Concentrated Advanced Course on

Lévy Processes

Lectures by Ken-iti Sato (Nagoya University)

and mini-course on

Fractional Calculus

Lectures by Francesco Mainardi (University of Bologna)
and Rudolf Gorenflo (Free University of Berlin)

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Auditorium G1, Building 532

Department of Mathematical Sciences, University of Aarhus

Centre for Mathematical Physics and Stochastics — MaPhySto

Program and other information for the Course participants

Contents

1 Course Program

time	Monday	Tuesday	Wednesday	Thursday	Friday
9.00-9.50	registration	Mainardi	Mainardi	Gorenflo	Gorenflo
10.10-11.00	Sato	Sato	Sato	Sato	Sato
11.10-12.00	Sato	Sato	Sato	Sato	Sato
12.30-14.00	lunch	lunch	lunch	lunch	lunch (sign up!)
14.00-14.50	Rüdiger	Levendorskiǐ	McCrorie/exercises	exercises	
15.20-16.20	exercises	exercises	exercises	exercises	
16.30-18.00	reception				

2 Summary of main course on Lévy Processes

Lévy processes are stochastic processes on the Euclidean space, stochastically continuous and with stationary independent increments. Examples are Brownian motion, Poisson processes, stable processes (such as Cauchy processes), and subordinators (such as Gamma-processes). They form a basic class in stochastic analysis. This course aims at giving an introduction to elementary properties of Lévy process and to transformations between Lévy processes. Familiarity with the method of characteristic functions and some knowledge of Brownian motion, Poisson processes, and infinitely divisible distributions are expected. The following are the main contents of the lectures.

1. Characterization of Lévy processes by the Lévy-Khintchine representation of infinitely divisible distributions. Probabilistic meaning of the characterization.

2. Transformations of Lévy processes to Lévy processes. Especially, the subordination invented by Bochner and the density transformation (mutual absolute continuity in the path space in finite time) of Skorohod, Newman, and Kunita-S.Watanabe will be discussed in detail.

3. Large time behaviors of Lévy processes. Especially recurrence, transience, and oscillation are characterized. (Chung-Fuchs, Spitzer, Port-Stone, Shepp, Kesten, Erickson)

4. Time evolution of unimodality and multimodality of the distributions of Lévy processes on the line. (Wolfe, Yamazato, Sato, Toshiro Watanabe)

3 Summary of mini-course on Fractional Calculus in the Theory of Probability and Stochastic Processes

As an introduction we briefly review the essentials of the fractional calculus according to different approaches that can be useful for our applications in the theory of probability and stochastic processes.

We discuss the linear operators of fractional integration and fractional differentiation, which were introduced in pioneering works by Abel, Liouville, Riemann, Weyl, Marchaud, M. Riesz, Feller and Caputo. Particular attention is devoted to the techniques of Fourier and Laplace transforms for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor.

Furthermore, we discuss the approach based on limit of difference quotients, formerly introduced by Grünwald and Letnikov, which provides a discrete view-point to the fractional calculus. Such approach is very useful for actual numerical computation and is complementary to the previous integral approaches, which provide the continuous view-point.

We also give some information on the transcendental functions of the Mittag-Leffler and Wright type which, together with the most common Eulerian functions, turn out to play a fundamental role in the theory and applications of the fractional calculus.

Fractional calculus allows one to generalize the linear, one-dimensional, diffusion equation by replacing either the second space derivative or the first time derivative by a space or time derivative of fractional order, α or β, respectively. Correspondingly, the generalized equation is referred to as the *space-fractional* diffusion equation of order α or the *time-fractional* diffusion equation of order β, provided that its fundamental solution for the Cauchy problem can be interpreted as a (no longer Gaussian) probability density evolving in time.

For the *space-fractional* diffusion equation of order α $(0 < \alpha \le 2)$ we generate the class of Lévy stable densities of index α according to the Feller parameterization. We thus obtain a special class of Markovian processes, called stable Lévy motions, which for $\alpha \neq 2$ exhibit infinite variance associated to the possibility of arbitrarily large jumps (*Lévy flights*).

