"Local Realism", Bell's Theorem and Quantum "Locally Realistic" Inequalities

Elena R. Loubenets*
Department of Mathematical Sciences
University of Aarhus, Denmark

December 18, 2003

Abstract

Based on the general probabilistic framework of randomness, introduced in [6], we analyze in mathematical terms the link between the CHSH inequality and the physical concept of "local realism". We prove that the violation of this inequality in the quantum case has no connection with the violation of "local realism".

In the most general settings, we formulate mathematically a condition on "local realism" under a joint experiment and consider examples of quantum "locally realistic" joint experiments. For any quantum state of a bipartite quantum system, we derive quantum analogs of Bell's inequality under quantum "locally realistic" joint experiments of the Alice/Bob type.

In view of our results, we argue that the violation of the CHSH inequality in the quantum case cannot be a valid argument in the discussion on locality or non-locality of quantum interactions.

Contents

*This work is partially funded by MaPhySto - A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation

1 Introduction

The Bell inequality [1-3] and the CHSH inequality [4] describe the relation between the statistical data observed under joint experiments. The original derivations of these inequalities (and their further numerous generalizations and strengthenings) are based on the structure of probability theory, associated with the formalism of random variables. The latter probabilistic formalism is often referred to as *classical* probability.

The sufficient mathematical condition, used for the derivation of Bell's inequality and the CHSH inequality in the classical probabilistic frame, is usually linked with the physical concept of "local realism". The latter refers (see, for example, [5], page 160) to those situations where, under a joint experiment, set-ups of marginal experiments are chosen independently.

In the quantum case, Bell's inequality is violated and Bell's theorem states that a "locally realistic" model cannot describe statistics under quantum joint experiments.

In this paper, we analyze this statement from the point of view of the general probabilistic formalism, introduced in [6] and based on the notions of an information state and a generalized observable.

For joint experiments upon a system of any type, represented initially by an information state, we formulate a general condition, which is sufficient for the derivation of the (original) CHSH inequality and, more generally, any CHSH-type inequality. This sufficient condition is valid for any initial information state of a system and concerns only a factorizable form of generalized observables describing these joint experiments.

Under this sufficient condition, the CHSH inequality is true whether or not an initial information state of a system is entangled.

We further specify the conditions for the derivation of the (original) Bell inequality and show that Bell's correlation restrictions on the observed outcomes (used for the derivation of this inequality in [1,3]) represent particular cases of the condition, introduced in this paper.

We discuss possible mathematical reasons for the violation of the CHSH inequality and point out that the sufficient condition for its derivation does not, in general, represent mathematically the physical concept of "local realism".

For a joint experiment upon a system of any type, we formulate mathematically in a very general setting a condition on "local realism" and consider examples of "locally realistic" joint experiments. In particular, we introduce examples of quantum "locally realistic" joint experiments.

From our presentation it follows that, for "locally realistic" joint experiments, the CHSH inequality and Bell's inequality may be violated whenever generalized observables, describing these joint experiments, do not have a factorizable form. The latter is just the situation in the quantum case.

For any quantum state of a bipartite system, we derive in a very general setting quantum analogs of Bell's inequality under quantum joint experiments of the Alice/Bob type. As an example, we consider inequalities under joint experiments on the two-qubit system.

In view of our results, we argue that the violation of the CHSH inequality and Bell's inequality in the quantum case cannot be a valid argument in the discussion on locality or non-locality of quantum interactions.

2 Description of joint experiments

Let $(\Lambda_i, \mathcal{F}_{\Lambda_i})$, $i = 1, 2$, be some measurable spaces of outcomes.

Consider a joint experiment, with outcomes in a product space

$$(\Lambda_1 \times \Lambda_2,\ \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2}) \tag{1}$$

and performed on a system $\mathcal{S}$ *of any type.*

Let $\mathcal{S}$ be initially represented by any of the *information states* (see [6], for details)

$$\mathcal{I} = (\Theta, \mathcal{F}_\Theta, \pi), \quad \forall \pi, \tag{2}$$

where $(\Theta, \mathcal{F}_\Theta)$ is a measurable space and π is a probability distribution on $(\Theta, \mathcal{F}_\Theta)$.

We call $(\Theta, \mathcal{F}_\Theta)$ an information space of S and denote by $\mu(\cdot;\mathcal{I})$ the probability distribution of outcomes in the case where $\mathcal{S}$ is initially in a state $\mathcal{I}$.

Let Π be a *generalized observable*, with an outcome space (1), uniquely representing this joint experiment on an information space $(\Theta, \mathcal{F}_\Theta)$ (cf. [6]).

Then, for any initial state $\mathcal{I}$ of $\mathcal{S}$, the probability distribution of outcomes is given by:

$$\mu(D;\mathcal{I}) = \int_\Theta (\Pi(D))(\theta)\pi(d\theta), \tag{3}$$

$\forall D \in \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2}$.

For a generalized observable Π, the marginal generalized observable

$$\Pi_1(B_1) := \Pi(B_1 \times \Lambda_2), \quad \forall B_1 \in \mathcal{F}_{\Lambda_1}, \tag{4}$$

with the outcome space $(\Lambda_1, \mathcal{F}_{\Lambda_1})$, and the marginal generalized observable

$$\Pi_2(B_2) := \Pi(\Lambda_1 \times B_2), \quad \forall B_2 \in \mathcal{F}_{\Lambda_2}, \tag{5}$$

with the outcome space $(\Lambda_2, \mathcal{F}_{\Lambda_2})$, describe the experimental situations where the outcomes in $(\Lambda_2, \mathcal{F}_{\Lambda_2})$ and in $(\Lambda_1, \mathcal{F}_{\Lambda_1})$, respectively, are ignored completely.

With respect to Π_1 and Π_2, the generalized observable Π is called *joint.*

Let, under a joint experiment, the outcomes be real-valued and, for simplicity, suppose they are bounded, that is:

$$\Lambda_i = \{\lambda \in \mathbb{R} : \quad |\lambda| \leq \mathrm{C}_i\}, \tag{6}$$

with some $\mathrm{C}_i > 0$, $i = 1, 2$.

