On the scattering operator for the Schrödinger equation with a time-dependent potential

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1 Introduction

In this paper we give some results on the scattering operator for the Schrödinger equation with a time-dependent potential. We consider the free Schrödinger equation

$$i\partial_t u(t,x) = -\Delta_x u(t,x), \qquad u(s,x) = u_0(x), \tag{1}$$

and the full Schrödinger equation

$$i\partial_t v(t,x) = -\Delta_x v(t,x) + V(t,x)v(t,x), \qquad v(s,x) = v_0(x). \tag{2}$$

Here V is a potential depending explicitly on time. The solution to (1) is given by $u(t) = U_0(t-s)u_0 = e^{-i(t-s)H_0}u_0$, where $H_0 = -\Delta_x$ with domain the usual Sobolev space of order 2, $\mathcal{D}(H_0) = H^2(\mathbf{R}^d)$. If we assume $V(t,x)$ a real-valued function, such that $V \in L^1(\mathbf{R}; L^\infty(\mathbf{R}^d))$, then associated with (2) is a unitary propagator on $L^2(\mathbf{R}^d)$, denoted by $U(t,s)$, such that the solution to (2) is given by $v(t) = U(t,s)v_0$, see for example [8, 9] and references therein. More precisely, $v(t)$ solves the equation in the sense that v satisfies the integral equation

$$v(t) = U_0(t-s)v_0 - i\int_s^t U_0(t-\tau)V(\tau)v(\tau)\,d\tau, \tag{3}$$

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i.e. v is a *mild* solution to the Cauchy problem (2). The propagator satisfies $U(t,t) = 1$ and $U(t,s)U(s,r) = U(t,r)$ for all $t,s,r \in \mathbf{R}$. Furthermore, $(t,s) \mapsto U(t,s)$ is strongly continuous.

For this class of V one has a scattering theory associated with the solutions to the equations (1), (2). The wave operators are given by

$$W_{\pm}(s) = \operatorname*{s\text{-}lim}_{t\to\pm\infty} U(s,t)U_0(t-s). \tag{4}$$

The limits exist on all of $L^2(\mathbf{R}^d)$ and are unitary. The scattering operator is given by

$$S(s) = W_{+}(s)^{-1}W_{-}(s). \tag{5}$$

In the paper [3] we studied the scattering problem in the space-time framework, i.e. we considered the problem in the spaces $L^r(\mathbf{R}; L^q(\mathbf{R}^d))$ for a certain range of q, r. This approach was first used by Kato in [4] to study a class of nonlinear Schrödinger equations. In [3, Theorem 4.10] we obtained a representation formula for the scattering operator, using a purely time-dependent method of proof. In this paper we give a different derivation of this formula. We use the stationary scattering theory, in the formulation due to Kuroda [5, 6], combined with the stationary formulation of scattering theory for explicitly time-dependent potentials, in the form given by Howland [2]. For the case of potentials periodic in time the stationary scattering theory has been applied in [7] to derive a representation formula for the associated scattering matrix. The difference with the case considered here is that without a periodicity assumption there is no scattering matrix associated with the original problem, since $U_0(t)$ and $S(s)$ do not commute for any $t \neq 0$. There is a large literature on scattering theory for Schrödinger operators with time-dependent potentials. See for example [1] and references therein.

2 Preliminaries

We start by defining various spaces and operators needed to formulate the problem. We write $\mathcal{H} = L^2(\mathbf{R}^d)$ and introduce

$$\mathcal{K} = L^2(\mathbf{R}) \otimes \mathcal{H} \cong L^2(\mathbf{R}; \mathcal{H}) \cong L^2(\mathbf{R}^{d+1}). \tag{6}$$

We use the identifications of the three spaces without comment in the sequel. Let

$$\widetilde{K}_0 = -i\frac{d}{dx} \otimes I + I \otimes H_0 \tag{7}$$

with domain $H^1(\mathbf{R}) \otimes_{\text{alg}} H^2(\mathbf{R}^d)$. Then $\widetilde{K}_0$ is essentially self-adjoint on this domain. The closure is denoted by K_0.

