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Non-perturbative Lorentzian Quantum Gravity, Causality and Topology Change

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Abstract

We formulate a non-perturbative lattice model of two-dimensional Lorentzian quantum gravity by performing the path integral over geometries with a causal structure. The model can be solved exactly at the discretized level. Its continuum limit coincides with the theory obtained by quantizing 2d continuum gravity in proper-time gauge, but it disagrees with 2d gravity defined via matrix models or Liouville theory. By allowing topology change of the compact spatial slices (i.e. baby universe creation), one obtains agreement with the matrix models and Liouville theory.

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1 Introduction

A moot point in non-perturbative Euclidean path-integral approaches to quantum gravity is the final interpretation of their results in terms of physical quantities defined in the Lorentzian sector of the theory. Even classically, it is known that the simple expedient of applying a Wick rotation $t \to \tau = it$ fails in all but a few special cases, for instance, when the space-time is static and thus admits a *global* choice of time x_0 such that the cross terms g_{0i}, $i \geq 1$ of the metric tensor vanish and its spatial components g_{ij} are time-independent. We do not know of a way to set up a 1-to-1 correspondence between generic solutions of the continuum Einstein equations with different signatures. If it exists, it is likely to be technically involved (see, for example, [1] for a recent proposal).

From this point of view it is unclear if one can expect to obtain the correct Lorentzian theory by first performing a path integral over general Euclidean metric configurations and then analytically continuing in some way. Related questions have been discussed in [2], where it was suggested that in the path integral for gravity with an action in square-root form, and using the rather unconventional weights $e^{\sqrt{i}S}$, it may be necessary to sum over both Lorentzian and Euclidean metrics in order to obtain a unitary evolution.

To investigate these issues further, we formulate a theory of gravity in two space-time dimensions where the summation is restricted to metric configurations with a *causal* structure, defined on each contribution to the sum over states in the discrete model under consideration. In this way one encodes at least part of the Lorentzian structure into the Euclidean path integral. In addition to that, we will consider a suitable analytic continuation of our results.[3] Curiously, our final continuum results show some similarity with the approach advocated in [2], although our path-integral construction proceeds along entirely different lines.

The idea that a notion of causality should be built into each history that contributes to the path integral amplitude goes back at least to Teitelboim [4], and has more recently been advocated in [5, 6]. However, to our knowledge a concrete implementation in a well-defined, non-perturbative model for quantum gravity has so far been missing. An ideal testing ground for this idea is in two dimensions, where the Euclidean path-integral construction leads to a non-trivial gravitational quantum theory. Its properties have been explored in great detail, both by analytical and numerical methods. The main reason why this model can be solved analytically, even at the discretized level, is that the action in 2d gravity is trivial. The Einstein-Hilbert term is a topological invariant and does not contribute unless we consider topology changes of space-time. Moreover, for fixed space-time volume the partition function is purely entropic and given by the number of different geometries. Exactly this fact is used in the formalism of dynamical triangulations (or

[3] Another way of making 2d gravity more Lorentzian (by allowing for both time- and spacelike edges in a dynamically triangulated model, but without causal structures) was studied in [3], but no significantly different behaviour from the Euclidean theory was found.

equivalently, matrix models) to construct the non-perturbative path integral. One counts the number of inequivalent triangulations which can be constructed from a given number of triangles of unit volume, and with a fixed topology for the resulting simplicial complex. Inequivalent triangulations (appropriately defined) can be related to different geometries [7, 8, 9, 10]. In this way the following quantities have been calculated in pure Euclidean 2d gravity:

(1) the partition function on the sphere, $Z^{(eu)}(\Lambda)$, as a function of the renormalized cosmological[7] constant Λ,

$$Z^{(eu)}(\Lambda) \sim \Lambda^{2-\gamma} + \text{ terms less singular in } \Lambda, \qquad \gamma = -\frac{1}{2}, \tag{1}$$

where γ is the so-called string susceptibility exponent;

(2) the Hartle-Hawking wave functional $W_\Lambda(L)$ as a function of the length L of the spatial boundary (with a marked point) [11],

$$W_\Lambda^{(eu)}(L) \propto \frac{1}{L^{5/2}}(1+\sqrt{\Lambda}\,L)\,\mathrm{e}^{-\sqrt{\Lambda}\,L}, \tag{2}$$

(more generally, one can calculate "multi-loop" correlators $W_\Lambda^{(eu)}(L_1, \ldots, L_n)$ [12]);

(3) the average Euclidean space-time volume, $B_V(R)$, of geodesic balls of radius R in the ensemble of universes of Euclidean space-time volume V [13, 14, 15],

$$B_V(R) \propto R^4 F\Big(\frac{R^4}{V}\Big), \tag{3}$$

where $F(0) = 1$ and $F(x) \propto \mathrm{e}^{-x^{1/3}}$ for $x \to \infty$. Relation (3) implies that the intrinsic Hausdorff dimension d_H of Euclidean 2d quantum gravity is four (and not two, as one naïvely might have expected).[4]

In the following we will show that for a universe with cylinder topology, restricting the path integral to configurations admitting a causal structure leads to a theory with $\gamma = 1/2$ (although the definition of γ turns out to be ambiguous), a Hartle-Hawking wave function with the same exponential decay (in the Euclidean sector) as (2), but a different functional form, and an intrinsic Hausdorff dimension of two, and not four as in (3). We will further show that once we allow for topology changes of space, i.e. the creation of baby universes, we are led to a theory satisfying (1)–(3).

[4] It can be shown, using only general arguments [16], that Euclidean space-time, with an assumed fractal dimension d_H, is characterized by a function like (3):

$$B_V(R) \sim R^{d_H} F\Big(R^{d_H}/V\Big), \quad F(0) = 1, \quad F(x) = \mathrm{e}^{-x^{1/(d_H-1)}} \quad \text{for } x \gg 1. \tag{4}$$

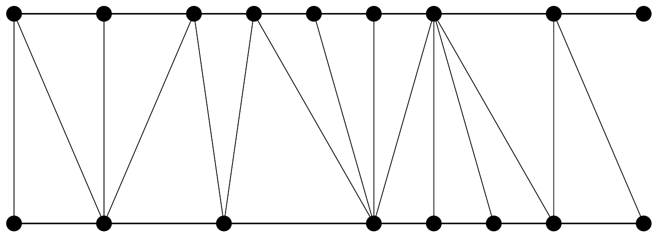

Figure 1: The propagation of a spatial slice from step t to step $t+1$. The ends of the strip should be joined to form a band with topology $S^1 \times [0,1]$.

2 The discrete model

As mentioned in the introduction, the solution of two-dimensional quantum gravity amounts to counting geometries. While this counting problem has been solved in Euclidean gravity, it seems non-trivial if the space-time has Lorentzian signature. Counting in a field theoretical context usually amounts to the introduction of a regularization (a discretization) which makes the counting procedure well defined. After the counting has been performed one may attempt to take the continuum limit of the discretized theory. It is unclear which class of geometries to include if the signature is not Euclidean. We propose here a model where a causal structure is explicitly present in all the geometries included in the path integral.

The model is defined as follows. The topology of the underlying manifold is taken to be $S^1 \times [0,1]$, with "space" represented by the closed manifold S^1. We consider the evolution of this space in "time". No topology change of space is allowed at this stage, but we will return to this issue in sec. 6.

The geometry of each spatial slice is uniquely characterized by the length assigned to it. In the discretized version, the length L will be quantized in units of a lattice spacing a, i.e. $L = l \cdot a$ where l is an integer. A slice will thus be defined by l vertices and l links connecting them. To obtain a 2d geometry, we will evolve this spatial loop in discrete steps. This leads to a preferred notion of (discrete) "time" t, where each loop represents a slice of constant t. The propagation from time-slice t to time-slice $t+1$ is governed by the following rule: each vertex i at time t is connected to k_i vertices at time $t+1$, $k_i \geq 1$, by links which are assigned length $-a$. The k_i vertices, $k_i > 1$, at time-slice $t+1$ will be connected by $k_i - 1$ consecutive space-like links, thus forming $k_i - 1$ triangles. Finally the right boundary vertex in the set of k_i vertices will be identified with the left boundary vertex of the set of k_{i+1} vertices. In this way we get a total of $\sum_{i=1}^{l}(k_i - 1)$ vertices (and also links) at time-slice $t+1$ and the two spatial slices are connected by $\sum_{i=1}^{l} k_i \equiv l_t + l_{t+1}$ triangles. See fig. 1.

The elementary building blocks of a geometry are therefore triangles with one space- and two time-like edges. We define them to be flat in the interior. A consistent way of assigning interior angles to such Minkowskian triangles is described in [17]. The angle between two time-like edges is $\gamma_{tt} = -\arccos\frac{3}{2}$, and between a space-and a time-like edge $\gamma_{st} = \frac{\pi}{2} + \frac{1}{2}\arccos\frac{3}{2}$, summing up to $\gamma_{tt} + 2\gamma_{st} = \pi$. The sum over all angles around a vertex with j incoming and k outgoing time-like edges (by definition $j, k \geq 1$) is given by $2\pi + (4 - j - k)\arccos\frac{3}{2}$. The regular triangulation of flat Minkowski space corresponds to $j = k = 2$ at all vertices. The volume of a single triangle is given by $\frac{\sqrt{5}}{4}a^2$.

One may view these geometries as a subclass of all possible triangulations that allow for the introduction of a causal structure. Namely, if we think of all time-like links as being future-directed, a vertex v' lies in the future of a vertex v iff there is an oriented sequence of time-like links leading from v to v'. Two arbitrary vertices may or may not be causally related in this way.

