Stationary and Selfsimilar Processes Driven by Lévy Processes

Ole E. Barndorff-Nielsen and Victor Pérez-Abreu

MaPhySto[1] CIMAT

Abstract

Using bivariate Lévy processes, stationary and selfsimilar processes, with prescribed one-dimensional marginal laws of type G, are constructed. In the case of square integrability the arbitrary spectral distribution of the stationary process can be chosen so that the corresponding selfsimilar process has second order stationary increments. The spectral distribution in question, which yields fractional Brownian motion when the driving Lévy process is the bivariate Brownian motion, is shown to possess a density, and an explicit expression for the density is derived.

1. Introduction

Laws of approximate or exact selfsimilarity, and more generally laws of scaling, are attracting interest in many applied fields, most recently in finance, see Guillaume, Dacorogna, Davé, Müller, Olsen and Pictet (1997). In finance, turbulence and other fields it is of importance to develop models that in addition to such laws have certain other key features. As a particular case one may ask whether, given a certain family of probability distributions, there exists a selfsimilar or approximately selfsimilar process whose marginal laws belong to that family. Apart from the field of stable processes, this is a largely unexplored area; in particular, there is no simple characterisation of the possible families of one-dimensional marginal laws of strictly selfsimilar processes with stationary increments (the latter property is, at least in some approximate form, essential for most applications).

To exemplify by a concrete question: Given $H \in (0,1)$ ($H \neq \frac{1}{2}$), does there exist a H-selfsimilar process with stationary increments whose one-dimensional

[1]MaPhySto - Centre for Mathematical Physics and Stochastics, funded by a grant from The Danish National Research Foundation

marginals distributions are normal inverse Gaussian (for a discussion of the normal inverse Gaussian distributions and their role in finance and turbulence see Barndorff-Nielsen (1998)). We have not been able to decide this question. However, as a special case of the results presented in the following we show that there does exist a H-selfsimilar process with second order stationary increments having the desired type of marginal laws.

Section 2 summarizes a variety of known results needed for the sequel. These results concern: (i) independently scattered measures of Lévy type (ii) type G random variables, i.e. random variables of the form $\sigma\varepsilon$ where σ^2 is infinitely divisible and ε is standard (multivariate) normal (iii) weakly stationary processes. Stationary processes driven by Lévy processes are discussed in Section 3. By a result due to Lamperti (1962) the class of strictly stationary processes is in one-to-one correspondence with the class of H-selfsimilar processes via a simple transformation of time and scale. This allows us, in Section 4, to construct H-selfsimilar processes with second order stationary increments, driven by Lévy processes and having marginal distributions of type G. In case the processes, to which the Lamperti transformation applies, are square integrable a necessary condition for stationary increments of the selfsimilar versions is that their covariance functions are the same as for the fractional Brownian motion. We translate this condition to the corresponding one on the correlation function r of the associated stationary processes and determine explicitly the probability distribution F of which r is the Fourier transform. The non-L^2 case is also considered in Section 4.

As a standard notation we shall write $\mathrm{C}\{\zeta \ddagger y\}$ for the cumulant (generating) function of a random vector y, i.e.

$$\mathrm{C}\{\zeta \ddagger y\} = \log \mathrm{E}\{e^{i\langle \zeta, y\rangle}\}.$$

Similarly,

$$\mathrm{L}\{\theta \ddagger y\} = \mathrm{E}\{e^{\langle \theta, y\rangle}\}$$

will denote the Laplace transform of y, and we let

$$\bar{\mathrm{L}}\{\theta \ddagger y\} = \mathrm{L}\{-\theta \ddagger y\} = \mathrm{E}\{e^{-\langle \theta, y\rangle}\}$$

$$\mathrm{K}\{\theta \ddagger y\} = \log \mathrm{L}\{\theta \ddagger y\}$$

$$\bar{\mathrm{K}}\{\theta \ddagger y\} = \mathrm{K}\{-\theta \ddagger y\} \ .$$

2. Background

2.1. Independently scattered measures of Lévy type

In this subsection we review some basic facts about infinitely divisible random measures and integration of non-random functions with respect to such measures (cf. Rajput and Rosinski, 1989).

Let $T = \mathbf{R}^d$ and $\mathcal{S}$ be a $\sigma-$ring of T (i.e. countable unions of sets in $\mathcal{S}$ belong to $\mathcal{S}$ and if A and B are sets in $\mathcal{S}$ with $A \subset B$ then $B \backslash A$ is also in $\mathcal{S}$). The σ-algebra generated by $\mathcal{S}$ will be denoted by $\sigma(\mathcal{S})$. A multiparameter process $z = \{z(A); A \in \mathcal{S}\}$ defined on a probability space is said to be an *independently scattered random measure* (i.s.r.m.) if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $z(A_n), n = 1, 2, ..,$ are independent and if

$$z(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} z(A_n) \text{ a.s.}$$

whenever $\cup_{n=1}^{\infty} A_n \in \mathcal{S}$. We shall be interested in the case when z is infinitely divisible, that is, for each $A \in \mathcal{S}$, $z(A)$ is an infinitely divisible random variable whose cumulant function can be written as

$$\mathrm{C}\{\zeta \ddagger X(A)\} = i\zeta m_0(A) - \frac{1}{2}\zeta^2 m_1(A) + \int_{\mathbf{R}} (e^{i\zeta u} - 1 - i\zeta\tau(u))\bar{q}(A, du), \qquad (2.1)$$

where m_0 is a signed measure, m_1 is a positive measure, $\bar{q}(A, du)$ is a measure in $\mathcal{B}(\mathbf{R})$ without atoms at 0 such that $\int_{\mathbf{R}} \min(1, |u|^2)\bar{q}(A, du) < \infty$ and where

$$\tau(u) = \begin{cases} u \text{ if } |u| \leq 1 \\ \frac{u}{|u|} \text{ if } |u| > 1 \end{cases} .$$

In this case we say that z has the Lévy characteristics $(m_0, m_1, \bar{q})$ and $\bar{q}$ is called the Lévy measure. There is a one to one correspondence between infinitely divisible i.s.r.m. and the class of parameters m_0, m_1 and $\bar{q}$.

For later use we note that the above definition of $\tau(u)$ has an immediate extension to the case where u, and hence $\tau(u)$, are d-dimensional vectors with $|u|$ denoting the Euclidean norm of u.

If $|\cdot|$ denotes Lebesgue measure in $\mathbf{R}^d$ and if $m_0 \sim |\cdot|, m_1 \sim |\cdot|$ and $\bar{q}(A, du) = |A| q(du)$, for q a classical Lévy measure, we say that z is an *homogeneous independently scattered measure* or a *multiparameter Lévy process with characteristics* (m_0, m_1, q). When $T = \mathbf{R}$, z is a Lévy process.

