will deal only with the first derivative of the function (7.99). The higher derivatives will be used in Section 7.2.

Proof. Let $\epsilon > 0$ be fixed and $z \notin \sigma(H_\epsilon)$. First note that

$$\|NG_\epsilon(z)\| < C. \tag{7.101}$$

In fact, by Lemma 7.3 $\|(H_{\epsilon,\mathrm{fr}} + \mathrm{i})G_\epsilon(z)\|$ is bounded and hence (7.101) follows from the bound

$$\|NG_\epsilon(z)\| \leq \|N(H_{\epsilon,\mathrm{fr}} + \mathrm{i})^{-1}\|\|(H_{\epsilon,\mathrm{fr}} + \mathrm{i})G_\epsilon(z)\|.$$

Next we note that for $|h| \leq \frac{\epsilon}{2}$ we have

$$\|(V_{\epsilon+h} - V_\epsilon)N^{-1}\| \leq C\frac{|h|}{|\epsilon|}\|\alpha\| \max_{t_1,t_2}|t_1\xi'(t_1 + t_2)|. \tag{7.102}$$

Hence, for $|h| \leq \frac{\epsilon}{2}$

$$\|(H_{\epsilon+h} - H_\epsilon)G_\epsilon(z)\| = \|(-\mathrm{i}hN + V_{\epsilon+h} - V_\epsilon)G_\epsilon(z)\| \leq C_1 h.$$

Therefore, for small enough h, $z \notin \sigma(H_{\epsilon+h})$ and $h \mapsto G_{\epsilon+h}(z)$ is norm continuous at $h = 0$.

We will now show that the function (7.99) is differentiable and that Relation (7.100) holds for $n = 1$. An easy inductive argument then yields that this function is infinitely differentiable and that Relation (7.100) holds for all n.

We have

$$G_{\epsilon+h}(z) - G_\epsilon(z) - hG_\epsilon(z)H_\epsilon^{(1)}G_\epsilon(z) = \mathrm{I} + \mathrm{II}, \tag{7.103}$$

where

$$\begin{aligned}
\mathrm{I} &:= \left(G_{\epsilon+h}(z) - G_\epsilon(z)\right)\left(H_{\epsilon+h} - H_\epsilon\right)G_\epsilon(z) \\
&= G_{\epsilon+h}(z)(-\mathrm{i}hN + \lambda V_{\epsilon+h} - \lambda V_\epsilon)G_\epsilon(z)(-\mathrm{i}hN + \lambda V_{\epsilon+h} - \lambda V_\epsilon)G_\epsilon(z), \\
\mathrm{II} &:= G_\epsilon(z)(H_{\epsilon+h} - H_\epsilon - hH_\epsilon^{(1)})G_\epsilon(z) \\
&= G_\epsilon(z)(\lambda V_{\epsilon+h} - \lambda V_\epsilon - h\lambda V_\epsilon^{(1)})G_\epsilon(z)
\end{aligned}$$

For $|h| \leq \frac{\epsilon}{2}$, we have

$$\|(V_{\epsilon+h} - V_\epsilon - hV_\epsilon^{(1)})N^{-1}\| \leq C\frac{|h|^2}{\epsilon^2}|\lambda|\|\alpha\| \sup_{t_1,t_2} |t_1^2 \xi''(t_1 + t_2)|. \tag{7.104}$$

Using (7.101), (7.102) and (7.104) we see that I and II are less than Ch^2. This ends the proof of the lemma for $n = 1$. □

We proceed to derive an alternative expression for $\frac{\mathrm{d}}{\mathrm{d}\epsilon}G_\epsilon(z)$ which will play an important role in the sequel.

The commutator $[S, H_\epsilon]$, defined as a quadratic form on $\mathcal{D}(S) \cap \mathcal{D}(H_\epsilon)$ is equal to

$$[S, H_\epsilon] = -\mathrm{i}N + \frac{\lambda}{\sqrt{2}}\left(-a(s\xi(-\epsilon s)\alpha) + a^*(s\xi(\epsilon s)\alpha)\right). \tag{7.105}$$

If we assume S(0), then it is easy to show that $\frac{\lambda}{\sqrt{2}}\left(-a(s\xi(-\epsilon s)\alpha) + a^*(s\xi(\epsilon s)\alpha)\right)$ is an infinitesimal perturbation of $-\mathrm{i}N$. Hence the right hand side of (7.105) defines a closed operator with domain $\mathcal{D}(N)$. By a slight abuse of notation, this operator will be also denoted by $[S, H_\epsilon]$.

Lemma 7.5 *Assume that Hypothesis S(0) holds. Let $\epsilon \neq 0$, $z \notin \sigma(H_\epsilon)$ and $m \geq 1$ be given. Then, $[S, G_\epsilon^m(z)]$, defined as a quadratic form on $\mathcal{D}(S)$, extends by continuity to a bounded operator on $\mathcal{H}$ equal to*

$$[S, G_\epsilon^m(z)] = \sum_{k=0}^{m-1} G_\epsilon^{k+1}(z)[S, H_\epsilon]G_\epsilon^{m-k}(z).$$

Remark. In this section, we will use Lemmas 7.5 and 7.6 with $m = 1$. The cases $m > 1$ will be used in the next section.

Proof. For t real we define

$$\begin{aligned} H_{\epsilon,t} &:= \mathrm{e}^{\mathrm{i}tS} H_\epsilon \mathrm{e}^{-\mathrm{i}tS} \\ &= H_{\epsilon,\mathrm{fr}} + tN + \tfrac{\lambda}{\sqrt{2}}(a(\mathrm{e}^{\mathrm{i}ts}\xi(-\epsilon s)\alpha) + a^*(\mathrm{e}^{\mathrm{i}ts}\xi(\epsilon s)\alpha)). \end{aligned}$$

Let

$$\begin{aligned} G_{\epsilon,t} &:= \mathrm{e}^{\mathrm{i}tS}(z - H_\epsilon)^{-1}\mathrm{e}^{-\mathrm{i}tS} \\ &= (z - H_{\epsilon,t})^{-1}. \end{aligned}$$

Arguing as in the proof of Lemma 7.4, one shows that the function

$$\mathbf{R} \ni t \mapsto G_{\epsilon,t}$$

is differentiable and that

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\epsilon,t}\big|_{t=0} = G_\epsilon(z)\mathrm{i}[S, H_\epsilon]G_\epsilon(z).$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} G^m_{\epsilon,t}\Big|_{t=0} = \sum_{k=0}^{m-1} G^{k+1}_\epsilon(z)\mathrm{i}[S, H_\epsilon]G^{m-k}_\epsilon(z). \tag{7.106}$$

On the other hand, in the quadratic form sense on $\mathcal{D}(S)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} G^m_{\epsilon,t}\Big|_{t=0} = \mathrm{i}[S, G^m_\epsilon(z)]. \tag{7.107}$$

Combining (7.106) and (7.107) we derive the statement. □

Let ζ be as in (7.92) and let

$$\eta(t) := \mathrm{e}^t t\zeta'(t) = t(\xi'(t) - \xi(t)). \tag{7.108}$$

We set

$$K_\epsilon := \frac{\lambda\epsilon^{-1}}{\sqrt{2}}\left(a(\eta(-\epsilon s)\alpha) + a^*(\eta(\epsilon s)\alpha)\right). \tag{7.109}$$

Note that $0 \notin \operatorname{supp}\eta$ (this fact will play an important role in the sequel) and that

$$H^{(1)}_\epsilon - [S, H_\epsilon] = K_\epsilon. \tag{7.110}$$

Lemma 7.6 *Assume that Hypothesis S(0) holds and let $z \notin \sigma(H_\epsilon)$ be fixed. Then, for any $m \geq 1$,*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} G^m_\epsilon(z) = [S, G^m_\epsilon(z)] + \sum_{k=0}^{m-1} G^{k+1}_\epsilon(z) K_\epsilon G^{m-k}_\epsilon(z). \tag{7.111}$$

Proof. Note that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}G_\epsilon^m(z) &= \textstyle\sum_{k=0}^{m-1} G_\epsilon^k(z)\left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}G_\epsilon(z)\right)G_\epsilon^{m-1-k}(z)\\ &= \textstyle\sum_{k=0}^{m-1} G_\epsilon^{k+1}(z)H_\epsilon^{(1)}G_\epsilon^{m-k}(z).\end{aligned}$$

The result follows from the identity (7.110) and Lemma 7.5. □

From now on we strengthen the hypothesis on the interaction and we will assume S(1). Note that the following inequality plays the role of the Mourre estimate.

Lemma 7.7 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$. Then, for any $\Psi \in \mathfrak{D}$,*

$$-\frac{1}{2\mathrm{i}\epsilon}(\Psi|(H_\epsilon - H_\epsilon^*)\Psi) \geq (1 - \sqrt{2}|\lambda|\|\xi'\|_\infty\|s\alpha\|)(\Psi|N\Psi).$$

Remark. Since $\xi'(t) = \mathrm{e}^t$ around 0, we have that $\|\xi'\|_\infty \geq 1$. On the other hand, by an appropriate choice of the function ζ, we can make $\|\xi'\|_\infty$ as close to 1 as we wish.

Proof. Let

$$\xi_1(t) = \frac{1}{2\mathrm{i}t}(\xi(-t) - \xi(t)).$$

Then on $\mathfrak{D}$,

$$\begin{aligned}-\tfrac{1}{2\mathrm{i}\epsilon}(H_\epsilon - H_\epsilon^*) &= N + \tfrac{\lambda}{2\mathrm{i}\epsilon}(V_{-\epsilon} - V_\epsilon)\\ &= N + \lambda\varphi(\xi_1(\epsilon s)s\alpha)\\ &= N^{\frac{1}{2}}\left(1 + \lambda N^{-\frac{1}{2}}\varphi(\xi_1(\epsilon s)s\alpha)N^{-\frac{1}{2}}\right)N^{\frac{1}{2}}.\end{aligned}$$

The result follows from this identity and the elementary estimate

$$\begin{aligned}\|N^{-\frac{1}{2}}\varphi(\xi_1(\epsilon s)s\alpha)N^{-\frac{1}{2}}\| &\leq \sqrt{2}\|\xi_1(\epsilon s)s\alpha\|\\ &\leq \sqrt{2}\|\xi_1\|_\infty\|s\alpha\| \leq \sqrt{2}\|\xi'\|_\infty\|s\alpha\|.\end{aligned}$$

□

In the sequel we choose $\Lambda_1 > 0$ and $C_0 > 0$ such that

$$\begin{aligned}\Lambda_1 &< (\sqrt{2}\|s\alpha\|)^{-1},\\ C_0 &< 1 - \sqrt{2}\Lambda_1\|s\alpha\|.\end{aligned} \tag{7.112}$$

It follows from Lemma 7.7 that we can choose ζ in (7.92) so that for $|\lambda| \leq \Lambda_1$ and $\Psi \in \mathfrak{D}$,

$$-\frac{1}{2\mathrm{i}\epsilon}(\Psi|(H_\epsilon - H_\epsilon^*)\Psi) \geq C_0(\Psi|N\Psi). \tag{7.113}$$

All the results in the sequel will hold for real λ such that $|\lambda| \leq \Lambda_1$, uniformly in λ.

