On Weighted $L^2(\Omega)$-Spaces, their Duals and Itô Integration

Fred Espen Benth*

Department of Mathematical Sciences
University of Aarhus, DK-8000 Aarhus C, Denmark

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Abstract

We construct spaces of smooth and generalized random variables which can be considered as weighted $L^2(\Omega)$-spaces. Itô integration for generalized stochastic processes is defined. The construction follows closely the standard $L^2(\Omega)$-case, except for the norms which are weighted. As an application of our results, we derive a Clark-Ocone formula.

1 Introduction

In stochastic analysis it is well-known that random variables with finite variance (i.e. which belongs to $L^2(\Omega)$) admit a chaos expansion in terms of iterated Wiener integrals. The integrands will be symmetric square integrable real-valued functions which are uniquely defined by the random variable. The variance of these variables can be represented as an infinite series of $L^2(\mathbb{R}^n)$-norms. In this paper we weight these norms with appropriate functions in order to construct new Hilbert spaces of smooth random variables. The weight functions will be a composition of a real valued (with values greater than one) function with the Number Operator. By introducing a family of weight functions, different countably Hilbert spaces are defined. The duals

*MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from the Danish National Research Foundation. E-mail: fredb@mi.aau.dk

of these spaces will consist of generalized random variables which only admit a formal chaos expansion. Typically, the Donsker δ-function will be an element of such spaces. In the literature there exists several examples of such constructions (see e.g. [PT]), however, a general treatment of weighted $L^2(\Omega)$-spaces seems to be lacking.

In [BP] a natural generalization of the Itô integral to a space of generalized stochastic processes was introduced. The definition of the integral followed the lines of construction in $L^2(\Omega)$. However, different norms where used since the processes belonged to the inductive limit of a sequence of "weighted" Hilbert spaces (or rather their duals). This integral was shown to coincide with the Hitsuda-Skorohod integral (see e.g. [HKPS] for a treatment of the Hitsuda-Skorohod integral for Hida distributions). We will follow the ideas in [BP] to construct an Itô integral for the spaces of generalized stochastic processes we introduce in this paper.

As an application of our results we derive the Clark-Ocone representation formula. For a class of spaces of generalized random variables this is shown to hold. Other works have been done to extend the Clark-Ocone representation, see e.g. [U], [AaØU] and [V]. Our representation is a true generalization of the results in these papers.

The article is organized as follows: In the next section we introduce some necessary notation and mathematical theory. In section 3 families of weighted $L^2(\Omega)$-spaces and their duals are constructed. For stochastic processes living in these spaces of generalized random variables we construct the Itô integral. This is done in section 4. Finally, in section 5, we derive the Clark-Ocone formula. For this purpose we need to extend the Malliavin derivative and conditional expectation.

2 Mathematical preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let T denote the interval $[0, t_0]$ or $\mathbb{R}_+$. Introduce the Hilbert space $H = L^2(T, \mathcal{B}, \mu)$ where T denotes the interval $[0, t_0]$ or $\mathbb{R}_+$ and $\mathcal{B}$ is the Borel σ-algebra. We have used the notation μ for the Lebesgue measure on $(T, \mathcal{B})$. Following Nualart, [N], $\{W(h), h \in H\}$ is a centered Gaussian family of random variables with variance $|h|^2_{L^2(T)}$. This family is characterized by the random variables $W(A) = W(\mathbf{1}_A)$ which takes independent values on disjoint subsets of T. Note that $W(h) = \int_T h dW$ is the Wiener integral. In the sequel we shall use the notation $L^2(\Omega)$ for the space $L^2(\Omega, \mathcal{G}, P)$ where $\mathcal{G}$ is the σ-algebra generated by $\{W(A), A \in \mathcal{B}\}$. Elements of $L^2(\Omega)$ can be expanded into a series of multiple Wiener integrals (the so-called chaos expansion of the random variable);

Theorem 2.1. *Let $f \in L^2(\Omega)$. Then*

$$f = \sum_{n=0}^{\infty} I_n(f_n) \tag{1}$$

where I_n is the n-fold Wiener integral and $f_n \in L^2(T^n)$ is symmetric. The functions f_n are uniquely defined by f. Moreover,

$$\|f\|^2 = \sum_{n=0}^{\infty} n! |f_n|^2_{L^2(T^n)} < \infty \tag{2}$$

where $\| \cdot \|$ denotes the usual norm in $L^2(\Omega)$.

