CHANGE OF TIME AND MEASURE FOR LÉVY PROCESSES

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With 50 Exercises

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Contents

1 Introduction

The aim of these notes is to describe some aspects of the change of time and the change of measure for Lévy processes. The problems under consideration are closely connected with the stock price modelling.

Section 2 contains a very brief overview of some classical and modern stock price models used in the mathematical finance. In particular, we consider the exponential Lévy models

$$S_t = S_0 e^{X_t}, \quad t \le T, \tag{1.1}$$

where $(X_t)_{t \le T}$ is a Lévy process, and the time-changed exponential Lévy models

$$S_t = S_0 e^{(X \circ \tau)_t}, \quad t \le T, \tag{1.2}$$

where $(X_t)_{t \ge 0}$ is a Lévy process and $(\tau_t)_{t \le T}$ is an increasing càdlàg process that is independent of X. Here, $(X \circ \tau)_t := X_{\tau_t}$. We also compare the adequacy of different models (see Table 1 on page 4).

Section 3 deals with the following problem: which processes X can be represented as $X \overset{\text{law}}{=} B \circ \tau$, where B is a Brownian motion? This problem is considered in 5 different settings:

I. B is an $(\mathcal{F}_t)$-Brownian motion, and τ is an $(\mathcal{F}_t)$-time-change;

II. B is an $(\mathcal{F}_t)$-Brownian motion, and τ is a continuous $(\mathcal{F}_t)$-time-change;

III. B is a Brownian motion, and τ is an independent increasing càdlàg process;

IV. B is a Brownian motion, and τ is an independent increasing continuous process;

V. B is a Brownian motion, and τ is an independent subordinator.

The answer to the above problem in Settings I, II, and IV is known (see Subsection 3.1). The answer to this problem in Setting V is presented in these notes (see Subsection 3.2). The problem in Setting III remains open.

In Section 4, we first cite some known general results related to the change of measure for Lévy processes (see Subsection 4.1). Then we apply these results to derive the necessary and sufficient conditions for the absence of arbitrage and for the completeness of models (1.1) (see Subsection 4.2).

In Section 5, we first describe some nice properties of models (1.2): stationarity of increments, persistence of volatility,... (see Subsection 5.1). Then we derive the necessary and sufficient conditions for the absence of arbitrage and for the completeness of models (1.2) (see Subsection 5.2).

The Appendix contains some known definitions and facts that are used in the notes as well as the statements needed to solve some exercises.

Each of Sections 2–5 contains exercises. The following notation is used:

$\circ$ — a simple exercise;

— an important exercise;

$*$ — a difficult exercise.

Each of Sections 3–5 contains an open problem.

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2 Some Financial Models

2.1 Definitions of Models

In what follows, S means the discounted price of an asset. We will consider one-dimensional models with a finite time horizon T.

The most classical model is

The Bachelier model:

$$S_t = S_0 + B_t, \quad t \leq T,$$

where $(B_t)_{t \leq T}$ is a Brownian motion.

The main disadvantage of this model is that the process S here can take negative values. This is overcome by

The Black-Scholes-Samuelson model:

$$S_t = S_0 e^{\mu t + \sigma B_t}, \quad t \leq T,$$

where $(B_t)_{t \leq T}$ is a Brownian motion and $\mu, \sigma \in \mathbb{R}$.

There are two main disadvantages of this model:

- the increments of $\ln S$ are Gaussian;
- the increments of $\ln S$ over disjoint intervals are independent.

The first of these disadvantages is overcome by

The exponential Lévy model:

$$S_t = S_0 e^{X_t}, \quad t \leq T, \tag{2.1}$$

where $(X_t)_{t \leq T}$ is a Lévy process.

Actually (2.1) is a whole class of models. It includes, in particular, the following models with a finite number of parameters:

- X is a variance gamma (VG) process;
- X is a Carr-Geman-Madan-Yor ($CGMY$) process;
- X is a normal inverse Gaussian (NIG) process;
- X is a hyperbolic (HYP) process;

The definitions of these processes are given in Subsection 3.1.

The main disadvantage of model (2.1) is that the increments of $\ln S$ over disjoint intervals are independent. In other words, this model does not tackle the phenomenon of the persistence of volatility (clustering). This disadvantage is overcome by

The time-changed exponential Lévy model (Carr, Geman, Madan, Yor [3]):

$$S_t = S_0 e^{(X \circ \tau)_t}, \quad t \leq T, \tag{2.2}$$

where $(X_t)_{t \geq 0}$ is a Lévy process and $(\tau_t)_{t \leq T}$ is an increasing càdlàg process that is independent of X. Here,

$$(X \circ \tau)_t := X_{\tau_t}, \quad t \leq T.$$

Actually (2.2) is a whole class of models. It includes, in particular, the following models with a finite number of parameters:

- X is a variance gamma (*VG*) process;
- X is a Carr-Geman-Madan-Yor (*CGMY*) process;
- X is a normal inverse Gaussian (*NIG*) process;
- X is a hyperbolic (*HYP*) process;
- τ is a Cox-Ingersoll-Ross (*CIR*) process, i.e.

$$\tau_t = \int_0^t y_s ds, \quad t \le T,$$

where y is a solution of the stochastic differential equation

$$dy_t = \theta(\eta \Leftrightarrow y_t)dt + \sigma\sqrt{y_t}dB_t.$$

Another modification of model (2.1) that tackles the phenomenon of the volatility persistence is

The exponential Lévy model with the stochastic integrals (Eberlein, Kallsen, Kirsten [8]):

$$S_t = S_0 e^{(\sigma \bullet X)_t}, \quad t \le T, \tag{2.3}$$

where $(X_t)_{t \le T}$ is a Lévy process and $(\sigma_t)_{t \le T}$ is an X-integrable process that is independent of X. Here,

$$(\sigma \bullet X)_t := \int_0^t \sigma_s dX_s, \quad t \le T.$$

2.2 Comparison of Different Models

Below is a list of some desirable properties of a financial model:

1. The marginal distributions of the increments of $\ln S$ are skewed.
2. The marginal distributions of the increments of $\ln S$ have heavy tails.
3. The increments of $\ln S$ are stationary in time.
4. The increments of $\ln S$ over disjoint intervals are not correlated.
5. The absolute values of the increments of $\ln S$ over disjoint intervals are positively correlated (the effect of "clustering", "volatility persistence").
6. The model is arbitrage free.
7. The model depends on a small number of parameters.

Table 1 shows which of these properties are satisfied for the models introduced above. For example, if we consider model (2.1), where X is a *VG*, *CGMY*, *NIG*, or a *HYP* process, then this model satisfies conditions 1, 2, 3, 4, 6 (to be more precise, this can be achieved by an appropriate choice of the model parameters). This model does not satisfy condition 5, and the number of the parameters is 3 for the *VG* model and 4 for the *CGMY*, *NIG*, or *HYP* model.

A model	1	2	3	4	5	6	7
$S_t = S_0 e^{\mu t + \sigma B_t}$	$\Leftrightarrow$	$\Leftrightarrow$	+	+	$\Leftrightarrow$	+	2
$S_t = S_0 e^{X_t}$, X is *VG*, *CGMY*, *NIG*, or *HYP*	+	+	+	+	$\Leftrightarrow$	+	3,4
$S_t = S_0 e^{(X \circ \tau)_t}$, X is *VG*, *CGMY*, *NIG*, or *HYP*, τ is *CIR*	+	+	+	+	+	+	6,7
$S_t = S_0 e^{(\sigma \bullet X)_t}$, X is *VG*, *CGMY*, *NIG*, or *HYP*, σ is *CIR*	+	+	+	+	+	+	6,7

Table 1: Comparison of different models

2.3 Exercises

$\circ$ **Exercise 2.1.** Give an example of a Brownian motion B and an independent increasing continuous process τ such that the process $B \circ \tau$ cannot be represented as $B \circ \tau \stackrel{\text{law}}{=} \sigma \bullet W$, where W is a Brownian motion and σ is an independent W-integrable process.

$\circ$# **Exercise 2.2.** Let W be a Brownian motion and σ be an independent W-integrable process, i.e.

$$\forall t \geq 0, \int_0^t \sigma_s^2 ds < \infty \quad \text{a.s.}$$

Prove that there exists a Brownian motion B and in independent increasing process τ such that $\sigma \bullet W \stackrel{\text{law}}{=} B \circ \tau$.

(Hint: Use Proposition 3.6.)

3 Change of Time for a Brownian Motion

3.1 Various Time-Changes

In this section, we consider the following problem.

The main problem. *Which processes* $(X_t)_{t\geq 0}$ *(we assume that* $X_0 = 0$*) can be represented as* $X \stackrel{\text{law}}{=} B \circ \tau$*, where* $(B_t)_{t\geq 0}$ *is a Brownian motion and* $(\tau_t)_{t\geq 0}$ *is an increasing càdlàg process with* $\tau_0 = 0$*?*

Making this problem precise leads to 5 possible settings.

Setting I. B *is an* $(\mathcal{F}_t)$*-Brownian motion (i.e.* B *is a Brownian motion,* B *is* $(\mathcal{F}_t)$*-adapted and, for any* $s \leq t$*, the increment* $B_t - B_s$ *is independent of* $\mathcal{F}_s$*), and* τ *is an* $(\mathcal{F}_t)$*-time-change (i.e. each* τ_t *is an* $(\mathcal{F}_t)$*-stopping time,* $\tau_0 = 0$*, and the maps* $t \mapsto \tau_t$ *are a.s. increasing and càdlàg), where* $(\mathcal{F}_t)$ *is an arbitrary filtration.*

Proposition 3.1. *If* Z *is an* $(\mathcal{F}_t)$*-semimartingale and* τ *is an* $(\mathcal{F}_t)$*-time-change, then* $Z \circ \tau$ *is an* $(\mathcal{F}_{\tau_t})$*-semimartingale.*

For the proof, see [18; Ch. 4, §7].

Proposition 3.2. *If* Z *is an* $(\mathcal{F}_t)$*-semimartingale and* $(\mathcal{G}_t)$ *is a filtration such that* $\mathcal{F}_t^Z \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$*, then* Z *is a* $(\mathcal{G}_t)$*-semimartingale.*

For the proof, see [18; Ch. 4, §6].

Proposition 3.3. *A process* X *can be represented as* $X \stackrel{\text{law}}{=} B \circ \tau$ *in the sense of Setting I if and only if* X *is a semimartingale (with respect to its own filtration).*

`Proof`. The "only if" part. It follows from Propositions 3.1 and 3.2 that $B \circ \tau$ is a semimartingale with respect to its own filtration. Hence, X is also a semimartingale with respect to its own filtration.

The "if" part was proved by I. Monroe [19]. □

Setting II. B *is an* $(\mathcal{F}_t)$*-Brownian motion, and* τ *is a continuous* $(\mathcal{F}_t)$*-time-change, where* $(\mathcal{F}_t)$ *is an arbitrary filtration.*

Proposition 3.4. (Dambis-Dubins-Schwarz.) *If* X *is a continuous local martingale, then there exists a Brownian motion* B *(that may be defined on a possibly enlarged probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathsf{P}}) = (\Omega, \mathcal{F}, \mathsf{P}) \times (\Omega', \mathcal{F}', \mathsf{P}')$*) and a filtration* $(\mathcal{G}_t)$ *such that* B *is a* $(\mathcal{G}_t)$*-Brownian motion,* $\langle X \rangle$ *is a* $(\mathcal{G}_t)$*-time-change, and* $X = B \circ \langle X \rangle$*.*

For the proof, see [21; Ch. V, Theorems 1.6, 1.7].

Proposition 3.5. *A process* X *can be represented as* $X \stackrel{\text{law}}{=} B \circ \tau$ *in the sense of Setting II if and only if* X *is a continuous local martingale (with respect to its own filtration).*

Proof. The "only if" part. Let B be an $(\mathcal{F}_t)$-Brownian motion and τ be a continuous $(\mathcal{F}_t)$-time-change such that $X \overset{\text{law}}{=} B\circ\tau$. Set $\sigma_n = \inf\{t \geq 0 : |B\circ\tau|_t \geq n\}$. Obviously, $(B\circ\tau)^{\sigma_n} = B^{\rho_n}\circ\tau$, where $\rho_n = \inf\{t \geq 0 : |B_t| \geq n\}$. The process B^{ρ_n} is a uniformly integrable $(\mathcal{F}_t)$-martingale. By the optional stopping theorem (see [21; Ch. II, Theorem 3.2]), for any $s \leq t$, we have

$$\mathsf{E}\big(B^{\rho_n}_{\tau_t} \mid \mathcal{F}_{\tau_s}\big) = B^{\rho_n}_{\tau_s}.$$

Consequently,

$$\mathsf{E}\big(B^{\rho_n}_{\tau_t} \mid \mathcal{F}^{B\circ\tau}_s\big) = B^{\rho_n}_{\tau_s}.$$

This means that $B\circ\tau$ is a local martingale. Hence, X is also a local martingale.

The "if" part follows from the Dambis-Dubins-Schwarz theorem. □

Remark. Note that any continuous local martingale X can actually be represented as $X \overset{\text{a.s.}}{=} B\circ\tau$ (possibly, on an enlarged probability space), and not only as $X \overset{\text{law}}{=} B\circ\tau$. □

Setting III. *B is a Brownian motion, and τ is an independent increasing càdlàg process.*

We do not know the answer to the problem in this setting.

Setting IV. *B is a Brownian motion, and τ is an independent increasing continuous process.*

Proposition 3.6. (Ocone.) *The following conditions are equivalent:*

(i) *X can be represented as $X \overset{\text{law}}{=} B\circ\tau$ in the sense of Setting IV;*

(ii) *X is a continuous local martingale, and, for any $a \geq 0$, we have*

$$\left(\int_0^t H^a_s dX_s;\, t \geq 0\right) \overset{\text{law}}{=} (X_t;\, t \geq 0),$$

where $H^a_t = I(t \leq a) - I(t > a)$;

(iii) *X is a continuous local martingale, and, for any $(\mathcal{F}^X_t)$-predictable process H with $|H| = 1$, we have*

$$\left(\int_0^t H_s dX_s;\, t \geq 0\right) \overset{\text{law}}{=} (X_t;\, t \geq 0);$$

(iv) *X is a continuous local martingale, and if we set $\mathsf{Q}_\varphi = \mathrm{Law}(X_t; t \geq 0 | \langle X\rangle = \varphi)$, then, for a.e. φ (with respect to the measure $\mathrm{Law}(\langle X\rangle_t;\, t \geq 0)$), the coordinate process Z on $C(\mathbb{R}_+)$ is a Q_φ-local martingale with $\langle Z\rangle = \varphi$.*

For the proof, see [20].

Definition 3.7. A process X that satisfies the conditions of Proposition 3.6 is called the *Ocone martingale*.

Remark. Note that any Ocone martingale X can actually be represented as $X \overset{\text{a.s.}}{=} B\circ\tau$ (possibly, on an enlarged probability space), and not only as $X \overset{\text{law}}{=} B\circ\tau$. □

Setting V. *B is a Brownian motion, and τ is an independent subordinator (i.e. an increasing Lévy process).*

The answer to the problem in this setting is provided by Theorem 3.17.

3.2 Time-Change by an Independent Subordinator

This subsection is devoted to the solution of the above problem in Setting V.

Lemma 3.8. *If X is a Lévy process and τ is an independent subordinator, then $X \circ \tau$ is a Lévy process.*

The proof is Exercise 3.32.

Example 3.9. (The Cauchy process.) *Let (B^1, B^2) be a two-dimensional Brownian motion. Set*

$$\tau_t = \inf\{s \geq 0 : B^1_s > t\}, \quad t \geq 0.$$

Then $B^2 \circ \tau$ is a Cauchy process, (i.e. $(B^2 \circ \tau)_1$ has the standard Cauchy distribution).

Proof. We can assume that $(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_1, \mathcal{F}_1, \mathsf{P}_1) \times (\Omega_2, \mathcal{F}_2, \mathsf{P}_2)$ and $B^1 = B^1(\omega_1)$, $B^2 = B^2(\omega^2)$. Then, for any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \mathsf{E}e^{i\lambda(B^2 \circ \tau)_1} &= \int_{\Omega_1} \int_{\Omega_2} \exp\{i\lambda B^2_{\tau_1(\omega_1)}(\omega_2)\} \mathsf{P}_2(d\omega_2)\mathsf{P}_1(d\omega_1) \\ &= \int_{\Omega_1} e^{-\frac{\lambda^2}{2}\tau_1(\omega_1)} \mathsf{P}_1(d\omega_1) = \mathsf{E}e^{-\frac{\lambda^2}{2}\tau_1}. \end{aligned} \tag{3.1}$$

The process

$$\exp\left\{|\lambda| B^1_{t\wedge\tau_1} \Leftrightarrow \frac{\lambda^2}{2} t \wedge \tau_1\right\}, \quad t \geq 0$$

is a uniformly integrable martingale (note that it is bounded). Hence,

$$\mathsf{E}\exp\left\{|\lambda| B^1_{\tau_1} \Leftrightarrow \frac{\lambda^2}{2}\tau_1\right\} = 1.$$

Since $B^1_{\tau_1} = 1$, we get $\mathsf{E}e^{-\frac{\lambda^2}{2}\tau_1} = e^{-|\lambda|}$. This, combined with (3.1), completes the proof. □

Example 3.10. (The *NIG* process.) *Let $\alpha \in \mathbb{R}_+$, $\beta \in [\Leftrightarrow\alpha, \alpha]$, $\delta \in \mathbb{R}_+$. Let (B^1, B^2) be a two-dimensional Brownian motion with the drift $(\sqrt{\alpha^2 \Leftrightarrow \beta^2}, \beta)$. Set*

$$\tau_t = \inf\{s \geq 0 : B^1_s > \delta t\}, \quad t \geq 0.$$

Then, by the definition, $B^2 \circ \tau$ is a normal inverse Gaussian (NIG) process with parameters $\alpha, \beta, 0, \delta$. (A NIG$(\alpha, \beta, \mu, \delta)$ process is obtained by adding the drift μt to this process.)

