On a connection between singular stochastic control and optimal stopping

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Abstract

We show that the value function of a singular stochastic control problem is equal to the integral of the value function of an associated optimal stopping problem. The relation is proved to hold for a general class of diffusions using the method of viscosity solutions.

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1 Introduction

If $V(t,x)$ is the value function of a stochastic singular control problem, it was observed by Bather and Chernoff [BC] that $V_x(t,x) = U(t,x)$ where $U(t,x)$ is the value function of an associated optimal stopping problem. They studied a monotone follower problem for Brownian motion, later treated by Benes et al. [BSW], and Karatzas [K1, K2]. The connection between singular control and optimal stopping was established rigorously by Karatzas and Shreve, [KS1, KS2], for the case of controlling a Brownian motion (so-called monotone and reflected follower problems). Their analysis relied on probabilistic methods. This approach was generalized by Baldursson and Karatzas [BaK] to treat a class of geometric Brownian motions. We also mention the interesting work by El Karoui and Karatzas [EK] where they use a Skorohod problem approach to prove the relation between singular control and optimal stopping for Brownian motion. Recently Boetius and Kohlmann [BoK] applied comparison results from stochastic analysis to prove the connection for a class of diffusion processes with a general drift but only time-dependent diffusion term.

We will in this paper establish the relation between singular stochastic control and optimal stopping under mild conditions for a general class of diffusions. We shall rely in our approach on the notion of viscosity solutions. For a wide class of stochastic control problems possessing a dynamical programming principle, the value function is a viscosity solution of a Bellman equation, see Crandall et al. [CIL] and Fleming and Soner [FS]. For the singular problem in question the Bellman equation takes the form of a variational inequality. Fleming and Soner [FS, Ch. VIII], Haussmann and Suo [HS1, HS2], Ma [M1, M2] and Zhu [Z] have studied this problem in detail and showed that $V(t,x)$ is the (unique) viscosity solution under rather general conditions on the parameters in the problem.

The analytical method applied in our paper proves that $\int_{-\infty}^{x} U(t,z)\,dz$ is a viscosity solution of the variational inequality associated to the singular control problem. We rely on dynamic programming principles for optimal stopping problems in the proof. Since the variational inequality permits a unique viscosity solution, we can conclude that $V(t,x) = \int_{-\infty}^{x} U(t,z)\,dz$. We remark that Myhre [Mh] recently obtained such a connection for a different singular stochastic control problem using verification theorems. His approach *assumes* smooth solutions to the variational inequalities. Within the framework of viscosity solutions, we need only continuity of the value functions. Conditions for when this holds are given. To prove the relation we need to impose a condition on the structure of the continuation region for the optimal stopping problem.

The value function $U(t,x)$ of the associated stopping problem can itself be considered as a viscosity solution of a variational inequality. We refer to Pham [P] and Øksendal and Reikvam [ØR] for details. This establishes a relation between the solutions of two different variational inequalities, which in itself may be interesting. However, this will not be investigated further here.

We also want to mention that different kinds of singular stochastic control problems with applications to finance and biology have been studied by several authors. We mention a few works: Akian et al. [AMS], Alvarez [A], Alvarez and Shepp [AS], Jeanblanc-Picque and Shiryayev [J-PS] and Lungu and Øksendal [LØ1, LØ2]. See also Ch. VIII in Fleming and Soner [FS] and the references therein.

The paper is organized as follows: In Section 2 we formulate the singular stochastic control problem and state the main assumptions on the parameters. The associated optimal stopping problem is considered in Section 3 where we state various required dynamic programming principles known from the literature. We also prove uniform continuity and integrability of the value function and state the assumption on the structure of the continuation region. Finally, in Section 4, we derive the connection between optimal stopping and singular stochastic control.

2 The singular stochastic control problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F}_t$ for $0 \leq t \leq T$ be the σ-algebra generated by the (standard) Brownian motion B_s, $0 \leq s \leq t$. $T < \infty$ is a fixed time horizon. We assume $\mathcal{F}_t$ to satisfy the standard conditions with $\mathcal{F}_T = \mathcal{F}$. Denote by $\mathcal{A}(t)$ the class of $\mathbb{R}$-valued $\mathcal{F}_s$-adapted processes $\xi = \{\xi(s) : t \leq s \leq T\}$ such that a.s. ω

(i) $\xi(t, \omega) = 0$,

(ii) $s \to \xi(s, \omega)$ is nondecreasing and left-continuous with right limits.

The state process to be controlled is

$$X_s^{t,x,\xi} = x + \int_t^s b(u, X_u^{t,x,\xi})du + \int_t^s \sigma(u, X_u^{t,x,\xi})dB_u - \xi(s) \tag{2.1}$$

$b, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ are assumed to be real-valued functions, continuously differentiable in x with bounded derivatives. In addition, we assume linear growth,

$$|\sigma(t,x)| + |b(t,x)| \leq K(1 + |x|) \tag{2.2}$$

where K is independent of t.

