MaPhySto
The Danish National Research Foundation:
Network in Mathematical Physics and Stochastics

Research Report<br>no. 25 November 2004

# Ole E. Barndorff-Nielsen and Steen Thorbjørnsen: Bicontinuity of the Upsilon Transformations 

# Bicontinuity of the Upsilon transformations 

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#### Abstract

In the papers [BT3] and [BT4], the authors introduced and studied one-to-one mappings $\Upsilon$ and $\Upsilon^{\alpha}(\alpha \in] 0,1[)$ from the class $\mathcal{J D}(*)$ of infinitely divisible probability measures on $\mathbb{R}$ into itself. In particular it was proved that these mappings are continuous, when $\mathcal{J D}(*)$ is endowed with the topology corresponding to weak convergence. In the present note we prove that the $\Upsilon$-mappings are homeomorphisms onto their ranges, which are closed subsets of $\mathfrak{J D}(*)$.


## 1 Introduction.

The paper [BT3] introduced a mapping $\Upsilon: \mathcal{J D}(*) \rightarrow \mathcal{J D}(*)$, where $\mathcal{J D}(*)$ denotes the class of infinitely divisible probability measures on the real line. (Here the $*$ refers to the usual convolution of probability measures). For a measure $\mu$ in $\mathcal{J D}(*), \Upsilon(\mu)$ may be characterized as the measure in $\mathfrak{J D}(*)$ satisfying

$$
\begin{equation*}
C_{\Upsilon(\mu)}(y)=\int_{0}^{\infty} C_{\mu}(y x) \mathrm{e}^{-x} \mathrm{~d} x, \quad(y \in \mathbb{R}), \tag{1.1}
\end{equation*}
$$

where, for any measure $\nu$ from $\mathfrak{J D}(*), C_{\nu}$ denotes the (classical) cumulant transform of $\nu$, i.e. the logarithm of the characteristic function of $\nu$.

As consequences of (1.1), the mapping $\Upsilon$ can be seen to have the following properties
(i) $\Upsilon$ is injective, but not onto.
(ii) For any measures $\mu, \nu$ in $\mathcal{J D}(*), \Upsilon(\mu * \nu)=\Upsilon(\mu) * \Upsilon(\nu)$.
(iii) For any measure $\mu$ in $\mathcal{J D}(*)$ and any constant $c$ in $\mathbb{R}, \Upsilon\left(D_{c} \mu\right)=D_{c} \Upsilon(\mu)$.
(iv) For any constant $c$ in $\mathbb{R}, \Upsilon\left(\delta_{c}\right)=\delta_{c}$ (where $\delta_{c}$ is the Dirac measure at $c$ ).
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(v) $\Upsilon$ is continuous with respect to weak convergence.

In (ii) above, $D_{c} \mu$ is the Dirac measure at 0 , when $c=0$, and, if $c \neq 0$, it is the measure given by $D_{c} \mu(B)=\mu\left(c^{-1} B\right)$, for any Borel-set $B$. It is immediate from the properties (ii)(iv) that $\Upsilon$ preserves the concepts of stability and selfdecomposability. In fact, one may verify (cf. [BT3]) that $\Upsilon(\mathcal{S}(*))=\mathcal{S}(*)$, where $\mathcal{S}(*)$ denotes the class of stable distributions on $\mathbb{R}$ and, in the subsequent paper [BT4] we proved that $\Upsilon$ maps the class $\mathcal{L}(*)$ of selfdecomposable distributions on $\mathbb{R}$ onto the subclass $\mathcal{T}(*)$; the so-called Thorin class.
The results mentioned above indicate that $\Upsilon$ has intrinsic interest within classical probability. The original motivation for introducing $\Upsilon$ was, however, its connection to free probability. Thus, $\Upsilon$ has the property that for any measure $\mu$ in $\mathcal{J D}(*)$,

$$
\begin{equation*}
C_{\Upsilon(\mu)}(y)=\mathfrak{C}_{\Lambda(\mu)}(\mathrm{i} y), \quad(y \in \mathbb{R}) \tag{1.2}
\end{equation*}
$$

where $\mathcal{C}_{\nu}$ denotes the free cumulant transform of a probability measure $\nu$, and $\Lambda$ is the Bercovici-Pata bijection from $\mathcal{J D}(*)$ onto its counterpart $\mathcal{J D}(\boxplus)$ in free probability. We refer to [BT1] for background material on free probability theory but mention at the same time, that there will be no direct use of free probability in the present note.
Another feature of $\Upsilon$, which has lead to further studies (see e.g. [BNMS]), is the fact, established in [BT3], that $\Upsilon(\mu)$ may be realized as the distribution of the stochastic integral

$$
\begin{equation*}
\int_{0}^{1}-\log (1-t) \mathrm{d} X_{t} \tag{1.3}
\end{equation*}
$$

where $\left(X_{t}\right)$ is the Lévy process satisfying that the distribution of $X_{1}$ is $\mu$.
In the paper [BT4] we also introduced a one-parameter family $\left(\Upsilon^{\alpha}\right)_{\alpha \in[0,1]}$ of mappings $\Upsilon^{\alpha}: \mathcal{J D}(*) \rightarrow \mathcal{J D}(*)$, such that $\Upsilon^{0}=\Upsilon$ and $\Upsilon^{1}$ is the identity mapping on $\mathcal{J D}(*)$. For each $\alpha$, the mapping $\Upsilon^{\alpha}$ has properties similar to those mentioned above for $\Upsilon$; in particular it has a realization in terms of a stochastic integral similar to (1.3).
It was proved in [BT1] that the Bercovici-Pata bijection $\Lambda: \mathcal{J D}(*) \rightarrow \mathcal{J D}(\boxplus)$ is a homeomorphism with respect to weak convergence. In view of (1.2), it is natural to ask whether property (v) above can be strengthened to the statement that $\Upsilon$ is a homeomorphism onto its range (which is the so-called Goldie-Steutel-Bondesson class, $\mathcal{B}(*)$, as proved in [BNMS]). In Section 3 below, we answer this question in the affirmative. We prove furthermore, in Section 4, that for each $\alpha$ in $] 0,1\left[\right.$, the mapping $\Upsilon^{\alpha}: \mathcal{J D}(*) \rightarrow \mathcal{J D}(*)$ is also a homemorphism onto its range, and the range is a closed subset of $\mathcal{J D}(*)$ with respect to weak convergence. The arguments in Section 4 actually work in the case $\alpha=0$ as well. We have chosen, however, to treat this case separately in Section 3, as this is the case of most interest, and since the involved calculations in this case are much more direct. In Section 2, we provide some background material on the mappings $\Upsilon$ and $\Upsilon^{\alpha}$.