The *control measure* m defined as

$$m(A) = |m_0|(A) + m_1(A) + \int_{\mathbf{R}} \min\{1, x^2\} q(A, dx) \qquad (2.2)$$

is such that $m(A_n) \to 0$ implies that $z(A_n) \to 0$ in probability. This measure is important to characterize the class of non-random functions that are integrable with respect to z (see Rajput and Rosinski, 1989). Namely, for a real simple function $f = \sum_{j=1}^{n} x_j 1_{A_j}$ on T, where $A_j \in \mathcal{S}$, define for every $A \in \sigma(\mathcal{S})$, such that $A \cap A_j \in \mathcal{S}, j = 1, ..., n$,

$$\int_A f dz = \sum_{j=1}^{n} x_j z(A \cap A_j). \tag{2.3}$$

In general, a function $f : (T, \sigma(\mathcal{S})) \to (R, \mathcal{B}(\mathbf{R}))$ is said to be $z-$integrable if there exists a sequence $\{f_n\}$ of simple functions as above, such that $f_n \to f$ a.e. $[m]$ and for every $A \in \sigma(\mathcal{S})$, the sequence $\{\int_A f_n dz\}$ converges in probability as $n \to \infty$. If f is $z-$integrable, we write

$$\int_A f dz = \text{p}-\lim_{n\to\infty} \int_A f_n dz. \tag{2.4}$$

The integral $\int_A f dz$ is well defined (does not depend on the approximating sequence) and

$$\text{C}\{\zeta \ddagger \int_A f dz\} = \int_A H(\zeta f(\lambda), \lambda) m(d\lambda), \tag{2.5}$$

for

$$H(t, \lambda) = ita(\lambda) - \frac{1}{2} t^2 \sigma^2(\lambda) + \int_{\mathbf{R}} (e^{itx} - 1 - it\tau(x)) \rho(\lambda, dx) \tag{2.6}$$

where $a(\lambda) = \frac{dm_0}{dm}, \sigma^2(\lambda) = \frac{dm_1}{dm}$ and $\rho : T \times \mathcal{B}(\mathbf{R}) \to [0, \infty]$ is such that (i) $\rho(\lambda, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbf{R})$, for every $\lambda \in T$, (ii) $\rho(\cdot, B)$ is a Borel measurable function, for every $B \in \mathcal{B}(\mathbf{R})$ and (iii)

$$\int_{T\times\mathbf{R}} h(\lambda, x) q(d\lambda, dx) = \int_T \left[\int_{\mathbf{R}} h(\lambda, x) \rho(\lambda, dx) \right] m(d\lambda), \tag{2.7}$$

for every $\sigma(\mathcal{S}) \times \mathcal{B}(\mathbf{R})-$ measurable function $h : \mathcal{S} \times \mathbf{R} \to [0, \infty)$. Thus, in essence,

$$\bar{q}(d\lambda, dx) = \rho(\lambda, dx) m(d\lambda) \tag{2.8}$$

In particular, when z is a multiparameter Lévy process with characteristics $(0, 0, q)$, we have $m(d\lambda) = d\lambda \int_{\mathbf{R}} \min\{1, x^2\} q(dx)$ and

$$d\lambda q(dx) = \rho(\lambda, dx) d\lambda \int_{\mathbf{R}} \min\{1, u^2\} q(du). \tag{2.9}$$

We now describe the class of z-integrable functions (see Rajput and Rosinski, 1989; Theorem 2.7) when $m_0 = 0$ and $m_1 = 0$. In this case, f is $z-$integrable if and only if the following two conditions hold:

(i) $\int_T |U(f(\lambda), \lambda)| \, m(d\lambda) < \infty$

(ii) $\int_T |V_0(f(\lambda), \lambda)| \, m(d\lambda) < \infty$

where

$$U(u,\lambda)=\int_{\mathbf{R}}\left(\tau(xu)-u\tau(x)\right)\rho(\lambda,dx)$$

$$V_0(u,\lambda)=\int_{\mathbf{R}}\min\{1,|xu|^2\}\rho(\lambda,dx).$$

The following facts will be used in the sequel. Let z be a Lévy process with characteristics $(0,0,q)$, let F be a distribution function and let $v(d\lambda)=z(F(d\lambda))$. Then, v is an independently scattered random measure with control measure

$$m(d\lambda)=F(d\lambda)\int_{\mathbf{R}}\min\{1,x^2\}q(dx), \tag{2.10}$$

and

$$F(d\lambda)q(dx)=\rho(\lambda,dx)F(d\lambda)\int_{\mathbf{R}}\min\{1,x^2\}q(dx) \tag{2.11}$$

in which case bounded measurable functions are $v-$integrable. Furthermore,

$$\mathrm{C}\{\zeta \ddagger \int_{\mathbf{R}} f dv\}=\int_{\mathbf{R}}\int_{\mathbf{R}}\left(e^{i\zeta f(\lambda)x}-1-i\zeta f(\lambda)\tau(x)\right)q(dx)F(d\lambda). \tag{2.12}$$

More generally, suppose $z=(z_1,...,z_d)$ is a d-dimensional Lévy process with characteristics $(0,0,q)$ and let $v_i(\lambda)=z_i(F(\lambda))$, $i=1,...,d$. Then, for $\zeta=(\zeta_1,...,\zeta_d)\in\mathbf{R}^d$, $x=(x_1,...,x_d)$ and bounded measurable functions $f_1,...,f_d$ we have

$$\mathrm{C}\{\zeta \ddagger \int_{\mathbf{R}} f_1 dv_1,...,\int_{\mathbf{R}} f_d dv_d\}=\int_{\mathbf{R}}\int_{\mathbf{R}^d} E(\zeta,\lambda,x)q(dx)F(d\lambda) \tag{2.13}$$

where

$$\mathrm{E}(\zeta,\lambda,x)=e^{i\sum_{v=1}^{d}\zeta_\nu f_\nu(\lambda)x_\nu}-1-i\sum_{\nu=1}^{d}\zeta_\nu f_\nu(\lambda)\tau_\nu(x)$$

and with $\tau(x)=(\tau_1(x),...,\tau_d(x))$ as defined earlier.