We introduce the unitary operator

$$(\Upsilon f)(t,x) = (U_0(t)f(t,\cdot))(x) \tag{8}$$

on $\mathcal{K}$. Furthermore, we introduce the partial Fourier transform in the t-variable

$$(\Phi f)(\tau,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} f(t,x)\,dt. \tag{9}$$

These operators are combined to yield $F_0 = \Phi\Upsilon^*$. Viewing this operator as a map from $\mathcal{K}$ to $L^2(\mathbf{R}_\tau;\mathcal{H})$, we see that it defines a spectral representation for K_0. This means that we have

$$(F_0K_0f)(\tau) = \tau \cdot (F_0f)(\tau), \tag{10}$$

initially for $f \in H^1(\mathbf{R}) \otimes_{\text{alg}} H^2(\mathbf{R}^d)$. Note that this result implies $\mathcal{D}(K_0) = \{f \in \mathcal{K} \,|\, F_0f \in L^{2,1}(\mathbf{R};\mathcal{H})\}$. Here $L^{2,s}$ denotes the usual weighted space in the t-variable.

One further ingredient in Kuroda's formulation of stationary scattering theory is the trace operator. We define

$$(\gamma(\tau)f)(x) = (F_0f)(\tau,x). \tag{11}$$

This operator is well-defined on the space

$$\mathcal{K}^s = \Upsilon(L^{2,s}(\mathbf{R};\mathcal{H})). \tag{12}$$

for any $s > 1/2$. It maps $\mathcal{K}^s$ boundedly into $\mathcal{H}$. For $s < 0$ we let $\mathcal{K}^s = (\mathcal{K}^{-s})^*$ and use the natural duality induced by the scalar product on $\mathcal{K}$ to get a scale ($s > 0$) $\mathcal{K}^s \hookrightarrow \mathcal{K} \hookrightarrow \mathcal{K}^{-s}$. For the sake of consistency we write $\mathcal{K} = \mathcal{K}^0$.

The limiting absorption principle holds for K_0. Given the comments above, the proof is well-known, and is omitted.

Proposition 1. *Assume $s > 1/2$. Then the boundary values*

$$(K_0 - \tau \mp i0)^{-1} = \lim_{\varepsilon \downarrow 0}(K_0 - \tau \mp i\varepsilon)^{-1} \tag{13}$$

exist in operator norm on $\mathcal{B}(\mathcal{K}^s,\mathcal{K}^{-s})$. The boundary values are Hölder-continuous in τ.

Some of the results from [3] will be needed. We recall the necessary definitions, using the same notation for ease of reference. We introduce the spaces

$$L(B) = L^\infty(\mathbf{R};\mathcal{H}), \qquad L(B') = L^1(\mathbf{R};\mathcal{H}). \tag{14}$$

There is a natural duality between these two spaces, obtained from the inner product on $L^2(\mathbf{R};\mathcal{H})$.

We note the following result. The proof is a simple consequence of the definitions and is omitted.

Lemma 2. *Let $s > 1/2$. We then have the continuous embeddings $\mathcal{K}^s \hookrightarrow L(B')$ and $L(B) \hookrightarrow \mathcal{K}^{-s}$.*

For each $s \in \mathbf{R}$ define an operator $\Gamma_0(s) \in \mathcal{B}(\mathcal{H}, L(B))$ by

$$\Gamma_0(s)\varphi = U_0(t-s)\varphi. \tag{15}$$

The adjoint relative to the duality mentioned above is

$$\Gamma_0(s)^* f = \int_{-\infty}^{\infty} U_0(s-t)f(t)\,dt, \tag{16}$$

and $\Gamma_0(s)^* \in \mathcal{B}(L(B'),\mathcal{H})$.