## 2 Background.

## Lévy-Khintchine representations

A probability measure $\mu$ on $\mathbb{R}$ belongs to the class $\mathcal{J D}(*)$ of infinitely divisible probability measures, if there exists, for each positive integer $n$, a probability measure $\mu_{n}$ on $\mathbb{R}$, such that

$$
\mu=\underbrace{\mu_{n} * \mu_{n} * \cdots * \mu_{n}}_{n \text { terms }} .
$$

The measures in $\mathcal{J D}(*)$ are characterized as those probability measures $\mu$ for which the cumulant transform $C_{\mu}$ (i.e., the logarithm of the characteristic function) admits the Lévy-Khintchine representation:

$$
\begin{equation*}
C_{\mu}(u)=\mathrm{i} \eta u-\frac{1}{2} a u^{2}+\int_{\mathbb{R}}\left(\mathrm{e}^{\mathrm{i} u t}-1-\mathrm{i} u t 1_{[-1,1]}(t)\right) \rho(\mathrm{d} t), \quad(u \in \mathbb{R}), \tag{2.1}
\end{equation*}
$$

where $\eta$ is a real constant, $a$ is a non-negative constant and $\rho$ is a measure on $\mathbb{R}$ satisfying the conditions:

$$
\rho(\{0\})=0 \quad \text { and } \quad \int_{\mathbb{R}} \min \left\{1, t^{2}\right\} \rho(\mathrm{d} t)<\infty,
$$

i.e. $\rho$ is a Lévy measure. The triplet $(a, \rho, \eta)$ is uniquely determined and is called the characteristic triplet for $\mu$.

In the present paper, we shall often work with the classical version of the Lévy-Khintchine representation, namely

$$
\begin{equation*}
C_{\mu}(u)=\mathrm{i} \gamma u+\int_{\mathbb{R}}\left(\mathrm{e}^{\mathrm{i} u t}-1-\frac{\mathrm{i} u t}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}} \sigma(\mathrm{~d} t), \quad(u \in \mathbb{R}), \tag{2.2}
\end{equation*}
$$

where $\gamma$ is a real constant and $\sigma$ is a finite measure on $\mathbb{R}$. Again, the pair $(\gamma, \sigma)$ is uniquely determined, and it is termed the generating pair for $\mu$.
The relationship between the two representations (2.1) and (2.2) is as follows:

$$
\begin{align*}
a & =\sigma(\{0\}), \\
\rho(\mathrm{d} t) & =\frac{1+t^{2}}{t^{2}} \cdot 1_{\mathbb{R} \backslash\{0\}}(t) \sigma(\mathrm{d} t),  \tag{2.3}\\
\eta & =\gamma+\int_{\mathbb{R}} t\left(1_{[-1,1]}(t)-\frac{1}{1+t^{2}}\right) \rho(\mathrm{d} t) .
\end{align*}
$$

## The mapping $\Upsilon$.

In the paper [BT3], we defined the mapping $\Upsilon: \mathcal{J D}(*) \rightarrow \mathfrak{J D}(*)$ as follows:
2.1 Definition. Let $\mu$ be a probability measure in $\mathcal{J D}(*)$ with characteristic triplet $(a, \rho, \eta)$. Then $\Upsilon(\mu)$ is the measure in $\mathcal{J D}(*)$ with generating triplet $(2 a, \tilde{\rho}, \tilde{\eta})$, where

$$
\begin{equation*}
\tilde{\eta}=\eta+\int_{0}^{\infty}\left(\int_{\mathbb{R}} t\left(1_{[-1,1]}(t)-1_{[-x, x]}(t)\right) D_{x} \rho(\mathrm{~d} t)\right) \mathrm{e}^{-x} \mathrm{~d} x \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\rho}=\int_{0}^{\infty}\left(D_{x} \rho\right) \mathrm{e}^{-x} \mathrm{~d} x \tag{2.5}
\end{equation*}
$$

By a tedious but straightforward calculation (see [BT3]), one may verify that the LévyKhintchine representation for $\Upsilon(\mu)$ takes the form:

$$
\begin{equation*}
C_{\Upsilon(\mu)}(y)=\mathrm{i} \eta y-a y^{2}+\int_{\mathbb{R}}\left[\frac{1}{1-\mathrm{i} y t}-1-\mathrm{i} y t 1_{[-1,1]}(t)\right] \rho(\mathrm{d} t), \quad(y \in \mathbb{R}) \tag{2.6}
\end{equation*}
$$

where $(a, \rho, \eta)$ is the characteristic triplet for $\mu$. As the right hand side of (2.6) is exactly the free Lévy-Khintchine representation for $\mathcal{C}_{\Lambda(\mu)}$ (see [BT2]), that same calculation also verifies the relation (1.2).

### 2.1 The mappings $\Upsilon^{\alpha}$.

For each $\alpha$ in $[0,1]$, we introduced in [BT4] the mapping $\Upsilon^{\alpha}: \mathcal{J D}(*) \rightarrow \mathcal{J D}(*)$ as follows:
2.2 Definition. For a probability measure $\mu$ in $\mathcal{J D}(*)$ with characteristic triplet ( $a, \rho, \eta$ ), we let $\Upsilon^{\alpha}(\mu)$ denote the measure in $\mathcal{J D}(*)$ with characteristic triplet $\left(c_{\alpha} a, \tilde{\rho}_{\alpha}, \eta_{\alpha}\right)$, where

$$
\begin{equation*}
\tilde{\rho}_{\alpha}=\int_{0}^{\infty}\left(D_{x} \rho\right) \zeta_{\alpha}(x) \mathrm{d} x \tag{2.7}
\end{equation*}
$$

and

$$
c_{\alpha}=\frac{2}{\Gamma(2 \alpha+1)},
$$

while

$$
\begin{equation*}
\eta_{\alpha}=\frac{\eta}{\Gamma(\alpha+1)}+\int_{0}^{\infty}\left(\int_{\mathbb{R}} t\left(1_{[-1,1]}(t)-1_{[-x, x]}(t)\right) D_{x} \rho(\mathrm{~d} t)\right) \zeta_{\alpha}(x) \mathrm{d} x \tag{2.8}
\end{equation*}
$$

For $\alpha$ in $] 0,1\left[\right.$, the function $\zeta_{\alpha}$ appearing in (2.7) and (2.8) is the probability density given by

$$
\zeta_{\alpha}(x)=\alpha^{-1} x^{-1-1 / \alpha} \sigma_{\alpha}\left(x^{-1 / \alpha}\right),
$$

where $\sigma_{\alpha}$ denotes the density function of the positive stable law with index $\alpha$ and Laplace transform $\exp \left(-\theta^{\alpha}\right)$. For $\alpha=0, \zeta_{\alpha}(x)=\mathrm{e}^{-x}$, and for $\alpha=1, \zeta_{\alpha}(x) \mathrm{d} x$ should be interpreted as the Dirac measure at 1 . With these conventions, it is apparent that $\Upsilon^{0}=\Upsilon$, whereas $\Upsilon^{1}$ is the identity mapping on $\mathfrak{J D}(*)$. Therefore the family $\left(\Upsilon^{\alpha}\right)_{\alpha \in[0,1]}$ provides a kind of smooth interpolation between $\Upsilon$ and the identity mapping on $\mathcal{J D}(*)$.