Due to (3), for any initial information state $\mathcal{I}$, the mean values are given by the formulae:

$$\begin{aligned} \langle \lambda_i \rangle_{\mathcal{I}}^{(\Pi)} &= \int_{\Lambda_i} \int_\Theta \lambda_i (\Pi_i(d\lambda_i))(\theta)\pi(d\theta) \\ &= \int_\Theta f_i(\theta)\pi(d\theta), \qquad i = 1, 2, \end{aligned} \tag{7}$$

$$\begin{aligned} \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi)} &= \int_{\Lambda_1 \times \Lambda_2} \int_\Theta \lambda_1 \lambda_2 (\Pi(d\lambda_1 \times d\lambda_2))(\theta)\pi(d\theta) \\ &= \int_\Theta f_{joint}(\theta)\pi(d\theta). \end{aligned} \tag{8}$$

In (7) and (8),

$$f_i(\theta) := \int_{\Lambda_i} \lambda_i(\Pi_i(d\lambda_i))(\theta), \qquad \forall\theta \in \Theta, \quad i = 1, 2, \tag{9}$$

$$f_{joint}(\theta) := \int_{\Lambda_1\times\Lambda_2} \lambda_1\lambda_2(\Pi(d\lambda_1 \times d\lambda_2))(\theta), \quad \forall\theta \in \Theta \tag{10}$$

are real-valued measurable functions[1] on $(\Theta, \mathcal{F}_\Theta)$ and, for any $\theta \in \Theta$,

$$|f_i(\theta)| \leq \mathrm{C}_i, \quad i = 1, 2; \qquad |f_{joint}(\theta)| \leq \mathrm{C}_1\mathrm{C}_2. \tag{11}$$

Remark 1 *From (7) it follows that, under a joint experiment on a system $\mathcal{S}$ of any type, represented initially by an information state $\mathcal{I}$ on $(\Theta, \mathcal{F}_\Theta)$, each mean value*

$$\langle\lambda_i\rangle_{\mathfrak{I}}^{(\Pi)}, \quad i = 1, 2, \tag{12}$$

is expressed in terms of the corresponding random variable f_i on $(\Theta, \mathcal{F}_\Theta)$. However, in contrast to the formalism of classical probability, the values of these random variables do not, in general, represent outcomes observed under this joint experiment.
Moreover, due to (8), the mean value

$$\langle\lambda_1\lambda_2\rangle_{\mathcal{I}}^{(\Pi)} \tag{13}$$

is represented via the random variable f_{joint}, which is, in general, different from the product $f_1 f_2$.

Remark 2 *The values of the random variables f_i do represent outcomes, observed under a joint experiment, if and only if this experiment is described by an "image" generalized observable (see in [6]):*

$$(\Pi(B_1 \times B_2))(\theta) = \chi_{f_1^{-1}(B_1)\cap f_2^{-1}(B_2)}(\theta), \tag{14}$$

$\forall\theta \in \Theta$, $\forall B_i \in \mathcal{F}_{\Lambda_i}$, $i = 1, 2$, where $\chi_F(\theta)$ is the indicator function of a subset $F \in \mathcal{F}_\Theta$ and, for any $i = 1, 2$,

$$f_i^{-1}(B_i) = \{\theta \in \Theta : f_i(\theta) \in B_i\} \tag{15}$$

is the preimage of a subset $B_i \in \mathcal{F}_{\Lambda_i}$.in $\mathcal{F}_\Theta$.
"Image" generalized observables (14) describe the special type of experiments, called classical measurements[2]. For classical measurements $f_{joint} = f_1 f_2$.

Remark 3 *As we discuss this in detail in section 3, the relation $f_{joint} = f_1 f_2$ may be valid not only under classical measurements but, for any joint experiment, described by a product generalized observable (21). However, for any product generalized observable, the values of f_1 and f_2 do not, in general, represent the observed outcomes.*

[1] Called random variables in probability theory.
[2] On the notion of a classical measurement, see [6] and the references therein.

Let two joint experiments on $\mathcal{S}$ be described on $(\Theta, \mathcal{F}_\Theta)$ by generalized observables $\Pi^{(1)}$ and $\Pi^{(2)}$, with an outcome space (1).

From (8) it follows that

$$\langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} \pm \langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})} = \int_\Theta \{ f_{joint}^{(1)}(\theta) \pm f_{joint}^{(2)}(\theta) \} \pi(d\theta), \tag{16}$$

where, to each $\Pi^{(n)}$, $n = 1, 2$, the corresponding random variable $f_{joint}^{(n)}$ is defined by (10).

Due to the inequality:

$$|x - y| \leq 1 - xy, \tag{17}$$

valid for any real numbers $|x| \leq 1$, $|y| \leq 1$, we derive

$$\left| \langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} \pm \langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})} \right| \leq \mathrm{C}_1\mathrm{C}_2 \pm \frac{1}{\mathrm{C}_1\mathrm{C}_2} \langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} \langle \lambda_1\lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})}. \tag{18}$$

3 Factorizable generalized observables

Let a generalized observables Π, with an outcome space (1), have the factorizable form:

$$\Pi = \int_\Omega \Pi_{1,\omega} \times \Pi_{2,\omega} \; \nu(d\omega), \tag{19}$$

where $(\Omega, \mathcal{F}_\Omega)$ is a measurable space and ν is a probability measure on $(\Omega, \mathcal{F}_\Omega)$.

The relation (19) means that, for any $\theta \in \Theta$ and any subsets $B_1 \in \mathcal{F}_{\Lambda_1}, B_2 \in \mathcal{F}_{\Lambda_2}$,

$$(\Pi(B_1 \times B_2))(\theta) = \int_\Omega (\Pi_{1,\omega}(B_1))(\theta)(\Pi_{2,\omega}(B_2))(\theta) \; \nu(d\omega). \tag{20}$$

If, in particular, ν is a Dirac measure then Π has the product form.

3.1 Product generalized observables

For two joint experiments on $\mathcal{S}$, with outcomes in (1), let each generalized observable $\Pi^{(n)}$, $n = 1, 2$, on $(\Theta, \mathcal{F}_\Theta)$ have the *product* form

$$\Pi^{(n)} = \Pi_1^{(n)} \times \Pi_2^{(n)}. \tag{21}$$

Suppose further that $\Pi_1^{(1)} = \Pi_1^{(2)} = \Pi_1$ and, hence,

$$\Pi^{(n)} = \Pi_1 \times \Pi_2^{(n)}, \quad n = 1, 2. \tag{22}$$

Under the joint experiments, described by (22), we have the following expressions:

$$\widetilde{f}_1(\theta) = \int_{\Lambda_1} \lambda(\Pi_1(d\lambda))(\theta), \tag{23}$$

$$\widetilde{f}_2^{(n)}(\theta) = \int_{\Lambda_2} \lambda(\Pi_2^{(n)}(d\lambda))(\theta), \qquad n = 1, 2, \tag{24}$$

$$\begin{aligned} \widetilde{f}_{joint}^{(n)}(\theta) &= \int_{\Lambda_1 \times \Lambda_2} \lambda_1\lambda_2 \, (\Pi_1(d\lambda_1)) \, (\theta) \; (\Pi_2^{(n)}(d\lambda_2))(\theta) \\ &= \widetilde{f}_1(\theta)\widetilde{f}_2^{(n)}(\theta), \qquad\qquad n = 1, 2, \end{aligned} \tag{25}$$

for the random variables, introduced by the formulae (9), (10). Notice that

$$|\widetilde{f}_1(\theta)| \leq \mathrm{C}_1, \quad |\widetilde{f}_2^{(n)}(\theta)| \leq \mathrm{C}_2, \quad n = 1, 2, \tag{26}$$

for any $\theta \in \Theta$.