In quantum gravity we are instructed to sum over all geometries connecting, say, two spatial boundaries of length L_1 and L_2, with the weight of each geometry g given by

$$\mathrm{e}^{iS[g]}, \qquad S[g] = \Lambda \int \sqrt{-g} \quad \text{(in 2d)}, \tag{5}$$

where Λ is the cosmological constant. In our discretized model the boundaries will be characterized by integers l_1 and l_2, the number of vertices or links at the two boundaries. The path integral amplitude for the propagation from geometry l_1 to l_2 will be the sum over all interpolating surfaces of the kind described above, with a weight given by the discretized version of (5). Let us call the corresponding amplitude $G^{(1)}_\lambda(l_1, l_2)$. Thus we have

$$\begin{aligned}
G^{(1)}_\lambda(l_1, l_2) &= \sum_{t=1}^{\infty} G^{(1)}_\lambda(l_1, l_2; t), & (6)\\
G^{(1)}_\lambda(l_1, l_2; t) &= \sum_{l=1}^{\infty} G^{(1)}_\lambda(l_1, l; 1)\; l\; G^{(1)}_\lambda(l, l_2, t-1), & (7)\\
G^{(1)}_\lambda(l_1, l_2; 1) &= \frac{1}{l_1} \sum_{\{k_1,\ldots,k_{l_1}\}} \mathrm{e}^{i\lambda a^2 \sum_{i=1}^{l_1} k_i}, & (8)
\end{aligned}$$

where λ denotes the *bare* cosmological constant[5] (we have absorbed the finite triangle volume factor), and where t denotes the total number of time-slices connecting l_1 and l_2.

From a combinatorial point of view it is convenient to mark a vertex on the entrance loop in order to get rid of the factors l and $1/l$ in (7) and (8), that is,

$$G_\lambda(l_1, l_2; t) \equiv l_1 G^{(1)}_\lambda(l_1, l_2; t) \tag{9}$$

(the unmarking of a point may be thought of as the factoring out by (discrete) spatial diffeomorphisms). Note that $G_\lambda(l_1, l_2; 1)$ plays the role of a transfer matrix,

[5] One obtains the renormalized (continuum) cosmological constant Λ in (5) by an additive renormalization, see below.

satisfying

$$G_\lambda(l_1, l_2, t_1 + t_2) = \sum_l G_\lambda(l_1, l; t_1)\, G_\lambda(l, l_2; t_2) \tag{10}$$

$$G_\lambda(l_1, l_2; t + 1) = \sum_l G_\lambda(l_1, l; 1)\, G_\lambda(l, l_2; t). \tag{11}$$

Knowing $G_\lambda(l_1, l_2; 1)$ allows us to find $G_\lambda(l_1, l_2; t)$ by iterating (11) t times. This program is conveniently carried out by introducing the generating function for the numbers $G_\lambda(l_1, l_2; t)$,

$$G_\lambda(x, y; t) \equiv \sum_{k,l} x^k\, y^l\, G_\lambda(k, l; t), \tag{12}$$

which we can use to rewrite (10) as

$$G_\lambda(x, y; t_1 + t_2) = \oint \frac{dz}{2\pi i\, z}\, G_\lambda(x, z^{-1}; t_1) G_\lambda(z, y; t_2), \tag{13}$$

where the contour should be chosen to include the singularities in the complex z–plane of $G_\lambda(x, z^{-1}; t_1)$ but not those of $G_\lambda(z, y; t_2)$.

One can either view the introduction of $G_\lambda(x, y; t)$ as a purely technical device or take x and y as boundary cosmological constants,

$$x = \mathrm{e}^{i\lambda_i a}, \qquad y = \mathrm{e}^{i\lambda_o a}, \tag{14}$$

such that $x^k = \mathrm{e}^{i\lambda_i a\, k}$ becomes a boundary cosmological term, and similarly for $y^l = \mathrm{e}^{i\lambda_o a\, l}$. Let us for notational convenience define

$$g = \mathrm{e}^{i\lambda a^2}. \tag{15}$$

For the technical purpose of counting we view x, y and g as variables in the complex plane. In general the function

$$G(x, y; g; t) \equiv G_\lambda(x, y; t) \tag{16}$$

will be analytic in a neighbourhood of $(x, y, g) = (0, 0, 0)$.

From the definitions (8) and (9) it follows by standard techniques of generating functions that we may associate a factor g with each triangle, a factor x with each vertex on the entrance loop and a factor y with each vertex on the exit loop, leading to

$$G(x, y; g; 1) = \sum_{k=0}^{\infty} \left(gx \sum_{l=0}^{\infty} (gy)^l \right)^k - \sum_{k=0}^{\infty} (gx)^k = \frac{g^2 xy}{(1 - gx)(1 - gx - gy)}. \tag{17}$$

Formula (17) is simply a book-keeping device for all possible ways of evolving from an entrance loop of any length in one step to an exit loop of any length. The subtraction of the term $1/(1 - gx)$ has been performed to exclude the degenerate cases where either the entrance or the exit loop is of length zero.

From (17) and eq. (13), with $t_1 = 1$, we obtain

$$G(x,y;g;t) = \frac{gx}{1-gx}\, G(\frac{g}{1-gx}, y; g; t-1). \tag{18}$$

This equation can be iterated and the solution written as

$$G(x,y;g;t) = F_1^2(x)F_2^2(x)\cdots F_{t-1}^2(x)\frac{g^2xy}{[1-gF_{t-1}(x)][1-gF_{t-1}(x)-gy]}, \tag{19}$$

where $F_t(x)$ is defined iteratively by

$$F_t(x) = \frac{g}{1-gF_{t-1}(x)}, \quad F_0(x) = x. \tag{20}$$

Let F denote the fixed point of this iterative equation. By standard techniques one readily obtains

$$F_t(x) = F\,\frac{1-xF+F^{2t-1}(x-F)}{1-xF+F^{2t+1}(x-F)}, \qquad F = \frac{1-\sqrt{1-4g^2}}{2g}. \tag{21}$$

Inserting (21) in eq. (19), we can write

$$G(x,y;g,t) = \frac{F^{2t}(1-F^2)^2\, xy}{(A_t - B_t x)(A_t - B_t(x+y) + C_t xy)} \tag{22}$$

$$= \frac{F^{2t}(1-F^2)^2\, xy}{\Big[(1-xF)-F^{2t+1}(F-x)\Big]\Big[(1-xF)(1-yF)-F^{2t}(F-x)(F-y)\Big]}\,, \tag{23}$$

where the time-dependent coefficients are given by

$$A_t = 1-F^{2t+2}, \quad B_t = F(1-F^{2t}), \quad C_t = F^2(1-F^{2t-2}). \tag{24}$$

The combined region of convergence to the expansion in powers $g^k x^l y^m$, valid for all t is

$$|g| < \frac{1}{2}, \quad |x| < 1, \quad |y| < 1. \tag{25}$$

The asymmetry between x and y in the expressions (22) and (23) is due to the marking of the entrance loop. If we also mark the exit loop we have to multiply $G_\lambda(l_1, l_2; t)$ by l_2. We define

$$G_\lambda^{(2)}(l_1, l_2; t) \equiv l_2\, G_\lambda(l_1, l_2; t) = l_1 l_2 G_\lambda^{(1)}(l_1, l_2; t). \tag{26}$$

The corresponding generating function $G^{(2)}(x,y;g;t)$ is obtained from $G(x,y;g;t)$ by acting with $y\frac{d}{dy}$,

$$G^{(2)}(x,y;g;t) = \frac{F^{2t}(1-F^2)^2\, xy}{(A_t - B_t(x+y) + C_t xy)^2}. \tag{27}$$

We can compute $G_\lambda(l_1,l_2;t)$ from $G(x,y;g;t)$ by a (discrete) inverse Laplace transformation

$$G_\lambda(l_1,l_2;t) = \oint \frac{dx}{2\pi i\, x} \oint \frac{dy}{2\pi i\, y} \frac{1}{x^{l_1}} \frac{1}{y^{l_2}} G(x,y;g;t), \tag{28}$$

where the contours should be chosen in the region where $G(x,y;g;t)$ is analytic. A more straightforward method is to rewrite the right-hand side of (22) as a power series in x and y, yielding

$$G_\lambda(l_1,l_2;t) = \frac{F^{2t}(1-F^2)^2 B^{l_1+l_2}}{l_2 \;\; A^{l_1+l_2+2}} \sum_{k=0}^{\min(l_1,l_2)-1} \frac{l_1+l_2-k-1}{k!(l_1-k-1)!(l_2-k-1)!} \left(-\frac{A_t C_t}{B_t^2}\right)^k. \tag{29}$$

which, as expected, is symmetric with respect to l_1 and l_2 after division by l_1.

In the next section we will give explicit expressions for $G_\lambda(l_1,l_2;t)$, $G_\lambda(l_1,l_2)$ and $G_\lambda(x,y)$ (the integral of $G_\lambda(x,y;t)$ over t) in a certain continuum limit.

