Quantum Scattering for homogeneous of degree zero potentials: Absence of channels at local maxima and saddle points.

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1 Introduction

In this paper we consider Hamiltonians $H = 2^{-1}p^2 + V_0, p = -i\nabla$, on $L^2(\mathbf{R}^n_z)$, $n \geq 2$, where V_0 is assumed to be real-valued, smooth outside zero and homogeneous of degree zero. We do include perturbations from a large class of compactly supported potentials, but for simplicity of presentation we shall only discuss the strictly homogeneous case in this introduction. In the same spirit let us assume that the set of critical points in S^{n-1} for V_0, denoted $\mathcal{C}_0$, is finite. Then it was proved by one of us (I.H.) that

$$(1.1) \qquad I \;=\; \sum_{z_j \in \mathcal{C}_0} E_j^+,$$

where $E_j^+ = E_j^+(H)$ are projections defined as follows: Pick any family $\{\chi_j | z_j \in \mathcal{C}_0\}$ of smooth functions on S^{n-1} with $\chi_j(z_i) = \delta_{ij}$ (the Kronecker symbol). Then

$$E_j^+ = s - \lim_{t \to +\infty} e^{itH} \chi_j(\hat{z}) e^{-itH}; \; \hat{z} = \frac{z}{|z|},$$

see [He] and [ACH]. Furthermore it was proved that there exists an asymptotic momentum p^+ and its relationship to the above projections was shown.

This paper is devoted to the following vanishing problem: Suppose $z_j \in \mathcal{C}_0$ corresponds to a local maximum or a saddle point. Show (possibly together with other conditions) that then

$$(1.2) \qquad E_j^+ \;=\; 0.$$

Although there is a simple physical intuition for (1.2) (see below) this problem turns out to be rather difficult even in the simplest possible case. While we obtain a fairly satisfactory result for local maxima (including some degenerate cases) our conditions for saddle points are somewhat restrictive. One could suspect that the latter fact is due to technical obstacles only.

Another natural problem motivated by (1.1) is to analyse the evolution for "open channels", that is at local minima. Is there a "simple" comparison dynamics for which one can show the existence of the Møller wave operator and completeness? We devote this problem to another paper, [HS].

To explain some ideas of our proof yielding (1.2) let us recall that (1.1) relied on the following weak type estimate

$$(1.3) \qquad \int_0^\infty \langle f(H)\psi, |z||\nabla V_0(z)|^2 f(H)\psi \rangle dt \leq C_f ||\psi||^2; \; f \in C_0^\infty,$$

[He, Theorem 3.6]. Suppose for simplicity that $z_0 \in \mathcal{C}_0$ corresponds to a non-degenerate local maximum. Let us split the coordinates as $z = xz_0 \oplus y$. In conjunction with the trivial bound $|\partial_x V_0(z)| \leq C|z|\,|\nabla V_0(z)|^2$ valid in a conical neighbourhood of z_0 (1.3) clearly indicates that the momentum $\xi = -i\frac{\partial}{\partial x}$ approaches a limit along a state starting at $t = 0$ in $f(H)\psi \in E_j^+ L^2$. Standard two-body techniques show that $x \approx t\xi$ along such state. So we expect that the dynamics can be approximated by a simpler one, namely the one generated by time-dependent Hamiltonians given by replacing x by $t\xi$ in the potential (or part of the potential). Although we only accomplish this in a very weak sense let us for the moment look at the following model Hamiltonians:

$$H(t) = 2^{-1}\big(\eta^2 + \xi^2\big) + \tfrac{y\cdot\nabla_y^2 V_0(z_0)y}{2t^2\xi^2};\ \eta = -i\nabla_y,\ t > 1. \tag{1.4}$$

Do there exist states for the model dynamics with $y = o(t)$ for $t \to +\infty$? The answer is negative as we want. It can readily be seen in this case by a little Fourier analysis, but to gain some intuition, it is fruitful to look at the corresponding classical problem:

Since ξ is preserved and the Hessian is negative (making the potential repulsive) we need to solve the Hamilton equations for $h_l(y_l) = 2^{-1}\eta_l^2 + q_l\frac{y_l^2}{2t^2},\ q_l < 0$. The solutions are $y_l = c^- t^{\alpha_l^-} + c^+ t^{\alpha_l^+};\ \alpha_l^- = 2^{-1}\big(1 - \sqrt{1-4q_l}\big),\ \alpha_l^+ = 2^{-1}\big(1 + \sqrt{1-4q_l}\big)$. Since $\alpha_l^+ > 1$ and we want $y_l = o(t)$ for $t \to +\infty$ we infer that $c^+ = 0$. On the other hand $t^{\alpha_l^-} = o\big(t^0\big)$ certainly is a sublinear solution.

Returning to the quantum case we notice that the above classical orbits are confined to a too small region in phase space to support a state: *The uncertainty principle* should exclude the existence of such states.

To implement the above ideas it is important to establish a better bound on y along the evolution of an "exceptional state" than just $y = o(t)$ for $t \to +\infty$. A similar problem occurs in Dereziński's proof of completeness for long-range N-body Hamiltonians, [D]. Since we are dealing with a repulsive potential (w.r.t. y) it turns out that we can adapt the method used by Dereziński to our case (even though our potential is extremely long-range). In this way we obtain $y = O\big(t^{1-\nu}\big)$ for some small $\nu > 0$.

In the saddle point case we can get this bound for the part of y that corresponds to negative eigenvalues of the Hessian. We can then derive a similar bound on the other part of y as well as on the difference $x - t\xi$ (the latter is needed for local maxima as well) by a differential inequality method. This method may be considered as a quantum version of the method of Liapunov known from the well-known theory of stability near fixed points of dynamical systems.

Putting these bounds together we have localized the evolution to a somewhat smaller region, than the definition of E_j^+ a priori prescribes. The next step is to change the potential outside this region as to obtain modifications of (1.4) essentially on the form

$$H(t) = 2^{-1}\big(\eta^2 + \xi^2\big) + \tfrac{y\cdot\nabla_y^2 V_0(z_0)y}{2t^2\xi^2} + R(t);\ ||R(t)|| = O\big(t^{-2\nu}\big),\ t > 1. \tag{1.5}$$

It should be remarked that the "error" that appears on the right hand side of (1.5) depends on x. Consequently ξ is not preserved for the evolution generated by the family (1.5). The approximate evolution is shown to be good enough for constructing the relative Møller wave operator. Although we show the existence of this operator by a well-known technique in scattering theory it is obtained only by means of complicated and even some delicate estimates. In conclusion it suffices to show that any nonzero state evolving in accordance with the dynamics for (1.5) obeys

$$(1.6)\qquad y \neq O\big(t^{1-\epsilon}\big) \text{ for } t \to +\infty \text{ for any } \epsilon > 0.$$

The next step is to change coordinates in a time-dependent manner to obtain essentially time-independent quadratic Hamiltonians with small time-dependent perturbations. The analysis of propagation properties for the corresponding evolution is the most interesting part of this paper. First we need to implement the uncertainty principle. For that we apply a global Mourre estimate (based on the repulsiveness of the leading part of the potential) in the extended Hilbert space in which t is considered as a space-variable, cf. [Ho]. The estimate is used in a singular variant of Mourre's method, cf. [Sk1]. Noticing that the resulting global smoothness for a certain observable yields smoothness in the previous picture we obtain in this way a basic tool for setting up a good calculus: We construct appropriate "propagation observables" yielding in the end a propagation estimate that implies (1.6) (cf. the precise statement (1.8) given below).

The result of the last step of this paper may have some independent interest. A modification of it reads:

Consider on the space $L^2\big(\mathbf{R}^m_y\big)$ a family of time-dependent Hamiltonians given by

$$(1.7)\qquad \begin{aligned} &H(t) = -2^{-1}\triangle_y + 2^{-1}t^{-2}y\cdot Qy + R(t,y);\ t \geq 1,\\ &Q = \text{diag}(q_1,\cdots,q_m) < 4^{-1}I, \text{ and for some } \nu > 0\\ &|\nabla_y R(t,y)| \leq C\Big(t^{-(2+\nu)}|y| + t^{-\left(\frac{3}{2}+\nu\right)}\Big),\\ &|\partial_y^\alpha R(t,y)| \leq Ct^{-2},\ |\alpha| = 2. \end{aligned}$$

Consider a corresponding propagator $U(t)$ assumed to preserve a sufficiently nice domain (and therefore in particular define solutions to the corresponding Schrödinger equation).

Under these conditions let $\psi(t) = U(t)\psi$ be *any* orbit. Then for all $\epsilon > 0$

$$(1.8)\qquad F\left(|y| < t^{2^{-1}\left(1+\sqrt{1-4\max(q_j)}\right)-\epsilon}\right)\psi(t) = o\big(t^0\big) \text{ for } t \to +\infty.$$

We notice that (1.8) in general is false if (1.7) is relaxed by requiring only $\nu = 0$. (This would admit $Q > 4^{-1}I$.) Although we shall not elaborate a modification of our proof allows relaxing (1.7) by replacing the decaying factors $t^{-\nu}$ by any vanishing function $f(t) = o\big(t^0\big)$ (paying though the price of demanding more bounded higher derivatives).

We refer to [DG, Theorem 3.4.1] for a similar result in the case $Q = 0$ (proved by other techniques).

The basic theme of this paper may be phrased as absence of certain quantum mechanical states that do have appearing counterparts in the corresponding classical model. For another example of this theme we refer to [Ge1].

This paper is organized as follows: Our main results (Theorems 2.2 and 2.3) and preliminaries are put in Section 2. Sections 3 and 4 are devoted to bounding variables orthogonal to the critical direction in question. In Section 5 we introduce the simplified dynamics. In Section 6 we introduce the change of variables. Section 7 is devoted to the basic uncertainty estimate. In Section 8 we construct appropriate propagation observables and complete the proof of Theorem 2.2. Finally we have included an Appendix A in which certain technicalities needed in the course of the proof are discussed in some detail.

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2 Main results and preliminaries

We consider a real-valued potential $V = V_0 + V_c$ on $\mathbf{R}^n$, $n \geq 2$, with the properties $V_0 \in C^\infty$ being homogeneous of degree zero in $\left\{z \in \mathbf{R}^n \,\middle|\, |z| > \frac{1}{4}\right\}$ and V_c being compactly supported and relatively compact to the Laplace-operator on $L^2(\mathbf{R}^n)$.

Let $\mathcal{C}_0$ be the set of critical points in S^{n-1} for V_0. We consider an isolated point in $\mathcal{C}_0$ for which we shall impose the following conditions.

Assumptions 2.1 *Let z_0 be isolated in $\mathcal{C}_0$. There exists an orthogonal decomposition of the variable in $\mathbf{R}^n$*

$$z = xz_0 \oplus y = xz_0 \oplus y_- \oplus y_+ \in \mathbf{R}z_0 \oplus Y_- \oplus Y_+ = \mathbf{R}^n,$$

with $\dim Y_- \geq 1$ (and possibly $Y_+ = \{0\}$) such that

(2.1) $\quad y_- \cdot \nabla_{y_-} V_0(z) \leq 0$ *on a neighborhood of* z_0,

and if $Y_+ \neq \{0\}$ then for some $\delta > 0$

(2.2) $\quad y_+ \cdot \nabla^2 V_0(z_0) y_+ \geq \delta |y_+|^2.$

Moreover for some $C > 0$

(2.3) $\quad |\partial_x V_0(z)| \leq C\Big(|z|\,|\nabla V_0(z)|^2 - \frac{y_-}{|y_-|} \cdot \nabla_{y_-} V_0(z)\Big)$ *on a neighborhood of* z_0.

We notice that (2.1) and (2.2) together imply that the Hessian takes the form

(2.4) $\quad \nabla^2 V_0(z_0) = 0 \oplus Q_- \oplus Q_+,$

where Q_- is a non-positive form on Y_- while Q_+ is strictly positive on Y_+.

Notice also that (2.1) implies

(2.5) $\quad \partial_{y_-} V_0(z_0 + y_+) = 0$ for $|y_+|$ small.

Moreover we remark that in the non-degenerate case (i.e. Q_- be strictly negative) (2.3) follows from the previous conditions. A simple example of a degenerate critical point that fulfils Assumptions 2.1 is V_0 given on a neighborhood of some $z_0 \in S^{n-1}$ by $y_+^2 x^{-2} - y_-^4 x^{-4}$.

Clearly if the Hessian is maximally negative at some isolated point $z_0 \in \mathcal{C}_0$ (i.e. having $n-1$ negative eigenvalues, counted with multiplicity, at a critical point $z_0 \in S^{n-1}$) then Assumptions 2.1 are fulfilled.

We consider the Hamiltonian $H = 2^{-1}p^2 + V, p = -i\nabla$, on $L^2(\mathbf{R}^n)$. Clearly the momentum operator is decomposed in agreement with the above splitting as $p = \xi \oplus \eta_- \oplus \eta_+$. We shall use the notation $\langle z \rangle = \left(1 + |z|^2\right)^{\frac{1}{2}}$, $\hat{z} = \frac{z}{|z|}$; $z \in \mathbf{R}^n$. Our main results are the following theorems.

Theorem 2.2 *Suppose Assumptions 2.1 for the part V_0 of the potential $V = V_0 + V_c$. Suppose $I_0 = [a, b] \subset (V_0(z_0), \infty)$ does not contain eigenvalues of H, critical values of V_0 (i.e. $I_0 \cap (\sigma_{pp}(H) \cup V_0(\mathcal{C}_0)) = \emptyset$) nor eigenvalues of $2Q_+ + V_0(z_0)I$, and that for a $\psi \in L^2(\mathbf{R}^n)$ on the form $\psi = f(H)\phi$ for some $f \in C_0^\infty((a, b))$ and $\phi \in L^2(\mathbf{R}^n)$*

(2.6) $$\lim_{t \to +\infty} ||\chi(\hat{z})e^{-itH}\psi|| = 0,$$

for all smooth functions χ on S^{n-1} with $\chi(z_0) = 0$.

Then $\psi = 0$.

Theorem 2.3 *Suppose Assumptions 2.1 for V_0 and that $V_0(\mathcal{C}_0)$ is countable. Let for $\epsilon > 0$ $\mathcal{N}_\epsilon = \left\{z \in S^{n-1} | \, |z - z_0| < \epsilon\right\}$. Then for $\epsilon > 0$ small enough*

(2.7) $$\lim_{t \to +\infty} ||1_{\mathcal{N}_\epsilon}(\hat{z})e^{-itH}\psi|| = 0,$$

for all $\psi \in L^2(\mathbf{R}^n)$.

Remark 2.4 Both of the above theorems hold upon adding to the potential V a third term of the "usual" long-range type, say $V_{-\epsilon}$, satisfying the estimate

(2.8) $$\partial_z^\alpha V_{-\epsilon} = O\Big(|z|^{-\epsilon-|\alpha|}\Big) \text{ for } |z| \to \infty$$

for some $\epsilon > 0$. For convenience we confine ourselves to the case $V_{-\epsilon} = 0$.

Our second result Theorem 2.3 is a easy consequence of Theorem 2.2, the Mourre estimate [ACH, Theorem C.1] and a "separation of channels" result from [He]. Therefore we shall from now on only be concerned with proving Theorem 2.2.

Definition 2.5 *Let $\mathcal{F}_+$ denote the largest set of $F = F_+ \in C^\infty(\mathbf{R})$, such that $0 \le F \le 1$, $F' = \frac{d}{dx}F \ge 0, F' \in C_0^\infty\left(\left(\frac{1}{2}, \frac{3}{4}\right)\right)$, $F\left(\frac{1}{2}\right) = 0$, $F\left(\frac{3}{4}\right) = 1$ and*

$\sqrt{1-F}, \sqrt{F}, \sqrt{F'} \in C^\infty$, *which is stable under the maps* $F \to F^m$ *and* $F \to 1-(1-F)^m$; $m \in \mathbf{N}$. *Let* $\mathcal{F}_-$ *denote the set of functions* $F_- = 1 - F_+$ *where* $F_+ \in \mathcal{F}_+$.

The first step is to localize the potential: Consider for some fixed $F_+ \in \mathcal{F}_+$, $F_- \in \mathcal{F}_-$ and small $\delta > 0$

$$V^\delta(z) = (V_0(z) - \min V_0) F_-\Big(\delta^{-1} \tfrac{y^2}{x^2}\Big) F_+(x) + \min V_0. \tag{2.9}$$

For convenience we shall assume that

$$V_0(z_0) = 0. \tag{2.10}$$

Clearly V^δ is of the same form as the potential V of Theorem 2.2 in fact with the homogeneous part V_0^δ obeying $V_0^\delta = V_0$ on a neighborhood of z_0. Moreover if δ is small enough

$$y_- \cdot \nabla_{y_-} V^\delta(z) \le 0 \text{ for all } z \in \mathbf{R}^n. \tag{2.11}$$

Clearly, if $C = C(\delta) > 0$ is large enough

$$|\partial_x V^\delta(z)| \le C\Big(|z|\,|\nabla V_0^\delta(z)|^2 - \tfrac{y_-}{|y_-|} \cdot \nabla_{y_-} V^\delta(z)\Big), \tag{2.12}$$

for all z with $|z| \ge 1, |\hat{z} - \hat{z}_0| \le C^{-1}$.

Due to (2.11) we readily obtain (by the virial theorem) that $H^\delta = 2^{-1}p^2 + V^\delta$ does not have eigenvalues: Notice

$$i\big[H^\delta, \tfrac{1}{2}(y_- \cdot \eta_- + \eta_- \cdot y_-)\big] \ge \eta_-^2. \tag{2.13}$$

Since $I_0 \cap V_0^\delta\big(S^{n-1}\big) = \emptyset$ (assuming δ small) we can therefore show the existence of

$$\lim_{t \to +\infty} e^{itH^\delta} \chi(\hat{z}) e^{-itH} \psi \tag{2.14}$$

for the ψ in Theorem 2.2 and any $\chi \in C^\infty\big(S^{n-1}\big)$ being supported near z_0 and $= 1$ on a neighborhood of this point (cf. [ACH] and [He], or (2.15) and (2.16) stated below).

We conclude that it suffices to prove Theorem 2.2 for $H = H^\delta$.

From this point we drop the superscript δ since we shall only deal with H^δ. We shall use the following collection of basic estimates. For a family of bounded self-adjoint operators $\{B(t) | t \in \mathbf{R}\}$ we use the notation $\langle B(t) \rangle_t$ to denote the expectation $\langle \psi(t), B(t)\psi(t) \rangle$ in a state $\psi(t) = e^{-itH}\psi$, $\psi = f(H)\phi$ where $f \in C_0^\infty(I_0)$ and $\phi \in L^2(\mathbf{R}^n)$.