Based on the definition of $\Upsilon^{\alpha}$ given above, one may verify (see [BT4]) that the LévyKhintchine representation for $\Upsilon^{\alpha}(\mu)$ is given by

$$
\begin{equation*}
C_{\Upsilon^{\alpha}(\mu)}(y)=\frac{\mathrm{i} \eta y}{\Gamma(\alpha+1)}-\frac{1}{2} c_{\alpha} a y^{2}+\int_{\mathbb{R}}\left(E_{\alpha}(\mathrm{i} y t)-1-\mathrm{i} y \frac{t}{\Gamma(\alpha+1)} 1_{[-1,1]}(t)\right) \rho(\mathrm{d} t), \quad(y \in \mathbb{R}), \tag{2.9}
\end{equation*}
$$

where $(a, \rho, \eta)$ is the characteristic triplet for $\mu$ and $E_{\alpha}$ is the Mittag-Leffler function given by

$$
\begin{equation*}
E_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)}, \tag{2.10}
\end{equation*}
$$

which enters the picture because of the relationship

$$
\begin{equation*}
E_{\alpha}(\mathrm{i} t)=\int_{0}^{\infty} \mathrm{e}^{\mathrm{i} t x} \zeta_{\alpha}(x) \mathrm{d} x, \quad(t>0) . \tag{2.11}
\end{equation*}
$$

Combining (2.9) and (2.11), one obtains the identity (cf. formula (5.5) in [BT4]):

$$
\begin{equation*}
\int_{0}^{\infty} x^{k} \zeta_{\alpha}(x) \mathrm{d} x=\frac{k!}{\Gamma(k \alpha+1)}, \quad\left(k \in \mathbb{N}_{0}\right) \tag{2.12}
\end{equation*}
$$

## 3 Bicontinuity of $\Upsilon$.

3.1 Lemma. Let $\mu$ be a measure in $\mathfrak{J D}(*)$ with generating pair $(\gamma, \sigma)$. We then have

$$
C_{\Upsilon(\mu)}(y)=\mathrm{i} \gamma y+\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma(\mathrm{~d} t), \quad(y \in \mathbb{R}) .
$$

Proof. Let $(a, \rho, \eta)$ be the characteristic triplet for $\mu$. According to (2.6) we have for any $y$ in $\mathbb{R}$ that

$$
\begin{align*}
C_{\Upsilon(\mu)}(y) & =\mathrm{i} \eta y-a y^{2}+\int_{\mathbb{R}}\left[\frac{1}{1-\mathrm{i} y t}-1-\mathrm{i} y t 1_{[-1,1]}(t)\right] \rho(\mathrm{d} t) \\
& =\mathrm{i} y\left(\eta+\int_{\mathbb{R}} t\left(\frac{1}{1+t^{2}}-1_{[-1,1]}(t)\right) \rho(\mathrm{d} t)\right)-a y^{2}+\int_{\mathbb{R}}\left[\frac{1}{1-\mathrm{i} y t}-1-\frac{\mathrm{i} y t}{\left(1+t^{2}\right)}\right] \rho(\mathrm{d} t) \\
& =\mathrm{i} \gamma y-\sigma(\{0\}) y^{2}+\int_{\mathbb{R} \backslash\{0\}}\left[\frac{1}{1-\mathrm{i} y t}-1-\frac{\mathrm{i} y t}{\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}} \sigma(\mathrm{~d} t), \tag{3.1}
\end{align*}
$$

where we have used the relationship between $(a, \rho, \eta)$ and $(\gamma, \sigma)$ given in (2.3). Note here that

$$
\left(\frac{1}{1-\mathrm{i} y t}-1-\frac{\mathrm{i} y t}{\left(1+t^{2}\right)}\right) \frac{1+t^{2}}{t^{2}}=\mathrm{i} y t\left(\frac{1}{1-i y t}-\frac{1}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}}=\frac{\mathrm{i} y}{t}\left(\frac{1+t^{2}}{1-\mathrm{i} y t}-1\right)=\frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t},
$$

which combined with (3.1) yields that

$$
C_{\Upsilon(\mu)}(y)=\mathrm{i} \gamma y+\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma(\mathrm{~d} t),
$$

as desired.
3.2 Proposition. Let $\left(\mu_{n}\right)$ be a sequence of measures from $\mathcal{J D}(*)$ with generating pairs $\left(\gamma_{n}, \sigma_{n}\right)$.
(i) If $\lim _{n \rightarrow \infty} C_{\Upsilon(\mu)}\left(y_{0}\right)$ exists in $\mathbb{C}$ for some non-zero real number $y_{0}$, then

$$
\sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R})<\infty .
$$

(ii) Assume that there exist $\epsilon>0$ such that

$$
C(y):=\lim _{n \rightarrow \infty} C_{\Upsilon(\mu)}(y) \quad \text { exists in } \mathbb{C} \text { for all } y \text { in }[0, \epsilon[.
$$

If the limit function $y \mapsto C(y)$ is continuous at 0 , then the sequence $\left(\sigma_{n}\right)$ is tight.