3 The continuum limit

The path integral formalism we are using here is very similar to the one used to represent the free particle as a sum over paths. Also there one performs a summation over geometric objects (the paths), and the path integral itself serves as the propagator. From the particle case it is known that the bare mass undergoes an additive renormalization (even for the free particle), and that the bare propagator is subject to a wave-function renormalization (see [16] for a review). The same is true in two-dimensional gravity, treated in the formalism of dynamical triangulations [16]. The coupling constants with positive mass dimension, i.e. the cosmological constant and the boundary cosmological constants, undergo an additive renormalization, while the partition function itself (i.e. the Hartle-Hawking-like wave functions) undergoes a multiplicative wave-function renormalization. We therefore expect the bare coupling constants λ, λ_i and λ_0 to behave as

$$\lambda = \frac{C_\lambda}{a^2} + \tilde{\Lambda}, \quad \lambda_i = \frac{C_{\lambda_i}}{a} + \tilde{X}, \quad \lambda_o = \frac{C_{\lambda_o}}{a} + \tilde{Y}, \tag{30}$$

where $\tilde{\Lambda}, \tilde{X}, \tilde{Y}$ denote the renormalized cosmological and boundary cosmological constants. If we introduce the notation

$$g_c = \mathrm{e}^{iC_\lambda}, \quad x_c = \mathrm{e}^{iC_{\lambda_i}}, \quad y_c = \mathrm{e}^{iC_{\lambda_o}}, \tag{31}$$

for critical values of the coupling constants, it follows from (14) and (15) that

$$g = g_c\, \mathrm{e}^{ia^2\tilde{\Lambda}}, \quad x = x_c\, \mathrm{e}^{ia\tilde{X}}, \quad y = y_c\, \mathrm{e}^{ia\tilde{Y}}. \tag{32}$$

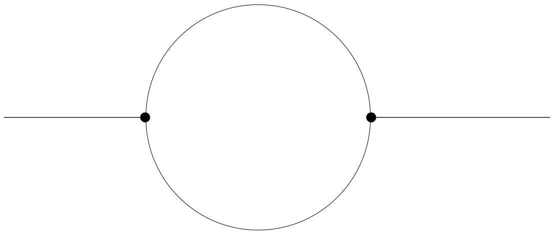

Figure 2: The circle of convergence in the complex g plane (radius 1/2), and the critical lines, ending in $g = \pm 1/2$.

The wave-function renormalization will appear as a multiplicative cut-off dependent factor in front of the bare "Green's function" $G(x, y; g; t)$,

$$G_{\tilde{\Lambda}}(\tilde{X}, \tilde{Y}; T) = \lim_{a \to 0} a^{\eta} G(x, y; g; t), \tag{33}$$

where $T = a\,t$, and where the critical exponent η should be chosen so that the right-hand side of eq. (33) exists. In general this will only be possible for particular choices of g_c, x_c and y_c in (33).

The basic relation (10) can survive the limit (33) only if $\eta = 1$, since we have assumed that the boundary lengths L_1 and L_2 have canonical dimensions and satisfy $L_i = a\,l_i$.

From eqs. (22) and (24) it is clear that we can only obtain a non-trivial continuum limit if $|F| \to 1$. This leads to a one-parameter family of possible choices

$$g_c = \frac{1}{2\cos\alpha} \quad \text{for} \quad F = \mathrm{e}^{i\alpha}, \quad \alpha \in \mathbb{R}, \tag{34}$$

for critical values of g. It follows from (15) that most values of g_c correspond to a complex *bare* cosmological constant λ. However, the renormalized cosmological constant $\tilde{\Lambda}$ in (30) (depending on how we approach g_c in the complex plane) could in principle still be real.

A closer analysis reveals that only at $g_c = \pm 1/2$, corresponding to $\alpha = 0, \pi$, is there any possibility of obtaining an interesting continuum limit. Note that these two values are the only ones which can be reached from a region of convergence of $G(x, y; g; t)$ (see fig. 2). Note also that requiring the bare λ to lie inside the region of convergence when $g \to g_c$ leads to a restriction $\mathrm{Im}\,\tilde{\Lambda} > 0$ on the *renormalized* cosmological constant $\tilde{\Lambda}$, since $|g| < \frac{1}{2} \Rightarrow \mathrm{Im}\ \lambda > \ln 2$.

Without loss of generality, we will consider the critical value $g_c = 1/2$. It corresponds to a purely imaginary bare cosmological constant $\lambda_c := C_\lambda / a^2 = -i \ln 2 / a^2$. If we want to approach this point from the region in the complex g-plane where $G(x, y; g; t)$ converges it is natural to choose the renormalized coupling $\tilde{\Lambda}$ imaginary as well, $\tilde{\Lambda} = i\Lambda$, i.e.

$$\lambda = i\, \frac{\ln \frac{1}{2}}{a^2} + i\Lambda. \tag{35}$$

One obtains a well-defined scaling limit (corresponding to $\Lambda \in \mathbb{R}$) by letting $\lambda \to \lambda_c$ along the imaginary axis. The Lorentzian form for the continuum propagator is obtained by an analytic continuation $\Lambda \to -i\Lambda$ in the *renormalized* coupling of the resulting Euclidean expressions.

At this stage it may seem that we are surreptitiously reverting to a fully Euclidean model. We could of course equivalently have conducted the entire discussion up to this point in the "Euclidean sector", by omitting the factor of $-i$ in the exponential (5) of the action, choosing λ positive real and taking all edge lengths equal to 1. However, from a purely Euclidean point of view there would not have been any reason for restricting the state sum to a subclass of geometries admitting a causal structure. The associated preferred notion of a discrete "time" allows us to define an "analytic continuation in time". Because of the simple form of the action in two dimensions, the rotation

$$\int dx\ dt \sqrt{-g_{lor}} \to i \int dx\ dt_{eu} \sqrt{g_{eu}} \tag{36}$$

to Euclidean metrics in our model is equivalent to the analytic continuation of the cosmological constant Λ.

From (22) or (23) it follows that we can only get macroscopic loops in the limit $a \to 0$ if we simultaneously take $x, y \to 1$. (For $g_c = -1/2$, one needs to take $x, y \to -1$. The continuum expressions one obtains are identical to those for $g_c = 1/2$.) Again the critical points correspond to purely imaginary bare boundary cosmological coupling constants. We will allow for such imaginary couplings and thus approach the critical point $\lambda_i = \lambda_o = 0$ from the region of convergence of $G(x, y; g; t)$, i.e. via real, positive X, Y where

$$\lambda_i = iXa, \qquad \lambda_o = iYa. \tag{37}$$

Again X and Y have an obvious interpretation as positive boundary cosmological constants in a Euclidean theory, which may be analytically continued to imaginary values to reach the Lorentzian sector.

Summarizing, we have

$$g = \frac{1}{2}\mathrm{e}^{-\Lambda a^2} \to \frac{1}{2}(1 - \frac{1}{2}\Lambda a^2), \quad (\text{i.e. } \ F = 1 - a\sqrt{\Lambda}) \tag{38}$$

as well as

$$x = \mathrm{e}^{-Xa} \to 1 - aX, \qquad y = \mathrm{e}^{-aY} \to 1 - aY, \tag{39}$$

where the arrows $\to$ in (38) and (39) should be viewed as analytic coupling constant redefinitions of Λ, X and Y, which we have performed to get rid of factors of 1/2 etc. in the formulas below. With the definitions (38) and (39) it is straightforward to perform the continuum limit of $G(x, y; g, t)$ as $(x, y, g) \to (x_c, y_c, g_c) = (1, 1, 1/2)$, yielding

$$G_\Lambda(X, Y; T) \ = \ \frac{4\Lambda\ \mathrm{e}^{-2\sqrt{\Lambda}T}}{(\sqrt{\Lambda} + X) + \mathrm{e}^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda} - X)}$$

$$\times \frac{1}{(\sqrt{\Lambda}+X)(\sqrt{\Lambda}+Y) - \mathrm{e}^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)(\sqrt{\Lambda}-Y)}. \quad (40)$$

For $T \to \infty$ one finds

$$G_\Lambda(X,Y;T) \overset{T\to\infty}{\longrightarrow} \frac{4\Lambda\ \mathrm{e}^{-2\sqrt{\Lambda}T}}{(X+\sqrt{\Lambda})^2(Y+\sqrt{\Lambda})}. \quad (41)$$

From $G_\Lambda(X,Y;T)$ we can finally calculate $G_\Lambda(L_1,L_2;T)$, the continuum amplitude for propagation from a loop of length L_1, with one marked point, at time-slice $T=0$ to a loop of length L_2 at time-slice T, by an inverse Laplace transformation,

$$G_\Lambda(L_1,L_2;T) = \int_{-i\infty}^{i\infty} dX \int_{-i\infty}^{i\infty} dY\ \mathrm{e}^{XL_1}\ \mathrm{e}^{YL_2}\ G_\Lambda(X,Y;T). \quad (42)$$

This transformation can be viewed as the limit of (28) for $a \to 0$. The continuum version of (13) thus reads

$$G_\Lambda(X,Y;T_1+T_2) = \int_{-i\infty}^{i\infty} dZ\ G_\Lambda(X,-Z;T_1)\ G_\Lambda(Z,Y;T_2), \quad (43)$$

where it is understood that the complex contour of integration should be chosen to the left of singularities of $G_\Lambda(X,-Z;T_1)$, but to the right of those of $G_\Lambda(Z,Y,T_2)$. By an inverse Laplace transformation we get in the limit $T \to \infty$

$$G_\Lambda(L_1,L_2;T) \overset{T\to\infty}{\longrightarrow} 4L_1 \mathrm{e}^{-\sqrt{\Lambda}(L_1+L_2)}\ \mathrm{e}^{-2\sqrt{\Lambda}T}, \quad (44)$$

where the origin of the factor L_1 is the marking of a point in the entrance loop. For $T \to 0$ we obtain

$$G_\Lambda(X,Y;T) \overset{T\to 0}{\longrightarrow} \frac{1}{X+Y}, \quad (45)$$

in agreement with the expectation that the inverse Laplace transform should behave like

$$G_\Lambda(L_1,L_2;T) \overset{T\to 0}{\longrightarrow} \delta(L_1-L_2). \quad (46)$$