For a proof of the chaos expansion, see e.g. [N].

In our study of weighted $L^2(\Omega)$-spaces, we shall use the Number operator N: If $f \in L^2(\Omega)$ has the chaos expansion $f = \sum_{n=0}^{\infty} I_n(f_n)$, the application of N on f is defined as

$$Nf = \sum_{n=0}^{\infty} n I_n(f_n) \tag{3}$$

The domain of this operator, denoted $Dom(N)$, is easily seen to be the subspace of $L^2(\Omega)$ for which $\sum_{n=0}^{\infty} n! n |f_n|^2 < \infty$.

3 Weighted $L^2(\Omega)$-spaces

Define the following weighted $L^2(\Omega)$-space:

Definition 3.1. *Let $\gamma : \mathbb{N}_0 \to [1, \infty)$ and define the space $(L^2)_\gamma$ to be the domain of the operator $\gamma(N)$ in $L^2(\Omega)$. Equip this space with the scalar product $(\cdot, \cdot)_\gamma$ given by*

$$(f, g)_\gamma := \sum_{n=0}^{\infty} n! \gamma^2(n) (f_n, g_n)_{L^2(T^n)} \tag{4}$$

Denote the norm induced by $(\cdot, \cdot)_\gamma$ for $\| \cdot \|_\gamma$.

Since $\gamma \geq 1$ we have $\| \cdot \| \leq \| \cdot \|_\gamma$. Thus $(L^2)_\gamma$ is a continuously embedded subspace of $L^2(\Omega)$. The following result is straightforward:

Corollary 3.2. *The space* $(L^2)_\gamma$ *equipped with the norm* $\|\cdot\|_\gamma$ *is a Hilbert space.*

We have the following characterization of $(L^2)_\gamma$:

Lemma 3.3. *Let* $f \in L^2(\Omega)$. *Then* $f \in (L^2)_\gamma$ *if and only if* $\gamma(N)f \in L^2(\Omega)$.

Proof. Straightforward calculation gives:

$$\|\gamma(N)f\|^2 = \sum_{n=0}^{\infty} n!\gamma^2(n)|f_n|^2 = \|f\|_\gamma^2$$

□

The topological dual of $(L^2)_\gamma$ is denoted $(L^2)_\gamma^*$. We have the following result:

Proposition 3.4. *The space* $(L^2)_\gamma^*$ *can be identified with the Hilbert space* $(L^2)_{\gamma^{-1}}$ *of formal sums* $F = \sum_{n=0}^{\infty} I_n(F_n)$ *with the inner product norm*

$$\|F\|_{\gamma^{-1}}^2 := \sum_{n=0}^{\infty} n!\gamma^{-2}(n)|F_n|_{L^2(T^n)}^2 < \infty \tag{5}$$

Proof. Suppose $F \in (L^2)_\gamma^*$. We first show that F has a formal chaos representation: Define a linear functional F_n on the symmetric $L^2(T^n)$ by

$$\langle F_n, f_n\rangle := \langle\langle F, \frac{1}{n!} I_n(f_n)\rangle\rangle$$

F_n is a symmetric continuous linear functional on $L^2(T^n)$, and hence itself an element of $L^2(T^n)$ (since the dual of $L^2(T^n)$ can be identified by itself). We can write the formal representation $F = \sum_{n=0}^{\infty} I_n(F_n)$. Moreover,

$$\begin{aligned} |F_n|_{L^2(T^n)} &= \sup_{|f_n|_{L^2(T^n)}=1} |(F_n, f_n)| \\ &\leq \sup_{|f_n|_{L^2(T^n)}=1} |\langle\langle F, \frac{1}{n!} I_n(f_n)\rangle\rangle| \\ &\leq \frac{1}{n!}\|F\|_* \end{aligned}$$

where we have denoted the operator norm in $(L^2)_\gamma^*$, $\|\cdot\|_*$. We show that $F \in (L^2)_{\gamma^{-1}}$: Define $f = \sum_{n=0}^{\infty} I_n(\gamma^{-2}(n)F_n)$. Since $\gamma(n) \geq 1$,

$$\begin{aligned} \|f\|_\gamma &= \sum_{n=0}^{\infty} n!\gamma^{-2}(n)|F_n|_{L^2(T^n)}^2 \\ &\leq \|F\|_*^2 \sum_{n=0}^{\infty} \frac{1}{n!}\gamma^{-2}(n) \\ &\leq K\|F\|_*^2 \end{aligned}$$