Let $h : [0,T] \times I\!R \to [0,\infty)$, $f : [0,T] \to [0,\infty)$ and $g : I\!R \to [0,\infty)$ be measurable functions. The value function of our singular control problem is defined as:

$$V(t,x) = \inf_{\xi \in \mathcal{A}(t)} \mathrm{E}^{t,x} \left[\int_t^T h(s, X_s)\, ds + \int_{[t,T]} f(s)\, d\xi(s) + g(X_T) \right] \tag{2.3}$$

The variational inequality associated to the singular control problem is:

$$\min \left[\mathcal{L}V(t,x) + h(t,x); f(t) - \frac{\partial V}{\partial x}(t,x) \right] = 0 \tag{2.4}$$

$$V(T,x) = g(x) \tag{2.5}$$

where $\mathcal{L}$ is the generator for the diffusion $X_s^{t,x,0}$ known to be $\mathcal{L} = \frac{\partial}{\partial t} + b(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}$. We define the notion of viscosity solution for (2.4) and (2.5), (see Crandall et al. [CIL]): Denote by $C^{1,2}([0,T] \times I\!R)$ the space of functions $\phi : [0,T] \times I\!R \to I\!R$ which are once continuously differentiable in t and twice continuously differentiable in x.

Definition 2.1. Assume $V(t,x)$ is continuous on $[0,T] \times I\!R$ and $V(T,x) = g(x)$:
(i) $V(t,x)$ is a *viscosity subsolution* of (2.4) if for every $\phi \in C^{1,2}([0,T] \times I\!R)$

$$\min \left(\mathcal{L}\phi(\bar{t},\bar{x}) + h(\bar{t},\bar{x}), f(\bar{t}) - \frac{\partial \phi}{\partial x}(\bar{t},\bar{x}) \right) \geq 0$$

where $(\bar{t},\bar{x}) \in [0,T) \times I\!R$ is the maximizer of $V(t,x) - \phi(t,x)$.
(ii) $V(t,x)$ is a *viscosity supersolution* of (2.4) if for every $\phi \in C^{1,2}([0,T] \times I\!R)$

$$\min \left(\mathcal{L}\phi(\bar{t},\bar{x}) + h(\bar{t},\bar{x}), f(\bar{t}) - \frac{\partial \phi}{\partial x}(\bar{t},\bar{x}) \right) \leq 0$$

where $(\bar{t},\bar{x}) \in [0,T) \times I\!R$ is the minimizer of $V(t,x) - \phi(t,x)$.
(iii) $V(t,x)$ is a *viscosity solution* of (2.4) and (2.5) if it is both a viscosity subsolution and supersolution.

From [HS2, Th. 5.5] we have the following result for the connection between the value function $V(t,x)$ defined in (2.3) and the variational inequality (2.4) and (2.5):

Theorem 2.1. *Assume $f(\cdot) > 0$ and Lipschitz continuous and $h(\cdot,\cdot)$ bounded and Lipschitz continuous in both variables. If $g(\cdot) \equiv 0$ then $V(t,x)$ is the unique viscosity solution to (2.4) for which $V(T,x) = 0$.*

Note that in the paper of [HS2] they consider a singular stochastic control problem involving a continuous control in addition to the control $\xi(t)$. Our problem is a special case of their formulation. [M1, M2], [FS] and [Z] have also treated the singular stochastic control problem (2.3) within the framework of viscosity solutions.

We end this section by stating the conditions on the parameter functions which will be assumed throughout the paper:

A1: The functions $h(\cdot,\cdot)$ and $g(\cdot)$ are continuously differentiable in x where the derivatives $h_x(\cdot,\cdot)$ and $g'(\cdot)$ are positive-valued functions. $h(\cdot,\cdot)$ and $g(\cdot)$ are of polynomial growth in x ($h(\cdot,\cdot)$ uniformly in t). In addition, $h_x(\cdot,\cdot)$ and $g'(\cdot)$ are Lipschitz continuous, ($h_x(\cdot,\cdot)$ uniformly in t);

$$|h_x(t,x) - h_x(t,y)| + |g'(x) - g'(y)| \le C|x-y|$$

for $x, y \in \mathbb{R}$ and $t \in [0,T]$. $f(\cdot)$ is Lipschitz continuous,

$$|f(t) - f(s)| \le C|t-s|$$

for $t, s \in [0,T]$.

A2: We assume the following relation between $f(\cdot)$ and $g'(\cdot)$:

$$\sup_{\mathbb{R}} g'(x) \le \inf_{t\in[0,T]} f(t) \tag{2.6}$$

A3: Both $h(\cdot,\cdot)$ and $g(\cdot)$ vanish at $-\infty$, i.e.

$$\lim_{x\to-\infty} h(t,x) = 0 = \lim_{x\to-\infty} g(x) \tag{2.7}$$

for every $t \in [0,T]$.