Lemma 7.8 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$. If $\mathrm{Im}z > -C_0\epsilon$ then $z \notin \sigma(H_\epsilon)$ and*

$$\|G_\epsilon(z)\| \leq \frac{1}{\mathrm{Im}z + C_0\epsilon}.$$

Proof. It follows from Relation (7.113) that the numerical range of the operator H_ϵ is contained in the region $\mathrm{Im}z \leq -C_0\epsilon$. Since $\mathcal{D}(H_\epsilon) = \mathcal{D}(H_\epsilon^*)$, the statement follows from Proposition 2.9. □

Before we make use of the last result, we need one additional lemma.

Lemma 7.9 *Assume that Hypothesis S(1) holds and let $\epsilon > 0$ and $z \in \mathbf{C}_+$ be given. Then, for any $\Psi \in \mathcal{H}$,*

$$\begin{aligned} \|N^{\frac{1}{2}} G_\epsilon(z)\Psi\| &\leq (C_0\epsilon)^{-\frac{1}{2}} |(\Psi|G_\epsilon(z)\Psi)|^{\frac{1}{2}}, \\ \|N^{\frac{1}{2}} G_\epsilon^*(z)\Psi\| &\leq (C_0\epsilon)^{-\frac{1}{2}} |(\Psi|G_\epsilon(z)\Psi)|^{\frac{1}{2}}. \end{aligned}$$

Proof. We prove the first relation. A similar argument yields the second. We have

$$\begin{aligned} \|N^{\frac{1}{2}} G_\epsilon(z)\Psi\|^2 &= (\Psi|G_\epsilon^*(z) N G_\epsilon(z)\Psi)| \\ &\leq (C_0\epsilon)^{-1} (\Psi|G_\epsilon^*(z)(\mathrm{Im}H_\epsilon + \mathrm{Im}z) G_\epsilon(z)\Psi) \\ &= (\mathrm{i}2C_0\epsilon)^{-1} (\Psi|(G_\epsilon^*(z) - G_\epsilon(z)\Psi) \\ &\leq (C_0\epsilon)^{-1} |(\Psi|G_\epsilon(z)\Psi)|, \end{aligned}$$

where in the first estimate we used Lemma 7.7. □

¿From now on ρ will denote a Schwartz function. Set

$$\begin{aligned} \langle S\rangle_{\epsilon,\rho}^{-\mu} &:= \langle S\rangle^{-\mu} \rho(\epsilon S), \\ F_{\epsilon,\rho}(z) &:= \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}. \end{aligned} \tag{7.114}$$

Note that Lemma 7.4 yields that the function $\mathbf{R}_+ \ni \epsilon \mapsto F_{\epsilon,\rho}(z)$ is infinitely differentiable. We are now ready to prove one of the key technical results of this section.

Lemma 7.10 *Assume that Hypothesis $S(\nu)$ holds with $\nu \geq 1$ and let $\mu > 0$. Set $\gamma(\mu) = \min(\mu, 1)$. Then, for any $z \in \mathbf{C}_+$,*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\epsilon,\rho}(z) \right\| \leq C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)} \|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}} + |\lambda| C_2 \epsilon^{-2+\nu} \|F_{\epsilon,\rho}(z)\|. \tag{7.115}$$

Proof. It follows from Lemma 7.5 that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}F_{\epsilon,\rho}(z) = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} &:= \left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\langle S\rangle_{\epsilon,\rho}^{-\mu}\right) G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu} + \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z)\left(\tfrac{\mathrm{d}}{\mathrm{d}\epsilon}\langle S\rangle_{\epsilon,\rho}^{-\mu}\right),\\
\mathrm{II} &:= \langle S\rangle_{\epsilon,\rho}^{-\mu}[S, G_\epsilon(z)]\langle S\rangle_{\epsilon,\rho}^{-\mu},\\
\mathrm{III} &:= \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) K_\epsilon G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}.
\end{aligned}$$

Note that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle S\rangle_{\epsilon,\rho}^{-\mu}\right\| \leq \epsilon^{-1+\gamma(\mu)}\|\rho'\|_\infty.$$

Note also that Lemma 7.9 yields the estimates

$$\begin{aligned}
\|G_\epsilon^*(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| &\leq \|N^{\frac{1}{2}}G_\epsilon^*(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| \leq (C_0\epsilon)^{-\frac{1}{2}}\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}},\\
\|G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| &\leq \|N^{\frac{1}{2}}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| \leq (C_0\epsilon)^{-\frac{1}{2}}\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}.
\end{aligned} \tag{7.116}$$

Thus, the term I is estimated as follows:

$$\begin{aligned}
\|\mathrm{I}\| &\leq \epsilon^{-1+\gamma(\mu)}\|\rho'\|_\infty\left(\|G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| + \|G_\epsilon^*(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\right)\\
&\leq 2C_0^{-\frac{1}{2}}\|\rho'\|_\infty\epsilon^{-\frac{3}{2}+\gamma(\mu)}\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}.
\end{aligned} \tag{7.117}$$

Since for any $\epsilon > 0$,

$$\|S\langle S\rangle_{\epsilon,\rho}^{-\mu}\| = \|\langle S\rangle_{\epsilon,\rho}^{-\mu}S\| \leq \epsilon^{-1+\gamma(\mu)}\|\rho\|_\infty,$$

the estimates (7.116) yield

$$\begin{aligned}
\|\mathrm{II}\| &\leq \epsilon^{-1+\gamma(\mu)}\|\rho\|_\infty\left(\|G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\| + \|G_\epsilon^*(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\right)\\
&\leq 2C_0^{-\frac{1}{2}}\|\rho\|_\infty\epsilon^{-\frac{3}{2}+\gamma(\mu)}\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}}.
\end{aligned} \tag{7.118}$$

The term III is estimated as

$$\begin{aligned}
\|\mathrm{III}\| &\leq \|N^{\frac{1}{2}}G_\epsilon^*(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\|N^{\frac{1}{2}}G_\epsilon(z)\langle S\rangle_{\epsilon,\rho}^{-\mu}\|\|N^{-\frac{1}{2}}K_\epsilon N^{-\frac{1}{2}}\|\\
&\leq C_0^{-1}|\lambda|\epsilon^{-2}(\|\eta(-\epsilon s)\alpha\| + \|\eta(\epsilon s)\alpha\|)\|F_{\epsilon,\rho}(z)\|\\
&\leq 2C_0^{-1}|\lambda|\|s^\nu\alpha\|\sup_t|t^{-\nu}\eta(t)|\epsilon^{-2+\nu}\|F_{\epsilon,\rho}(z)\|.
\end{aligned} \tag{7.119}$$

Note that since $0 \notin \operatorname{supp}\eta$, $\sup_t|t^{-\nu}\eta(t)| < \infty$. Combining the estimates (7.117), (7.118) and (7.119) we derive that Relation (7.115) holds with

$$\begin{aligned}
C_1 &= 2C_0^{-\frac{1}{2}}(\|\rho\|_\infty + \|\rho'\|_\infty),\\
C_2 &= 2C_0^{-1}\|s^\nu\alpha\|\sup_t|t^{-\nu}\eta(t)|.
\end{aligned} \tag{7.120}$$

□

Lemma 7.11 *Assume that Hypothesis $S(\nu)$ holds for some $\nu > 1$ and let $\mu > \frac{1}{2}$. Then,*
(i) $\sup_{(\epsilon,z)\in\mathbf{R}_+\times\mathbf{C}_+} \|F_{\epsilon,\rho}(z)\| \leq C$.
(ii) *For any $z \in \mathbf{C}_+$, the norm-limit $\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z)$ exist.*

Proof. Since

$$\|F_{\epsilon,\rho}(z)\|^{\frac{1}{2}} \leq 1 + \|F_{\epsilon,\rho}(z)\|,$$

it follows from (7.115) that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\epsilon,\rho}(z) \right\| \leq a_\epsilon \|F_{\epsilon,\rho}(z)\| + b_\epsilon, \tag{7.121}$$

where

$$\begin{aligned} a_\epsilon &:= C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)} + |\lambda| C_2 \epsilon^{-2+\nu}, \\ b_\epsilon &:= C_1 \epsilon^{-\frac{3}{2}+\gamma(\mu)}. \end{aligned}$$

Since $\mu > \frac{1}{2}$, $\nu > 1$, we have

$$\int_0^1 a_\tau \mathrm{d}\tau < \infty, \quad \int_0^1 b_\tau \mathrm{d}\tau < \infty.$$

Note also that Lemma 7.8 yields that for $\epsilon \geq 1$, $\|F_{\epsilon,\rho}(z)\| \leq C_0^{-1}$ for all $z \in \mathbf{C}_+$. Thus by the Gronwall inequality (see e.g. [DG], Proposition A.1.1) we obtain for all $z \in \mathbf{C}_+$ and $\epsilon > 0$,

$$\|F_{\epsilon,\rho}(z)\| \leq C, \tag{7.122}$$

where

$$C := \exp\left(\int_0^1 a_\tau d\tau \right) \left(C_0^{-1} \int_0^1 a_\tau \mathrm{d}\tau + \int_0^1 b_\tau d\tau \right). \tag{7.123}$$

This yields Part (i) of the lemma.