Example 3.11. (The *VG* process.) *Let B be a Brownian motion with a drift and τ be an independent gamma process (i.e. τ_1 has gamma distribution). Then, by the definition, $B \circ \tau$ is a variance gamma (VG) process.*

Definition 3.12. A Lévy process X *has the characteristics* $(b,c,\nu)_h$ if

$$\mathsf{E}e^{i\lambda X_t} = \exp\bigg\{t\bigg[i\lambda b \Leftrightarrow \frac{\lambda^2}{2}c + \int_{\mathbb{R}}\big(e^{i\lambda x} \Leftrightarrow 1 \Leftrightarrow i\lambda h(x)\big)\nu(dx)\bigg]\bigg\}.$$

Here, h is a truncation function. (We will also take $h=0$ if $\int_{\mathbb{R}}|x|\wedge 1\,\nu(dx)<\infty$.) In what follows, H denotes the "canonical" truncation function, i.e.

$$H(x) = xI(|x|\le 1). \tag{3.2}$$

Lemma 3.13. (Transformation of the characteristics under subordination.) *Let B be a Brownian motion and τ be an independent subordinator with the characteristics $(\beta,0,\eta)_0$. Then $B\circ\tau$ has the characteristics $(0,\beta,\nu)_H$, where*

$$\nu = \int_0^\infty \mathsf{Q}_z\eta(dz),$$

i.e. for any $A\in\mathcal{B}(\mathbb{R})$, $\nu(A)=\int_0^\infty \mathsf{Q}_z(A)\eta(dz)$. Here, Q_z denotes the normal distribution with the zero mean and with the variance x.

`Proof`. For any $\lambda\in\mathbb{R}$, we have

$$\begin{aligned}
&\mathsf{E}e^{i\lambda(B\circ\tau)_1} = \mathsf{E}e^{-\frac{\lambda^2}{2}\tau_1}\\
&= \exp\bigg\{\Leftrightarrow\frac{\lambda^2}{2}\beta + \int_0^\infty\big(e^{-\frac{\lambda^2}{2}z}\Leftrightarrow 1\big)\eta(dz)\bigg\}\\
&= \exp\bigg\{\Leftrightarrow\frac{\lambda^2}{2}\beta + \int_0^\infty\int_{\mathbb{R}}\big(e^{i\lambda x}\Leftrightarrow 1\big)\mathsf{Q}_z(dx)\eta(dz)\bigg\}\\
&= \exp\bigg\{\Leftrightarrow\frac{\lambda^2}{2}\beta + \int_0^\infty\int_{\mathbb{R}}\big(e^{i\lambda x}\Leftrightarrow 1\Leftrightarrow i\lambda xI(|x|\le 1)\big)\mathsf{Q}_z(dx)\eta(dz)\bigg\}\\
&= \exp\bigg\{\Leftrightarrow\frac{\lambda^2}{2}\beta + \int_{\mathbb{R}}\big(e^{i\lambda x}\Leftrightarrow 1\Leftrightarrow i\lambda xI(|x|\le 1)\big)\nu(dx)\bigg\}.
\end{aligned}$$

In the first equality, we have used Fubini's theorem (compare with (3.1)); in the last equality, we have used the estimates

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}}\big|e^{i\lambda x}\Leftrightarrow 1\Leftrightarrow i\lambda xI(|x|\le 1)\big|\mathsf{Q}_z(dx)\eta(dz)\\
&\le c_\lambda\int_0^\infty\int_{\mathbb{R}} x^2\wedge 1\,\mathsf{Q}_z(dx)\eta(dz)\le c_\lambda\int_0^\infty x\wedge 1\,\eta(dx)<\infty.
\end{aligned}$$

□

`Remark`. An extension of this lemma to general Lévy processes instead of a Brownian motion can be found in [22; Theorem 30.1]. □

Example 3.14. (Lévy measure of the VG process.) *Let B be a Brownian motion and τ be an independent gamma process with parameters λ,θ, i.e. τ_1 has the density*

$$\frac{x^{\lambda-1}e^{-x/\theta}}{\theta^\lambda,\,(\lambda)}I(x>0).$$

Then $B \circ \tau$ *has the characteristics* $(0,0,\nu)_H$, *where*

$$\frac{\nu(dx)}{dx} = \frac{\lambda}{|x|} e^{-\sqrt{2\theta}|x|}.$$

Proof. The process τ has the characteristics $(0,0,\eta)_0$, where

$$\frac{\eta(dx)}{dx} = \frac{\lambda e^{-\theta x} I(x>0)}{x}$$

(see [22; Example 8.10]). By Lemma 3.13, $B \circ \tau$ has the characteristics $(0,0,\nu)_H$, where

$$\begin{aligned} \frac{\nu(dx)}{dx} &= \int_0^\infty \frac{\lambda}{\sqrt{2\pi z}} \exp\left\{ \Leftrightarrow \frac{x^2}{2z} \right\} \frac{e^{-\theta z}}{z} dz \\ &= \frac{2\lambda}{\sqrt{2\pi}} \int_0^\infty \exp\left\{ \Leftrightarrow \frac{x^2 y^2}{2} \Leftrightarrow \frac{\theta}{y^2} \right\} dy = \frac{\lambda}{|x|} e^{-\sqrt{2\theta}|x|}. \end{aligned}$$

In the last equality, we have applied [12; (3.325)]. □

We now turn to the solution of the main problem in Setting V.

Definition 3.15. A function $\varphi : (0,\infty) \to (0,\infty)$ is *completely monotone* if φ is infinitely differentiable, $\varphi' < 0$ on $(0,\infty)$, $\varphi'' > 0$ on $(0,\infty)$ $\varphi''' < 0$ on $(0,\infty)$, and so on.

Proposition 3.16. (Bernstein.) *A function* $\varphi : (0,\infty) \to (0,\infty)$ *is completely monotone if and only if there exists a positive (not necessarily finite) measure* μ *on* $[0,\infty)$ *such that*

$$\varphi(x) = \int_0^\infty e^{-xy} \mu(dy), \quad x > 0. \tag{3.3}$$

For the proof, see [10; Ch. XIII, §4].

Remark. A measure μ satisfying (3.3) is unique (see [10; Ch. XIII, §1]). □

Theorem 3.17. (a) *Let* X *be a Lévy process with the characteristics* $(b,c,\nu)_H$. *Then* X *can be represented as* $X \overset{\text{law}}{=} B \circ \tau$ *in the sense of Setting V if and only if*

(i) $b = 0$;

(ii) ν *is symmetric, absolutely continuous with respect to the Lebesgue measure, and the density* $q(x) = \frac{\nu(dx)}{dx}$ *can be chosen in such a way that* $q(\sqrt{x})$, $x > 0$ *is completely monotone.*

(b) *Suppose that conditions* (i) *and* (ii) *are satisfied. Then there exists a unique positive measure* μ *on* $(0,\infty)$ *such that*

$$q(\sqrt{x}) = \int_0^\infty e^{-xy} \mu(dy), \quad x > 0.$$

Then a subordinator τ *satisfying the condition* $X \overset{\text{law}}{=} B \circ \tau$ *should have the characteristics* $(\beta, 0, \eta)_0$, *where*

$$\beta = c, \quad \eta(dx) = \sqrt{2\pi x}(\mu \circ \theta^{-1})(dx).$$

Here, $\theta : x \mapsto \frac{1}{2x}$.

`Remark.` The distribution of τ is determined uniquely by the distribution of X (see Exercise 3.24). □

`Proof of Theorem 3.17.` **(a)** We first make the following observation. If η is a positive measure on $(0,\infty)$ and the measure

$$\nu := \int_0^\infty \mathsf{Q}_z \eta(dz)$$

satisfies

$$\int_{\mathbb{R}} x^2 \wedge 1\, \nu(dx) < \infty,$$

then

$$\int_0^\infty x \wedge 1\, \eta(dx) < \infty.$$

In order to prove this, we write

$$\int_{\mathbb{R}} x^2 \wedge 1\, \nu(dx) = \int_0^\infty \int_{\mathbb{R}} x^2 \wedge 1\, \mathsf{Q}_z(dx)\eta(dz) = \int_0^\infty \varphi(z)dz$$

and note that

$$\varphi(z) \sim \begin{cases} z, & z \to 0, \\ 1, & z \to \infty. \end{cases}$$

We now proceed as follows:

X can be represented as $X \overset{\text{law}}{=} B \circ \tau$

$\iff b = 0$ and $\nu = \int_0^\infty \mathsf{Q}_z \eta(dz)$, where η is a positive measure on $(0,\infty)$ such that $\int_0^\infty x \wedge 1\, \eta(dx) < \infty$

$\iff b = 0$ and $\nu = \int_0^\infty \mathsf{Q}_z \eta(dz)$, where η is a positive measure on $(0,\infty)$

$\iff b = 0$ and ν has the density $q(x) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{x^2}{2z}} \eta(dz)$, where η is a positive measure on $(0,\infty)$

$\iff b = 0$ and ν has a symmetric density $q(x)$ such that, for $x > 0$, $q(\sqrt{x}) = \int_0^\infty e^{-xy}\mu(dy)$, where μ is a positive measure on $(0,\infty)$

$\iff b = 0$ and ν has a symmetric density $q(x)$ such that $q(\sqrt{x})$, $x > 0$ is completely monotone.

In the first equivalence we have applied Lemma 3.13; in the last equivalence we have used Proposition 3.16 and the fact that a measure μ satisfying the equality $q(\sqrt{x}) = \int_0^\infty e^{-xy}\mu(dy)$ has no mass at zero (otherwise $\nu([1,\infty))$ would be infinite).

(b) This statement easily follows from the reasoning above. □

`Remark.` The above proof also shows that a probability distribution on $\mathbb{R}$ can be represented as a mixture of the normal distributions with the zero mean if and only if it is symmetric and admits a density $q(x)$ such that $q(\sqrt{x})$, $x > 0$ is completely monotone. □

Lemma 3.18. **(a)** *If φ and ψ are completely monotone, then $\varphi\psi$ is completely monotone.*

(b) *If φ is completely monotone and $\psi : (0, \infty) \to (0, \infty)$ has a completely monotone derivative, then $\varphi \circ \psi$ is completely monotone.*

`Proof.` **(a)** This is a direct consequence of the equality

$$(\varphi\psi)^{(n)} = \varphi^{(n)}\psi + C_n^1 \varphi^{(n-1)}\psi^{(1)} + \dots + C_n^{n-1}\varphi^{(1)}\psi^{(n-1)} + \varphi\psi^{(n)}.$$

Here, $\varphi^{(n)}$ denotes the n-th derivative of φ.

(b) The proof is Exercise 3.36. □

Definition 3.19. A *CGMY* process is a Lévy process with the zero diffusion coefficient and with the Lévy measure

$$\frac{\nu(dx)}{dx} = \frac{Ce^{-G|x|}I(x<0) + Ce^{-M|x|}I(x>0)}{|x|^{Y+1}}.$$

Here, $C > 0$, $G > 0$, $M > 0$, $Y < 2$.

Theorem 3.17 and Lemma 3.18 yield

Corollary 3.20. *A CGMY process X with $G = M$, $0 \le Y < 2$ can be represented as $X \overset{\text{law}}{=} B \circ \tau$ in the sense of Setting V.*

3.3 Exercises

Exercise 3.21. (On $(\mathcal{F}_t)$-Brownian motions.) Let B be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and $(\mathcal{F}_t)$ be a filtration on this space. Prove that the following conditions are equivalent:

(i) B is an $(\mathcal{F}_t)$-local martingale;

(ii) B is an $(\mathcal{F}_t)$-martingale;

(iii) B is $(\mathcal{F}_t)$-adapted, and, for any $s \le t$, the increment $B_t \Leftrightarrow B_s$ is independent of $\mathcal{F}_s$.

(`Hint`: Use Exercise 4.9.)

○ **Exercise 3.22.** *Let B be a Brownian motion. Construct an $(\mathcal{F}_t^B)$-time-change τ such that $(B \circ \tau)_t = t$, $t \ge 0$.*

Exercise 3.23. Let B be an $(\mathcal{F}_t)$-Brownian motion and H be an $(\mathcal{F}_t)$-predictable process such that

$$\forall t \ge 0, \quad \int_0^t H_s^2 ds < \infty \quad \text{a.s.}$$

Set

$$M_t = \exp\biggl\{\int_0^t H_s dB_s \Leftrightarrow \frac{1}{2}\int_0^t H_s^2 ds\biggr\}, \quad t \ge 0.$$

Prove that

$$\Bigl\{\lim_{t\to\infty} M_t = 0\Bigr\} = \biggl\{\int_0^\infty H_s^2 ds = \infty\biggr\} \quad \text{a.s.},$$

$$\Bigl\{\lim_{t\to\infty} M_t \in (0,\infty)\Bigr\} = \biggl\{\int_0^\infty H_s^2 ds = \infty\biggr\} \quad \text{a.s.}$$

\# **Exercise 3.24. (Revealing the law of the time-change.)** Let B be a Brownian motion and τ, $\widetilde{\tau}$ be increasing càdlàg processes that are independent of B. Supppose that $B\circ\tau \stackrel{\text{law}}{=} B\circ\widetilde{\tau}$. Prove that $\tau \stackrel{\text{law}}{=} \widetilde{\tau}$.

Example 3.25. Give an example of a Brownian motion B and an independent increasing càdlàg process τ such that $B\circ\tau$ is not a local martingale.

○ **Exercise 3.26.** Give an example of a continuous local martingale started at zero that is not an Ocone martingale.

○ **Exercise 3.27.** Give an example of an Ocone martingale that is not a martingale.

\# **Exercise 3.28. (Determining the law of a martingale by the law of its bracket.)** **(a)** Give an example of two continuous local martingales X and Y started at zero such that $\langle X\rangle \stackrel{\text{law}}{=} \langle Y\rangle$, but X and Y have different laws.

(b) Let X and Y be two Ocone martingales such that $\langle X\rangle \stackrel{\text{law}}{=} \langle Y\rangle$. Prove that $X \stackrel{\text{law}}{=} Y$.

Exercise 3.29. Let X be an Ocone martingale and H be an $(\mathcal{F}_t^X)$-predictable process with $|H|=1$. Prove that

$$\left(\int_0^t H_s dX_s, \langle X\rangle_t;\, t\geq 0\right) \stackrel{\text{law}}{=} (X_t, \langle X\rangle_t;\, t\geq 0).$$

Exercise 3.30. (An extension of P. Lévy's theorem to Ocone martingales.) Let X be an Ocone martingale. Set $S_t=\sup_{u\leq t} X_u$ and let L denote the local time of X at zero. Prove that

$$(S - X, S) \stackrel{\text{law}}{=} (|X|, L).$$

(`Hint:` Use P. Lévy's theorem; see [21; Ch. VI, Theorem 2.3].)

* **Exercise 3.31.** Prove the implication (i)$\Rightarrow$(iii) in Proposition 3.6.

\# **Exercise 3.32.** Prove Lemma 3.8.

\# **Exercise 3.33. (Change of time for a Brownian motion with a drift.)** Let B^α be a Brownian motion with a drift α and let τ be an independent subordinator with the characteristics $(\beta,0,\eta)_0$. Prove that the diffusion coefficient of $B^\alpha\circ\tau$ equals α and the Lévy measure of $B^\alpha\circ\tau$ equals $\int_0^\infty \mathsf{Q}_z\eta(dz)$, where Q_z is the normal distribution with the mean αz and the variance z.

Exercise 3.34. (Lévy measure of the *VG* process.) Let B^α be a Brownian motion with a drift α and τ be an independent gamma process with parameters λ,θ (see Example 3.14). Prove that the Lévy measure of $B^\alpha\circ\tau$ is given by

$$\frac{\nu(dx)}{dx}=\frac{\lambda}{|x|}e^{-(\sqrt{\alpha^2+2\theta}+\alpha)|x|}I(x<0)+\frac{\lambda}{|x|}e^{-(\sqrt{\alpha^2+2\theta}-\alpha)|x|}I(x>0).$$

(`Hint:` Use Exercise 3.33.)

\# **Exercise 3.35.** Let X be a Lévy process with the characteristics $(b,c,\nu)_H$. Prove that X can be represented as

$$(X_t;\, t \geq 0) \stackrel{\text{law}}{=} (\gamma t + (B^\alpha \circ \tau)_t;\, t \geq 0),$$

where $\alpha, \gamma \in \mathbb{R}$, B^α is a Brownian motion with a drift α and τ is an independent subordinator, if and only if ν admits the density $e^{\alpha x} q(x)$, where $q(x)$ is symmetric and $q(\sqrt{x})$, $x > 0$ is completely monotone.

\# **Exercise 3.36.** Prove Lemma 3.18 (b).

\# **Exercise 3.37.** Let $(\Lambda, \mathcal{A}, \nu)$ be a measurable space with a positive measure ν. Let $f(\lambda, x)$ be an $\mathcal{A} \times \mathcal{B}(\mathbb{R}_+)$-measurable function such that, for any $\lambda \in \Lambda$, the function $f(\lambda, \cdot)$ is completely monotone.

(a) Let μ_λ denote the (unique) measure such that

$$f(\lambda, x) = \int_0^\infty e^{-xy} \mu_\lambda(dy), \quad x > 0.$$

Prove that, for any $A \in \mathcal{B}(\mathbb{R}_+)$, the map $\lambda \mapsto \mu_\lambda(A)$ is $\mathcal{A}$-measurable.

(b) Suppose that the function

$$f(x) := \int_0^\infty f(\lambda, x) \nu(d\lambda)$$

is finite for any $x > 0$. Prove that f is completely monotone.

$\circ$# **Exercise 3.38. (Generalized hyperbolic distributions.)** The *generalized hyperbolic distribution* (*GHYP*) with parameters $\lambda, \alpha, \beta, \delta, \mu$ (here, $\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta \in (\Leftrightarrow\alpha, \alpha)$, $\delta > 0$, $\mu \in \mathbb{R}$) is known to be infinitely divisible with the Lévy measure

$$\frac{\nu(dx)}{dx} = \begin{cases} \dfrac{e^{\beta x}}{|x|}\left(\displaystyle\int_0^\infty \frac{\exp\{\Leftrightarrow\sqrt{2y+\alpha^2}|x|\}}{\pi^2 y (J_\lambda^2(\delta\sqrt{2y}) + Y_\lambda^2(\delta\sqrt{2y}))}\, dy + \lambda e^{-\alpha|x|}\right) & \text{if } \lambda \geq 0, \\ \dfrac{e^{\beta x}}{|x|}\displaystyle\int_0^\infty \frac{\exp\{\Leftrightarrow\sqrt{2y+\alpha^2}|x|\}}{\pi^2 y (J_{-\lambda}^2(\delta\sqrt{2y}) + Y_{-\lambda}^2(\delta\sqrt{2y}))}\, dy & \text{if } \lambda < 0 \end{cases} \tag{3.4}$$

(see [7; (3.10),(3.11)]). Here, J_λ and Y_λ are the Bessel functions of the first and the second kind, respectively. This class of distributions includes *NIG* and *HYP* as particular cases.