3 The optimal stopping problem

Consider the state process $X_s^{t,x,0}$ which we from now on simply denote $X_s^{t,x}$,

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})\,du + \int_t^s \sigma(u, X_u^{t,x})\,dB_u$$

It will be convenient to study the following optimal stopping problem in connection with our singular control problem:

$$U(t,x,y) = \inf_{t\le\tau\le T} \mathrm{E}^{t,x,y}\left[\int_t^\tau h_x(s,X_s)Y_s\,ds + f(\tau)Y_\tau \mathbf{1}_{\tau<T} + g'(X_\tau)Y_\tau \mathbf{1}_{\tau=T}\right] \tag{3.1}$$

where the τ's are stopping times with respect to $\mathcal{F}_s$. Later we shall see that $U(t,x,1)$ is the derivative with respect to x of $V(t,x)$ defined in (2.3). The process $Y_s^{t,y}$ solves the stochastic differential equation

$$Y_s^{t,y} = y + \int_t^s b_x(u, X_u^{t,x}) Y_u^{t,y}\, du + \int_t^s \sigma_x(u, X_u^{t,x}) Y_u^{t,y}\, dB_u$$

Note that $Y_s^{t,1} = \frac{\partial}{\partial x} X_s^{t,x}$ (see e.g. [IW]). Obviously, $Y_s^{t,1} \geq 0$ for all $s \in [t,T]$. We shall frequently use the notation

$$\tilde{g}(t,x,y) = (f(t)\mathbf{1}_{t<T} + g'(x)\mathbf{1}_{t=T})\, y \tag{3.2}$$

From the positivity conditions on the parameter functions we obviously have that $U(t,x,y) \geq 0$ whenever $y \geq 0$ for all $(t,x) \in [0,T] \times I\!\!R$. Note also that by choosing $\tau = t$ we get

$$U(t,x,y) \leq \tilde{g}(t,x,y)$$

For $t = T$ we easily see that $U(T,x,y) = g'(x)y$.

3.1 Bellman principles for optimal stopping

By the regularity assumptions on g' and f we have the following Bellman principle for the optimal stopping problem (see e.g. [S, Ch. 3]): Let for $\epsilon \geq 0$

$$\tau_\epsilon := \tau_\epsilon^{t,x,y} := \inf\left\{ s \geq t\, ;\, U(s, X_s^{t,x}, Y_s^{t,y}) \geq \tilde{g}(s, X_s^{t,x}, Y_s^{t,y}) + \epsilon \right\} \tag{3.3}$$

The τ_ϵ's will be ϵ-optimal stopping times. For all stopping times $\tau \leq \tau_\epsilon$,

$$U(t,x,y) = \mathrm{E}^{t,x,y}\left[\int_t^\tau h_x(u, X_u) Y_u\, du + U(\tau, X_\tau, Y_\tau) \right] \tag{3.4}$$

For a general stopping time $\tau \in [t,T]$ we have

$$U(t,x,y) \leq \mathrm{E}^{t,x,y}\left[\int_t^\tau h_x(u, X_u) Y_u\, du + U(\tau, X_\tau, Y_\tau) \right] \tag{3.5}$$

Define the continuation region D to be

$$D = \left\{ (t,x,y) \in [0,T) \times I\!\!R^2\, ;\, U(t,x,y) < f(t)y \right\} \tag{3.6}$$

Observe that τ_0 is the exit time from the continuation region D for the process $(t+s, X_s^{t,x}, Y_s^{t,y})$. (3.4) and (3.5) imply a dynamical programming principle for the optimal stopping problem which is due to [Kr]:

Proposition 3.1. *For any stopping time $\theta \in [t, T]$ we have,*

$$U(t,x,y) = \inf_{t \le \tau \le T} E^{t,x,y}\left[\int_t^{\tau\wedge\theta} h_x(u, X_u)Y_u\,du + \tilde{g}(\tau, X_\tau, Y_\tau)\mathbf{1}_{\tau<\theta}\right.$$

$$\left. + U(\theta, X_\theta, Y_\theta)\mathbf{1}_{\theta\le\tau}\right] \tag{3.7}$$

Proof. The following argument is taken from [Kr, p. 135]. We state it here for completeness: Define $U_1(t,x,y)$ to be the right-hand side of (3.7). Then,

$$\begin{aligned}
U_1(t,x,y) &\le \mathrm{E}^{t,x,y}\left[\int_t^{\tau\wedge\tau_\epsilon} h_x(u,X_u)Y_u\,du + \tilde{g}(\tau, X_\tau, Y_\tau)\mathbf{1}_{\tau<\tau_\epsilon}\right. \\
&\quad \left. + U(\tau_\epsilon, X_{\tau_\epsilon}, Y_{\tau_\epsilon})\mathbf{1}_{\tau_\epsilon\le\tau}\right] \\
&\le \mathrm{E}^{t,x,y}\left[\int_t^{\tau\wedge\tau_\epsilon} h_x(u,X_u)Y_u\,du + U(\tau, X_\tau, Y_\tau)\mathbf{1}_{\tau<\tau_\epsilon}\right. \\
&\quad \left. + U(\tau_\epsilon, X_{\tau_\epsilon}, Y_{\tau_\epsilon})\mathbf{1}_{\tau_\epsilon\le\tau}\right] - \epsilon \\
&= \mathrm{E}\left[\int_t^{\tau\wedge\tau_\epsilon} h_x(u,X_u)Y_u\,du + U(\tau\wedge\tau_\epsilon, X_{\tau\wedge\tau_\epsilon}, Y_{\tau\wedge\tau_\epsilon})\right] - \epsilon
\end{aligned}$$

where we have used that

$$\tilde{g}(\tau_\epsilon, X_{\tau_\epsilon}, Y_{\tau_\epsilon}) \le U(\tau_\epsilon, X_{\tau_\epsilon}, Y_{\tau_\epsilon}) - \epsilon$$

But since $\tau\wedge\tau_\epsilon \le \tau_\epsilon$, we obtain from (3.4)

$$U_1(t,x,y) \le U(t,x,y) - \epsilon$$

for all $\epsilon \ge 0$. Hence, $U_1(t,x,y) \le U(t,x,y)$.