From (8) and (25) it follows that, under two joint experiments, described by (22),

$$\langle \lambda_1 \lambda_2 \rangle_\sigma^{(\Pi^{(n)})} = \int_\Theta \widetilde{f}_1(\theta) \widetilde{f}_2^{(n)}(\theta) \pi(d\theta), \qquad n = 1, 2, \tag{27}$$

and, hence, in (16),

$$\langle \lambda_1 \lambda_2 \rangle_\jmath^{(\Pi^{(1)})} \pm \langle \lambda_1 \lambda_2 \rangle_\jmath^{(\Pi^{(2)})} = \int_\Theta \widetilde{f}_1(\theta) \left\{ \widetilde{f}_2^{(1)}(\theta) \pm \widetilde{f}_2^{(2)}(\theta) \right\} \pi(d\theta). \tag{28}$$

We formulate and prove the following proposition.

Proposition 1 *Let product generalized observables* $\Pi^{(n)}$, $n = 1, 2$, *be specified by (22). Then, for any initial information state* $\mathcal{I}$,

$$\left| \langle \lambda_1 \lambda_2 \rangle_\mathcal{I}^{(\Pi^{(1)})} \pm \langle \lambda_1 \lambda_2 \rangle_\mathcal{I}^{(\Pi^{(2)})} \right| \leq \mathrm{C}_1 \mathrm{C}_2 \pm \frac{\mathrm{C}_1}{\mathrm{C}_2} \langle \lambda_2' \lambda_2 \rangle_\mathcal{I}^{(\Pi)}, \tag{29}$$

where

$$\begin{aligned} \langle \lambda_2' \lambda_2 \rangle_\mathcal{I}^{(\Pi)} &:= \int_\Theta \widetilde{f}_2^{(1)}(\theta) \widetilde{f}_2^{(2)}(\theta) \pi(d\theta) \\ &= \int_\Theta \int_{\Lambda_2 \times \Lambda_2} \lambda_2' \lambda_2 (\Pi(d\lambda_2' \times d\lambda_2))(\theta) \pi(d\theta) \end{aligned} \tag{30}$$

and

$$\Pi := \Pi_2^{(1)} \times \Pi_2^{(2)} \tag{31}$$

is a generalized observable on $(\Theta, \mathcal{F}_\Theta)$ *with the outcome space* $(\Lambda_2 \times \Lambda_2, \mathcal{F}_{\Lambda_2} \otimes \mathcal{F}_{\Lambda_2})$.

Proof. The proof is based on (28), bounds (26), the inequality (17) and notation (30):

$$\begin{aligned} \left| \langle \lambda_1 \lambda_2 \rangle_\mathcal{I}^{(\Pi^{(1)})} \pm \langle \lambda_1 \lambda_2 \rangle_\mathcal{I}^{(\Pi^{(2)})} \right| &\leq \int_\Theta \left| \widetilde{f}_1(\theta) \{ \widetilde{f}_2^{(1)}(\theta) \pm \widetilde{f}_2^{(2)}(\theta) \} \right| \pi(d\theta) \\ &\leq \mathrm{C}_1 \int_\Theta \left| \widetilde{f}_2^{(1)}(\theta) \pm \widetilde{f}_2^{(2)}(\theta) \right| \pi(d\theta) \\ &\leq \mathrm{C}_1 \mathrm{C}_2 \int_\Theta (1 \pm \frac{1}{\mathrm{C}_2^2} \widetilde{f}_2^{(1)}(\theta) \widetilde{f}_2^{(2)}(\theta)) \pi(d\theta) \\ &= \mathrm{C}_1 \mathrm{C}_2 \pm \frac{\mathrm{C}_1}{\mathrm{C}_2} \langle \lambda_2' \lambda_2 \rangle_\mathcal{I}^{(\Pi)}. \end{aligned} \tag{32}$$

■

3.2 The CHSH inequality

Consider now four joint experiments, with outcomes in (1), described by the product generalized observables

$$\Pi^{(1)} = \Pi_1^{(a)} \times \Pi_2^{(b)}, \qquad \Pi^{(2)} = \Pi_1^{(a)} \times \Pi_2^{(d)}, \tag{33}$$
$$\Pi^{(3)} = \Pi_1^{(c)} \times \Pi_2^{(b)}, \qquad \Pi^{(4)} = \Pi_1^{(c)} \times \Pi_2^{(d)},$$

where a, b, c, d are parameters of any nature characterizing set-ups of marginal experiments.

The following statement follows from proposition 1.

Corollary 1 (The CHSH inequality) *For product generalized observables (33), the inequality*

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} + \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})} + \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(3)})} - \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(4)})} \right| \leq 2\mathrm{C}_1\mathrm{C}_2 \tag{34}$$

is valid for any initial information state $\mathcal{I}$.

In view of the obvious correspondence

$$(1) \to (a, b), \quad (2) \to (a, d), \quad (3) \to (c, b), \quad (4) \to (c, d), \tag{35}$$

for $\mathrm{C}_1 = \mathrm{C}_2 = 1$, the inequality (34) coincides in form with the original CHSH inequality [4]

We further refer to (34) *as the CHSH inequality.*

3.3 Bell's inequality

Consider now two joint experiments, with real-valued outcomes in

$$(\Lambda \times \Lambda, \mathcal{F}_\Lambda \otimes \mathcal{F}_\Lambda) \tag{36}$$

(with $|\lambda| \leq \mathrm{C}$, $\forall \lambda \in \Lambda$) and described by the product generalized observables

$$\Pi^{(1)} = \Pi_1^{(a)} \times \Pi_2^{(b)}, \quad \Pi^{(2)} = \Pi_1^{(a)} \times \Pi_2^{(d)}. \tag{37}$$

In view of proposition 1, we have the following statement.