The general expression for $G_\Lambda(L_1,L_2;T)$ can be computed as the inverse Laplace transform of formula (40), yielding

$$G_\Lambda(L_1,L_2;T) = \frac{\mathrm{e}^{-[\coth\sqrt{\Lambda}T]\sqrt{\Lambda}(L_1+L_2)}}{\sinh\sqrt{\Lambda}T}\ \frac{\sqrt{\Lambda L_1 L_2}}{L_2}\ I_1\left(\frac{2\sqrt{\Lambda L_1 L_2}}{\sinh\sqrt{\Lambda}T}\right), \quad (47)$$

where $I_1(x)$ is a modified Bessel function of the first kind. The asymmetry between L_1 and L_2 arises because the entrance loop has a marked point, whereas the exit loop has not. The amplitude with both loops marked is obtained by multiplying with L_2, while the amplitude with no marked loops is obtained after dividing (47) by L_1. Quite remarkably, our highly non-trivial expression (47) agrees with the loop propagator obtained from a bona-fide continuum calculation in proper-time gauge of pure 2d gravity by Nakayama [18]. More precisely, his propagator A_m for $m = 0$ (m is a winding number introduced in a somewhat ad-hoc manner in [18]) is related to ours by $G_\Lambda(L_1, L_2; T) = \frac{L_1}{L_2} A_0(L_1, L_2; T)$, which just reflects the fact that the exit instead of the entrance loop has been marked. Also additional ambiguities in the continuum formulation, involving shifts $m \to m + 1/2$ due to renormalization, are fixed in our approach.

To obtain the propagator of the Lorentzian theory, we substitute $\Lambda \to -i\Lambda$ in (47). As a consequence, the amplitude becomes complex and the hyperbolic functions pick up oscillatory contributions. Both the real and the imaginary parts continue to be exponentially damped for large T. What is at first puzzling about the functional form of (47) is that the naïve analytical continuation in "time", $T \to -iT$, leads to a drastically different (and highly singular) result. However, this is an incorrect choice, which can be understood as follows. The combination $\sqrt{\Lambda}T$ appearing as arguments in (47) arises in taking the continuum limit of powers of the form F^t in expressions like (22), (23), where F is defined in (21).

There are two aspects to a possible analytic continuation of F^t. The power t in F^t should clearly not be continued, since it is simply an integer counting the number of iterations of the transfer matrix. However, the function F itself does refer to the action, because the dimensionless coupling constant $g = \mathrm{e}^{i\lambda a_t a_l}$ is the action for a single Lorentzian triangle. (For added clarity we have distinguished between the lattice spacings in time- and space-directions, and called them a_t and a_l.) From the expression for F in terms of g in (21), we have $F = 1 - \sqrt{a_t a_l \Lambda}$. The analytic continuation of F in time, from Euclidean to Lorentzian time, corresponds to the substitution $a_t \to -i\ a_t$ under the square-root sign, and thus becomes equivalent to the continuation $\Lambda \to -i\Lambda$ in the cosmological constant, as already remarked below eq. (36). The subtleties associated with the analytical continuation in the "time"-parameter T appearing in a transfer-matrix formulation of quantum gravity were first discussed in [2, 19] in the context of a square-root action formulation. They will be present also in more complicated theories, where the analytic continuation from Euclidean metrics to Lorentzian metrics cannot be absorbed by a similar continuation in Λ [2].

Finally, we compute the amplitude describing the transition from L_1 to L_2 for an arbitrary "time"-separation of the slices by integrating over T. From (47), multiplied by L_2 in order to arrive at the symmetric propagator where both loops are marked, we get

$$G^{(2)}_\Lambda(L_1, L_2) = \int_0^\infty dT\ G^{(2)}_\Lambda(L_1, L_2; T) = \frac{\mathrm{e}^{-\sqrt{\Lambda}|L_1 - L_2|} - \mathrm{e}^{-\sqrt{\Lambda}(L_1+L_2)}}{2\sqrt{\Lambda}}. \tag{48}$$

One of course obtains the same result by first integrating $G_\Lambda(X, Y; T)$ with respect to T, and then doing the inverse Laplace transform. Again the analytic continuation $\Lambda \to -i\Lambda$ leads to a complex amplitude.

4 The differential equation

The basic result (40) for $G_\Lambda(X,Y;T)$ can be derived by taking the continuum limit of the recursion relation (18). By inserting (38) and (39) in eq. (18) and expanding to first order in the lattice spacing a we obtain

$$\frac{\partial}{\partial T} G_\Lambda(X,Y;T) + \frac{\partial}{\partial X}\Big[(X^2-\Lambda)G_\Lambda(X,Y;T)\Big] = 0. \tag{49}$$

This is a standard first order partial differential equation which should be solved with the boundary condition (45) at $T=0$, since this expresses the natural condition (46) on $G_\Lambda(L_1,L_2)$. The solution is thus

$$G_\Lambda(X,Y;T) = \frac{\bar{X}^2(T;X)-\Lambda}{X^2-\Lambda}\,\frac{1}{\bar{X}(T;X)+Y}, \tag{50}$$

where $\bar{X}(T;X)$ is the solution to the characteristic equation

$$\frac{d\bar{X}}{dT} = -(\bar{X}^2-\Lambda), \qquad \bar{X}(T=0)=X. \tag{51}$$

It is readily seen that the solution is indeed given by (40) since we obtain

$$\bar{X}(T) = \sqrt{\Lambda}\,\frac{(\sqrt{\Lambda}+X) - \mathrm{e}^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)}{(\sqrt{\Lambda}+X) + \mathrm{e}^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)}. \tag{52}$$

If we interpret the propagator $G_\Lambda(L_1,L_2;T)$ as the matrix element between two boundary states of a Hamiltonian evolution in "time" T,

$$G_\Lambda(L_1,L_2;T) = < L_1|\mathrm{e}^{-\hat{H}T}|L_2 > \tag{53}$$

we can, after an inverse Laplace transformation, read off the functional form of the Hamiltonian operator $\hat{H}$ from (49),

$$\hat{H}(L,\frac{\partial}{\partial L}) = -\frac{\partial^2}{\partial L^2} + \Lambda. \tag{54}$$

Using (48), it is now straightforward to check that

$$\hat{H}(L_1,\frac{\partial}{\partial L_1})\; G^{(2)}_\Lambda(L_1,L_2) = \delta(L_1-L_2), \tag{55}$$

as is expected for the propagator. The corresponding Hamiltonian for the propagator of unmarked loops is given by

$$\hat{H}_u(L,\frac{\partial}{\partial L}) = -L\frac{\partial^2}{\partial L^2} - 2\frac{\partial}{\partial L} + \Lambda L. \tag{56}$$

A solution to the "Wheeler-DeWitt equation" is only obtained if we integrate the expression (47) over the entire T-axis (as observed a long time ago in [4]):

$$\hat{H}(L_1, \frac{\partial}{\partial L_1}) \int_{-\infty}^{\infty} dT\; G_\Lambda^{(2)}(L_1, L_2; T) = \hat{H}(L_1, \frac{\partial}{\partial L_1})\; \frac{\sinh \sqrt{\Lambda}(L_1 + L_2)}{\sqrt{\Lambda}} = 0. \tag{57}$$

The above construction refers to the evolution of the system with respect to the "time"-parameter T appearing in the transfer-matrix approach. However, we have argued earlier that one should *not* simply analytically continue $T \to -iT$ to relate the Euclidean and Lorentzian sectors of the theory. Not taking T seriously as a time-parameter presumably implies that also the operator $\hat{H}$ appearing in (53) is not the physically relevant Hamiltonian.

At any rate, our choice of analytic continuation does not seem to lead to a self-adjoint Hamiltonian if one uses the prescription (53) for the Lorentzian case. A possible way out may be to use weights of the form $\mathrm{e}^{\sqrt{i}S}$ *and* to sum over a class of both Lorentzian and Euclidean geometries, as advocated in [2]. Like in this approach, we are also encountering the factor $\sqrt{i}$, but it is presently unclear to us whether the two can be related.

5 Observables

We will now compare the predictions coming from the Euclidean sector of our model with those from 2d quantum gravity as defined by matrix models (or Liouville theory). As mentioned in the introduction, this amounts to calculating the string susceptibility exponent γ, the Hartle-Hawking wave function $W_\Lambda(L)$ and the intrinsic Hausdorff dimension d_H. We must first *define* what we mean by the disc amplitude $W_\Lambda(L)$ in our model. A natural definition is given by

$$W_\Lambda(L) := G_\Lambda(L, L_2{=}0) = \mathrm{e}^{-\sqrt{\Lambda}L}, \tag{58}$$

where the last equality follows from (48). We have contracted the exit loop to length zero in order to produce a disc. This procedure leaves a "mark" at the very endpoint of the universe, contrary to the usual definition of $W_\Lambda(L)$ in Euclidean 2d quantum gravity. Had we applied a definition analogous to (58) in Euclidean gravity, setting $L_2 = 0$ in the propagator would also have led to a marking inside the disc. To relate it to the usual disc amplitude $W_\Lambda^{(eu)}(L)$, formula (58) would have had to be replaced by

$$-\frac{\partial}{\partial \Lambda} W_\Lambda^{(eu)}(L) = G_\Lambda^{(eu)}(L, L_2{=}0) \quad (\propto \frac{1}{\sqrt{L}}\, \mathrm{e}^{-\sqrt{\Lambda}L}). \tag{59}$$