On the other hand,

$$\begin{aligned}
U_1(t,x,y) &= \inf_{t\le\tau\le T} \mathrm{E}^{t,x,y}\left[\int_t^{\tau\wedge\theta} h_x(u,X_u)Y_u\,du + \tilde{g}(\tau, X_\tau, Y_\tau)\mathbf{1}_{\tau<\theta}\right. \\
&\quad \left. + U(\theta, X_\theta, Y_\theta)\mathbf{1}_{\theta\le\tau}\right] \\
&\ge \inf_{t\le\tau\le T} \mathrm{E}^{t,x,y}\left[\int_t^{\tau\wedge\theta} h_x(u,X_u)Y_u\,du + U(\tau, X_\tau, Y_\tau)\mathbf{1}_{\tau\le\theta}\right. \\
&\quad \left. + U(\theta, X_\theta, Y_\theta)\mathbf{1}_{\theta<\tau}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{t\le\tau\le T} \mathrm{E}^{t,x,y}\left[\int_t^{\tau\wedge\theta} h_x(u,X_u)Y_u\,du + U(\tau\wedge\theta, X_{\tau\wedge\theta}, Y_{\tau\wedge\theta})\right] \\
&\ge U(t,x,y)
\end{aligned}
$$

where we have used (3.5) and the fact that $U(t,x,y) \le \tilde{g}(t,x,y)$. Hence, $U_1(t,x,y) = U(t,x,y)$. □

3.2 Properties of the value function

We have the following factorization result which will be useful later:

Lemma 3.2. *The value function $U(t,x,y)$ satisfies*

$$
U(t,x,y) = yU(t,x,1) \tag{3.8}
$$

Proof. Note that $Y_s^{t,y} = yY_s^{t,1}$. A direct calculation shows,

$$
\begin{aligned}
U(t,x,y) &= \mathrm{E}^{t,x,y}\left[\int_t^{\tau} h_x(s,X_s)Y_s\,ds + \tilde{g}(\tau,X_\tau,Y_\tau)\right] \\
&= \mathrm{E}\left[\int_t^{\tau} h_x(s,X_s^{t,x})Y_s^{t,y}\,ds + \tilde{g}(\tau,X_\tau^{t,x},Y_\tau^{t,y})\right] \\
&= \mathrm{E}\left[\int_t^{\tau} h_x(s,X_s^{t,x})Y_s^{t,1}\,ds + \tilde{g}(\tau,X_\tau^{t,x},Y_\tau^{t,1})\right]\cdot y \\
&= yU(t,x,1)
\end{aligned}
$$

□

We shall from now on only consider the stopping problem with $y=1$, i.e. the value function $U(t,x) := U(t,x,1)$. Note that by choosing $\tau = t$ we get $U(t,x) \le \tilde{g}(t,x,1)$. But when $t < T$, $\tilde{g}(t,x,1)$ equals $f(t)$, while for $t = T$ it is $g'(x)$. Recall the assumption (2.6), which yields

$$
U(t,x) \le \sup_{t\in[0,T]} f(t) \tag{3.9}
$$

We prove that $U(t,x)$ is uniformly continuous in t and x (the proof is inspired by [P]):

Proposition 3.3. *$U(t,x)$ is Lipschitz continuous in x uniformly in t.*

Proof. Fix $t\in[0,T]$ and let $x,y\in I\!R$. Then, by the Lipschitz continuity of $h_x(s,\cdot)$ and $g'(\cdot)$,

$$
|U(t,x)-U(t,y)| \le \sup_{t\le\tau\le T}\Big|\mathrm{E}[\int_t^{\tau}\left(h_x(s,X_s^{t,x})Y_s^{t,1} - h_x(s,X_s^{t,y})Y_s^{t,1}\right)\,ds
$$

$$
\begin{aligned}
&+ \mathbf{1}_{\tau=T}\{\left(g'(X_\tau^{t,x}) - g'(X_\tau^{t,y})\right) Y_\tau^{t,1}\}]\Big| \\
&\le C \sup_{t\le\tau\le T} \mathrm{E}\Big[\int_t^\tau |X_s^{t,x} - X_s^{t,y}||Y_s^{t,1}|\, ds \\
&+ \mathbf{1}_{\tau=T}|X_T^{t,x} - X_T^{t,y}||Y_T^{t,1}|\Big]
\end{aligned}
$$