Lemma 2.6 *Let* $\tilde{g} \in C_0^\infty((0,\infty))$ *be given such that* $\tilde{g} = 1$ *on a neighborhood of* I_0 *and* $0 \le \tilde{g} \le 1$. *Define* $g(\xi) = \tilde{g}\big(2^{-1}\xi^2\big) 1_{(0,\infty)}(\xi)$. *There exist* $\chi \in C^\infty\big(S^{n-1}\big)$ *being supported near* z_0 *with* $\chi = 1$ *on a neighborhood* z_0 *and* $0 \le \chi \le 1$, *and* $\kappa_0 > 0$, *such* *that for all $\kappa \in (0,\kappa_0)$, $F_- \in \mathcal{F}_-, F_+ \in \mathcal{F}_+$ and all states of the form $\psi = f(H)\phi$ for $f \in C_0^\infty(I_0)$ and $\phi \in L^2(\mathbf{R}^n)$*

$$\int \left\langle \langle z\rangle\, |\nabla V_0(z)|^2 \right\rangle_t dt \le C||\psi||^2, \tag{2.15}$$

$$\int \left\langle \langle z\rangle^{-\mu} \right\rangle_t dt \le C_\mu ||\psi||^2;\ \mu > 1, \tag{2.16}$$

$$\int_1^\infty \left\langle |\partial_x V| F_+\big(\kappa^{-1}\tfrac{x}{t}\big)\chi(\hat z)\right\rangle_t dt \le C_\kappa ||\psi||^2, \tag{2.17}$$

$$\int_1^\infty t^{-1}\left\langle g(\xi)\chi(\hat z)F_+'\big(\kappa^{-1}\tfrac{x}{t}\big)\chi(\hat z)g(\xi)\right\rangle_t dt \le C_\kappa ||\psi||^2, \tag{2.18}$$

$$\int_1^\infty t^{-1}\left\langle -g(\xi)\chi(\hat z)F_+\big(\kappa^{-1}\tfrac{x}{t}\big)F_-'\big(\kappa\tfrac{x}{t}\big)\chi(\hat z)g(\xi)\right\rangle_t dt \le C_\kappa ||\psi||^2, \tag{2.19}$$

$$\begin{aligned} &\int_1^\infty t^{-1}\left\langle S(t)^\star\Big(\xi - \frac{x}{t}\Big)^2 S(t)\right\rangle_t dt \le C_\kappa ||\psi||^2;\\ &S(t) = F_-\Big(\kappa\frac{x}{t}\Big)F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat z)g(\xi), \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\int_1^\infty t^{-1}\left\langle -T(t)^\star\Big(\xi - \frac{x}{t}\Big)^2 F_-'\Big(\epsilon^{-1}\Big(\xi-\frac{x}{t}\Big)^2\Big)T(t)\right\rangle_t dt \le C_{\kappa,\epsilon}||\psi||^2;\\ &T(t) = F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat z)g(\xi), \epsilon > 0. \end{aligned} \tag{2.21}$$

Proof

For (2.15) and (2.16) we refer to [ACH] and [He].

As for (2.17)–(2.21) we assume that the support of χ is so small that $\mathrm{supp}(\partial^\alpha\chi) \cap \mathcal{C}_0 = \emptyset$ for $|\alpha| \ge 1$. For (2.17) we need in addition that the support of χ is contained in the neighborhood $\mathcal{N}_{C^{-1}}$ (using notation of Theorem 2.3) with C specified in (2.12) and in fact another smallness condition (see (2.26) and (2.28)).

We shall use a standard argument, [D, Lemma A.1 (b)], which is based on constructing a uniformly bounded "propagation observable" $\Phi(t)$ whose Heisenberg derivative $\mathbf{D}\Phi = \frac{d}{dt}\Phi + i[H,\Phi]$ up to integrable errors has a definite sign.

For (2.17) we let $\Phi(t) = \tilde g(H) b_t \tilde g(H)$ with b_t introduced at (3.1) in Section 3. With the parameter $\delta \in \big(2^{-1},1\big)$ (3.3) yields

$$\int_1^\infty \left\langle -\big\langle \tfrac{y_-}{t^\delta}\big\rangle^{-1}\tfrac{y_-}{t^\delta}\cdot\nabla_{y_-}V\right\rangle_t dt \le C_\delta||\psi||^2. \tag{2.22}$$

Combining (2.12), (2.15), (2.16) and (2.22) we get

$$\int_1^\infty \left\langle |\partial_x V| F_+\Big(|\frac{y_-}{t^\delta}|\Big)\chi(\hat z)\right\rangle_t dt \le C||\psi||^2,$$

so it remains to prove that

$$(2.23)\quad \int\limits_1^\infty \left\langle |\partial_x V| F_+\left(\kappa^{-1}\tfrac{x}{t}\right) F_-\left(|\tfrac{y_-}{t^\delta}|\right)\chi(\hat z)\right\rangle_t dt \le C||\psi||^2.$$

We compute

$$(2.24)\quad \partial_x V = -\tfrac{y_-}{x}\cdot\nabla_{y_-}V - \tfrac{y_+}{x}\cdot\nabla_{y_+}V + O\Big(\langle z\rangle^{-2}\Big).$$

Clearly the third (remainder) term can be treated by (2.16). For the contribution from the first term we estimate

$$(2.25)\quad |\tfrac{y_-}{x}\cdot\nabla_{y_-}V|F_+\left(\kappa^{-1}\tfrac{x}{t}\right)F_-\left(|\tfrac{y_-}{t^\delta}|\right) = O\left(t^{\delta-2}\right),$$

which clearly is integrable.

For the middle term we notice the bound

$$(2.26)\quad \tfrac{y_+^2}{x^3} \le C|z||\nabla_{y_+}V|^2(y_-=0);\ |\tfrac{y_+}{x}| \le C^{-1}, x\ge 1,$$

(cf. (2.2)).

By the fundamental theorem of calculus

$$(2.27)\quad \left(|\nabla_{y_+}V|^2(y_-=0) - |\nabla_{y_+}V|^2\right)\chi(\hat z) = O\left(\tfrac{y_-}{x^3}\right) \text{ as } x\to+\infty.$$

Combining (2.26) and (2.27) give the estimate

$$(2.28)\quad \begin{aligned} &|\frac{y_+}{x}\cdot\nabla_{y_+}V|F_+\Big(\kappa^{-1}\frac{x}{t}\Big)F_-\Big(|\frac{y_-}{t^\delta}|\Big)\chi(\hat z) \le 2^{-1}\Big(\frac{y_+^2}{x^3}+x|\nabla_{y_+}V|^2\Big)F_+F_-\chi \\ &\le C\Big(|\frac{y_-}{x^2}|+\langle z\rangle|\nabla V|^2\Big)F_+F_-\chi \le O\Big(t^{\delta-2}\Big)+C\langle z\rangle|\nabla V|^2. \end{aligned}$$

Clearly (2.15) and (2.28) yield integrability and hence (2.23).

For (2.18) introduce

$$\Phi(t) = g(\xi)\chi(\hat z)F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat z)g(\xi).$$

Then $\mathbf{D}\Phi(t)$ is a sum of various terms. The contribution (error) from commuting with the two χ's can be dealt with using (2.15). The one from commuting with the two $g(\xi)$'s can be treated by using (2.17) as can be seen as follows. We compute (cf. (2.50) and (2.52) given below)

$$(2.29)\quad \begin{aligned} &i[H,g(\xi)]\chi(\hat z)F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat z)g(\xi)+h.c. \\ &= -g'(\xi)\chi(\hat z)\partial_x V F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat z)g(\xi)+h.c.+O\left(t^{-2}\right). \end{aligned}$$

Let for $\mu \in \left(1, \frac{3}{2}\right)$

(2.30) $\quad V_\mu(t) = \left(|\partial_x V|^2 + t^{-2\mu}\right)^{\frac{1}{2}}.$

We substitute in (2.29)

$$\text{(2.31)} \quad \begin{aligned} &\partial_x V F_+\left(\kappa^{-1}\frac{x}{t}\right)\chi(\hat{z})^2 = \left(V_\mu(t)^{\frac{1}{2}} F_+\left(\kappa^{-1}\frac{x}{t}\right)^{\frac{1}{2}}\chi(\hat{z})\right) \\ &\left(V_\mu(t)^{-\frac{1}{2}}\partial_x V V_\mu(t)^{-\frac{1}{2}}\right)\left(V_\mu(t)^{\frac{1}{2}} F_+\left(\kappa^{-1}\frac{x}{t}\right)^{\frac{1}{2}}\chi(\hat{z})\right), \end{aligned}$$

and commute the first factor to the left and the third to the right (i.e. we symmetrize). The middle term is bounded uniformly w.r.t. t. Since uniformly w.r.t. z

$$\text{(2.32)} \quad \partial_z^\alpha\left(V_\mu(t)^{\frac{1}{2}} F_+\left(\kappa^{-1}\tfrac{x}{t}\right)^{\frac{1}{2}}\chi(\hat{z})\right) = O\left(t^{(2\mu-3)|\alpha|}\right)V_\mu(t)^{\frac{1}{2}} = O\left(t^{(2\mu-3)|\alpha|-\frac{1}{2}}\right),$$

the errors from the commutators are $O\left(t^{2\mu-4}\right)$ (cf. Corollary 2.9), and hence integrable. Now by estimating the factors involving ξ and the middle term we reduce to integrating $\chi(\hat{z})V_\mu(t)F_+\left(\kappa^{-1}\frac{x}{t}\right)\chi(\hat{z})$, which in turn can be done by (2.17).

We are left with the contribution from commuting with $F_+\left(\kappa^{-1}\frac{x}{t}\right)$, that is the expression

$$T = g(\xi)\chi(\hat{z})\frac{1}{2\kappa t}\left(F_+'\left(\kappa^{-1}\frac{x}{t}\right)\left(\xi - \frac{x}{t}\right) + \left(\xi - \frac{x}{t}\right)F_+'\left(\kappa^{-1}\frac{x}{t}\right)\right)\chi(\hat{z})g(\xi).$$

Since there exists a constant $C = C(\kappa) > 0$ for small enough $\kappa > 0$, such that $T \geq C^{-1}t^{-1}g(\xi)\chi(\hat{z})F_+'\left(\kappa^{-1}\frac{x}{t}\right)\chi(\hat{z})g(\xi) + O\left(t^{-2}\right)$ (2.18) follows.

The argument for (2.19) is similar using

$$\Phi(t) = g(\xi)\chi(\hat{z})F_+\left(\kappa^{-1}\frac{x}{t}\right)F_-\left(\kappa\frac{x}{t}\right)\chi(\hat{z})g(\xi).$$

As for (2.20) we define (cf. [E], [Gr])

$$\Phi(t) = S(t)^\star\left(\xi - \frac{x}{t}\right)^2 S(t),$$

compute the Heisenberg derivative and use the previous estimates.

As for (2.21) we introduce

$$\Phi(t) = T(t)^\star F_-\left(\epsilon^{-1}\left(\xi - \frac{x}{t}\right)^2\right)T(t)$$

and proceed similarly using Corollary 2.9 (stated below).

□

Due to Lemma 2.6 we can define (cf. a standard argument [D, Lemma A.2]) the strong limits

(2.33) $\quad W_\kappa^1 = s - \lim_{t\to+\infty} e^{itH}\bar{f}(H)T(t)^\star T(t)f(H)e^{-itH},$

(2.34) $W^2_\kappa = s - \lim_{t\to+\infty} e^{itH}\bar f(H)S(t)^\star S(t)f(H)e^{-itH},$

for $f, g, \chi, S(t)$ and $T(t)$ as in Lemma 2.6 and for all $\kappa > 0$ small enough.

By a standard argument (cf. [Gr] or [He]) for any $\psi \in L^2(\mathbf{R}^n)$

(2.35) $W^2_\kappa\psi = W^1_\kappa\psi + o(\kappa^0)$ for $\kappa \to 0.$

Since for small κ, $\psi = W^1_\kappa\psi$ for the ψ of Theorem 2.2 if we pick $f \in C_0^\infty(I_0)$ above such that $\psi = f(H)\psi$ and $0 \le f \le 1$ (seen by using the Mourre estimate of [ACH]), we obtain from (2.35) that this ψ obeys

(2.36) $\psi = W^2_\kappa\psi + o(\kappa^0)$ for $\kappa \to 0.$

Let for κ as above, $\epsilon > 0$ and $S(t)$ as in (2.20)

(2.37) $W^3_{\kappa,\epsilon} = s - \lim_{t\to+\infty} e^{itH}\bar f(H)S(t)^\star F_-\Big(\epsilon^{-1}\big(\xi - \tfrac{x}{t}\big)^2\Big)S(t)f(H)e^{-itH}.$

An application of (2.20) yields that in fact

(2.38) $W^2_\kappa = W^3_{\kappa,\epsilon}.$

Combining (2.36), (2.38) and Lemma 2.10 (stated below) we readily obtain the following result.

Lemma 2.7 *Let ψ be given as in Theorem 2.2. Then for g and χ as in Lemma 2.6, and for all $\epsilon > 0$ and sufficiently small $\kappa > 0$*

(2.39)
$$e^{-itH}\psi = L(t)e^{-itH}\psi + o(t^0);$$
$$L(t) = L_{\kappa,\epsilon}(t) = F_-\left(\epsilon^{-1}\left(\xi - \frac{x}{t}\right)^2\right)F_+\left(\kappa^{-1}\frac{x}{t}\right)\chi(\hat z)^2 g(\xi),$$

for $t \to +\infty$.

We shall need a modification of the abstract result [D, Lemma A.3 (b)]. The proof will be skipped. The last part (c) is based on a technique from [Ge2, Appendix] representing functions with "power-like" behavior at infinity as a limit of a sequence of C_0^∞-functions each of which in turn is represented in terms of an almost analytic extension, cf. (2.41). (In Section 8 we shall use this procedure to some functions outside the class $\mathcal{F}_+ \cup \mathcal{F}_-$ treated below.) The condition (2.40) is a technical condition reminiscent of [M] used below to justify (2.42).

Lemma 2.8 *Suppose H and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, and that $\{B(t)|\, t > 1\}$ is a family of self-adjoint operators on the same space with a common* *domain $\mathcal{D}(B(t)) = \mathcal{D}(B)$ and with $\mathcal{D} := \mathcal{D}(B) \cap \mathcal{D}(H)$ dense in $\mathcal{D}(B)$. Suppose that either H is bounded or*

$$\sup_{|s|\leq 1} ||Be^{isH}\psi|| < \infty \text{ for all } \psi \in \mathcal{D}(B), \tag{2.40}$$

that the commutator form $i[H, B(t)]$ extends from $\mathcal{D}$ to a symmetric operator with domain $\mathcal{D}(B)$ and that the $\mathcal{B}(\mathcal{H})$-valued function $B(t)(B-i)^{-1}$ is continuously differentiable.

(a) For any given $F \in C_0^\infty(\mathbf{R})$ we let $\tilde{F} \in C_0^\infty(\mathbf{C})$ denote an almost analytic extension. In particular

$$F(B(t)) = \frac{1}{\pi}\int_{\mathbf{C}} \left(\bar{\partial}\tilde{F}\right)(w)(B(t)-w)^{-1}dudv,\ w = u+iv. \tag{2.41}$$

The $\mathcal{B}(\mathcal{H})$-valued function $F(B(t))$ is continuously differentiable, and introducing the Heisenberg derivative $\mathbf{D} = \frac{d}{dt} + i[H,\cdot]$ we can extend the form $\frac{d}{dt}F(B(t)) + i[H, F(B(t))]$ from $\mathcal{D}(H)$ to the bounded operator

$$\mathbf{D}F(B(t)) = -\frac{1}{\pi}\int_{\mathbf{C}} \left(\bar{\partial}\tilde{F}\right)(w)(B(t)-w)^{-1}\mathbf{D}B(t)(B(t)-w)^{-1}dudv. \tag{2.42}$$

In particular if $\mathbf{D}B(t)$ is bounded then for any $\epsilon > 0$

$$||\mathbf{D}F(B(t))|| \leq C_\epsilon \sup_{w\in\mathbf{C}} \left(\langle w\rangle^{\epsilon+2}|Im w|^{-2}|\left(\bar{\partial}\tilde{F}\right)(w)|\right)||\mathbf{D}B(t)||. \tag{2.43}$$

(b) Suppose in addition that we can split $\mathbf{D}B(t) = D(t) + D_r(t)$, where $D(t)$ and $D_r(t)$ are symmetric on $\mathcal{D}(B)$ and that the form $i^k ad^k_{B(t)}(D(t)) = i\left[i^{k-1}ad^{k-1}_{B(t)}(D(t)), B(t)\right]$ for $k = 1$ extends from $\mathcal{D}(B)$ to a symmetric operator on $\mathcal{D}(B)$; $ad^0_{B(t)}(D(t)) = D(t)$. (No assumption is made for the form when $k = 2$.) Then the contribution from $D(t)$ to (2.42) can be written as

$$\begin{aligned} &-\frac{1}{\pi}\int_{\mathbf{C}} \left(\bar{\partial}\tilde{F}\right)(w)(B(t)-w)^{-1}D(t)(B(t)-w)^{-1}dudv \\ &= \frac{1}{2}\big(F'(B(t))D(t) + D(t)F'(B(t))\big) + R_1(t); \\ &R_1(t) = \frac{1}{2\pi}\int_{\mathbf{C}} \left(\bar{\partial}\tilde{F}\right)(w)(B(t)-w)^{-2}ad^2_{B(t)}(D(t))(B(t)-w)^{-2}dudv. \end{aligned} \tag{2.44}$$

In particular if $ad_{B(t)}(D(t))$ is bounded then for any $\epsilon > 0$

$$||R_1(t)|| \leq C_\epsilon \sup_{w\in\mathbf{C}} \left(\langle w\rangle^{\epsilon+2}|Im w|^{-3}|\left(\bar{\partial}\tilde{F}\right)(w)|\right)||ad_{B(t)}(D(t))||. \tag{2.45}$$

For any $f \in C_0^\infty(\mathbf{R})$

$$\begin{aligned}
&\frac{1}{2}\big(f^2(B(t))D(t) + D(t)f^2(B(t))\big)\\
&= f(B(t))D(t)f(B(t)) + R_2(t);\\
&R_2(t) = 2^{-1}\pi^{-2}\int_{\mathbf{C}}\int_{\mathbf{C}} \Big(\bar\partial\tilde f\Big)(w_2)\Big(\bar\partial\tilde f\Big)(w_1)(B(t)-w_2)^{-1}(B(t)-w_1)^{-1}\\
&ad^2_{B(t)}(D(t))(B(t)-w_1)^{-1}(B(t)-w_2)^{-1}du_1dv_1du_2dv_2.
\end{aligned}\tag{2.46}$$

(c) Suppose $D_r(t) = 0$ *and in fact the stronger conditions: For all* $t > 1$ $B(t)$ *is bounded and these operators constitutes a continuously differentiable* $\mathcal{B}(\mathcal{H})$*-valued function. Moreover both of the forms* $L(t) = i^k ad^k_{B(t)}(D(t))$*,* $k = 0, 1$*, extend from* $\mathcal{D}$ *to bounded self-adjoint operators. Then for any* $F \in \mathcal{F}_+$ *the* $\mathcal{B}(\mathcal{H})$*-valued function* $F(B(t))$ *is continuously differentiable, and there is an almost analytic extension with*

$$|\Big(\bar\partial\tilde F\Big)(w)| \le C_k\langle w\rangle^{-1-k}|Im z|^k;\ k \in \mathbf{N},\tag{2.47}$$

yielding the representation

$$\mathbf{D}F(B(t)) = F'^{\frac{1}{2}}(B(t))D(t)F'^{\frac{1}{2}}(B(t)) + R_1(t) + R_2(t),\tag{2.48}$$

where $R_1(t)$ *is given by (2.44), and* $R_2(t)$ *by (2.46) with* $f = \sqrt{F'}$*.*

Similar statements hold for $F \in \mathcal{F}_-$*.*

For these cases

$$\begin{aligned}
&||R_1(t)||, ||R_2(t)|| \le C||(B(t)+i)^{-1}ad^2_{B(t)}(D(t))(B(t)-i)^{-1}||\\
&\le 2C||ad_{B(t)}(D(t))||;\ C = C(F).
\end{aligned}\tag{2.49}$$

We shall often use Lemma 2.8 in the situation where $B = B(t)$. In that case the lemma provides a calculus for H and certain functions of B. Some well-known basic examples are given in the following result.