Proof.
(i) Note first that for any $y$ in $\mathbb{R}$,

$$
\operatorname{Re}\left(\frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t}\right)=\frac{-y^{2}\left(1+t^{2}\right)}{1+t^{2} y^{2}}
$$

and hence Lemma 3.1 yields that

$$
-\operatorname{Re}\left(C_{\Upsilon\left(\mu_{n}\right)}(y)\right)=\int_{\mathbb{R}} \frac{y^{2}\left(1+t^{2}\right)}{1+t^{2} y^{2}} \sigma_{n}(\mathrm{~d} t),
$$

for any $n$ in $\mathbb{N}$. For fixed $y$, note next that

$$
\inf _{t \in \mathbb{R}} \frac{y^{2}\left(1+t^{2}\right)}{1+y^{2} t^{2}}=\min \left\{1, y^{2}\right\}
$$

and therefore

$$
\left|C_{\Upsilon\left(\mu_{n}\right)}(y)\right| \geq \int_{\mathbb{R}} \min \left\{1, y^{2}\right\} \sigma_{n}(\mathrm{~d} t)=\min \left\{1, y^{2}\right\} \sigma_{n}(\mathbb{R})
$$

Assume now that $\lim _{n \rightarrow \infty} C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right)$ exists in $\mathbb{C}$ for some $y_{0}$ in $\mathbb{R} \backslash\{0\}$. Then by the estimate above

$$
\sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R}) \leq \max \left\{1, y^{-2}\right\} \sup _{n \in \mathbb{N}}\left|C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right)\right|<\infty,
$$

as desired.
(ii) Assume that the limit function $y \mapsto C(y)$ is continuous at 0 , and let $\delta>0$ be given. Since $C(0)=0$, we may choose $y_{0}$ in $] 0, \epsilon\left[\right.$ such that $\left|C\left(y_{0}\right)\right| \leq \delta$ and, since $C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right) \rightarrow C\left(y_{0}\right)$ as $n \rightarrow \infty$, we may subsequently choose $N$ in $\mathbb{N}$ such that

$$
\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right| \leq 2 \delta, \quad \text { whenever } n \geq N
$$

As in the proof of (i), we have for each $n$ in $\mathbb{N}$ that

$$
\begin{equation*}
-\operatorname{Re}\left(C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right)\right)=\int_{\mathbb{R}} \frac{y_{0}^{2}\left(1+t^{2}\right)}{1+t^{2} y_{0}^{2}} \sigma_{n}(\mathrm{~d} t), \tag{3.2}
\end{equation*}
$$

and here $\frac{y_{0}^{2}\left(1+t^{2}\right)}{1+t^{2} y_{0}^{2}} \rightarrow 1$ as $|t| \rightarrow \infty$. Hence, we may choose $T_{1}>0$ such that

$$
\frac{y_{0}^{2}\left(1+t^{2}\right)}{1+t^{2} y_{0}^{2}} \geq \frac{1}{2}, \quad \text { whenever } t \in\left[-T_{1}, T_{1}\right]^{c} .
$$

It then follows from (3.2) that

$$
\left|C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right)\right| \geq \int_{\left[-T_{1}, T_{1}\right]^{c}} \frac{y_{0}^{2}\left(1+t^{2}\right)}{1+t^{2} y_{0}^{2}} \sigma_{n}(\mathrm{~d} t) \geq \frac{1}{2} \sigma_{n}\left(\left[-T_{1}, T_{1}\right]^{c}\right)
$$

for all $n$ in $\mathbb{N}$, and consequently

$$
\sigma_{n}\left(\left[-T_{1}, T_{1}\right]^{c}\right) \leq 2\left|C_{\Upsilon\left(\mu_{n}\right)}\left(y_{0}\right)\right| \leq 4 \delta, \quad \text { whenever } n \geq N .
$$

Since a finite family of finite measures is automatically tight, we may subsequently choose $T_{2}>0$, such that also

$$
\max _{1 \leq n<N} \sigma_{n}\left(\left[-T_{2}, T_{2}\right]^{c}\right) \leq 4 \delta .
$$

Setting $T=\max \left\{T_{1}, T_{2}\right\}$, it follows that

$$
\sup _{n \in \mathbb{N}} \sigma_{n}\left([-T, T]^{c}\right) \leq 4 \delta,
$$

and since $\delta>0$ was arbitrary, we have proved that $\left(\sigma_{n}\right)$ is tight.

Before stating the main result about the mapping $\Upsilon$, we recall that for probability measures $\mu, \mu_{1}, \mu_{2}, \mu_{3}, \ldots$ on $\mathbb{R}$, we use the notation " $\mu_{n} \xrightarrow{\mathbf{w}} \mu$ as $n \rightarrow \infty$ " to express that the sequence ( $\mu_{n}$ ) is weakly convergent to $\mu$.
3.3 Theorem. Let $\left(\mu_{n}\right)$ be a sequence of measures in $\mathfrak{J D}(*)$ and assume that

$$
\Upsilon\left(\mu_{n}\right) \xrightarrow{\mathrm{w}} \nu, \quad \text { as } n \rightarrow \infty,
$$

for some measure $\nu$ in $\mathcal{J D}(*)$. Then there exists a measure $\mu$ in $\mathcal{J D}(*)$ such that

$$
\mu_{n} \xrightarrow{\mathrm{w}} \mu, \text { as } n \rightarrow \infty \quad \text { and } \quad \nu=\Upsilon(\mu) .
$$

Proof. For each $n$, let $\left(\gamma_{n}, \sigma_{n}\right)$ denote the generating pair for $\mu_{n}$. Since $\Upsilon\left(\mu_{n}\right) \xrightarrow{\mathbf{w}} \nu$ as $n \rightarrow \infty$, we have

$$
C_{\Upsilon\left(\mu_{n}\right)}(y) \longrightarrow C_{\nu}(y), \quad(y \in \mathbb{R})
$$

(cf. [Sa, Lemma 7.7]), where $C_{\nu}$ is continuous at 0 . Hence Proposition 3.2 asserts that

$$
\left(\sigma_{n}\right) \text { is tight } \quad \text { and } \quad \sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R})<\infty,
$$

so in particular the family $\left\{\sigma_{n} \mid n \in \mathbb{N}\right\}$ is conditionally compact (cf. [GK, $\S 9$, Theorem 3 bis]).
The main task of the proof is to show the existence of a real number $\gamma$ and a finite measure $\sigma$ on $\mathbb{R}$, such that

$$
\lim _{n \rightarrow \infty} \gamma_{n}=\gamma, \quad \text { and } \quad \sigma_{n} \xrightarrow{\mathrm{w}} \sigma, \quad \text { as } n \rightarrow \infty .
$$