Proposition 2 (Bell's inequality) *Let two product generalized observables (37) satisfy the relation*

$$\int_\Lambda \lambda \Pi_2^{(b)}(d\lambda) = \pm \int_\Lambda \lambda_1 \Pi_1^{(b)}(d\lambda_1). \tag{38}$$

Then, for any information state $\mathcal{I}$,

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(a,b)} - \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(a,d)} \right| \leq \mathrm{C}^2 \mp \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(b,d)}, \tag{39}$$

where

$$\langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(b,d)} := \int_{\Lambda \times \Lambda} \int_\Theta \lambda_1 \lambda_2 (\Pi(d\lambda_1 \times d\lambda_2))(\theta) \pi(d\theta) \tag{40}$$

and

$$\Pi := \Pi_1^{(b)} \times \Pi_2^{(d)} \tag{41}$$

is a generalized observable on $(\Theta, \mathcal{F}_\Theta)$ with the outcome space (36).

If C= 1 then *the inequality* (39) *coincides in form with the original Bell inequality*[3] [1,3].

[3]The sign $(-)$ corresponds to the perfect correlation form while the sign $(+)$ - to the anti-correlation form of the original Bell inequality.

We further refer to (39) *as the Bell inequality.*

Notice that the original Bell inequality and its further numerous generalizations and strengthenings have been proved with restrictions on the number and the character of observed outcomes (the so-called "measurement result"restrictions).

We derive the Bell inequality in a very general setting without such "measurement result" restrictions.

Moreover, we derive the Bell inequality under the condition (38), which is more general than the Bell correlation restrictions on the observed outcomes[4] and reduces to the latter only for classical measurements.

3.4 Sufficient condition

Let us specify the sufficient condition, used in corollary 1 for the derivation of the CHSH inequality.

Condition 1 *Under joint experiments on a system $\mathcal{S}$ of any type, the product form (33) of generalized observables represents a sufficient condition for the CHSH inequality to hold for any initial information state $\mathcal{I}$ of $\mathcal{S}$.*

Moreover, this condition is sufficient for the derivation of any CHSH-type inequality.

Notice also that, based on the general framework, introduced in [6], we have derived the CHSH inequality and Bell's inequality in very general settings and without any reference to a local hidden variable (LHV) model [1-3].

Remark 4 (On sufficient condition and "local realism") *Although all joint experiments, described by (33) and (37),* may be *"locally realistic", the arguments based on the physical concept of "local realism" are not essential for the derivation of the CHSH inequality.*
We discuss this in detail in section 4, after example 1.
There, we show that if, for example, in (33) the parameter a *depends on parameters* b*,* d*, then the corresponding joint experiments are not "locally realistic". However, for these joint experiments, the setting of corollary 1 is fulfilled and, hence, the CHSH inequality (34) is satisfied for any information state.*

Consider now a more general sufficient condition for the CHSH inequality to hold.

Let $\Pi^{(n)}$, $n = 1, 2$, be factorizable[5] generalized observables of the form:

$$\Pi^{(n)} = \int_{\Omega} \Pi_{1,\omega} \times \Pi^{(n)}_{2,\omega} \ \nu(d\omega). \tag{42}$$

Proposition 3 *For factorizable generalized obserables $\Pi^{(n)}$, $n = 1, 2$, specified by (42), the inequality*

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} \pm \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})} \right| \leq \mathrm{C}_1 \mathrm{C}_2 \pm \frac{C_1}{C_2} \langle \lambda_2' \lambda_2 \rangle_{\mathcal{I}}^{(\Pi)} \tag{43}$$

is valid for any initial information state $\mathcal{I}$. Here,

$$\langle \lambda_2' \lambda_2 \rangle_{\mathcal{I}}^{(\Pi)} := \int_{\Lambda_2 \ \times \Lambda_2} \int_{\Theta} \lambda_2' \lambda_2 (\Pi(d\lambda_2' \times d\lambda_2))(\theta) \pi(d\theta) \tag{44}$$

[4] Used for the derivation of Bell's inequality in [1,3].

[5] See (19).

and

$$\Pi = \int_{\Omega} \Pi_{2,\omega}^{(1)} \times \Pi_{2,\omega}^{(2)} \nu(d\omega) \tag{45}$$

is a generalized observable on $(\Theta, \mathcal{F}_{\Theta})$ *with the outcome space* $(\Lambda_2 \times \Lambda_2, \mathcal{F}_{\Lambda_2} \otimes \mathcal{F}_{\Lambda_2})$.

Consider four factorizable generalized observables, with an outcome space (1) and having the form:

$$\Pi^{(1)} = \int_{\Omega} \Pi_{1,\omega}^{(a)} \times \Pi_{2,\omega}^{(b)} \ \nu(d\omega), \qquad \Pi^{(2)} = \int_{\Omega} \Pi_{1,\omega}^{(a)} \times \Pi_{2,\omega}^{(b)} \ \nu(d\omega), \tag{46}$$
$$\Pi^{(3)} = \int_{\Omega} \Pi_{1,\omega}^{(c)} \times \Pi_{2,\omega}^{(b)} \ \nu(d\omega), \qquad \Pi^{(4)} = \int_{\Omega} \Pi_{1,\omega}^{(c)} \times \Pi_{2,\omega}^{(d)} \ \nu(d\omega),$$

where a, b, c, d are some parameters.

Corollary 2 *For factorizable generalized observables (46), the CHSH inequality*

$$\left| \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(1)})} + \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(2)})} + \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(3)})} - \langle \lambda_1 \lambda_2 \rangle_{\mathcal{I}}^{(\Pi^{(4)})} \right| \leq 2\mathrm{C}_1\mathrm{C}_2 \tag{47}$$

is valid for any initial information state $\mathcal{I}$.

Hence, a more general condition for the CHSH inequality to hold constitutes:

Condition 2 *The factorizable form (46) of generalized observables represents a sufficient condition for the CHSH inequality to hold for any initial information state* $\mathcal{I}$ *of* $\mathcal{S}$.

Remark 5 (On initial correlation) *Under this sufficient condition, the CHSH inequality is valid whether or not an initial information state of a compound system* $\mathcal{S}$ *is entangled.*

Remark 6 (On possible reasons for the violation) *The violation of the CHSH inequality may happen whenever joint generalized observables, describing the corresponding joint experiments, are not of the factorizable type.*
This is, for example, the case where, for a system of some concrete type, the generalized observables must satisfy a superselection rule and this rule excludes product (hence, factorizable) generalized observables from consideration (see [6]). The latter is just the general situation under quantum joint experiments and we discuss this in sections 4, 5.