Corollary 2.9 *Let* $F \in C_0^\infty(\mathbf{R})$*.*

There exists $C > 0$ *such that for all* $G \in C^2(\mathbf{R})$ *with bounded derivatives*

$$\begin{aligned}
&i[G(x), F(\xi)] + \frac{1}{2}\big(G'(x)F'(\xi) + F'(\xi)G'(x)\big) = R_1;\\
&R_1 = \frac{1}{2\pi}\int_{\mathbf{C}} \Big(\bar\partial\tilde F\Big)(w)(\xi-w)^{-2}G''(x)(\xi-w)^{-2}dudv,
\end{aligned}\tag{2.50}$$

$$i\Big[G(x), F\Big(\xi - \frac{x}{t}\Big)\Big] + \frac{1}{2}\Big(G'(x)F'\Big(\xi - \frac{x}{t}\Big) + F'\Big(\xi - \frac{x}{t}\Big)G'(x)\Big) = R_2;$$

(2.51)

$$R_2 = \frac{1}{2\pi}\int_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(w)\Big(\xi - \frac{x}{t} - w\Big)^{-2} G''(x)\Big(\xi - \frac{x}{t} - w\Big)^{-2} dudv,\ t > 1,$$

and

(2.52) $||R_1||, ||R_2|| \le C||G''||.$

For any $\psi \in \mathcal{D}(\xi^2)$, $F(\xi - \frac{x}{t})\psi \in \mathcal{D}(\xi^2)$ *and*

(2.53) $i\xi^2 F(\xi - \frac{x}{t})\psi = iF(\xi - \frac{x}{t})\xi^2\psi - F'(\xi - \frac{x}{t})\frac{2\xi}{t}\psi - i\frac{1}{t^2}F''(\xi - \frac{x}{t})\psi.$

For any $\psi \in \mathcal{D}(\xi^2)$ *the* $L^2 -$ *valued function* $F(\xi - \frac{x}{t})\psi$ *is continuously differentiable w.r.t.* $t > 1$ *and*

(2.54) $\frac{d}{dt}F(\xi - \frac{x}{t})\psi + i\Big[\frac{\xi^2}{2}, F(\xi - \frac{x}{t})\Big]\psi = -F'(\xi - \frac{x}{t})\frac{\xi - \frac{x}{t}}{t}\psi.$

Proof The statements (2.50) and (2.51) are examples of (2.44).

The estimate (2.52) follows from the representations (cf. (2.49)).

As for (2.53) we put $B = B(t) = \xi - t^{-1}x$ and $H = \xi$. Then we use (2.42) repeatedly to pull H^2 to the right.

As for the formula (2.54) we represent for $\psi \in \mathcal{D}(\xi^2)$

(2.55) $F(\xi - \frac{x}{t})\psi = e^{-it\frac{\xi^2}{2}}F(-\frac{x}{t})e^{it\frac{\xi^2}{2}}\psi,$

which combined with (2.53) yields the differentiability as well as the formula.

□

Lemma 2.10 *Suppose* $A_1, \cdots, A_m$ *are self-adjoint operators on a Hilbert space* $\mathcal{H}$, $0 \le A_1, \cdots, A_m \le I$ *and for some* $\phi \in \mathcal{H}$ *with* $||\phi|| = 1$

(2.56) $\mathrm{Re}\langle\phi, A_1 \cdots A_m\phi\rangle \ge 1 - \epsilon;\ \epsilon > 0.$

Then

(2.57) $||(A_j - I)\phi|| \le 2^j\epsilon^{3^{-j}};\ j = 1, \cdots, m.$

Proof Clearly we may assume that $\epsilon < 2^{-3}$. We decompose for $\delta \in (0,1)$ with $\delta \ge \delta^2 + 2\epsilon$

(2.58) $\phi = \phi_\delta + \bar{\phi}_\delta;\ \phi_\delta = 1_{[0,1-\delta]}(A_1)\phi,\ \bar{\phi}_\delta = 1_{(1-\delta,1]}(A_1)\phi.$

Then writing $A_1 \cdots A_m = A_1 B$, and using the Cauchy Schwarz inequality and the spectral theorem (for the decomposition above) we get the bound

$$(2.59) \quad ||(A_1 - I)\phi||^2 \leq \delta^2 + \tfrac{2\epsilon}{\delta}.$$

Pick $\delta = \epsilon^{\frac{1}{3}}$ in (2.59). Then clearly we get (2.57) for $j = 1$.

Using (2.56) and the bound just proven we obtain (by another application of the Cauchy Schwarz inequality) that

$$(2.60) \quad \mathrm{Re}\langle \phi, A_2 \cdots A_m \phi \rangle \geq 1 - \epsilon - 2\epsilon^{3^{-1}} \geq 1 - \epsilon_2;\ \epsilon_2 = 2^2 \epsilon^{3^{-1}},$$

which allows iteration.

□

Lemma 2.11 *Suppose A and B are bounded self-adjoint operators on a Hilbert space and $B \geq c$ for some $c > 0$. Then*

$$(2.61) \quad \left|\left|\left[\sqrt{B}, A\right]\right|\right| \leq C c^{-\frac{1}{2}} ||[B, A]||.$$

Proof From the representation

$$\sqrt{B} = C_1 \int_0^\infty s^{-\frac{1}{2}} B(B+s)^{-1} ds$$

we get

$$\left[\sqrt{B}, A\right] = C_1 \int_0^\infty s^{\frac{1}{2}} (B+s)^{-1} [B, A] (B+s)^{-1} ds,$$

yielding

$$\left|\left|\left[\sqrt{B}, A\right]\right|\right| \leq C_1 ||[B, A]|| \int_0^\infty s^{\frac{1}{2}} ||(B+s)^{-1}||^2 ds.$$

We complete the proof (for $C = C_1 C_2$) by noticing that

$$\int_0^\infty s^{\frac{1}{2}} ||(B+s)^{-1}||^2 ds \leq \int_0^\infty s^{\frac{1}{2}} (c+s)^{-2} ds = c^{-\frac{1}{2}} C_2.$$

□

3 Localization of y_-

In this section we shall obtain a bound of the size of y_- along the evolution $\psi(t) = e^{-itH}\psi$ of the state ψ in Theorem 2.2 (with the potential given by (2.9)).

We shall follow the procedure of Dereziński [D, Section 5], see also [DG, Section 6.12]. Let for $t > 1$ and $\delta \in \big(2^{-1}, 1\big)$ $r_t(y_-) = t^\delta r\big(t^{-\delta} y_-\big)$; $r(y_-) = \langle y_- \rangle$, and let $\mathbf{D}$ denote the Heisenberg derivative as defined in the proof of Lemma 2.6. We compute (formally)

$$
\begin{aligned}
b_t &:= \mathbf{D} r_t = \frac{1}{2}\Big((\nabla_{y_-} r)\Big(t^{-\delta} y_-\Big) \cdot \eta_- + \eta_- \cdot (\nabla_{y_-} r)\Big(t^{-\delta} y_-\Big)\Big) + \delta t^{\delta - 1} d_t; \\
d_t &= r\Big(t^{-\delta} y_-\Big) - t^{-\delta} y_- \cdot (\nabla_{y_-} r)\Big(t^{-\delta} y_-\Big),
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\mathbf{D} b_t &= \mathbf{D}^2 r_t = t^{-\delta} c_t + \delta(\delta - 1) t^{\delta - 2} d_t - t^{-3\delta} e_t - (\nabla_{y_-} r)\Big(t^{-\delta} y_-\Big) \cdot \nabla_{y_-} V; \\
c_t &= \Big(\eta_- - \delta t^{-\delta} y_-\Big) \cdot \Big(\nabla^2_{y_-} r\Big)\Big(t^{-\delta} y_-\Big)\Big(\eta_- - \delta t^{-\delta} y_-\Big), \\
e_t &= 4^{-1}\big(\triangle^2 r\big)\Big(t^{-\delta} y_-\Big).
\end{aligned}
\tag{3.2}
$$

The right hand sides of (3.1) and (3.2) are H-bounded. Also we notice the non-negativity of c_t and the uniform boundedness (with respect to t) of the terms d_t and e_t. For the specified range of δ's we thus have

$$
\mathbf{D} b_t \geq -(\nabla_{y_-} r)\big(t^{-\delta} y_-\big) \cdot \nabla_{y_-} V - C_\delta t^{-\mu} \geq -C_\delta t^{-\mu};\ \mu = 2 - \delta > 1. \tag{3.3}
$$

For δ as above and small $\sigma > 0$, cf. (3.20), we consider

$$
\tilde{b}_t = N_t^{-1} b_t N_t^{-1};\ N_t = 1 + t^{-2\sigma} \eta_-^2. \tag{3.4}
$$

Clearly $\tilde{b}_t = O(t^\sigma)$ as $t \to +\infty$.

For $c > 0$, $F_+ \in \mathcal{F}_+$ and $L(t)$ given in Lemma 2.7 we will consider

$$
\tilde{\psi}(t) = F_+^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) L(t) \psi(t). \tag{3.5}
$$

We know that for any value of c

$$
\tilde{\psi}(t) = o\big(t^0\big) \text{ for } t \to +\infty, \tag{3.6}
$$

and want to show the same result for $c = t^\rho$ for some $\rho > 0$.

Following [D] we shall proceed by showing approximate positivity of the Heisenberg derivative of $\tilde{F}(t) := L(t)^\star F_+\Big(c\tilde{b}_t\Big) L(t)$.

By (2.48)

$$
\mathbf{D} F_+\Big(c\tilde{b}_t\Big) = F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) c\Big(\mathbf{D}\tilde{b}_t\Big) F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) + R_1(t) + R_2(t). \tag{3.7}
$$

We look at the first term on the right hand side of (3.7):

Clearly

$$
\text{(3.8)}\quad
\begin{aligned}
&\mathbf{D}N_t^{-1} = 2\sigma t^{-2\sigma-1}\eta_-^2 N_t^{-2} + N_t^{-1} i\big[t^{-2\sigma}\eta_-^2, V\big]N_t^{-1}\\
&\ = (T_1(t)+T_2(t))N_t^{-1};\\
&T_1(t) = 2\sigma t^{-2\sigma-1}\eta_-^2 N_t^{-1},\\
&T_2(t) = t^{-2\sigma}N_t^{-1}(\eta_-\cdot\nabla_{y_-}V + \nabla_{y_-}V\cdot\eta_-),
\end{aligned}
$$

so that

$$
\text{(3.9)}\quad
\begin{aligned}
&\mathbf{D}\tilde{b}_t = \big(\mathbf{D}N_t^{-1}\big)b_t N_t^{-1} + N_t^{-1}(\mathbf{D}b_t)N_t^{-1} + N_t^{-1}b_t\mathbf{D}N_t^{-1}\\
&\ = (T_1(t)+T_2(t))\tilde{b}_t + N_t^{-1}(\mathbf{D}b_t)N_t^{-1} + \tilde{b}_t(T_1(t)+T_2(t))^\star
\end{aligned}
$$

with $\mathbf{D}b_t$ computed in (3.2).

To estimate the contribution from the term $T_1(t)$ in (3.9) to the right hand side of (3.7) we define $G(x) = (xF_+(2x))^{\frac{1}{2}}F_-\big(2^{-1}x\big)$; $F_-\in\mathcal{F}_-$, and notice that by (2.42) (with $B = B(t)$)

$$
\text{(3.10)}\quad
\begin{aligned}
&F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)\Big(T_1(t)c\tilde{b}_t + c\tilde{b}_tT_1(t)\Big)F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) = F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)T_3(t)F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) + R;\\
&T_3(t) = 2G\Big(c\tilde{b}_t\Big)T_1(t)G\Big(c\tilde{b}_t\Big) \geq 0,\\
&||R|| \leq C_1\Big|\Big|\Big[G\Big(c\tilde{b}_t\Big), T_1(t)\Big]\Big|\Big| \leq C_2ct^{-1-\delta}.
\end{aligned}
$$

Clearly the same bounds hold upon multiplying by a factor $L(t)^\star$ to the left and $L(t)$ from the right.

We estimate the contribution from the term $T_2(t)$ writing

$$
\text{(3.11)}\quad T_2(t) = (t^\sigma\langle x\rangle T_2(t))t^{-\sigma}\langle x\rangle^{-1}.
$$

Since the first factor is bounded uniformly w.r.t. t and $\langle x\rangle^{-1}L(t) = O\big(t^{-1}\big)$ we thus get the bound

$$
\text{(3.12)}\quad L(t)^\star F'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)\Big(T_2(t)c\tilde{b}_t + c\tilde{b}_tT_2(t)^\star\Big)F'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)L(t) = O\big(t^{-\sigma-1}\big),
$$

uniformly w.r.t. c.

By (3.3) the contribution from the middle term on the right hand side of (3.9) obeys

$$
\text{(3.13)}\quad F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)N_t^{-1}(c\mathbf{D}b_t)N_t^{-1}F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big) \geq -Cct^{\delta-2}.
$$

We conclude from (3.10), (3.12) and (3.13)

$$
\text{(3.14)}\quad L(t)^\star F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)c\Big(\mathbf{D}\tilde{b}_t\Big)F_+'^{\frac{1}{2}}\Big(c\tilde{b}_t\Big)L(t) \geq -C\big(t^{-\sigma-1} + ct^{\delta-2}\big)
$$

(using that $\delta > 1/2$).

The contributions from $R_1(t)$ and $R_2(t)$ are bounded by

$$||L(t)^\star (R_1(t) + R_2(t))L(t)|| \leq Cc^2 t^{2\sigma-2\delta}, \tag{3.15}$$

as can be seen as follows:

The contribution to $ad_{c\tilde{b}_t}\Big(\mathbf{D}c\tilde{b}_t\Big)$ coming from those terms in (3.9), say $D_1(t)$; $\mathbf{D}\tilde{b}_t = D_1(t) + D_2(t)$, that do not involve the potential can be estimated by

$$||ad_{c\tilde{b}_t}(cD_1(t))|| \leq Cc^2 t^{2\sigma-2\delta}, \tag{3.16}$$

which by the formulas in Lemma 2.8 (cf. (2.49)) gives the bound (3.15) for the corresponding term.

For the terms $D_2(t)$ that do involve derivatives of the potential we substitute in those formulas

$$\partial^\alpha_{y_-} V = \langle x\rangle^{-|\alpha|}\Big(\langle x\rangle^{|\alpha|}\partial^\alpha_{y_-} V\Big) \tag{3.17}$$

(cf. (3.11)) and pull the factors $\langle x\rangle^{-|\alpha|}$ to the factor $L(t)$ appearing to the right. The final contribution from these terms is $O\big(c^2 t^{-\delta-1}\big)$ and hence in particular in agreement with (3.15).

Combining (3.7), (3.14) and (3.15) we get

$$\begin{aligned} &\mathbf{D}\tilde{F}(t) \geq T(t) + T(t)^\star - C\Big(t^{-\sigma-1} + ct^{\delta-2} + c^2 t^{2\sigma-2\delta}\Big);\\ &T(t) = L(t)^\star F_+\Big(c\tilde{b}_t\Big)\mathbf{D}L(t). \end{aligned} \tag{3.18}$$

By (3.6) and (3.18)

$$||\tilde{\psi}(t)||^2 \leq -\int_t^\infty \big\langle T(s) + T(s)^\star\big\rangle_s ds + C\int_t^\infty \big(s^{-\sigma-1} + cs^{\delta-2} + c^2 s^{2\sigma-2\delta}\big)ds. \tag{3.19}$$

We assume that $\sigma > 0$ is chosen so small that

$$\sigma < \delta - \tfrac{1}{2}, \tag{3.20}$$

and consider positive ρ_1 with

$$\rho_1 < \min\big(1-\delta, \delta - \tfrac{1}{2} - \sigma\big). \tag{3.21}$$

For a later application we notice that (3.21) implies that

$$\rho_1 < \tfrac{1}{4}(1-2\sigma). \tag{3.22}$$

With the constraint (3.21) and with $c = t^{\rho_1}$ the second term on the right hand side of (3.19) goes to zero for $t \to \infty$. We claim the same property for the first term. For that we compute $\mathbf{D}L(t)$. After symmetrizing various terms we can use Lemma 2.6. (Notice the energy-localization of the state ψ.) It remains to treat the error terms arising from this procedure:

The errors coming from symmetrizing $\mathbf{D}g(\xi)$ come from commutators with $V_\mu(t)^{\frac{1}{2}} F_+\big(\kappa^{-1}\frac{x}{t}\big)^{\frac{1}{2}}\chi(\hat{z})$ (see (2.30) and (2.31)). Commuting this factor through the middle term $F_+\Big(c\tilde{b}_t\Big)$ pick up the bound

$$Cct^{(2\mu-3)-1}, \tag{3.23}$$

as may be seen by using (2.43) (which is uniform for a time-dependent family of C_0^∞-cutoff-functions with $\epsilon = 1$) and (2.32) (for $|\alpha| = 1$). This gives the constraint

$$\rho_1 + 2\mu - 3 < 0. \tag{3.24}$$

We choose

$$\mu \in \big(1, 1 + \tfrac{\delta}{2}\big), \tag{3.25}$$

and observe that (3.24) follows from (3.21) and (3.25).

The error terms coming from symmetrizing $\mathbf{D}F_-\Big(\epsilon^{-1}\big(\xi - \frac{x}{t}\big)^2\Big)$ also yield to the bound (3.23) (when commuting the potential part through the middle term $F_+\Big(c\tilde{b}_t\Big)$).

The error terms coming from symmetrizing $\mathbf{D}\chi(\hat{z})$ and $\mathbf{D}F_+\big(\kappa^{-1}\frac{x}{t}\big)$ may be treated similarly and do not give new constraints.

We summarize:

Lemma 3.1 *Under the constraint (3.21)*

$$F_+^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)L(t)\psi(t) = o\big(t^0\big) \text{ for } t \to +\infty. \tag{3.26}$$

The next step is to consider

$$\tilde{\psi}(t) = F_-^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)F_+(cr_t)N_t^{-1}L(t)\psi(t) \tag{3.27}$$

for $c > 0$ and N_t as above (cf. (3.4)). Obviously for fixed $t > 1$, $||\tilde{\psi}(t)|| \to 0$ for $c \to 0$. We want to show the same result for c depending of t as given by $c = t^{\rho_2 - 1}$ for $\rho_2 \in (0, \rho_1)$. For that we proceed by showing approximate negativity of the expectation-value of the Heisenberg derivative of $\tilde{F}(t) := \big(F_+(cr_t)N_t^{-1}L(t)\big)^\star F_-\Big(t^{\rho_1}\tilde{b}_t\Big)F_+(cr_t)N_t^{-1}L(t)$ in the state $\psi(t)$. We shall use the arguments that lead to Lemma 3.1.

We compute (cf. (3.7))

$$\mathbf{D}F_-\Big(t^{\rho_1}\tilde b_t\Big) = -\big(-F_-'\big)^{\frac12}\Big(t^{\rho_1}\tilde b_t\Big)\Big(\mathbf{D}\Big(t^{\rho_1}\tilde b_t\Big)\Big)\big(-F_-'\big)^{\frac12}\Big(t^{\rho_1}\tilde b_t\Big)+R_1(t)+R_2(t), \tag{3.28}$$

where (as above) the terms $R_1(t)$ and $R_2(t)$ are specified in Lemma 2.8.