To establish Part (ii), note that Relations (7.121) and (7.122) yield that for any $0 < \epsilon_1 < \epsilon_2$,

$$\begin{aligned} \|F_{\epsilon_2,\rho}(z) - F_{\epsilon_1,\rho}(z)\| &\leq \int_{\epsilon_1}^{\epsilon_2} \left\| \tfrac{\mathrm{d}}{\mathrm{d}\tau} F_{\tau,\rho}(z) \right\| \mathrm{d}\tau \\ &\leq \int_{\epsilon_1}^{\epsilon_2} (C a_\tau + b_\tau) \mathrm{d}\tau. \end{aligned}$$

Thus, if $\epsilon_n \to 0$ then the sequence $F_{\epsilon_n,\rho}(z)$ is Cauchy and this yields the statement. □

Assume now that $\rho(0) = 1$. Clearly,

$$F_{0,\rho}(z) = \langle S \rangle^{-\mu} (z - H)^{-1} \langle S \rangle^{-\mu}.$$

Thus, to finish the proof of Theorem 7.1 it suffices to show that

$$\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.124}$$

(Note that Part (ii) of Lemma 7.11 guarantees the existence of the limit in (7.124) but says nothing about its value). The proof of Relation (7.124) resolves the infrared problem we have discussed in the introduction. In the physics language, there is no infrared problem as long as $\epsilon > 0$ – this constant plays a role of a "complex boson mass". In our approach, the infrared problem appears in the limit $\epsilon \downarrow 0$ and is due to the fact that domains of H and H_ϵ with $\epsilon > 0$ are different. This difficulty is resolved below. We remark that an argument similar to ours has been used in [JP1]. The reader may consult [JP3] for additional discussion of this point.

The following technical result plays a key role in the resolution of the infrared problem. Recall that $G_0(z) = (z - H)^{-1}$.

Lemma 7.12 *Assume that Hypothesis $S(1)$ holds and let $z \in \mathbf{C}_+$ be given. Let $\beta = \pm\frac{1}{2}$. Then,*
(i) *For any $\epsilon > 0$,*

$$\|N^{-\beta} G_\epsilon(z) N^\beta\| \leq \frac{2}{\mathrm{Im} z} \left(1 + \frac{\sqrt{2}\|\alpha\||\lambda|}{\mathrm{Im} z}\right). \tag{7.125}$$

(ii) *Assume in addition that Hypothesis* A *is satisfied. Then, (7.125) is true also for $\epsilon = 0$.*

Proof. Let N_δ be as in Lemma 4.2. For any $\epsilon \geq 0$ consider the operator

$$H_{\epsilon,\delta,\beta} := H_\epsilon + \lambda N_\delta^{-\beta} V_\epsilon N_\delta^\beta - \lambda V_\epsilon.$$

It follows from Lemma 4.2 that

$$\|N_\delta^{-\beta} V_\epsilon N_\delta^\beta - V_\epsilon\| \leq C\sqrt{\delta},$$

where we use the shorthand $C := \sqrt{2}\|\alpha\|$. Therefore, if $\epsilon > 0$, $H_{\epsilon,\delta,\beta}$ is a closed operator on $\mathfrak{D}$. If $\epsilon = 0$, Hypothesis A yields that $H_{\epsilon,\delta,\beta}$ is essentially self-adjoint on $\mathfrak{D}$. Note also that for any $\epsilon \geq 0$,

$$\mathfrak{N}(H_\epsilon) \subset \{z : \ \mathrm{Im} z \leq -C_0\epsilon\},$$

(recall the estimate (7.113)), therefore

$$\mathfrak{N}(H_{\epsilon,\delta,\beta}) \subset \{z : \ \mathrm{Im} z \leq -C_0\epsilon + C|\lambda|\sqrt{\delta}\}.$$

Thus, if $\mathrm{Im} z > -C_0\epsilon + C|\lambda|\sqrt{\delta}$ then the operator $z - H_{\epsilon,\delta,\beta}$ is invertible and

$$\|(z - H_{\epsilon,\delta,\beta})^{-1}\| \leq \left(\mathrm{Im} z + C_0\epsilon - C|\lambda|\sqrt{\delta}\right)^{-1}.$$

¿From now on we fix $z \in \mathbf{C}_+$ and choose δ such that

$$C|\lambda|\sqrt{\delta} < \mathrm{Im} z.$$

Let

$$\mathfrak{D}_{\rm fin} := \mathfrak{D} \cap (\mathcal{K} \otimes \Gamma_{\rm fin}).$$

For any $\Psi \in \mathfrak{D}_{\rm fin}$ we have

$$H_{\epsilon,{\rm fr}}\Psi = N_\delta^{-\beta} H_{\epsilon,{\rm fr}} N_\delta^\beta \Psi.$$

Similarly, for $\Psi \in \mathfrak{D}_{\rm fin}$ we have

$$(N_\delta^{-\beta} V_\epsilon N_\delta^\beta - V_\epsilon)\Psi = N_\delta^{-\beta} V_\epsilon N_\delta^\beta \Psi - V_\epsilon \Psi.$$

Thus, on $\mathfrak{D}_{\rm fin}$,

$$\begin{aligned} H_{\epsilon,\delta,\beta} &= H_{\epsilon,{\rm fr}} + N_\delta^{-\beta} V_\epsilon N_\delta^\beta \\ &= N_\delta^{-\beta} H_\epsilon N_\delta^\beta. \end{aligned} \tag{7.126}$$

One easily shows that $\mathfrak{D}_{\rm fin}$ is a core for $H_{\epsilon,{\rm fr}}$. If $\epsilon > 0$, V_ϵ is an infinitesimal perturbation of $H_{\epsilon,{\rm fr}}$, and therefore $\mathfrak{D}_{\rm fin}$ is also a core for H_ϵ and $H_{\epsilon,\delta,\beta}$. It follows from Hypothesis A that $\mathfrak{D}_{\rm fin}$ is a core of H_0. Therefore, for $\epsilon \geq 0$,

$$\tilde{\mathfrak{D}}_{\rm fin} := (z - H_{\epsilon,\delta,\beta})\mathfrak{D}_{\rm fin}, \tag{7.127}$$

is dense in $\mathcal{H}$ and on $\tilde{\mathfrak{D}}_{\rm fin}$,

$$(z - H_{\epsilon,\delta,\beta})^{-1} = N_\delta^{-\beta}(z - H_\epsilon)^{-1} N_\delta^\beta.$$

Next, we note that

$$\|N_\delta^\beta N^{-\beta}\| = \|N^{-\beta} N_\delta^\beta\| \leq \begin{cases} (1+\delta)^\beta, & \beta > 0, \\ \delta^\beta, & \beta \leq 0. \end{cases}$$

Therefore, for any $\epsilon \geq 0$ and $\Psi \in \tilde{\mathfrak{D}}_{\rm fin}$,

$$\|N^{-\beta}(z - H_\epsilon)^{-1} N^\beta \Psi\| \leq \left(\frac{1+\delta}{\delta}\right)^{|\beta|} \Big({\rm Im}z + C_0\epsilon - C|\lambda|\sqrt{\delta}\Big)^{-1} \|\Psi\|.$$

Taking $\delta = \left(\frac{{\rm Im}z}{2C|\lambda|}\right)^2$, we derive the statements of the lemma. □

Lemma 7.13 *Assume that Hypotheses* A *and* $S(1)$ *hold. Then*

$$\lim_{\epsilon \downarrow 0} N^{-\frac{1}{2}} G_\epsilon(z) N^{-\frac{1}{2}} = N^{-\frac{1}{2}}(z - H)^{-1} N^{-\frac{1}{2}}. \tag{7.128}$$

Proof. Let $\epsilon > 0$. We have

$$N^{-\frac{1}{2}}(G_\epsilon(z) - G_0(z))N^{-\frac{1}{2}} = N^{-\frac{1}{2}}G_0(z)(H_\epsilon - H)G_\epsilon(z)N^{-\frac{1}{2}} \tag{7.129}$$

$$= N^{-\frac{1}{2}}G_0(z)N^{\frac{1}{2}}\left(-\mathrm{i}\epsilon + \lambda N^{-\frac{1}{2}}(V_\epsilon - V)N^{-\frac{1}{2}}\right)N^{\frac{1}{2}}G_\epsilon(z)N^{-\frac{1}{2}}. \tag{7.130}$$

This identity and Lemma 7.12 yield that

$$\|N^{-\frac{1}{2}}(G_\epsilon(z) - G_0(z))N^{-\frac{1}{2}}\| \leq C_z^2\epsilon\left(1 + 2|\lambda|\|\alpha\|\sup_{t_1,t_2}|t_1\xi'(t_1+t_2)|\right), \tag{7.131}$$

where C_z is the constant on the right-hand side in (7.125). Clearly, this estimate yields (7.128). □

We are now ready to finish

Proof of Theorem 7.1. As we have already remarked, it follows from Lemma 7.11 that to prove Theorem 7.1 it suffices to show that for any $z \in \mathbf{C}_+$,

$$\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.132}$$

Since we know that the limit on the right hand side exist, it suffices to show that

$$\mathrm{w}-\lim_{\epsilon\downarrow 0} F_{\epsilon,\rho}(z) = F_{0,\rho}(z). \tag{7.133}$$

This relation follows from (7.128). □

7.2 Hölder continuity

This section is devoted to the proof of the following theorem:

Theorem 7.14 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $n \in \mathbf{N}$, $0 < \theta \leq 1$, *and* $\mu \geq \frac{1}{2}$ *be such that* $\nu \geq \mu + \frac{1}{2} = 1 + n + \theta$. *Let* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then, for* $|\lambda| \leq \Lambda_1$, *the function*

$$\mathbf{C}_+ \ni z \mapsto \langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu} \tag{7.134}$$

extends by continuity to $\overline{\mathbf{C}}_+$ *and is of the class* $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$ *uniformly in* λ.

The rest of this section is devoted to the proof of Theorem 7.14. We will freely use the results and notation of Section 7.1. In particular, we drop superscripts $\overline{\mathrm{vv}}$ until the end of this section. We fix $\Lambda_1 > 0$, $C_0 > 0$ and ζ such that Relations (7.112) and (7.113) hold. All the results in the sequel will hold for real λ such that $|\lambda| \leq \Lambda_1$, uniformly in λ.

For any $\epsilon \geq 0$, the function

$$\mathbf{C}_+ \ni z \mapsto G_\epsilon(z),$$

is analytic and

$$\partial_z^l G_\epsilon(z) = (-1)^l l! G_\epsilon^{l+1}(z).$$

It follows from Lemma 7.4 that for any l and m the mixed derivatives

$$\partial_z^l \partial_\epsilon^m G_\epsilon(z)$$

exists on $\mathbf{C}_+ \times \mathbf{R}_+$, and that they are linear combinations of the terms

$$G_\epsilon^{l_1}(z) H_\epsilon^{(m_1)} G_\epsilon^{l_2}(z) \dots G_\epsilon^{l_k}(z) H_\epsilon^{(m_k)} G_\epsilon^{l_{k+1}}(z), \tag{7.135}$$

where l_k's and m_k's are positive integers such that

$$\sum_{j=1}^{k} m_j = m, \qquad \sum_{j=1}^{k+1} l_j = l + k + 1. \tag{7.136}$$

We proceed to study in some detail these mixed derivatives.