The Cauchy-Schwarz' inequality gives,

$$
\begin{aligned}
|U(t,x) - U(t,y)| &\le C\mathrm{E}\Big[\int_t^T |X_s^{t,x} - X_s^{t,y}|\, ds \cdot \sup_{t\le s\le T} |Y_s^{t,1}|\Big] \\
&+ C \sup_{t\le\tau\le T} \mathrm{E}\Big[|Y_T^{t,1}||X_T^{t,x} - X_T^{t,y}|\Big] \\
&\le C\mathrm{E}\Big[(\sup_{t\le s\le T} Y_s^{t,1})^2\Big]^{1/2} \mathrm{E}\Big[(\int_t^T |X_s^{t,x} - X_s^{t,y}|\, ds)^2\Big]^{1/2} \\
&+ C\mathrm{E}\Big[(Y_T^{t,1})^2\Big]^{1/2} \mathrm{E}\Big[|X_T^{t,x} - X_T^{t,y}|^2\Big]^{1/2}
\end{aligned}
$$

The Lipschitz continuity now follows by moment estimates for diffusions, see e.g. [GS, IW]. □

The Lipschitz continuity in x is applied to prove the uniform continuity in t of $U(t,x)$:

Proposition 3.4. *U(t,x) is uniformly continuous on $[0,T]$ for each $x \in I\!\!R$. In fact, there exists a positive constant C such that*

$$
|U(t,x) - U(s,x)| \le C\left(|t-s|^{1/2} + |t-s|\right)
$$

Proof. First note that $U(t,x)$ is nondecreasing in t. Use (3.7) with $\theta = s$, where $0 \le t < s \le T$:

$$
\begin{aligned}
&0 \le U(s,x) - U(t,x) \\
&= U(s,x) - \inf_\tau \mathrm{E}\Big[[\int_t^{\tau\wedge s} h_x(u, X_u^{t,x}) Y_u^{t,1}\, du + \tilde{g}(\tau, X_\tau^{t,x}, Y_\tau^{t,1})\mathbf{1}_{\tau<s} \\
&\quad + \mathbf{1}_{s\le\tau} U(s, X_s^{t,x}) Y_s^{t,1}\Big] \\
&= \sup_\tau \mathrm{E}\Big[-\int_s^{\tau\wedge s} h_x(u, X_u) Y_u\, du - \mathbf{1}_{\tau<s}\tilde{g}(\tau, X_\tau^{t,x}, Y_\tau^{t,1})
\end{aligned}
$$

$$
\begin{aligned}
&- \mathbf{1}_{s\le\tau} U(s, X_s^{t,x}) Y_s^{t,1} + U(s,x)\Big] \\
&\le \sup_\tau \mathrm{E}\Big[\mathbf{1}_{s\le\tau}\{U(s,x) - U(s, X_s^{t,x}) Y_s^{t,1}\} + \mathbf{1}_{\tau<s} U(s,x) \\
&- \mathbf{1}_{\tau<s}\tilde{g}(\tau, X_\tau^{t,x}, Y_\tau^{t,1})\Big]
\end{aligned}
$$

where we have used the assumption $h_x \ge 0$ in the last inequality. Furthermore, by rewriting, and observing that $\tau < s$ implies $\tau < T$,

$$
\begin{aligned}
U(s,x) - U(t,x) &\le \sup_\tau \mathrm{E}\Big[\mathbf{1}_{s\le\tau}\{U(s,x) - U(s, X_s^{t,x}) Y_s^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{\tilde{g}(t,x,1) - \tilde{g}(\tau, X_\tau^{t,x}, Y_\tau^{t,1})\} \\
&+ \mathbf{1}_{\tau<s}\{U(s,x) - \tilde{g}(t,x,1)\}\Big] \\
&= \sup_\tau \mathrm{E}\Big[\mathbf{1}_{s\le\tau}\{U(s,x) - U(s, X_s^{t,x}) Y_s^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{f(t) - f(\tau) Y_\tau^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{U(s,x) - \tilde{g}(s,x,1)\} \\
&+ \mathbf{1}_{\tau<s}\{\tilde{g}(s,x,1) - \tilde{g}(t,x,1)\}\Big]
\end{aligned}
$$

By choosing $\tau = s$ we get $U(s,x) \le \tilde{g}(s,x,1)$. Hence, $U(s,x) - \tilde{g}(s,x,1) \le 0$. Furthermore, by the assumption $\sup_x g'(x) \le \inf_{t\in[0,T]} f(t)$,

$$
\tilde{g}(s,x,1) - \tilde{g}(t,x,1) = \tilde{g}(s,x,1) - f(t) \le f(s) - f(t)
$$

Hence,

$$
\begin{aligned}
U(s,x) - U(t,x) &\le \sup_\tau \mathrm{E}\Big[\mathbf{1}_{s\le\tau}\{U(s,x) - U(s, X_s^{t,x}) Y_s^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{(f(t) - f(\tau)) Y_\tau^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{f(t) - f(t) Y_\tau^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{f(s) - f(t)\}\Big] \\
&\le \sup_\tau \mathrm{E}\Big[\mathbf{1}_{s\le\tau}\{U(s,x) - U(s, X_s^{t,x}) Y_s^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{(f(t) - f(\tau)) Y_\tau^{t,1}\} \\
&+ \mathbf{1}_{\tau<s}\{(1 - Y_\tau^{t,1}) \sup_{t\in[0,T]} f(t)\} \\
&+ \mathbf{1}_{\tau<s}\{f(s) - f(t)\}\Big]
\end{aligned}
$$