4 General "local realism" condition

As we explain in introduction, the physical concept of "local realism" under a joint experiment refers to the situation where set-ups of marginal experiments (possibly remote) are independent.

Based on the notion of a joint generalized observable, representing on $(\Theta, \mathcal{F}_{\Theta})$ a joint experiment on $\mathcal{S}$, we proceed to express the physical concept of "local realism" mathematically in the most general settings.

Suppose that a set-up of a joint experiment is characterized by a pair (a, b) of parameters of any nature: $a, b \in \Gamma$. Then the generalized observable, describing this joint experiment, depends on (a, b) and we denote this by $\Pi^{(a,b)}$.

Let $\Pi_1^{(a,b)}$ and $\Pi_2^{(a,b)}$ be the marginal generalized observables of $\Pi^{(a,b)}$, defined by (4) and (5).

Definition 1 (”Local realism” condition) *We say that a joint generalized observable* $\Pi^{(a,b)}$*, with outcomes in a product space*

$$(\Lambda_1 \times \Lambda_2, \mathcal{F}_{\Lambda_1} \otimes \mathcal{F}_{\Lambda_2}), \tag{48}$$

satisfies the ”local realism” condition if parameters a *and* b *are functionally independent and if both marginal generalized observables of* $\Pi^{(a,b)}$*, that is,* $\Pi_1^{(a,b)}$*, with outcomes in* $(\Lambda_1, \mathcal{F}_{\Lambda_1})$*, and* $\Pi_2^{(a,b)}$*, with outcomes in* $(\Lambda_2, \mathcal{F}_{\Lambda_2})$*, satisfy one of the following relations:*

$$\Pi_1^{(a,b)} = \Pi_1^{(a)}, \qquad \Pi_2^{(a,b)} = \Pi_2^{(b)}, \tag{49}$$

or

$$\Pi_1^{(a,b)} = \Pi_1^{(b)}, \qquad \Pi_2^{(a,b)} = \Pi_2^{(b)}. \tag{50}$$

We call a joint experiment ”locally realistic” if it is described by this type of a joint generalized observable.

Since the parameters a and b are functionally independent, the relation (49) (or (50)) implies, in the most general settings, that set-ups of individual experiments are independent.

Example 1 (”Locally realistic”product generalized observables) *From definition 1 it follows that a product generalized observable*

$$\Pi^{(a,b)} = \Pi_1^{(a,b)} \times \Pi_2^{(a,b)} \tag{51}$$

on $(\Theta, \mathcal{F}_\Theta)$ *is ”locally realistic” iff, in (51), the marginal generalized observables have one of the following forms:*

$$\Pi_1^{(a,b)} = \Pi_1^{(a)}, \quad \Pi_2^{(a,b)} = \Pi_2^{(b)}, \tag{52}$$

or

$$\Pi_1^{(a,b)} = \Pi_1^{(b)}, \quad \Pi_2^{(a,b)} = \Pi_2^{(a)}. \tag{53}$$

with parameters a, b *being functionally independent.*

However, in general, a ”locally realistic” joint generalized observable does not need to be of the product type.

In corollary 1, the four product generalized observables (33) are all ”locally realistic” iff parameters a, b, c, d are functionally independent. If this is not the case and, for example,

$$a = a(b, d), \qquad c = c(b, d), \tag{54}$$

then, in (33), each of joint experiments is not ”locally realistic”. However, the settings of corollary 1 are valid, and, that is why, for these joint experiments, the CHSH inequality (34) holds.

Hence, *the ”local realism” condition (although may be fulfilled) is not essential for the derivation of CHSH-type inequalities.*

Further, consider an example where a ”locally realistic” joint experiment is not described by a factorizable generalized observable.

Let $\mathcal{K}$ be a separable complex Hilbert space of a quantum system $\mathcal{S}_Q$ and $\mathcal{R}_\mathcal{K}$ be the set of all density operators ρ on $\mathcal{K}$. On the information space[6]

$$(\mathcal{R}_\mathcal{K}, \mathcal{F}_{\mathcal{R}_\mathcal{K}}), \tag{55}$$

of a quantum system $\mathcal{S}_Q$, any joint experiment on $\mathcal{S}_Q$ is uniquely represented by the quantum generalized observable

$$(\Pi_Q(\cdot))(\rho) = \mathrm{tr}[\rho M(\cdot)], \quad \forall \rho \in \mathcal{R}_\mathcal{H}, \tag{56}$$

where M is a POV measure of this quantum experiment.

Suppose that a set-up of a quantum joint experiment is specified by parameters (α, β).

Then, in (56), a generalized observable and the corresponding POV measure have the forms: $\Pi_Q^{(\alpha,\beta)}$ and $M^{(\alpha,\beta)}$.

The marginal POV measure $M_1^{(\alpha,\beta)}$ and $M_2^{(\alpha,\beta)}$ of $M^{(\alpha,\beta)}$ are defined by the relations:

$$\begin{aligned} M_1^{(\alpha,\beta)}(B_1) &= M^{(\alpha,\beta)}(B_1 \times \Lambda_2), \\ M_1^{(\alpha,\beta)}(B_2) &= M^{(\alpha,\beta)}(\Lambda_1 \times B_2), \end{aligned} \tag{57}$$

for any subset B_1 of Λ_1 and any subset B_2 of Λ_2.

According to definition 1, a quantum joint experiment is "locally realistic" iff parameters α and β are functionally independent and

$$M_1^{(\alpha,\beta)} = M_1^{(\alpha)}, \qquad M_2^{(\alpha,\beta)} = M_2^{(\beta)}, \tag{58}$$

or

$$M_1^{(\alpha,\beta)} = M_1^{(\beta)}, \qquad M_2^{(\alpha,\beta)} = M_1^{(\alpha)}.$$

Example 2 (Quantum "locally realistic" experiments) *Consider, for example, a bipartite quantum system* $\mathcal{S}_q^{(1)} + \mathcal{S}_q^{(2)}$*, described in terms of a separable complex Hilbert space*

$$\mathcal{K} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{59}$$

Suppose that the POV measure of a joint experiment on $\mathcal{S}_q^{(1)} + \mathcal{S}_q^{(2)}$ *is given by*

$$M^{(\alpha,\beta)}(B_1 \times B_2) = M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta)}(B_2), \tag{60}$$

for any subsets $B_1 \in \mathcal{F}_{\Lambda_1}$, $B_2 \in \mathcal{F}_{\Lambda_2}$, *with parameters* α *and* β *being functionally independent. Then, the POV measure (60) satisfies the relations (58) and, hence, represents a quantum "locally realistic" joint experiment.*
For convenience, Λ_1 *and* Λ_2 *may be thought as sets of outcomes on the "sides" of Alice and Bob, respectively.*