Lemma 7.15 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Then,*

$$\begin{aligned} \sup_{z\in\mathbf{C}_+} \left\| \langle S\rangle_{\epsilon,\rho}^{-\mu} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) \right) N^{\frac{1}{2}} \right\| &\le C\epsilon^{-\frac{1}{2}-l-m}, \\ \sup_{z\in\mathbf{C}_+} \left\| N^{\frac{1}{2}} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) \right) \langle S\rangle_{\epsilon,\rho}^{-\mu} \right\| &\le C\epsilon^{-\frac{1}{2}-l-m}. \end{aligned} \tag{7.137}$$

Proof. We will prove the first relation. A similar argument yields the second. We write $\partial_z^l \partial_\epsilon^m G_\epsilon(z)$ as a linear combinations of the terms (7.135). After inserting $\langle S\rangle_{\epsilon,\rho}^{-\mu}$ and $N^{\frac{1}{2}}$, we estimate the norm of each term by

$$\begin{aligned} &\|\langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) N^{\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_1-1} \|N^{-\frac{1}{2}} H_\epsilon^{(m_1)} N^{-\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_2} \dots \\ &\qquad \dots \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_k} \|N^{-\frac{1}{2}} H_\epsilon^{(m_k)} N^{-\frac{1}{2}}\| \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\|^{l_{k+1}}. \end{aligned} \tag{7.138}$$

It follows from Lemmas 7.9 and 7.11 that

$$\begin{aligned} \|\langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) N^{\frac{1}{2}}\| &\le C_0^{-\frac{1}{2}} \epsilon^{-\frac{1}{2}} \|F_{\epsilon,\rho}(z)\| \le C_0^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} C^{\frac{1}{2}}, \\ \|N^{\frac{1}{2}} G_\epsilon(z) N^{\frac{1}{2}}\| &\le C_0^{-1} \epsilon^{-1}. \end{aligned} \tag{7.139}$$

Furthermore,

$$\|N^{-\frac{1}{2}} H_\epsilon^{(m_j)} N^{-\frac{1}{2}}\| \le c_{m_j} \epsilon^{1-m_j}, \tag{7.140}$$

where

$$c_{m_j} = \delta_{1m_j} + 2|\lambda| \sup_t |t^{m_j-1} \xi^{(m_j)}(t)| \|s\alpha\|. \tag{7.141}$$

Combining these estimates and using (7.136) we bound (7.138) with

$$\epsilon^{-\frac{1}{2}-l-m} C_0^{\frac{1}{2}-l-k} C^{\frac{1}{2}} \prod c_{m_j},$$

Summing over the terms (7.135) we derive the Relation (7.137). □

Lemma 7.16 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Let $k = k_1 + k_2$. Then*

$$\sup_{z\in\mathbf{C}_+} \left\| \langle S\rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z) \right) S^{k_2} \langle S\rangle_{\epsilon,\rho_2}^{-\mu} \right\| \leq C\epsilon^{-l-m-k-2+\nu}. \tag{7.142}$$

Proof. Note that

$$\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z)$$

is a linear combination of terms

$$(\partial_z^{l_1} \partial_\epsilon^{m_1} G_\epsilon(z))(\partial_\epsilon^k K_\epsilon)(\partial_z^{l_2} \partial_\epsilon^{m_2} G_\epsilon(z)),$$

where $l = l_1 + l_2$, $m = m_1 + m_2 + k$. After inserting $\langle S\rangle_{\epsilon,\rho_1}^{-\mu}$ and $\langle S\rangle_{\epsilon,\rho_2}^{-\mu}$, we bound the norms of these terms by

$$\left\| \langle S\rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_z^{l_1} \partial_\epsilon^{m_1} G_\epsilon(z) \right) N^{\frac{1}{2}} \right\| \, \| N^{-\frac{1}{2}} \partial_\epsilon^k K_\epsilon N^{-\frac{1}{2}} \| \, \left\| N^{\frac{1}{2}} \left(\partial_z^{l_2} \partial_\epsilon^{m_2} G_\epsilon(z) \right) \langle S\rangle_{\epsilon,\rho_2}^{-\mu} \right\|. \tag{7.143}$$

We estimate

$$\begin{aligned} \| N^{-\frac{1}{2}} \partial_\epsilon^k K_\epsilon N^{-\frac{1}{2}} \| &\leq |\lambda| \epsilon^{-1} \left(\| s^k \eta^{(k)}(\epsilon s)\alpha \| + \| s^k \eta^{(k)}(-\epsilon s)\alpha \| \right) \\ &\leq 2|\lambda| \sup_t |t^{k-\nu} \eta^{(k)}(t)| \| s^\nu \alpha \| \epsilon^{-1+\nu-k}. \end{aligned} \tag{7.144}$$

Note that since $0 \notin \operatorname{supp} \eta$, the constant $\sup_t |t^{k-\nu}\eta^{(k)}(t)|$ is finite. Combining the estimate (7.144) with Lemma 7.15, we obtain

$$\sup_{z\in\mathbf{C}_+} \| \langle S\rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z) \right) \langle S\rangle_{\epsilon,\rho_2}^{-\mu} \| \leq C\epsilon^{-l-m-2+\nu}. \tag{7.145}$$

Next note that we can find Schwartz functions γ_i, $\tilde{\gamma}_i$ such that $\rho_i = \gamma_i \tilde{\gamma}_i$, for $i = 1, 2$. Clearly

$$\langle S\rangle_{\epsilon,\rho_i}^{-\mu} = \tilde{\gamma}_i(\epsilon S) \langle S\rangle_{\epsilon,\gamma_i}^{-\mu}, \quad i = 1, 2.$$

Therefore, we can estimate the left-hand side of (7.142) by

$$\| \tilde{\gamma}_1(\epsilon S) S^{k_1} \| \left\| \langle S\rangle_{\epsilon,\gamma_2}^{-\mu} \left(\partial_z^l \partial_\epsilon^m G_\epsilon(z) K_\epsilon G_\epsilon(z) \right) \langle S\rangle_{\epsilon,\gamma_2}^{-\mu} \right\| \| S^{k_2} \tilde{\gamma}_2(\epsilon S) \|.$$

Thus, Relation (7.142) follows from (7.145) and the estimates

$$\| \tilde{\gamma}_i(\epsilon S) S^{k_i} \| \leq \epsilon^{-k_i} \sup_t |t^{k_i} \tilde{\gamma}_i(t)|, \quad i = 1, 2. \tag{7.146}$$

□

Lemma 7.17 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$. Let $\mu > \frac{1}{2}$, $k \geq 2\mu$, $k = k_1 + k_2$. Then,*

$$\sup_{z\in\mathbf{C}_+} \|\partial_z^l \langle S\rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1} G_\epsilon(z) S^{k_2} \langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C\epsilon^{-k-l-\frac{1}{2}+\mu}. \tag{7.147}$$

Proof. We use the splitting $\rho_2 = \tilde{\gamma}_2\gamma_2$, as in the proof of the previous lemma. If $k_1 \geq k_2$, then $k_1 \geq \mu$ and we use the estimates

$$\begin{aligned}
\|\langle S\rangle_{\epsilon,\rho_1}^{-\mu} S^{k_1}\| &\leq \epsilon^{-k_1+\mu} \sup_t |t^{k_1-\mu}\rho_1(t)|,\\
\|S^{k_2}\tilde{\gamma}_2(\epsilon S)\| &\leq \epsilon^{-k_2} \sup_t |t^{k_2}\tilde{\gamma}_2(t)|,\\
\|\partial_z^l G_\epsilon(z)\langle S\rangle_{\epsilon,\gamma_2}^{-\mu}\| &\leq C\epsilon^{-l-\frac{1}{2}},
\end{aligned}$$

where we used Lemma 7.15 in the last estimate. If $k_1 \leq k_2$, one interchanges the roles of k_1 and k_2 and argues similarly. □

We recall that for any operator A,

$$\mathbf{S}A = [S, A],$$

is the quadratic form defined on $\mathcal{D}(S)\cap\mathcal{D}(A)$. If A is bounded, one can define the multiple commutators $\mathbf{S}^p A$ for any positive integer p as quadratic forms on $\mathcal{D}(S^p)$.

In the following lemma it will be convenient to use the following function:

$$\omega(\nu,\epsilon) := \begin{cases} \epsilon^\nu, & \nu < 0,\\ 1 + \log(1+\epsilon^{-1}), & \nu = 0,\\ 1, & \nu > 0. \end{cases} \tag{7.148}$$

Let us note the following properties of this function:

$$\begin{aligned}
&\textstyle\int_\epsilon^1 \omega(\nu,\tau)\mathrm{d}\tau \leq C\omega(\nu+1,\epsilon),\\
&\omega(\theta-1,\epsilon) = \epsilon^{-1}\ell_\theta(\epsilon), \quad 0 < \theta \leq 1,
\end{aligned} \tag{7.149}$$

where the function $\ell_\theta(\epsilon)$ was defined in (2.27).

Lemma 7.18 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 1$ and let $\mu > \frac{1}{2}$. Then, for some constants C_1 and C_2,*

$$\begin{aligned}
&\sup_{z\in\mathbf{C}_+} \|\partial_z^l \partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}\|\\
&\leq C_1\omega(-k-l-\tfrac{1}{2}+\mu,\epsilon) + C_2\omega(-k-l-1+\nu,\epsilon).
\end{aligned} \tag{7.150}$$

Proof. Note that

$$\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu},$$

is the sum of the terms

$$\left(\partial_\epsilon^{k_1} \langle S\rangle_{\epsilon,\rho}^{-\mu}\right)\left(\partial_\epsilon^{k_2} G_\epsilon(z)\right)\left(\partial_\epsilon^{k_3} \langle S\rangle_{\epsilon,\rho}^{-\mu}\right), \tag{7.151}$$

where $k_1 + k_2 + k_3 = k$. ¿From the formula

$$\partial_\epsilon G_\epsilon(z) = \mathbf{S} G_\epsilon(z) + G_\epsilon(z) K_\epsilon G_\epsilon(z),$$

(recall Lemma 7.5) one easily shows by induction that

$$\partial_\epsilon^{k_2} G_\epsilon(z) = \mathbf{S}^{k_2} G_\epsilon(z) + \sum_{p+r+1=k_2} \mathbf{S}^p \partial_\epsilon^r G_\epsilon(z) K_\epsilon G_\epsilon(z), \tag{7.152}$$

as a quadratic form on $\mathcal{D}(S^k)$.