For the *time-fractional* diffusion equation of order β $(0 < \beta < 2)$ we generate a class of symmetric densities whose moments of order $2n$ are proportional to the $n\,\beta$ power of time. We thus obtain a class of stochastic processes which for $\beta \neq 1$ are non-Markovian and exhibit a variance consistent with anomalous diffusion.

We also briefly consider the *space-time-fractional* diffusion equation, namely the cases with $\alpha \neq 2$ and $\beta \neq 1$.

In the *space-fractional* case we approximate these processes by random walk models among which we roughly distinguish four types:

(a) discrete in space, discrete in time,

(b) discrete in space, continuous in time,

(c) continuous in space, discrete in time,

(d) continuous in space, continuous in time.

In type (a) jumps of an integer multiple of a basic step-length occur at equidistant instants of time. In type (b) such jumps can occur at any instant of time, the waiting time from one jump to the next being characterized by an exponential waiting time distribution. In type (c) jumps (obeying a continuous jump-width distribution) to any point in space occur at equidistant time instants. In type (d) jumps to any point on space can occur at any instant of time and again there is exponentially distributed waiting time between two successive jumps.

We give examples for these four types and show how, via properly scaled transition to vanishing space or time step or by proper scaling of the independent variables, these four types are related to space-fractional diffusion processes and among each other. Then, as an example of a more exotic model we discuss the famous *Weierstrass* random walk.

Finally, we present a model of type (a) for the *time-fractional* diffusion process.

4 Abstracts of additional talks

Sergei Levendorskiĭ (with S.I. Boyarchenko)

Rostov State Academy of Economics

Generalization of the Black-Scholes equation for truncated Lévy processes, with applications

ABSTRACT: We define a class of generalized truncated Lévy processes (GTLP), which contains variance gamma processes, hyperbolic processes, processes of normal inverse gaussian type and Lévy processes of Koponen's family. With a market of a riskless bond and a stock, whose returns follow a GTLP, and an equivalent martingale measure (EMM), we associate GTLP-analogs of the Black-Scholes formula and equation; the GTLP-analog of the Black-Scholes equation is a pseudo-differential equation. We discuss possible ways of fitting parameters of GTLP and EMM to data, and study main properties of GTLP-analogs of the Black-Scholes equation. We show that these properties essentially depend on a process but not on a choice of EMM. We construct a locally risk-minimizing portfolio and produce numerical examples to show how option prices and hedging ratios depend on characteristics of a process and on a choice of EMM. We apply the generalized Black-Scholes Equations to Pricing of the Perpetual American Put and Barrier Options.

Roderick McCrorie

University of Essex

Applications of the fractional calculus to econometrics

ABSTRACT: In this talk we explain how the fractional calculus has been used to solve the problem of finding the exact distribution of the Wald statistic and how it underlies a method of constructing efficient tests of non-stationary hypotheses. In statistical distribution theory, the Weyl definition of a fractional integral (or derivative) has been more useful than the Riemann–Liouville definition.

Barbara Rüdiger

Institut für Angewandte Mathematik, Universität Bonn

Construction by subordination of processes with jumps on infinite dimensional state spaces and corresponding non local Dirichlet forms

Abstract: We show how symmetric non local, quasi-regular Dirichlet forms on infinite dimensional state spaces can be constructed by subordination. We start by considering any symmetric Markov semigroup $(T_t)_{t\geq 0}$ on $L^2(X,m)$, with (X,m) any measure space, and consider the subordinate Markov semigroup $(T_t^f)_{t\geq 0}$ with subordinator corresponding to any Bernstein function f. We construct the generator on its whole domain. We analyze properties of essentially self-adjointness of the subordinate generators, and properties of closability and quasi-regularity of the subordinate Dirichlet forms. We then show how these results can be applied to construct processes with jumps on infinite dimensional state spaces and apply them as a particular example to the subordinate of Ornstein-Uhlenbeck processes.

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