The derivative appears because marking a point in Euclidean gravity is equivalent to differentiating with respect to the cosmological constant. However, in our model the mark in (58) does not correspond to a differentiation $\partial/\partial\Lambda$, since unlike for

Euclidean gravity the mark cannot be located anywhere in the bulk. In fact, from the definition of $G_\Lambda(L_1, L_2; T)$ it follows that

$$W_\Lambda(L) = \int_0^\infty \mathrm{d}T \; G_\Lambda(L, L_2 = 0; T), \tag{60}$$

which implies that the mark is always at the latest proper time T. Although quite similar, (58) differs from (59), a fact which becomes more obvious when we consider their respective Laplace transforms, whose regular parts are given by

$$W_\Lambda(X) = \frac{1}{X + \sqrt{\Lambda}}, \qquad W_\Lambda^{(eu)}(X) = (X - \frac{1}{2}\sqrt{\Lambda})\sqrt{X + \sqrt{\Lambda}}. \tag{61}$$

From $W_\Lambda^{(eu)}(L)$ we can extract γ as follows: by its very definition, γ is the critical exponent controlling the leading *non-analytical* behaviour of the partition function $Z^{(eu)}(\Lambda)$ as a function of the cosmological constant Λ. From definition (1) and the arguments given above, differentiating $Z^{(eu)}(\Lambda)$ twice with respect to Λ leads to the partition function where we sum over closed surfaces of spherical topology with two marked points. In (59) we have already marked one point on the disc by differentiating with respect to Λ. All that remains to be done in order to create closed surfaces with two marked points is to divide $W_\Lambda^{(eu)}(L)$ by L to remove a factor proportional to the boundary length L (which was originally introduced for combinatorial convenience), and to contract the boundary loop L to zero. This gives

$$-\frac{\partial}{\partial\Lambda} \frac{W_\Lambda^{(eu)}(L)}{L} = \frac{1}{L^{3/2}} - \frac{\sqrt{\Lambda}}{L^{1/2}} + \cdots. \tag{62}$$

Expanding this expression in L in the limit $L \to 0$, and extracting the first non-analytic power of Λ leads to the critical exponent

$$\gamma = -\frac{1}{2}. \tag{63}$$

Following the analogous procedure for the disc amplitude (58) leads to $\gamma = 1/2$.

However, the physical interpretation of γ in our model cannot be the same as in Liouville gravity. Firstly, the standard interpretation of $\gamma = 1/2$ would be that we are dealing with objects with the fractal structure of so-called branched polymers [20], which is obviously not the case. We also remind the reader that apart from characterizing the leading singularity in the partition function, in Euclidean quantum gravity the exponent γ governs the rate of baby universe creation [21, 22, 23]. This is implicit in the expansion (62) in that $\sqrt{\Lambda}$ is multiplied by a divergent power of L. This singularity reflects the proliferation of baby universes at the cut-off scale. By contrast, the analogous term in the expansion

$$\frac{W_\Lambda(L)}{L} = \frac{1}{L} - \sqrt{\Lambda} + \cdots \tag{64}$$

for (58) is non-singular, in correspondence with the fact that by construction our model contains no baby universes.

The proliferation of baby universes is closely related to the intrinsic Hausdorff dimension $d_H = 4$ in Euclidean 2d quantum gravity, as decribed by the relation (3). As a consequence of this fractal dimensionality, the geodesic distance R, or equivalently the "time T" in the amplitude $G^{(eu)}_\Lambda(L_1, L_2; T)$ has anomalous length dimension $[\mathrm{L}]^{1/2}$. This also implies that for large T the average "spatial" volume of a slice at some intermediate time will have an anomalous dimension. In fact, in Euclidean 2d quantum gravity we have

$$G^{(eu)}_\Lambda(L_1, L_2; T) \propto \mathrm{e}^{-\sqrt[4]{\Lambda}T} \quad \text{for} \quad T \to \infty, \tag{65}$$

from which one can calculate the average *two-dimensional* volume $V(T)$ in the ensemble of universes with two boundaries separated by a geodesic distance T,

$$\langle V(T)\rangle = -\frac{1}{G^{(eu)}_\Lambda(L_1, L_2; T)} \frac{\partial}{\partial\Lambda} \, G^{(eu)}_\Lambda(L_1, L_2; T) \propto \frac{T}{\Lambda^{3/4}}. \tag{66}$$

For large T we therefore expect the average spatial volume L_{space} at intermediate T's to behave like

$$\langle L_{space}\rangle = \frac{\langle V(T)\rangle}{T} \propto \frac{1}{\Lambda^{3/4}}. \tag{67}$$

In the present model, according to (44), the amplitude behaves for large T like

$$G_\Lambda(L_1, L_2; T) \propto \mathrm{e}^{-\sqrt{\Lambda}T}, \tag{68}$$

which simply means that the dimension of T in this case is [L]. We therefore obtain instead of (67) the dependence

$$\langle L_{space}\rangle \propto \frac{1}{\sqrt{\Lambda}}. \tag{69}$$

This reflects the fact that the quantum space-time of our model does not have an anomalous fractal dimension, and thus differs drastically from the average space-time in the usual two-dimensional Euclidean quantum gravity.

It is possible to calculate explicitly $\langle L_{space}(T_0)\rangle$ for the spatial volume at time $T_0 < T$, even at the discretized level. The details are given in the appendix.

6 Topology changes

In our non-perturbative regularization of 2d quantum gravity we have so far not included the possibility of topology changes of space. At the same time we found disagreement with the theory of Euclidean two-dimensional quantum gravity, whose properties we summarized in the introduction. We will now show that *if* one allows for spatial topology changes, one is led in an essentially unambiguous manner to the theory of two-dimensional quantum gravity, as defined by dynamical triangulations or Liouville theory.

By a topology change of *space* in our Lorentzian setting we have in mind the following: a baby universe may branch off at some time T and develop in the future, where it will eventually disappear in the vacuum, but it is not allowed to rejoin the "parent" universe and thus change the overall topology of the two-dimensional manifold. This is a restriction we impose to be able to compare with the analogous calculation in usual 2d Euclidean quantum gravity.

It is well-known that such a branching leads to additional complications, compared with the Euclidean situation, in the sense that, in general, no continuum Lorentzian metrics which are smooth and non-degenerate everywhere can be defined on such space-times (see, for example, [24] and references therein). These considerations do not affect the cosmological term in the action, but lead potentially to contributions from the Einstein-Hilbert term at the singular points where a branching or pinching occurs.

We have so far ignored the curvature term in the action since it gives merely a constant contribution in the absence of topology change. We will continue to do so in the slightly generalized setting just introduced. The continuum results of [24] suggest that the contributions from the two singular points associated with each branching of a baby universe (one at the branching point and one at the tip of the baby universe where it contracts to a point) cancel in the action. The physical geometry of these configurations may seem slightly contrived, but they may well be important in the quantum theory of gravity and deserve further study. However, for the moment our main motivation for introducing them is to make contact with the usual non-perturbative Euclidean path-integral results.

We will use the rest of this section to demonstrate the following: once we allow for spatial topology changes,

(1) this process completely dominates and changes the critical behaviour of the discretized theory, and

(2) the disc amplitude $W_\Lambda(L)$ (the Hartle-Hawking wave function) is uniquely determined, almost without any calculations.

Our starting point will be the discretized model introduced in sec. 2. Its disc amplitudes will be denoted by $w^{(b)}(l,g)$ and $w^{(b)}(x,g)$, where the superscript ${}^{(b)}$ indicates the "bare" model without spatial topology changes. Similarly, the transfer matrices will be labelled by $G^{(b)}_\lambda(l_1,l_2;t=1)$ and $G_\lambda(l_1,l_2;t=1)$, and the continuum amplitudes by $W^{(b)}_\Lambda(L)$, $W^{(b)}_\Lambda(X)$.

There are a number of ways to implement the creation of baby universes, some more natural than others, but they all agree in the continuum limit, as will be clear from the general arguments provided below. We mention just two ways of implementing such a change. The first is a simple generalization of the forward step

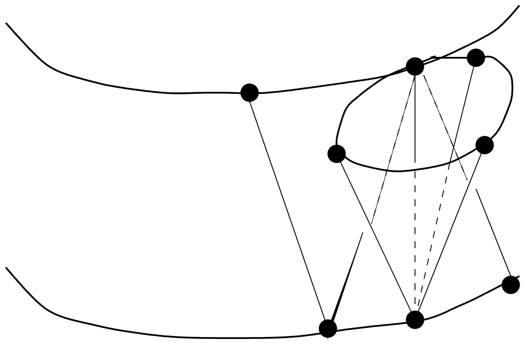

Figure 3: A "baby universe" branches off locally in one time-step.

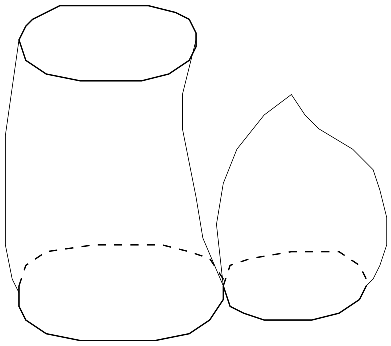

Figure 4: An alternative representation of the process in fig. 3: a "baby universe" is created by a global pinching.

we have used in the original model, where each vertex at time t could connect to n vertices at time $t+1$. We now allow in addition that these sets of n vertices (for $n > 2$) may form a baby universe with closed spatial topology S^1, branching off from the rest. The process is illustrated in fig. 3. An alternative and technically somewhat simpler way to implement the topology change is shown in fig. 4: stepping forward from t to $t+1$ from a loop of length l_1 we create a baby universe of length $l < l_1$ by pinching it off non-locally from the main branch. We have checked that the continuum limit is the same in both cases. For simplicity we only present the derivation in the latter case.