Using (7.111) and the identity $\partial_\epsilon^{k_1} \langle S\rangle_{\epsilon,\rho}^{-\mu} = S^{k_1} \langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu}$, we can write (7.151) as a linear combination of the terms

$$\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu} S^{k_1+j_1} G_\epsilon(z) S^{j_2+k_3} \langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu}, \tag{7.153}$$

where $j_1 + j_2 = k_2$, and of the terms

$$\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu} S^{k_1+l_1} \left(\partial_\epsilon^r G_\epsilon(z) K_\epsilon G_\epsilon(z)\right) S^{l_2+k_3} \langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu}, \tag{7.154}$$

where $l_1 + l_2 + r + 1 = k_2$. After applying ∂_z^l to terms (7.153) and (7.154) we estimate them with the help of Lemmas 7.17 and 7.16. This yields (7.150) if $k \geq 2\mu$.

Next note that (7.135), (7.139) and (7.140) yield that for any k and l there is a constant C such that

$$\sup_{z\in\mathbf{C}_+,\epsilon=1} \|\partial_z^l \partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}\| \leq C.$$

Therefore, if Relation (7.150) holds for $k+1$, integrating the function

$$\tau \mapsto \partial_z^l \partial_\tau^{k+1} \langle S\rangle_{\tau,\rho}^{-\mu} G_\tau(z) \langle S\rangle_{\tau,\rho}^{-\mu},$$

over $[\epsilon, 1[$ and using (7.149) we derive that Relation (7.150) also holds for k. The proof of Lemma 7.18 is complete. □

We are now ready to finish

Proof of Theorem 7.14. Assume that $\rho(0) = 1$. Let $m = 0, \ldots, n$. We have

$$\|\partial_\epsilon \partial_z^m \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}\| \leq C\omega(-1+\theta+n-m, \epsilon). \tag{7.155}$$

Since $-1 < -1 + \theta + n - m$, the right hand side of (7.155) is integrable around 0. This implies the uniform convergence of

$$\lim_{\epsilon \downarrow 0} \partial_z^m \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}, \quad m = 0, \ldots, n.$$

Hence, by the well known calculus lemma we can interchange the order of the limit and the differentiation in the following formula:

$$\begin{aligned} \partial_z^n \langle S \rangle^{-\mu} (z - H)^{-1} \langle S \rangle^{-\mu} &= \partial_z^n \lim_{\epsilon \downarrow 0} \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu} \\ &= \lim_{\epsilon \downarrow 0} \partial_z^n \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu}, \end{aligned}$$

where in the first step we used (7.132).

It follows from Lemma 7.18 that for some constant C,

$$\sup_{z \in \mathbf{C}_+} \| \partial_\epsilon^k \partial_z^l \partial_z^n \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu} \| \leq C \omega(-1 + \theta, \epsilon) = C \epsilon^{-1} \ell_\theta(\epsilon), \quad k + l \leq 1.$$

Therefore, the function

$$[0, 1[\times \mathbf{C}_+ \ni (\epsilon, z) \mapsto \partial_z^n \langle S \rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S \rangle_{\epsilon,\rho}^{-\mu},$$

satisfies all the conditions of Proposition 2.3. It follows that the function

$$\mathbf{C}_+ \ni z \mapsto \partial_z^n \langle S \rangle^{-\mu} (z - H)^{-1} \langle S \rangle^{-\mu}, \tag{7.156}$$

is in the class $C_{\mathrm{u}}^{0,\theta}(\mathbf{C}_+)$. Therefore the function (7.156) with $n = 0$ satisfies the conditions of Proposition 2.4. The proof of Theorem 7.14 is complete. □

7.3 Properties of $w(z)$

In this section we prove Theorem 6.1. Let us first state a version of Theorem 7.14 for the free Hamiltonian. Setting $\lambda = 0$ in Theorem 7.14 we derive

Theorem 7.19 *Let $n \in \mathbf{N}$, $0 < \theta \leq 1$, $\nu = \frac{1}{2} + n + \theta$. Then, the function*

$$\mathbf{C}_+ \ni z \mapsto \langle S^{\overline{\mathrm{vv}}} \rangle^{-\nu} (z \mathbf{1}^{\overline{\mathrm{vv}}} - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1} \langle S^{\overline{\mathrm{vv}}} \rangle^{-\nu} \tag{7.157}$$

extends by continuity to $\overline{\mathbf{C}}_+$ and is in the class $C_{\mathrm{u}}^{n,\theta}(\overline{\mathbf{C}}_+)$.

Proof of Theorem 6.1. Note that if Hypothesis $S(\nu)$ holds then the operators

$$\varphi(\alpha)^{\mathrm{v}\overline{\mathrm{v}}} \langle S^{\overline{\mathrm{vv}}} \rangle^{\nu} \qquad \text{and} \qquad \langle S^{\overline{\mathrm{vv}}} \rangle^{\nu} \varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}},$$

are bounded and their norms are less than or equal to $\| \langle s \rangle^{\nu} \alpha \| / \sqrt{2}$. Since

$$w(z) = \varphi(\alpha)^{\mathrm{v}\overline{\mathrm{v}}} \langle S^{\overline{\mathrm{vv}}} \rangle^{\nu} \left(\langle S^{\overline{\mathrm{vv}}} \rangle^{-\nu} (z \mathbf{1}^{\overline{\mathrm{vv}}} - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1} \langle S^{\overline{\mathrm{vv}}} \rangle^{-\nu} \right) \langle S^{\overline{\mathrm{vv}}} \rangle^{\nu} \varphi(\alpha)^{\overline{\mathrm{v}}\mathrm{v}}, \tag{7.158}$$

we derive the result from Theorem 7.19.

7.4 Comparison with the free resolvent

In this section we estimate the difference of the free and the full resolvent on the radiation sector.

Theorem 7.20 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu = \nu - \frac{1}{2}$ *and* $0 < \Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$. *Then*

$$\sup_{(\lambda,z)\in[-\Lambda_1,\Lambda_1]\times\overline{\mathbf{C}}_+} \left\|\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}\Big((z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1} - (z\mathbf{1}^{\overline{\mathrm{vv}}} - H_{\mathrm{fr}}^{\overline{\mathrm{vv}}})^{-1}\Big)\langle S^{\overline{\mathrm{vv}}}\rangle^{-\mu}\right\| \leq C|\lambda|^{(\nu-1)/\nu} \tag{7.159}$$

For notational simplicity, in the sequel we drop the superscripts $\overline{\mathrm{vv}}$. It follows from Theorem 7.14 that it suffices to take in (7.159) supremum over $z \in \mathbf{C}_+$. We choose again $\Lambda_1 > 0$, $C_0 > 0$ and ζ such that Relations (7.112) and (7.113) hold. All the results in the sequel will hold for real λ such that $|\lambda| \leq \Lambda_1$.

In what follows we will denote by the same letter C various constants which depend only on the constants introduced in the previous section, but do not depend on λ. The values of these constants may change from estimate to estimate.

Lemma 7.21 *Assume that Hypotheses* A *and* $S(\nu)$ *hold with* $\nu > 1$. *Let* $\mu > \frac{1}{2}$. *Then*

$$\sup_{z\in\mathbf{C}_+} \|\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho_1}^{-\mu}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C|\lambda|\epsilon^{-k-1}.$$

Proof. Using

$$G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z) = \lambda G_\epsilon(z)V_\epsilon G_{\mathrm{fr},\epsilon}(z)$$

we can write

$$\langle S\rangle_{\epsilon,\rho_1}^{-\mu}\left(\partial_\epsilon^m(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\right)\langle S\rangle_{\epsilon,\rho_2}^{-\mu}$$

as a linear combination the terms

$$\begin{aligned}&\langle S\rangle_{\epsilon,\rho_1}^{-\mu} G_\epsilon(z)H_\epsilon^{(m_1)}G_\epsilon(z)\dots G_\epsilon(z)H_\epsilon^{(m_{k-1})}G_\epsilon(z)\\ &\times\lambda V_\epsilon^{(m_k)}G_{\mathrm{fr},\epsilon}(z)H_{\mathrm{fr},\epsilon}^{(m_{k+1})}G_{\mathrm{fr},\epsilon}(z)\dots G_{\mathrm{fr},\epsilon}(z)H_{\mathrm{fr},\epsilon}^{(m_l)}G_{\mathrm{fr},\epsilon}(z)\langle S\rangle_{\epsilon,\rho_2}^{-\mu},\end{aligned} \tag{7.160}$$

where $\sum_{j=1}^l m_j = m$. Using

$$\|N^{-\frac{1}{2}}H_\epsilon^{(m_j)}N^{-\frac{1}{2}}\| \leq C\epsilon^{1-m_j}, \qquad \|N^{-\frac{1}{2}}H_{\mathrm{fr},\epsilon}^{(m_j)}N^{-\frac{1}{2}}\| \leq C\epsilon^{1-m_j},$$

$$\|N^{-\frac{1}{2}}V_\epsilon^{(m_j)}N^{-\frac{1}{2}}\| \leq C\epsilon^{1-m_j},$$

$$\|\langle S\rangle_{\epsilon,\rho_1}^{-\mu}G_\epsilon(z)N^{\frac{1}{2}}\| \leq C\epsilon^{-\frac{1}{2}}, \qquad \|N^{\frac{1}{2}}G_{\mathrm{fr},\epsilon}(z)\langle S\rangle_{\epsilon,\rho_2}^{-\mu}\| \leq C\epsilon^{-\frac{1}{2}},$$

$$\|N^{\frac{1}{2}}G_\epsilon(z)N^{\frac{1}{2}}\| \le C\epsilon^{-1}, \qquad \|N^{\frac{1}{2}}G_{\mathrm{fr},\epsilon}(z)N^{\frac{1}{2}}\| \le C\epsilon^{-1},$$

we see that (7.160) can be estimated by $|\lambda|\epsilon^{-m}$. This shows

$$\sup_{z\in\mathbf{C}_+} \left\| \langle S\rangle_{\epsilon,\rho_1}^{-\mu} \left(\partial_\epsilon^m (G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))\right) \langle S\rangle_{\epsilon,\rho_2}^{-\mu} \right\| \le C|\lambda|\epsilon^{-m-1}. \tag{7.161}$$