From (56) it follows that, for any joint experiment, represented by a POV measure (60), the value of the corresponding quantum generalized observable

$$\begin{aligned} (\Pi_Q(B_1 \times B_2))(\rho) &= \mathrm{tr}\{\rho M^{(\alpha,\beta)}(B_1 \times B_2)\} \\ &= \mathrm{tr}\{\rho (M_1^{(\alpha)}(B_1) \otimes M_2^{(\beta)}(B_2))\} \end{aligned} \tag{61}$$

[6] See in [6].

on a separable state

$$\rho_{sep} = \sum_m \gamma_m \rho_1^{(m)} \otimes \rho_2^{(m)}, \qquad \gamma_m > 0, \quad \sum_m \gamma_m = 1, \tag{62}$$

has the factorizable form:

$$(\Pi_Q(B_1 \times B_2))(\rho_{sep}) = \sum_m \gamma_m \mathrm{tr}[\rho_1^{(m)} M_1^{(\alpha)}(B_1)]\ \mathrm{tr}[\rho_2^{(m)} M_2^{(\beta)}(B_2)], \tag{63}$$

$\forall B_1 \in \mathcal{F}_{\Lambda_1}$, $\forall B_2 \in \mathcal{F}_{\Lambda_2}$.

However, the relation (63) is valid only on separable density operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and, moreover, for a separable state ρ_{sep}, a representation (62) is not, in general, unique. That is why (63) does not mean that, on the information space $(\mathcal{R}_{\mathcal{K}}, \mathcal{F}_{\mathcal{R}_{\mathcal{K}}})$, the quantum generalized observable (61) is of the factorizable type.

A quantum "locally realistic" generalized observable (61) *does not satisfy the sufficient condition 2, and the CHSH inequality does not need to hold for any quantum state* ρ *of* $\mathcal{S}_q^{(1)} + \mathcal{S}_q^{(2)}$.

5 Quantum analogs of Bell's inequality

In this section, we proceed to introduce in a very general setting quantum analogs of Bell's inequality for quantum joint experiments of the Alice/Bob type (60).

We suppose that

$$\mathcal{S}_q^{(1)} = \mathcal{S}_q^{(2)} = \mathcal{S}_q. \tag{64}$$

In this case, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and, for both Boson and Fermi statistics, a quantum state of $\mathcal{S}_q + \mathcal{S}_q$ is represented by a density operator ρ on $\mathcal{H} \otimes \mathcal{H}$, satisfying the relation

$$\mathrm{S}_2 \rho = \rho, \tag{65}$$

where S_2 is the symmetrization operator on $\mathcal{H} \otimes \mathcal{H}$. Below, an initial density operator ρ of $\mathcal{S}_q + \mathcal{S}_q$ satisfies this condition.

Furthermore, in the case of identical quantum sub-systems, for any joint experiment of the Alice/Bob type, the POV measure of each marginal experiment must have a symmetrized tensor product form and be specified by a set of outcomes on the "side" of Alice or Bob but not by the "side" of the tensor product.

The latter means that, for any Alice/Bob joint experiment on $\mathcal{S}_q + \mathcal{S}_q$, the joint POV measure must have the form:

$$M(B_1 \times B_2) = \{M_1(B_1) \otimes M_2(B_2)\}_{sym} := \frac{1}{2}\{M_1(B_1) \otimes M_2(B_2) + M_2(B_2) \otimes M_1(B_1)\}, \tag{66}$$

for any subsets $B_1 \in \mathcal{F}_{\Lambda_1}$, $B_2 \in \mathcal{F}_{\Lambda_2}$.

Consider two joint experiments on $\mathcal{S}_q + \mathcal{S}_q$, with real-valued outcomes in

$$(\Lambda \times \Lambda, \mathcal{F}_\Lambda \otimes \mathcal{F}_\Lambda) \tag{67}$$

(where $|\lambda| \leq \mathrm{C}$, $\forall \lambda \in \Lambda$) and described by the POV measures

$$\begin{aligned} M^{(\alpha,\beta_1)}(B_1 \times B_2) &= \{M^{(\alpha)}(B_1) \otimes M^{(\beta_1)}(B_2)\}_{sym}, \\ M^{(\alpha,\beta_2)}(B_1 \times B_2) &= \{M^{(\alpha)}(B_1) \otimes M^{(\beta_2)}(B_2)\}_{sym}, \end{aligned} \tag{68}$$

for any subsets $B_1, B_2 \in \mathcal{F}_\Lambda$. In (68), the parameters $\alpha, \beta_1, \beta_2 \in \Gamma$ are of any nature and characterize set-ups of marginal experiments.

Introduce by

$$A^{(\alpha)} = \int_{\not\geq} \lambda M^{(\alpha)}(d\lambda), \qquad A^{(\beta)} = \int_{\not\geq} \lambda M_2^{(\beta)}(d\lambda) \tag{69}$$

self-adjoint bounded linear operators on $\mathcal{H}$ with the operator norms

$$||A^{(\alpha)}|| \leq \mathrm{C}, \quad ||A^{(\beta)}|| \leq \mathrm{C}. \tag{70}$$

Let ρ be an initial quantum state of $\mathcal{S}_q+\mathcal{S}_q$.

Due to (61) and (69), under joint experiments, described by the POV measures (68) and performed on a bipartite quantum system $\mathcal{S}_q+\mathcal{S}_q$ in a state ρ, we have the following expressions for the mean values[7]:

$$\begin{aligned}\langle\lambda\rangle_\rho^{(\alpha)} &= \mathrm{tr}[\rho\{A^{(\alpha)} \otimes I\}_{sym}] = \mathrm{tr}[\rho(A^{(\alpha)} \otimes I)], \\ \langle\lambda\rangle_\rho^{(\beta_i)} &= \mathrm{tr}[\rho\{I \otimes A^{(\beta_i)}\}_{sym}] = \mathrm{tr}[\rho(I \otimes A^{(\beta_i)})], \quad i = 1, 2,\end{aligned} \tag{71}$$

$$\langle\lambda_1\lambda_2\rangle_\rho^{(\alpha,\beta_i)} = \mathrm{tr}[\rho\{A^{(\alpha)} \otimes A^{(\beta_i)}\}_{sym}] = \mathrm{tr}[\rho(A^{(\alpha)} \otimes A^{(\beta_i)})], \qquad i = 1, 2. \tag{72}$$

For a state ρ, consider a representation of the form

$$\begin{aligned}\rho &= \sum_j \frac{\alpha_j}{2}\{\tau_j \otimes \widetilde{\tau}_j + \widetilde{\tau}_j \otimes \tau_j\} + \sigma_\rho^{(\tau)}, \\ \alpha_j &> 0, \quad \sum_j \alpha_j = 1,\end{aligned} \tag{73}$$

where $\{\tau_j\}_{j=1}^N$ and $\{\widetilde{\tau}_j\}_{j=1}^N$, $N \leq \infty$, are any families of density operators on $\mathcal{H}$.