Thus $f \in (L^2)_\gamma$. Let $\tilde{f} := f/\|f\|_\gamma$. Then

$$\begin{aligned}\|F\|_* &\geq \frac{1}{\|f\|_\gamma}\left|\sum_{n=0}^{\infty} n!\gamma^{-2}(n)|F_n|^2_{L^2(T^n)}\right| \\ &= \left(\sum_{n=0}^{\infty} n!\gamma^{-2}(n)|F_n|^2\right)^{1/2} \\ &= \|F\|_{\gamma^{-1}}\end{aligned}$$

Hence, $(L^2)^*_\gamma$ is a subspace of $(L^2)_{\gamma^{-1}}$.

Let $F \in (L^2)_{\gamma^{-1}}$. Then we can define the application of F on $f \in (L^2)_\gamma$ by

$$\langle\langle F, f\rangle\rangle := \sum_{n=0}^{\infty} n!(F_n, f_n)_{L^2(T^n)}$$

F will obviously define a linear operator on $(L^2)_\gamma$. Moreover, application of the Cauchy-Schwarz inequality twice implies that

$$\begin{aligned}|\langle\langle F, f\rangle\rangle| &\leq \sum_{n=0}^{\infty} n!|(F_n, f_n)_{L^2(T^n)}| \\ &\leq \left(\sum_{n=0}^{\infty} n!\gamma^2(n)|f_n|^2_{L^2(T^n)}\right)^{1/2}\left(\sum_{n=0}^{\infty} n!\gamma^{-2}(n)|F_n|^2_{L^2(T^n)}\right)^{1/2} \\ &= \|f\|_\gamma \cdot \|F\|_{\gamma^{-1}}\end{aligned}$$

This shows that F is a continuous linear operator, and hence in $(L^2)^*_\gamma$. Furthermore, $\|F\|_* \leq \|f\|_{\gamma^{-1}}$. Hence, $\|\cdot\|_* = \|\cdot\|_{\gamma^{-1}}$, which completes the proof of the proposition. □

From now on we will refer to $(L^2)_{\gamma^{-1}}$ as the dual of $(L^2)_\gamma$.

Example: The Donsker δ function. In this example we will give sufficient conditions on γ to ensure that $\delta_0(B_t) \in (L^2)_{\gamma^{-1}}$. It is well-known (see e.g. [PT]) that $\delta_0(B_t)$ has a chaos expansion

$$\delta_0(B_t) = \frac{1}{\sqrt{4\pi t}}\sum_{n=0}^{\infty} I_{2n}\left(\frac{(-1)^n}{(2t)^n n!}\mathbf{1}^{\otimes 2n}_{[0,t)}\right) \tag{6}$$

Assume $\lim_{n\to\infty} \gamma(n)/\gamma(n+1) < 1$. Then it is easily seen by the ratio test that

$$\|\delta_0(B_t)\|^2_{\gamma^{-1}} = \frac{1}{2\pi t}\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2\gamma^2(2n)} < \infty$$

and thus $\delta_0(B_t) \in (L^2)_{\gamma^{-1}}$.

Example: Domain of N. With $\gamma(n) = \sqrt{n}$ for $n \geq 1$ and zero otherwise, we see that $(L^2)_\gamma$ coincides with the domain of N. Hence, $Dom(N)$ can be realized as a weighted $L^2(\Omega)$-space.

We continue the section with introducing countably Hilbert space structures for different choices of families of weight functions γ: For every $q \in \mathbb{N}_0$, suppose we have a mapping $\gamma_q : \mathbb{N}_0 \to [1, \infty)$. Furthermore, for every $n \in \mathbb{N}_0$ assume

$$\gamma_q(n) \leq \gamma_{q+1}(n) \tag{7}$$

For notational simplicity we henceforth denote $(L^2)_{\gamma_q}$ by $(L^2)_q$. Its topological dual is denoted $(L^2)_{-q}$. The corresponding norms will be written $\|\cdot\|_q$ and $\|\cdot\|_{-q}$, respectivly.