2.2. Type G distributions

In this section we present basic facts about multivariate type G distributions. Let y_Δ be a random variable of the form $y_\Delta=\sigma\varepsilon_\Delta$ where $\sigma>0$ and ε_Δ are independent random variables with $\varepsilon\sim N_m(0,\Delta)$ (the multivariate normal m-dimensional distribution), that is the distribution of y_Δ is a normal variance mixture. When $\Delta=I$ (the standard normal m-dimensional distribution) we simply write $y=\sigma\varepsilon$. The characteristic function of y_Δ is

$$\mathrm{E}\{e^{i\langle\zeta,y_\Delta\rangle}\}=\mathrm{E}\{e^{-\frac{1}{2}\zeta\Delta\zeta^\top\sigma^2}\}=\bar{\mathrm{L}}\{\frac{1}{2}\zeta\Delta\zeta^\top \ddagger \sigma^2\}$$

and we have

$$\mathrm{C}\{\zeta \ddagger y_\Delta\} = \bar{\mathrm{K}}\{\frac{1}{2}\zeta\Delta\zeta^\top \ddagger \sigma^2\}. \tag{2.14}$$

Thus the cumulant function $\mathrm{C}\{\zeta \ddagger y_\Delta\}$ depends on ζ through $\zeta\Delta\zeta^\top$ only. In case the distribution of σ^2 is infinitely divisible, y_Δ and its distribution are said to be of *type G*. We recall that when the distribution of σ^2 is infinitely divisible, the cumulant transform has the representation (see, for instance, Feller (1971; p. 450))

$$\bar{\mathrm{K}}\{\omega \ddagger \sigma^2\} = -\int_0^\infty (1 - e^{-\omega\xi})\mathrm{Q}(\mathrm{d}\xi), \tag{2.15}$$

for Q a measure on $[0, \infty)$. Consequently,

$$\mathrm{C}\{\zeta \ddagger y_\Delta\} = -\int_0^\infty \left(1 - e^{-\frac{1}{2}\xi\zeta\Delta\zeta^\top}\right)\mathrm{Q}(\mathrm{d}\xi) \tag{2.16}$$

The measure Q is, in fact, identical to the Lévy measure of the Lévy-Khintchine representation for σ^2.

A multivariate type G distribution is infinitely divisible and its Lévy measure is absolutely continuous with respect to Lebesgue measure on $\mathbf{R}^d$ as shown by the following result.

Proposition 1 The distribution of a multivariate type G random variable y_Δ is infinitely divisible with Lévy-Khintchine representation

$$\mathrm{C}\{\zeta \ddagger y_\Delta\} = -\frac{1}{2}\mathrm{Q}(\{0\})\zeta'\Delta\zeta + \int_{\mathbf{R}^d}(\cos(\langle\zeta, u\rangle) - 1)g_\Delta(u)du \tag{2.17}$$

where the Lévy density $g_\Delta(u)$ is given by

$$g_\Delta(u) = \int_{0+}^\infty \phi_\Delta(\xi^{-1/2}u)\mathrm{Q}(\mathrm{d}\xi) \tag{2.18}$$

and where $\phi_\Delta(u)$ is the multivariate normal density

$$\phi_\Delta(u) = (2\pi)^{-d/2}\,|\Delta|^{-d/2}\,e^{-\frac{1}{2}u\Delta^{-1}u^\top}$$

and $\int_{\mathbf{R}^d}(\min(1, |u|^2)g_\Delta(u)du < \infty$. □

Proof We do the proof on the lines of the one-dimensional case as presented in Rosinski (1991). Let $q_\Delta(du)$ be the (symmetric) Lévy measure of the type G random vector y_Δ. Then the log Lévy-Khintchine representation is

$$\mathrm{C}\{\zeta \ddagger y_\Delta\} = -\frac{1}{2}\zeta\Sigma\zeta^\top + \int_{\mathbf{R}^d}(\cos(\langle\zeta, u\rangle - 1)q_\Delta(du).$$

for Σ a $d\times d$ positive definite matrix. Using (2.16) and the fact that $Q(\{0\})\Delta = \Sigma$ we have

$$\int_{\mathbf{R}^d} (\cos(\langle\zeta,u\rangle)-1)q_\Delta(du) = -\int_{0+}^\infty \left(1-e^{-\frac{1}{2}\zeta\Delta\zeta^\top\xi}\right) \mathrm{Q}(\mathrm{d}\xi).$$

Since

$$e^{-\frac{1}{2}\zeta'\Delta\zeta\xi} = \int_{\mathbf{R}^d} (\cos(\xi^{1/2}\langle\zeta,u\rangle)-1)\phi_\Delta(u)du,$$

we obtain

$$\int_{\mathbf{R}^d} (\cos(\langle\zeta,u\rangle)-1)q_\Delta(du) = \int_{0+}^\infty \int_{\mathbf{R}^d} \left(\cos(\xi^{1/2}\langle\zeta,u\rangle)-1\right)\phi_\Delta(u)du\mathrm{Q}(\mathrm{d}\xi)$$

Then, by the uniqueness of the Lévy measure we have $q_\Delta(du) = g_\Delta(\xi^{-1/2}u)du$ where $g_\Delta(u)$ is given by (2.18). $\square$

A number of examples of type G distributions will now be presented.

Example 1 *Generalized hyperbolic distributions* Suppose the law of σ^2 is the generalized inverse Gaussian distribution $GIG(\lambda,\delta,\gamma)$, given in terms of its density by

$$\frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{\lambda-1} e^{-\frac{1}{2}(\delta^2 x^{-1}+\gamma^2 x)}, \tag{2.19}$$

and let $y=\mu+\sigma^2\Delta\beta+\sigma u$ where u follows the m-dimensional normal distribution with mean 0 and variance matrix Δ. (For parametric identifiability, Δ is assumed to have determinant 1.) The probability density of x is then

$$\frac{(\gamma/\delta)^{m/2}\alpha^{m/2-\lambda}}{(2\pi)^{m/2}K_\lambda(\delta\gamma)}\{\delta^2+R\}^{(\lambda-m/2)/2}K_{\lambda-m/2}(\alpha\{\delta^2+R\}^{1/2})e^{\langle\beta,x-\mu\rangle} \tag{2.20}$$

where $\alpha=\{\gamma^2+\beta^T\Delta\beta\}^{1/2}$ and

$$R=(x-\mu)^T\Delta^{-1}(x-\mu)\ .$$

This class of distributions is closed under marginalization and conditioning (with respect to subvectors of x), and when $\beta=0$ and $\Delta=I$ the distributions are of type G. The class of all normal inverse Gaussian distributions is obtained for $\lambda=-1/2$, while the class of hyperbolic laws correspond to $\lambda=(m+1)/2$. These special types have been applied in a variety of contexts, in particular geology and finance (see Barndorff-Nielsen 1977, 1978, 1979, 1982, 1986, 1997a,b, 1998; Barndorff-Nielsen, Blæsild, Jensen and Sørensen 1985; Barndorff-Nielsen and Christiansen (1985); Barndorff-Nielsen, Jensen and Sørensen 1989, 1990, 1993; Eberlein and

Keller 1995; Küchler, Neumann, Sørensen and Streller (1994); Rydberg 1996a,b,c) and they are considered further in the following two examples.