(a) Prove that a symmetric generalized hyperbolic Lévy process X can be represented as $X \stackrel{\text{law}}{=} B \circ \tau$ in the sense of Setting V.

(`Hint:` Use Exercise 3.37.)

(b) Prove that any generalized hyperbolic Lévy process X can be represented as

$$(X_t;\, t \geq 0) \stackrel{\text{law}}{=} (\gamma t + (B^\beta \circ \tau)_t;\, t \geq 0),$$

where $\gamma \in \mathbb{R}$, B^β is a Brownian motion with a drift β and τ is an independent subordinator.

(`Hint:` Use Exercises 3.35 and 3.37.)

Exercise 3.39. Let B be a Brownian motion and τ be an independent subordinator. Prove that the following conditions are equivalent:

(i) $B \circ \tau$ is a σ-martingale (see Definition A.9);

(ii) $B \circ \tau$ is a martingale;

(iii) for any $t \geq 0$, $\mathsf{E}\sqrt{\tau_t} < \infty$;

(iv) the Lévy measure η of τ satisfies $\int_1^\infty \sqrt{x}\eta dx < \infty$.

(`Hint:` For the implication (i)$\Rightarrow$(ii), use Exercise 4.22. For the equivalence (iii)$\Leftrightarrow$(iv), use Proposition A.2.)

Exercise 3.40. Let B be a Brownian motion and τ be an independent subordinator. Prove that the following conditions are equivalent:

(i) $B \circ \tau$ is a process of finite variation;

(ii) the Lévy measure η of τ satisfies $\int_0^1 \sqrt{x}\eta(dx) < \infty$.

(`Hint:` Use Proposition A.6.)

An open problem is

Problem 3.41. Which processes X can be represented as $X \overset{\text{law}}{=} B \circ \tau$ in the sense of Setting III?

4 Change of Measure for Lévy Processes

4.1 General Theorems

Throughout this section, T is a fixed positive number meaning the time horizon.

Definition 4.1. Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathsf{P}\big)$ be a filtered probability space. A process $(X_t)_{t\le T}$ is called an $(\mathcal{F}_t)$-*Lévy process* if X is a Lévy process, X is $(\mathcal{F}_t)$-adapted, and, for any $s \le t$, the increment $X_t \Leftrightarrow X_s$ is independent of $\mathcal{F}_s$.

Lemma 4.2. *If $(X_t)_{t\le T}$ is an $(\mathcal{F}_t)$-Lévy process, then, for any $s \le T$, the σ-field $\sigma(X_t \Leftrightarrow X_s;\, t \in [s,T])$ is independent of $\mathcal{F}_s$.*

The proof is Exercise 4.10.

Lemma 4.3. (Change of measure for compound Poisson processes.) *Let $(X_t)_{t\le T}$ be an $(\mathcal{F}_t, \mathsf{P})$-Lévy process with the characteristics $(0,0,\nu)_0$, where ν is a finite measure (i.e. X is a compound Poisson process). Let $\widetilde{\nu}$ be a finite measure such that $\widetilde{\nu} \ll \nu$. Consider*

$$M_t = \exp\bigg\{ t(\nu(\mathbb{R}) \Leftrightarrow \widetilde{\nu}(\mathbb{R})) + \sum_{s\le t} \ln \rho(\Delta X_s) \bigg\}, \quad t \le T,$$

where $\rho = \frac{d\widetilde{\nu}}{d\nu}$ (we set $\rho(0) = 0$). Then M is an $(\mathcal{F}_t, \mathsf{P})$-martingale. If we set $\widetilde{\mathsf{P}} = M_T \mathsf{P}$, then X is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process with the characteristics $(0,0,\widetilde{\nu})_0$.

`Proof`. As $\nu(\mathbb{R}) < \infty$, the process X has a.s. only a finite number of jumps, and thus, M is defined correctly. For any $s \le t$, we have

$$\begin{aligned}\mathsf{E}(M_t \mid \mathcal{F}_s) &= M_s \exp\{(t \Leftrightarrow s)(\nu(\mathbb{R}) \Leftrightarrow \widetilde{\nu}(\mathbb{R}))\}\, \mathsf{E} \prod_{s<r\le t} \rho(\Delta X_r) \\ &= M_s \exp\{\Leftrightarrow (t \Leftrightarrow s)\widetilde{\nu}(\mathbb{R})\} \sum_{k=0}^{\infty} \frac{((t \Leftrightarrow s)\nu(\mathbb{R}))^k}{k!} \bigg(\int_{\mathbb{R}} \frac{\rho(x)}{\nu(\mathbb{R})} \nu(dx) \bigg)^k = M_s.\end{aligned}$$

In the first equality, we have applied Lemma 4.2.

In order to prove that X is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process with the prescribed characteristics, it is sufficient to note that, for any $s \le t$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathsf{E}_{\widetilde{\mathsf{P}}^u}\big(e^{i\lambda(X_t - X_s)} \,\big|\, \mathcal{F}_s\big) &= \mathsf{E}\bigg(e^{i\lambda(X_t - X_s)} \frac{M_t}{M_s} \,\bigg|\, \mathcal{F}_s \bigg) \\ &= \mathsf{E} \exp\bigg\{ \sum_{s<r\le t} i\lambda \Delta X_r + (t \Leftrightarrow s)(\nu(\mathbb{R}) \Leftrightarrow \widetilde{\nu}(\mathbb{R})) + \sum_{s<r\le t} \ln \rho(\Delta X_r) \bigg\} \\ &= \exp\{\Leftrightarrow (t \Leftrightarrow s)\widetilde{\nu}(\mathbb{R})\} \sum_{k=0}^{\infty} \frac{((t \Leftrightarrow s)\nu(\mathbb{R}))^k}{k!} \bigg(\int_{\mathbb{R}} \frac{e^{i\lambda x + \ln \rho(x)}}{\nu(\mathbb{R})} \nu(dx) \bigg)^k \\ &= \exp\bigg\{ \Leftrightarrow (t \Leftrightarrow s)\widetilde{\nu}(\mathbb{R}) + (t \Leftrightarrow s) \int_{\mathbb{R}} e^{i\lambda x + \ln \rho(x)} \nu(dx) \bigg\} \\ &= \exp\bigg\{ (t \Leftrightarrow s) \int_{\mathbb{R}} (e^{i\lambda x} \Leftrightarrow 1) \widetilde{\nu}(dx) \bigg\}.\end{aligned}$$

In the first equality, we have used the Bayes formula; in the second equality, we have applied Lemma 4.2. Consequently, $X_t \Leftrightarrow X_s$ is $\widetilde{\mathsf{P}}$-independent of $\mathcal{F}_s$ (see Exercise 4.9) and has the prescribed characteristic function. □

Proposition 4.4. (Change of measure for Lévy processes.) *Let* $(X_t)_{t\le T}$ *and* $(\widetilde{X}_t)_{t\le T}$ *be Lévy processes with the characteristics* $(b,c,\nu)_H$ *and* $(\widetilde{b},\widetilde{c},\widetilde{\nu})_H$*, respectively. Then* $\text{Law}(X_t;\, t\le T) \sim \text{Law}(\widetilde{X}_t;\, t\le T)$ *if and only if the following conditions are satisfied:*

(i) *either* $c>0$*, or* $c=0$ *and*

$$\widetilde{b} = b + \int_{\{|x|\le 1\}} x(\nu \Leftrightarrow \widetilde{\nu})(dx); \tag{4.1}$$

(ii) $\widetilde{c} = c$;

(iii) $\widetilde{\nu} \sim \nu$ *and*

$$\int_{\mathbb{R}} \bigl(\sqrt{\rho(x)} \Leftrightarrow 1\bigr)^2 \nu(dx) < \infty, \tag{4.2}$$

where $\rho = \frac{d\widetilde{\nu}}{d\nu}$.

For the proof, see [22; Theorem 33.1].

`Remark`. Condition (4.2) guarantees that

$$\int_{\{|x|\le 1\}} |x| d\text{Var}(\nu \Leftrightarrow \widetilde{\nu})(dx) < \infty,$$

and thus, the rigth-hand side of (4.2) is defined correctly (see Exercise 4.12). □

Proposition 4.5. *Let* $(X_t)_{t\le T}$ *and* $(\widetilde{X}_t)_{t\le T}$ *be Lévy processes with the characteristics* $(b,c,\nu)_H$ *and* $(\widetilde{b},\widetilde{c},\widetilde{\nu})_H$*, respectively. Suppose that* $\widetilde{\nu}\sim\nu$*. Then the distributions* $\text{Law}(X_t;\, t\le T)$ *and* $\text{Law}(\widetilde{X}_t;\, t\le T)$ *are either equivalent or singular.*

For the proof, see [2].

4.2 No Arbitrage and Completeness of Exponential Lévy Models

Let $\bigl(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\le T},\mathsf{P}\bigr)$ be a filtered probability space. Let $(X_t)_{t\le T}$ be an $(\mathcal{F}_t)$-Lévy process. Consider an *exponential Lévy model* for the discounted stock price:

$$S_t = S_0 e^{X_t}, \quad t\le T, \tag{4.3}$$

where $S_0>0$.

In the following theorems we exclude the trivial case $X\equiv 0$.

Theorem 4.6. (No arbitrage.) (a) *Model* (4.3) *does not satisfy the* (*NFLVR*) *condition* (*see Definition A.12*) *only in the following cases:*

(i) S *is increasing;*

(ii) S *is decreasing.*

(b) *Moreover, if the* (*NFLVR*) *condition is satisfied, then there exists a measure* $\widetilde{\mathsf{P}}\sim\mathsf{P}$ *such that* S *is an* $(\mathcal{F}_t,\widetilde{\mathsf{P}})$*-martingale and* X *is an* $(\mathcal{F}_t,\widetilde{\mathsf{P}})$*-Lévy process.*

Theorem 4.7. (Completeness.) *Suppose that model* (4.3) *satisfies the* (*NFLVR*) *condition and* $\mathcal{F}_t = \mathcal{F}_t^S$. *Then the model is complete* (*see Definition A.14*) *only in the following cases:*

(i) $X_t = \mu t + \sigma B_t$, *where* B *is a Brownian motion and* $\sigma \neq 0$;

(ii) $X_t = \mu t + \sigma N_{\lambda t}$, *where* N *is a standard Poisson process and* $\mu\sigma < 0$.

`Remarks`. (i) Theorem 4.6 (a) was proved in the paper [15] by P. Jakubėnas. We will give here a different proof.

(ii) M. Yor and J. de Sam Lazaro [26; Appendix] proved the following result. Suppose that $(X_t)_{t\geq 0}$ is a martingale such that, for any $s \geq 0$, $\mathrm{Law}(X_t;\, t \geq 0) = \mathrm{Law}(X_{t+s} \Leftrightarrow X_s;\, t \geq 0)$. Set $\mathcal{F}_t = \mathcal{F}_t^X$. Then any $(\mathcal{F}_t^X)$-local martingale started at zero can be represented as a stochastic integral with respect to X if and only if X is a Brownian motion or a compensated Poisson process (compare with Exercise 4.24). □

In what follows, we use the notation

$$H_a(x) = xI(|x| \leq a), \tag{4.4}$$

where a is a positive real number.

The expectation sign with no subscript will always mean the expectation with respect to the original measure P.

Lemma 4.8. *Let* $a > 0$ *and* $(X_t)_{t\leq T}$ *be an* $(\mathcal{F}_t)$-*Lévy process with the characteristics* $(b, c, \nu)_{H_a}$. *Let* $\widetilde{\nu}$ *be a positive measure such that* $\widetilde{\nu} = \nu$ *on* $\{|x| \leq a\}$, $\widetilde{\nu} \sim \nu$ *on* $\{|x| > a\}$, *and* $\widetilde{\nu}(\{|x| > a\}) < \infty$. *Then there exists a measure* $\widetilde{\mathsf{P}} \sim \mathsf{P}$ *such that* X *is an* $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-*Lévy process with the characteristics* $(b, c, \widetilde{\nu})_{H_a}$.

`Proof`. Consider the processes

$$X_t^1 = \sum_{s\leq t} \Delta X_s I(|\Delta X_s| > a), \quad X_t^2 = X_t \Leftrightarrow X_t^1, \quad t \leq T.$$

Then X^1 and X^2 are independent Lévy processes with the characteristics $\left(0, 0, \nu|_{\{|x|>a\}}\right)_0$ and $\left(b, c, \nu|_{\{|x|\leq a\}}\right)_{H_a}$, respectively. Moreover, the two-dimensional process (X^1, X^2) is an $(\mathcal{F}_t)$-Lévy process (see Exercise 4.20).

Set

$$M_t = \exp\left\{t(\nu(\{|x| > a\}) \Leftrightarrow \widetilde{\nu}(\{|x| > a\})) + \sum_{s\leq t} \ln \rho(\Delta X_s^1)\right\}, \quad t \leq T,$$

where $\rho = \frac{d\widetilde{\nu}}{d\nu}$. Similarly to the proof of Lemma 4.3, we check that M is an $(\mathcal{F}_t, \mathsf{P})$-martingale. Consider $\widetilde{\mathsf{P}} = M_T \mathsf{P}$. Then, by Lemma 4.3, X^1 is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process with the characteristics $\left(0,0,\widetilde{\nu}|_{\{|x|>a\}}\right)_0$. Furthermore, for any $s \le t$, $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathsf{E}\big(e^{i\lambda(X_t-X_s)} \mid \mathcal{F}_s\big) = \mathsf{E}\left(e^{i\lambda(X_t-X_s)}\frac{M_t}{M_s}\,\middle|\, \mathcal{F}_s\right)\\
&= \mathsf{E}e^{i\lambda(X_t-X_s)}\frac{M_t}{M_s} = \mathsf{E}e^{i\lambda(X_t^1-X_s^1)}\frac{M_t}{M_s}\,\mathsf{E}e^{i\lambda(X_t^2-X_s^2)}\\
&= \exp\left\{(t-s)\int_{\{|x|>a\}}\big(e^{i\lambda x}-1\big)\widetilde{\nu}(dx)\right\} \qquad (4.5)\\
&\quad\times \exp\left\{(t-s)\left[i\lambda b-\frac{\lambda^2}{2}c+\int_{\{|x|\le a\}}\big(e^{i\lambda x}-1-i\lambda x\big)\nu(dx)\right]\right\}\\
&= \exp\left\{(t-s)\left[i\lambda b-\frac{\lambda^2}{2}c+\int_{\mathbb{R}}\big(e^{i\lambda x}-1-i\lambda H_a(x)\big)\nu(dx)\right]\right\}.
\end{aligned}$$

Consequently, X_t-X_s is $\widetilde{\mathsf{P}}$-independent of $\mathcal{F}_s$ (see Exercise 4.9) and has the prescribed characteristic function. □

`Proof of Theorem 4.6.` Throughout the proof, c denotes the diffusion coefficient of X and ν denotes the Lévy measure of X. We will prove the theorem by considering several cases.

Case I. Suppose that there exists $a>0$ such that $\nu((-\infty,-a))>0$ and $\nu((a,\infty))>0$. Let b be the first characteristic of X with respect to H_a. There exists a positive measure $\widetilde{\nu}$ such that

$$\widetilde{\nu}=\nu \text{ on } \{|x|\le a\}, \qquad (4.6)$$

$$\widetilde{\nu}\sim\nu \text{ on } \{|x|>a\}, \qquad (4.7)$$

$$\widetilde{\nu}(\{|x|>a\})<\infty, \qquad (4.8)$$

$$\int_{\{|x|>a\}} e^x\widetilde{\nu}(dx)<\infty, \qquad (4.9)$$

$$b+\frac{c}{2}+\int_{\mathbb{R}}\big(e^x-1-xI(|x|\le a)\big)\widetilde{\nu}(dx)=0. \qquad (4.10)$$

In order to construct such a measure, it is sufficient to take first a rapidly decreasing at infinity function $\overline{\rho}$ such that $\overline{\rho}>0$, $\overline{\rho}=1$ on $[-a,a]$, and the measure $\overline{\nu}=\overline{\rho}\nu$ satisfies conditions (4.6)–(4.9). Then, using the density of the form $\widetilde{\rho}=\alpha I(x\le a)+\beta I(x>a)$ with $\alpha,\beta>0$, one can construct a measure $\widetilde{\nu}=\widetilde{\rho}\overline{\nu}$ that satisfies conditions (4.6)–(4.10). By Lemma 4.8, there exists a measure $\widetilde{\mathsf{P}}\sim\mathsf{P}$ such that X is an $(\mathcal{F}_t,\widetilde{\mathsf{P}})$-Lévy process with the characteristics $(b,c,\widetilde{\nu})_{H_a}$. It follows from Proposition A.5 that $\mathsf{E}_{\widetilde{\mathsf{P}}}e^{X_t}=1$, $t\le T$. Now, it is easy to see that e^X is an $(\mathcal{F}_t,\widetilde{\mathsf{P}})$-martingale.

Case II. Suppose that ν is concentrated on $(0,\infty)$ and $\int_0^1 x\nu(dx)=\infty$. Let $b(a)$ denote the first characteristic of X with respect to H_a. Then, for $a\in(0,1)$,

$$b(a)=b(1)-\int_{\{a<x\le 1\}}x\nu(dx),$$

and, due to the condition $\int_0^1 x\nu(dx)=\infty$, we can choose $a>0$ such that $\nu(\{x>a\})>0$ and

$$b(a)+\frac{c}{2}+\int_{\{0<x\le a\}}\big(e^x-1-x\big)\nu(dx)<0.$$

Obviously, there exists a positive measure $\widetilde{\nu}$ that satisfies conditions (4.6)–(4.10). We now proceed as in Case I.