For this, it suffices to show the existence of $\gamma$ and $\sigma$, such that any subsequence ( $\gamma_{n^{\prime}}, \sigma_{n^{\prime}}$ ) of ( $\sigma_{n}, \gamma_{n}$ ) has a subsequence $\left(\gamma_{n^{\prime \prime}}, \sigma_{n^{\prime \prime}}\right)$, which converges (coordinate-wise) to $(\gamma, \sigma)$. So let $\left(\gamma_{n^{\prime}}, \sigma_{n^{\prime}}\right)$ be an arbitrary subsequence of $\left(\gamma_{n}, \sigma_{n}\right)$. Since the family $\left\{\sigma_{n} \mid n \in \mathbb{N}\right\}$ is conditionally compact, there exists a subsequence $\left(\sigma_{n^{\prime \prime}}\right)$ of ( $\sigma_{n^{\prime}}$ ) and a finite measure $\sigma$ on $\mathbb{R}$, such that $\sigma_{n^{\prime \prime}} \xrightarrow{\mathbf{w}} \sigma$. By Lemma 3.1, we have for each $n$ in $\mathbb{N}$ that

$$
C_{\Upsilon\left(\mu_{n}\right)}(y)=\mathrm{i} \gamma_{n} y+\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma_{n}(\mathrm{~d} t), \quad(y \in \mathbb{R}) .
$$

For fixed $y$ in $\mathbb{R}$, the function $t \mapsto \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t}$ is bounded and continuous, and hence

$$
\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma_{n^{\prime \prime}}(\mathrm{d} t) \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma(\mathrm{~d} t), \quad \text { as } n \rightarrow \infty
$$

Since also

$$
\mathrm{i} \gamma_{n^{\prime \prime}} y+\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma_{n^{\prime \prime}}(\mathrm{d} t)=C_{\Upsilon\left(\mu_{n^{\prime \prime}}\right)}(y) \longrightarrow C_{\nu}(y), \quad \text { as } n \rightarrow \infty
$$

we conclude that the subsequence $\left(\gamma_{n^{\prime \prime}}\right)$ must converge to some real number $\gamma$, which then has to satisfy the equation:

$$
\begin{equation*}
C_{\nu}(y)=\mathrm{i} \gamma y+\int_{\mathbb{R}} \frac{\mathrm{i} y(t+\mathrm{i} y)}{1-\mathrm{i} y t} \sigma(\mathrm{~d} t), \quad \text { for all } y \text { in } \mathbb{R} \tag{3.3}
\end{equation*}
$$

Now, let $\mu$ be the measure in $\mathcal{J D}(*)$ with generating pair $(\gamma, \sigma)$. Then by Lemma 3.1 and (3.3),

$$
C_{\Upsilon(\mu)}(y)=C_{\nu}(y), \quad(y \in \mathbb{R})
$$

and hence $\Upsilon(\mu)=\nu$. Since $\Upsilon$ is injective, this implies, in addition, that the pair $(\gamma, \sigma)$ is uniquely determined (as the generating pair for $\Upsilon^{-1}(\nu)$ ). In summary, we have singled out a real number $\gamma$ and a finite measure $\sigma$ on $\mathbb{R}$ with the property that any subsequence of $\left(\gamma_{n}, \sigma_{n}\right)$ has a subsequence converging (coordinate-wise) to $(\gamma, \sigma)$. As mentioned above, this means that the whole sequence ( $\gamma_{n}, \sigma_{n}$ ) converges (coordinate-wise) to ( $\gamma, \sigma$ ). Appealing finally to Gnedenko's Theorem (cf. [GK, §19, Theorem 1]), we may deduce that $\mu_{n} \xrightarrow{\mathrm{~W}} \mu$, as $n \rightarrow \infty$, and this completes the proof.
3.4 Corollary. Let $\mathcal{B}(*)$ denote the range of the mapping $\Upsilon$, i.e. $\mathcal{B}(*)=\Upsilon(\mathcal{J D}(*))$. Then $\mathcal{B}(*)$ is a closed subset of $\mathcal{J} \mathcal{D}(*)$ with respect to weak convergence, and the mapping $\Upsilon: \mathcal{J D}(*) \rightarrow \mathcal{B}(*)$ is a homeomorphism with respect to weak convergence.

Proof. It was proved in [BT3] that $\Upsilon$ is continuous. The remaining assertions follow immediately from Theorem 3.3.
3.5 Remark. As previously mentioned, it was proved in $[\mathrm{BNMS}]$ that $\mathcal{B}(*)$ is the socalled Goldie-Steutel-Bondesson class. As this class is, by definition, closed in the topology for weak convergence, the cited result from [BNMS] also shows that $\mathcal{B}(*)$ is closed with respect to weak convergence.

## 4 Bicontinuity of $\left.\Upsilon^{\alpha}, \alpha \in\right] 0,1[$.

Recall that $E_{\alpha}$ denotes the Mittag-Leffler function (cf. (2.10)).
4.1 Lemma. Let $\mu$ be a measure in $\mathfrak{J D}(*)$ with characteristic triplet $(\gamma, \sigma)$. Then for any $\alpha$ in $] 0,1[$ we have

$$
C_{\Upsilon^{\alpha}(\mu)}(y)=\frac{\mathrm{i} \gamma y}{\Gamma(\alpha+1)}+\int_{\mathbb{R}} g_{\alpha}(t, y) \sigma(\mathrm{d} t), \quad(y \in \mathbb{R})
$$

where $g_{\alpha}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ is the function given by

$$
g_{\alpha}(t, y)= \begin{cases}{\left[E_{\alpha}(\mathrm{i} y t)-1-\frac{\mathrm{i} y t}{\Gamma(\alpha+1)\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}},} & \text { if } t \neq 0,  \tag{4.1}\\ -\frac{1}{2} c_{\alpha} y^{2}, & \text { if } t=0,\end{cases}
$$

and $c_{\alpha}=\frac{2}{\Gamma(2 \alpha+1)}$.
Proof. Let $(a, \rho, \eta)$ be the characteristic triplet for $\mu$. According to (2.9) we have for any $y$ in $\mathbb{R}$ that