We impose the following

Assumption 3. *Let $V \in L^1(\mathbf{R}; L^\infty(\mathbf{R}^d))$ be a real-valued function.*

Then, as mentioned in the introduction, we have a propagator $U(t,s)$ associated with (2). We define four operators $G^0_\pm, G_\pm \in \mathbf{B}(L(B'), L(B))$ by

$$(G^0_\pm f)(t) = \int_{\pm\infty}^{t} U_0(t-s)f(s)\,ds, \tag{17}$$

$$(G_\pm f)(t) = \int_{\pm\infty}^{t} U(t,s)f(s)\,ds. \tag{18}$$

Lemma 4. *Let V satisfy Assumption 3. Then $1 + iG^0_- V$ is invertible in $\mathcal{B}(L(B))$ with inverse given by $1 - G_- V$. Similarly, $1 + iG^0_+ V$ is invertible with inverse given by $1 - G_+ V$.*

Proof. First we note that Assumption 3 implies $V \in \mathcal{B}(L(B), L(B'))$. Then we use that the following identities hold in $\mathcal{B}(L(B'), L(B))$:

$$G^0_- - G_- = iG^0_- V G_- = iG_- V G^0_-, \tag{19}$$

$$G^0_+ - G_+ = iG^0_+ V G_+ = iG_+ V G^0_+, \tag{20}$$

see [3, Lemma 3.7], whose proof is valid also under Assumption 3. The remainder of the proof is now a straightforward computation. □

3 Stationary scattering theory

We now briefly outline the stationary scattering theory applied to our problem. We need the following lemma, cf. [2, equation (1.8)].

Lemma 5. *Let* $f \in \mathcal{K}^s$, $s > 1/2$, $\sigma \in \mathbf{R}$, *and* $\varepsilon > 0$. *Then we have*

$$\big((K_0 - \sigma - i\varepsilon)^{-1} f\big)(t) = i \int_{-\infty}^{t} e^{i\sigma(t-t')} e^{-\varepsilon(t-t')} U_0(t-t') f(t')\, dt'. \tag{21}$$

Proof. We note $(K_0 - \sigma - i\varepsilon)^{-1} f = \Upsilon\Phi^*(\tau - \sigma - i\varepsilon)^{-1}\Phi\Upsilon^* f$. Since $f \in L(B')$ by Lemma 2, the result then follows using well-known results on the Fourier transform and convolutions. □

We will introduce a slightly stronger assumption on V, in order to be able to apply the stationary scattering theory.

Assumption 6. *Let* $V(t,x)$ *be a real-valued function such that for some* $\beta > 1$ *we have* $(1+|t|)^\beta V(t,x) \in L^\infty(\mathbf{R}; L^\infty(\mathbf{R}^d))$.

Let V satisfy Assumption 6. We define $K = K_0 + V$ on $\mathcal{K}$, with domain $\mathcal{D}(K) = \mathcal{D}(K_0)$. Since V is a bounded self-adjoint operator on $\mathcal{K}$, K is self-adjoint on this domain. Briefly stated, Howland's method [2] consists in applying the scattering theory to the pair K_0, K.

To establish the connection with the results in [3], we need the following lemma.

Lemma 7. *Let* V *satisfy Assumption* 6. *Assume* $1/2 < s < \beta - 1/2$. *Then we have the following results.*
(i) *The operators* $1 + iVG^0_\pm$ *are invertible on* $\mathcal{K}^s$.
(ii) *Let* $\sigma \in \mathbf{R}$. *Then* $1 + V(K_0 - \sigma \mp i0)^{-1}$ *are invertible on* $\mathcal{K}^s$.

Proof. It suffices to consider one of the cases. Assumption 6 and the restriction $1/2 < s < \beta - 1/2$ imply that $1 + iVG^0_-$ is bounded on $\mathcal{K}^s$. Assume $f \in \mathcal{K}^s$ and $(1 + iVG^0_-)f = 0$. Then Lemmas 2 and 5 imply $f = 0$. Let $g \in \mathcal{K}^s$. By Lemma 5 there exists $f \in L(B')$ such that $(1 + iVG^0_-)f = g$. But then $f = -iVG^0_- f + g$ shows that $f \in \mathcal{K}^s$. To prove (ii) we start by taking limits in (21) to get

$$(K_0 - i0)^{-1} f = iG^0_- f \tag{22}$$

for $f \in \mathcal{K}^s$, $s > 1/2$. Let $\mathcal{M}_\sigma$ denote the unitary operator of multiplication by $e^{-it\sigma}$ on $\mathcal{K}^s$, $s \in \mathbf{R}$. We note that