Case III. Suppose that ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx)<\infty$, and $c>0$. By the Lévy-Itô decomposition (see Proposition A.7), X can be represented as $X=X^1+X^2$, where X^1 and X^2 are independent Lévy processes with the characteristics $(0,0,\nu)_H$ and $(b,c,0)_H$, respectively. (Here, b denotes the first characteristic of X with respect to H.) The two-dimensional process (X^1,X^2) is an $(\mathcal{F}_t)$-Lévy process (see Exercise 4.19). There exists a positive measure $\widetilde{\nu}$ that satisfies conditions (4.6)–(4.9) with $a=1$. Take

$$\widetilde{b}=-\frac{c}{2}-\int_{\mathbb{R}}(e^x-1-H(x))\nu(dx)$$

and set

$$M_t=\exp\Big\{t(\nu(\{|x|>1\}-\widetilde{\nu}(\{|x|>1\}))+\sum_{s\le t}\ln\rho(\Delta X_s^1)\Big\}$$
$$\times\exp\Big\{\frac{\widetilde{b}-b}{\sigma^2}X_t^2-\frac{(\widetilde{b}-b)^2}{2\sigma^2}t\Big\},\quad t\le T,$$

where $\rho=\frac{d\widetilde{\nu}}{d\nu}$. Computations similar to (4.5) show that M is an $(\mathcal{F}_t,\mathsf{P})$-martingale, and X is an $(\mathcal{F}_t,\widetilde{\mathsf{P}})$-Lévy process with the characteristics $(\widetilde{b},c,\widetilde{\nu})_H$, where $\widetilde{\mathsf{P}}=M_T\mathsf{P}$. We now proceed as in Case I.

Case IV. Suppose that ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx)<\infty$, $c=0$, and $b<0$, where b is the first characteristic of X with respect to the zero truncation function. The first characteristic $b(a)$ with respect to H_a is given by

$$b(a)=b+\int_{\{0<x\le a\}}x\nu(dx).$$

We can find $a>0$ such that $\nu((a,\infty))>0$ and

$$b(a)+\int_{\{0<x\le a\}}(e^x-1-x)\nu(dx)<0.$$

We now proceed as in Case I.

Case V. Suppose that ν is concentrated on $(0,\infty)$, $\int_0^1 x\nu(dx)<\infty$, $c=0$, and $b\ge 0$, where b is the first characteristic of X with respect to the zero truncation function. In this case X is a subordinator (see [22; Theorem 21.5]), and hence, S is increasing.

Case VI. Suppose that $\nu=0$. In this case the desired statement follows from Girsanov's theorem.

The cases, where ν is concentrated on $(-\infty,0)$, are considered similarly to Cases II–V. □

`Proof of Theorem 4.7.` *Part I.* Let us first prove that in cases (i), (ii) the model is complete.

In case (i) this is the Black-Scholes-Samuelson model, and its completeness is widely known.

In case (ii) we have, by Itô's formula (see [14; Ch. I, Theorem 4.57]),

$$\begin{aligned} S_t &= S_0 + \int_0^t e^{X_{s-}} dX_s + \sum_{s\le t} \left(e^{X_s} - e^{X_{s-}} - e^{X_{s-}} \Delta X_s\right) \\ &= S_0 + \mu \int_0^t e^{X_{s-}} ds + \sigma \int_0^t e^{X_{s-}} dN_{\lambda s} + \sum_{s\le t} e^{X_{s-}} \left(e^{\sigma \Delta N_{\lambda s}} - 1 - \sigma \Delta N_{\lambda s}\right) \\ &= S_0 + \mu \int_0^t e^{X_{s-}} ds + \sum_{s\le t} e^{X_{s-}} \left(e^{\sigma \Delta N_{\lambda s}} - 1\right) \\ &= S_0 + \mu \int_0^t e^{X_{s-}} ds + (e^{\sigma} - 1) \int_0^t e^{X_{s-}} dN_{\lambda s} \\ &= S_0 + (e^{\sigma} - 1) \int_0^t e^{X_{s-}} \, d\left(N_{\lambda s} - \frac{\mu}{1 - e^{\sigma}} s\right), \quad t \le T. \end{aligned} \tag{4.11}$$

In view of Lemma 4.3, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that with respect to $\widetilde{\mathsf{P}}$ the process $(N_{\lambda t})_{t\le T}$ is a Poisson process with intensity $\frac{\mu}{1-e^{\sigma}}$ (note that $\frac{\mu}{1-e^{\sigma}} < 0$ since $\mu\sigma < 0$). Then $\frac{\mu}{1-e^{\sigma}} t$ is the $\widetilde{\mathsf{P}}$-compensator of $N_{\lambda t}$, and it is known that any $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingale M (recall that $\mathcal{F}_t = \mathcal{F}_t^S = \mathcal{F}_t^N$) started at zero can be represented as

$$M_t = \int_0^t K_s \, d\left(N_{\lambda s} - \frac{\mu}{1 - e^{\sigma}} s\right)$$

(see [14; Ch. III, Theorem 4.37]). Hence, M can also be represented as a stochastic integral with respect to S. Now, it follows from the Second Fundamental Theorem of Asset Pricing (see Proposition A.15) that the model is complete.

Part II. Let us prove that model (4.3) is complete only in cases (i) and (ii). Let ν denote the Lévy measure of the process X. Suppose that the support of ν contains more than one point. The analysis of the proof of Theorem 4.6 shows that in this case one can construct two different measures $\widetilde{\mathsf{P}}$ and $\widetilde{\mathsf{P}}'$ that are equivalent to P and such that S is a martingale with respect to both of them. So, in this case the model is not complete.

Suppose now that the Lévy measure of X is concentrated at one point. Then X can be represented as

$$X_t = \mu t + \sigma B_t + \delta N_{\lambda t}, \quad t \le T, \tag{4.12}$$

where $\mu, \sigma, \delta \in \mathbb{R}$, $\lambda > 0$, B is a Brownian motion, and N is a standard Poisson process that is independent of B. By Itô's formula (compare with (4.11)),

$$S_t = S_0 + \int_0^t e^{X_{s-}} d\left(\mu s + \frac{\sigma^2}{2} s + \sigma B_s + (e^{\delta} - 1) N_{\lambda s}\right), \quad t \le T. \tag{4.13}$$

Using Girsanov's theorem and Lemma 4.8, we can, for each $a \in \mathbb{R}$, $b > 0$, construct a measure $\widetilde{\mathsf{P}}_{ab} \sim \mathsf{P}$ such that with respect to this measure B is a Brownian motion with the drift a and N is a Poisson process with the intensity b that is independent of B. If

$$\mu + \frac{\sigma^2}{2} + \sigma a + (e^{\delta} - 1)\lambda b = 0, \tag{4.14}$$

then, in view of (4.13), the process S is an $(\mathcal{F}_t, \widetilde{\mathsf{P}}_{ab})$-local martingale.

Suppose that $\sigma, \delta \neq 0$. Then there exist different pairs (a, b) satisfying (4.14) and hence, different equivalent local martingale measures for S. As a result, model (4.3) with X given by (4.12) can be complete only if $\sigma = 0$ or $\delta = 0$. But these are exactly cases (i) and (ii). □

4.3 Exercises

Exercise 4.9. (On conditional characteristic functions.) Let ξ be a d-dimensional random variable on $(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathcal{G} \subseteq \mathcal{F}$. Suppose that

$$\forall \lambda \in \mathbb{R}^d, \ \ \mathsf{E}\big(e^{i\langle\lambda,\xi\rangle} \mid \mathcal{G}\big) = \mathsf{E}e^{i\langle\lambda,\xi\rangle}.$$

Prove that ξ is independent of $\mathcal{G}$.

Exercise 4.10. Prove Lemma 4.2.

(`Hint`: Use Exercise 4.9.)

○ **Exercise 4.11.** Let $(X_t)_{t\le T}$ and $(\widetilde{X}_t)_{t\le T}$ be compound Poisson processes with Lévy measures ν and $\widetilde{\nu}$, respectively. Prove that $\mathrm{Law}(\widetilde{X}_t;\, t \le T) \ll \mathrm{Law}(X_t;\, t \le T)$ if and only if $\widetilde{\nu} \ll \nu$.

○ **Exercise 4.12.** Prove the remark following Proposition 4.4.

Exercise 4.13. (Revealing the drift of a Lévy process.) Let $(X_t)_{t\le T}$ be a Lévy process with the diffusion coefficient c. Let $a \neq 0$.

(a) Prove that if $c \neq 0$, then $\mathrm{Law}(X_t + at;\, t \le T) \sim \mathrm{Law}(X_t;\, t \le T)$.

(b) Prove that if $c = 0$, then $\mathrm{Law}(X_t + at;\, t \le T) \perp \mathrm{Law}(X_t;\, t \le T)$.

Exercise 4.14. (Revealing the time scale of a Lévy process.) Let $(X_t)_{t\ge 0}$ be a Lévy process. Let $a > 0$ and $a \neq 1$.

(a) Prove that if X is a compound Poisson process, then $\mathrm{Law}(X_{at};\, t \le T) \sim \mathrm{Law}(X_t;\, t \le T)$.

(b) Prove that if X is not a compound Poisson process, then $\mathrm{Law}(X_{at};\, t \le T) \perp \mathrm{Law}(X_t;\, t \le T)$.

Exercise 4.15. (Singularity of distributions of Lévy processes with different diffusion coefficients.) Let $(X_t)_{t\le T}$ and $(\widetilde{X}_t)_{t\le T}$ be two Lévy processes with different diffusion coefficients. Prove that $\mathrm{Law}(\widetilde{X}_t;\, t \le T) \perp \mathrm{Law}(X_t;\, t \le T)$.

(`Hint`: Use the Lévy-Itô decomposition; see Proposition A.7.)

Exercise 4.16. (Singularity of distributions of different stable processes.) Let $(X_t)_{t\le T}$ and $(\widetilde{X}_t)_{t\le T}$ be two stable Lévy processes with different distributions. Prove that $\mathrm{Law}(\widetilde{X}_t;\, t \le T) \perp \mathrm{Law}(X_t;\, t \le T)$.

Exercise 4.17. (Singularity of distributions of different Lévy processes.) Let $(X_t)_{t\ge 0}$ and $(\widetilde{X}_t)_{t\ge 0}$ be two Lévy processes with different distributions. Prove that $\mathrm{Law}(\widetilde{X}_t;\, t \ge 0) \perp \mathrm{Law}(X_t;\, t \ge 0)$.

○ **Exercise 4.18.** Give an example of two Lévy processes $(X_t)_{t\le T}$ and $(\widetilde{X}_t)_{t\le T}$ such that neither of the following conditions is satisfied:

$$\begin{aligned}
&\mathrm{Law}(\widetilde{X}_t;\, t\le T) \ll \mathrm{Law}(X_t;\, t\le T),\\
&\mathrm{Law}(X_t;\, t\le T) \ll \mathrm{Law}(\widetilde{X}_t;\, t\le T),\\
&\mathrm{Law}(\widetilde{X}_t;\, t\le T) \perp \mathrm{Law}(X_t;\, t\le T).
\end{aligned}$$

\# **Exercise 4.19.** Let $(X_t)_{t\ge 0}$ be an $(\mathcal{F}_t)$-Lévy process and X^1, X^2 be the processes given by the Lévy-Itô decomposition (see Proposition A.7). Prove that the two-dimensional process (X^1, X^2) is an $(\mathcal{F}_t)$-Lévy process.

\# **Exercise 4.20.** Let $a > 0$ and $(X_t)_{t\ge 0}$ be an $(\mathcal{F}_t)$-Lévy process with the characteristics $(b, c, \nu)_{H_a}$. Set

$$X_t^1 = \sum_{s\le t} \Delta X_s I(|\Delta X_s| > a), \quad X_t^2 = X_t \Leftrightarrow X_t^1, \quad t \ge 0.$$

(a) Prove that X^1 and X^2 are independent Lévy processes with the characteristics $\big(0, 0, \nu|_{\{|x|>a\}}\big)_0$ and $\big(b, c, \nu|_{\{|x|\le a\}}\big)_{H_a}$, respectively.

(`Hint`: Use the Lévy-Itô decomposition; see Proposition A.7.)

(b) Prove that the two-dimensional process (X^1, X^2) is an $(\mathcal{F}_t)$-Lévy process.

\# **Exercise 4.21. (No arbitrage for a linear Lévy model.)** Let $(X_t)_{t\le T}$ be an $(\mathcal{F}_t)$-Lévy process. Consider a *linear Lévy model*

$$S_t = S_0 + X_t, \quad t \le T,$$

where $S_0 \in \mathbb{R}$.

(a) Prove that this model does not satisfy the (*NFLVR*) condition only in the following cases:

(i) S is increasing;

(ii) S is decreasing.

(b) Prove that if the (*NFLVR*) condition is satisfied, then there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that X is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale and an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-Lévy process.

(`Hint`: Use Proposition A.4.)

* **Exercise 4.22. (On Lévy martingales.)** **(a)** Let $(X_t)_{t\ge 0}$ be an $(\mathcal{F}_t)$-Lévy process. Prove that the following conditions are equivalent:

(i) X is an $(\mathcal{F}_t)$-σ-martingale (see Definition A.9);

(ii) X is an $(\mathcal{F}_t)$-martingale.

(`Hint`: Use Propositions A.4 and A.11.)

(b) Let $(X_t)_{t\ge 0}$ be an $(\mathcal{F}_t)$-Lévy process. Prove that the following conditions are equivalent:

(i) e^X is an $(\mathcal{F}_t)$-σ-martingale;

(ii) e^X is an $(\mathcal{F}_t)$-martingale.

(`Hint`: Use Propositions A.5 and A.11.)

\# **Exercise 4.23. (Completeness for a linear Lévy model.)** Suppose that a linear Lévy model introduced in Exercise 4.21 satisfies the $(NFLVR)$ condition and $\mathcal{F}_t = \mathcal{F}_t^S$. Prove that the model is complete only in the following cases:

(i) $X_t = \mu t + \sigma B_t$, where B is a Brownian motion and $\sigma \neq 0$;

(ii) $X_t = \mu t + \sigma N_{\lambda t}$, where N is a standard Poisson process and $\mu\sigma < 0$.

\# **Exercise 4.24. (Predictable representation property for Lévy processes.)** Let $(X_t)_{t\geq 0}$ be a Lévy process. Suppose that X is a martingale and any $(\mathcal{F}_t^X)$-local martingale started at zero can be represented as a stochastic integral with respect to X. Prove that either $X_t = \sigma B_t$, where B is a Brownian motion, or $X_t = \sigma N_{\lambda t} \Leftrightarrow \sigma\lambda t$, where N is a standard Poisson process.

(`Hint`: Use Exercise 4.23 and the Second Fundamental Theorem of Asset Pricing; see Proposition A.15.)

Exercise 4.25. (On the condition $\mathcal{F}_t = \mathcal{F}_t^S$.) Give an example of an $(\mathcal{F}_t)$-Brownian motion B and an $(\mathcal{F}_t)$-local martingale started at zero that cannot be represented as a stochastic integral with respect to B.

Exercise 4.26. (On the condition $\mathcal{F}_t = \mathcal{F}_t^S$.) Give an example of an $(\mathcal{F}_t)$-Brownian motion B such that $(\mathcal{F}_t)$ is strictly larger than $(\mathcal{F}_t^B)$, but any $(\mathcal{F}_t)$-local martingale started at zero can be represented as a stochastic integral with respect to B.

An open problem is

Problem 4.27. (Exponential Lévy models with an infinite time horizon.) Consider model (4.3) with an infinite time horizon. In which cases does this model satisfy the $(NFLVR)$ condition?

5 Change of Time for Lévy Processes

5.1 Nice Properties of Time-Changed Exponential Lévy Models

Throughout this section, T is a fixed positive number meaning the time horizon.

Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, \mathsf{P})$ be a filtered probability space and $(X_t)_{t\ge 0}$ be a $(\mathcal{G}_t)$-Lévy process. Let $(\tau_t)_{t\le T}$ be an increasing càdlàg process that is $\mathcal{G}_0$-adapted (in particular, X and τ are independent; see Lemma 4.2). Consider a *time-changed exponential Lévy model* for the discounted stock price:

$$S_t = S_0 e^{(X\circ\tau)_t}, \quad t \le T, \tag{5.1}$$

where $S_0 > 0$. The filtration $(\mathcal{F}_t)_{t\le T}$ is an arbitrary filtration such that $\mathcal{F}_t^S \subseteq \mathcal{F}_t \subseteq \mathcal{G}_{\tau_t}$.

Definition 5.1. A process Z has stationary increments if the distribution of $Z_t \Leftrightarrow Z_s$ depends only on $t \Leftrightarrow s$.

Lemma 5.2. *Let $(X_t)_{t\ge 0}$ be a Lévy process and $(\tau_t)_{t\le T}$ be an independent increasing càdlàg process with stationary increments. Then $X\circ\tau$ has stationary increments.*

`Proof`. We can assume that $(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_1, \mathcal{F}_1, \mathsf{P}_1)\times(\Omega_2, \mathcal{F}_2, \mathsf{P}_2)$ and $X = X(\omega_1)$, $\tau = \tau(\omega_2)$. Denoting by $\Phi(\lambda)$ the characteristic exponent of X (i.e. $\mathsf{E}e^{i\lambda X_t} = e^{t\Phi(\lambda)}$), we get, for any $s \le t$, $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathsf{E}e^{i\lambda((X\circ\tau)_t-(X\circ\tau)_s)} &= \int_{\Omega_2}\int_{\Omega_1} \exp\bigl\{i\lambda\bigl(X_{\tau_t(\omega_2)}(\omega_1) \Leftrightarrow X_{\tau_s(\omega_2)}(\omega_1)\bigr)\bigr\}\mathsf{P}_1(d\omega_1)\mathsf{P}_2(d\omega_2)\\ &= \int_{\Omega_2} \exp\bigl\{(\tau_t(\omega_2) \Leftrightarrow \tau_s(\omega_2))\Phi(\lambda)\bigr\}\mathsf{P}_2(d\omega_2) = \mathsf{E}e^{i(\tau_t-\tau_s)\Phi(\lambda)},\end{aligned}$$

and this quantity depends only on $t \Leftrightarrow s$ and λ. □

Lemma 5.3. *Let $(X_t)_{t\ge 0}$ be a Lévy process such that, for any $t \ge 0$, $\mathsf{E}X_t^2 < \infty$ and $\mathsf{E}X_t = 0$. Let $(\tau_t)_{t\le T}$ be an independent increasing càdlàg process such that, for any $t \ge 0$, $\mathsf{E}\tau_t < \infty$. Then, for any $t \ge 0$, $\mathsf{E}(X\circ\tau)_t^2 < \infty$ and $\mathsf{E}(X\circ\tau)_t = 0$. Moreover, the increments of $X\circ\tau$ over disjoint intervals are not correlated.*

The proof is Exercise 5.8.