Clearly

$$\mathbf{D}\Big(t^{\rho_1}\tilde b_t\Big) \;=\; \rho_1 t^{-1}\Big(t^{\rho_1}\tilde b_t\Big) \,+\, t^{\rho_1}\mathbf{D}\tilde b_t. \tag{3.29}$$

The first term on the right hand side of (3.29) contributes when inserting into the first term on the right hand side of (3.28) by a non-positive piece. Similarly the second term contributes by a term which tends to be negative. Precisely we have the upper bound

$$C\big(t^{-\sigma-1} + t^{\rho_1}t^{\delta-2}\big) \tag{3.30}$$

for its contribution to the Heisenberg derivative of $\tilde F(t)$ by the arguments that lead to (3.14).

The contribution from the error terms in (3.28) is bounded by

$$Ct^{2\rho_1}t^{2\sigma-2\delta}, \tag{3.31}$$

cf. (3.15).

We notice (assuming (3.21)) that the bounds in (3.30) and (3.31) can be integrated to infinity.

Next we look at the contribution

$$D(t) = \big(F_+(cr_t)N_t^{-1}L(t)\big)^\star F_-\Big(t^{\rho_1}\tilde b_t\Big)(\mathbf{D}(F_+(cr_t)))N_t^{-1}L(t) + h.c.. \tag{3.32}$$

Using (3.1) we compute

$$\mathbf{D}(F_+(cr_t)) = c\big(b_tF_+'(cr_t) + c2^{-1}iF_+''(cr_t)|\nabla_{y_-}r_t|^2\big). \tag{3.33}$$

The first term on the right hand side contributes after symmetrizing (cf. (2.43) and (2.45)) by

$$\begin{aligned} &D_1(t) = 2ct^{-\rho_1}T(t)^\star F_-\Big(t^{\rho_1}\tilde b_t\Big)t^{\rho_1}\tilde b_tT(t)\\ &\;+cO\Big(t^{\rho_1+\sigma-\delta}\Big) + c^2O\big(t^0\big) + c^3O\Big(t^{\rho_1-\delta}\Big);\\ &T(t) = F_+^{\frac12}(cr_t)F_+'^{\frac12}(cr_t)L(t). \end{aligned} \tag{3.34}$$

Clearly for the first term

$$2ct^{-\rho_1}T(t)^\star F_-\Big(t^{\rho_1}\tilde b_t\Big)t^{\rho_1}\tilde b_tT(t) \,\le\, Cct^{-\rho_1}. \tag{3.35}$$

The second term on the right hand side of (3.33) contributes by $c^2 O\big(t^0\big)$.

The contribution to the Heisenberg derivative of $\tilde{F}(t)$ coming from the two factors $\mathbf{D}N_t^{-1}$ is estimated by

$$Ct^{-2\sigma-1} \tag{3.36}$$

as can seen by using (3.8) and the energy-localization of the state ψ.

We summarize, for $t > t_0 > 1$ and $C > 0$ independent of t, t_0 and c

$$\begin{aligned}
&||\tilde{\psi}(t)||^2 - ||\tilde{\psi}(t_0)||^2 \leq \int_{t_0}^{t} \big\langle T(s) + T(s)^\star \big\rangle_s ds \\
&+ C\Big(t_0^{-\sigma} + t_0^{\rho_1+\delta-1} + t_0^{2\rho_1+2\sigma-2\delta+1}\Big) \\
&+ C\Big(c\Big(t^{\rho_1+\sigma-\delta+1} + t^{1-\rho_1}\Big) + c^2 t + c^3 t\Big); \\
&T(t) = \big(F_+(cr_t)N_t^{-1}L(t)\big)^\star F_-\Big(t^{\rho_1}\tilde{b}_t\Big) F_+(cr_t)N_t^{-1}\mathbf{D}L(t).
\end{aligned} \tag{3.37}$$

We treat the first term involving $T(t)$ as we did for a similar term in the proof of Lemma 3.1. We get the estimate

$$\int_{t_0}^{t} \big\langle T(s) + T(s)^\star \big\rangle_s ds \; = \; o\big(t_0^0\big), \tag{3.38}$$

uniformly w.r.t. $t > t_0$ and $c > 0$.

By combining (3.21), (3.22), (3.37) and (3.38) we obtain by first choosing t_0 large that indeed for any $\rho_2 \in (0, \rho_1)$

$$||\tilde{\psi}(t)|| \to 0 \text{ for } t \to \infty \text{ with } c = t^{\rho_2 - 1}. \tag{3.39}$$

In conjunction with Lemma 3.1 (3.39) yields:

Lemma 3.2 *Under the constraint (3.21) and with* $\rho_2 \in (0, \rho_1)$

$$L(t)\psi(t) - F_-\Big(t^{\rho_1}\tilde{b}_t\Big) F_-\big(t^{\rho_2-1}r_t\big)N_t^{-1}L(t)\psi(t) = o\big(t^0\big) \textit{ for } t \to +\infty. \tag{3.40}$$

We can use the computations that lead to Lemmas 3.1 and 3.2 to supplement the list of propagation estimates in Lemma 2.6. Although we shall only use these new estimates for the state ψ of Theorem 2.2 they hold for any state localized w.r.t. energy as in Lemma 2.6.

Lemma 3.3 *Suppose (3.21) and* $\rho_2 \in (0, \rho_1)$. *Then under the conditions of Lemma 2.6, and with* $L(t)$ *given as in Lemma 2.7,* $\tilde{c}_t = N_t^{-1} c_t N_t^{-1}$ *and* $C > 0$ *depending only on the parameters involved*

$$\begin{aligned}
&\int_1^\infty \Big\langle G(t)^\star \big(-F_-^{2\prime}\big)^{\frac12}\Big(t^{\rho_1}\tilde{b}_t\Big)\tilde{f}_t\big(-F_-^{2\prime}\big)^{\frac12}\Big(t^{\rho_1}\tilde{b}_t\Big)G(t)\Big\rangle_t dt \le C||\psi||^2;\\
&\tilde{f}_t = t^{\rho_1-\delta}\tilde{c}_t + t^{-1}I - t^{\rho_1}N_t^{-1}(\nabla_{y_-}r)\Big(t^{-\delta}y_-\Big)\cdot\nabla_{y_-}VN_t^{-1},\\
&G(t) = G_+(t) = F_+\big(t^{\rho_2-1}r_t\big)N_t^{-1}L(t) \textit{ or}\\
&G(t) = G_-(t) = F_-\big(t^{\rho_2-1}r_t\big)N_t^{-1}L(t),
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
&\int_1^\infty \Big\langle H(t)^\star F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)\Big(t^{\rho_2-1-\rho_1}\Big(I - t^{\rho_1}\tilde{b}_t\Big) + t^{-1}I\Big)H(t)\Big\rangle_t dt \le C||\psi||^2;\\
&H(t) = H_+(t) = \big(F_+^{2\prime}\big)^{\frac12}\big(t^{\rho_2-1}r_t\big)L(t) \textit{ or}\\
&H(t) = H_-(t) = \big(-F_-^{2\prime}\big)^{\frac12}\big(t^{\rho_2-1}r_t\big)L(t).
\end{aligned} \tag{3.42}$$

Proof For (3.41) we notice that the estimate with $G(t) = N_t^{-1}L(t)$ follows by considering the propagation observable

$$\Phi(t) = \big(N_t^{-1}L(t)\big)^\star F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)N_t^{-1}L(t). \tag{3.43}$$

We use the computation (3.29) and the arguments that follow giving the bound (3.30).

To obtain (3.41) for $G(t) = G_-(t)$ it thus suffices to show the statement for $G(t) = G_+(t)$. We show the latter and (3.42) for $H(t) = H_+(t)$ in one stroke by considering

$$\Phi(t) = G_+(t)^\star F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)G_+(t). \tag{3.44}$$

We use the computations (3.33) and (3.34) with $c = t^{\rho_2-1}$ and notice that there is an extra term from $\mathbf{D}c$ with the same favourable sign.

To obtain (3.42) for $H(t) = H_-(t)$ we notice that for any given $F_- \in \mathcal{F}_-$ we can find a positive constant C and $F_+ \in \mathcal{F}_+$ such that

$$CF_+^{2\prime} \ge -F_-^{2\prime}. \tag{3.45}$$

After a commutation in (3.42) with $H(t) = H_+(t)$ (using (2.43)) we use (3.45) and obtain (after commuting back) (3.42) for $H(t) = H_-(t)$.

(Alternatively we consider $\Phi(t) = G_-(t)^\star F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)G_-(t)$ under use of similar computations as for the first estimate of (3.42), and (3.41) for $G(t) = G_-(t)$.)

□

4 Localization of y_+

In this section we shall obtain a bound of the size of y_+ along the evolution $\psi(t) = e^{-itH}\psi$ of the state ψ in Theorem 2.2. Together with the bound of y_- of the previous section this will give a total bound of y of the form $y = O\big(t^{1-\nu}\big)$ for a positive ν.

We fix $\epsilon = 4^{-1}$ and $\kappa > 0$ small in agreement with Lemma 2.7 and combine Lemmas 2.7 and 3.2 to obtain

$$\begin{aligned} &\psi(t) = F(t)\psi(t) + o\big(t^0\big); \\ &F(t) = F_-\Big(t^{\rho_1}\tilde{b}_t\Big)F_-\big(t^{\rho_2-1}r_t\big)N_t^{-1}F_-\bigg(4\Big(\xi - \frac{x}{t}\Big)^2\bigg)F_+\Big(\kappa^{-1}\frac{x}{t}\Big)\chi(\hat{z})^2 g(\xi), \end{aligned} \tag{4.1}$$

under the conditions of these lemmas.

We write the part of the Hessian on Y_+ as $Q_+ = \text{diag}(q_{+,1}, \cdots, q_{+,m})$. Clearly we can assume that $2q_{+,j} \notin \text{supp}(\tilde{g})$ where $\tilde{g} \in C_0^\infty((0,\infty))$ defines the g in (4.1) (cf. Lemma 2.6). We pick a positive function $h \in C^\infty(\mathbf{R}\backslash\{0\})$ with $h = 1$ on a neighborhood of the support of g. Suppose furthermore that the function $x^2(2h(x))^{-1}$ takes values in $\mathbf{R}_+\backslash\{2q_{+,j} | 1 \leq j \leq m\}$, and that it is constant on a neighborhood of zero and locally constant outside a bounded set.

With these conventions we define

$$\begin{aligned} &\alpha_j^+ = \frac{1}{2}\bigg(1 + \sqrt{1 - 4q_{+,j}\xi^{-2}h(\xi)}\bigg), \\ &\alpha_j^- = \frac{1}{2}\bigg(1 - \sqrt{1 - 4q_{+,j}\xi^{-2}h(\xi)}\bigg); \text{ if } 2q_{+,j} < a, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\alpha_j^+ = \frac{1}{2}\bigg(1 + i\sqrt{4q_{+,j}\xi^{-2}h(\xi) - 1}\bigg), \\ &\alpha_j^- = \frac{1}{2}\bigg(1 - i\sqrt{4q_{+,j}\xi^{-2}h(\xi) - 1}\bigg); \text{ if } 2q_{+,j} > b. \end{aligned} \tag{4.3}$$

We pick non-negative real-valued $g_1, g_2 \in C_0^\infty(\mathbf{R}\backslash\{0\})$ with $g_k = 1$ on a neighborhood of the support of g_{k-1}, $g_0 := g$, and such that $h = 1$ on the support of g_2.

For $t > 1$ we define

$$\begin{aligned} &B(t) = \gamma_0^\star\gamma_0 + \sum_{j=1}^m \Big(\gamma_j^{+\star}\gamma_j^+ + \gamma_j^{-\star}\gamma_j^-\Big); \\ &\gamma_0 = \Big(\xi - \frac{x}{t}\Big)F_-\bigg(\Big(\xi - \frac{x}{t}\Big)^2\bigg)g_2(\xi), \\ &\gamma_j^+ = \Big(\eta_{+,j} - \alpha_j^+\frac{y_{+,j}}{t}\Big), \\ &\gamma_j^- = \Big(\eta_{+,j} - \alpha_j^-\frac{y_{+,j}}{t}\Big). \end{aligned} \tag{4.4}$$

We notice that $\mathcal{D}(B(t)) = \mathcal{D}\big(\eta_+^2 + y_+^2\big) = \mathcal{D}\big(\eta_+^2\big) \cap \mathcal{D}\big(y_+^2\big)$,

$$\sup_{t>1} ||\eta_+^2 B(t)^{-1}||,\ \sup_{t>1} ||\big(\tfrac{y_+}{t}\big)^2 B(t)^{-1}||,\ \sup_{t>1} ||\gamma_0^\star\gamma_0 B(t)^{-1}|| < \infty, \tag{4.5}$$

and for some $C > 0$ independent of $t > 1$

$$C^{-1}(B(t) - \gamma_0^\star\gamma_0) \le \eta_+^2 + \big(\tfrac{y_+}{t}\big)^2 \le C(B(t) - \gamma_0^\star\gamma_0). \tag{4.6}$$

A consequence of (4.6) is that

$$||B(t)^{-1}|| \le Ct. \tag{4.7}$$

We modify the potential by defining $V_t = VF_+\big(4\kappa^{-1}\frac{x}{t}\big)$, and correspondingly put $H_t = 2^{-1}p^2 + V_t$ and $\mathbf{D}_t = \frac{d}{dt} + i[H_t, \cdot]$.

The $B(t)$'s and H_t (the latter for any fixed $t > 1$) obey the conditions of Lemma 2.8 (b) with $B = \eta_+^2 + y_+^2$ and

$$\begin{aligned}
&D(t) = S_1 + \cdots + S_4;\\
&S_1 = -2t^{-1}\left(\gamma_0^\star\gamma_0 + \sum_{j=1}^m \Big(\mathrm{Re}\Big(\alpha_j^+\Big)\gamma_j^{+\star}\gamma_j^+ + \mathrm{Re}\Big(\alpha_j^-\Big)\gamma_j^{-\star}\gamma_j^-\Big)\right),\\
&S_2 = -t^{-1}\gamma_0^\star\beta_0 + h.c.;\ \beta_0 = t\partial_x V_t(y_- = 0)F_-\left(\Big(\xi - \frac{x}{t}\Big)^2\right)g_2(\xi),\\
&S_3 = t^{-1}\sum_{j=1}^m \Big(\gamma_j^{+\star}\beta_j^+ + h.c.\Big);\ \beta_j^+ = ti\left[V_t(y_- = 0) - \frac{h(\xi)y_+\cdot Q_+y_+}{2t^2\xi^2}, \gamma_j^+\right],\\
&S_4 = t^{-1}\sum_{j=1}^m \Big(\gamma_j^{-\star}\beta_j^- + h.c.\Big);\ \beta_j^- = ti\left[V_t(y_- = 0) - \frac{h(\xi)y_+\cdot Q_+y_+}{2t^2\xi^2}, \gamma_j^-\right],
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&D_r(t) = T_1 + \cdots + T_5;\\
&T_1 = i\left[V_t(z) - V_t(y_- = 0), \sum_{j=1}^m \Big(\gamma_j^{+\star}\gamma_j^+ + \gamma_j^{-\star}\gamma_j^-\Big)\right],\\
&T_2 = -\gamma_0^\star(\partial_x V_t(z) - \partial_x V_t(y_- = 0))F_-\left(\Big(\xi - \frac{x}{t}\Big)^2\right)g_2(\xi) + h.c.,\\
&T_3 = \gamma_0^\star\Big(\xi - \frac{x}{t}\Big)\mathbf{D}_t\left(F_-\left(\Big(\xi - \frac{x}{t}\Big)^2\right)g_2(\xi)\right) + h.c.,\\
&T_4 = -t^{-1}\sum_{j=1}^m \left(\left(q_{+,j}\frac{h(\xi)}{\xi^2} - \alpha_j^+ + \alpha_j^{+2}\right)\gamma_j^{+\star}\frac{y_{+,j}}{t} + h.c.\right),\\
&T_5 = -t^{-1}\sum_{j=1}^m \left(\left(q_{+,j}\frac{h(\xi)}{\xi^2} - \alpha_j^- + \alpha_j^{-2}\right)\gamma_j^{-\star}\frac{y_{+,j}}{t} + h.c.\right).
\end{aligned} \tag{4.9}$$

By the definitions (4.2) and (4.3) the terms T_4 and T_5 are zero.

Notice also that the "ξ–symbol" of the operators Re$\big(\alpha_j^+\big)$ and Re$\big(\alpha_j^-\big)$ in both cases has a positive lower bound. Hence for small enough $\epsilon_1 > 0$

$$\begin{aligned} D(t) &\le S_1 + t^{-1}\left(\epsilon_1 B(t) + \epsilon_1^{-1}\left(\beta_0^\star\beta_0 + \sum_{j=1}^m \left(\beta_j^{+\star}\beta_j^+ + \beta_j^{-\star}\beta_j^-\right)\right)\right) \\ &\le -C^{-1}t^{-1}B(t) + Ct^{-1}\left(\beta_0^\star\beta_0 + \sum_{j=1}^m \left(\beta_j^{+\star}\beta_j^+ + \beta_j^{-\star}\beta_j^-\right)\right); \\ C &= C(\epsilon_1) > 0. \end{aligned} \tag{4.10}$$

In order to estimate the second term on the right hand side of (4.10) we need localization and (conveniently) the Taylor formula

$$V(y_- = 0) = \tfrac{y_+\cdot Q_+ y_+}{2x^2} + \int_0^1 2^{-1}(1-s)^2 \tfrac{d^3}{ds^3}V\big(z_0 + s\tfrac{y_+}{x}\big)ds;\ x > 1, \tag{4.11}$$

cf. (2.10).

We abbreviate

$$\begin{aligned} &\beta_\epsilon = \beta F_\epsilon; \\ &F_\epsilon = F_-\left(2\Big(\xi - \frac{x}{t}\Big)^2\right)F_+\Big(2\kappa^{-1}\frac{x}{t}\Big)g_1(\xi)F_-\left(\frac{1}{2\epsilon}B(t)\right),\ \epsilon > 0, \\ &\beta = \beta_0, \beta_j^+ \text{ or } \beta_j^-. \end{aligned} \tag{4.12}$$

We obtain by various commutation and estimation (cf. (4.5), Lemma 2.8, (4.7), and (4.25) and (4.26) stated below) that in all cases

$$\beta_\epsilon = O\big(t^0\big)B(t)F_\epsilon + O\big(t^{-1}\big) = O\big(t^0\big)B(t)F_\epsilon$$

uniformly in ϵ. Upon inserting

$$I = F_-\left(\frac{1}{4\epsilon}B(t)\right) + F_+\left(\frac{1}{4\epsilon}B(t)\right)$$

after the factor $O\big(t^0\big)$ we obtain by another commutation that

$$\beta_\epsilon = O\big(t^0\big)F_-\big(\tfrac{1}{4\epsilon}B(t)\big)B(t)F_\epsilon + O\big(t^{-1}\big) \tag{4.13}$$

with the same factor $O\big(t^0\big)$ as before.