Next we note that

$$\partial_\epsilon^k \langle S\rangle_{\epsilon,\rho}^{-\mu} (G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z)) \langle S\rangle_{\epsilon,\rho}^{-\mu},$$

is a linear combination of the terms

$$S^{k_1}\langle S\rangle_{\epsilon,\rho^{(k_1)}}^{-\mu} (\partial_\epsilon^{k_2}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))) \langle S\rangle_{\epsilon,\rho^{(k_3)}}^{-\mu} S^{k_3}, \tag{7.162}$$

where $k_1+k_2+k_3 = k$. Then we write $\rho^{(k_i)} = \gamma_i\tilde{\gamma}_i$, $i = 1, 2$ for some Schwartz functions $\gamma_i\tilde{\gamma}_i$ and we rewrite (7.162) as

$$\tilde{\gamma}_1(\epsilon S) S^{k_1}\langle S\rangle_{\epsilon,\gamma_1}^{-\mu} (\partial_\epsilon^{k_2}(G_\epsilon(z) - G_{\mathrm{fr},\epsilon}(z))) \langle S\rangle_{\epsilon,\gamma_2}^{-\mu} S^{k_3}\tilde{\gamma}_2(\epsilon S). \tag{7.163}$$

Now (7.161) combined with (7.146) yields the statement. □

We are now ready to finish

Proof of Theorem 7.20. Let n be the integer such that $n+1 > \nu$, and let ρ be a fixed Schwartz function such that $\rho(0) = 1$. We will use the shorthand

$$R(\epsilon) = \langle S\rangle_{\epsilon,\rho}^{-\mu} G_\epsilon(z) \langle S\rangle_{\epsilon,\rho}^{-\mu}.$$

It follows from Lemma 7.18 and the choice of n and μ that

$$\sup_{z\in\mathbf{C}_+} \|\partial_\epsilon^n R(\epsilon)\| \le C\epsilon^{-n-1+\nu}. \tag{7.164}$$

Furthermore, it follows from Theorem 7.14 and Taylor's formula that for any $\epsilon > 0$

$$R(0) = \sum_{k=0}^{n-1} (-1)^k \frac{\epsilon^k}{k!} \partial_\epsilon^k R(\epsilon) + \frac{(-1)^n}{(n-1)!} \int_0^\epsilon (\epsilon-\tau)^{n-1} \partial_\tau^n R(\tau) \mathrm{d}\tau.$$

(This formula is derived using Taylor's expansion of the function $R(\epsilon-\delta)$ in the variable δ and then taking $\delta \uparrow \epsilon$.) Setting $\lambda = 0$, we get a similar expansion for $R_{\mathrm{fr}}(0)$:

$$R_{\mathrm{fr}}(0) = \sum_{k=0}^{n-1} (-1)^k \frac{\epsilon^k}{k!} \partial_\epsilon^k R_{\mathrm{fr}}(\epsilon) + \frac{(-1)^n}{(n-1)!} \int_0^\epsilon (\epsilon-\tau)^{n-1} \partial_\tau^n R_{\mathrm{fr}}(\tau) \mathrm{d}\tau.$$

It follows from (7.164) that the error terms in both expansions are estimated by $C\epsilon^{\nu-1}$. Combining this estimate with Lemma 7.21 we derive that

$$\begin{aligned} \|R(0) - R_{\mathrm{fr}}(0)\| &\le \textstyle\sum_{k=0}^{n-1} \frac{\epsilon^k}{k!} \|\partial_\epsilon^k (R(\epsilon) - R_{\mathrm{fr}}(\epsilon))\| + C\epsilon^{\nu-1} \\ &\le C(|\lambda|\epsilon^{-1} + \epsilon^{\nu-1}). \end{aligned} \tag{7.165}$$

This estimate is optimized for $\epsilon = (|\lambda|/\nu - 1)^{1/\nu}$. Substituting this value into (7.165) we complete the proof of Theorem 7.20. □

8 Proofs of the main theorems

Throughout this chapter we assume that Hypotheses A and $S(\nu)$ hold with $\nu > 1$ and that $n \in \mathbf{N}$, $0 < \theta \leq 1$ and $\mu > \frac{1}{2}$ satisfy $\nu \geq \mu + \frac{1}{2} = 1 + n + \theta$.
Notation. In this and the next section we adopt the following shorthand. Let I be an interval, $\Omega \subset \mathbf{C}$, $\Omega \times I \ni (z, \lambda) \mapsto A_\lambda(z)$ an operator-valued function, and $f(\lambda)$ a positive function on I. We will write $A_\lambda(z) = O(f(\lambda))$ *for* $z \in \Omega$ if there exist constant C such that $\forall (z, \lambda) \in \Omega \times I$, $\|A_\lambda(z)\| \leq C f(\lambda)$. As customary, we will suppress the variable λ in the operator-valued functions, and write $A(z)$ for $A_\lambda(z)$, etc.

8.1 Proof of Limiting Absorption Principle away from $\sigma(K)$

In this section we prove Theorem 6.2. We fix $\Lambda_1 > 0$ such that $\Lambda_1 < (\sqrt{2}\|s\alpha\|)^{-1}$.

In what follows we assume that $|\lambda| \leq \Lambda_1$. Recall that the self-energy $W_{\rm v}(z)$ and the resonance function $G_{\rm v}(z)$ are defined by (3.41).

Lemma 8.1 *The function $\mathbf{C}_+ \ni z \mapsto W_{\rm v}(z)$ extends by continuity to $\overline{\mathbf{C}}_+$ and is of the class $C^{n,\theta}_{\rm u}(\overline{\mathbf{C}}_+)$ uniformly in λ. Furthermore, there exist β_1 such that*

$$\sup \lambda^{-2} \|W_{\rm v}(z)\| < \beta_1. \tag{8.166}$$

where the supremum is taken over $|\lambda| \leq \Lambda_1$ and $z \in \overline{\mathbf{C}}_+$.

Proof. Since

$$W_{\rm v}(z) = \lambda^2 \varphi(\alpha)^{\overline{\rm vv}} \langle S^{\overline{\rm vv}} \rangle^{\mu} \left(\langle S^{\overline{\rm vv}} \rangle^{-\mu} (z \mathbf{1}^{\overline{\rm vv}} - H^{\overline{\rm vv}})^{-1} \langle S^{\overline{\rm vv}} \rangle^{-\mu} \right) \langle S^{\overline{\rm vv}} \rangle^{\mu} \varphi(\alpha)^{\overline{\rm v}{\rm v}},$$

the result follows from Theorem 7.14. □

An immediate consequence of the previous lemma is that for $z \in \overline{\mathbf{C}}_+$, $G_{\rm v}(z)$ is a well-defined closed operator with domain $\mathcal{D}(K)$.

Lemma 8.2 *The operators $G_{\rm v}(z)$ are invertible for z in $\overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2 \beta_1)$ and the function $G_{\rm v}^{-1}(z)$ is of the class $C^{n,\theta}_{\rm u}$ of this set.*

Proof. Since $G_{\rm v}(z) = z\mathbf{1}^{\rm vv} - K - W_{\rm v}(z)$, the estimate (8.166) and Proposition 2.8 yield that $G_{\rm v}(z)$ is invertible and

$$\|G_{\rm v}^{-1}(z)\| = O(\lambda^{-2}). \tag{8.167}$$

for $z \in \overline{\mathbf{C}}_+ \setminus B(\sigma(K), \lambda^2 \beta_1)$. The regularity properties of $G_{\rm v}(z)$ are inferred by induction from the identity

$$G_{\rm v}^{-1}(z_1) - G_{\rm v}^{-1}(z_2) = G_{\rm v}^{-1}(z_1) \left((z_2 - z_1) \mathbf{1}^{\rm vv} - (W_{\rm v}(z_2) - W_{\rm v}(z_1)) \right) G_{\rm v}^{-1}(z_2), \tag{8.168}$$

and Lemma 8.1. □

Proof of Theorem 6.2. The theorem follows from Theorem 7.14 and Lemma 8.2 after sandwiching the Feshbach formula

$$\begin{aligned}(z-H)^{-1} &= (z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\\ &\quad+(\mathbf{1}^{\mathrm{v}\mathrm{v}}+(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}})G_{\mathrm{v}}^{-1}(z)(H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}+\mathbf{1}^{\mathrm{v}\mathrm{v}}),\end{aligned} \tag{8.169}$$

with $\langle S\rangle^{-\mu}$, and after inserting $\langle S\rangle^{-\mu}\langle S\rangle^{\mu}$ in front of (after) $H^{\overline{\mathrm{v}}\mathrm{v}}$ ($H^{\mathrm{v}\overline{\mathrm{v}}}$). (Note that $\langle S\rangle^{-\mu}=\mathbf{1}^{\mathrm{v}\mathrm{v}}\oplus\langle S^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\rangle^{-\mu}$.) □

In the next two sections, similar elementary arguments based on the insertion of various powers of $\langle S\rangle$ at appropriate places will be skipped.