5.2 No Arbitrage and Completeness of Time-Changed Exponential Lévy Models

In the following theorems we exclude the trivial case $X \equiv 0$.

Theorem 5.4. (No arbitrage.) (a) *Model* (5.1) *does not satisfy the* (*NFLVR*) *condition* (*see Definition A.12*) *only in the following cases:*

(i) *S is increasing;*

(ii) *S is decreasing.*

(b) *Moreover, if the* (*NFLVR*) *condition is satisfied, then there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that S is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale and*

$$\mathrm{Law}((X\circ\tau)_t;\ t \le T \mid \widetilde{\mathsf{P}}) = \mathrm{Law}((Z\circ\sigma)_t;\ t \le T),$$

where Z is a Lévy process and σ is an independent increasing càdlàg process.

Theorem 5.5. (Completeness.) *Suppose that model* (5.1) *satisfies the* (*NFLVR*) *condition and* $\mathcal{F}_t = \mathcal{F}_t^S$. *Then the model is complete* (*see Definition A.14*) *only in the following cases:*

(i) $X_t = \mu t + \sigma B_t$, *where* B *is a Brownian motion,* $\sigma \neq 0$, *and* τ *is a continuous deterministic function;*

(ii) $X_t = \mu t + \sigma N_{\lambda t}$, *where* N *is a standard Poisson process,* $\mu\sigma < 0$, *and* τ *is a continuous deterministic function.*

`Proof of Theorem 5.4.` **(a)** The conditions of the theorem imply that X is neither increasing nor decreasing. The analysis of the proof of Theorem 4.6 shows that there exists an $(\mathcal{F}_t^X)$-adapted $(\mathcal{G}_t, \mathsf{P})$-martingale $(M_t)_{t\geq 0}$ such that, for any $u \geq 0$, the process $(X_t)_{t\leq u}$ is a $(\mathcal{G}_t, \widetilde{\mathsf{P}}_u)$-Lévy process and $(e^{X_t})_{t\leq u}$ is a $(\mathcal{G}_t, \widetilde{\mathsf{P}}_u)$-martingale, where $\widetilde{\mathsf{P}}_u = M_u\mathsf{P}$. Denote

$$\begin{aligned} \mathsf{Q} &= \mathrm{Law}(X_t;\, t \geq 0),\\ \widetilde{\mathsf{Q}}_u &= \mathrm{Law}(X_t;\, t \geq 0 \mid \widetilde{\mathsf{P}}_u), \quad u \geq 0,\\ \mathsf{R} &= \mathrm{Law}(\tau_t;\, t \leq T), \end{aligned}$$

so that Q, $\widetilde{\mathsf{Q}}_u$ are measures on $D(\mathbb{R}_+)$ and R is a measure on $D([0,T])$. Since $(X_t)_{t\leq u}$ is a $\widetilde{\mathsf{P}}_u$-Lévy process and $\mathsf{E}_{\widetilde{\mathsf{P}}_u} e^{X_t} = 1$ for $t \leq u$, we conclude by Proposition A.3 that $\mathsf{E}_{\widetilde{\mathsf{P}}_u} \sup_{t\leq u} e^{X_t} < \infty$.

Obviously, there exists a measure $\widetilde{\mathsf{R}} \sim \mathsf{R}$ such that the density $\rho = \frac{d\widetilde{\mathsf{R}}}{d\mathsf{R}}$ is bounded and

$$\int_{D([0,T])} \int_{D(\mathbb{R}_+)} \sup_{t\leq \tau_T} e^{X_t}\, \widetilde{\mathsf{Q}}_{\tau_T}(dX)\, \widetilde{\mathsf{R}}(d\tau) < \infty. \tag{5.2}$$

(In order to construct such a measure, it is sufficient to consider the density of the form $\rho(\tau) = \varphi(\tau_T)$, where φ is a bounded rapidly decreasing at infinity function.)

Set $\widetilde{\mathsf{P}} = \rho(\tau)M_{\tau_T}(X)\mathsf{P}$. It follows from the equalities

$$\mathsf{E}\rho(\tau)M_{\tau_T}(X) = \int_{D([0,T])} \widetilde{\rho}(\tau) \int_{D(\mathbb{R}_+)} M_{\tau_T}(X)\, \mathsf{Q}(dX)\, \mathsf{R}(d\tau) = \int_{D([0,T])} \rho(\tau)\, \mathsf{R}(d\tau) = 1$$

and

$$\begin{aligned} \mathsf{P}\big(\rho(\tau)M_{\tau_T}(X) > 0\big) &= \int_{D([0,T])} I(\rho(\tau) > 0) \int_{D(\mathbb{R}_+)} I(M_{\tau_T}(X) > 0)\, \mathsf{Q}(dX)\, \mathsf{R}(d\tau)\\ &= \int_{D([0,T])} I(\rho(\tau) > 0)\, \mathsf{R}(d\tau) = 1 \end{aligned}$$

that $\widetilde{\mathsf{P}}$ is a probability measure and $\widetilde{\mathsf{P}} \sim \mathsf{P}$.

Set $Y_t = X_{t\wedge\tau_T}$. Then

$$\begin{aligned} \mathsf{E}_{\widetilde{\mathsf{P}}} \sup_{t\geq 0} e^{Y_t} = \mathsf{E}_{\widetilde{\mathsf{P}}} \sup_{t\leq\tau_T} e^{X_t} &= \int_{D([0,T])} \rho(\tau) \int_{D(\mathbb{R}_+)} \sup_{t\leq\tau_T} e^{X_t} M_{\tau_T}(X)\, \mathsf{Q}(dX)\, \mathsf{R}(d\tau)\\ &= \int_{D([0,T])} \int_{D(\mathbb{R}_+)} \sup_{t\leq\tau_T} e^{X_t}\, \widetilde{\mathsf{Q}}_{\tau_T}(dX)\, \widetilde{\mathsf{R}}(d\tau) < \infty \end{aligned} \tag{5.3}$$

(see (5.2)). For any $u \geq 0$, the process $(M_t e^{X_t})_{t \leq u}$ is a $(\mathcal{G}_t, \mathsf{P})$-martingale (see [14; Ch. III, Proposition 3.8]). Hence, $(M_t e^{X_t})_{t \geq 0}$ is a $(\mathcal{G}_t, \mathsf{P})$-martingale. Consequently, $(M_{t \wedge \tau_T} e^{Y_t})_{t \geq 0}$ is a $(\mathcal{G}_t, \mathsf{P})$-martingale. Since $\rho(\tau)$ is bounded and $\mathcal{G}_0$-measurable, the process $(\rho(\tau) M_{t \wedge \tau_T} e^{Y_t})_{t \geq 0}$ is a $(\mathcal{G}_t, \mathsf{P})$-martingale. Notice that $(\rho(\tau) M_{t \wedge \tau_T})_{t \geq 0}$ is the density process of $\widetilde{\mathsf{P}}$ with respect to P. Thus, $(e^{Y_t})_{t \geq 0}$ is a $(\mathcal{G}_t, \widetilde{\mathsf{P}})$-martingale (see [14; Ch. III, Proposition 3.8]). Combining this with (5.3), we conclude that $(e^{Y_t})_{t \geq 0}$ is a uniformly integrable $(\mathcal{G}_t, \widetilde{\mathsf{P}})$-martingale. The theory of martingales ensures that there exists a random variable ξ such that, for any $t \leq T$, $e^{(Y \circ \tau)_t} = \mathsf{E}_{\widetilde{\mathsf{P}}}(\xi \mid \mathcal{G}_{\tau_t})$. This implies that the process $e^{Y \circ \tau}$ is a $(\mathcal{G}_{\tau_t}, \widetilde{\mathsf{P}})$-martingale. Obviously, $e^{Y \circ \tau} = e^{X \circ \tau}$, and hence, $e^{X \circ \tau}$ is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale.

(b) The analysis of the proof of Theorem 4.6 shows that there exists a Lévy process $(Z_t)_{t \geq 0}$ such that, for any $u \geq 0$,

$$\operatorname{Law}(X_t;\, t \leq u \mid \widetilde{\mathsf{P}}_u) = \operatorname{Law}(Z_t;\, t \leq u).$$

Obviously,

$$\operatorname{Law}((X \circ \tau)_t;\, t \leq T \mid \widetilde{\mathsf{P}}) = \operatorname{Law}((Z \circ \sigma)_t;\, t \leq T),$$

where Z, σ are independent and $\operatorname{Law}(\sigma_t;\, t \leq T) = \widetilde{\mathsf{R}}$. □

`Proof of Theorem 5.5.` *Part I.* Let us prove that in case (i) the model is complete. Obviously, there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that $X_t = \alpha \widetilde{B}_t \Leftrightarrow \frac{\alpha^2}{2} t$, where $(\widetilde{B}_t)_{t \leq \tau_T}$ is a Brownian motion with respect to $\widetilde{\mathsf{P}}$ (recall that τ is deterministic).

Let $(M_t)_{t \leq T}$ be an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale started at zero. Set $\sigma_t = \inf\{s \geq 0 : \tau_s > t\}$ for $t < \tau_T$ and $\sigma_{\tau_T} = T$. Then σ is an increasing right-continuous function. The process $(M_{\sigma_t})_{t \leq \tau_T}$ is an $(\mathcal{F}_{\sigma_t}, \widetilde{\mathsf{P}})$-martingale. Note that $\mathcal{F}_t = \mathcal{F}_t^S = \mathcal{F}_{\tau_t}^X = \mathcal{F}_{\tau_t}^{\widetilde{B}}$, $t \leq T$. In view of the continuity of τ, we have $\mathcal{F}_{\sigma_t} = \mathcal{F}_{\tau_{\sigma_t}}^{\widetilde{B}} = \mathcal{F}_t^{\widetilde{B}}$, $t \leq \tau_T$. Hence, there exists an $(\mathcal{F}_t^{\widetilde{B}})$-predictable $\widetilde{B}$-integrable process $(H_t)_{t \leq \tau_T}$ such that

$$M_{\sigma_t} = \int_0^t H_s d\widetilde{B}_s, \quad t \leq \tau_T.$$

In view of the equality

$$e^{X_t} = 1 + \alpha \int_0^t e^{X_s} d\widetilde{B}_s, \quad t \geq 0,$$

we have

$$M_{\sigma_t} = \frac{1}{\alpha S_0} \int_0^t H_u e^{-X_u} d(S_0 e^{X_u}) = \int_0^t \widetilde{H}_u d(S_0 e^{X_u}), \quad t \leq \tau_T.$$

Using the time-change formula for stochastic integrals (see [21; Ch. V, Proposition 1.5]), we deduce that

$$M_{\sigma_{\tau_t}} = \int_0^t \widetilde{H}_{\tau_u} dS_u, \quad t \leq T.$$

Let $[a, b]$ be an interval of constancy of τ, i.e. $\tau_a = \tau_b$. Then $\mathcal{F}_a = \mathcal{F}_b$ up to P-null sets, and hence, $M_a = M_b$ a.s. Since M is càdlàg, this means that almost all the paths of M are constant over all the intervals of constancy of τ. Hence, $M_{\sigma_{\tau_t}} = M_t$ for $t \le T$.

Thus, any $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale started at zero can be represented as a stochastic integral with respect to S. Then this is also true for all the $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingales. By the Second Fundamental Theorem of Asset Pricing, the model is complete.

The proof of the completeness for case (ii) is similar.

Part II. Suppose that the model is complete. Assume first that τ is not deterministic. Then there exists $r \in [0,T]$ such that the support of $\operatorname{Law}(\tau_r)$ contains at least two points a, b. We can choose a sequence of bounded densities $\rho_n(\tau)$ such that they satisfy (5.2) and

$$\operatorname{Law}(\tau_r \mid \rho_n \mathsf{R}) \xrightarrow[n\to\infty]{w} \delta_a.$$

Here, $\mathsf{R} = \operatorname{Law}(\tau_t;\, t \le T)$ and δ_a is the Dirac measure at the point a. There also exists a sequence $\rho_n'(\tau)$ satisfying the same conditions and such that

$$\operatorname{Law}(\tau_r \mid \rho_n' \mathsf{R}) \xrightarrow[n\to\infty]{w} \delta_b.$$

If we set $\widetilde{\mathsf{P}}_n = \rho_n(\tau) M_{\tau_T}(X)\mathsf{P}$, $\widetilde{\mathsf{P}}_n' = \rho_n'(\tau) M_{\tau_T}(X)\mathsf{P}$, where M the same as in the previous proof, then $(S_t)_{t\le T}$ is a martingale with respect to all the measures $\widetilde{\mathsf{P}}_n$, $\widetilde{\mathsf{P}}_n'$ (see the proof of Theorem 5.4).

There exists a Lévy process $(Z_t)_{t\ge 0}$ such that, for any $u \ge 0$,

$$\operatorname{Law}(X_t;\, t \le u \mid \widetilde{\mathsf{P}}_u) = \operatorname{Law}(Z_t;\, t \le u),$$

where $\widetilde{\mathsf{P}}_u = M_u \mathsf{P}$. Then

$$\operatorname{Law}((X \circ \tau)_r \mid \widetilde{\mathsf{P}}_n) \xrightarrow[n\to\infty]{w} \operatorname{Law}(Z_a),$$
$$\operatorname{Law}((X \circ \tau)_r \mid \widetilde{\mathsf{P}}_n') \xrightarrow[n\to\infty]{w} \operatorname{Law}(Z_b),$$

which shows that there exists n such that

$$\operatorname{Law}((X \circ \tau)_r \mid \widetilde{\mathsf{P}}_n) \ne \operatorname{Law}((X \circ \tau)_r \mid \widetilde{\mathsf{P}}_n').$$

Hence, there exist different equivalent martingale measures for S. By the Second Fundamental Theorem of Asset Pricing, the model is not complete. Thus, τ is a deterministic function.

Now, the arguments used in the proof of Theorem 4.6 show that the model can be complete only if X is a Brownian motion with a drift or a Poisson process with a drift.

Finally, let us prove that τ is continuous. Suppose that there exists $r \in [0,T]$ such that $\tau_{r-} \ne \tau_r$. Then, by considering densities of the form $\rho(X_{\tau_r} - X_{\tau_{r-}})$, one can construct different martingale measures for S. □

5.3 Exercises

Exercise 5.6. Let $(X_t)_{t\ge 0}$ be a Lévy process and $(\tau_t)_{t\le T}$ be an independent increasing càdlàg process, whose increments are stationary in the following sense: if a collection $(t_0', \dots, t_n')$ is obtained from $(t_0, \dots, t_n)$ by a shift, then

$$\operatorname{Law}\big(\tau_{t_1'} - \tau_{t_0'}, \dots, \tau_{t_n'} - \tau_{t_{n-1}'}\big) = \operatorname{Law}\big(\tau_{t_1} - \tau_{t_0}, \dots, \tau_{t_n} - \tau_{t_{n-1}}\big).$$

Prove that the increments of $X \circ \tau$ are stationary in the same sense.

Exercise 5.7. Let $(X_t)_{t\geq 0}$ be a Lévy process such that, for any $t \geq 0$, $\mathsf{E}X_t^2 < \infty$ and $\mathsf{E}X_t = 0$. Prove that there exists $k \geq 0$ such that $\mathsf{E}X_t^2 = kt$, $t \geq 0$.

Exercise 5.8. Prove Lemma 5.3.
(`Hint`: Use Exercise 5.7.)

○ **Exercise 5.9.** Let $(X_t)_{t\geq 0}$ be a strictly α-stable Lévy process and $(\tau_t)_{t\leq T}$ be an independent strictly β-stable subordinator. Prove that $X \circ \tau$ is a strictly $\alpha\beta$-stable Lévy process.

(`Hint`: Use the fact that a Lévy process X is strictly α-stable if and only if for any $k, t \geq 0$, $X_{kt} \overset{\text{law}}{=} k^\alpha X_t$.)

##**Exercise 5.10.** Let X be a symmetric Lévy process that belongs to one of the following classes of processes: *VG*, *CGMY*, *GHYP*. Let $(\tau_t)_{t\leq T}$ be an independent increasing càdlàg process. Prove that $X \circ \tau$ can be represented as $X \circ \tau \overset{\text{law}}{=} B \circ \sigma$, where B is a Brownian motion and σ is an independent increasing càdlàg process.

Exercise 5.11. (No arbitrage for a time-changed linear Lévy model.) Consider a *time-changed linear Lévy model*

$$S_t = S_0 + (X \circ \tau)_t, \quad t \leq T,$$

where $S_0 \in \mathbb{R}$, and X, τ, $(\mathcal{F}_t)$ are the same as in (5.1).

(a) Prove that this model does not satisfy the (*NFLVR*) condition only in the following cases:

(i) S is increasing;

(ii) S is decreasing.

(b) Prove that if the (*NFLVR*) condition is satisfied, then there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that S is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-martingale and

$$\text{Law}((X \circ \tau)_t;\, t \leq T \mid \widetilde{\mathsf{P}}) = \text{Law}((Z \circ \sigma)_t;\, t \leq T),$$

where Z is a Lévy process and σ is an independent increasing càdlàg process.

(`Hint`: Use Exercise 4.21.)

Exercise 5.12. (Completeness for a time-changed linear Lévy model.) Suppose that a time-changed linear Lévy model introduced in Exercise 5.11 satisfies the (*NFLVR*) condition and $\mathcal{F}_t = \mathcal{F}_t^S$. Prove that the model is complete only in the following cases:

(i) $X_t = \mu t + \sigma B_t$, where B is a Brownian motion, $\sigma \neq 0$, and τ is a continuous deterministic function;

(ii) $X_t = \mu t + \sigma N_{\lambda t}$, where N is a standard Poisson process, $\mu\sigma < 0$, and τ is a continuous deterministic function.

Exercise 5.13. Let κ be a Poisson random variable with parameter λ. Let $(\eta_m)_{m=1}^\infty$ be a sequence of independent random variables (that are also independent of κ) with

$$\mathsf{P}(\eta_m = 1) = p, \quad \mathsf{P}(\eta_m = 0) = 1 - p.$$

Prove that $\xi := \sum_{m=1}^\kappa \eta_m$ has the Poisson distribution with the parameter $p\lambda$.