$$
\begin{aligned}
C_{\Upsilon^{\alpha}(\mu)}(y)= & \frac{\mathrm{i} \eta y}{\Gamma(\alpha+1)}-\frac{1}{2} c_{\alpha} a y^{2}+\int_{\mathbb{R}}\left[E_{\alpha}(\mathrm{i} y t)-1-\frac{\mathrm{i} y t}{\Gamma(\alpha+1)} 1_{[-1,1]}(t)\right] \rho(\mathrm{d} t) \\
= & \frac{\mathrm{i} y}{\Gamma(\alpha+1)}\left(\eta+\int_{\mathbb{R}} t\left(\frac{1}{1+t^{2}}-1_{[-1,1]}(t)\right) \rho(\mathrm{d} t)\right)-\frac{1}{2} c_{\alpha} a y^{2} \\
& +\int_{\mathbb{R}}\left[E_{\alpha}(\mathrm{i} y t)-1-\frac{\mathrm{i} y t}{\Gamma(\alpha+1)\left(1+t^{2}\right)}\right] \rho(\mathrm{d} t) \\
= & \frac{\mathrm{i} \gamma y}{\Gamma(\alpha+1)}-\frac{1}{2} c_{\alpha} \sigma(\{0\}) y^{2}+\int_{\mathbb{R} \backslash\{0\}}\left[E_{\alpha}(\mathrm{i} y t)-1-\frac{\mathrm{i} y t}{\Gamma(\alpha+1)\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}} \sigma(\mathrm{~d} t) \\
= & \frac{\mathrm{i} \gamma y}{\Gamma(\alpha+1)}+\int_{\mathbb{R}} g_{\alpha}(t, y) \sigma(\mathrm{d} t),
\end{aligned}
$$

where we have used the relationship between $(a, \rho, \eta)$ and $(\gamma, \sigma)$ given in (2.3).
4.2 Lemma. Let $\alpha$ be a number in $] 0,1\left[\right.$, and consider the Mittag-Leffler function $E_{\alpha}$. Consider further the constant $c_{\alpha}=\frac{2}{\Gamma(2 \alpha+1)}$. We then have
(i) For any real number $y$,

$$
\left[E_{\alpha}(\mathrm{i} y t)-1-\frac{\mathrm{i} y t}{\Gamma(\alpha+1)\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}} \longrightarrow-\frac{1}{2} c_{\alpha} y^{2} \quad \text { for }|t| \searrow 0
$$

(ii) For any real number $y$,

$$
\left[1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)\right] \frac{1+t^{2}}{t^{2}} \longrightarrow \frac{1}{2} c_{\alpha} y^{2} \quad \text { for }|t| \searrow 0
$$

(iii) For any $y$ in $\mathbb{R} \backslash\{0\}$ we have

$$
E_{\alpha}(\mathrm{i} y t) \longrightarrow 0, \quad \text { as }|t| \rightarrow \infty
$$

(iv) For any $y$ in $\mathbb{R} \backslash\{0\}$ we have

$$
\inf _{t \in \mathbb{R} \backslash\{0\}}\left[1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)\right] \frac{1+t^{2}}{t^{2}}>0 .
$$

## Proof.

(i) We recall first that $s \mapsto E_{\alpha}(\mathrm{i} s)$ is the characteristic function for $\zeta_{\alpha}(x) \mathrm{d} x$ (cf. (2.11)), so that

$$
E_{\alpha}(\mathrm{i} y t)=\int_{0}^{\infty} \mathrm{e}^{\mathrm{i} y t x} \zeta_{\alpha}(x) \mathrm{d} x=\int_{0}^{\infty} \cos (y t x) \zeta_{\alpha}(x) \mathrm{d} x+\mathrm{i} \int_{0}^{\infty} \sin (y t x) \zeta_{\alpha}(x) \mathrm{d} x
$$

for any $t, y$ in $\mathbb{R}$. Recalling further that $\int_{0}^{\infty} x^{k} \zeta_{\alpha}(x) \mathrm{d} x=\frac{k!}{\Gamma(k \alpha+1)}$ for all $k$ in $\mathbb{N}_{0}$ (cf. (2.12)), we thus find that

$$
\begin{align*}
& {\left[E_{\alpha}(\mathrm{i} t y)-1-\frac{\mathrm{i} t y}{\Gamma(\alpha+1)\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}}} \\
& \quad=\left(1+t^{2}\right) \int_{0}^{\infty}\left[\frac{\cos (y t x)-1}{t^{2}}\right] \zeta_{\alpha}(x) \mathrm{d} x+\mathrm{i} \int_{0}^{\infty}\left[\sin (y t x)-\frac{y t x}{\left(1+t^{2}\right)}\right] \frac{1+t^{2}}{t^{2}} \zeta_{\alpha}(x) \mathrm{d} x . \tag{4.2}
\end{align*}
$$

By second order Taylor expansion we have for fixed $x$ in $\mathbb{R}$ that

$$
\frac{\cos (y t x)-1}{t^{2}} \longrightarrow-\frac{1}{2} y^{2} x^{2}, \quad \text { as }|t| \searrow 0
$$

and that

$$
\left|\frac{\cos (y t x)-1}{t^{2}}\right| \leq \frac{1}{2} y^{2} x^{2}, \quad \text { for all } t \text { in } \mathbb{R} \backslash\{0\}
$$

Hence, by dominated convergence,

$$
\begin{align*}
\left(1+t^{2}\right) \int_{0}^{\infty}\left[\frac{\cos (y t x)-1}{t^{2}}\right] \zeta_{\alpha}(x) \mathrm{d} x & \underset{t \rightarrow 0}{\longrightarrow} \int_{0}^{\infty}-\frac{1}{2} x^{2} y^{2} \zeta_{\alpha}(x) \mathrm{d} x=-\frac{1}{2} y^{2} \int_{0}^{\infty} x^{2} \zeta_{\alpha}(x) \mathrm{d} x \\
& =-\frac{1}{2} y^{2} \frac{2}{\Gamma(2 \alpha+1)}=-\frac{1}{2} c_{\alpha} y^{2} \tag{4.3}
\end{align*}
$$

for each fixed $y$ in $\mathbb{R}$. Regarding the second term in (4.2), note first that for fixed $x$ in $\mathbb{R}$ and $t$ in $\mathbb{R} \backslash\{0\}$

$$
\left(\sin (y t x)-\frac{y t x}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}}=\frac{\sin (y t x)-y t x}{t^{2}}+\sin (y t x) .
$$

By second order Taylor expansion, it thus follows that

$$
\left(\sin (y t x)-\frac{y t x}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}} \longrightarrow 0, \quad \text { as }|t| \searrow 0
$$

and that

$$
\left|\left(\sin (y t x)-\frac{y t x}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}}\right| \leq \frac{1}{2} y^{2} x^{2}+1, \quad(t \in \mathbb{R} \backslash\{0\})
$$

Hence, by dominated convergence,

$$
\begin{equation*}
\int_{0}^{\infty}\left(\sin (y t x)-\frac{y t x}{1+t^{2}}\right) \frac{1+t^{2}}{t^{2}} \zeta_{\alpha}(x) \mathrm{d} x \longrightarrow 0, \quad \text { as }|t| \searrow 0 \tag{4.4}
\end{equation*}
$$