The one-dimensional generalized hyperbolic distributions are selfdecomposable, a result due to Halgreen (1979). In the multivariate case, the distributions are selfdecomposable if and only if $\beta = 0$; note that this provides examples of multivariate non-selfdecomposable laws all of whose one-dimensional marginals are selfdecomposable (Shanbhag and Sreehari, 1979). □

Example 2 *NIG distributions of type G* Suppose that σ^2 follows the inverse Gaussian distribution $IG(\delta, \gamma)$, with density

$$\frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} e^{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)}, \tag{2.21}$$

(i.e. the special case of (2.19) for $\lambda = -\frac{1}{2}$) the corresponding Q measure being

$$\mathrm{Q}(\mathrm{d}\xi) = (2\pi)^{-1/2} \delta \xi^{-3/2} e^{-\gamma^2 \xi/2} \mathrm{d}\xi.$$

Then the cumulant function of $y = \sigma\bullet$ is

$$\mathrm{C}\{\zeta \ddagger y\} = \delta\gamma \left[1 - \left\{1 + \gamma^{-2} \langle \zeta, \zeta \rangle \right\}^{1/2}\right]$$

and this is the cumulant function of the m-dimensional distribution with density

$$\begin{aligned} &\delta^{-m} 2 (2\pi)^{-(m+1)/2} (\delta\gamma)^{(m+1)/2} e^{\delta\gamma} \\ &\cdot \{\delta^2 + \langle x, x \rangle\}^{-(m+1)/4} \mathrm{K}_{(m+1)/2}(\gamma \{\delta^2 + \langle x, x \rangle\}^{1/2}) \end{aligned} \tag{2.22}$$

□

Example 3 *Hyperbolic distributions of type G* The general m-dimensional hyperbolic distribution has density

$$\frac{(\gamma/\delta)^{m/2} \alpha^{-1/2}}{(2\pi)^{m/2} K_{(m+1)/2}(\delta\gamma)} \exp\{-\alpha(\delta^2 + R)^{1/2} + \langle \beta, x - \mu \rangle\} \tag{2.23}$$

The graph of the logaritm of this density function is an m-dimensional hyperboloid, in particular a hyperbola when $m = 1$, whence the name hyperbolic laws. In the special case of $m = 2$ the expression (2.23) becomes fully explicit since

$$K_{(m+1)/2}(s) = \sqrt{\pi/2} s^{-1/2} (1 + s^{-1}) e^{-s} \tag{2.24}$$

(cf. Barndorff-Nielsen, 1977; Blæsild, 1981; Blæsild and Jensen, 1981).

The m-dimensional spherically symmetric hyperbolic distribution with density

$$\frac{(\gamma/\delta)^{m/2}\gamma^{-1/2}}{(2\pi)^{m/2}K_{(m+1)/2}(\delta\gamma)}\exp(-\gamma\{\delta^2+\langle x,x\rangle\}^{1/2}) \tag{2.25}$$

is of type G . □

Example 4 *Student distributions* The multivariate Student distributions on ν degrees of freedom occur as the special cases of the generalized hyperbolic distributions (2.20) corresponding to $\beta=\gamma=0$ and $\lambda=-\nu/2$. □

Example 5 *Symmetric α stable distributions* For $0<\alpha<2$, let σ^2 be a positive $\alpha/2$-stable random variable with Laplace transform

$$\mathrm{E}e^{-\theta\sigma}=e^{-c\theta^{\alpha/2}},\quad \theta>0,$$

i.e.

$$\bar{\mathrm{K}}\{\theta\ddagger\sigma^2\}=-c\theta^{\alpha/2} \tag{2.26}$$

(c a positive constant). Then (see Samorodnitsky and Taqqu (1994; p. 77-84)), the distribution of the random vector $y=\sigma\varepsilon$ is symmetric α stable ($S\alpha S$) in $\mathbf{R}^d$. Thus $S\alpha S$ random variables are of type G.

More generally we have that

$$e^{(\delta\gamma)^\nu-\delta^\nu(\gamma^2-\theta)^{\nu/2}},$$

(where $\theta<\gamma^2$) is the Laplace transform of a positive random variable σ^2 having density of the form

$$a(\delta,\gamma,\nu)p_{\nu/2}(x/\delta)e^{-\gamma^2x}$$

where $p_{\nu/2}(x)$ denotes the density of a positive $\nu/2-$stable random variable and $\delta>0$ and $\gamma\geq0$ are parameters. For $\gamma>0$ all moments of σ^2 exist.

The corresponding type G law has cumulant function

$$\mathrm{C}\{\zeta\ddagger y\}=(\delta\gamma)^\nu\left[1-\left\{1+\frac{1}{2}\gamma^{-2}\langle\zeta,\zeta\rangle\right\}^{\nu/2}\right].$$

□

As discussed in Rosinski (1991), if $z(t)$ is a univariate Lévy process with $z(1)$ of type G, i.e. $z\left(1\right)\stackrel{\mathcal{L}}{=}\sigma\varepsilon$, then

$$\{z(s):0\leq s\leq1\}\stackrel{\mathcal{L}}{=}\{\bar{z}(s):0\leq s\leq1\} \tag{2.27}$$

with $\bar{z}(t)$ defined by

$$\bar{z}(t) = \sum_{n=1}^{\infty} \varepsilon_n \bar{\mathrm{Q}}^{-1}(\Gamma_n) 1_{[0,t]}(u_n) \tag{2.28}$$

where $\{\varepsilon_n\}$ is an i.i.d. sequence of $N(0,1)$ variates, $\Gamma_1, ..., \Gamma_n, ...$ are the arrival times of a Poisson process with intensity 1, and $\{u_n\}$ is an i.i.d. sequence of uniform random variables on $[0,1]$, and $\{\varepsilon_n\}$, $\{\Gamma_n\}$ and $\{u_n\}$ are mutually independent. Furthermore, $\bar{\mathrm{Q}}^{-1}$ is the inverse function of the function $\bar{\mathrm{Q}}(x) = \mathrm{Q}(x, \infty)$, Q being the Lévy measure of $z(1)$. (Here we have assumed, for simplicity, that $\sup\{x : P\{\sigma^2 \geq x\} = 1\} = 0$. If this is not the case, a slightly more involved definition of $\bar{\mathrm{Q}}^{-1}$ is needed.) The infinite series in (2.28) converges almost surely uniformly for $t \in [0,1]$.

Furthermore, if $f(s,\tau)$ is a function such that $f(\cdot,\tau)$ is integrable on the interval $[0,1]$ with respect to the process z for each $\tau \in T$, T an arbitrary set, then the process $y(t)$, $t \in T$, given by

$$y(\tau) = \int_0^1 f(s,\tau) z(\mathrm{d}s) \tag{2.29}$$

is representable in law as

$$\{y(\tau) : \tau \in T\} \stackrel{\mathcal{L}}{=} \{\bar{y}(\tau) : \tau \in T\} \tag{2.30}$$

where

$$\bar{y}(\tau) = \sum_{n=1}^{\infty} \varepsilon_n \bar{\mathrm{Q}}^{-1}(\Gamma_n) f(u_n, \tau) \tag{2.31}$$

cf. Rosinski (1991).