$*$ **Exercise 5.14. (Filtering the time-change.)** Let $(X_t)_{t\geq 0}$ be a Lévy process with the characteristics $(b, c, \nu)_H$ and $(\tau_t)_{t\leq T}$ be an independent increasing continuous process.

(a) Suppose that $\nu(\mathbb{R}) = \infty$. Find disjoint sets $(A_n)_{n=1}^{\infty}$ such that $\nu(A_n) \in [1, \infty)$. Let $(\eta_m^n)_{m,n=1}^{\infty}$ be independent random variables (that are also independent of (X, τ)) with

$$\mathsf{P}(\eta_m^n = 1) = \frac{1}{\nu(A_n)}, \quad \mathsf{P}(\eta_m^n = 0) = 1 \Leftrightarrow \frac{1}{\nu(A_n)}.$$

Set

$$\kappa_n = \sum_{t\leq T} I(\Delta(X \circ \tau)_t \in A_n), \quad \xi_n = \sum_{m=1}^{\kappa_n} \eta_m^n.$$

Prove that

$$\tau_T = (\text{a.s.}) \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \xi_n.$$

(b) Suppose that $c > 0$. Prove that

$$\tau_T = \frac{1}{c}\biggl([X \circ \tau]_T \Leftrightarrow \sum_{s\leq T} \Delta(X \circ \tau)_s^2\biggr).$$

(c) Suppose that $\nu(\mathbb{R}) < \infty$, $c = 0$, but X is not a compound Poisson process. Then X can be represented as

$$X_t = \gamma t + \sum_{s\leq t} \Delta X_s, \quad t \leq T.$$

Prove that

$$\tau_T = \frac{1}{\gamma}\biggl((X \circ \tau)_T \Leftrightarrow \sum_{t\leq T} \Delta(X \circ \tau)_t\biggr).$$

$**$ **Exercise 5.15. (Conditional law of the time-change.)** Let $(X_t)_{t\geq 0}$ be a Lévy process and $(\tau_t)_{t\leq T}$ be an independent increasing càdlàg process with independent increments. Set $\mathsf{Q}_\varphi = \text{Law}(\tau_t;\, t \leq T \mid X \circ \tau = \varphi)$. Prove that, for a.e. φ (with respect to the measure $\text{Law}((X \circ \tau)_t;\, t \leq T)$), the canonical process Z on $D([0, T])$ has independent increments with respect to Q_φ.

An open problem is

Problem 5.16. (Time-changed exponential Lévy model with an infinite time horizon). Consider model (5.1) with an infinite time horizon. In which cases does this model satisfy the (*NFLVR*) condition?

Appendix

A.1 Lévy Processes

Definition A.1. A function $f : \mathbb{R} \to \mathbb{R}_+$ is called *submultiplicative* if there exists a constant a such that, for any $x, y \in \mathbb{R}$, we have $f(x+y) \le af(x)f(y)$.

`Remark`. The functions e^x and $|x|^\alpha \vee 1$ with $\alpha > 0$ are submultiplicative (see [22; Proposition 25.4]. □

Proposition A.2. *Let X be a Lévy process and $f(x)$ be a submultiplicative function. Then the following conditions are equivalent:*
(i) $\mathsf{E}f(X_t) < \infty$ *for some* $t > 0$;
(ii) $\mathsf{E}f(X_t) < \infty$ *for any* $t > 0$;
(iii) *the Lévy measure ν of X satisfies the condition*

$$\int_{\{|x|>1\}} f(x)\nu(dx) < \infty.$$

For the proof, see [22; Theorem 25.3].

Proposition A.3. *Let X be a Lévy process and f be a continuous submultiplicative function increasing to ∞ as $x \to \infty$. Then the following conditions are equivalent:*
(i) $\mathsf{E}f(|X_t|) < \infty$ *for some* $t > 0$;
(ii) $\mathsf{E}f(|X_t|) < \infty$ *for any* $t > 0$;
(iii) $\mathsf{E}f\big(\sup_{s\le t} |X_s|\big) < \infty$ *for some* $t > 0$;
(iv) $\mathsf{E}f\big(\sup_{s\le t} |X_s|\big) < \infty$ *for any* $t > 0$.

For the proof, see [22; Theorem 25.18].

Proposition A.4. (The first moment of a Lévy process.) *Let $a > 0$ and X be a Lévy process with the characteristics $(b, c, \nu)_{H_a}$ (H_a is defined in (4.4)). Suppose that*

$$\int_{\{|x|>a\}} |x|\nu(dx) < \infty.$$

Then, for any $t \ge 0$, $\mathsf{E}|X_t| < \infty$ and

$$\mathsf{E}X_t = t\left(b + \int_{\{|x|>a\}} x\nu(dx)\right), \quad t \ge 0.$$

For the proof, see [22; Example 25.12].

Proposition A.5. (The exponential moment of a Lévy process.) *Let $a > 0$ and X be a Lévy process with the characteristics $(b, c, \nu)_{H_a}$ (H_a is defined in (4.4)). Suppose that*

$$\int_{\{|x|>a\}} e^x\nu(dx) < \infty.$$

Then

$$\mathsf{E}e^{X_t} = \exp\left\{t\left[b + \frac{c}{2} + \int_{\mathbb{R}}\Big(e^x \Leftrightarrow 1 \Leftrightarrow H_a(x)\Big)\nu(dx)\right]\right\}, \quad t \ge 0.$$

For the proof, see [22; Example 25.17].

Proposition A.6. (Lévy processes of finite variation.) *Let X be a Lévy process with the characteristics $(b, c, \nu)_H$ (H is defined in (3.2)). Then X has finite variation if and only if $c = 0$ and*

$$\int_{\mathbb{R}} |x| \wedge 1 \, \nu(dx) < \infty.$$

For the proof, see [22; Theorem 21.9].

Proposition A.7. (Lévy-Itô decomposition.) *Let $(X_t)_{t \geq 0}$ be a Lévy process with the characteristics $(b, c, \nu)_h$, where h is a truncation function.*

(a) *The random measure*

$$N(\omega, B) := \sum_{s \geq 0} I(\Delta X_s \neq 0, \ (s, \Delta X_s) \in B), \quad B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$$

is a Poisson random measure with the intensity measure $dt \times \nu(dx)$.

(b) *With probability one the limit*

$$X_t^1 := \lim_{\varepsilon \downarrow 0} \Biggl(\sum_{s \leq t} \Delta X_s I(|\Delta X_s| > \varepsilon) - t \int_{\{|x| > \varepsilon\}} h(x) \nu(dx) \Biggr), \quad t \geq 0$$

is defined for all $t \geq 0$, and the convergence is uniform in t on every compact interval. If we set $X^2 = X - X^1$, then X^1 and X^2 are independent Lévy processes with the characteristics $(0, 0, \nu)_h$ and $(b, c, 0)_h$, respectively.

For the proof, see [22; Theorem 19.2].

A.2 Sigma-Martingales

Proposition A.8. *Let $(X_t)_{t \leq T}$ be a semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathsf{P})$. The following conditions are equivalent:*

(i) *there exists a sequence of predictable sets $D_n \subseteq \Omega \times [0, T]$ such that $D_n \subseteq D_{n+1}$, $\bigcup_n D_n = \Omega \times [0, T]$, and , for each n, the process $\int_0^\cdot I_{D_n} dX_s$ is an $(\mathcal{F}_t)$-martingale;*

(ii) *there exist an $(\mathcal{F}_t)$-local martingale M and an M-integrable process H such that*

$$X_t = X_0 + \int_0^t H_s dM_s, \quad t \leq T.$$

For the proof, see [24; Lemma 5.1].

Definition A.9. A process X that satisfies the equivalent conditions of Proposition A.8 is called a σ-*martingale.*

This class of processes was introduced by C.S. Chou [4] and M. Émery [9] under the name "semimartingales de la classe Σ_m" (they defined a σ-martingale as a stochastic integral with respect to a local martingale). F. Delbaen and W. Schachermayer [6] called these processes "σ-martingales". The description of σ-martingales through predictable sets D_n was proposed by T. Goll and J. Kallsen [11]. For more information on σ-martingales, see also the paper by J. Kallsen [17].

Obviously, any local martingale is a σ-martingale. The reverse is not true. A corresponding example was constructed by M. Émery [9].

Proposition A.10. (Ansel, Stricker.) *A σ-martingale that is bounded below is a local martingale.*

For the proof, see [1].

Let $(X_t)_{t \le T}$ be an $(\mathcal{F}_t)$-semimartingale with the characteristics $(B, C, \nu)_h$, where h is a truncation function. Then there exist predictable processes b, c, a transition kernel $K(\omega, t, dx)$ from $(\Omega \times [0, T], \mathcal{P})$ (here, $\mathcal{P}$ stands for the predictable σ-field) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and an increasing predictable process A such that

$$B_t = \int_0^t b_s dA_s, \quad C_t = \int_0^t c_s dA_s, \quad \nu(\omega, dt, dx) = K(\omega, t, dx) dA_t(\omega)$$

(see [14; Ch. II, Proposition 2.9]).

Proposition A.11. *The process X is a σ-martingale if and only if for $\mathsf{P} \times dA$-a.e. (ω, t), we have*

$$\int_{\{|x|>1\}} |x| K(\omega, t, dx) < \infty$$

and

$$b(\omega, t) + \int_{\mathbb{R}} (x \Leftrightarrow h(x)) K(\omega, t, dx) = 0.$$

For the proof, see [16; Lemma 3].

A.3 Fundamental Theorems of Asset Pricing

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \le T}, \mathsf{P}; (S_t)_{t \le T}\right)$ be a *model* of a financial market. Here, S is an $(\mathcal{F}_t)$-semimartingale. From the financial point of view, S is the discounted price of an asset. Recall that a *strategy* is a pair (x, H), where x is $\mathcal{F}_0$-measurable and $(H_t)_{t \le T}$ is an S-integrable process (for the definition of integrability, see, for instance, [24]). The discounted *capital* of this strategy is the process $x + \int_0^{\cdot} H_u dS_u$.

The following notion of the *no free lunch with vanishing risk* was introduced by F. Delbaen and W. Schachermayer [5]. It serves as an adequate continuous-time analogue of the no arbitrage condition.

Definition A.12. A sequence of strategies (x^n, H^n) realizes *free lunch with vanishing risk* if

(i) for each n, $x^n = 0$;

(ii) for each n, there exists $a_n \in \mathbb{R}$ such that

$$\mathsf{P}\left(\forall t \le T, \;\; x + \int_0^t H_u dS_u \ge a_n\right) = 1;$$

(iii) for each n,

$$\int_0^T H_u^n dS_u \ge \Leftrightarrow \frac{1}{n} \quad \text{a.s.};$$

(iv) there exists $\delta > 0$ such that, for each n,

$$\mathsf{P}\left(\int_0^T H_u^n dS_u > \delta\right) > \delta.$$

A model satisfies the *no free lunch with vanishing risk* condition if such a sequence of strategies does not exist. Notation: (*NFLVR*).

Proposition A.13. (First Fundamental Theorem of Asset Pricing.) *A model satisfies the (NFLVR) condition if and only if there exists an equivalent σ-martingale measure, i.e. a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that S is an $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-$\sigma$-martingale.*

Statement (a) was proved by F. Delbaen and W. Schachermayer [6]; compare with Yu.M. Kabanov [16].

`Remark.` If the process S is nonnegative, then the word "σ-martingale" in the above theorem can be replaced by the word "local martingale". This follows from Proposition A.10.

Definition A.14. A model is *complete* if for any bounded $\mathcal{F}$-measurable function f, there exists a strategy (x, H) such that

(i) there exist constants a, b such that

$$\mathsf{P}\left(\forall t \le T,\ \ a \le x + \int_0^t H_u dS_u \le b\right) = 1;$$

(ii) $f = x + \int_0^T H_u dS_u$ a.s.

Proposition A.15. (Second Fundamental Theorem of Asset Pricing.) *Suppose that a model satisfies the (NFLVR) condition and $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}_t$. Then the following conditions are equivalent:*

(i) *the model is complete;*

(ii) *the equivalent σ-martingale measure is unique;*

(iii) *there exists an equivalent σ-martingale measure $\widetilde{\mathsf{P}}$ such that any $(\mathcal{F}_t, \widetilde{\mathsf{P}})$-local martingale started at zero can be represented as a stochastic integral with respect to S.*

This statement follows from [6; Theorem 5.14]. It can also be derived from [1] or [13; Théorème 11.2]. An explicit proof of the Second Fundamental Theorem of Asset Pricing in this form can be found in [24].

`Remark.` If the process S is nonnegative, then the word "σ-martingale" in the above theorem can be replaced by the word "local martingale". This follows from Proposition A.10.

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Solutions of the Exercises

Section 2

Exercise 2.1. If τ is not absolutely continuous, then $B \circ \tau$ cannot be represented as $B \circ \tau \overset{\text{law}}{=} \sigma \bullet W$ since $\langle B \circ \tau \rangle = \tau$, while $\langle \sigma \bullet W \rangle = \int_0^\cdot \sigma_s^2 ds$.

Exercise 2.2. The process $\sigma \bullet W$ obviously satisfies condition (iii) of Proposition 3.6.

Section 3

Exercise 3.21. It will suffice to prove the implication (i)$\Rightarrow$(iii). By Itô's formula, for any $\lambda \in \mathbb{R}$, the process

$$e^{i\lambda B_t + \frac{\lambda^2}{2}t}, \quad t \ge 0$$

is an $(\mathcal{F}_t)$-local martingale. Being (locally) bounded, it is a martingale. Hence, for any $s \le t$, we have

$$\mathsf{E}\big(e^{i\lambda(B_t - B_s)} \mid \mathcal{F}_s\big) = e^{-(t-s)\frac{\lambda^2}{2}}.$$

We now apply Lemma 4.2.

Exercise 3.22. Consider $\tau_t = \inf\{s \ge 0 : B_s = t\}$.

Exercise 3.23. Set

$$N_t = \int_0^t H_s dB_s, \quad t \ge 0.$$

By the Dambis-Dubins-Schwarz theorem, N can be represented as $N \overset{\text{a.s.}}{=} W \circ \langle N \rangle$, where W is a Brownian motion (possibly defined on an enlarged probability space). Then, obviously, $M = \mathcal{E}(W) \circ \langle N \rangle$, where $\mathcal{E}(W)_t = e^{W_t - t/2}$. Now, the statement follows from the equality

$$\langle N \rangle_\infty = \int_0^\infty H_s^2 ds$$

and the fact that $\lim_{t \to \infty} \mathcal{E}(B) = 0$ a.s.

Exercise 3.24. Fix $0 = t_0 \le t_1 \le \dots \le t_n$. Then, using Fubini's theorem (compare with (3.1)), we get, for any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$,

$$\begin{aligned} &\mathsf{E} \exp\{i\lambda_1((B \circ \tau)_{t_1} - (B \circ \tau)_{t_0}) + \dots + i\lambda_n((B \circ \tau)_{t_n} - (B \circ \tau)_{t_{n-1}})\} \\ &= \mathsf{E} \exp\Big\{-\frac{\lambda_1^2}{2}(\tau_{t_1} - \tau_{t_0}) - \dots - \frac{\lambda_n^2}{2}(\tau_{t_n} - \tau_{t_{n-1}})\Big\}. \end{aligned}$$

Hence, the multidimensional Laplace transform of $((\tau_{t_1} - \tau_{t_0}), \dots, (\tau_{t_n} - \tau_{t_{n-1}}))$ is uniquely determined by the distribution of $B \circ \tau$.

For any random variable $(\xi_1, \dots, \xi_n)$ with positive components, its distribution is uniquely determined by the multidimensional Laplace transform

$$\varphi(\mu_1, \dots, \mu_n) = \mathsf{E} \exp\{-\mu_1 \xi_1 - \dots - \mu_n \xi_n\}, \quad \mu_1, \dots, \mu_n \in \mathbb{R}_+.$$

Indeed,

$$\mathsf{E} \exp\{-\mu_1 \xi_1 - \dots - \mu_n \xi_n\} = \mathsf{E} \eta_1^{\mu_1} \dots \eta_n^{\mu_n},$$

where $\eta_k = e^{-\xi_k}$. By the Weierstrass approximation theorem, any continuous function on $[0,1]^n$ can be uniformly approximated by the linear combinations of the functions having the form $x_1^{\mu_1} \dots x_n^{\mu_n}$. Thus, the distribution of $(\eta_1, \dots, \eta_n)$ as well as the distribution of $(\xi_1, \dots, \xi_n)$ are uniquely determined by φ.

Exercise 3.25. Set $\tau_t = \xi I(t \ge 1)$, where ξ is a positive random variable with $\mathsf{E}\sqrt{\xi} = \infty$. Then $(B \circ \tau)_t = B_\xi I(t \ge 1)$. Using Fubini's theorem, one can verify that $\mathsf{E}|B_\xi| = \sqrt{2/\pi}\mathsf{E}\sqrt{\xi} = \infty$. Furthermore, for any $(\mathcal{F}_t^{B\circ\tau})$-stopping time σ, we have either $\sigma < 1$ or $\sigma \ge 1$ since $\mathcal{F}_{1-}^{B\circ\tau}$ is trivial. Hence, $B \circ \tau$ is not a local martingale.

Exercise 3.26. Let B be a Brownian motion. Set

$$X_t = \int_0^t \bigl(I(t \le 1) + I(t > 1, B_1 > 0)\bigr) dB_s, \quad t \ge 0.$$

Then X and $\Leftrightarrow X$ have different distributions, and therefore, X is not an Ocone martingale.

Exercise 3.27. If B is a Brownian motion and τ is an independent increasing continuous process with $\mathsf{E}\sqrt{\tau_1} = \infty$, then, by Fubini's theorem, $\mathsf{E}|B \circ \tau|_1 = \sqrt{2/\pi}\mathsf{E}\sqrt{\tau_1} = \infty$.

Exercise 3.28. **(a)** It will suffice to consider X constructed in the solution of Exercise 3.26 and to take $Y = \Leftrightarrow X$.

(b) Since X is an Ocone martingale, it can be represented as $X \overset{\text{law}}{=} B \circ \tau$, where B is a Brownian motion and τ is an independent increasing continuous process. We have $\langle X \rangle \overset{\text{law}}{=} \langle B \circ \tau \rangle = \tau$ (see [21; Ch. V, Proposition 1.5]). Consequently, $X \overset{\text{law}}{=} W \circ \langle X \rangle$, where W is a Brownian motion that is independent of $\langle X \rangle$. This shows that the distribution of X is uniquely determined by the distribution of $\langle X \rangle$.