For a later application we notice that without the factor F_ϵ we have the bounds

$$\beta = O\big(t^0\big)B(t)^{\frac{1}{2}},\ D(t) = O\big(t^{-1}\big)B(t). \tag{4.14}$$

By using the Cauchy Schwarz inequality and the spectral theorem it follows from (4.13) that

$$\beta_\epsilon^\star \beta_\epsilon \le C\epsilon F_\epsilon^\star B(t) F_\epsilon + C(\epsilon) t^{-2}. \tag{4.15}$$

Using (4.15) for a small ϵ we can estimate the (localized) right hand side of (4.10) to obtain

$$F_\epsilon^\star D(t) F_\epsilon \le -C^{-1} t^{-1} F_\epsilon^\star B(t) F_\epsilon + C t^{-3};\ C = C(\epsilon) > 0. \tag{4.16}$$

By a result of [He] (combined with Lemma 2.7) we may for any $\epsilon > 0$ and $F_- \in \mathcal{F}_-$ multiply $F_-\big(\epsilon^{-1} B(t)\big)$ to the right hand side of (4.1) and still have the statement. In particular if we define, with $F = F(t)$ given by (4.1),

$$H_{k,\epsilon,\nu}(t) = F_-\Big(\epsilon^{-1}\big(\tfrac{t}{k}\big)^{2\nu} B(t)\Big) F;\ k \in \mathbf{N},\ \epsilon, \nu > 0, \tag{4.17}$$

then

$$\psi(t) = H_{k,\epsilon,\nu}(t)\psi(t) + o\big(t^0\big) \text{ along } t = k \to \infty. \tag{4.18}$$

We fix an $\epsilon > 0$ in (4.18) in agreement with (4.16) and want to prove the existence of $\nu > 0$ such that

$$\psi(t) = H_{k,\epsilon,\nu}(t)\psi(t) + o_t\big(k^0\big), \tag{4.19}$$

where the term $o_t\big(k^0\big)$ vanishes uniformly w.r.t. $t \ge k$ as $k \to \infty$.

We show (4.19) by proving approximate positivity of the Heisenberg derivative of $H_{k,\epsilon,\nu}(t)^\star H_{k,\epsilon,\nu}(t)$ in the state $\psi(t)$ and then invoking (4.18) and Lemma 2.10. Precisely we need to show that

$$\begin{aligned} &\big\langle H_{k,\epsilon,\nu}(t)^\star H_{k,\epsilon,\nu}(t)\big\rangle_t - \big\langle H_{k,\epsilon,\nu}(k)^\star H_{k,\epsilon,\nu}(k)\big\rangle_k \\ &= \int_k^t \langle \mathbf{D}_t(H^\star H)\rangle_s ds + \int_k^t \langle i[(V - V_t, H^\star H)]\rangle_s ds \\ &= \int_k^t \langle \mathbf{D}_{\mathrm{t}}(H^\star H)\rangle_s ds + o_t\big(k^0\big) \ge o_t\big(k^0\big). \end{aligned} \tag{4.20}$$

To see that this is possible we first focus on the Heisenberg derivative of “the new factor” $F_-^2\Big(\epsilon^{-1}\big(\tfrac{t}{k}\big)^{2\nu} B(t)\Big)$. The contributions from the derivatives of all other factors can be treated by Lemmas 2.6 and 3.3.

Clearly we need to use Lemma 2.8 for $B(t)$ multiplied by $\epsilon^{-1}\big(k^{-1}t\big)^{2\nu}$ (not for $B(t)$ itself) and therefore the discussion and some formulas above need slight modification. With $D(t)$ and $D_r(t)$ given by (4.8) and (4.9), respectively, we compute

$$
\begin{aligned}
&\mathbf{D}_t\Big(\tilde{B}(t)\Big) = \tilde{D}(t) + \tilde{D}_r(t);\\
&\tilde{B}(t) = \epsilon^{-1}\bigg(\frac{t}{k}\bigg)^{2\nu} B(t),\\
&\tilde{D}(t) = \epsilon^{-1}\bigg(\frac{t}{k}\bigg)^{2\nu} \big(2\nu t^{-1}B(t) + D(t)\big),\\
&\tilde{D}_r(t) = \epsilon^{-1}\bigg(\frac{t}{k}\bigg)^{2\nu} D_r(t).
\end{aligned}
\tag{4.21}
$$

We demand $\nu \in \big(0, \frac{1}{4}\big)$ and $2\nu \leq C^{-1}$ where C is the constant on the right hand side of (4.16). By the latter estimate we then obtain

$$
\begin{aligned}
&F^\star G_\epsilon^\star F_\epsilon^\star \tilde{D}(t) F_\epsilon G_\epsilon F \leq C\bigg(\frac{t}{k}\bigg)^{2\nu} t^{-3};\\
&G_\epsilon = \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(\tilde{B}(t)\Big),\ C>0.
\end{aligned}
\tag{4.22}
$$

Clearly the right hand side integrated obeys

$$
\int_k^t C\big(\tfrac{s}{k}\big)^{2\nu} s^{-3}ds \ =\ -o_t\big(k^0\big), \tag{4.23}
$$

in agreement with (4.20).

By (2.42) and (2.48) (the latter for $F_-^2 \in \mathcal{F}_-$) we obtain from (4.22) the bound

$$
\begin{aligned}
&\int_k^t \Big\langle F^\star\Big(\mathbf{D}_t F_-^2\Big(\tilde{B}(s)\Big)\Big)F\Big\rangle_s ds \geq o_t\big(k^0\big) + \int_k^t \Big\langle \tilde{R}_1(s)+\cdots+\tilde{R}_4(s)\Big\rangle_s ds;\\
&\tilde{R}_3(t) = F^\star G_\epsilon^\star F_\epsilon^\star \tilde{D}(t) F_\epsilon G_\epsilon F - F^\star G_\epsilon^\star \tilde{D}(t) G_\epsilon F,\\
&\tilde{R}_4(t) = -\frac{1}{\pi}F^\star \int_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}_-^2\Big)(w)\Big(\tilde{B}(t)-w\Big)^{-1}\tilde{D}_r(t)\Big(\tilde{B}(t)-w\Big)^{-1} dudvF,
\end{aligned}
\tag{4.24}
$$

where the function $\tilde{F}_-^2$ in the expression $\tilde{R}_4(t)$ is in $C_0^\infty(\mathbf{C})$ and given as an almost analytic extension of F_-^2 from a neighborhood of the positive real axis, and the expressions $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ are specified by (2.44) and (2.46) with $C_0^\infty(\mathbf{C})$ – extensions.

To deal with $\tilde{R}_3(t)$ we need various commutator estimates:

By (2.50) and (2.51)

$$
\Big|\Big|\Big[g_1(\xi), F_-\Big(4\big(\xi - \tfrac{x}{t}\big)^2\Big)\Big]\Big|\Big| = \Big|\Big|\Big[g_1(\xi), F_-\Big(4\big(\tfrac{x}{t}\big)^2\Big)\Big]\Big|\Big| \leq Ct^{-1}, \tag{4.25}
$$

(4.26) $\quad ||\Big[F_+\big(2\kappa^{-1}\frac{x}{t}\big), F_-\Big(4\big(\xi - \frac{x}{t}\big)^2\Big)\Big]|| \leq Ct^{-1}.$

Similarly, using (2.42),

(4.27) $\quad ||[g_1(\xi), G_\epsilon]|| \leq C\big(\frac{t}{k}\big)^\nu t^{-1},$

and

(4.28) $\quad ||\big[F_+\big(2\kappa^{-1}\frac{x}{t}\big), G_\epsilon\big]||, ||\Big[F_-\Big(2\big(\xi - \frac{x}{t}\big)^2\Big), G_\epsilon\Big]|| \leq C\big(\frac{t}{k}\big)^\nu t^{-1}.$

We conclude from (4.25)-(4.28) that

(4.29) $\quad ||(F_\epsilon - I)G_\epsilon F|| \leq Ct^{\nu-1},$

for $C > 0$ independent of k.

Upon checking the representation formula (2.42) used for (4.27) and (4.28) we also obtain the bound

(4.30) $\quad ||\tilde{B}(t)(F_\epsilon - I)G_\epsilon F|| \leq Ct^{\nu-1},$

and therefore (cf. (4.14))

(4.31) $\quad ||\tilde{D}(t)(F_\epsilon - I)G_\epsilon F|| \leq Ct^{\nu-2},$

again both estimates being uniformly in k.

By (4.31) we get

(4.32) $\quad ||\tilde{R}_3(t)|| \leq Ct^{\nu-2}.$

Clearly we obtain from (4.32) that

(4.33) $\quad \int_k^t \Big\langle \tilde{R}_3(s) \Big\rangle_s ds = o_t\big(k^0\big).$

To obtain the similar estimate

(4.34) $\quad \int_k^t \Big\langle \tilde{R}_1(s) + \tilde{R}_2(s) \Big\rangle_s ds = o_t\big(k^0\big),$

we notice the bound

(4.35) $\quad ||ad_{\tilde{B}(t)}\Big(\tilde{D}(t)\Big)\Big(\tilde{B}(t) - i\Big)^{-1}|| \leq Ct^{3\nu-2},$

which by the defining formulas of Lemma 2.8 yields (4.34).

It remains to show

$$\int_k^t \left\langle \tilde{R}_4(s) \right\rangle_s ds \;=\; o_t\big(k^0\big). \tag{4.36}$$

We skip the details, but remark that we use the "support-properties" of the factor F after commutation. Notice for the contributions from the terms T_1 and T_2 that the y_- – localization comes into play: We use (5.4) (stated below) yielding the bound

$$||\tilde{R}_4(t)|| \;\le\; Ct^{\nu-2\rho_2-1}. \tag{4.37}$$

Clearly (4.37) implies (4.36) if

$$\tfrac{\nu}{2} \;<\; \rho_2. \tag{4.38}$$

We also skip most of the discussion of contributions from the Heisenberg derivative of all other factors in the product $H_{k,\epsilon,\nu}(t)^\star H_{k,\epsilon,\nu}(t)$. However we shall elaborate on the contribution from the factors $F_-\Big(t^{\rho_1}\tilde{b}_t\Big)$:

The result Lemma 3.3 is only applicable after a symmetrizing procedure due to the factor $F_-^2\Big(\epsilon^{-1}\big(\tfrac{t}{k}\big)^{2\nu}B(t)\Big)$. We notice that by (3.21) and other previous constraints

$$\rho_1 + \nu \;<\; \tfrac{1}{2},\; \rho_1 - \delta + 2\sigma \;\le\; 0. \tag{4.39}$$

To explain how (4.39) is used we look at the expression

$$\cdots F_-\Big(t^{\rho_1}\tilde{b}_t\Big)F_-^2\left(\epsilon^{-1}\left(\frac{t}{k}\right)^{2\nu}B(t)\right)\mathbf{D}_tF_-\Big(t^{\rho_1}\tilde{b}_t\Big)\cdots + h.c..$$

Up to an integrable term this is given by the same expression except that $\mathbf{D}_tF_-\Big(t^{\rho_1}\tilde{b}_t\Big)$ is replaced by

$$\begin{aligned}
&-\big(-F_-'\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)T\big(-F_-'\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big);\\
&T = t^{\rho_1-\delta}\tilde{c}_t + \rho_1 t^{\rho_1-1}\tilde{b}_t + T_3(t)\\
&-t^{\rho_1}N_t^{-1}(\nabla_{y_-}r)\Big(t^{-\delta}y_-\Big)\cdot\nabla_{y_-}V_tN_t^{-1},
\end{aligned}$$

where the term $T_3(t)$ is specified in (3.10) (with $c = t^{\rho_1}$).

Now we use (3.21) to get the expression (up to an integrable term uniformly bounded in k)

$$\cdots - F_-\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)T\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)F_-\cdots;\; F_- = F_-\left(\epsilon^{-1}\left(\frac{t}{k}\right)^{2\nu}B(t)\right).$$

Next we substitute the upper bound

$$B = t^{\rho_1-\delta}\tilde{c}_t + (\rho_1+6\sigma)t^{-1}I - t^{\rho_1}N_t^{-1}(\nabla_{y_-}r)\Big(t^{-\delta}y_-\Big)\cdot\nabla_{y_-}V_tN_t^{-1}$$

for T and commute $\sqrt{B}\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)$ through F_- using Lemma 2.11 (with $c = (\rho_1+6\sigma)t^{-1}$) and (4.39) to obtain the expression

$$\cdots - \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)\sqrt{B}F_-^2\left(\epsilon^{-1}\left(\frac{t}{k}\right)^{2\nu}B(t)\right)\sqrt{B}\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(t^{\rho_1}\tilde{b}_t\Big)\cdots.$$

Notice that the second condition of (4.39) assures $\sqrt{B} = O\big(t^0\big)$.

Finally we replace the middle factor by the upper bound I and invoke (3.41).

We remark that the above arguments treating the derivative of $F_-\Big(t^{\rho_1}\tilde{b}_t\Big)$ will by used again in Section 5.

By choosing $\nu > 0$ sufficiently small we can thus get (4.19). Picking it slightly smaller we eliminate the parameters ϵ and k and have as a conclusion the following result.

Lemma 4.1 *Suppose (3.21). Then for the state $\psi(t)$ in Theorem 2.2 and with $F(t)$ given by* (4.1) *and $B(t)$ by* (4.4) *there exists $\nu_0 \in \big(0,\frac{1}{4}\big)$ independent of $\rho_1 > \rho_2$ such that if in addition $\nu \in (0,\nu_0)$ and (4.38) holds then*

(4.40) $\quad \psi(t) = F_-\big(t^{2\nu}B(t)\big)F(t)\psi(t) + o\big(t^0\big)$ *for* $t \to +\infty$.

Moreover our proof gives the following bound (cf. (4.22)), which is similar to the ones of Lemmas 2.6 and 3.3.

Lemma 4.2 *Under the conditions of Lemmas 3.3 and 4.1, for any localized state $\psi(t)$ as in Lemmas 2.6 and 3.3*

(4.41) $\quad \int_1^\infty t^{-1}\big\langle -F(t)^\star\big(F_-^{2\prime}\big)\big(t^{2\nu}B(t)\big)F(t)\big\rangle_t dt \le C||\psi||^2.$

5 A comparison dynamics

In this section we introduce some new "variables" similar to the γ's considered in Section 4 and we construct a comparison time-dependent propagator.

Let g and g_1 be given as in Section 4. Pick a positive function $h_- \in C^\infty(\mathbf{R}\backslash\{0\})$ with $h_- = 1$ on a neighborhood of the support of g_1. Suppose furthermore that the function $x^2(2h_-(x))^{-1}$ is constant on a neighborhood of zero and constant outside a bounded set. (Notice that a similar function appeared in Section 4.) We write the part of the Hessian on Y_- as $Q_- = \text{diag}(q_{-,1}, \cdots, q_{-,m_-})$ and define

$$(5.1)\quad \begin{aligned} \alpha^+_{-,j} &= \frac{1}{2}\Big(1 + \sqrt{1 - 4q_{-,j}\xi^{-2}h_-(\xi)}\Big), \\ \alpha^-_{-,j} &= \frac{1}{2}\Big(1 - \sqrt{1 - 4q_{-,j}\xi^{-2}h_-(\xi)}\Big), \\ \gamma^+_{-,j} &= \Big(\eta_{-,j} - \alpha^+_{-,j}\frac{y_{-,j}}{t}\Big), \\ \gamma^-_{-,j} &= \Big(\eta_{-,j} - \alpha^-_{-,j}\frac{y_{-,j}}{t}\Big);\ t > 1. \end{aligned}$$

Let

$$(5.2)\quad \begin{aligned} N^+_t &= \sum_{j=1}^{m_-}\Big(\gamma^+_{-,j}\Big)^2, \\ N^-_t &= \sum_{j=1}^{m_-}\Big(\gamma^-_{-,j}\Big)^2. \end{aligned}$$

We may multiply the first term on the right hand side of (4.40) by $F_-\big(N^-_t\big)F_-\big(N^+_t\big)$ for any given $F_- \in \mathcal{F}_-$ and still have the statement, cf. [He], i.e.

$$(5.3)\quad \begin{aligned} &\psi(t) = G(t)\psi(t) + o\big(t^0\big) \text{ for } t \to +\infty; \\ &G(t) = F_-\big(N^-_t\big)F_-\big(N^+_t\big)F_-\big(t^{2\nu}B(t)\big)F(t), \end{aligned}$$

with the parameters given as in Lemma 4.1.

We shall find the comparison dynamics by suitably changing the potential outside the “support” of $G(t)$. Notice that by the Taylor formula and (2.5)

$$(5.4)\quad \begin{aligned} &V(z) = V(y_- = 0) + \frac{y_- \cdot Q_-\big(\frac{y_+}{x}\big)y_-}{2x^2} \\ &+ \int_0^1 2^{-1}(1-s)^2\frac{d^3}{ds^3}V\Big(z_0 + s\frac{y_-}{x} + \frac{y_+}{x}\Big)ds; \\ &Q_-\Big(\frac{y_+}{x}\Big) := \nabla^2_{y_-}V\Big(z_0 + \frac{y_+}{x}\Big),\ x > 1. \end{aligned}$$

Motivated by (5.4) we consider for $t \geq 1$ the “potential”

$$
\begin{aligned}
&\check{V}(t) = V_-(t, y_-) + R(t, y_-);\\
&V_-(t, y_-) = \frac{h_-(\xi) y_- \cdot Q_- y_-}{2t^2\xi^2},\\
&R(t, y_-) = R = R_1 + R_2 + R_3,\ R_j = F_j^\star W_j F_j,\\
&W_1 = V(y_- = 0),\\
&F_1 = F_-\big(t^{-2} y_-^2\big) F_+\Big(2\kappa^{-1}\frac{x}{t}\Big) F_-\bigg(2\Big(\xi - \frac{x}{t}\Big)^2\bigg) g_1(\xi) F_-\big(2^{-1} t^{2\nu} B(t)\big),\\
&W_2 = \frac{y_- \cdot Q_-\big(\frac{y_+}{x}\big) y_-}{2x^2} - \frac{h_-(\xi) y_- \cdot Q_- y_-}{2t^2\xi^2},\ F_2 = F_-\big(2^{-1} t^{2\rho_2 - 2} y_-^2\big) F_1,\\
&W_3 = \int_0^1 2^{-1}(1-s)^2 \frac{d^3}{ds^3} V\Big(z_0 + s\frac{y_-}{x} + \frac{y_+}{x}\Big) ds,\ F_3 = F_2.
\end{aligned}
\tag{5.5}
$$

We notice that for some $\epsilon > 0$

$$
\begin{aligned}
&||(F_j(t, y_-) - I) G(t)^\star G(t)|| = O\big(t^{-1-\epsilon}\big),\\
&||G(t)^\star G(t)(F_j(t, y_-) - I)|| = O\big(t^{-1-\epsilon}\big);\ F_j(t, y_-) = F_j,
\end{aligned}
\tag{5.6}
$$

and that the second term $R(t, y_-)$ is bounded. In fact we have the following bounds for $k \in \mathbf{N} \cup \{0\}$:

$$||ad_\xi^k\big(\partial_{y_-}^\alpha R_1\big)|| = O\Big(t^{-2\nu-(1-\nu)k-|\alpha|}\Big). \tag{5.7}$$

$$\tfrac{\partial}{\partial y_{-,j}} R_1 = \tfrac{|y_-|}{t} R_{1,j}(t, y_-);\ ||R_{1,j}|| = O\big(t^{-2\nu-1}\big). \tag{5.8}$$

$$||ad_\xi^k\big(\partial_{y_-}^\alpha R_2\big)|| = O\Big(t^{-2\rho_2-\nu-(1-\nu)k-(1-\rho_2)|\alpha|}\Big). \tag{5.9}$$

$$\tfrac{\partial}{\partial y_{-,j}} R_2 = \tfrac{|y_-|}{t} R_{2,j}(t, y_-);\ ||R_{2,j}|| = O\big(t^{-\nu-1}\big). \tag{5.10}$$

$$||ad_\xi^k\big(\partial_{y_-}^\alpha R_3\big)|| = O\Big(t^{-3\rho_2-(1-\nu)k-(1-\rho_2)|\alpha|}\Big). \tag{5.11}$$

$$\tfrac{\partial}{\partial y_{-,j}} R_3 = \tfrac{|y_-|}{t} R_{3,j}(t, y_-);\ ||R_{3,j}|| = O\big(t^{-\rho_2-1}\big). \tag{5.12}$$

We shall need the following propagation estimates for the full dynamics.

