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A Change-of-Variable Formula with Local Time on Surfaces

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Abstract**

Let $X = (X^1, \dots, X^n)$ be a continuous semimartingale and let $b : I\!R^{n-1} \to I\!R$ be a continuous function such that the process $b^X = b(X^1, \dots, X^{n-1})$ is a semimartingale. Setting $C = \{ (x_1, \dots, x_n) \in I\!R^n \mid x_n < b(x_1, \dots, x_{n-1}) \}$ and $D = \{ (x_1, \dots, x_n) \in I\!R^n \mid x_n > b(x_1, \dots, x_{n-1}) \}$ suppose that a continuous function $F : I\!R^n \to I\!R$ is given such that F is $C^{i_1, \dots, i_n}$ on $\bar{C}$ and F is $C^{i_1, \dots, i_n}$ on $\bar{D}$ where each i_k equals 1 or 2 depending on whether X^k is of bounded variation or not. Then the following change-of-variable formula holds:

$$
\begin{aligned}
F(X_t) = F(X_0) &+ \sum_{i=1}^{n} \int_0^t \frac{1}{2} \left(\frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^n +) + \frac{\partial F}{\partial x_i}(X_s^1, \dots, X_s^n -) \right) dX_s^i \\
&+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{1}{2} \left(\frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \dots, X_s^n +) + \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \dots, X_s^n -) \right) d\langle X^i, X^j \rangle_s \\
&+ \frac{1}{2} \int_0^t \left(\frac{\partial F}{\partial x_n}(X_s^1, \dots, X_s^n +) - \frac{\partial F}{\partial x_n}(X_s^1, \dots, X_s^n -) \right) I(X_s^n = b_s^X)\; d\ell_s^b(X)
\end{aligned}
$$

where $\ell_s^b(X)$ is the local time of X on the surface b given by:

$$
\ell_s^b(X) = I\!P - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s I(-\varepsilon < X_r^n - b_r^X < \varepsilon)\; d\langle X^n - b^X, X^n - b^X \rangle_r
$$

and $d\ell_s^b(X)$ refers to the integration with respect to $s \mapsto \ell_s^b(X)$. The analogous formula extends to general semimartingales X and b^X as well. A version of the same formula under weaker conditions on F is derived for the semimartingale $((t, X_t, S_t))_{t \ge 0}$ where $(X_t)_{t \ge 0}$ is an Itô diffusion and $(S_t)_{t \ge 0}$ is its running maximum.

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**A full version of the paper can be obtained at the website home.imf.au.dk/goran.

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