8.2 Proof of Limiting Absorption Principle around $k\in\sigma(K)$

In this section we prove Theorem 6.3. We introduce new splittings of the Hilbert space $\mathcal{H}$,

$$\mathcal{H}=\mathcal{H}^{k}\oplus\mathcal{H}^{\overline{k}}=\mathcal{H}^{k}\oplus\mathcal{H}^{\underline{k}}\oplus\mathcal{H}^{\overline{\mathrm{v}}}, \tag{8.170}$$

where

$$\mathcal{H}^{k}:=\mathrm{Ran}p_k,\quad \mathcal{H}^{\overline{k}}:=\mathrm{Ran}(\mathbf{1}-p_k),\quad \mathcal{H}^{\underline{k}}:=\mathcal{H}^{\overline{k}}\cap\mathcal{H}^{\mathrm{v}}.$$

In our argument we will apply several times the Feshbach formula with respect to these decompositions. To that end we introduce some additional notation. The matrix form of the operator H with respect to the decomposition $\mathcal{H}=\mathcal{H}^{k}\oplus\mathcal{H}^{\overline{k}}$ is denoted by

$$H=\begin{bmatrix} H^{kk} & H^{k\overline{k}}\\ H^{\overline{k}k} & H^{\overline{k}\overline{k}}\end{bmatrix}. \tag{8.171}$$

The operator $H^{\overline{k}\overline{k}}$ acts on $\mathcal{H}^{\underline{k}}\oplus\mathcal{H}^{\overline{\mathrm{v}}}$, and its matrix form is denoted by

$$H^{\overline{k}\overline{k}}=\begin{bmatrix} H^{\underline{k}\underline{k}} & H^{\underline{k}\overline{\mathrm{v}}}\\ H^{\overline{\mathrm{v}}\underline{k}} & H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\end{bmatrix}. \tag{8.172}$$

(Arguing as in the beginning of Chapter 6 one easily shows that these matrix representations are well-defined and that the formalism and results of Chapter 3 can be applied.) We employ the same notation for other operators. Note that $p_k=\mathbf{1}^{kk}$. Note also that

$$H^{\overline{k}k}=H^{\overline{\mathrm{v}}k},\qquad H^{k\overline{k}}=H^{k\overline{\mathrm{v}}}. \tag{8.173}$$

For $z\in\overline{\mathbf{C}}_{+}$ we set

$$\begin{aligned}W_{\underline{k}}(z)&:=H^{\underline{k}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}H^{\overline{\mathrm{v}}\underline{k}},\\ G_{\underline{k}}(z)&:=z\mathbf{1}^{\underline{k}\underline{k}}-H^{\underline{k}\underline{k}}-W_{\underline{k}}(z).\end{aligned} \tag{8.174}$$

Note that if $W_{\rm v}(z)$ and $G_{\rm v}(z)$ are the usual self-energy and resonance function, then

$$W_{\underline{k}}(z) = W_{\rm v}^{\underline{k}\underline{k}}(z), \qquad G_{\underline{k}}(z) = G_{\rm v}^{\underline{k}\underline{k}}(z). \tag{8.175}$$

In the next lemma we assume that $|\lambda| < \Lambda_1$.

Lemma 8.3 *The function $W_{\underline{k}}(z)$ belongs to $C_{\rm u}^{n,\theta}(\overline{\mathbf{C}}_+)$ uniformly in λ and we have*

$$\sup \lambda^{-2} \|W_{\underline{k}}(z)\| < \beta_1, \tag{8.176}$$

where the supremum is taken over $|\lambda| \leq \Lambda_1$ and $z \in \overline{\mathbf{C}}_+$.

Proof. We apply Lemma 8.1 and (8.175). □

Let $\delta_1 > 0$ be such that

$$\delta_1 < \operatorname{dist}(k, \sigma(K) \setminus \{k\}).$$

We choose $\Lambda_2 > 0$ such that

$$\Lambda_2 \leq \Lambda_1, \qquad \beta_1 \Lambda_2^2 < \delta_1. \tag{8.177}$$

Until the end of this section we assume that $|\lambda| \leq \Lambda_2$.

Lemma 8.4 *The operators $G_{\underline{k}}(z)$ are invertible for $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ and the function $G_{\underline{k}}^{-1}(z)$ is in the class $C_{\rm u}^{n,\theta}$ of this set uniformly in λ.*

Proof. The invertibility of $G_{\underline{k}}(z)$ follows from Proposition 2.8, Lemma 8.3 and the choice of Λ_2. The regularity properties of $G_{\underline{k}}^{-1}(z)$ follow by induction from Lemma 8.3 and an identity similar to (8.169). □

For $z \in \mathbf{C}_+$ we set

$$\begin{aligned} W_k(z) &:= H^{k\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}k}, \\ G_k(z) &:= z\mathbf{1}^{kk} - H^{kk} - W_k(z). \end{aligned} \tag{8.178}$$

Lemma 8.5 *The function*

$$\mathbf{C}_+ \ni z \mapsto \langle S \rangle^{-\mu} (z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} \langle S \rangle^{-\mu}, \tag{8.179}$$

extends by continuity to $\overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ and is in the class $C_{\rm u}^{n,\theta}$ of this set uniformly in λ. The same result holds for the function $W_k(z)$.

Proof. The Feshbach formula yields that for $z \in \mathbf{C}_+$,

$$\begin{aligned}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} &= (z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1} \\ &\quad +(\mathbf{1}^{\underline{kk}} + (z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}H^{\overline{\vee}\underline{k}})G_{\underline{k}}^{-1}(z)(\mathbf{1}^{\underline{kk}} + H^{\underline{k}\overline{\vee}}(z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}).\end{aligned} \tag{8.180}$$

We derive the result sandwiching this identity with $\langle S\rangle^{-\mu}$ and invoking the previous lemma and Theorem 7.14. □

We will now make use of the Feshbach formula with respect to the decomposition $\mathcal{H} = \mathcal{H}^k \oplus \mathcal{H}^{\overline{k}}$. Note that in the notation we have adopted, $w^{kk}(z) = p_k w(z) p_k$. In the sequel we set $\kappa := (\nu - 1)/\nu$. Recall that in (6.89) we defined

$$w_k := w^{kk}(k + \mathrm{i}0).$$

Lemma 8.6 *There is a constant β_2 such that if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$, then*

$$\sup |\lambda|^{-2-\kappa}\|W_k(z) - \lambda^2 w_k\| < \beta_2, \tag{8.181}$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$, $|\lambda| \le \Lambda_2$.

Proof. It follows from (8.180) that

$$\begin{aligned}W_k(z) &= H^{k\overline{k}}(z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}H^{\overline{k}k} \\ &\quad +H^{k\overline{k}}(z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}H^{\overline{\vee}\underline{k}}G_{\underline{k}}^{-1}(z)H^{\underline{k}\overline{\vee}}(z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}H^{\overline{k}k}.\end{aligned} \tag{8.182}$$

Since $\beta_1\Lambda_2^2 < \delta_1$, Lemma 8.4 yields that for $z \in \mathbf{C}_+ \cap \overline{B}(k, \lambda^2\beta_1)$ the second term on the right-hand side is $O(\lambda^4)$. It follows from Theorem 7.20 that the first term equals

$$H^{k\overline{k}}(z\mathbf{1}^{\overline{\vee\vee}} - H^{\overline{\vee\vee}})^{-1}H^{\overline{k}k} = H^{k\overline{k}}(z\mathbf{1}^{\overline{\vee\vee}} - H_{\mathrm{fr}}^{\overline{\vee\vee}})^{-1}H^{\overline{k}k} + O(|\lambda|^{2+\kappa}).$$

Recall that in (2.27) we defined the function $\ell_\theta(\tau)$. We extend the definition of this function for $\theta \in [0, \infty[$ as follows:

$$\ell_\theta(\tau) := \begin{cases} \tau^\theta, & 0 < \theta < 1 \\ \tau\,(1 + \ln(1 + \tau^{-1})) & \theta = 1, \\ \tau, & \theta > 1. \end{cases}$$

Then, by Theorem 7.19,

$$H^{k\overline{k}}(z\mathbf{1}^{\overline{\vee\vee}} - H_{\mathrm{fr}}^{\overline{\vee\vee}})^{-1}H^{\overline{k}k} = \lambda^2 w^{kk}(z) = \lambda^2 w_k^{kk} + R(z),$$

where

$$\begin{aligned}\|R(z)\| &\le C_1\lambda^2\ell_{\nu-\frac{1}{2}}(|z-k|) \\ &\le C_2\lambda^2\ell_{\nu-\frac{1}{2}}(\beta_1\lambda^2) \le C_3|\lambda|^{2+\kappa},\end{aligned}$$

where we used $2\nu - 1 > \kappa$ and $2 > \kappa$ in the last step. □

The operator w_k is dissipative and by the assumption,

$$\mathcal{T}_k = \sigma(w_k) \cap \mathbf{R} \subset \sigma_{\rm disc}(w_k).$$

Therefore, Propositions 3.3 and 3.4 can be applied. Let

$$c := \sup_{z \in \overline{\mathbf{C}}_+} \|(z - w_k)^{-1} \mathbf{1}_{\sigma(w_k) \setminus \mathbf{R}}(w_k)\|, \tag{8.183}$$

(which is the constant c in Proposition 3.3 applied to w_k). In addition to (8.177), in the sequel we assume that Λ_2 satisfies

$$\beta_2 \Lambda_2^\kappa c < 1. \tag{8.184}$$

Lemma 8.7 *The operators $G_k(z)$ are invertible for*

$$z \in \overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2 \beta_1) \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2) \right),$$

and the function $G_k^{-1}(z)$ is in the class $C_{\rm u}^{n,\theta}$ of this set.

Proof. It follows from the definition of $G_k(z)$ and Lemma 8.6 that if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2 \beta_1)$, then

$$\begin{aligned} G_k(z) &= (z - k)\mathbf{1}^{kk} - W_k(z) \\ &= (z - k)\mathbf{1}^{kk} - \lambda^2 w_k + R(z) \\ &= \lambda^2 \left(\lambda^{-2}(z - k)\mathbf{1}^{kk} - w_k + \lambda^{-2} R(z) \right), \end{aligned} \tag{8.185}$$

where $\|R(z)\| < \beta_2 |\lambda|^{2+\kappa}$. If c is defined by (8.183), then $\|\lambda^{-2} R(z)\| c < 1$ and

$$\sup \operatorname{dist}(\sigma(\lambda^{-2} k + w_k) \cap \mathbf{R}, \lambda^{-2} z) > |\lambda|^\kappa \beta_2,$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2)$. Therefore, it follows from Proposition 3.4 that $G_k(z)$ is invertible for

$$z \in \overline{\mathbf{C}}_+ \cap \left(\overline{B}(k, \lambda^2 \beta_1) \setminus B(k + \lambda^2 \mathcal{T}_k, |\lambda|^{2+\kappa} \beta_2) \right).$$

The regularity properties of $G_k^{-1}(z)$ are inferred from Lemma 8.5 and an identity similar to (8.169). □

Proof of Theorem 6.3. Since (ii) ⇒ (i), we have only to prove (ii). We choose Λ_2 such that (8.177) and (8.184) hold. The Feshbach formula yields

$$\begin{aligned} (z - H)^{-1} &= (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} \\ &\quad + (\mathbf{1}^{kk} + (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} H^{\overline{kk}}) G_k^{-1}(z) (\mathbf{1}^{kk} + H^{k\overline{k}} (z \mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}). \end{aligned}$$

Sandwiching this formula with $\langle S \rangle^{-\mu}$, we derive (ii) from Lemmas 8.5 and 8.7. □

8.3 Proof of Limiting Absorption Principle around $k + \lambda^2 m$

In this section we prove Theorem 6.4. We will freely use the notation and the results of the previous section. We will indicate the place in our argument where we require that Hypothesis $S(\nu)$ with $\nu > 2$ holds.