In extension of (2.27)-(2.28), if $z(t) = (z_1(t), z_2(t))$ is a bivariate Lévy process with $z(1) \stackrel{\mathcal{L}}{=} \sigma\varepsilon = \sigma(\varepsilon_1, \varepsilon_2)$ of type G then

$$\{z(s) : 0 \leq s \leq 1\} \stackrel{\mathcal{L}}{=} \{\bar{z}(s) : 0 \leq s \leq 1\}$$

with

$$\bar{z}(t) = (\bar{z}_1(t), \bar{z}_2(t)) = \sum_{n=1}^{\infty} (\varepsilon_{1n}, \varepsilon_{2n}) \bar{\mathrm{Q}}^{-1}(\Gamma_n) 1_{[0,t]}(u_n) \tag{2.32}$$

where $\{\varepsilon_{1n}\}$ and $\{\varepsilon_{2n}\}$ are two independent an i.i.d. sequences of $N(0,1)$ variates and $\{\Gamma_n\}$ and $\{u_n\}$ are as above. This is simple to verify from the earlier result, by verifying that $\{z(s) : 0 \leq s \leq 1\}$ and $\{\bar{z}(s) : 0 \leq s \leq 1\}$ have the same characteristic functions for the finite dimensional marginal distributions.

2.3. Second order stationary processes

Any real second order stationary process $\{x(t)\}_{\mathbf{R}}$ which is continuous in quadratic mean and has mean 0 is representable as

$$x(t) = \int_{-\infty}^{\infty} \cos(\lambda t) v(\mathrm{d}\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) w(\mathrm{d}\lambda). \tag{2.33}$$

Here $\{v(t)\}_R$ and $\{w(t)\}_{\mathbf{R}}$ are mean 0 and square-integrable real processes which are mutually orthogonal and have orthogonal increments, and the integrals are defined in the L^2 sense (see for instance Cramér and Leadbetter, 1967; p. 137). Conversely, if v and w are two mutually orthogonal processes with orthogonal increments then (2.33) determines a second order stationary process.

The correlation function of $x(t)$ satisfies

$$r(u) = \int_{-\infty}^{\infty} e^{iu\lambda} F(\mathrm{d}\lambda), \tag{2.34}$$

where F is a probability distribution function such that

$$F(\mathrm{d}\lambda) = \frac{1}{2}\mathrm{E}\{\cos^2(\lambda t) v(\mathrm{d}\lambda)^2 + \sin^2(\lambda t) w(\mathrm{d}\lambda)^2\}. \tag{2.35}$$

The measure $F(\mathrm{d}\lambda)$ is, in fact, symmetric around 0 (follows from $x(t)$ being real) and hence $r(u)$ satisfies

$$r(u) = \int_{-\infty}^{\infty} \cos(\lambda u) F(\mathrm{d}\lambda). \tag{2.36}$$

The processes v and w may be expressed in terms of x by

$$v(\lambda) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} t^{-1} \sin \lambda t \, x(t) dt$$

$$w(\lambda) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} t^{-1}(1 - \cos \lambda t) \, x(t) dt. \tag{2.37}$$

3. Stationary processes driven by type G Lévy processes

The representation (2.33) holds in particular for square integrable processes x that are strictly stationary. By the theory of independently scattered measures of Lévy type, discussed in Section 2, we may conclude that the representation defines a (strictly) stationary process also in certain cases where $(v(\lambda), w(\lambda))$ does not satisfy the above-mentioned requirements, in particular the square integrability. We shall be especially interested in stationary processes that correspond to a bivariate innovation process $(v(\lambda), w(\lambda))$ of the form

$$(v(\lambda), w(\lambda)) = z(F(\lambda)) = (z_1(F(\lambda)), z_2(F(\lambda))), \tag{3.1}$$

where $z(\lambda)$ is a bivariate Lévy process $z(1)$ of type G.

Theorem 1 Let F be an arbitrary distribution and suppose $(v(\lambda), w(\lambda))$ is of the form $z(F(\lambda))$ where the bivariate Lévy process z is of type G, i.e. $z(1) \overset{\mathcal{L}}{=} \sigma\varepsilon$. Then a process x is welldefined by

$$x(t) = \int_{-\infty}^{\infty} \cos(\lambda t) v(\mathrm{d}\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) w(\mathrm{d}\lambda) \tag{3.2}$$

and

(i) $\{x(t) : t \in \mathbf{R}\}$ is (strictly) stationary and infinitely divisible

(ii) the law of $x(t)$ is given by

$$\mathrm{C}\{\eta \ddagger x(t)\} = \bar{\mathrm{K}}\{\frac{1}{2}\eta^2 \ddagger \sigma^2\}$$

(iii) if $\mathrm{E}\sigma^2 < \infty$ then x is square integrable and its correlation function is given by $r(u) = \int_{-\infty}^{\infty} \cos(\lambda u) F(\mathrm{d}\lambda)$.

□

PROOF The existence of x follows from the theory outlined in Subsection 2.1.

To prove the stationarity we calculate the joint characteristic function of $x(t_1), ..., x(t_n)$ for arbitrary $n = 1, 2, ...$ and $t_1 < ... < t_n$. From (2.13) we have

$$\mathrm{E}\left\{\exp(i \sum_{k=1}^{n} \eta_k x(t_k))\right\}$$

$$= \exp\left\{\int_{-\infty}^{\infty} \mathrm{C}\{(\sum_{k=1}^{n} \eta_k \cos(\lambda t_k), \sum_{k=1}^{n} \eta_k \sin(\lambda t_k)) \ddagger z(1)\} F(\mathrm{d}\lambda)\right\}.$$

Since $z(1)$ is of type G its cumulant function is of the form

$$\mathrm{C}\{\zeta \ddagger z(1)\} = \bar{K}\{\frac{1}{2}\langle\zeta, \zeta\rangle \ddagger \sigma^2\},$$

Thus

$$\mathrm{E}\left\{\exp(i \sum_{k=1}^{n} \eta_k x(t_k))\right\}$$

$$= \exp\left[\int_{-\infty}^{\infty} \bar{\mathrm{K}}\left\{\frac{1}{2}\sum_{k=1}^{n}\eta_k^2 + \sum_{k<k'}\eta_k\eta_{k'}\cos(\lambda(t_{k'}-t_k)) \ddagger \sigma^2\right\} F(\mathrm{d}\lambda)\right] \tag{3.3}$$

and because this depends on $t_1, ..., t_m$ through the differences $t_{k'} - t_k$ only, the process $x(t)$ is stationary.

The infinite divisibility of the process and the statement (ii) follow immediately from the latter formula and the infinite divisibility of σ^2.

And direct calculation yields (iii). □

Remark It seems likely that $\{z(t)\}_{\mathbf{R}}$ being type G generated as in Theorem 1 is not only sufficient but also necessary for strict stationarity of $\{x(t)\}_{\mathbf{R}}$.