Lemma 5.1 *Under the conditions of Lemma 4.2 and*

$$\rho_1 + 2\sigma - 2\delta < -\tfrac{3}{2}, \tag{5.13}$$

and with $H(t) := F_-\big(t^{2\nu} B(t)\big) F(t)$

$$\int_1^\infty t^{-1}\big\langle -H(t)^\star F_-'\big(N_t^+\big) H(t)\big\rangle_t dt \le C||\psi||^2, \tag{5.14}$$

(5.15) $\int\limits_1^\infty t^{-1}\left\langle H(t)^\star \sum\limits_{j=1}^{m_-} \alpha^-_{-,j}\Big(\gamma^-_{-,j}\Big)^2 F'_-\big(N^-_t\big) H(t)\right\rangle_t dt \le C||\psi||^2.$

Proof Consider for (5.14)

$$\Phi(t) = H(t)^\star F_-\big(N^+_t\big) H(t),$$

while for (5.15)

$$\Phi(t) = H(t)^* F_-\big(N^-_t\big) H(t).$$

Computing the Heisenberg derivative give factors that can be treated using Lemmas 3.3 and 4.2 after symmetrizing, cf. the arguments after (4.39) combined with (5.13), and new terms from taking the derivative of $F_-\big(N^+_t\big)$ and $F_-\big(N^-_t\big)$. To treat the latter we can replace the potential by the one of (5.5) (cf. (5.6)). So we need only to consider the derivatives

$$\mathbf{D}_t F_-\big(N^+_t\big) \text{ and } \mathbf{D}_t F_-\big(N^-_t\big);\ \mathbf{D}_t = \frac{d}{dt} + i\Big[2^{-1}p^2 + \breve{V}(t), \cdot\Big].$$

Notice that these derivatives have a natural interpretation as forms on $\mathcal{D}\big(p^2\big) \cap \mathcal{D}\big(y_-^2\big)$ and that with the splitting

$$\mathbf{D}_t = \mathbf{D}_{t-} + i[R(t, y_-), \cdot];\ \mathbf{D}_{t-} = \frac{d}{dt} + i\big[2^{-1}p^2 + V_-(t, y_-), \cdot\big],$$

we may use the proof of (2.54) and Lemma 2.8 to justify the computations

(5.16) $\mathbf{D}_{t-} F_-\big(N^+_t\big) = -2t^{-1} \sum\limits_{j=1}^{m_-} \alpha^+_{-,j}\Big(\gamma^+_{-,j}\Big)^2 F'_-\big(N^+_t\big),$

(5.17) $\mathbf{D}_{t-} F_-\big(N^-_t\big) = -2t^{-1} \sum\limits_{j=1}^{m_-} \alpha^-_{-,j}\Big(\gamma^-_{-,j}\Big)^2 F'_-\big(N^-_t\big),$

(5.18) $i\big[R(t, y_-), F_-\big(N^+_t\big)\big] = O\Big(t^{-(1+\nu)}\Big),$

and

(5.19) $i\big[R(t, y_-), F_-\big(N^-_t\big)\big] = O\Big(t^{-(1+\nu)}\Big).$

(For (5.18) and (5.19) we used (5.7), (5.9) and (5.11).)

We obtain (5.14) and (5.15) from (5.16)-(5.19) and the fact that $\alpha^+_{-,j} \ge 1$.

□

Now let $\breve{U}(t)$ be the propagator for the Hamiltonian $\breve{H}(t) = 2^{-1}p^2 + \breve{V}(t)$, that is

(5.20) $$i\frac{d}{dt}\breve{U}(t) = \breve{H}(t)\breve{U}(t)$$
$$\breve{U}(1) = I.$$

(Consult Appendix A for basic properties.)

With ψ given as in Theorem 2.2 we want to show the existence of

$$\breve{\psi} = \lim_{t \to +\infty} \breve{U}(t)^{\star}\psi(t) = \lim_{t \to +\infty} \breve{U}(t)^{\star} G(t)^{\star} G(t)\psi(t). \tag{5.21}$$

For that we need the following propagation estimates for $\breve{U}(t)$.

Lemma 5.2 *Under the conditions of Lemma 5.1, the conditions*

$$\rho_1 - \rho_2 < \min(\nu, \rho_2), \tag{5.22}$$

$$2\nu > \rho_1 > \rho_2 > \tfrac{\nu}{2}, \tag{5.23}$$

and with $\langle A(t)\rangle_t$ *used to denote the expectation* $\langle \phi(t), A(t)\phi(t)\rangle$ *in any state* $\phi(t) = \breve{U}(t)\phi$, $\phi \in L^2(\mathbf{R}^n)$,

$$\int_1^\infty t^{-1}\left\langle -F_-'\left(N_t^+\right)\right\rangle_t dt \le C||\phi||^2, \tag{5.24}$$

$$\int_1^\infty t^{-1}\left\langle \sum_{j=1}^{m_-} \alpha_{-,j}^- \left(\gamma_{-,j}^-\right)^2 F_-'\left(N_t^-\right)\right\rangle_t dt \le C||\phi||^2, \tag{5.25}$$

$$\begin{aligned} &\int_1^\infty \left\langle H_3(t)^* \left(-F_-^{2\prime}\right)^{\frac{1}{2}}\left(t^{\rho_1}\tilde{b}_t\right)\tilde{f}_t\left(-F_-^{2\prime}\right)^{\frac{1}{2}}\left(t^{\rho_1}\tilde{b}_t\right) H_3(t)\right\rangle_t dt \le C||\phi||^2;\\ &\tilde{f}_t = t^{\rho_1-\delta}\tilde{c}_t + t^{-1}I - t^{\rho_1}(N_t)^{-1}(\nabla_{y_-}r)\left(t^{-\delta}y_-\right)\cdot\left(\frac{h_-(\xi)}{t^2\xi^2}Q_-y_-\right)(N_t)^{-1},\\ &H_3(t) = (N_t)^{-1}F_-\left(N_t^-\right)F_-\left(N_t^+\right), \end{aligned} \tag{5.26}$$

$$\begin{aligned} &\int_1^\infty \left\langle H_4(t)^{\star} F_-^2\left(t^{\rho_1}\tilde{b}_t\right)\left(t^{\rho_2-1-\rho_1}\left(I - t^{\rho_1}\tilde{b}_t\right) + t^{-1}I\right)H_4(t)\right\rangle_t dt \le C||\phi||^2;\\ &H_4(t) = \left(-F_-^{2\prime}\right)^{\frac{1}{2}}\left(t^{\rho_2-1}r_t\right)F_-\left(N_t^-\right)F_-\left(N_t^+\right), \end{aligned} \tag{5.27}$$

$$\begin{aligned} &\int_1^\infty t^{-1}\left\langle -H_5(t)^{\star}\left(F_-^{2\prime}\right)\left(t^{2\nu}B(t)\right)H_5(t)\right\rangle_t dt \le C||\phi||^2;\\ &H_5(t) = F_-\left(t^{\rho_1}\tilde{b}_t\right)F_-\left(t^{\rho_2-1}r_t\right)H_3(t)F_-\left(4\left(\xi - \frac{x}{t}\right)^2\right)F_+\left(\kappa^{-1}\frac{x}{t}\right)g(\xi), \end{aligned} \tag{5.28}$$

$$\int_1^\infty \Big\langle H_6(t)^*\big(-F_-^{2\prime}\big)^{\frac12}\Big(t^{\rho_1}\tilde b_t\Big)\tilde f_t\big(-F_-^{2\prime}\big)^{\frac12}\Big(t^{\rho_1}\tilde b_t\Big)H_6(t)\Big\rangle_t dt \le C||\phi||^2;$$

(5.29) $$\tilde f_t = t^{\rho_1-\delta}\tilde c_t + t^{-1}I - t^{\rho_1}(N_t)^{-1}(\nabla_{y_-}r)\Big(t^{-\delta}y_-\Big)\cdot\nabla_{y_-}V(N_t)^{-1},$$

$$H_6(t) = F_-\big(t^{2\nu}B(t)\big)F_-\big(t^{\rho_2-1}r_t\big)H_3(t)F_-\left(4\Big(\xi-\frac{x}{t}\Big)^2\right)F_+\Big(\kappa^{-1}\frac{x}{t}\Big)g(\xi).$$

Sketch of proof Consider the following propagation observables in the indicated order to obtain the bounds in the listed order:

(5.30) $\Phi_1(t) = F_-\big(N_t^+\big).$

(5.31) $\Phi_2(t) = F_-\big(N_t^-\big).$

(5.32) $\Phi_3(t) = H_3(t)^*F_-^2\Big(t^{\rho_1}\tilde b_t\Big)H_3(t).$

(5.33) $\Phi_4(t) = H(t)^*H(t);\ H(t) = F_-\Big(t^{\rho_1}\tilde b_t\Big)F_-\big(t^{\rho_2-1}r_t\big)H_3(t).$

(5.34) $\Phi_5(t) = H_5(t)^*F_-^2\big(t^{2\nu}B(t)\big)H_5(t).$

(5.35) $\Phi_6(t) = H_6(t)^\star F_-^2\Big(t^{\rho_1}\tilde b_t\Big)H_6(t).$

The computation and treatment of the Heisenberg derivatives are similar to previously discussed derivatives:

We remark that $\breve U(t)$ preserves $\mathcal D = \mathcal D\big(p^2\big)\cap\mathcal D\big(y_-^2\big)$, cf. Appendix A. Therefore we conveniently consider expectations for $\phi\in\mathcal D$.

For (5.30) and (5.31) we refer to the proof of Lemma 5.1.

For (5.32) and (5.33) we use the proof of Lemma 3.3, (5.22), (5.23), (5.24) and (5.25). Notice the bounds

$$\sup_{t>1}||\eta_-^2F_-\big(N_t^-\big)F_-\big(N_t^+\big)||,\ \sup_{t>1}||\Big(\frac{y_-}{t}\Big)^2F_-\big(N_t^-\big)F_-\big(N_t^+\big)||<\infty,$$

which compensate for energy-localization, and that $t^{\rho_1}||\partial_{y_-}^\alpha R(t,y_-)||$ (by (5.22) and (5.23)) is integrable for $|\alpha|=1$.

To treat the contribution from $\mathbf D_tF_-^2\big(t^{2\nu}B(t)\big)$ to the derivative of the observable (5.34) we may replace $\mathbf D_t$ by $\mathbf D$ (i.e. use the time-independent potential), cf. (5.6), and then use the proof of Lemma 4.2. For other derivatives we use either (5.24)-(5.27) or the fact that the observable in question contains a full weight yielding integrability without need for symmetrization. The argument for (5.35) is similar.

□

Lemma 5.3 *Under the assumptions of Lemma 5.2 there exists the limit (5.21).*

Sketch of proof Using (5.6) it suffices to show integrability of the Heisenberg derivative $\mathbf{D}$ of $G(t)^\star G(t)$, which in turn amounts to integrating the contribution from the derivative of each of the various factors. Some of the latter are (when combined with the other factors) $O\big(t^{-1-\epsilon}\big)$ for some $\epsilon > 0$ and therefore integrable. An example is the contribution from $\mathbf{D}g(\xi)$ which by (2.24) and the presence of a full weight is $O\big(t^{-1-\epsilon}\big)$.

Other factors contribute by an expression that needs to be symmetrized. After symmetrizing we then invoke the propagation estimates in Lemmas 3.3, 4.2 and 5.1 for e^{-itH} and (5.24), (5.25), (5.27), (5.28) and (5.29) for $\breve{U}(t)$.

□

Obviously, by Lemma 5.3, Theorem 2.2 follows if we can prove that for any $\breve{\psi} \in L^2(\mathbf{R}^n)$ and $\epsilon > 0$

(5.36) $\quad F_-\big(t^{\epsilon-1}|y_-|\big)\breve{U}(t)\breve{\psi} = o(1)$ for $t \to +\infty$.

In the Sections 6–8 we shall prove (5.36).

6 A change of variables

In this section we shall suitably rescale the y_- −variable (and the time-variable) and thereby get a new formulation of (5.36). This transformation appears in [DG, Section 3.8.3], see also [Y].

Define for $\phi_- \in L^2\big(\mathbf{R}^{m_-}_{y_-}\big)$ or $L^2(\mathbf{R}^n_z)$ and $t \geq 1$

(6.1)
$$(U_0\phi)(y_-) = e^{i4^{-1}y_-^2}\phi(y_-),$$
$$(T_t\phi)(y_-) = t^{-4^{-1}m_-}\phi\Big(t^{-2^{-1}}y_-\Big);\ m_- = \dim Y_-.$$

Clearly letting

(6.2) $\quad A = \frac{1}{2}(\eta_- \cdot y_- + y_- \cdot \eta_-)$

we have

(6.3) $\quad T_t = e^{-i\frac{1}{2}(\ln t)A}.$

We define

(6.4) $\quad \tilde{U}(t) = U_0^{-1}T_t^{-1}\breve{U}(t)U_0,$

and "compute"

$$
(6.5)\quad
\begin{aligned}
&i\frac{d}{dt}\tilde{U}(t) = \tilde{H}(t)\tilde{U}(t);\\
&\tilde{H}(t) = 2^{-1}\big(\xi^2 + \eta_+^2\big) + t^{-1}\Big(2^{-1}\eta_-^2 - 2^{-1}y_- \cdot Py_- + tR\Big(t, t^{\frac{1}{2}}y_-\Big)\Big),\\
&P = 4^{-1}I - \frac{h_-(\xi)}{\xi^2}Q_-,
\end{aligned}
$$

where the "remainder" term R is given by (5.5).

Next we substitute $t \to e^t$ defining $U(t) = \tilde{U}\big(e^t\big)$ so that for $t \geq 0$

$$
(6.6)\quad
\begin{aligned}
&i\frac{d}{dt}U(t) = H(t)U(t);\\
&H(t) = H_+(t) + H_- + R(t),\\
&H_+(t) = 2^{-1}e^t\big(\xi^2 + \eta_+^2\big),\\
&H_- = 2^{-1}\big(\eta_-^2 - y_- \cdot Py_-\big),\\
&R(t) = e^tR\Big(e^t, e^{2^{-1}t}y_-\Big).
\end{aligned}
$$

In Appendix A we define $U(t)$ independently in terms of (6.6) and examine some basic properties.

The second term H_- will be essential for the proof of (5.36). Notice the lower bound

$$
(6.7)\quad P \;\geq\; 4^{-1}I.
$$

Obviously the first term $H_+(t)$ commutes with the variables y_-, η_- and ξ which are the only variables that will be used in various constructions of "propagation observables". The third term $R(t)$ commutes with y_- but not with η_- and ξ. The commutators with the latter can be read off from (5.7)-(5.12):

For simplicity we take here and in Sections 7 and 8

$$
(6.8)\quad \rho_2 \;=\; \nu,
$$

which is in agreement with the assumptions of Lemma 5.2.

$$
(6.9)\quad ||ad_\xi^k\big(\partial_{y_-}^\alpha R(t)\big)|| = O\Big(e^{t\big(1-2\nu-(1-\nu)k-\frac{1}{2}|\alpha|\big)}\Big) + O\Big(e^{t\big(1-3\nu-(1-\nu)k-\big(\frac{1}{2}-\nu\big)|\alpha|\big)}\Big).
$$

$$
(6.10)\quad \tfrac{\partial}{\partial y_{-,j}}R(t) \;=\; |y_-|R_j(t);\; ||R_j(t)|| \;=\; O\big(e^{-\nu t}\big).
$$

7 Implementing the uncertainty principle

We shall prove the following estimate which constitutes the basic "quantum input" in our proof. Let A and $U(t)$ be given by (6.2) and (6.6), respectively.

Proposition 7.1 *For any* $\varepsilon > 0$ *and* $\psi \in L^2(\mathbf{R}^n)$

$$\int_0^\infty ||\langle A\rangle^{-\varepsilon} U(t)\psi||^2 dt \le C||\psi||^2. \tag{7.1}$$

The remaining part of this section is devoted to proving Proposition 7.1 under the assumption $\varepsilon < \frac{1}{2}$.

Formally, by (6.7) and (6.10)

$$\begin{aligned} i[H(t), A] &= \eta_-^2 + y_- \cdot P y_- - y_- \cdot \nabla_{y_-} R(t) \\ &\ge \eta_-^2 + \frac{1}{4} y_-^2 + O\big(e^{-\nu t}\big) y_-^2, \end{aligned} \tag{7.2}$$

and therefore for $t > T$ for T sufficiently large

$$i[H(t), A] \ge \eta_-^2 + \tfrac{1}{5} y_-^2 \ge \tfrac{1}{5}. \tag{7.3}$$

We define for some $F_+ \in \mathcal{F}_+$ and with $R(t)$ given by (6.6)

$$R_T(t) = F_+\Big((2T)^{-1} t\Big) R(t), \tag{7.4}$$

and let $U_T(t)$ denote the propagator given by replacing accordingly in (6.6).

Obviously (7.1) follows from

$$\int_{-\infty}^\infty ||\langle A\rangle^{-\varepsilon} U_T(t)\psi||^2 dt \le C||\psi||^2, \tag{7.5}$$

which in turn will be proven for T large by extending the Hilbert space as in [Ho].

So we introduce on the Hilbert space $\mathcal{H} = L^2\big(\mathbf{R}_t, L^2(\mathbf{R}_z^n)\big)$

$$K_T = \tau + H_T(t), \tag{7.6}$$

where $\tau = -i\frac{d}{dt}$ and $H_T(t)$ is the Hamiltonian corresponding to $U_T(t)$.

We claim that (7.5) will follow from the bound

$$\int_{-\infty}^\infty ||\langle A\rangle^{-\varepsilon} e^{-i\sigma K_T} f||^2 d\sigma \le C||f||^2;\ f \in \mathcal{H}. \tag{7.7}$$

To show that, we pick $f = g \otimes \phi$; $g \in L^2(\mathbf{R}_t)$, $\phi \in L^2(\mathbf{R}_z^n)$ and use [Ho, (1.3)] to obtain

$$\begin{aligned} &\int_{-\infty}^\infty d\sigma \int_{-\infty}^\infty |g(t-\sigma)|^2 ||\langle A\rangle^{-\varepsilon} U_T(t, t-\sigma)\phi||^2_{L^2(\mathbf{R}^n)} dt \\ &\le C||g||^2_{L^2(\mathbf{R})} ||\phi||^2_{L^2(\mathbf{R}^n)}. \end{aligned} \tag{7.8}$$

By a change of variables the left hand side of (7.8) may be written

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} |g(s)|^2 ||\langle A\rangle^{-\varepsilon} U_T(t,s)\phi||^2 ds. \tag{7.9}$$

Next we pick a normalized sequence $(g_j) \subset L^2(\mathbf{R})$ with $|g_j|^2 \to \delta$. By substituting (7.9) in (7.8) for the elements of this sequence, and using the strong continuity of the propagator (in s) and Fatou's Lemma, we obtain (7.5) by letting $j \to \infty$.

It remains to prove (7.7) for T large. We shall proceed by using a variant of the Mourre method, [M], based on the "global" Mourre estimate

$$M_T := i[K_T, A] = \eta_-^2 + y_- \cdot \big(P + O\big(e^{-\nu t}\big)\big) y_- \geq \eta_-^2 + \tfrac{1}{5} y_-^2 \geq \tfrac{1}{5} \tag{7.10}$$

(cf. (7.2) and (7.3)). Clearly $\mathcal{D}(M_T) = \mathcal{D}\big(\eta_-^2\big) \cap \mathcal{D}\big(y_-^2\big)$.