For concreteness, we further refer to (73) as a τ-representation of ρ.

The operator $\sigma_\rho^{(\tau)}$ on $\mathcal{H} \otimes \mathcal{H}$, defined to each τ-representation (73), is self-adjoint , trace class and satisfies (65). Below, we denote by $||\sigma_\rho^{(\tau)}||_1$ the trace norm of $\sigma_\rho^{(\tau)}$.

Lemma 1 *For any state ρ of $\mathcal{S}_q+\mathcal{S}_q$, the inequality*

$$\begin{aligned}&\left|\langle\lambda_1\lambda_2\rangle_\rho^{(\alpha,\beta_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(\alpha,\beta_2)} - \langle z(\rho)\rangle\right| \\ &\leq \mathrm{C}^2 - \sum_j \frac{\alpha_j}{2}\left\{\mathrm{tr}[\tau_j A^{(\beta_1)}]tr[\tau_j A^{(\beta_2)}] + \mathrm{tr}[\widetilde{\tau}_j A^{(\beta_1)}]tr[\widetilde{\tau}_j A^{(\beta_2)}]\right\}\end{aligned} \tag{74}$$

is valid for every τ-representation (73) of ρ. Here,

$$\begin{aligned}\langle z(\rho)\rangle &= \mathrm{tr}[\sigma_\rho^{(\tau)}(A^{(\alpha)} \otimes (A^{(\beta_1)} - A^{(\beta_2)}))] \\ &\leq 2||\sigma_\rho^{(\tau)}||_1\mathrm{C}^2.\end{aligned} \tag{75}$$

Proof. We use (72), (73), the bound

$$|tr\{\eta W\}| \leq ||\eta||_1||W||, \tag{76}$$

[7] A density operator ρ satisfies (65).

valid for any trace class operator η any bounded linear operator W on $\mathcal{H}$, then we use (70), (71), and, finally, the inequality (17). ■

If bounded linear operators A_α, A_β are Hilbert-Schmidt, then, in lemma 1, the bound for $|\langle z(\rho)\rangle|$ can be easily specified in terms of the operator norms $||\cdot||_2$ on the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$.

Denote further by

$$\widetilde{\rho}_\rho^{(\tau)} := \sum_j \frac{\alpha_j}{2}\{\tau_j \otimes \tau_j + \widetilde{\tau}_j \otimes \widetilde{\tau}_j\} \tag{77}$$

the density operator on $\mathcal{H}\otimes\mathcal{H}$, arising in (74) and defined to each τ-representation (73) of ρ.

Due to lemma 1, it is easy to prove the following statement.

Proposition 4 (Quantum analogs of Bell's inequality) *For any state ρ of $\mathcal{S}_q{+}\mathcal{S}_q$ and any τ-representation (73) of ρ, the inequality*

$$\left|\langle\lambda_1\lambda_2\rangle_\rho^{(\alpha,\beta_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(\alpha,\beta_2)}\right| \le \gamma_\rho^{(\tau)}\mathrm{C}^2 - \langle\lambda_1\lambda_2\rangle_{\widetilde{\rho}_\rho^{(\tau)}}^{(\beta_1,\beta_2)}, \tag{78}$$

with

$$\begin{aligned}\gamma_\rho^{(\tau)} &= 1 + \mathrm{C}^{-1}||\sigma_\rho^{(\tau)}||_1||A^{(\beta_1)} - A^{(\beta_2)}|| \\ &\le 1 + 2||\sigma_\rho^{(\tau)}||_1,\end{aligned} \tag{79}$$

holds under all Alice/Bob joint experiments (68). In (78),

$$\langle\lambda_1\lambda_2\rangle_{\widetilde{\rho}_\rho^{(\tau)}}^{(\beta_1,\beta_2)} = \mathrm{tr}[\widetilde{\rho}_\rho^{(\tau)}\{A^{(\beta_1)} \otimes A^{(\beta_2)}\}_{sym}] \tag{80}$$

is the mean value in the state $\widetilde{\rho}_\rho^{(\tau)}$ under the joint experiment described by the POV measure

$$M^{(\beta_1,\beta_1)}(B_1 \times B_2) = \{M^{(\beta_1)}(B_1) \otimes M^{(\beta_2)}(B_2)\}_{sym}, \tag{81}$$

$\forall B_1, B_2 \in \mathcal{F}_\Lambda$.

Let a quantum state ρ' admit a separable "symmetrical" approximation

$$\begin{aligned}\rho' &= \sum_j \alpha_j\tau_j \otimes \tau_j + \sigma_\rho, \qquad ||\sigma_\rho||_1 = 0, \\ \alpha_j &> 0, \qquad \sum_i \alpha_j = 1.\end{aligned} \tag{82}$$

From proposition 4, we have:

Corollary 3 *Any quantum state ρ', which admits a separable "symmetrical" approximation (82), satisfies the perfect correlation form of the Bell inequality, that is:*

$$\left|\langle\lambda_1\lambda_2\rangle_{\rho'}^{(\alpha,\beta_1)} - \langle\lambda_1\lambda_2\rangle_{\rho'}^{(\alpha,\beta_2)}\right| + \langle\lambda_1\lambda_2\rangle_{\rho'}^{(\beta_1,\beta_2)} \le \mathrm{C}^2. \tag{83}$$

As an example, let us consider "locally realistic" joint measurements upon the two-qubit quantum system, presented usually in connection with the violation of Bell's inequality in the quantum case (see, for example, [5], page 164).