Example 6 With σ^2 chosen to follow the inverse Gaussian law (2.21), the theorem implies the existence of a stationary stochastic process having normal inverse Gaussian one-dimensional marginals and spectral measure F. □

Example 7 If σ^2 is $\alpha/2$-stable with $\bar{\mathrm{K}}\{\theta \ddagger \sigma^2\} = -c\theta^{\alpha/2}$ $(0 < \alpha < 2)$ then the joint law of $x(t_1), ..., x(t_n)$ has cumulant function

$$\mathrm{C}\{\eta \ddagger x(t_1), ..., x(t_n)\} = -c\int_{-\infty}^{\infty}\left|\frac{1}{2}\sum_{k=1}^{n}\eta_k^2 + \sum_{k<k'}\eta_k\eta_{k'}\cos(\lambda(t_{k'}-t_k))\right|^{\alpha/2} F(\mathrm{d}\lambda).$$

The joint laws are stable and, in fact, the process $x(t)$ is identical in law to the real part of the complex harmonizable process discussed in Samorodnitsky and Taqqu (1994; Example 6.3.6). □

Suppose the distribution function F is continuous and strictly increasing on R, with inverse F^{-1}. Then $x(t)$, given by (3.2), may be reexpressed as

$$x(t) = \int_0^1 \cos(F^{-1}(\xi)t)z_1(\mathrm{d}\xi) + \int_0^1 \sin(F^{-1}(\xi)t)z_2(\mathrm{d}\xi) \tag{3.4}$$

Hence, by (2.32), we further have

$$\{x(t) : 0 \leq t \leq 1\}$$

$$\overset{\mathcal{L}}{=} \sum_{n=1}^{\infty}(\varepsilon_{1n}\cos(F^{-1}(u_n)t) + \varepsilon_{2n}\sin(F^{-1}(u_n)t))R_0(\Gamma_n) : 0 \leq t \leq 1\} \tag{3.5}$$

a type of representation that is particularly useful for simulation purposes, due to the rapid convergence of the infinite series.

4. Selfsimilar processes

A stochastic process $x^*(t)$ on the interval $[0,\infty)$ is selfsimilar with exponent $H>0$ if $x^*(0)=0$ and if for any $c>0$

$$\{x^*(ct)\}_{\mathbf{R}_+} \stackrel{\mathcal{L}}{=} \{c^H x^*(t)\}_{\mathbf{R}_+}.$$

Such a process is said to be H-selfsimilar, for short.

By a result due to Lamperti (1962), if a process $\{x^*(t)\}_{\mathbf{R}_+}$ with $x^*(0)=0$ is selfsimilar then the derived process $\{x(t)\}_{\mathbf{R}}$ where

$$x(t) = e^{-Ht}x^*(e^t) \tag{4.1}$$

is strictly stationary; and conversely, if $\{x(t)\}_{\mathbf{R}}$ is a strictly stationary process then $x^*(t)$ defined by

$$x^*(t) = t^H x(\log t) \tag{4.2}$$

is selfsimilar. Note that, in this case, $x^*(t)$ may and may not have strictly stationary increments.

Suppose now that the strictly stationary process $x(t)$ is given by (3.2) with $(v(\lambda), w(\lambda))$ of the form $z(F(\lambda))$ where F is an arbitrary distribution function and z is a bivariate Lévy process of type G, such that $z(1) \stackrel{\mathcal{L}}{=} \sigma\varepsilon$. For the associated self-similar process $x^*(t) = t^H x(\log t)$ we find, from (3.3), that

$$\begin{aligned} \log \mathrm{E}\left\{\exp(i\sum_{k=1}^{n}\eta_k x^*(t_k)\right\} &= \log E\left\{\exp(i\sum_{k=1}^{n}\eta_k t_k^H x(\log t_k))\right\} \\ &= \int_{\mathbf{R}} \bar{K}\{\frac{1}{2}\underline{\eta}'\Delta_\lambda^*(\underline{t})\underline{\eta} \ddagger \sigma^2\}F(\mathrm{d}\lambda), \end{aligned} \tag{4.3}$$

where

$$\Delta_\lambda^*(\underline{t}) = \left(t_j^H t_k^H \cos(\lambda(\log\frac{t_j}{t_k}))\right)_{j,k=1}^{n}. \tag{4.4}$$

In particular

$$\begin{aligned} \mathrm{C}\{\eta \ddagger (x^*(t) - x^*(s))\} &= -\int_{\mathbf{R}_+}\int_{\mathbf{R}}\left(1-\exp\left\{-\frac{1}{2}\xi\eta^2 W_\lambda(s,t,)\right\}\right)\mathrm{Q}(\mathrm{d}\xi)F(\mathrm{d}\lambda) \\ &= \int_{\mathbf{R}} \bar{K}\{\frac{1}{2}\eta^2 W_\lambda(s,t,) \ddagger \sigma^2\}F(\mathrm{d}\lambda) \end{aligned} \tag{4.5}$$

where

$$W_\lambda(s,t) = s^{2H} + t^{2H} - 2s^H t^H \cos(\lambda \log\frac{t}{s}). \tag{4.6}$$

For use below we note that (4.5) may be rewritten as

$$\mathrm{C}\{\eta \ddagger (x^*(t) - x^*(s))\} = -\int_{\mathbf{R}_+} M(-\frac{1}{2}\xi\eta^2; s,t)\mathrm{Q}(\mathrm{d}\xi) \tag{4.7}$$

where

$$M(\theta; s,t) = \int_{\mathbf{R}} (1 - \exp\{\theta W_\lambda(s,t)\})\, F(\mathrm{d}\lambda). \tag{4.8}$$

4.1. Selfsimilar processes with second order stationary increments

We shall now show that in the square integrable case, processes $x^*(t)$ of the type considered in Theorem 1 have second order stationary increments for a special choice of F, specifically for F equal to the distribution function associated to the covariance function of fractional Brownian motion.

If $x^*(t)$ is H-selfsimilar with second order stationary increments then its covariance function is necessarily of the form

$$\mathrm{E}\{x^*(s)x^*(t)\} = \frac{1}{2}\{s^{2H} + t^{2H} - (t-s)^{2H}\}\mathrm{E}\left\{x^*(1)^2\right\} \tag{4.9}$$

for some $H \in (0,1)$ and $s < t$. (This is well known and follows easily from the identity $\mathrm{E}\{(x^*(t) - x^*(s))^2\} = \mathrm{E}\{x^*(t-s)^2\}$). Equivalently, the correlation function of the associated stationary process $x(t)$ must be of the form

$$\begin{aligned} r(u) &= \cosh(Hu) - 2^{2H-1}\sinh^{2H}(u/2) \\ &= \cosh(Hu) - 2^{-(1-H)}(\cosh u - 1)^{2H} \end{aligned} \tag{4.10}$$

for $u > 0$.

In fact,

$$\begin{aligned} \mathrm{E}\{x(t)x(t+u)\} &= e^{-H(2t+u)}E\left\{x^*(e^t)x^*(e^{t+u})\right\} \\ &= \frac{1}{2}e^{-H(2t+u)}\left\{e^{2Ht} + e^{2H(t+u)} - \left(e^{t+u} - e^t\right)^{2H}\right\} \\ &\quad \cdot E\left\{x^*(1)^2\right\} \\ &= \frac{1}{2}\left\{e^{-Hu} + e^{Hu} - \left(e^{u/2} - e^{-u/2}\right)^{2H}\right\} E\left\{x^*(1)^2\right\} \\ &= \left\{\cosh(Hu) - 2^{2H-1}\sinh^{2H}(u/2)\right\} E\left\{x(0)^2\right\}. \end{aligned}$$

Let F_H denote the (uniquely defined) distribution function satisfying

$$r(u) = \int_{-\infty}^{\infty} \cos(\lambda u) F_H(\mathrm{d}\lambda)$$

with $r(u)$ given by (4.10).