Lemma 3.5. *Let $q \leq p$, then the space $(L^2)_p$ is continuously embedded into $(L^2)_q$.*

Proof. Since γ_q is increasing in q we have $\|\cdot\|_q \leq \|\cdot\|_p$. □

Define the space

$$(L^2)_\infty := \bigcap_{q \in \mathbb{N}_0} (L^2)_q \tag{8}$$

equipped with the projective limit topology. Since $\gamma_q \geq 1$, $(L^2)_\infty \subset L^2(\Omega)$. $(L^2)_\infty$ becomes a countably Hilbert space in the sense of [GV], where its dual is represented as

$$(L^2)_{-\infty} = \bigcup_{q \in \mathbb{N}_0} (L^2)_{-q} \tag{9}$$

with the inductive limit topology. $(L^2)_\infty$ and $(L^2)_{-\infty}$ will be spaces of smooth and generalized random variables with respect to a given family of weight functions γ_q. It will always be clear from the context which family of γ_q's we use. Note that Brownian motion B_t is a smooth random variable since it has only first chaos, and thus will have finite γ_q-norm for any q.

Example: Let $\gamma_q(n) = \exp(qn)$. Then we obtain the spaces $(\mathcal{G})$ and $(\mathcal{G})^*$ studied in [PT].

4 The generalized Itô integral

We extend the Itô integral to $(L^2)_{-\infty}$. Our approach is analogous to the classical construction in $L^2(\Omega)$, but now with different norms. We note that the extended Itô integral will coincide with the Hitsuda-Skorokhod integral.

Fix a family γ_q which satisfies the assumption (7) above. In addition we suppose that γ_q is increasing in n, i.e.

$$\gamma_q(n) \leq \gamma_q(n+1) \tag{10}$$

Let $\tau \leq t_0$. A process $F_t \in (L^2)_{-q}$ for $t \in [0,\tau]$ will be called a *generalized simple function* if

$$F_t = \sum_{i=0}^{M} F^{(i)} \cdot \mathbf{1}_{[t_i,t_{i+1})}(t) \tag{11}$$

where $\{t_i\}_i$ is a partition of the interval $[0,\tau]$ and $F^{(i)} \in (L^2)_{-q}$. In addition we assume F_t is adapted, i.e. $\operatorname{supp}\{F_n(t,\cdot)\} \subset [0,t]^n$, where F_n is the n'th chaos. Note that q is assumed to be independent of t. Introduce the Itô integral for generalized simple functions as

$$\int_0^\tau F_t\, dB_t = \sum_{i=0}^{M} F^{(i)} \cdot \left(B_{t_{i+1}} - B_{t_i}\right) \tag{12}$$

The integral satisfies an isometry property:

Proposition 4.1. *Let $F_t \in (L^2)_{-q}$ be a simple function. Then*

$$\|\int_0^\tau F_t\, dB_t\|_{-q}^2 = \int_0^\tau \|\delta_q(N) F_t\|_{-q}^2\, dt \tag{13}$$

where $\delta_q(n) := \gamma_q(n)/\gamma_q(n+1)$.

Proof. Since F_t is adapted, we have by strong independence (see e.g. [BP])

$$F^{(i)} \cdot \left(B_{t_{i+1}} - B_{t_i}\right) = \sum_{n=0}^{\infty} I_{n+1}\left(F_n^{(i)} \widehat{\otimes} \mathbf{1}_{[t_i,t_{i+1})}\right)$$

Therefore,

$$\begin{aligned}\|\int_0^\tau F_t\, dB_t\|_{-q}^2 &= \|\gamma_q(N)^{-1} \int_0^\tau F_t\, dB_t\|^2 \\ &= \|\sum_{i=0}^{M} \gamma_q(N)^{-1}\left(F^{(i)} \cdot (B_{t_{i+1}} - B_{t_i})\right)\|^2\end{aligned}$$

But,

$$\begin{aligned}\gamma_q(N)^{-1}\left(F^{(i)}\cdot\left(B_{t_{i+1}}-B_{t_i}\right)\right) \\ &= \sum_{n=0}^{\infty}\gamma_q(n+1)^{-1}I_{n+1}\left(F_n^{(i)}\widehat{\otimes}\mathbf{1}_{[t_i,t_{i+1})}\right) \\ &= \left(\delta_q(N)\gamma_q(N)^{-1}F^{(i)}\right)\cdot\left(B_{t_{i+1}}-B_{t_i}\right)\end{aligned}$$