Use the Lipschitz continuity of U and f to get,

$$\begin{aligned}
|U(s,x)-U(t,x)| &\leq \sup_{t\leq\tau\leq T}\Big\{C\mathrm{E}\Big[|x-X_s^{t,x}||Y_s^{t,1}|+C|\tau-t||Y_\tau^{t,1}|\mathbf{1}_{\tau<s} \\
&\quad + \sup_{t\in[0,T]} f(t)|1-Y_\tau^{t,1}\mathbf{1}_{\tau<s}\Big]\Big\}+C|t-s| \\
&\leq C\sup_{t\leq\tau\leq T}\mathrm{E}\Big[|Y_s^{t,1}||x-X_s^{t,x}|\Big]+C|t-s|\sup_{t\leq\tau\leq s}\mathrm{E}\Big[Y_\tau^{t,1}\Big] \\
&\quad + \sup_{t\in[0,T]} f(t)\sup_{s\leq\tau\leq T}\mathrm{E}\Big[|1-Y_\tau^{t,1}|\Big]+C|t-s|
\end{aligned}$$

By invoking known estimates regarding the dependence on the initial condition for diffusions, we get the desired continuity in t (see e.g. [GS, IW]):

$$|U(s,x)-U(t,x)| \leq C\left(|t-s|^{1/2}+|t-s|\right)$$

□

We conclude,

Proposition 3.5. *There exists a positive constant C such that for $t,s\in[0,T]$ and $x,y\in I\!R$,*

$$|U(t,x)-U(s,y)| \leq C\left(|t-s|^{1/2}+|t-s|+|x-y|\right)$$

Since we want to prove that the integral of $U(t,z)$ with respect to z is a viscosity solution of (2.4), we need to show that $U(t,z)$ is integrable on $(-\infty,x]$ for every $x\in I\!R$. The next proposition proves this. But first we need a lemma which shows that the process $X_s^{t,x}$ goes to minus infinity when the intial condition x does, a.s. ω:

Lemma 3.6. *For every $0\leq t\leq s\leq T$*

$$\lim_{t\to-\infty} X_s^{t,x}(\omega)=-\infty$$

a.s. $\omega\in\Omega$.

Proof. Fix $0\leq t\leq s\leq T$. From the Comparison Theorem VI.1.1 in [IW] the mapping $x\to X_s^{t,x}$ is nondecreasing a.s. ω. Since the coeffiecients $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ in the equation for $X_s^{t,x}$ are of linear growth and continuously differentiable with bounded derivatives, $x\to X_s^{t,x}(\omega)$ is a C^1-diffeomorphism of $I\!R$ a.s. ω (see [Ku] or the comment following Th. V.2.3 in [IW]). Hence, for every constant $N>0$ there exists a $K>0$ such that $X_s^{t,x}\leq -N$ for all $x\leq -K$, a.s. ω. Note that N and K depends on ω. This proves the lemma. □

Proposition 3.7. *For every* $(t,x) \in [0,T] \times I\!\!R$ *we have that*

$$\int_{-\infty}^{x} U(t,z)\,dz < \infty$$

Proof. First note that due to the polynomial growth conditions on $h(\cdot,\cdot)$ and $g(\cdot)$ we have

$$\mathrm{E}\left[\int_t^T h(s,X_s^{t,x})\,ds + g(X_T^{t,x})\right] \le C(1+|x|^m)$$

This is proven by using standard moment estimates diffusions (see e.g. [GS, IW]). Choose the stopping time $\tau = T$ to obtain

$$0 \le U(t,z,1) \le \mathrm{E}^{t,z,1}\left[\int_t^T h_x(s,X_s)Y_s\,ds + g'(X_T)Y_T\right]$$

Consider $\int_t^T h_x(s,X_s^{t,z})Y_s^{t,1}\,ds + g'(X_T^{t,z})Y_T^{t,1}$. For an $x \in I\!\!R$ we apply the Fubini-Tonelli theorem to obtain

$$\begin{aligned}
&\int_{-\infty}^{x}\int_t^T h_x(s,X_s^{t,z})Y_s^{t,1}\,ds\,dz + \int_{-\infty}^{x} g'(X_T^{t,z})Y_T^{t,1}\,dz \\
&\quad = \int_t^T \left(\int_{-\infty}^{x} h_x(s,X_s^{t,z})Y_s^{t,1}\,dz\right)ds + \int_{-\infty}^{x} g'(X_T^{t,z})Y_T^{t,1}\,dz \\
&\quad = \int_t^T \left(\int_{-\infty}^{X_s^{t,x}} h_x(s,u)\,du\right)ds + \int_{-\infty}^{X_T^{t,x}} g'(u)\,du \\
&\quad = \int_t^T h(s,X_s^{t,x})\,ds + g(X_T^{t,x})
\end{aligned}$$