Combined with Theorem 1 the above discussion implies the validity of

Theorem 2 Suppose that $\{x(t)\}_{\mathbf{R}}$ is stationary with representation (3.2) where $v(\lambda)$ and $w(\lambda)$ are defined (as in (3.1)) from a square integrable bivariate Lévy process $z(\lambda)$ that is type G generated (i.e. $z(1) \stackrel{\mathcal{L}}{=} \sigma(\varepsilon_1, \varepsilon_2)$) and for which $F = F_H$. Then the correlation function $r(u)$ of $x(t)$ satisfies

$$r(u) = \cosh(Hu) - 2^{2H-1} \sinh^{2H}(u/2) \tag{4.11}$$

and the associated selfsimilar process $x^*(t) = t^H x(\log t)$ has second order stationary increments. □

Example 8 In particular, the above construction gives a new integral representation of the Fractional Brownian motion B_H. Let W_1 and W_2 be independent standard Brownian motions. Then

$$B_H(t) = t^H \int_{-\infty}^{\infty} \cos(\lambda \log t) W_1(F_H(\mathrm{d}\lambda)) + t^H \int_{-\infty}^{\infty} \sin(\lambda \log t) W_2(F_H(\mathrm{d}\lambda)) \tag{4.12}$$

is a fractional Brownian motion. □

In the general case of type G distributions with second moment, although the selfsimilar process x^* does not have stationary increments, the associated fractional noise keeps several properties of the fractional Gaussian noise. For $j = 0, 1, 2, ...,$ let $y_j^* = x^*(j+1) - x^*(j)$. Then the fractional sequence $\{y_j^*\}$ is second order stationary and has autocovariance function

$$r^*(j) = \mathrm{E}\{y_0^* y_j^*\} = \frac{1}{2}\left[\mid j+1 \mid^{2H} -2 \mid j \mid^{2H} + \mid j-1 \mid^{2H}\right]. \tag{4.13}$$

Then, from Proposition 7.2.10 in Samorodnitsky and Taqqu (1994), for $H \neq 1/2$

$$r^*(j) \sim H(2H-1) j^{2H-2} \quad \text{as } j \to \infty. \tag{4.14}$$

We conclude this section by showing that $r(u)$ possesses a spectral density f_H and by deriving a formula for f_H. Together with the representation (3.5), this formula will be useful for simulation studies.

Theorem 3 The correlation function

$$r(u) = \cosh(Hu) - 2^{2H-1} \sinh^{2H}(u/2) \tag{4.15}$$

has, for $0 < H < 1$ a spectral density of the form

$$f_H(\lambda) = (2\pi)^{-1} \sum_0^\infty (-1)^{j-1} \binom{2H}{j} (j-H)\{(j-H)^2 + \lambda^2\}^{-1} \tag{4.16}$$

i.e. a weighted sum of Cauchy densities. □

PROOF First note that

$$\begin{aligned} r(u) &= \cosh(Hu) - 2^{2H-1} \sinh^{2H}(u/2) \\ &= \frac{1}{2} e^{Hu} \left\{ 1 + e^{-2HU} - \left(1 - e^{-u}\right)^{2H} \right\} \\ &= \frac{1}{2} e^{Hu} \left\{ e^{-2HU} + \sum_1^\infty (-1)^{j-1} \binom{2H}{j} e^{-ju} \right\}. \end{aligned}$$

Consequently, for $u \to \infty$ we have

$$r(u) \left\{ \begin{array}{lll} \sim & \frac{1}{2} e^{-Hu} & \text{for } 0 < H < \frac{1}{2} \\ = & \frac{1}{2} e^{-u/2} & \text{for } H = \frac{1}{2} \\ \sim & H e^{-(1-H)u} & \text{for } \frac{1}{2} < H < 1 \end{array} \right.$$

showing that $F_H(\lambda)$ is absolutely continuous with a density $f_H(\lambda)$.
Further, since $r(u)$ is symmetric we may reexpress $r(u)$ as

$$r(u) = \frac{1}{2} \left\{ e^{-H|u|} + \sum_1^\infty (-1)^{j-1} \binom{2H}{j} e^{-(j-H)|u|} \right\}. \tag{4.17}$$

From this expression it is possible to develop the series representation for the density $f_H(\lambda)$ of $F_H(\lambda)$. In fact, using that $\exp\{-c|u|\}$ is the characteristic function of the Cauchy density $\pi^{-1} c\{c^2 + x^2\}^{-1}$ we find by Fourier inversion that

$$\begin{aligned} f_H(\lambda) &= (2\pi)^{-1} \left\{ H\{H^2 + \lambda^2\}^{-1} + \sum_1^\infty (-1)^{j-1} \binom{2H}{j} (j-H)\{(j-H)^2 + \lambda^2\}^{-1} \right\} \\ &= (2\pi)^{-1} \sum_0^\infty (-1)^{j-1} \binom{2H}{j} (j-H)\{(j-H)^2 + \lambda^2\}^{-1}. \end{aligned}$$

□

4.2. Selfsimilar processes without assumption of square integrability

Without the assumption of square integrability of the processes v and w one can, of course, not speak of second order stationarity. However, there is in any case approximate stationarity in the sense that the coefficient $c_1(s,t)$ in the below Proposition 2 depends on s and t through $t-s$ only. We shall refer to this as weak stationarity of the increments.

Lemma 1 The following identity holds

$$2s^H t^H \int_{-\infty}^{\infty} \cos(\lambda \log \frac{t}{s}) F_H(\mathrm{d}\lambda) = s^{2H} + t^{2H} - (t-s)^{2H}. \tag{4.18}$$

□

PROOF This follows from formula (4.9) in conjunction with the relation (4.2).
□

Now, recall the formulae (4.7) and (4.8).