In the last equality we have again used strong independence. Thus

$$\begin{aligned}\|\int_0^{\tau} F_t\,dB_t\|_{-q}^2 &= \sum_{i,j=0}^{M}\mathrm{E}\left[\delta_q(N)\gamma_q(N)^{-1}F^{(i)}\cdot\delta_q(N)\gamma_q(N)^{-1}F^{(j)}\cdot\right. \\ &\quad \left.(B_{t_{i+1}}-B_{t_i})\cdot(B_{t_{j+1}}-B_{t_j})\right] \\ &= \sum_{i=0}^{M}\mathrm{E}\left[\left(\delta_q(N)\gamma_q(N)^{-1}F^{(i)}\right)^2\right]\triangle t_i \\ &= \int_0^{\tau}\|\delta_q(N)F_t\|_{-q}^2\,dt\end{aligned}$$

The isometry is thus proved. □

Let F_t be an adapted process in $(L^2)_{-q}$ such that

$$\int_0^T \|F_t\|_{-q}^2\,dt < \infty \tag{14}$$

Then $\gamma_q(N)^{-1}F_t$ is Itô integrable in the (L^2)-sense. Hence there exists a sequence of simple stochastic processes $\{\phi_t^k\}_k$ in $L^2(\Omega)$ such that

$$\int_0^{\tau}\|\phi_t^k-\gamma_q(N)^{-1}F_t\|^2\,dt\to 0,\ \ k\to\infty$$

i.e.,

$$\int_0^{\tau}\|\gamma_q(N)\phi_t^k-F_t\|_{-q}^2\,dt\to 0,\ \ k\to\infty$$

It is easy to see that $\gamma_q(N)\phi_t^k$ defines a sequence of simple processes in $(L^2)_{-q}$. Since $\delta_q(n)\le 1$ by assumption (10), $\|\delta_q(N)\cdot\|_{-q}\le\|\cdot\|_{-q}$. Hence, by the Itô isometry the sequence

$$\left\{\int_0^{\tau}\gamma_q(N)\phi_t^k\,dB_t\right\}_k$$

is Cauchy in $(L^2)_{-q}$. By completeness of this space we can define the Itô integral of F_t as

$$\int_0^\tau F_t\, dB_t := \lim_{k\to\infty} \int_0^\tau \gamma_q(N)\phi_t^k\, dB_t \tag{15}$$

(limit in $(L^2)_{-q}$). Note that the Itô isometry implies that the definition of the integral is independent of the choice of simple processes.

We prove a connection between the Itô integral in the $L^2(\Omega)$-sense and our generalized Itô integral:

Proposition 4.2. *Let $F_t \in (L^2)_{-q}$ be adapted and $\int_0^\tau \|\delta_q(N)F_t\|_{-q}^2\, dt < \infty$. Then*

$$\int_0^\tau F_t\, dB_t = \gamma_q(N-1)\int_0^\tau \gamma_q(N)^{-1} F_t\, dB_t \tag{16}$$

where the Itô integral on the r.h.s. is understood in the (L^2)-sense.

Remark: The proposition tells us that we can integrate F_t with respect to Brownian motion by first pulling F_t back to $L^2(\Omega)$ with the Number Operator and integrating in the usual Itô sense. The resulting integral is then pushed back to $(L^2)_{-q}$.

Proof. First note that by definition

$$\| \int_0^\tau \phi_t^k\, dB_t - \int_0^\tau \gamma_q(N)^{-1} F_t\, dB_t \|^2 \to 0$$

Thus

$$\|\gamma_q(N)\int_0^\tau \phi_t^k\, dB_t - \gamma_q(N)\int_0^\tau \gamma_q(N)^{-1} F_t\, dB_t\|_{-q}^2 \to 0$$

But

$$\begin{aligned}
\gamma_q(N)\int_0^\tau \phi_t^k\, dB_t &= \sum_{n=0}^{\infty}\sum_{i=1}^{M_k} \gamma_q(n+1) I_{n+1}\left(f_{n,i}^k \widehat{\otimes} \mathbf{1}_{[t_i,t_{i+1})}\right)\\
&= \sum_{n=0}^{\infty}\sum_{i=1}^{M_k} \delta_q(n)^{-1} I_{n+1}\left(\gamma_q(n) f_{n,i}^k \widehat{\otimes} \mathbf{1}_{[t_i,t_{i+1})}\right)\\
&= \delta_q(N-1)^{-1}\left(\sum_{n=0}^{\infty}\sum_{i=1}^{M_k} I_n(\gamma_q(n) f_{n,i}^k)\cdot (B_{t_{i+1}} - B_{t_i})\right)\\
&= \delta_q(N-1)^{-1}\int_0^\tau \gamma_q(N)\phi_t^k\, dB_t
\end{aligned}$$