In the second last equality we made a change of variables by substituting with $u(z) = X_s^{t,z}$. Note that $u'(z) = Y_s^{t,1}$. In addition, we have applied the assumption (2.7) and Lemma 3.6. Again appealing to the Fubini-Tonelli theorem,

$$\int_{-\infty}^{x} U(t,z,1)\,dz \le \mathrm{E}\left[\int_t^T h(s,X_s^{t,x})\,ds + g(X_T^{t,x})\right] < \infty$$

□

We introduce the continuation region for the stopping problem when $y = 1$:

$$D_1 = \Big\{(t,x) \in [0,T) \times I\!\!R\,;\; U(t,x) < f(t)\Big\} \tag{3.10}$$

D_1 is open by the Lipschitz condition on $f(\cdot)$ and the continuity of $U(\cdot,\cdot)$. Observe that $(t,x) \in D_1$ implies that $(t,x,1) \in D$. This region will be of crucial importance in our proof for the connection between optimal stopping and singular stochastic control. The following assumption is made from now on:

A4: The region D_1 defined in (3.10) is connected in the sense that if $(t,x) \in D_1$ then $(t,z) \in D_1$ for any $z \in (-\infty, x)$

Remark. In [BØ] they assume that the continuation region has the shape

$$\Big\{(t,x) : x < \alpha(t)\Big\}$$

for some function $\alpha(t)$. If we assume $\alpha(\cdot)$ to be positive, it is easily seen that condition (A4) holds.

4 The connection between optimal stopping and singular stochastic control problems

Let $V(t,x)$ be defined as in (2.3) and $U(t,x) = U(t,x,1)$ in (3.1). The first result links $U(t,x)$ to the variational inequality associated to the singular stochastic control problem:

Theorem 4.1. *Assume conditions (A1)-(A4) are satisfied. Then the function $\int_{-\infty}^{x} U(t,z)\,dz$ is a viscosity solution of (2.4) with $\int_{-\infty}^{x} U(T,z)\,dz = g(x)$.*

Proof. We first observe by Prop. 3.7 and Prop. 3.4 that $U(t,z)$ is integrable with respect to z and that the integral is continuous in both variables. The boundary condition $g(x)$ is obviously satisfied since,

$$\int_{-\infty}^{x} U(T,z)\,dz = \int_{-\infty}^{x} g'(z)\,dz = g(x)$$

where we have used the assumption $\lim_{x\to-\infty} g(x) = 0$.

We first treat the viscosity supersolution case: Let $\phi \in C^{1,2}([0,T] \times I\!R)$ and $(\bar{t},\bar{x})$ be a minimizer of $\int_{-\infty}^{x} U(t,z)\,dz - \phi(t,x)$. Without any loss of generality we may assume that

$$\phi(\bar{t},\bar{x}) = \int_{0}^{\bar{x}} U(\bar{t},z)dz \quad \text{and} \quad \phi(t,x) \le \int_{-\infty}^{x} U(t,z)\,dz$$

Suppose $(\bar{t},\bar{x}) \notin D_1$. Since $(\bar{t},\bar{x})$ is an optimum and both $\phi(t,x)$ and $\int_{-\infty}^{x} U(t,z)\,dz$ are differentiable at $(\bar{t},\bar{x})$, we have

$$\frac{\partial \phi}{\partial x}(\bar{t},\bar{x}) = U(\bar{t},\bar{x}) = f(\bar{t})$$

In the second equality we have used that $(\bar{t},\bar{x},1) \notin D$ since $(\bar{t},\bar{x}) \notin D_1$. Thus we see that the supersolution property is satisfied in this case independent of what $\mathcal{L}\phi + h$ must be. Suppose now that the maximum is inside the region D_1. Since D_1 is open and assumed to have the property that $(\bar{t},z) \in D_1$ for all $z \in (-\infty,\bar{x})$, there exists a stopping time τ which is smaller that or equal to $\tau_{D_1}(\bar{t},z)$ for all z and not being equal to $\bar{t}$. Namely $\tau = \inf\{\tau_{D_1}(\bar{t},z)\}$. (We use the notation τ_{D_1} for the first exit time of D_1). The Fubini-Tonelli theorem and the dynamical programming principle in D give

$$\begin{aligned}
\phi(\bar{t},\bar{x}) &= \int_{-\infty}^{\bar{x}} U(\bar{t},z)\,dz \\
&= \int_{-\infty}^{\bar{x}} \left(\mathrm{E}^{\bar{t},z,1}\left[\int_{\bar{t}}^{\tau} h_x(s,X_s)Y_s\,ds + Y_\tau U(\tau,X_\tau) \right] \right) dz \\
&= \mathrm{E}\left[\int_{\bar{t}}^{\tau} \left(\int_{-\infty}^{\bar{x}} h_x(s,X_s^{\bar{t},z})Y_s^{\bar{t},1} \right) dt \right] + \mathrm{E}\left[\int_{-\infty}^{\bar{x}} Y_\tau^{\bar{t},1} U(\tau,X_\tau^{\bar{t},z})\,dz \right]
\end{aligned}$$

The substitution $u = X_s^{\bar{t},z}$ in the two integrals above yields (since $du = Y_s^{\bar{t},1}dz$)

$$\int_{-\infty}^{\bar{x}} h_x(s,X_s^{\bar{t},z})Y_s^{\bar{t},1}\,ds = \int_{-\infty}^{X_s^{\bar{t},\bar{x}}} h_x(s,u)\,du = h(s,X_s^{\bar{t},\bar{x}})$$