Proposition 2 For $F = F_H$ we have

$$\begin{aligned} M(\theta; s,t) &= \int_{\mathbf{R}} (1 - \exp\{\theta W_\lambda(s,t)\}) F_H(\mathrm{d}\lambda) \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} c_j(s,t) \frac{\theta^j}{j!} \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} c_j(s,t) &= \sum_{m=0}^{j} (-1)^m \binom{j}{m} (s^{2H} + t^{2H})^{j-m} \sum_{q=0}^{[\frac{m}{2}]} \binom{m}{q} (st)^{2qH} \\ &\quad \cdot \{ s^{(m-2q)2H} + t^{(m-2q)2H} - (t^{m-2q} - s^{m-2q})^{2H} \} \end{aligned} \tag{4.20}$$

In particular,

$$c_1(s,t) = (t-s)^{2H} \tag{4.21}$$

$$c_2(s,t) = (t-s)^{2H}\{2(s^{2H} + t^{2H}) - (s+t)^{2H}\} + (s^{2H} + t^{2H})^2 + 2(st)^{2H} \tag{4.22}$$

i.e. $c_1(s,t)$, but not $c_2(s,t)$, depends on $t-s$ only. □

PROOF From Gradshteyn and Ryzhik (1965; p. 31) we have

$$\cos^m(x) = \frac{1}{2^{m-1}} \sum_{q=0}^{[\frac{m}{2}]} \binom{m}{q} \cos((m-2q))x. \tag{4.23}$$

It follows that

$$\begin{aligned} c_j(s,t) &= \int_{\mathbf{R}} \{W_\lambda(s,t)\}^j F_H(\mathrm{d}\lambda) \\ &= \sum_{m=0}^{j} (-1)^m 2^m \binom{j}{m} (s^{2H}+t^{2H})^{j-m}(st)^{mH} \\ &\quad \cdot \int_{\mathbf{R}} \cos^m(\lambda \log \frac{t}{s}) F_H(\mathrm{d}\lambda) \end{aligned} \tag{4.24}$$

and, using (4.23) and (4.18), we find

$$\begin{aligned} \int_{\mathbf{R}} \cos^m(\lambda \log \frac{t}{s}) F_H(\mathrm{d}\lambda) &= 2^{-m+1} \sum_{q=0}^{[\frac{m}{2}]} \binom{m}{q} \int_{\mathbf{R}} \cos(\lambda \log \frac{t^{m-2q}}{s^{m-2q}}) F_H(\mathrm{d}\lambda) \\ &= 2^{-m} \sum_{q=0}^{[\frac{m}{2}]} \binom{m}{q} (st)^{-(m-2q)H} \\ &\quad \cdot \left\{ t^{(m-2q)2H} + s^{(m-2q)2H} - (t^{m-2q} - s^{m-2q})^{2H} \right\} \end{aligned}$$

Inserting this in (4.24) we obtain (4.20). □

References

[1] Barndorff-Nielsen, O.E. (1977): Exponentially decreasing distributions for the logarithm of particle size. *Proc. R. Soc. London* A 353, 401-419.

[2] Barndorff-Nielsen, O.E. (1978): Hyperbolic distributions and distributions on hyperbolae. *Scand. J. Statist.* 5, 151-157.

[3] Barndorff-Nielsen, O.E. (1979): Models for non-Gaussian variation; with applications to turbulence. *Proc. R. Soc. London* A 368, 501-520.

[4] Barndorff-Nielsen, O.E. (1982): The hyperbolic distribution in statistical physics. *Scand. J. Statist.* 9, 43-46.

[5] Barndorff-Nielsen, O.E. (1986): Sand, wind and statistics. *Acta Mechanica* 64, 1-18.

[6] Barndorff-Nielsen, O.E. (1997a): Probability and Statistics: selfdecomposability, finance and turbulence. Proceedings of the Conference "Probability towards 2000", Centre for Applied Probability, Cornell University, October 1995. (To appear.)

[7] Barndorff-Nielsen, O.E. (1997b): Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.* 24, 1-14.

[8] Barndorff-Nielsen, O.E. (1998): Processes of normal inverse Gaussian type. *Finance and Stochastics* 2, 41-68.

[9] Barndorff-Nielsen, O.E., Blæsild, P. Jensen, J.L. and Sørensen, M. (1985): The fascination of sand. In A.C.Atkinson and S. E. Fienberg (Eds.): *A Celebration of Statistics.* Centenary Volume of the ISI. Heidelberg: Springer-Verlag. Pp. 57-87.

[10] Barndorff-Nielsen, O.E. and Christiansen, C. (1985): Erosion, deposition and size distributions of sand. *Proc. R. Soc. London* A 417, 335-352.

[11] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1989): Wind shear and hyperbolic distributions. *Boundary-Layer Meteorology* 46, 417-431.

[12] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1990): Parametric modelling of turbulence. *Phil. Trans. R. Soc. London* A 332, 439-455.

[13] Barndorff-Nielsen, O.E., Jensen, J.L. and Sørensen, M. (1993): A statistical model for the streamwise component of a turbulent wind field. *Ann. Geophysicae* 11, 99-103.

[14] Blæsild, P. (1981): The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data. *Biometrika* 68, 251-263

[15] Blæsild, P. and Jensen, J.L. (1981): Multivariate distributions of hyperbolic type. In C. Taillie, G. P. Patil and B. A. Baldessari (Eds.): *Statistical Distributions in Scientific Work.* Vol. 4. Dordrecht: Reidel. pp. 45-66.

[16] Cramér, H. and Leadbetter, M. R. (1967). *Stationary and Related Stochastic Processes.* John Wiley and Sons, New York.

[17] Eberlein, E. and Keller, U. (1995): Hyperbolic distributions in finance. *Bernoulli* 1, 281-299.

[18] Feller, W. (1971): *An Introduction to Probability Theory and Its Applications.* Vol. 2 (Sec. Ed.)

[19] Gradshteyn, I.S. and Ryzhik, I.W. (1965): *Table of Integrals, Series and Products.* New York: Academic Press.

[20] Guillaume, D.M., Dacorogna, M.M., Davé, R.R., Müller, U.A., Olsen, R.B. and Pictet, O.V. (1997): From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets. *Finance and Stochastics* 1, 95-129.

[21] Halgreen, C. (1979): Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 47, 13-17.

[22] Lamperti, J.W. (1962): Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* 104, 62-78.

[23] Rajput, B. and J. Rosiński (1989). Spectral representations of infinite divisible processes. *Probab. Th. Rel. Fields* 82, 451-487.

[24] Rosinski, J. (1991): On a class of infinitely divisible processes represented as mixtures of Gaussian processes. In S. Cambanis, G. Samorodnitsky and M.S. Taqqu (Eds.): *Stable Processes and Related Topics.* Basel: Birkhäuser. Pp. 27-41.

[25] Rydberg, T. (1996a): The normal inverse Gaussian Lévy Process: simulation and approximation. Research Report 344, Dept. Theor. Statistics, Aarhus University. *Stochastic Models.* (To appear.)

[26] Rydberg, T. (1996b): Generalized hyperbolic diffusions with applications towards finance. Research Report 342, Dept. Theor. Statistics, Aarhus University. Submitted to *Mathematical Finance.*

[27] Rydberg, T. (1996c): Existence of unique equivalent martingale measures in a Markovian setting. *Finance and Stochastics* **1**, 251-257.

[28] Samorodnitsky, G. and Taqqu, M.S. (1994): *Stable Non-Gaussian Random Processes.* New York: Chapman and Hall.

[29] Shanbhag, D.N. and Sreehari, M. (1979): An extension of Goldie's result and further results in infinite divisibility. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 47, 13-17.