Example 3 (Two-qubit system) *Let $\mathcal{H} = \mathbb{C}^2$ and, under joint experiments, the self-adjoint operators $A^{(\beta_i)}$, $A^{(\alpha)}$ are given by:*

$$A^{(\beta_i)} = \{|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\}\cos 2\beta_i + \{|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\}\sin 2\beta_i, \qquad i = 1, 2, \tag{84}$$

$$A^{(\alpha)} = \{|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\}\cos 2\alpha + \{|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\}\sin 2\alpha.$$

In (84), the parameters α, β_1, β_2 characterizing set-ups of marginal experiments, represent angles from some axis, and, by symbols $|\uparrow\rangle$ and $|\downarrow\rangle$, we denote the eigenvectors of the self-adjoint operator $A^{(0)}$ with eigenvalues $(+1)$ and (-1), respectively.
For the considered joint experiment, the constant C= 1 and

$$||A^{(\beta_1)} - A^{(\beta_2)}|| = 2|\sin(\beta_1 - \beta_2)|. \tag{85}$$

Take, the non-separable pure initial state

$$\rho_0 = \frac{1}{2}|\uparrow\otimes\uparrow + \downarrow\otimes\downarrow\rangle\langle\uparrow\otimes\uparrow + \downarrow\otimes\downarrow|, \tag{86}$$

which has a τ-representation

$$\begin{aligned}\rho_0 &= \frac{1}{2}\{\,|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle|\uparrow| + |\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|\} \\ &+ \frac{1}{2}\{|\uparrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\uparrow|\}.\end{aligned} \tag{87}$$

To this τ-representation,

$$\begin{aligned}\widetilde{\rho}_{\rho_0} &= \frac{1}{2}\{\,|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle|\uparrow| + |\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|\}, \\ \sigma_{\rho_0} &= \frac{1}{2}\{|\uparrow\rangle\langle\downarrow| \otimes |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| \otimes |\downarrow\rangle\langle\uparrow|\},\end{aligned} \tag{88}$$

and, hence,

$$\begin{aligned}\|\sigma_{\rho_0}\|_1 &= \mathrm{tr}[\sqrt{\sigma^2_{\rho_0}}] \\ &= \frac{1}{2}\mathrm{tr}\{|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle|\uparrow| + |\downarrow\rangle\langle\downarrow| \otimes |\downarrow\rangle\langle\downarrow|\} = 1.\end{aligned} \tag{89}$$

Substituting (89) into the inequality (78), we have

$$\begin{aligned}&\left|\langle\lambda_1\lambda_2\rangle^{(\alpha,\beta_1)}_{\rho_0} - \langle\lambda_1\lambda_2\rangle^{(\alpha,\beta_2)}_{\rho_0}\right| + \langle\lambda_1\lambda_2\rangle^{(\beta_1,\beta_2)}_{\widetilde{\rho}_{\rho_0}} \\ &\leq 1 + 2|\sin(\beta_1 - \beta_2)|.\end{aligned} \tag{90}$$

To check that (90) is valid for all α, β_1, β_2, let us substitute the mean values

$$\begin{aligned}\langle\lambda\rangle^{(\alpha)}_{\rho_0} &= \langle\lambda\rangle^{(\beta_i)}_{\rho_0} = 0, \quad i = 1, 2, \\ \langle\lambda_1\lambda_2\rangle^{(\alpha,\beta_i)}_{\rho_0} &= \cos 2(\alpha - \beta_i), \qquad i = 1, 2, \\ \langle\lambda_1\lambda_2\rangle^{(\beta_1,\beta_2)}_{\widetilde{\rho}_{\rho_0}} &= \cos 2\beta_1 \cos 2\beta_2\end{aligned} \tag{91}$$

into (90):

$$|\cos 2(\alpha-\beta_1)-\cos 2(\alpha-\beta_2)|+\cos 2\beta_1\cos 2\beta_2 \leq 1+2|\sin(\beta_1-\beta_2)|. \tag{92}$$

Since, for any α, β_1, β_2,

$$\begin{aligned}|\cos 2(\alpha-\beta_1)-\cos 2(\alpha-\beta_2)| &= 2\,|\sin(\beta_1-\beta_2)\sin(2\alpha-\beta_1-\beta_2)\ | \\ &\leq 2|\sin((\beta_1-\beta_2)|,\end{aligned} \tag{93}$$

the left hand-side of (92) has the following upper bound:

$$\begin{aligned}&|\cos 2(\alpha-\beta_1)-\cos 2(\alpha-\beta_2)|+\cos 2\beta_1\cos 2\beta_2 \\ &\leq 2|\sin(\beta_1-\beta_2)|+\cos 2\beta_1\cos 2\beta_2 \\ &\leq 2|\sin(\beta_1-\beta_2)|+1.\end{aligned} \tag{94}$$

This bound coincides with the right-hand side of (92). Hence, (90) is true for all angles α, β_1, β_2.
The cases of any other initial quantum states of the two-qubit can be considered similarly.

6 On locality and non-locality of quantum interactions

In the present paper, based on the general probabilistic framework, introduced in [6], we considered the description of joint experiments on a system of any type.

Mathematically, a joint experiment is described by the notion of a joint generalized observable, and this notion does not include any specifications on whether or not marginal experiments are separated in space and in time.

The main results of our paper indicate:

- For joint experiments, described by factorizable generalized observables, the CHSH inequality holds true whether or not an initial information state of a compound system is entangled;
- The sufficient condition for the CHSH inequality to hold does not, in general, represent mathematically the physical concept of "local realism";
- The physical concept of "local realism" can be expressed mathematically for a joint experiment on a system of any type. The mathematical specification of this concept in the frame of a LHV model corresponds to a particular case of classical measurements;
- There exist quantum "locally realistic" joint experiments. However, quantum "locally realistic" joint generalized observables do not have a factorizable form. That is why the CHSH inequality and Bell's inequality do not need to be true in the quantum case;
- The quantum analogs of Bell's inequality, derived in this paper, specify the relation between the statistical data under quantum joint experiments of the Alice/Bob type.

In the light of these results, we argue that the violation of the CHSH inequality and Bell's inequality in the quantum case cannot be a valid argument in the discussion on the problem of locality and non-locality of quantum interactions.

Acknowledgements. I am grateful to Klaus Molmer and Asher Peres for useful discussions.

References

[1] J.S. Bell. On the Einstein-Podolsky-Rosen paradox. Physics 1, 195-200 (1964)

[2] J.S. Bell. On the problem of hidden variables in quantum mechanics. *Rev. Mod. Phys.* **38**, 447-452 (1966)

[3] J.S. Bell. *Speakable and unspeakable in quantum mechanics.* Cambridge Univ. Press (1987)

[4] J. F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt. *Phys. Rev.Letters*, **23**, 880-884 (1969)

[5] A. Peres. *Quantum Theory: Concepts and Methods.* Kluwer, Dordrecht (1993)

[6] E.R. Loubenets. General probabilistic framework of randomness. *Research Report No* **8** (May 2003), pp. 36, MaPhySto, University of Aarhus, Denmark (e-print quant-ph/0305126).