Exercise 3.29. Set $Y_t = \int_0^t H_s dX_s$. Then Y obviously satisfies condition (iii) of Proposition 3.6, and hence, it is an Ocone martingale. Since $\langle Y \rangle = \langle X \rangle$, then, by Exercise 3.28, $Y \overset{\text{law}}{=} X$. This leads to the equality $(Y, \langle Y \rangle) \overset{\text{law}}{=} (X, \langle X \rangle)$, which is the desired statement.

Exercise 3.30. Without the loss of generality, we may assume that $X \overset{\text{a.s.}}{=} B \circ \tau$, where B is a Brownian motion and τ is an independent increasing continuous process. Set $S_t^B = \sup_{u \le t} B_u$ and let L^B denote the local time of B at zero. P. Lévy's theorem states that

$$(S^B \Leftrightarrow B, S^B) \overset{\text{law}}{=} (|B|, L^B). \tag{S.1}$$

Obviously, $S = S^B \circ \tau$. It follows from Tanaka's formula (see [21; Ch. VI, Theorem 1.2]) and the time-change formula for stochastic integrals (see [21; Ch. V, Proposition 1.5]) that $L = L^B \circ \tau$. Furthermore, τ is independent of both $(S^B \Leftrightarrow S, S^B)$ and $(|B|, L^B)$. Now, the desired result follows from (S.1).

Exercise 3.31. Without the loss of generality, we may assume that $X \overset{\text{a.s.}}{=} B \circ \tau$, where B is a Brownian motion and τ is an independent increasing continuous process. Consider the filtration $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{F}_\infty^\tau$. Then B is an $(\mathcal{F}_t)$-Brownian motion and X is $(\mathcal{F}_{\tau_t})$-adapted (see [21; Ch. V, Proposition 1.4]). Set

$$M_t = \int_0^t H_s dX_s, \quad t \ge 0$$

and consider

$$\sigma_t = \inf\{s \ge 0 : \tau_s > t\}, \quad t \ge 0$$

(here, $\inf \emptyset = \infty$). We have

$$\int_0^\infty \big(H_s \Leftrightarrow H_{\sigma_{\tau_s}} I(\sigma_{\tau_s} < \infty)\big)^2 d\langle X\rangle_s = \int_0^\infty \big(H_s \Leftrightarrow H_{\sigma_{\tau_s}} I(\sigma_{\tau_s} < \infty)\big)^2 d\tau_s$$
$$= \int_0^\infty \big(H_{\sigma_s} \Leftrightarrow H_{\sigma_s} I(\sigma_s < \infty)\big)^2 I(\sigma_s < \infty) ds = 0.$$

In the second equality, we have applied [21; Ch. 0, Proposition 4.9]. Therefore,

$$M_t = \int_0^t H_{\sigma_{\tau_s}} I(\sigma_{\tau_s} < \infty) dX_s = \int_0^t H_{\sigma_{\tau_s}} I(\sigma_{\tau_s} < \infty) dB_{\tau_s}$$
$$= \int_0^{\tau_t} H_{\sigma_s} I(\sigma_s < \infty) dB_s, \quad t \ge 0.$$

In the last equality, we have used [21; Ch. V, Proposition 1.5]. The process

$$W_t = \int_0^t \big(H_{\sigma_s} I(\sigma_s < \infty) + I(\sigma_s = \infty)\big) dB_s, \quad t \ge 0$$

is a Brownian motion and an $(\mathcal{F}_t)$-local martingale. It follows from Exercise 3.21 and Lemma 4.2 that W is independent of $\mathcal{F}_0$. In particular, W is independent of τ. Finally, we get $M = W \circ \tau \overset{\text{law}}{=} B \circ \tau = X$.

Exercise 3.32. Fix $0 = t_0 \le t_1 \le \dots \le t_n$. For any $\lambda_1, \dots \lambda_n \in \mathbb{R}$, we have

$$\mathsf{E}\exp\{i\lambda_1((X \circ \tau)_{t_1} \Leftrightarrow (X \circ \tau)_{t_0}) + \dots + i\lambda_n((X \circ \tau)_{t_n} \Leftrightarrow (X \circ \tau)_{t_{n-1}})\}$$
$$= \mathsf{E}\exp\{(\tau_{t_1} \Leftrightarrow \tau_{t_0})\Phi(\lambda_1) + \dots + (\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\Phi(\lambda_n)\}$$
$$= \mathsf{E}\exp\{(\tau_{t_1} \Leftrightarrow \tau_{t_0})\Phi(\lambda_1)\} \dots \mathsf{E}\exp\{(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\Phi(\lambda_n)\},$$

where $\Phi(\lambda)$ denotes the characteristic exponent of X (i.e $\mathsf{E}e^{i\lambda X_t} = e^{t\Phi(\lambda)}$).

Exercise 3.33. The proof is similar to the proof of Lemma 3.13.

Exercise 3.34. We have

$$\frac{\nu(dx)}{dx} = \int_0^\infty \frac{\lambda}{\sqrt{2\pi z}} \exp\Big\{\Leftrightarrow \frac{(x \Leftrightarrow \alpha z)^2}{2z}\Big\} \frac{e^{-\theta z}}{z} dz$$
$$= \frac{\lambda}{\sqrt{2\pi}} e^{\alpha x} \int_0^\infty \frac{1}{z^{3/2}} \exp\Big\{\Leftrightarrow \frac{x^2}{2z} \Leftrightarrow \frac{\alpha^2 z}{2} \Leftrightarrow \theta z\Big\} dz$$
$$= \frac{2\lambda}{\sqrt{2\pi}} e^{\alpha x} \int_0^\infty \exp\Big\{\Leftrightarrow \frac{x^2 y^2}{2} \Leftrightarrow \frac{\alpha^2}{2y^2} \Leftrightarrow \frac{\theta}{y^2}\Big\} dy$$
$$= \frac{\lambda}{|x|} e^{\alpha x - \sqrt{\alpha^2 + 2\theta}|x|}.$$

In the last equality, we have applied [12; (3.325)].

Exercise 3.35. The proof is similar to the proof of Theorem 3.17. The difference is that instead of Lemma 3.13, we should use Exercise 3.33, and instead of the equality

$$q(x) = \int_0^\infty \frac{1}{\sqrt{2\pi z}}\, e^{-\frac{x^2}{2z}}\eta(dz),$$

we will have the equality

$$q(x) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\Big\{-\frac{(x - \alpha z)^2}{2z}\Big\}\eta(dz) = e^{\alpha x}\int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{\alpha^2 z}{2}} e^{-\frac{x^2}{2z}}\eta(dz).$$

Exercise 3.36. Let us prove by the induction the following statement: for any completely monotone function φ and any $\psi : (0,\infty) \to (0,\infty)$ with a completely monotone derivative, we have $\operatorname{sgn}(\varphi\circ\psi)^{(n)} = (-1)^n$. To this end, write

$$\begin{aligned}&(\varphi\circ\psi)^{(n+1)} = ((\varphi\circ\psi)')^{(n)} = (\psi'(\varphi'\circ\psi))^{(n)}\\ &= \psi^{(n+1)}(\varphi'\circ\psi) + C_n^1\psi^{(n)}(\varphi'\circ\psi)^{(1)} + \dots + C_n^{n-1}\psi^{(2)}(\varphi'\circ\psi)^{(n-1)} + \psi^{(1)}(\varphi'\circ\psi)^{(n)}.\end{aligned}$$

We now apply the induction hypothesis to the terms $(\varphi'\circ\psi), \dots, (\varphi'\circ\psi)^{(n)}$.

Exercise 3.37. **(a)** Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function, whose support is compact and does not include 0. Then $h(x)$ can be uniformly approximated by a sequence (h_n), where each $h_n(x)$ has the form $\sum_{k=1}^m a_k e^{-xy_k}$ with $a_k \in \mathbb{R}$, $y_k > 0$. (In order to see this, it is sufficient to consider the transformation $[0,\infty) \ni x \mapsto e^{-x} \in [0,1)$ and to apply the Weierstrass approximation theorem.) Set

$$g(\lambda) = \int_0^\infty h(x)\mu_\lambda(dx), \quad g_n(\lambda) = \int_0^\infty h_n(x)\mu_\lambda(dx), \quad \lambda\in\Lambda.$$

Then each g_n is $\mathcal{A}$-measurable, and g is a pointwise limit of g_n. Hence, g is also $\mathcal{A}$-measurable. Now, the proof is easily completed.

(b) In view of (a), the measure

$$\mu := \int_\Lambda \mu_\lambda \nu(d\lambda)$$

is well defined. Obviously,

$$f(x) = \int_0^\infty e^{-xy}\mu(dy), \quad x > 0.$$

By Proposition 3.16, f is completely monotone.

Exercise 3.38. **(a)** The function under the integral in (3.4) is positive, and the result follows from Theorem 3.17, Lemma 3.18, and Exercise 3.37.

(b) The result follows from Exercise 3.35, Lemma 3.18, and Exercise 3.37.

Exercise 3.39. **(i)** ⇔ **(ii)** This equivalence is a direct consequence of Lemma 3.8 and Exercise 4.22.

(ii) ⇔ **(iii)** This equivalence follows from the equality $\mathsf{E}|B\circ\tau|_t = \sqrt{2/\pi}\,\mathsf{E}\sqrt{\tau_t}$ and the fact that a Lévy process with the zero mean is a martingale.

(iii) $\Leftrightarrow$ **(iv)** This equivalence follows from Proposition A.2.

Exercise 3.40. Let ν denote the Lévy measure of $B \circ \tau$. Using Lemma 3.13 and Proposition A.6, we can write

$$\int_{\mathbb{R}} |x| \wedge 1\, \nu(dx) = \int_0^\infty \int_{\mathbb{R}} |x| \wedge 1\, \mathsf{Q}_z(dx)\eta(dz) = \int_0^\infty \varphi(z)\eta(dz) < \infty,$$

where Q_z denotes the normal distribution with the zero mean and the variance z. The function φ is strictly positive on $(0, \infty)$ and $\varphi(z) \sim \sqrt{2z/\pi}$ as $z \to 0$. This leads to the desired statement.

Section 4

Exercise 4.9. Let η be a $\mathcal{G}$-measurable random variable. Then, for any $\lambda_1, \dots, \lambda_{d+1} \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathsf{E}\exp\{i\lambda_1\xi_1 + \dots + i\lambda_d\xi_d + i\lambda_{d+1}\eta\} \\
&\quad = \mathsf{E}\big(\mathsf{E}\big(\exp\{i\lambda_1\xi_1 + \dots + i\lambda_d\xi_d + i\lambda_{d+1}\eta\} \mid \mathcal{G}\big)\big) \\
&\quad = \mathsf{E}\big(\exp\{i\lambda_{d+1}\eta\}\mathsf{E}\big(\exp\{i\lambda_1\xi_1 + \dots + i\lambda_d\xi_d\} \mid \mathcal{G}\big)\big) \\
&\quad = \mathsf{E}\exp\{i\lambda_{d+1}\eta\}\mathsf{E}\exp\{i\lambda_1\xi_1 + \dots + i\lambda_n\xi_n\}.
\end{aligned}$$

Exercise 4.10. Fix $t_0 \le \dots \le t_n \in [s, T]$. For any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathsf{E}\big(\exp\{i\lambda_1(X_{t_1} \Leftrightarrow X_{t_0}) + \dots + i\lambda_n(X_{t_n} \Leftrightarrow X_{t_{n-1}})\} \mid \mathcal{F}_s\big) \\
&= \mathsf{E}\big(\mathsf{E}\big(\exp\{i\lambda_1(X_{t_1} \Leftrightarrow X_{t_0}) + \dots + i\lambda_n(X_{t_n} \Leftrightarrow X_{t_{n-1}})\} \mid \mathcal{F}_{t_{n-1}}\big) \mid \mathcal{F}_s\big) \\
&= \mathsf{E}\exp\{i\lambda_n(X_{t_n} \Leftrightarrow X_{t_{n-1}})\}\mathsf{E}\big(\exp\{i\lambda_1(X_{t_1} \Leftrightarrow X_{t_0}) + \dots + i\lambda_{n-1}(X_{t_{n-1}} \Leftrightarrow X_{t_{n-2}})\} \mid \mathcal{F}_s\big) \\
&= \dots = \mathsf{E}\exp\{i\lambda_1(X_{t_1} \Leftrightarrow X_{t_0})\} \dots \mathsf{E}\exp\{i\lambda_n(X_{t_n} \Leftrightarrow X_{t_{n-1}})\}.
\end{aligned}$$

We now apply Exercise 4.9.

Exercise 4.11. The "if" part follows from Lemma 4.3. In order to prove the "only if" part, suppose that $\widetilde{\nu}$ is not absolutely continuous with respect to ν. Find $A \in \mathcal{B}(\mathbb{R})$ such that $\widetilde{\nu}(A) > 0$, while $\nu(A) = 0$. Then $\mathsf{P}(\exists t \le T : \Delta\widetilde{X}_t \in A) > 0$, while $\mathsf{P}(\exists t \le T : \Delta X_t \in A) = 0$.

Exercise 4.12. Combining (4.2) with the inequality

$$\int_{\{|x| \le 1\}} x^2 \nu(dx) < \infty,$$

we get

$$\int_{\{|x| \le 1\}} |x(\sqrt{\rho(x)} \Leftrightarrow 1)|\nu(dx) < \infty,$$

and hence,

$$\int_{\{|x| \le 1\}} |x(\rho(x) \Leftrightarrow 1)|\nu(dx) < \infty,$$

which is the desired statement.

Exercise 4.13. **(a)** The proof of this statement follows the same arguments as the proof of Theorem 4.6 in Case III.

(b) This statement follows immediately from Propositions 4.4 and 4.5.

Exercise 4.14. **(a)** The process $(X_{at};\, t \le T)$ is a compound Poisson process with the Lévy measure $a\nu$. Now, the result follows immediately from Lemma 4.3.

(b) Let X have the characteristics $(b, c, \nu)_H$. Then $(X_{at})_{t \le T}$ has the characteristics $(ab, ac, a\nu)_H$. It follows from Proposition 4.5 that the measures $\mathrm{Law}(X_t;\, t \le T)$ and $\mathrm{Law}(X_{at};\, t \le T)$ are either equivalent or singular. Suppose that they are equivalent. Then it follows from Proposition 4.4 that $c = 0$ and $\nu(\mathbb{R}) < \infty$. Hence, X is a compound Poisson process with a nonzero drift. The proof is now easily completed.

Exercise 4.15. Let $(b, c, \nu)_H$ and $(\widetilde{b}, \widetilde{c}, \widetilde{\nu})_H$ denote the characteristics of X and $\widetilde{X}$, respectively. Let Z be the coordinate process on $D([0, T])$. Set $\mathsf{Q} = \mathrm{Law}(X_t;\, t \le T)$, $\widetilde{\mathsf{Q}} = \mathrm{Law}(\widetilde{X}_t;\, t \le T)$. By the Lévy-Itô decomposition, with Q-probability one there exists a limit

$$Z_t^1 := \lim_{\varepsilon \downarrow 0} \biggl(\sum_{s \le t} \Delta Z_s I(|\Delta Z_s| > \varepsilon) - t \int_{\{|x| > \varepsilon\}} H(x)\nu(dx) \biggr), \quad t \le T,$$

and the convergence is uniform in t. With $\widetilde{\mathsf{Q}}$-probability one there exists a limit

$$\widetilde{Z}_t^1 := \lim_{\varepsilon \downarrow 0} \biggl(\sum_{s \le t} \Delta Z_s I(|\Delta Z_s| > \varepsilon) - t \int_{\{|x| > \varepsilon\}} H(x)\widetilde{\nu}(dx) \biggr), \quad t \le T,$$

and the convergence is uniform in t.

If the limit

$$\lim_{\varepsilon \downarrow 0} \biggl(\int_{\{|x| > \varepsilon\}} H(x)\nu(dx) - \int_{\{|x| > \varepsilon\}} H(x)\widetilde{\nu}(dx) \biggr) \tag{S.2}$$

does not exist, then, obviously, $\widetilde{\mathsf{Q}} \perp \mathsf{Q}$.

Suppose that limit (S.2) exists and equals γ. Then the process $(\widetilde{Z}_t^1)_{t \le T}$ is well defined Q-a.s. and equals $(Z_t^1 - \gamma t)_{t \le T}$. Set $\widetilde{Z}^2 = Z - \widetilde{Z}^1$. Then $\widetilde{Z}^2$ is a Q-Lévy process with the characteristics $(b + \gamma, c, 0)_H$. Let (S_n) be a sequence of partitions of $[0, T]$ with $d(S_n) \to 0$. We can extract a subsequence (S_n') such that

$$\sum_{s_i \in S_n'} (\widetilde{Z}_{s_i}^2 - \widetilde{Z}_{s_{i-1}}^2)^2 \xrightarrow[n \to \infty]{\mathsf{Q}\text{-a.s.}} cT.$$

On the other hand, $\widetilde{Z}^2$ is a $\widetilde{\mathsf{Q}}$-Lévy process with the characteristics $(\widetilde{b}, \widetilde{c}, 0)_H$. Hence, we can extract a subsequence (S_n'') from (S_n') such that

$$\sum_{s_i \in S_n''} (\widetilde{Z}_{s_i}^2 - \widetilde{Z}_{s_{i-1}}^2)^2 \xrightarrow[n \to \infty]{\widetilde{\mathsf{Q}}\text{-a.s.}} \widetilde{c}T.$$

Consequently, $\widetilde{\mathsf{Q}} \perp \mathsf{Q}$.

Exercise 4.16. Let α and $\widetilde{\alpha}$ denote the indices of stability of X and $\widetilde{X}$, respectively.

If $\alpha = \widetilde{\alpha} = 2$, then the statement is proved by considering the quadratic variation of X and $\widetilde{X}$ (compare with the solution of Exercise 4.15).

If $\alpha = 2$ and $\widetilde{\alpha} < 2$, then the paths of X are continuous, while the paths of $\widetilde{X}$ are not continuous.