This implies that

$$\delta_q(N-1)^{-1}\int_0^\tau \gamma_q(N)\phi_t^k\,dB_t \to \gamma_q(N)\int_0^\tau \gamma_q(N)^{-1}F_t\,dB_t$$

in $(L^2)_{-q}$ when $k\to\infty$. But from the definition we have

$$\int_0^\tau \gamma_q(N)\phi_t^k\,dB_t \to \int_0^\tau F_t\,dB_t$$

Hence, the proposition follows. □

Example: Consider $\gamma_q(n)=\exp(qn)$. Then $\delta_q(n)=\exp(-q)$ and thus the Itô isometry has the particularly simple form

$$\|\int_0^\tau F_t\,dB_t\|_{-q}^2 = e^{-2q}\int_0^\tau \|F_t\|_{-q}^2\,dt$$

The Itô integral for this special choice of γ_q was treated in [BP]. Consider $\gamma_q(n)=\sqrt{n!}\exp(qn)$. We obtain a version of the Kondratiev test and generalized functionals with chaos kernels in $L^2(\mathbb{R}^n)$ (recently, [K] has developed a White Noise theory for these spaces. See also [KLS]). It is easily seen that $\delta_q(n)=e^{-q}(n+1)^{-1/2}$. The Itô isometry becomes

$$\|\int_0^\tau F_t\,dB_t\|_{-q}^2 = e^{-2q}\int_0^\tau \|(N+1)^{-1/2}F_t\|_{-q}^2\,dt$$

Corollary 4.3. *The extended Itô integral coincides with the Hitsuda-Skorokhod integral, i.e. if $F_t\in(L^2)_{-q}$ is adapted and $\int_0^\tau \|F_t\|_{-q}^2\,dt<\infty$ then*

$$\int_0^\tau F_t\,dB_t = \sum_{n=0}^\infty \frac{1}{n+1} I_{n+1}\left(\mathbf{1}_{[0,\tau)}^{\otimes n+1}\hat{F}_n\right) \tag{17}$$

where $F_t=\sum_{n=0}^\infty I_n(F_n(t,\cdot))$ and $\hat{F}_n$ is the symmetrization.

Proof. Using proposition 4.2 and the definition of the Hitsuda-Skorokhod integral (see e.g. [HKPS]), we get:

$$\begin{aligned}\int_0^\tau \gamma_q^{-1}(N)F_t\,dB_t &= \sum_{n=0}^\infty \frac{1}{n+1} I_n\left(\gamma_q^{-1}(n)\mathbf{1}_{[0,\tau)}^{\otimes n+1}\hat{F}_n\right)\\ &= \gamma_q^{-1}(N-1)\sum_{n=0}^\infty \frac{1}{n+1} I_n\left(\mathbf{1}_{[0,\tau)}^{\otimes n+1}\hat{F}_n\right)\end{aligned}$$

□

5 The Clark-Ocone representation formula

In this section we will prove a Clark-Ocone representation formula for generalized random variables. To obatian this representation, we need to extend the definitions of the Malliavin derivative and conditional expectation. Both definitions will be direct extensions from the $L^2(\Omega)$-case. Itô integration of generalized adapted stochastic processes has been defined in the previous section.

We fix a family of γ_q which is increasing in both q and n, i.e. which satisfies conditions (7) and (10). Consider the following extension of the Malliavin derivative to $(L^2)_{-\infty}$:

Definition 5.1. *Let $F \in (L^2)_{-\infty}$ with chaos expansion $F = \sum_{n=0}^{\infty} I_n(F_n)$. If for each $t \in T$ there exists a $p \in \mathbb{N}_0$ such that*

$$\sum_{n=1}^{\infty} n! n \gamma_p^{-2}(n-1) |F_n|^2_{L^2(T^n)} < \infty$$

we define the Malliavin derivative of F at $t \in T$ to be

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(F_n(t, \cdot)) \tag{18}$$

We observe that this definition coincides with the Malliavin derivative in (the domain of definition in) $L^2(\Omega)$. See e.g. [N] for more information of the Malliavin derivative in $L^2(\Omega)$. We shall be particularly interested in the case where the whole of $(L^2)_{-\infty}$ is differentiable in the sense of Malliavin ("uniformly" in t). A sufficient condition for this to hold is the following: For each $q \in \mathbb{N}_0$ there exists a $p \geq q$ such that

$$\frac{\gamma_p(n-1)}{\gamma_q(n)} \geq \sqrt{n} \tag{19}$$

If the family γ_q is chosen such that (19) holds, we have $D_t F \in (L^2)_{-p}$ for almost every $t \in T$: (Note that this p will be independent of t). In fact if