Inserting (4.3) and (4.4) in (4.2), assertion (i) follows readily.
(ii) This statement follows immediately by taking real parts in (i).
(iii) According to [HTF, Formula (7), page 207], we have the estimate

$$
E_{\alpha}(z)=\frac{-z^{-1}}{\Gamma(1-\alpha)}+O\left(|z|^{-2}\right), \quad \text { for }|z| \rightarrow \infty, \alpha \frac{\pi}{2}<|\arg (z)| \leq \pi
$$

Since $\alpha \in] 0,1[$, it follows in particular that

$$
E_{\alpha}(\mathrm{i} s)=\frac{\mathrm{i} s^{-1}}{\Gamma(1-\alpha)}+O\left(s^{-2}\right), \quad \text { for }|s| \rightarrow \infty
$$

and (iii) follows.
(iv) According to (ii) and (iii) we have for any $y$ in $\mathbb{R} \backslash\{0\}$ that

$$
\lim _{t \rightarrow 0}\left[1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)\right] \frac{1+t^{2}}{t^{2}}=\frac{1}{2} c_{\alpha} y^{2} \quad \text { and } \quad \lim _{|t| \rightarrow \infty}\left[1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)\right] \frac{1+t^{2}}{t^{2}}=1
$$

Hence, by continuity, it suffices to show that

$$
\begin{equation*}
\left[1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)\right] \frac{1+t^{2}}{t^{2}}>0, \quad(t, y \in \mathbb{R} \backslash\{0\}) \tag{4.5}
\end{equation*}
$$

But

$$
1-\operatorname{Re}\left(E_{\alpha}(\mathrm{i} y t)\right)=\int_{0}^{\infty}(1-\cos (y t x)) \zeta_{\alpha}(x) \mathrm{d} x
$$

where $(1-\cos (y t x)) \zeta_{\alpha}(x) \geq 0$ for all $x$, and since equality does not hold for almost all $x$ w.r.t. Lebesgue measure (assuming that $t, y \neq 0$ ), (4.5) follows readily.
4.3 Proposition. Let $\alpha$ be a fixed number in $] 0,1\left[\right.$ and, for each $n$ in $\mathbb{N}$, let $\mu_{n}$ be a measure in $\mathfrak{J D}(*)$ with generating pair $\left(\gamma_{n}, \sigma_{n}\right)$.
(i) If $\lim _{n \rightarrow \infty} C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)$ exists in $\mathbb{C}$ for some $y_{0}$ in $] 0, \infty\left[\right.$, then $\sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R})<\infty$.
(ii) Assume that there exists $\epsilon>0$ such that

$$
C(y):=\lim _{n \rightarrow \infty} C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}(y) \quad \text { exists in } \mathbb{C} \text { for all } y \text { in }[0, \epsilon[.
$$

If the limit function $y \mapsto C(y)$ is continuous at 0 , then the sequence $\left(\sigma_{n}\right)$ is tight.
Proof.
(i) For each $n$ in $\mathbb{N}$, it follows from Lemma 4.1 that

$$
\begin{equation*}
-\operatorname{Re}\left(C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right)=\int_{\mathbb{R}} f(t) \sigma_{n}(\mathrm{~d} t) \tag{4.6}
\end{equation*}
$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is the function given by

$$
f(t)= \begin{cases}{\left[1-\operatorname{Re}\left(E_{\alpha}\left(\mathrm{i} y_{0} t\right)\right)\right] \frac{1+t^{2}}{t^{2}},} & \text { if } t \neq 0  \tag{4.7}\\ \frac{1}{2} c_{\alpha} y_{0}^{2}, & \text { if } t=0\end{cases}
$$

According to Lemma 4.2, $f$ is continuous and $c_{0}:=\inf _{t \in \mathbb{R}} f(t)>0$. By (4.6) it follows that

$$
\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right| \geq \int_{\mathbb{R}} f(t) \sigma_{n}(\mathrm{~d} t) \geq c_{0} \sigma_{n}(\mathbb{R})
$$

for all $n$, and hence

$$
\sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R}) \leq \frac{1}{c_{0}} \sup _{n \in \mathbb{N}}\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right|<\infty
$$

assuming that $\lim _{n \rightarrow \infty} C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)$ exists in $\mathbb{C}$.
(ii) Assume that the limit function $y \mapsto C(y)$ is continuous at 0 , and let $\delta>0$ be given. Then we may choose $y_{0}$ in $] 0, \epsilon\left[\right.$ such that $\left|C\left(y_{0}\right)\right| \leq \delta$, and since $C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right) \rightarrow C\left(y_{0}\right)$ as $n \rightarrow \infty$, we may subsequently choose $N$ in $\mathbb{N}$ such that

$$
\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right| \leq 2 \delta, \quad \text { whenever } n \geq N
$$

As in the proof of (i), we have for each $n$ in $\mathbb{N}$ that

$$
\begin{equation*}
-\operatorname{Re}\left(C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right)=\int_{\mathbb{R}} f(t) \sigma_{n}(\mathrm{~d} t) \tag{4.8}
\end{equation*}
$$

with $f$ given in (4.7). By Lemma 4.2, $E_{\alpha}\left(\mathrm{i} y_{0} t\right) \rightarrow 0$ as $|t| \rightarrow \infty$, and hence we may choose $T>0$ such that

$$
f(t)=\left[1-\operatorname{Re}\left(E_{\alpha}\left(\mathrm{i} y_{0} t\right)\right)\right] \frac{1+t^{2}}{t^{2}}>\frac{1}{2}, \quad \text { for all } t \text { in }[-T, T]^{c} .
$$

Then, since $1-\operatorname{Re}\left(E_{\alpha}\left(\mathrm{i} y_{0} t\right)\right) \geq 0$ for all $t$, it follows from (4.8) that

$$
\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right| \geq \int_{[-T, T]^{c}} f(t) \sigma_{n}(\mathrm{~d} t) \geq \frac{1}{2} \sigma_{n}\left([-T, T]^{c}\right)
$$

for all $n$ in $\mathbb{N}$, and consequently

$$
\sigma_{n}\left([-T, T]^{c}\right) \leq 2\left|C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}\left(y_{0}\right)\right| \leq 4 \delta, \quad \text { whenever } n \geq N .
$$