Now, suppose that $\alpha, \widetilde{\alpha} \in (0,2)$. In this case the Lévy measures of X and $\widetilde{X}$ have the form

$$\frac{\nu(dx)}{dx} = \frac{m_1 I(x<0)}{|x|^{\alpha}} + \frac{m_2 I(x>0)}{|x|^{\alpha}},$$
$$\frac{\widetilde{\nu}(dx)}{dx} = \frac{\widetilde{m}_1 I(x<0)}{|x|^{\widetilde{\alpha}}} + \frac{\widetilde{m}_2 I(x>0)}{|x|^{\widetilde{\alpha}}},$$

respectively. If $m_1 > 0$ and $\widetilde{m}_1 = 0$, then X a.s. has infinitely many positive jumps, while $\widetilde{X}$ has no positive jumps. In a similar way, we consider the cases $m_1 = 0,\ \widetilde{m}_1 > 0;\ m_2 > 0,\ \widetilde{m}_2 = 0;\ m_2 = 0,\ \widetilde{m}_2 > 0$.

The remaining case is $\widetilde{\nu} \sim \nu$. By Proposition 4.5, in this case the measures $\mathrm{Law}(X_t;\, t \le T)$ and $\mathrm{Law}(\widetilde{X}_t;\, t \le T)$ are either equivalent or singular. Suppose that they are equivalent. Taking into account condition (iii) of Proposition 4.4, we conclude that $\nu = \widetilde{\nu}$. Hence, X and $\widetilde{X}$ differ by a drift. We now apply Exercise 4.13.

Exercise 4.17. Since the processes X and $\widetilde{X}$ have different distributions, the random variables X_1 and $\widetilde{X}_1$ have different distributions. Find $A \in \mathcal{B}(\mathbb{R})$ such that $p \neq \widetilde{p}$, where $p := \mathsf{P}(X_1 \in A)$, $\widetilde{p} := \mathsf{P}(\widetilde{X}_1 \in A)$. Then

$$\frac{1}{N}\sum_{n=1}^{N} I(X_n \Leftrightarrow X_{n-1} \in A) \xrightarrow[N\to\infty]{\text{a.s.}} p,$$

while

$$\frac{1}{N}\sum_{n=1}^{N} I(\widetilde{X}_n \Leftrightarrow \widetilde{X}_{n-1} \in A) \xrightarrow[N\to\infty]{\text{a.s.}} \widetilde{p}.$$

Exercise 4.18. It is sufficient to consider two compound Poisson processes with the Lévy measures ν and $\widetilde{\nu}$, respectively, such that neither of the following conditions is satisfied:

$$\widetilde{\nu} \ll \nu, \quad \nu \ll \widetilde{\nu}, \quad \widetilde{\nu} \perp \nu.$$

Exercise 4.19. For $\varepsilon > 0$, we set

$$Y_t^{\varepsilon} = \sum_{s \le t} \Delta X_s I(|\Delta X_s| > \varepsilon) \Leftrightarrow t \int_{\{|x|>\varepsilon\}} h(x)\nu(dx), \quad t \ge 0$$

and $Z^{\varepsilon} = X \Leftrightarrow Y^{\varepsilon}$. It follows from Lemma 4.2 that, for any $s \le t$ and any $\varepsilon > 0$, the random variable $(Y_t^{\varepsilon} \Leftrightarrow Y_s^{\varepsilon}, Z_t^{\varepsilon} \Leftrightarrow Z_s^{\varepsilon})$ is independent of $\mathcal{F}_s$. In order to complete the proof, note that

$$Y_t^{\varepsilon} \Leftrightarrow Y_s^{\varepsilon} \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} X_t^1 \Leftrightarrow X_s^1, \quad Z_t^{\varepsilon} \Leftrightarrow Z_s^{\varepsilon} \xrightarrow[\varepsilon\downarrow 0]{\text{a.s.}} X_t^2 \Leftrightarrow X_s^2.$$

Exercise 4.20. **(a)** Let Z^1 and Z^2 denote the processes given by the Lévy-Itô decomposition of X, i.e.

$$Z_t^1 := \lim_{\varepsilon\downarrow 0}\Big(\sum_{s \le t} \Delta X_s I(|\Delta X_s| > \varepsilon) \Leftrightarrow t \int_{\{|x|>\varepsilon\}} H_a(x)\nu(dx)\Big), \quad t \ge 0$$

and $Z^2 = X \Leftrightarrow Z^1$. Since Z^2 is continuous, we have

$$X_t^1 = \sum_{s \le t} \Delta Z_s^1 I(|\Delta Z_s^1| > a), \quad t \ge 0.$$

It follows from Proposition A.7 (a) that X^1 is a compound Poisson process with the Lévy measure $\nu|_{\{|x|>a\}}$. Furthermore, for any $\varepsilon > 0$, the process X^1 is independent of the process

$$\begin{aligned} Y_t^\varepsilon &:= \sum_{s \le t} \Delta Z_s^1 I(\varepsilon < |\Delta Z_s^1| \le a) \Leftrightarrow t \int_{\{|x|>\varepsilon\}} H_a(x)\nu(dx) \\ &= \sum_{s \le t} \Delta X_s I(|\Delta X_s| > \varepsilon) \Leftrightarrow t \int_{\{|x|>\varepsilon\}} H_a(x)\nu(dx) \Leftrightarrow X_t^1, \quad t \ge 0. \end{aligned}$$

Consequently, X^1 is also independent of $Y := Z^1 \Leftrightarrow X^1$. As X^1 and Y are functionals of Z^1 and since Z^1 is independent of Z^2, we conclude that X^1, Y, and Z^2 are independent.

It follows from Proposition A.7 (a) that, for each $\varepsilon > 0$, Y^ε is a Lévy process. Hence, $Y = \lim_{\varepsilon \downarrow 0} Y^\varepsilon$ is a Lévy process. Consequently, $Y + Z^2$ is a Lévy process. Obviously, $Y + Z^2 = X^2$. We already know that X^1 and X^2 are independent and X^1 has the characteristics $\left(0, 0, \nu|_{\{|x|>a\}}\right)_0$. It has the same characteristics with respect to H_a since $\int_{\{|x|>a\}} H_a(x)\nu(dx) = 0$. Now, we conclude that X^2 has the characteristics $\left(b, c, \nu|_{\{|x| \le a\}}\right)_{H_a}$.

(b) This is a consequence of Lemma 4.2.

Exercise 4.21. The proof is similar to the proof of Theorem 4.6. The only difference is that Proposition A.4 should be used instead of Proposition A.5.

Exercise 4.22. **(a)** Suppose that X is an $(\mathcal{F}_t)$-σ-martingale. Let $(b, c, \nu)_H$ be the characteristics of X. Then its semimartingale characteristics $(B, C, \mathcal{V})_H$ are given by $B_t = bt$, $C_t = ct$, $\mathcal{V}(dt, dx) = dt\nu(dx)$. It follows from Proposition A.11 that

$$\int_{\{|x|>1\}} |x|\nu(dx) < \infty$$

and

$$b + \int_{\{|x|>1\}} x\nu(dx) = 0.$$

By Proposition A.4, for any $t \ge 0$, we have $\mathsf{E}|X_t| < \infty$ and $\mathsf{E}X_t = 0$. Now, the proof is easily completed.

(b) Suppose that X is an $(\mathcal{F}_t)$-semimartingale. Let $(b, c, \nu)_H$ be the characteristics of X. Then the third characteristic $\mathcal{V}$ of the semimartingale e^X is given by $\mathcal{V}(\omega, dt, dx) = dtK(\omega, t, dx)$, where $K(\omega, t, \cdot)$ is the image of ν under the map

$$\mathbb{R} \ni x \Leftrightarrow\!\!\rightarrow e^{X_{t-}(\omega)}(e^x \Leftrightarrow 1) \in \mathbb{R}.$$

Using Proposition A.11, we deduce that

$$\int_{\{|x|>1\}} |e^x \Leftrightarrow 1|\nu(dx) < \infty.$$

It follows from Proposition A.5 that $\mathsf{E}e^{X_t} = e^{\alpha t}$ with some $\alpha \in \mathbb{R}$.

The process $M_t = e^{X_t - \alpha t}$ is an $(\mathcal{F}_t)$-martingale. By Itô's formula,

$$e^{X_t} = e^{\alpha t} M_t = 1 + \int_0^t \alpha e^{\alpha s} M_{s-} ds + \int_0^t e^{\alpha s} dM_s, \quad t \le T.$$

Since the integrand $e^{\alpha s}$ is bounded, the process $\int_0^\cdot e^{\alpha s} dM_s$ is a local martingale. By Proposition A.10, e^X is a local martingale. Therefore, $\int_0^\cdot \alpha e^{\alpha s} M_{s-} ds$ is a local martingale. But this is an increasing process. As a result, $\alpha = 0$. This completes the proof.

Exercise 4.23. The proof is similar to the proof of Theorem 4.7.

Exercise 4.24. This is an immediate consequence of Exercise 4.23 and the Second Fundamental Theorem of Asset Pricing.

Exercise 4.25. Let B, W be two independent Brownian motions. Consider $\mathcal{F}_t = \mathcal{F}_t^{B,W}$. Then B is an $(\mathcal{F}_t)$-Brownian motion and W is an $(\mathcal{F}_t)$-martingale. Suppose that W can be represented as $\int_0^\cdot H_s dB_s$. Then we would have

$$\langle W \rangle_t = \left\langle W, \int_0^\cdot H_s dB_s \right\rangle_t = \int_0^t H_s d\langle W, B \rangle_s = 0, \quad t \ge 0.$$

Exercise 4.26. Let W be a Brownian motion. Consider $B = \int_0^\cdot \operatorname{sgn} W_s dW_s$ and $\mathcal{F}_t = \mathcal{F}_t^W$. Then $\mathcal{F}_t^B$ is strictly smaller that $\mathcal{F}_t$ (see [21; Ch. VI, Corollary 2.2]). On the other hand, any $(\mathcal{F}_t)$-local martingale M started at zero can be represented as

$$M_t = \int_0^t H_s dW_s = \int_0^t H_s \operatorname{sgn} W_s dB_s, \quad t \ge 0.$$

Section 5

Exercise 5.6. Using Fubini's theorem (compare with the proof of Lemma 5.2), for any $0 \le t_0 \le \dots \le t_n$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we get

$$\begin{aligned} &\mathsf{E}\exp\{i\lambda_1((X \circ \tau)_{t_1} \Leftrightarrow (X \circ \tau)_{t_0}) + \dots + i\lambda_n((X \circ \tau)_{t_n} \Leftrightarrow (X \circ \tau)_{t_{n-1}})\} \\ &= \mathsf{E}\exp\{(\tau_{t_1} \Leftrightarrow \tau_{t_0})\Phi(\lambda_1) + \dots + (\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\Phi(\lambda_n)\}, \end{aligned}$$

where Φ is the characteristic exponent of X. The last quantity does not change under the shifts of the collection $(t_0, \dots, t_n)$.

Exercise 5.7. Obviously, for any $t \ge 0$ and any $n \in \mathbb{N}$, we have $\mathsf{E}X_{nt}^2 = n\mathsf{E}X_t^2$. Hence, for any $t \in \mathbb{Q}_+$, $\mathsf{E}X_t^2 = t\mathsf{E}X_1^2$. Using the monotonicity of the function $\varphi(t) = \mathsf{E}X_t^2$, we complete the proof.

Exercise 5.8. We can assume that $(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_1, \mathcal{F}_1, \mathsf{P}_1) \times (\Omega_2, \mathcal{F}_2, \mathsf{P}_2)$ and $X = X(\omega_1)$, $\tau = \tau(\omega_2)$. Then, for any $t \ge 0$, we have

$$\mathsf{E}(X \circ \tau)_t^2 = \int_{\Omega_2} \int_{\Omega_1} X_{\tau_t(\omega_2)}^2(\omega_1) \mathsf{P}_1(d\omega_1) \mathsf{P}_2(d\omega_2) = \int_{\Omega_2} k\tau_t(\omega_2) = k\mathsf{E}\tau_t < \infty,$$

where k is given by Exercise 5.7. The other statements are proved similarly.

Exercise 5.9. For any $k, t \ge 0$, we have

$$X_{\tau_{kt}} \overset{\text{law}}{=} X_{k^\beta \tau_t} \overset{\text{law}}{=} k^{\alpha\beta} X_{\tau_t}.$$

Exercise 5.10. This is an immediate consequence of Example 3.14, Corollary 3.20, and Exercise 3.38.

Exercise 5.11. The proof is similar to the proof of Theorem 5.4.

Exercise 5.12. The proof is similar to the proof of Theorem 5.5.

Exercise 5.13. Without the loss of generality, we may assume that $\eta_m = I(U_m \le p)$, where $(U_m)_{m=1}^\infty$ are independent uniformly distributed on $[0,1]$ random variables that are also independent of κ. Set

$$X_t = \sum_{m=1}^{\kappa} I(U_m \le t), \quad t \le 1$$

and let $(N_t)_{t \le 1}$ be a Poisson process with the intensity λ. Then $X_1 \overset{\text{law}}{=} N_1$, and, for any $0 = t_0 \le \dots \le t_n = 1$, $k \in \mathbb{N}$, we have

$$\begin{aligned}&\operatorname{Law}(X_{t_1} \Leftrightarrow X_{t_0}, \dots, X_{t_n} \Leftrightarrow X_{t_{n-1}} \mid X_1 = k)\\ &= \operatorname{Law}(N_{t_1} \Leftrightarrow N_{t_0}, \dots, N_{t_n} \Leftrightarrow N_{t_{n-1}} \mid N_1 = k)\end{aligned}$$

(see [22; Proposition 3.3]). Consequently, $\operatorname{Law}(X_t;\, t \le 1) = \operatorname{Law}(N_t;\, t \le 1)$. Now, note that $\xi = X_p \overset{\text{law}}{=} N_p$.

Exercise 5.14. (a) Suppose first that τ_T is a degenerate random variable. Then it follows from Proposition A.7 (a) that $\kappa_1, \kappa_2, \dots$ are independent Poisson random variables with the parameters $\tau_T \nu(A_1), \tau_T \nu(A_2), \dots$. By Exercise 5.13, $\xi_1, \xi_2, \dots$ are independent Poisson random variables with the parameter τ_T. The result follows from the strong law of large numbers.

Now, let τ_T be arbitrary. We can assume that $(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_1, \mathcal{F}_1, \mathsf{P}_1) \times (\Omega_2, \mathcal{F}_2, \mathsf{P}_2) \times (\Omega_3, \mathcal{F}_3, \mathsf{P}_3)$ and $X = X(\omega_1)$, $\tau = \tau(\omega_2)$, $\eta_m^n = \eta_m^n(\omega_3)$. Then

$$\begin{aligned}&\mathsf{P}\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \xi_n = \tau_T\Big)\\ &= \int_{\Omega_2} \int_{\Omega_1 \times \Omega_3} I\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N \sum_{m=1}^{\kappa_n(\omega_1,\omega_2,\omega_3)} \eta_m^n(\omega_3) = \tau_T(\omega_2)\Big) \mathsf{P}_1 \times \mathsf{P}_3(d\omega_1, d\omega_3) \mathsf{P}_2(d\omega_2),\end{aligned}$$

and the result follows from the above argument for a degenerate τ_T.

(b) We have

$$[X]_t = ct + \sum_{s \le t} \Delta X_s^2, \quad t \ge 0$$

(see [14; Ch. I, Theorem 4.52]). Using [14; Ch. I, Theorem 4.47] and keeping in mind that τ is continuous, we deduce that $[X \circ \tau]_T = ([X] \circ \tau)_T$. Hence,

$$[X \circ \tau]_T = c\tau_T + \sum_{s \le \tau_T} \Delta X_s^2 = c\tau_T + \sum_{s \le T} \Delta (X \circ \tau)_s^2.$$

(c) This statement is obvious.

Exercise 5.15. For a random process Y and times $s \leq t$, we set $\mathcal{F}^{Y}_{[s,t]} = \sigma(Y_r \Leftrightarrow Y_s;\, r \in [s,t])$. Fix $0 = t_0 \leq \cdots \leq t_n = T$. For any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we have

$$
\begin{aligned}
&\mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0}) + \cdots + i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\} \mid \mathcal{F}^{X\circ\tau}_T\big) \\
&= \mathsf{E}\big[\mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0}) + \cdots + i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\} \mid \mathcal{F}^{X\circ\tau}_{[0,t_1]} \vee \mathcal{F}^{X\circ\tau}_{[t_1,T]} \vee \mathcal{F}^{\tau}_{[t_1,T]}\big) \mid \mathcal{F}^{X\circ\tau}_T\big] \\
&= \mathsf{E}\big[\exp\{i\lambda_2(\tau_{t_2} \Leftrightarrow \tau_{t_1}) + \cdots + i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\} \\
&\qquad \mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0})\} \mid \mathcal{F}^{X\circ\tau}_{[0,t_1]} \vee \mathcal{F}^{X\circ\tau}_{[t_1,T]} \vee \mathcal{F}^{\tau}_{[t_1,T]}\big) \mid \mathcal{F}^{X\circ\tau}_T\big] \\
&= \mathsf{E}\big[\exp\{i\lambda_2(\tau_{t_2} \Leftrightarrow \tau_{t_1}) + \cdots + i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\}\mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0})\} \mid \mathcal{F}^{X\circ\tau}_{[0,t_1]}\big) \mid \mathcal{F}^{X\circ\tau}_T\big] \\
&= \mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0})\} \mid \mathcal{F}^{X\circ\tau}_T\big)\, \mathsf{E}\big(\exp\{i\lambda_2(\tau_{t_2} \Leftrightarrow \tau_{t_1}) + \cdots + i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\} \mid \mathcal{F}^{X\circ\tau}_T\big) \\
&= \cdots = \mathsf{E}\big(\exp\{i\lambda_1(\tau_{t_1} \Leftrightarrow \tau_{t_0})\} \mid \mathcal{F}^{X\circ\tau}_T\big) \dots \mathsf{E}\big(\exp\{i\lambda_n(\tau_{t_n} \Leftrightarrow \tau_{t_{n-1}})\} \mid \mathcal{F}^{X\circ\tau}_T\big).
\end{aligned}
$$

In the third equality we have used the following fact: if ξ is an integrable random variable and $\mathcal{A}$, $\mathcal{B}$ are σ-fields such that $\sigma(\xi) \vee \mathcal{A}$ is independent of $\mathcal{B}$, then $\mathsf{E}(\xi \mid \mathcal{A} \vee \mathcal{B}) = \mathsf{E}(\xi \mid \mathcal{A})$.