We introduce for $\epsilon \in \mathbf{R}\backslash\{0\}$

$$K_T(\epsilon) = K_T - \epsilon i M_T;\ \mathcal{D}(K_T(\epsilon)) = \mathcal{D}(K_T) \cap \mathcal{D}(M_T). \tag{7.11}$$

Lemma 7.2 *For T sufficiently large the resolvent $(K_T(\epsilon) - \zeta)^{-1}$ exists and*

$$||(K_T(\epsilon) - \zeta)^{-1}|| \leq |\mathrm{Im}\zeta + 5^{-1}\epsilon|^{-1}, \tag{7.12}$$

provided $\mathrm{Im}\zeta$ *and* ϵ *have the same sign.*

Proof We consider

$$\begin{aligned} &K_T^0(\epsilon) = K_T - \epsilon i M^0;\ M^0 = \eta_-^2 + y_- \cdot P y_-, \\ &\mathcal{D}\big(K_T^0(\epsilon)\big) = \mathcal{D}(K_T(\epsilon)). \end{aligned} \tag{7.13}$$

We prove in the last part of Appendix A that the form $i\big[M^0, K_T\big]$ defined on $\mathcal{D}(K_T) \cap \mathcal{D}\big(M^0\big)$ extends to an M^0-bounded operator with

$$||i\big[M^0, K_T\big]\big(M^0\big)^{-1}|| \leq C \text{ for } C \text{ independent of } T \tag{7.14}$$

(cf. (6.9) and (6.10)), and

$$\sup_{|\sigma|\leq 1} ||M^0 e^{i\sigma K_T}\big(M^0\big)^{-1}|| < \infty. \tag{7.15}$$

(The bound (7.15) follows from (A.5).)

Using these properties and a slight modification of [Sk1, Lemma 2.6] we conclude that

$$K_T^0(\epsilon)^\star = K_T^0(-\epsilon), \tag{7.16}$$

and therefore by the numerical range argument (cf. (7.2)) that $\big(K_T^0(\epsilon)-\zeta\big)^{-1}$ exists provided Imζ and ϵ have the same sign.

By (7.10)

$$||\big(M_T - M^0\big)\big(M^0\big)^{-1}|| \to 0 \text{ for } T \to +\infty. \tag{7.17}$$

We claim that

$$||\epsilon M^0\big(K_T^0(\epsilon)-\zeta\big)^{-1}|| \le 1, \tag{7.18}$$

for all sufficiently large T and $|\text{Im}\zeta|$ (provided Imζ and ϵ have the same sign). To see this we compute for any $f \in \mathcal{H}, g = \big(K_T^0(\epsilon)-\zeta\big)^{-1}f$,

$$\begin{aligned} &||f||^2 = ||\big(K_T^0(\epsilon)-\zeta\big)g||^2 \\ &= ||(K_T - \text{Re}\zeta)g||^2 + ||\big(\epsilon M^0 + \text{Im}\zeta\big)g||^2 + \epsilon\big\langle g, i\big[M^0, K_T\big]g\big\rangle \\ &\ge \epsilon^2||M^0 g||^2 + \epsilon(2\text{Im}\zeta - C)\big\langle g, M^0 g\big\rangle \ge \epsilon^2||M^0 g||^2, \end{aligned} \tag{7.19}$$

where we used (7.14) for the first inequality and the condition $|\text{Im}\zeta|$ being large for the last one.

We have proved (7.18).

By combining (7.17) and (7.18) with a standard perturbation argument we infer that $(K_T(\epsilon)-\zeta)^{-1}$ exists for T and $|\text{Im}\zeta|$ large and $\epsilon\text{Im}\zeta > 0$. Using now again the numerical range argument (cf. (7.10)) we readily conclude Lemma 7.2.

□

Following Mourre [M] (and [PSS]) we define for T large and ϵ, Im$\zeta > 0$

$$\begin{aligned} &F_\zeta(\epsilon) = D(\epsilon)R_\zeta(\epsilon)D(\epsilon) \\ &D(\epsilon) = \langle A\rangle^{-\varepsilon}\langle \epsilon A\rangle^{\varepsilon-\frac{1}{2}}, \\ &R_\zeta(\epsilon) = (K_T(\epsilon)-\zeta)^{-1}. \end{aligned} \tag{7.20}$$

We want to show that

$$\sup_{\epsilon,\text{Im}\zeta>0} ||F_\zeta(\epsilon)|| < \infty, \tag{7.21}$$

since then (as shown below) we may let $\epsilon \downarrow 0$ to obtain

$$\sup_{\text{Im}\zeta>0} ||\langle A\rangle^{-\varepsilon}(K_T-\zeta)^{-1}\langle A\rangle^{-\varepsilon}|| < \infty. \tag{7.22}$$

Clearly Kato's well-known theory of global smoothness and (7.22) yield (7.7).

To show that indeed (7.22) follows from (7.21) we need to show the convergence

$$R_\zeta(\epsilon)f \to (K_T-\zeta)^{-1}f \text{ for } \epsilon \downarrow 0 \tag{7.23}$$

for fixed $f \in \mathcal{H}$ and ζ with Im$\zeta > 0$. By (7.12) and an interpolation argument we may assume that Imζ is large and that $f \in \mathcal{D}\big(M^0\big)$.

Let

(7.24) $g(\epsilon) = \epsilon M_- R_\zeta(\epsilon) f;\ M_- = \eta_-^2 + y_-^2.$

We need to show that

(7.25) $||g(\epsilon)|| \to 0$ for $\epsilon \downarrow 0,$

uniformly on Im$\zeta = \kappa,\ \kappa$ large. But

(7.26) $g(\epsilon) = \epsilon R_\zeta(\epsilon) M_- f - \epsilon R_\zeta(\epsilon)[M_-, K_T - i\epsilon M_T] R_\zeta(\epsilon) f.$

The commutator on the right hand side of (7.26) is M_-–bounded (cf. (6.9), (6.10) and (7.14)). Using (7.12) again we may thus estimate

(7.27) $||g(\epsilon)|| \leq O(\epsilon) + O\big(\kappa^{-1}\big)||g(\epsilon)||,$

the latter bound on the right hand side being uniform in ϵ and ζ (with Im$\zeta = \kappa$).

Clearly, by a subtraction, we obtain (7.25) from (7.27).

We shall prove (7.21) by showing the differential inequality

(7.28) $||\frac{d}{d\epsilon} F_\zeta(\epsilon)|| \leq C\Big(\epsilon^{\varepsilon - 1}||F_\zeta(\epsilon)||^{\frac{1}{2}} + ||F_\zeta(\epsilon)||\Big).$

In conjunction with (7.12), (7.28) yields (7.21) (by a finite number of iterations).

So it remains to show (7.28). For that we compute

(7.29) $\frac{d}{d\epsilon} F_\zeta(\epsilon) = \big(\frac{d}{d\epsilon} D(\epsilon)\big) R_\zeta(\epsilon) D(\epsilon) + D(\epsilon)\big(\frac{d}{d\epsilon} R_\zeta(\epsilon)\big) D(\epsilon) + D(\epsilon) R_\zeta(\epsilon)\big(\frac{d}{d\epsilon} D(\epsilon)\big).$

We notice (for the first term on the right hand side) that

(7.30) $||D(\epsilon) A\big(M^0\big)^{-\frac{1}{2}}|| \leq C_1||D(\epsilon)|A|^{\frac{1}{2}}|| \leq C_2 \epsilon^{\varepsilon - \frac{1}{2}},$

and therefore

(7.31)
$$\begin{aligned} &||\Big(\frac{d}{d\epsilon} D(\epsilon)\Big) R_\zeta(\epsilon) D(\epsilon)|| \leq C_3||D(\epsilon) A R_\zeta(\epsilon) D(\epsilon)|| \\ &\leq C_3 C_2 \epsilon^{\varepsilon - \frac{1}{2}}||\big(M^0\big)^{\frac{1}{2}} R_\zeta(\epsilon) D(\epsilon)||. \end{aligned}$$

To estimate the right hand side of (7.31) we compute for any $f \in \mathcal{H}$

(7.32)
$$\begin{aligned} &||\big(M^0\big)^{\frac{1}{2}} R_\zeta(\epsilon) D(\epsilon) f||^2 \leq \epsilon^{-1} \mathrm{Im}\big\langle R_\zeta(\epsilon) D(\epsilon) f, i\epsilon M^0 R_\zeta(\epsilon) D(\epsilon) f\big\rangle \\ &\leq -C_4^2 \epsilon^{-1} \mathrm{Im}\big\langle R_\zeta(\epsilon) D(\epsilon) f, (K_T - i\epsilon M_T - \zeta) R_\zeta(\epsilon) D(\epsilon) f\big\rangle \\ &= -C_4^2 \epsilon^{-1} \mathrm{Im}\big\langle D(\epsilon) R_\zeta(\epsilon) D(\epsilon) f, f\big\rangle \\ &\leq C_4^2 \epsilon^{-1}||F_\zeta(\epsilon)||\,||f||^2. \end{aligned}$$

We take the square root and substitute into (7.31) yielding

(7.33) $$||\big(\tfrac{d}{d\epsilon}D(\epsilon)\big)R_\zeta(\epsilon)D(\epsilon)|| \le C\epsilon^{\varepsilon-1}||F_\zeta(\epsilon)||^{\frac12};\ C = C_2C_3C_4,$$

which agrees with (7.28).

For the for the last term on the right hand side of (7.29) we proceed similarly.

For for the middle term on the right hand side of (7.29) we compute

(7.34) $$\begin{aligned}&D(\epsilon)\left(\frac{d}{d\epsilon}R_\zeta(\epsilon)\right)D(\epsilon) = iDRM_TRD\\ &= iDR(i[K_T - i\epsilon M_T - \zeta, A] + i[i\epsilon M_T, A])RD\\ &= DRAD - DARD + \epsilon iDR\tilde{M}RD,\end{aligned}$$

where $\tilde{M}$ is M^0–bounded.

By inserting $I = \big(M^0\big)^{\frac12}\big(M^0\big)^{-\frac12} = \big(M^0\big)^{-\frac12}\big(M^0\big)^{\frac12}$ at various places on the right hand side of (7.34), and using (7.30), (7.32) and their adjoint analogues, we obtain

(7.35) $$\begin{aligned}&||D(\epsilon)\left(\frac{d}{d\epsilon}R_\zeta(\epsilon)\right)D(\epsilon)|| \le ||DR\big(M^0\big)^{\frac12}||C_2\epsilon^{\varepsilon-\frac12} + C_2\epsilon^{\varepsilon-\frac12}||\big(M^0\big)^{\frac12}RD||\\ &+ \epsilon||DR\big(M^0\big)^{\frac12}||C_5||\big(M^0\big)^{\frac12}RD||\\ &\le 2C_4C_2\epsilon^{\varepsilon-1}||F_\zeta(\epsilon)||^{\frac12} + C_4^2C_5||F_\zeta(\epsilon)||,\end{aligned}$$

which again agrees with (7.28).

We have completed the proof of (7.28) and hence the proof of Proposition 7.1.

8 Propagation estimates

We shall prove various propagation estimates, which at the end of the section will be used to establish (5.36) and therefore finish the proof of Theorem 2.2.

The matrix P in (6.5) is on the form

(8.1) $$P = P'^2;\ P' = \mathrm{diag}(q_1, \cdots, q_{m_-}),\ q_j = q_j(\xi) \ge \tfrac12.$$

Let

$$q_0 = \min(q_j),$$

and (with $\eta_j = \eta_{-,j}$ and $y_j = y_{-,j}$)

(8.2) $$\begin{aligned}&A = \sum_{j=1}^{m_-}\left(\frac{q_j}{q_0}\right)^2\frac12(\eta_j\cdot y_j + y_j\cdot\eta_j),\\ &N = q_0^{-1}\sum_{j=1}^{m_-}\big(\eta_j^2 + q_j^2y_j^2\big),\\ &B = N^{-1}A + AN^{-1}.\end{aligned}$$

We consider these operators as acting on $L^2(\mathbf{R}^n)$. Clearly B is a bounded self-adjoint operator.

We use in this section the notation $\mathbf{D}$ to denote the Heisenberg derivative w.r.t. the family of Hamiltonians $H(t)$ of (6.6).

By (6.9) and (6.10) we compute as a form on $\mathcal{D}$, the latter given in Appendix A,

$$\begin{aligned}\mathbf{D}A &= \sum_{j=1}^{m_-}\left(\frac{q_j}{q_0}\right)^2\big(\eta_j^2+q_j^2y_j^2\big)-O\big(e^{-\nu t}\big)y_-^2\\ &+\sum_{j=1}^{m_-} i\left[R(t),\left(\frac{q_j}{q_0}\right)^2\right]\frac{1}{2}(\eta_j\cdot y_j+y_j\cdot\eta_j)\\ &\geq q_0N-T;\\ T&=O\big(e^{-\nu t}\big)N \text{ or } T=NO\big(e^{-\nu t}\big).\end{aligned} \tag{8.3}$$

Similarly we compute

$$\mathbf{D}N = 4q_0A+i[R(t),N], \tag{8.4}$$

where the commutator

$$i[R(t),N]=i\big[R(t),q_0^{-1}\big]\eta_-^2+q_0^{-1}i\big[R(t),\eta_-^2\big]+\sum_{j=1}^{m_-} i\Big[R(t),\tfrac{q_j^2}{q_0}\Big]y_j^2 \tag{8.5}$$

by (6.9) and (6.10) has the form

$$i[R(t),N]=O\big(e^{-\nu t}\big)N \text{ or } i[R(t),N]=NO\big(e^{-\nu t}\big). \tag{8.6}$$

Combining (8.3), (8.4) and (8.6) and the fact that $i[N,A]$ is N-bounded we obtain

$$\begin{aligned}\mathbf{D}B &= \sum_{j=1}^{m_-}\left(\frac{q_j}{q_0}\right)^2\Big(N^{-1}\big(\eta_j^2+q_j^2y_j^2\big)+\big(\eta_j^2+q_j^2y_j^2\big)N^{-1}\Big)\\ &-N^{-1}4q_0AN^{-1}A-AN^{-1}4q_0AN^{-1}+O\big(e^{-\nu t}\big)\\ &\geq 2q_0\big(I-B^2\big)+N^{-1}O\big(t^0\big)N^{-1}+O\big(e^{-\nu t}\big).\end{aligned} \tag{8.7}$$

Lemma 8.1 *For any non-negative $f\in C_0^\infty((-1,1))$ and all $\psi\in L^2(\mathbf{R}^n)$*

$$\int_0^\infty \big\langle\big(f^2(B)\big)\big\rangle_t dt \leq C||\psi||^2,\ C=C(f). \tag{8.8}$$

(Here $\langle\cdot\rangle_t$ denotes expectation in the state $\psi(t)=U(t)\psi$.)

Proof Clearly we may assume that $\psi\in\mathcal{D}$. We consider the propagation observable

$$\Phi=F(B),\ F(x)=\int_{-\infty}^{x} f^2(s)ds.$$

By Lemma 2.8 and (8.7)

$$\begin{aligned}\mathbf{D}\Phi &= f(B)\mathbf{D}Bf(B) + R_1 + R_2 + O\big(e^{-\nu t}\big)\\ &\geq 2q_0 f^2(B)\big(I - B^2\big) + N^{-\frac{1}{2}}O\big(t^0\big)N^{-\frac{1}{2}} + O\big(e^{-\nu t}\big).\end{aligned} \tag{8.9}$$

We integrate (8.9) using the support property of f to treat the contribution from the first term on the right hand side and Proposition 7.1 to treat the middle term.

□

Lemma 8.2 *Let* $\varepsilon > 0$ *and* $F \in \mathcal{F}_+$. *Then for all* $\psi \in \mathcal{D}(N)$

$$\int_0^\infty \left\langle F\big(\varepsilon^{-1}(I - B)\big)\right\rangle_t dt \leq C||N\psi||^2. \tag{8.10}$$

Proof Consider for $F_+ \in \mathcal{F}_+$ and $\epsilon \in (0,1)$ the observable

$$\tilde{A} = -\tfrac{1}{2}\ln(N)F_+\big(-\epsilon^{-1}B\big) + h.c.,\ \mathcal{D}\Big(\tilde{A}\Big) = \mathcal{D}(N). \tag{8.11}$$

The idea of the proof is to write

$$-\Big\langle\tilde{A}\Big\rangle_{t=0} = -\Big\langle\tilde{A}\Big\rangle_t + \int_0^t \Big\langle\mathbf{D}\tilde{A}\Big\rangle_s ds \tag{8.12}$$

and then use the approximate positivity of both terms on the right hand side to get the bound

$$\int_0^\infty \left\langle F_+\big(-\epsilon^{-1}B\big)\right\rangle_t dt \leq C||N\psi||^2;\ \psi \in \mathcal{D}. \tag{8.13}$$

Clearly Lemma 8.1 and (8.13) yield (8.10).

So it suffices to show (8.13):

We claim that the Heisenberg derivative of the factor $\ln N$ is given for any $\sigma \in (0,1)$ by

$$\mathbf{D}\ln N = 2q_0 B + N^\sigma O\big(e^{-\nu t}\big) + N^{-\frac{1}{2}}O\big(t^0\big)N^{-\frac{1}{2}}. \tag{8.14}$$

To show (8.14) we notice that the representations (2.42) and (2.44) are valid for the derivative of approximations of the logarithm function (cf. (7.15)) and therefore in the limit (in the form sense): For some smooth function $\bar{\partial}\tilde{F}$ on $\mathbf{C}$ with

$$|\Big(\bar{\partial}\tilde{F}\Big)(w)| \leq C_{\delta,k}\langle w\rangle^{\delta-1-k}|\mathrm{Im}z|^k;\ \delta > 0, k \in \mathbf{N}, \tag{8.15}$$

$$\begin{aligned} & i[H(t), \ln N] \\ & = \frac{1}{2}\big(N^{-1}4q_0A + 4q_0AN^{-1}\big) + R_1(t) + R_r(t); \end{aligned}$$

(8.16) $$R_1(t) = \frac{1}{2\pi}\int_{\mathbf{C}} \Big(\bar\partial\tilde F\Big)(w)(N-w)^{-2}ad_N^2(4q_0A)(N-w)^{-2}dudv,$$

$$R_r(t) = -\frac{1}{\pi}\int_{\mathbf{C}} \Big(\bar\partial\tilde F\Big)(w)(N-w)^{-1}i[R(t),N](N-w)^{-1}dudv,$$

cf. (8.4).

Obviously the first term on the right hand side of (8.16) is equal to the first term on the right hand side of (8.14).

As for the term $R_1(t)$ we use boundedness of $N^{-\frac{1}{2}}\,ad_N^2(A)N^{-\frac{1}{2}}$ yielding the representation $R_1(t) = N^{-\frac{1}{2}}O\big(t^0\big)N^{-\frac{1}{2}}$.

To analyse the term $R_r(t)$ we use (8.6) and (8.15) to conclude that $N^{-\sigma}R_r(t) = O\big(e^{-\nu t}\big)$ is bounded. Consequently $R_r(t) = N^\sigma O\big(e^{-\nu t}\big)$, in agreement with (8.14).

We have shown (8.14).