In the last equality we have used that $\lim_{x\to-\infty} X_s^{t,x} = -\infty$ a.s. and the assumption $\lim_{x\to-\infty} h(t,x) = 0$. Equivalently, we have

$$\int_{-\infty}^{\bar{x}} Y_\tau^{\bar{t},1} U(\tau,X_\tau^{\bar{t},z})\,dz = \int_{-\infty}^{X_\tau^{\bar{t},\bar{x}}} U(\tau,u)\,du$$

Hence, using that $\phi(t,x) \le \int_{-\infty}^{x} U(t,z)\,dz$ we get

$$\begin{aligned}
\phi(\bar{t},\bar{x}) &= \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} h(s,X_s)\,ds \right] + \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{-\infty}^{X_\tau} U(\tau,u)\,du \right] \\
&\ge \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} h(s,X_s)\,ds + \phi(\tau,X_\tau) \right]
\end{aligned}$$

Dynkin's formula yields

$$0 \geq \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} \mathcal{L}\phi(s,X_s) + h(s,X_s)\,ds\right]$$

A limiting argument when $\tau \to \bar{t}$ gives that $\mathcal{L}\phi + h \leq 0$. Hence, $\int_{-\infty}^{x} U(t,z)\,dz$ is a viscosity supersolution.

Consider now the viscosity subsolution case: Let $\phi \in C^{1,2}([0,T] \times I\!R)$ and $(\bar{t},\bar{x})$ be a maxmizer of $\int_{-\infty}^{x} U(t,z)\,dz - \phi(t,x)$. Without any loss of generality we may assume that

$$\phi(\bar{t},\bar{x}) = \int_{0}^{\bar{x}} U(\bar{t},z)dz \quad \text{and} \quad \phi(t,x) \geq \int_{-\infty}^{x} U(t,z)\,dz$$

Since $U(t,x,y) \leq \tilde{g}(t,x,y)$ we know that $f(\bar{t}) - \frac{\partial\phi}{\partial x}(\bar{t},\bar{x}) \geq 0$. It remains to show that $\mathcal{L}\phi(\bar{t},\bar{x}) + h(\bar{t},\bar{x}) \geq 0$ in order to have the viscosity supersolution property. Using the dynamic programming principle and the same argumentation as above yields,

$$\begin{aligned}
\phi(\bar{t},\bar{x}) &= \int_{-\infty}^{\bar{x}} U(\bar{t},z)\,dz \\
&\leq \int_{-\infty}^{\bar{x}} \left(\mathrm{E}^{\bar{t},z,1}\left[\int_{\bar{t}}^{\tau} h_x(s,X_s)Y_s ds + Y_\tau U(\tau,X_\tau)\right]\right)\,dz \\
&= \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} h(s,X_s)\,ds + \int_{-\infty}^{X_\tau} U(\tau,u)\,du\right] \\
&\leq \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} h(s,X_s)\,ds + \phi(\tau,X_\tau)\right]
\end{aligned}$$

Dynkin's formula now yields,

$$0 \leq \mathrm{E}^{\bar{t},\bar{x}}\left[\int_{\bar{t}}^{\tau} \mathcal{L}\phi(s,X_s) + h(s,X_s)\,ds\right]$$

A limiting argument when $\tau \to \bar{t}$ gives that $\mathcal{L}\phi + h \geq 0$. Hence, $\int_{-\infty}^{x} U(t,z)\,dz$ is a viscosity subsolution. This completes the proof. □

We state the theorem which connects the optimal stopping problem to the singular stochastic control problem:

Theorem 4.2. *In addition to the conditions (A1)-(A4), assume that* $f(\cdot) > 0$ *for all* $t \in [0,T]$ *and that* $h(\cdot,\cdot)$ *is bounded and Lipschitz continuous in both variables. If* $g(\cdot) \equiv 0$*,*

$$V(t,x) = \int_{-\infty}^{x} U(t,z)\,dz$$

Proof. Under the above assumptions we know from Th. 2.1 that $V(t,x)$ is the unique viscosity solution of (2.4) such that $V(T,x) = 0$. From Th. 4.1 we can therefore conclude that $V(t,x) = \int_{-\infty}^{x} U(t,z)\,dz$. □

From this connection we immediately get that $V(t,x)$ is at least continuously differentiable in the space variable.

It is known that in many cases the optimal control $\xi^*(s)$ of the singular problem behaves like a local time on the boundary of some region in $[0,T) \times \mathbb{R}$, see e.g. the works of [LØ1, LØ2], [BaK], [KS1, KS2] and [FS, Ch. VIII]. This region is sometimes called the region of in-action, and is defined as

$$\tilde{D}_1 := \left\{(t,x) \in [0,T) \times \mathbb{R}\,;\, \frac{\partial V}{\partial x}(t,x) < f(t)\right\} \tag{4.1}$$

Under the assumption of Th. 4.2, we immediately get $\tilde{D}_1 = D_1$, i.e. the continuation region of the stopping problem coincides with the in-action region of the singular control problem.

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