$$K_0 - \sigma = \mathcal{M}_\sigma^* K_0 \mathcal{M}_\sigma. \tag{23}$$

Using this result, part (i), and a limiting argument, part (ii) follows. □

The modified trace operators are defined by

$$\gamma_{\pm}(\tau)f = \gamma(\tau)\left(1 + V(K_0 - \tau \mp i0)^{-1}\right)^{-1} f \tag{24}$$

on $\mathcal{K}^s$, $1/2 < s < \beta - 1/2$, and the modified spectral representations by

$$(F_{\pm}f)(\tau) = \gamma_{\pm}(\tau)f, \tag{25}$$

initially on the same space. A standard argument then shows that $F_{\pm}$ extend to unitary operators on $\mathcal{K}$.

In Howland's theory the connection between the wave operators defined in (4) and the wave operators $\mathsf{W}_{\pm} = s\text{-}\lim_{\sigma \to \pm\infty} e^{i\sigma K} e^{-i\sigma K_0}$ is given by $(\mathsf{W}_{\pm}f)(t) = W_{\pm}(t)f(t)$. The connection with the stationary theory presented here is summarized in the relation $\mathsf{W}_{\pm} = F_{\pm}^* F_0$. A careful examination of the proof in [6] shows that it applies to the present case. We will omit the details.

The connection with the scattering operator defined in (5) is then given by $(\mathsf{S}f)(t) = (\mathsf{W}_{+}^{-1}\mathsf{W}_{-}f)(t) = S(t)f(t)$. On the other hand, the stationary scattering theory yields a representation for the decomposition of S in the spectral representation for K_0 given by F_0. Using this connection we get the following result, which is the main result connecting Howland's theory with the space-time scattering theory from [3]. We have retained the formulation given in that paper.

Theorem 8. *Let V satisfy Assumption* 6. *Then the scattering operator $S(s)$ from* (5) *has a representation*

$$S(s) = 1 - i\Gamma_0(s)^* V(1 + iG_-^0 V)^{-1}\Gamma_0(s). \tag{26}$$

Proof. We have from the stationary scattering theory (see [5, 6]) that the scattering matrix given by $(F_0\mathsf{S}f)(\tau) = \mathsf{S}(\tau)(F_0 f)(\tau)$ is represented as

$$\mathsf{S}(\tau) = 1 - 2\pi i\gamma(\tau)\left(1 + V(K_0 - \tau - i0)^{-1}\right)^{-1} V\gamma(\tau)^*. \tag{27}$$

We now translate this representation into the terms used in [3]. Recalling the definitions of F_0 and $\gamma(\tau)$, we find that $\gamma(\tau) = \gamma(0)\mathcal{M}_\tau$. Combining this relation with (23) we find $\mathsf{S}(\tau) = \mathsf{S}(0)$, such that the scattering matrix is independent of the spectral parameter τ. As already observed by Howland [2, Remark (3), p. 325], the scattering matrix in our spectral representation is also given by multiplication by the constant operator $S(0)$. Thus we have the relation $\mathsf{S}(0) = S(0)$. Now for any $f \in \mathcal{K}^s$, $s > 1/2$,

$$\gamma(0)f = (F_0 f)(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U_0(-t)f(t)\,dt = \frac{1}{\sqrt{2\pi}}\Gamma_0(0)^* f. \tag{28}$$

Using this relation together with (22), we find

$$S(0) = 1 - i\Gamma_0(0)^*(1 + iVG_-^0)^{-1}V\Gamma_0(0). \tag{29}$$

From (15) follows $\Gamma_0(s) = \Gamma_0(0)U_0(-s)$. Furthermore, $S(s) = U_0(s)S(0)U_0(-s)$. Finally, we have $V(1+iG_-^0V)^{-1} = (1+iVG_-^0)^{-1}V$. Combining these results equation (26) follows. □

Some applications of the formula (26) are given in [3]. We need Assumption 6 to use the stationary scattering theory in our proof. The results in [3] show that the formula is valid also under Assumption 3.

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