Since any finite family of finite measures is automatically tight, it now follows as in the proof of Proposition 3.2(ii) above, that the sequence $\left(\sigma_{n}\right)$ is tight.
4.4 Theorem. Let $\alpha$ be fixed number in $] 0,1\left[\right.$, and let $\left(\mu_{n}\right)$ be a sequence of measures from $\operatorname{JD}(*)$. Assume that

$$
\Upsilon^{\alpha}\left(\mu_{n}\right) \xrightarrow{\mathrm{w}} \nu, \quad \text { as } n \rightarrow \infty,
$$

for some measure $\nu$ in $\mathfrak{J D}(*)$. Then there exists a measure $\mu$ in $\mathcal{J D}(*)$ such that

$$
\mu_{n} \xrightarrow{\mathrm{w}} \mu, \text { as } n \rightarrow \infty \quad \text { and } \quad \nu=\Upsilon^{\alpha}(\mu) .
$$

Proof. The proof is similar to that of Theorem 3.3, and we shall not repeat all details. For each $n$ in $\mathbb{N}$, let $\left(\gamma_{n}, \sigma_{n}\right)$ denote the generating pair for $\mu_{n}$. Since $\Upsilon^{\alpha}\left(\mu_{n}\right) \xrightarrow{\mathbf{w}} \nu$ as $n \rightarrow \infty$, we have

$$
\lim _{n \rightarrow \infty} C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}(y)=C_{\nu}(y), \quad(y \in \mathbb{R})
$$

where $C_{\nu}$ is continuous at 0 . It follows thus from Proposition 4.3 that

$$
\left(\sigma_{n}\right) \text { is tight } \quad \text { and } \quad \sup _{n \in \mathbb{N}} \sigma_{n}(\mathbb{R})<\infty
$$

and by $\left[\mathrm{GK}, \S 9\right.$, Theorem 3 bis] this implies that the family $\left\{\sigma_{n} \mid n \in \mathbb{N}\right\}$ is conditionally compact. As in the proof of Theorem 3.3, we show the existence of a real number $\gamma$ and a finite measure $\sigma$ on $\mathbb{R}$ such that

$$
\lim _{n \rightarrow \infty} \gamma_{n}=\gamma \quad \text { and } \quad \sigma_{n} \xrightarrow{\mathrm{w}} \sigma, \quad \text { as } n \rightarrow \infty .
$$

For that, it suffices to verify the existence of $(\gamma, \sigma)$ such that any subsequence $\left(\gamma_{n^{\prime}}, \sigma_{n^{\prime}}\right)$ has a subsequence $\left(\gamma_{n^{\prime \prime}}, \sigma_{n^{\prime \prime}}\right)$ satisfying that

$$
\lim _{n \rightarrow \infty} \gamma_{n^{\prime \prime}}=\gamma \quad \text { and } \quad \sigma_{n^{\prime \prime}} \xrightarrow{\mathrm{w}} \sigma, \quad \text { as } n \rightarrow \infty .
$$

So let $\left(\gamma_{n^{\prime}}, \sigma_{n^{\prime}}\right)$ be a given subsequence of $\left(\gamma_{n}, \sigma_{n}\right)$. Since the family $\left\{\sigma_{n} \mid n \in \mathbb{N}\right\}$ is conditionally compact, there exists a subsequence ( $\sigma_{n^{\prime \prime}}$ ) and a finite measure $\sigma$ on $\mathbb{R}$, such that $\sigma_{n^{\prime \prime}} \xrightarrow{\mathrm{w}} \sigma$ as $n \rightarrow \infty$. Recall now from Lemma 4.1 that

$$
C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}(y)=\frac{\mathrm{i} \gamma_{n} y}{\Gamma(\alpha+1)}+\int_{\mathbb{R}} g_{\alpha}(t, y) \sigma(\mathrm{d} t), \quad(y \in \mathbb{R})
$$

where $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the function given in (4.1). For fixed $y$ in $\mathbb{R}$, it follows from (i) and (iii) of Lemma 4.2 that the function $t \mapsto g(t, y): \mathbb{R} \rightarrow \mathbb{R}$ is bounded and continuous, and hence

$$
\int_{\mathbb{R}} g(t, y) \sigma_{n^{\prime \prime}}(\mathrm{d} t) \longrightarrow \int_{\mathbb{R}} g(t, y) \sigma(\mathrm{d} t) \quad \text { as } n \rightarrow \infty
$$

for each fixed $y$ in $\mathbb{R}$. At the same time we have for each $y$ in $\mathbb{R}$ that

$$
\frac{\mathrm{i} \gamma_{n} y}{\Gamma(\alpha+1)}+\int_{\mathbb{R}} g_{\alpha}(t, y) \sigma_{n}(\mathrm{~d} t)=C_{\Upsilon^{\alpha}\left(\mu_{n}\right)}(y) \underset{n \rightarrow \infty}{\longrightarrow} C_{\nu}(y),
$$

and hence we may conclude that $\gamma_{n^{\prime \prime}} \rightarrow \gamma$ as $n \rightarrow \infty$ for some real number $\gamma$, which then has to satisfy the identity

$$
\begin{equation*}
\frac{\mathrm{i} \gamma y}{\Gamma(\alpha+1)}+\int_{\mathbb{R}} g_{\alpha}(t, y) \sigma(\mathrm{d} t)=C_{\nu}(y) \tag{4.9}
\end{equation*}
$$

for each real number $y$. Now let $\mu$ be the measure in $\mathcal{J D}(*)$ with generating pair $(\gamma, \sigma)$. Then by Lemma 4.1, formula (4.9) asserts that

$$
C_{\Upsilon^{\alpha}(\mu)}(y)=C_{\nu}(y), \quad(y \in \mathbb{R}),
$$

and hence $\Upsilon^{\alpha}(\mu)=\nu$. Since $\Upsilon^{\alpha}$ is injective (cf. [BT4, Corollary 5.7]), this also means that $\mu$, and hence $(\gamma, \sigma)$, is independent of the considered subsequences.

From this point, the proof is completed exactly as that of Theorem 3.3.
4.5 Corollary. For each $\alpha$ in $] 0,1\left[\right.$, the full range $\mathcal{B}_{\alpha}(*):=\Upsilon^{\alpha}(\mathcal{J D}(*))$ of $\Upsilon^{\alpha}$ is a closed subset of $\mathfrak{J D}(*)$ with respect to weak convergence. Furthermore, the bijection $\Upsilon^{\alpha}: \mathfrak{J D}(*) \rightarrow \mathcal{B}_{\alpha}(*)$ is a homeomorphism with respect to weak convergence.

Proof. It was proved in [BT4] that $\Upsilon^{\alpha}$ is continuous with respect to weak convergence. The remaining assertions are immediate consequences of Theorem 4.4

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