$F \in (L^2)_{-q}$,

$$\int_T \|D_t F\|_p^2 \, dt = \int_T \sum_{n=1}^{\infty} (n-1)! n^2 \gamma_p^{-2}(n-1) |F_n(t,\cdot)|^2_{L^2(T^{n-1})} \, dt$$
$$= \sum_{n=1}^{\infty} n! n \gamma_p^{-2}(n-1) |F_n|^2_{L^2(T^n)}$$
$$\leq \sum_{n=1}^{\infty} n! \gamma_q^{-2}(n) |F_n|^2_{L^2(T^n)}$$
$$\leq \|F\|^2_{-q} < \infty$$

Observe also that since $\gamma_q(n) \geq 1$, (19) implies

$$\gamma_p(n) \geq \sqrt{n+1}\gamma_q(n+1) \geq \sqrt{n+1} \geq \sqrt{n}$$

for a $p \geq 0$. Since γ_p is increasing in p we can conclude that there exists a q_0 such that $\gamma_p(n) \geq \sqrt{n}$ for all $p \geq q_0$. Therefore $(L^2)_p \subset Dom(N)$ for all $p \geq q_0$, and hence $(L^2)_\infty$ is a subspace of the domain of definition of the classical Malliavin derivative (this domain is frequently denoted by $\mathbf{D}_{1,2}$).

Example: Consider $\gamma_q(n) = \exp(qn)$ or $\gamma_q(n) = \sqrt{n!}\exp(qn)$. Then (19) is satisfied in both cases.

Let $\mathcal{G}_t$ denote the σ-algebra generated by Brownian motion $\{B_s; 0 \leq s \leq t\}$. Conditional expectation with respect to $\mathcal{G}_t$ is defined in the following manner:

Definition 5.2. *Let $F \in (L^2)_{-\infty}$ with chaos expansion $F = \sum_{n=0}^{\infty} I_n(F_n)$. Then*

$$E[F|\mathcal{G}_t] = \sum_{n=0}^{\infty} I_n(F_n \cdot \mathbf{1}^{\otimes n}_{[0,t)}) \tag{20}$$

where $t \in T$.

This defintion is a straightforward generalization of the result in $L^2(\Omega)$. (See e.g. [H]).

Theorem 5.3. *(The Clark-Ocone representation formula). Suppose (19) holds and assume $F \in (L^2)_{-\infty}$. Then*

$$F = E[F] + \int_T E[D_t F|\mathcal{G}_t] \, dB_t \tag{21}$$

Proof. All the objects involved in the representation are well-defined. The proof is simply a calculation with chaos using the theory developed above. □

The Clark-Ocone formula has been extended to generalized random variables by [U], [AaØU] and [V]. [U] considered the class of Meyer-Watanabe distributions which is contained in the distribution space $(\mathcal{G})^*$ discussed in [PT]. Both [AaØU] and [V] treat generalized random variables from the spaces obtained with $\gamma_q(n) = \sqrt{n!}\exp(qn)$. Our general Clark-Ocone formula includes all these cases.

Example: Consider again the Donsker δ-function with chaos expansion as in (6). The kernel in the Clark-Ocone formula is (for $s < t_0$):

$$X_s := \mathrm{E}[D_s\delta_0(B_{t_0})|\mathcal{G}_s] = \frac{1}{\sqrt{4\pi t_0}}\sum_{n=1}^{\infty}(2n)I_{2n-1}\left(\frac{(-1)^n}{(2t_0)^n n!}\mathbf{1}_{[0,s)}^{\otimes 2n-1}\right)$$

Hence, the representation says

$$\delta_0(B_{t_0}) = \frac{1}{\sqrt{4\pi t_0}} + \int_0^{t_0} X_s\,dB_s \tag{22}$$

In [AaØU] the Donsker δ-function is considered in connection with pricing and hedging of digital options in mathematical finance.

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