In addition to (8.170) we introduce the following splittings of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}^m \oplus \mathcal{H}^{\overline{m}} = \mathcal{H}^m \oplus \mathcal{H}^{\underline{m}} \oplus \mathcal{H}^{\overline{k}},$$

where

$$\mathcal{H}^m := \mathrm{Ran} p_{k,m}, \quad \mathcal{H}^{\overline{m}} := \mathrm{Ran}(\mathbf{1} - p_{k,m}), \quad \mathcal{H}^{\underline{m}} := \mathcal{H}^k \cap \mathcal{H}^{\overline{m}}.$$

Note that by the definition of $p_{k,m}$, $\mathcal{H}^m \subset \mathcal{H}^k$, so the above splittings are well-defined. Note also that

$$H^{\overline{m}m} = H^{\overline{k}m} = H^{\overline{\nu}m}, \qquad H^{m\overline{m}} = H^{m\overline{k}} = H^{m\overline{\nu}}. \tag{8.186}$$

For these splittings we adopt the notation analogous to (8.171) and (8.172).

For $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \delta_1)$ we set

$$W_{\underline{m}}(z) := H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}\underline{m}},$$

$$G_{\underline{m}}(z) := z\mathbf{1}^{\underline{m}\underline{m}} - H^{\underline{m}\underline{m}} - W_{\underline{m}}(z).$$

Note that if $W_k(z)$ and $G_k(z)$ are given by (8.178) then

$$W_{\underline{m}}(z) = W_k^{\underline{m}\underline{m}}(z), \qquad G_{\underline{m}}(z) = G_k^{\underline{m}\underline{m}}(z). \tag{8.187}$$

We assume that $|\lambda| \leq \Lambda_2$.

Lemma 8.8 *Assume that $z \in \overline{\mathbf{C}}_+ \cap B(k, \lambda^2\beta_1)$. Then*

$$\begin{aligned} H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}m} &= O(|\lambda|^{2+\kappa}), \\ H^{m\overline{k}}(z\mathbf{1}^{\overline{k}\overline{k}} - H^{\overline{k}\overline{k}})^{-1} H^{\overline{k}\underline{m}} &= O(|\lambda|^{2+\kappa}). \end{aligned} \tag{8.188}$$

Moreover, we have

$$\sup |\lambda|^{-2-\kappa} \|W_{\underline{m}}(z) - \lambda^2 w_k^{\underline{m}\underline{m}}\| < \beta_2, \tag{8.189}$$

where the supremum is taken over $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$, $|\lambda| \leq \Lambda_2$.

Proof. To prove (8.188) we use $w_k^{\underline{m}\underline{m}} = (p_k - p_{k,m}) w_k p_{k,m} = 0$ and Lemma 8.6. (8.189) follows from Relation (8.187) and Lemma 8.6. □

Let

$$\delta_2 < \mathrm{dist}\left(\{m\}, \sigma(w_k) \setminus \{m\}\right).$$

We introduce $\Lambda_3 > 0$ such that

$$\Lambda_3 \leq \Lambda_2, \qquad \beta_2 \Lambda_3^\kappa \leq \delta_2. \tag{8.190}$$

¿From now on we assume that $|\lambda| \leq \Lambda_3$.

Lemma 8.9 *For $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$ the operators $G_{\underline{m}}(z)$ are invertible and the function $G_{\underline{m}}^{-1}(z)$ is in the class $C_{\mathrm{u}}^{n,\theta}$ of this set. Furthermore,*

$$\frac{\mathrm{d}^j}{\mathrm{d}z^j} G_{\underline{m}}^{-1}(z) = O(\lambda^{-2j-2}), \quad j = 0, \dots, n. \tag{8.191}$$

Proof. Clearly,

$$\begin{aligned} G_{\underline{m}}(z) &= (z-k)\mathbf{1}^{\underline{mm}} - W_{\underline{m}}(z) \\ &= (z-k)\mathbf{1}^{\underline{mm}} - \lambda^2 w_k^{\underline{mm}} + R^{\underline{mm}}(z) \\ &= \lambda^2 \left(\lambda^{-2}(z-k)\mathbf{1}^{\underline{mm}} - w_k^{\underline{mm}} + \lambda^{-2} R^{\underline{mm}}(z)\right), \end{aligned}$$

where $R(z) = W_k(z) - \lambda^2 w_k$ is the same as in (8.185). By Lemma 8.6,

$$\|R^{\underline{mm}}(z)\| \le \|R(z)\| \le \beta_2 |\lambda|^{2+\kappa},$$

if $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1)$. Moreover,

$$w_k^{\underline{mm}} \mathbf{1}_{\sigma(w_k^{\underline{mm}}) \setminus \mathbf{R}}(w_k^{\underline{mm}}) = w_k \mathbf{1}_{\sigma(w_k) \setminus \mathbf{R}}(w_k).$$

Therefore,

$$c = \sup_{z \in \overline{\mathbf{C}}_+} \|(z - w_k^{\underline{mm}})^{-1} \mathbf{1}_{\sigma(w_k^{\underline{mm}}) \setminus \mathbf{R}}(w_k^{\underline{mm}})\|$$

is the same as in (8.183). Thus, we can apply Proposition 3.4, which implies that $G_{\underline{m}}(z)$ is invertible and satisfies the bound (8.191) with $j = 0$.

The regularity properties of $G_{\underline{m}}^{-1}(z)$ are inferred from the formula analogous to (8.169) and Lemma 8.5. The bound (8.191) for $j = 0$ and induction yield (8.191) for all j. □

For $z \in \mathbf{C}_+$ we set

$$\begin{aligned} W_m(z) &:= H^{m\overline{m}}(z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} H^{\overline{m}m}, \\ G_m(z) &:= z\mathbf{1}^{mm} - H^{mm} - W_m(z). \end{aligned}$$

Lemma 8.10 *The function*

$$\mathbf{C}_+ \ni z \mapsto \langle S \rangle^{-\mu} (z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} \langle S \rangle^{-\mu},$$

extends by continuity to $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$ and is in the class $C_{\mathrm{u}}^{n,\theta}$ of this set. The same result holds for the function $W_m(z)$.

Proof. The Feshbach formula yields

$$\begin{aligned}(z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} &= (z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1} \\ &\quad +(\mathbf{1}^{\underline{mm}} + (z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{k}\underline{m}})G_{\underline{m}}^{-1}(z)(z\mathbf{1}^{\underline{mm}} + H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}),\end{aligned}$$

and the result follows from Lemmas 8.6 and 8.8. □

Lemma 8.11 *Assume that Hypothesis $S(\nu)$ holds with $\nu > 2$. There exist a constant γ such that for $z \in \overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k + \lambda^2 m, \lambda^{2+\kappa}\beta_2)$*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}z} W_m(z) \right\| \leq \gamma |\lambda|^{2\kappa}. \tag{8.192}$$

Proof. It follows from the Feshbach formula that

$$\begin{aligned}W_m(z) &= H^{m\overline{m}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{m}m} \\ &\quad +H^{m\overline{m}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{k}\underline{m}}G_{\underline{m}}^{-1}(z)H^{\underline{m}\overline{k}}(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}H^{\overline{m}m}.\end{aligned} \tag{8.193}$$

The derivative of the first term in (8.193) is $O(\lambda^2)$ by Lemma 8.8. When we differentiate the second term and the derivative hits $(z\mathbf{1}^{\overline{kk}} - H^{\overline{kk}})^{-1}$, then we get by Lemmas 8.8, 8.9 and (8.191)

$$O(\lambda^2)O(\lambda^{-2})O(|\lambda|^{2+\kappa}) = O(|\lambda|^{2+\kappa}).$$

When the derivative hits $G_{\underline{m}}^{-1}(z)$, then we get by the same lemmas and (8.191)

$$O(|\lambda|^{2+\kappa})O(\lambda^{-4})O(|\lambda|^{2+\kappa}) = O(|\lambda|^{2\kappa}).$$

Then we use $2\kappa \leq 2 + \kappa$. □

We are now ready to finish

Proof of Theorem 6.4. Let γ be given by (8.192). Recall that so far Λ_3 has to satisfy (8.190). We demand in addition that

$$\Lambda_3^{2\kappa} < 1/\gamma. \tag{8.194}$$

Assumption (8.194) and Lemma 8.11 imply that

$$\sup_{x \in \Theta(k,m)} \|W_m'(x + \mathrm{i}0)\| < 1.$$

Thus, the conditions of Corollary 3.13 are satisfied on the interval $\Theta(k, m)$ with respect to the decomposition $\mathcal{H} = \mathcal{H}^m \oplus \mathcal{H}^{\overline{m}}$. By Corollary 3.13,

$$\dim \mathbf{1}^{\mathrm{pp}}_{\Theta(k,m)} \leq \dim \mathcal{H}^m = \dim p_{k,m}.$$

This completes the proof of the Part (i).

Since (iii) $\Rightarrow$ (ii), it remains to prove (iii). First note that,by Lemma 8.10, $\Theta(k,m) \cap \sigma_{\rm pp}(H^{mm}) = \emptyset$. Therefore, by Proposition 3.7 and Theorem 3.8, $\sigma_{\rm pp}(H) \cap \Theta(k,m)$ coincides with $\{x \in \Theta(k,m) \ : \ 0 \in \sigma(G_m(x+{\rm i}0))\}$. Since $G_m(z)$ is a continuous function and $\overline{\mathbf{C}}_+ \cap \overline{B}(k, \lambda^2\beta_1) \cap \overline{B}(k+\lambda^2 m, \beta_2\lambda^{2+\kappa}) \setminus B(\sigma_{\rm pp}(H), \epsilon)$ is compact,

$$\|G_m^{-1}(z)\| \leq C.$$

on this set. A formula analogous to (8.169) and Lemma 8.10 yield that $G_m^{-1}(z) \in C_{\rm u}^{n,\theta}$ of this set.

The Feshbach formula yields that for $z \in \mathbf{C}_+$,

$$\begin{aligned}(z-H)^{-1} \ &= (z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1} \\ &\quad +(\mathbf{1}^{mm} + (z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1}H^{\overline{m}m})G_m^{-1}(z)(\mathbf{1}^{mm} + H^{m\overline{m}}(z\mathbf{1}^{\overline{mm}} - H^{\overline{mm}})^{-1}).\end{aligned}$$

Sandwiching this formula with $\langle S\rangle^{-\mu}$, we derive (iii) from Lemma 8.10 and the regularity properties of $G_m^{-1}(z)$ proven above. The proof of Theorem 6.4 is complete. $\Box$

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