Accounting for the new possibilities of evolution in each step according to fig. 4, the new and old transfer matrices are related by

$$G_\lambda(l_1, l_2; 1) = G_\lambda^{(b)}(l_1, l_2; 1) + \sum_{l=1}^{l_1-1} l_1 w(l_1 - l, g)\, G_\lambda^{(b)}(l, l_2; 1). \tag{70}$$

The factor l_1 in the sum comes from the fact that the "pinching" shown in fig. 4 can take place at any of the l_1 vertices. As before, the new transfer matrix leads to new amplitudes $G_\lambda(l_1,l_2;t)$, satisfying

$$G_\lambda(l_1,l_2;t_1+t_2) = \sum_l G_\lambda(l_1,l;t_1) G_\lambda(l,l_2;t_2), \tag{71}$$

and in particular

$$G_\lambda(l_1,l_2;t) = \sum_l G_\lambda(l_1,l;1)\; G_\lambda(l,l_2;t-1). \tag{72}$$

Performing a (discrete) Laplace transformation of eq. (72) leads to

$$G(x,y;g;t) = \oint \frac{dz}{2\pi i\, z} \left[G^{(b)}_\lambda(x,z^{-1};1) + x\frac{\partial}{\partial x}\Big(w(x;g) G^{(b)}_\lambda(x,z^{-1};1)\Big)\right] G(z,y;g;t–1), \tag{73}$$

or, using the explicit form of the transfer matrix $G^{(b)}_\lambda(x,z;1)$, formula (17),

$$G(x,y;g;t) = \Big[1 + x\frac{\partial w(x,g)}{\partial x} + xw(x,g)\frac{\partial}{\partial x}\Big] \frac{gx}{1-gx}\, G\Big(\frac{g}{1-gx}, y;g;t-1\Big). \tag{74}$$

At this point neither the disc amplitude $w(x,g)$ nor the amplitude $G(x,y;g;t)$ are known. We will now show that they are uniquely determined if we assume that the boundary length scales canonically with the lattice spacing, $L = a\,l$, implying a renormalized boundary cosmological constant X with the dimension of mass, $x = x_c(1-aX)$. In addition we assume that the dimension of the renormalized cosmological constant Λ is canonical too, $g = g_c(1-\frac{1}{2}\Lambda a^2)$. Somewhat related arguments have been presented in different settings in [25, 26].

It follows from relation (71) that we need

$$G_\lambda(l_1,l_2,t) \stackrel{a\to 0}{\longrightarrow} a\, G_\Lambda(L_1,L_2;T). \tag{75}$$

It is important for the following discussion that $G_\lambda(l_1,l_2;t)$ cannot contain a non-scaling part since from first principles (subadditivity) it has to decay exponentially in t. By a Laplace transformation, using $x = x_c(1-aX)$ in the scaling limit, we thus conclude that

$$G_\lambda(x,l_2,t) \stackrel{a\to 0}{\longrightarrow} G_\Lambda(X,L_2,T), \tag{76}$$

and further, by a Laplace transformation in L_2,

$$G_\lambda(x,y;t) \stackrel{a\to 0}{\longrightarrow} a^{-1} G_\Lambda(X,Y;T). \tag{77}$$

We will now show that the scaling of $w(x,g)$ is quite restricted too. The starting point is a combinatorial identity which the disc amplitude has to satisfy. The arguments are valid both for the disc amplitude in Euclidean quantum gravity and the disc amplitude we have introduced for our model in (58). The discretized version of formula (58) is

$$w^{(b)}(x,g) := \sum_t G^{(b)}(x,l_2\!=\!1;g;t) = G^{(b)}(x,l_2\!=\!1;g). \tag{78}$$

It follows from eq. (76) that

$$w^{(b)}(x,g) \to a^{-1} W^{(b)}_\Lambda(X). \tag{79}$$

This scaling is indeed very different from the scaling of the disc amplitude in Euclidean 2d gravity where one has

$$w^{(eu)}(x,g) = w^{(eu)}_{ns}(x,g) + a^{3/2} W^{(eu)}_\Lambda(X). \tag{80}$$

In relation (80), $w^{(eu)}_{ns}(x,g)$ is the non-scaling, analytic part of $w^{(eu)}(x,g)$, and $W^{(eu)}_\Lambda(X)$ is given by (61). We will assume the general form

$$w(x,g) = w_{ns}(x,g) + a^\eta W_\Lambda(X) + \text{less singular terms} \tag{81}$$

for the disc amplitude. In the case $\eta < 0$ the first term is considered absent (or irrelevant). However, if $\eta > 0$ a term like w_{ns} will generically be present, since any slight redefinition of coupling constants of the model will produce such a term if it was not there from the beginning.

We will introduce an explicit mark in the bulk of $w(x,g)$ by differentiating with respect to g. This leads to the combinatorial identity

$$g\,\frac{\partial w(x,g)}{\partial g} = \sum_t \sum_l G(x,l;g;t)\, l\, w(l,g), \tag{82}$$

or, after the usual Laplace transform,

$$g\,\frac{\partial w(x,g)}{\partial g} = \sum_t \oint \frac{dz}{2\pi i\, z}\, G(x,z^{-1};g;t)\,\frac{\partial w(z,g)}{\partial z}. \tag{83}$$

The situation is illustrated in fig. 5. A given mark has a distance t (T in the continuum) to the entrance loop. In the figure we have drawn all points which have the same distance to the entrance loop and which form a connected loop. In the bare model these are all the points at distance t. In the case where baby universes are allowed (which we have not included in the figure), there can be many disconnected loops at the same distance. Let us assume a general scaling

$$T = a^\varepsilon t, \qquad \varepsilon > 0, \tag{84}$$

for the time variable T in the continuum limit. Above we saw that the bare model without baby universe creation corresponded to $\varepsilon = 1$. With the generalization (84) we account for the fact that by allowing for baby universes we have introduced an explicit asymmetry between the time- and space-directions.

Inserting (81) and (84) into eq. (83) we obtain

$$\frac{\partial w_{ns}}{\partial g} - 2a^{\eta-2}\frac{\partial W_\Lambda(X)}{\partial \Lambda} = \frac{1}{a^\varepsilon}\int \mathrm{d}T \int dZ\; G_\Lambda(X,-Z;T)\Big[\frac{\partial w_{ns}}{\partial z} - a^{\eta-1}\frac{1}{z_c}\frac{\partial W_\Lambda(Z)}{\partial Z}\Big], \tag{85}$$

where $(x,g) = (x_c,g_c)$ in the non-singular part.

From eq. (85) and the requirement $\epsilon > 0$ it follows that the only consistent choices for η are

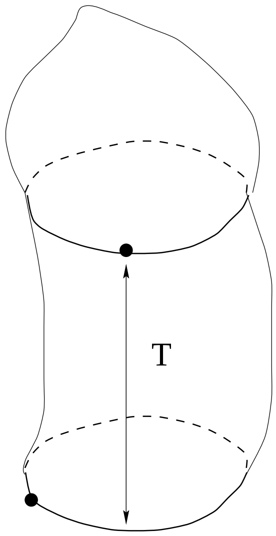

Figure 5: Marking a vertex in the bulk of $W_\Lambda(X)$. The mark has a distance T from the boundary loop, which itself has one marked vertex.

1. $\eta < 0$, i.e.

$$a^{\eta-2}\frac{\partial W_\Lambda(X)}{\partial \Lambda} = \frac{a^{\eta-1}}{2a^\varepsilon}\int \mathrm{d}T \int dZ \; G_\Lambda(X,-Z;T)\,\frac{1}{z_c}\frac{\partial W_\Lambda(Z)}{\partial Z}, \tag{86}$$

in which case we get $\varepsilon = 1$; and

2. $1 < \eta < 2$. Here formula (85) splits into the two equations

$$-\,a^{\eta-2}\frac{\partial W_\Lambda(X)}{\partial \Lambda} = \frac{1}{2a^\varepsilon}\left.\frac{\partial w_{ns}}{\partial z}\right|_{z=x_c}\int \mathrm{d}T \int dZ \; G_\Lambda(X,-Z;T), \tag{87}$$

and

$$\left.\frac{\partial w_{ns}}{\partial g}\right|_{g=g_c} = -\frac{a^{\eta-1}}{a^\varepsilon}\int \mathrm{d}T \int dZ \; G_\Lambda(X,-Z;T)\,\frac{1}{z_c}\frac{\partial W_\Lambda(Z)}{\partial Z}. \tag{88}$$

We are led to the conclusion that $\varepsilon = 1/2$ and $\eta = 3/2$, which are precisely the values found in Euclidean 2d gravity. Let us further remark that eq. (87) in this case becomes

$$-\,\frac{\partial W_\Lambda(X)}{\partial \Lambda} = \mathrm{const.}\; G_\Lambda(X, L_2 = 0), \tag{89}$$

which differs from (58), but agrees with (59). Finally, eq. (88) becomes

$$\int \mathrm{d}T \int dZ \; G_\Lambda(X,-Z;T)\,\frac{\partial W_\Lambda(Z)}{\partial Z} = \mathrm{const.}, \tag{90}$$

which will be satisfied automatically if $\eta = 3/2$ and $\varepsilon = 1/2$, as we will show below.

At first sight it appears surprising that we cannot have values $\eta > 2$, since it is known that the so-called multi-critical matrix models [27] have $\eta = m - 1/2$, $m > 2$ ($m = 2$ corresponds to pure gravity). However, in these situations a generic coupling constant g does *not* correspond to a cosmological constant, and differentiation with respect to g has a different meaning. The scaling in these theories is therefore different (they describe non-unitary matter coupled to 2d quantum gravity and have negative-dimensional operators which dominate over the cosmological constant term).