To treat the contribution from the middle term on the right hand side of (8.14) to the integral on the right hand side of (8.12) it will suffice to show that for $\sigma > 0$ small enough

(8.17) $$\int_0^\infty |\big\langle N^\sigma O\big(e^{-\nu t}\big)F_+\big(-\epsilon^{-1}B\big) + h.c.\big\rangle_t|dt \le C||N\psi||^2.$$

For that we use (A.3) and interpolation to obtain that for all $\sigma\in[0,1]$

(8.18) $$||N^\sigma U(t)\psi|| \le e^{\sigma C_2 t}||N\psi||.$$

We pick $\sigma < \nu C_2^{-1}$ in (8.18) yielding (8.17).

For the contribution from the last term on the right hand side of (8.14) to the integral on the right hand side of (8.12) we notice that $i[N,B]$ is bounded and therefore also $T := N^{-\frac{1}{2}}F_+\big(-\epsilon^{-1}B\big)N^{\frac{1}{2}}$. Therefore we can invoke Proposition 7.1 to obtain

(8.19) $$\int_0^\infty |\Big\langle N^{-\frac{1}{2}}O\big(t^0\big)TN^{-\frac{1}{2}} + h.c.\Big\rangle_t|dt \le C||N\psi||^2.$$

For the contribution from the first term on the right hand side of (8.14) to the integral we estimate (using $q_0 \ge 2^{-1}$)

(8.20) $$\begin{aligned} & \int_0^t \big\langle -2q_0BF_+\big(-\epsilon^{-1}B\big)\big\rangle_s ds \\ & \ge \frac{1}{2}\epsilon\int_0^t \big\langle F_+\big(-\epsilon^{-1}B\big)\big\rangle_s ds. \end{aligned}$$

Next we consider

$$(8.21)\quad \int_0^t \left\langle -\tfrac{1}{2}\ln(N)\mathbf{D}F_+\big(-\epsilon^{-1}B\big) + h.c.\right\rangle_s ds.$$

By Lemma 2.8, (8.7), (8.18), Proposition 7.1 and commutation

$$(8.22)\quad \int_0^t \left\langle -\tfrac{1}{2}\ln(N)\mathbf{D}F_+\big(-\epsilon^{-1}B\big) + h.c.\right\rangle_s ds \ge -C||N\psi||^2.$$

By two applications of (2.42) and (2.43) using again that $i[N,B]$ is bounded, we can show boundedness of

$$-\tilde{A} - F_+^{\frac{1}{2}}\big(-\epsilon^{-1}B\big)\ln(N)F_+^{\frac{1}{2}}\big(-\epsilon^{-1}B\big).$$

In particular

$$(8.23)\quad -\tilde{A} \ge -CI.$$

We combine (8.12), (8.17), (8.19), (8.20), (8.22) and (8.23) to obtain (8.13).

□

Lemma 8.3 *For all* $\psi \in L^2(\mathbf{R}^n)$, $\varepsilon \in (0,1)$ *and* $F_+ \in \mathcal{F}_+$

$$(8.24)\quad F_+\big(\varepsilon^{-1}(I-B)\big)\psi(t) = o\big(t^0\big) \text{ for } t \to +\infty.$$

Proof It is enough to show (8.24) for $\psi \in \mathcal{D}$. We use (8.9) to compute the derivative of $p(t) := ||F_+\big(\varepsilon^{-1}(I-B)\big)\psi(t)||^2$. It is integrable due to Lemma 8.1, whence we conclude that $p(t)$ has a limit $= p(\infty)$ as $t \to +\infty$. By Lemma 8.2 $p(\infty) = 0$.

□

Lemma 8.4 *For all* $\psi \in L^2(\mathbf{R}^n)$, $\varepsilon > 0$ *and* $F_- \in \mathcal{F}_-$

$$(8.25)\quad F_-\Big(e^{-(2q_0-\varepsilon)t}N\Big)\psi(t) = o\big(t^0\big) \text{ for } t \to +\infty.$$

Proof Again we can assume that $\psi \in \mathcal{D}$.

We consider for $\epsilon > 0$ the observable

$$(8.26)\quad \tilde{A} = \tilde{A}(t) = \ln(N) - 2q_0(1-\epsilon)\tau;\ \tau = t+1,\ t \ge 0.$$

The Heisenberg derivative is by (8.14) and (6.9) given for any $\sigma \in (0,1)$ by

$$\mathbf{D}\tilde{A} = D(t) + D_r(t);$$

$$D(t) = 2q_0(B - (1-\epsilon)I), \tag{8.27}$$

$$D_r(t) = N^{\sigma} O\big(e^{-\nu t}\big) N^{\sigma} + N^{-\frac{1}{2}} O\big(t^0\big) N^{-\frac{1}{2}}.$$

We pick with C_2 given in (8.18)

$$\sigma < \tfrac{\nu}{2C_2}. \tag{8.28}$$

Next we consider for $F_+ \in \mathcal{F}_+$ the composition

$$\hat{A} = F\Big(\tilde{A}, \tau\Big) = -\Big(-\tilde{A}\Big)^{\frac{1}{2}} F_+\Big(-2\epsilon^{-1}\tau^{-1}\tilde{A}\Big) \tag{8.29}$$

(assuming $\big(\partial_{\tilde{A}} F\big)^{\frac{1}{2}}(\cdot,\tau) \in C^\infty$), cf. [Sk2].

We shall follow the same scheme as used in the proof of Lemma 8.2 (cf. (8.12), that is we will show a lower bound for the integral of the Heisenberg derivative of $\hat{A}$. (Notice that the operator itself is non-positive.)

We compute the Heisenberg derivative using (8.27) and Lemma 2.8: For some smooth function $\bar{\partial}\tilde{F}$ on $\mathbf{C}$ with

$$|\Big(\bar{\partial}\tilde{F}\Big)(w)| \le C_k \langle w\rangle^{-\frac{1}{2}-k} |\mathrm{Im} z|^k;\ k \in \mathbf{N} \text{ and } C_k \text{ independent of } \tau, \tag{8.30}$$

we represent (cf. (8.16))

$$\begin{aligned} &\mathbf{D}\hat{A} = \partial_\tau F\Big(\tilde{A},\tau\Big) + \big(\partial_{\tilde{A}} F\big)^{\frac{1}{2}}\Big(\tilde{A},\tau\Big) 2q_0(B-(1-\epsilon)I)\big(\partial_{\tilde{A}} F\big)^{\frac{1}{2}}\Big(\tilde{A},\tau\Big)\\ &+ R_1(t) + R_2(t) + R_r(t);\\ &R_r(t) = -\frac{1}{\pi}\int_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(w)\Big(\tilde{A}-w\Big)^{-1} D_r(t)\Big(\tilde{A}-w\Big)^{-1} dudv. \end{aligned} \tag{8.31}$$

(The terms $R_1(t)$ and $R_2(t)$ are given as in Lemma 2.8.)

Clearly the first term on the right hand side of (8.31) is non-negative.

To treat the second term on the right hand side of (8.31) we insert

$$I = F_+\big(\epsilon^{-1}(I-B)\big) + F_-\big(\epsilon^{-1}(I-B)\big)$$

in the middle.

Since

$$(B-(1-\epsilon)I)F_-\big(\epsilon^{-1}(I-B)\big) \ge 0$$

only the contribution from the first term needs further consideration:

We factorize

$$F_+\big(\epsilon^{-1}(I-B)\big) = F_+ = \Big(F_+^{\frac{1}{2}}\Big)^2$$

and commute one factor to the right and the other to the left (through the factors $\big(\partial_{\tilde{A}}F\big)^{\frac{1}{2}}\Big(\tilde{A},\tau\Big)$). Then we invoke Lemma 8.2.

As for the remainders we notice that

$$\Big[\big(\partial_{\tilde{A}}F\big)^{\frac{1}{2}}\Big(\tilde{A},\tau\Big), F_+^{\frac{1}{2}}(B)\Big] = N^{-\frac{1}{4}}O(1)N^{-\frac{1}{4}},$$

which combined with Proposition 7.1 yields integrability.

Similarly we obtain integrability for the terms $R_1(t)$ and $R_2(t)$.

It remains to estimate the integral of the expectation of $R_1(t)$ on the right hand side of (8.31). Inserting the two terms of $D_r(t)$ on the right hand side of (8.27) only the contribution from the first need further examination:

By (8.18) and (8.28)

(8.32)
$$\begin{aligned}-\int_0^\infty e^{-\nu s}||N^\sigma\psi(s)||^2ds &\geq -\int_0^\infty e^{-(\nu-2\sigma C_2)s}ds||N\psi||^2\\ &\geq -C||N\psi||^2.\end{aligned}$$

We combine (8.32) and previously discussed estimates to obtain the bound

$$\Big\langle\Big(\frac{\epsilon}{4}\tau\Big)^{\frac{1}{2}}F_+\Big(-2\epsilon^{-1}\tau^{-1}\tilde{A}\Big)\Big\rangle_t \leq \Big\langle -F\Big(\tilde{A}(t),\tau\Big)\Big\rangle_t \leq C||N\psi||^2,$$

which upon inserting (8.26) and chosing $\epsilon > 0$ small yields

$$||F_-\Big(e^{-(2q_0-\varepsilon)t}N\Big)\psi(t)||^2 \leq Ct^{-\frac{1}{2}}||N\psi||^2,$$

and therefore in particular (8.25).

□

Lemma 8.5 *For all* $\psi \in L^2(\mathbf{R}^n)$, $\varepsilon > 0$ *and* $F_- \in \mathcal{F}_-$

(8.33) $$F_-\Big(e^{-(2q_0-\varepsilon)t}y_-^2\Big)\psi(t) = o\big(t^0\big) \text{ for } t \to +\infty.$$

Proof We introduce

$$F_{-,1} = F_-\Big(e^{-(2q_0-\varepsilon)t}y_-^2\Big),$$

$$F_{-,2} = F_-\big(\epsilon^{-1}(I-B)\big);\ \epsilon \in \Big(0,\frac{1}{2}\Big),$$

$$F_{+,3} = F_+\Big(e^{-\left(2q_0-\frac{\varepsilon}{2}\right)t}N\Big).$$

By Lemmas 8.3, 8.4 and a commutation using Lemma 2.8

$$F_{-,1}\psi(t) = \psi_1(t) + o\big(t^0\big) = \psi_2(t) + o\big(t^0\big) = \psi_3(t) + o\big(t^0\big);$$
$$\psi_1(t) = F_{-,1}F_{-,2}F_{+,3}\psi(t),$$
$$\psi_2(t) = F_{-,2}F_{-,1}F_{+,3}\psi(t),$$
$$\psi_3(t) = F_{+,3}F_{-,1}F_{-,2}\psi(t).$$

We compute using the formula $N^{-\frac{1}{2}} = C\int_0^\infty s^{-\frac{1}{2}}(N+s)^{-1}ds$ that

$$\begin{aligned}
&B-(1-2\epsilon)I\\
&= \frac{1}{2}N^{-1}\sum_{j=1}^{m_-}\left(\left(\frac{q_j}{q_0}\right)^2(\eta_j\cdot y_j + y_j\cdot\eta_j) - (1-2\epsilon)q_0^{-1}\big(\eta_j^2 + q_j^2y_j^2\big)\right) + h.c.\\
&= N^{-\frac{1}{2}}\sum_{j=1}^{m_-}\left(\left(\frac{q_j}{q_0}\right)^2(\eta_j\cdot y_j + y_j\cdot\eta_j) - (1-2\epsilon)q_0^{-1}\big(\eta_j^2 + q_j^2y_j^2\big)\right)N^{-\frac{1}{2}}\\
&+ N^{-\frac{1}{4}}O\big(t^0\big)N^{-\frac{1}{4}},
\end{aligned}$$

from which we obtain by commutation

$$\begin{aligned}
&||\psi_2(t)||^2 \le \epsilon^{-1}\langle\psi_2(t),(B-(1-2\epsilon)I)\psi_2(t)\rangle\\
&= \epsilon^{-1}\langle\psi_1(t),(B-(1-2\epsilon)I)\psi_1(t)\rangle + o\big(t^0\big)\\
&\le C_1\Big\langle N^{-\frac{1}{2}}\psi_1(t),\big(C_2y_-^2 - \eta_-^2\big)N^{-\frac{1}{2}}\psi_1(t)\Big\rangle + o\big(t^0\big)\\
&\le C_4\Big\langle\psi_1(t),N^{-1}e^{(2q_0-\varepsilon)t}\psi_1(t)\Big\rangle + o\big(t^0\big)\\
&= C_4\Big\langle\psi_3(t),N^{-1}e^{(2q_0-\varepsilon)t}\psi_3(t)\Big\rangle + o\big(t^0\big)\\
&\le C_5\Big\langle\psi_3(t),e^{-\frac{\varepsilon}{2}t}\psi_3(t)\Big\rangle + o\big(t^0\big) = o\big(t^0\big).
\end{aligned}$$

□

We can now prove (5.36):

Transforming back to the original frame (using (6.4)) the statement (8.33) is rephrased in terms of the $\breve{U}(t)$'s of Section 5 as

$$F_-\big(t^{-2q_0-1+\varepsilon}y_-^2\big)\breve{U}(t)\breve{\psi} = o(1) \text{ for } t\to+\infty,$$

which (since $q_0 \ge \frac{1}{2}$) obviously implies (5.36).

Appendix A

In this appendix we shall collect some basic facts about the propagator given by (5.20) and other used propagators, and some related technicalities that are needed in justifying various calculations.

We proceed by first studying (6.6):

On $L^2(\mathbf{R}_z^n)$ we introduce

$$B = p^2 + y_-^2,\ \mathcal{D} := \mathcal{D}(B),$$

$$U_0(t) = e^{-i\int_0^t (H_+(s)+H_-)ds},\ t \in \mathbf{R}.$$

By first estimating on the set of Schwartz functions, which is preserved by $U_0(t)$, it is readily proved (cf. (A.3) below) that $\mathcal{D}$ is preserved as well, and in fact that $BU_0(t)B^{-1}$ is a strongly continuous $\mathcal{B}\big(L^2(\mathbf{R}^n)\big)$-valued function.

Next we define the $U(t)$'s of (6.6) as the (unique) solution of the equation

$$U(t) = U_0(t) - iU_0(t)\int_0^t U_0(s)^{-1}R(s)U(s)ds. \tag{A.1}$$

Noticing that $BR(t)B^{-1}$ is a strongly continuous $\mathcal{B}\big(L^2(\mathbf{R}^n)\big)$-valued function (cf. (8.3), (8.4) and (8.6), it follows from (A.1) that $BU(t)B^{-1}$ solves a similar Volterra integral equation. Consequently we infer that $BU(t)B^{-1}$ and $BU(t)^{-1}B^{-1}$ are strongly continuous $\mathcal{B}\big(L^2(\mathbf{R}^n)\big)$-valued functions and that

$$i\tfrac{d}{dt}U(t)\psi = H(t)U(t)\psi,\ \psi \in \mathcal{D}. \tag{A.2}$$

We have now justified (6.6). Next we define $\breve{U}(t)$ by (6.4). Noticing that the operators T_t, U_0 and U_0^{-1} preserves $\mathcal{D}$ we obtain that

$$i\frac{d}{dt}\breve{U}(t)\psi = \breve{H}(t)\breve{U}(t)\psi,\ \psi \in \mathcal{D},$$

justifying (5.20) as well.

Returning to the study of $U(t)$ we notice the bounds

$$\begin{aligned} &||BU(t)B^{-1}|| \le e^{C_1 t}, \\ &||NU(t)N^{-1}|| \le e^{C_2 t};\ t \ge 0, \end{aligned} \tag{A.3}$$

where N is defined by (8.2). These can be proved by the Gronwall inequality, cf. [DG, p. 381]:

For the first estimate we consider for $\psi \in \mathcal{D}$ and $\epsilon > 0$ the expression $k_\epsilon(t) = ||\phi_\epsilon(t)||^2;\ \phi_\epsilon(t) = B(\epsilon B + 1)^{-1}\psi(t), \psi(t) = U(t)\psi$.

Writing $B(\epsilon B+1)^{-1} = \epsilon^{-1}\Big(I-(\epsilon B+1)^{-1}\Big)$ we compute

$$\frac{d}{dt}k_\epsilon(t) = 2\text{Re}\Big\langle \phi_\epsilon(t), i\Big[H(t), B(\epsilon B+1)^{-1}\Big]\psi(t)\Big\rangle$$
$$= 2\text{Re}\Big\langle \phi_\epsilon(t), (\epsilon B+1)^{-1} i[H(t),B]B^{-1}\phi_\epsilon(t)\Big\rangle_t,$$

whence we infer that

$$\frac{d}{dt}k_\epsilon(t) \leq 2||i[H(t),B]B^{-1}||k_\epsilon(t) \leq 2C_1 k_\epsilon(t);\ C_1 = \sup_{s\geq 0}||i[H(s),B]B^{-1}||,$$

and

$$||B\psi(t)||^2 = \lim_{\epsilon\downarrow 0} k_\epsilon(t) \leq \lim_{\epsilon\downarrow 0} e^{t2C_1}k_\epsilon(0) = e^{t2C_1}||B\psi||^2.$$

Similarly for the second estimate we consider for $\psi \in \mathcal{D}$ and $\epsilon > 0$ the expression $k_\epsilon(t) = ||\phi_\epsilon(t)||^2;\ \phi_\epsilon(t) = N(\epsilon N+1)^{-1}\psi(t), \psi(t) = U(t)\psi$. Using that $(\epsilon N+1)^{-1}$ preserves $\mathcal{D}$ we obtain similarly that

$$\frac{d}{dt}k_\epsilon(t) \leq 2C_2 k_\epsilon(t);\ C_2 = \sup_{s\geq 0}||i[H(s),N]N^{-1}||,$$

and therefore

$$||N\psi(t)||^2 \leq e^{t2C_2}||N\psi||^2. \tag{A.4}$$

Since $\mathcal{D}$ is a core for N we infer the second estimate of (A.3) from (A.4) by approximation.

The rest of this appendix is devoted to filling out some details needed in Section 7.

On the Hilbert space $\mathcal{H} = L^2\big(\mathbf{R}_t, L^2(\mathbf{R}_z^n)\big)$ we specify the invariant subspace

$$\mathcal{D}_{in} = C_0(\mathbf{R}_t;\mathcal{D}) \cap C^1\big(\mathbf{R}_t; L^2(\mathbf{R}_z^n)\big),$$

where that notation C_0 is used for continuous compactly supported functions. Notice that indeed $e^{-i\sigma K_T}$ leave $\mathcal{D}_{in}$ invariant as it follows from the formula

$$e^{-i\sigma K_T}f(t) = U_T(t)U_T(t-\sigma)^{-1}f(t-\sigma),$$

cf. [Ho, (1.3)]. Moreover the generator K_T is essentially self-adjoint on $\mathcal{D}_{in}$ where it acts according to (7.6).

By mimicking the proof of (A.3) first proving the corresponding estimates for $\psi \in \mathcal{D}_{in}$ we obtain

$$\begin{aligned} &||Be^{-i\sigma K_T}B^{-1}|| \leq e^{C_3|\sigma|}, \\ &||M^0 e^{-i\sigma K_T}\big(M^0\big)^{-1}|| \leq e^{C_4|\sigma|}, \end{aligned} \tag{A.5}$$

where M^0 is given by (7.13).

Clearly the form $i\big[M^0, K_T\big]$ extends from $\mathcal{D}_{in}$ to an M^0-bounded operator. Using this fact, the invariance of $\mathcal{D}_{in}$, the second estimate of (A.5) and [M, Proposition II.1] it follows that the form $i\big[M^0, K_T\big]$ extends from $\mathcal{D}(K_T)\cap\mathcal{D}\big(M^0\big)$ to an M^0-bounded operator. In conclusion (7.14) holds.

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