We will now analyze a possible scaling limit of (74), assuming the canonical scaling $x = x_c(1 - aX)$ and $g = g_c(1 - \frac{1}{2}\Lambda a^2)$. In order that the equation have a scaling limit at all, x_c, g_c and $w_{ns}(x_c, g_c)$ must satisfy two relations which can be determined straightforwardly from (74). The remaining continuum equation reads

$$\begin{aligned} a^{\varepsilon} \frac{\partial}{\partial T} G_\Lambda(X,Y;T) &= -a \frac{\partial}{\partial X}\Big[(X^2 - \Lambda) G_\Lambda(X,Y;T)\Big] \\ &\quad - a^{\eta - 1} \frac{\partial}{\partial X}\Big[W_\Lambda(X) G_\Lambda(X,Y;T)\Big]. \end{aligned} \tag{91}$$

The first term on the right-hand side of eq. (91) is precisely the one we have already encountered in our original model, while the second term is due to the creation of baby universes. Clearly the case $\eta < 0$ (in fact $\eta \leq 1$) is inconsistent with the presence of the second term, i.e. the creation of baby universes. However, since $\eta < 2$, the last term on the right-hand side of (91) will always dominate over the first term. *Once we allow for the creation of baby universes, this process will completely dominate the continuum limit.* In addition we get $\varepsilon = \eta - 1$, in agreement with (88). It follows that $\eta > 1$ and we conclude that $\varepsilon = 1/2, \eta = 3/2$ are the only possible scaling exponents if we allow for the creation of baby universes. These are precisely the scaling exponents obtained from two-dimensional Euclidean gravity in terms of dynamical triangulations, as we have already remarked. The topology changes of space have induced an anomalous dimension for T. If the second term on the right-hand side of (91) had been absent, this would have led to $\varepsilon = 1$, and the time T scaling in the same way as the spatial length L.

In summary, in the case $(\eta, \varepsilon) = (3/2, 1/2)$ eq. (91) leads to the continuum equation

$$\frac{\partial}{\partial T} G_\Lambda(X,Y;T) = -\frac{\partial}{\partial X}\Big[W_\Lambda(X) G_\Lambda(X,Y;T)\Big], \tag{92}$$

which, combined with eq. (89), determines the continuum disc amplitude $W_\Lambda(X)$. Integrating (92) with respect to T and using that $G_\Lambda(L_1, L_2; T\!=\!0) = \delta(L_1 - L_2)$, i.e.

$$G_\Lambda(X, L_2\!=\!0; T\!=\!0) = 1, \tag{93}$$

we obtain

$$-1 = \frac{\partial}{\partial X}\Big[W_\Lambda(X) \frac{\partial}{\partial \Lambda} W_\Lambda(X)\Big]. \tag{94}$$

Since $W_\Lambda(X)$ has length dimension –3/2, i.e. $W_\Lambda^2(X) = X^3 F(\sqrt{\Lambda}/X)$, the general solution must be of the form

$$W_\Lambda(X) = \sqrt{-2\Lambda X + b^2 X^3 + c^2 \Lambda^{3/2}}. \tag{95}$$

From the very origin of $W_\Lambda(X)$ as the Laplace transform of a disc amplitude $W_\Lambda(L)$ which is bounded, it follows that $W_\Lambda(X)$ has no singularities or cuts for $\mathrm{Re}\, X > 0$. This requirement fixes the constants b, c in (95) such that

$$W_\Lambda(X) = b\Big(X - \frac{\sqrt{2}}{b\sqrt{3}}\sqrt{\Lambda}\Big)\sqrt{X + \frac{2\sqrt{2}}{b\sqrt{3}}\sqrt{\Lambda}}, \tag{96}$$

where the constant b is determined by the model-dependent constant in (89). This expression for the disc amplitude agrees after a rescaling of the cosmological constant with $W_\Lambda^{(eu)}(X)$ from 2d Euclidean quantum gravity. With $W_\Lambda(X)$ substituted into (92), the resulting equation is familiar from the usual theory of 2d Euclidean quantum gravity, where it has been derived in various ways [28, 25, 26], with T playing the role of *geodesic distance* between the initial and final loop. In particular, the intrinsic Hausdorff dimension is $d_H = 4$ as soon as we allow for baby universe creation.

Let us finally comment on the difference between the equations for the amplitudes (86)-(88) for $(\eta, \varepsilon) = (-1, 1)$ and $(\eta, \varepsilon) = (3/2, 1/2)$ respectively. In the first case there are no baby universes and eq. (86) entails that only *macroscopic loops* at a distance T from the entrance loop are important (as illustrated by fig. 5). On the other hand, the term $\partial W_\Lambda(Z)/\partial Z$ which describes the presence of these macroscopic loops is absent in eq. (87). This is consistent with eq. (89), which shows explicitly that the length of the upper loop in fig. 5 remains at the cut-off scale, i.e. it never becomes macroscopic. This agrees with the dominance of baby universes: at any point in space-time the probability for creating a little “tip” of the size of the cut-off scale will dominate. At the same time the right-hand side of eq. (86), i.e. eq. (90) will play no role in the case $1 < \eta < 2$, being simply equal to a constant. This latter property is satisfied automatically, as can be seen by using an equation analogous to (92) for the exit instead of the entrance loop. Thus eq. (90) becomes proportional to

$$\int_0^\infty \mathrm{d}T \frac{\partial}{\partial T}\, G(X, L_2 \!=\! 0; T) = \text{const.}, \tag{97}$$

proving our previous assertion.

7 Conclusions

We have tried to construct a non-perturbative model of 2d Lorentzian quantum gravity for universes with cylindrical topology and compact space-like slices. The regularization of the model was performed in the spirit of dynamical triangulations, with fixed edge lengths for each triangle. To encode the light-cone structure of the Lorentzian geometries, we have restricted the sum over states to metric configurations with a discrete causal structure. A class of such causal triangulations was constructed by varying the connectivity according to the rules laid down in sec. 2.

If we regard the edge lengths as invariant geodesic distances on the piecewise linear manifold corresponding to the triangulation, each triangulation defines a geometry, i.e. an equivalence class of metrics, constructed according to Regge's prescription [29]. In this way the class of triangulations we are considering constitutes a grid in the set of all geometries allowing for a causal structure. As in the case of dynamical triangulations, we conjecture that this subset becomes uniformly dense at a critical point, where a continuum limit can be taken. As a consequence, neither a gauge-fixing nor a Faddeev-Popov determinant are needed. We should stress that this is an assumption, as it is in the framework of dynamical triangulations. In this latter approach it is corroborated by the fact that its results agree with the continuum theory, whenever they can be compared. Also in the present model the assumption seems justified, since we obtain agreement with continuum calculations, and, when allowing the creation of baby universes, with the dynamical triangulation results.

A continuum limit of our model exists if we permit an analytic continuation in the coupling constants. Our results then agree with the formal continuum calculations performed in the so-called proper-time gauge [18]. However, they disagree with those of dynamical triangulations, even if we stay "Euclidean", i.e. do not continue the cosmological constant $\Lambda \to -i\tilde{\Lambda}$ back to its original "Lorentzian" value. For example, we obtain $d_H = 2$ for the intrinsic Hausdorff dimension, which shows that the typical geometry of a configuration entering in the path integral is much smoother than in the usual Euclidean version of dynamical triangulations. We have therefore shown that there exists a consistent and non-trivial theory of pure two-dimensional quantum gravity, which can be defined as the continuum limit of a discrete path integral, and which does not lie in the universality class of the usual Liouville gravity. However, as discussed in sec. 6, once we admit baby universe creation such that the spatial topology can change, we are forced back into the universality class of field theories represented by Euclidean 2d quantum gravity.

The comparison between the Lorentzian and Euclidean sectors of our model is not without subtleties. One can define quantities in the Lorentzian sector that rely on the distinction between space- and time-like directions, and which do not possess an immediate analogue in the Euclidean theory. A related observation is that from a purely Lorentzian viewpoint, our extension to configurations with branching baby universes is not forced upon us from first principles. Apart from introducing points where a branch of the universe may disappear into nothing, the two-loop propagator $G_\Lambda(L_1, L_2; T)$ has a peculiar "acausal" property. Although the causal structures on the individual histories are still well defined in the case of topology change, the baby universes that contribute to the amplitude $G_\Lambda(L_1, L_2; T)$ consist entirely of vertices that do not lie in the past of any part of the central universe, (except for a single point of zero volume). Their branches can even extend to times $T' > T$.

We could of course equally well have worked in a "time-reflected" picture where the baby universes are branches coming from the past which join the central universe at some later stage. This would have avoided the problem with the propagator, but we would have had to allow for the spontaneous creation of baby universes. We do not think that the distinction matters for our present purposes, or that these features are a reason for serious concern at this stage. Our main motivation for the extension of our model was to understand which part of the construction needs to be modified in order to make contact with the usual Euclidean results. Since the Lorentzian configurations possess an additional, causal structure, one has to expect subtleties of the kind just mentioned.

Taking the possibility of topology changes more seriously would seem to require a *third quantized* version of gravity (equivalently, a string field theory), in order to deal in a consistent way with the creation and annihilation of universes of length L. In the context of (Euclidean) non-critical string field theory, much progress has been made, in particular for $c = 0$, that is, pure 2d Euclidean quantum gravity [25, 26, 30]. However, its relation with a Lorentzian theory is totally unclear. Is any such formulation consistent with a concept of causality? Can one implement a restriction to causal structures in the Lorentzian path integral? Will the pathologies of Lorentzian metrics at points of topology change [24] play an important role in the path integral? (Unlike in the special case of baby universes, one does not in general expect a cancellation among the action contributions coming from such curvature singularities.) Maybe one will be forced to include geometries of both Euclidean and Lorentzian signature in the path integral, as was suggested in [2]. This work was motivated by some subtleties associated with the analytic continuation in the proper time, similar to those we have encountered in our transfer-matrix approach.

One may summarize our results as follows: the problem of defining a theory of two-dimensional quantum gravity as the continuum limit of a discrete path integral does not have a unique answer. Addressing the problem in a Euclidean setting, one obtains Liouville gravity. In a Lorentzian framework, summing only over metrics with a causal structure, the simplest consistent model leads to a different, inequivalent theory (even modulo any analytic continuation). *If* one wants to obtain agreement between the two, one *must* extend the Lorentzian model by allowing for spatial topology changes. However, from a purely Lorentzian point of view it is unclear whether or not such configurations should be included in the path integral.

What can we conclude from the present work for the case of *real* four-dimensional Lorentzian gravity? One clearly can formulate similar causal restrictions on dynamically triangulated four-geometries, but we do not know whether this prescription alone would make the path integral *sufficiently* Lorentzian. The presence of the Einstein-Hilbert term makes the issue of analytic continuation more complicated. However, if our results in two dimensions are anything to go by, summing over geometries with a causal structure could in principle lead to a drastic change of the results. It may be worthwhile to test this idea by numerical simulations, which are well-developped for the case of Euclidean geometries (see, for instance, [31] and [32] for most recent developments).

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Appendix

In this appendix we calculate the average size of the spatial universe at time T_1, when the total time is given by $T_1 + T_2$. We will show that in the continuum limit the spatial slices are genuinely extended with a spatial volume $\sim 1/\sqrt{\Lambda}$. One can use various boundary conditions. The computation is simplest when one fixes the boundary cosmological constants X, Y and lets the boundary lengths fluctuate according to the distribution dictated by the choice of the boundary cosmological constants. The calculation can be done even at a discretized level. It is helpful to use the properties

$$A_{t_1}A_{t_2} - B_{t_1}B_{t_2} = (1-F^2)A_{t_1+t_2}, \qquad B_{t_1}B_{t_2} - C_{t_1}C_{t_2} = (1-F^2)C_{t_1+t_2} \tag{98}$$

$$B_{t_1}A_{t_2} - C_{t_1}B_{t_2} = A_{t_1}B_{t_2} - B_{t_1}C_{t_2} = (1-F^2)B_{t_1+t_2} \tag{99}$$

of the coefficients A_t, B_t and C_t, which follow directly from the fact that $G(x,y;g;t)$, parametrized as in (22) must satisfy the fundamental composition law (13).

For purely aesthetic reasons, we will consider the symmetric situation where both the entrance and exit loops are marked. We calculate the average size $\langle l(t_1)\rangle_{x,y}$ of a spatial universe at time-slice t_1 as

$$\langle l(t_1)\rangle_{x,y} = \frac{1}{G^{(2)}(x,y;g;t_1+t_2)} \oint \frac{dz}{2\pi i\, z}\, G(x,z^{-1};g;t_1)\; z\frac{d}{dz}G^{(2)}(z,y;g;t_2), \tag{100}$$

where the contour encloses the poles in the z-plane of $G(x,z^{-1};g;t_1)$ but not of $G^{(2)}(z,y;g;t_2)$. From (27) and (98)-(99) the result of the contour integral is

$$\langle l(t_1)\rangle_{x,y} = \frac{1}{1-F^2}\,\frac{(A_{t_1}-B_{t_1}x)(A_{t_2}-B_{t_2}y)+(B_{t_1}-C_{t_1}x)(B_{t_2}-C_{t_2}y)}{A_{t_1+t_2}-B_{t_1+t_2}(x+y)+C_{t_1+t_2}xy}. \tag{101}$$

If t_1 and t_2 go to infinity and a to zero in such a way that T_1 and T_2 stay finite, at the critical point $g_c = \frac{1}{2}$, but with x and y not going to 1 (the case of microscopic boundaries), we have

$$\langle l(t_1)\rangle_{x,y} = \frac{1}{a\sqrt{\Lambda}}\,\frac{(1-\mathrm{e}^{-2\sqrt{\Lambda}T_1})(1-\mathrm{e}^{-2\sqrt{\Lambda}T_2})}{1-\mathrm{e}^{-2\sqrt{\Lambda}(T_1+T_2)}}. \tag{102}$$

This shows that $\langle L(T_1)\rangle = \langle a\, l(t_1)\rangle_{x,y}$ possesses a continuum limit, independent of the size of the microscopic boundaries and independent of T_1 and T_2 if they are sufficiently large.

From (101) we can directly find the expression $\langle L(T_1)\rangle_{X,Y}$ in the case of macroscopic loops:

$$\langle L(T_1)\rangle_{X,Y} = \frac{1}{\sqrt{\Lambda}}\frac{\Big[(\sqrt{\Lambda}+X)+\mathrm{e}^{-2\sqrt{\Lambda}T_1}(\sqrt{\Lambda}-X)\Big]\Big[(\sqrt{\Lambda}+Y)+\mathrm{e}^{-2\sqrt{\Lambda}T_2}(\sqrt{\Lambda}-Y)\Big]}{(X+\sqrt{\Lambda})(Y+\sqrt{\Lambda})-\mathrm{e}^{-2\sqrt{\Lambda}(T_1+T_2)}(\sqrt{\Lambda}-X)(\sqrt{\Lambda}-Y)}. \tag{103}$$

The same expression could have been obtained starting directly from the continuum expression (40) or equivalently from

$$G^{(2)}_\Lambda(X,Y;T) = \frac{4\Lambda\ \mathrm{e}^{-2\sqrt{\Lambda}T}}{\Big[(X+\sqrt{\Lambda})(Y+\sqrt{\Lambda})-\mathrm{e}^{-2\sqrt{\Lambda}T}(\sqrt{\Lambda}-X)(\sqrt{\Lambda}-Y)\Big]^2}, \tag{104}$$

and by calculating

$$\begin{aligned}\langle L(T_1)\rangle_{X,Y} &= \frac{1}{G^{(2)}_\Lambda(X,Y;T_1+T_2)}\int_{-i\infty}^{i\infty} dZ\; G_\Lambda(X,-Z;T_1)\frac{d}{dZ}G^{(2)}_\Lambda(Z,Y;T_2)\\ &= \frac{1}{G^{(2)}_\Lambda(X,Y;T_1+T_2)}\int_{-i\infty}^{i\infty} dZ\; G^{(2)}_\Lambda(X,-Z;T_1)G^{(2)}_\Lambda(Z,Y;T_2)\end{aligned} \tag{105}$$

Let us finally compute $\langle L(T_1)\rangle_{L_1,L_2}$ with the boundary lengths L_1 and L_2 kept fixed. In this case there is no problem with the marking of boundary loops since the factors of L_1 and L_2 cancel in the normalization. Thus

$$\langle L(T_1)\rangle_{L_1,L_2} = \frac{1}{G_\Lambda(L_1,L_2;T_1+T_2)}\int_0^\infty dL\; G_\Lambda(L_1,L;T_1)\; L\; G_\Lambda(L,L_2;T_2) \tag{106}$$

Rather surprisingly, the integration can be performed explicitly using

$$\int_0^\infty dx\; x^3\,\mathrm{e}^{-ax^2} I_1(\beta x)I_1(\gamma x) = \frac{1}{8a^3}\,\mathrm{e}^{\frac{\beta^2+\gamma^2}{4a}}\Big[(\beta^2+\gamma^2)I_1\Big(\frac{\beta\gamma}{2a}\Big)+2\beta\gamma I_0\Big(\frac{\beta\gamma}{2a}\Big)\Big], \tag{107}$$

and we obtain (before normalization), with the notation $S_i=\sinh\sqrt{\Lambda}T_i$ and $C_i=\cosh\sqrt{\Lambda}T_i$, $i=1,2$,

$$\begin{aligned}\langle L(T_1)\rangle^{unnorm}_{L_1,L_2} &= \frac{\sqrt{\Lambda L_1L_2}}{L_2S^3_{1+2}}\,\mathrm{e}^{-\sqrt{\Lambda}(\frac{C_1}{S_1}L_1+\frac{C_2}{S_2}L_2)}\,\mathrm{e}^{\frac{\sqrt{\Lambda}\left(L_1\frac{S_2}{S_1}+L_2\frac{S_1}{S_2}\right)}{S_{1+2}}}\\ &\times\left[(L_1S_2^2+L_2S_1^2)I_1\Big(\frac{2\sqrt{\Lambda L_1L_2}}{S_{1+2}}\Big)+2S_1S_2\sqrt{L_1L_2}\,I_0\Big(\frac{2\sqrt{\Lambda L_1L_2}}{S_{1+2}}\Big)\right].\end{aligned} \tag{108}$$

Dividing by the normalization factor we obtain

$$\langle L(T_1)\rangle_{L_1,L_2} = \frac{1}{S_{1+2}^2}\left[(L_1S_2^2+L_2S_1^2)+2S_1S_2\sqrt{L_1L_2}\frac{I_0\left(\frac{2\sqrt{\Lambda L_1L_2}}{S_{1+2}}\right)}{I_1\left(\frac{2\sqrt{\Lambda L_1L_2}}{S_{1+2}}\right)}\right]. \tag{109}$$

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