Statistical Hausdorff dimension of labelled trees and quadrangulations.

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Abstract

Exploiting a bijective correspondence between planar quadrangulations and so-called well labelled trees, we define a random ensemble of infinite surfaces, as a limit of uniformly distributed ensembles of quadrangulations of fixed finite volume. The limit random surface can be described in terms of a birth and death process and a sequence of multitype Galton Watson trees. As a consequence, we find that the volume of the ball of radius r around a marked point in the limit random surface is $\Theta(r^4)$, leading to statistical Hausdorff dimension 4 for this random ensemble of infinite surfaces.

Key words. Random surface, statistical Hausdorff dimension, well labelled trees, Galton Watson trees, birth and death process, quantum gravity.

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Running head. Hausdorff dimension of trees and quadrangulations.

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1 Introduction

Beginning with the seminal article [25] by W.T. Tutte the combinatorics of planar maps have been a subject of continuing development, gaining further impetus in recent years after the realization of its importance in quantum field theory [10] and in string theory and two-dimensional quantum gravity [2, 13, 20]. In the latter case planar maps play the role of two-dimensional discretized (Euclidean) space-time manifolds, whose topology equals that of a sphere with a number of holes, see e.g. [3] for an overview. We shall frequently use the term planar surface instead of planar map.

Of primary interest up to now have been properties depending on the volume, i.e. the number of vertices, or the number and length of the boundary components, or local properties such as the distribution of vertex degrees, whereas quantities depending on the internal (graph) metric structure have received less attention, despite their obvious relevance in e.g. quantum gravity [4]. A relevant quantity to consider in this connection is the so-called *statistical Hausdorff dimension* of an ensemble of random planar maps. In this article we use the following definition of this quantity. Denoting by $B_r(\mathcal{M})$ the ball of radius r around a marked point in the surface $\mathcal{M}$, and by $|B_r(\mathcal{M})|$ its volume, i.e. the number of vertices in $B_r(\mathcal{M})$, the statistical Hausdorff dimension $\mathcal{D}_H$ is determined by

$$\langle |B_r| \rangle = \Theta(r^{\mathcal{D}_H}) \;, \tag{1}$$

assuming such a relation exists. Here $\langle \cdot \rangle$ is the expectation value with respect to the probability measure defining the ensemble, and we use the standard notation $\Theta(\phi(r))$ for a generic function bounded from above and below by positive constant multiple of $\phi(r)$ for r large enough. It is implicit in the definition that the ensemble consists of surfaces of infinite extent. We shall define such an ensemble as a limit of uniformly distributed ensembles of surfaces of fixed finite volume.

It is also possible, and more common, to define $\mathcal{D}_H$ directly in terms of an ensemble of finite surfaces. For instance, one can introduce the so-called two-point correlation function $G(r)$ defined as a certain integral over finite planar surfaces with two marked points a fixed (graph) distance r apart, whose exponential decay rate as a function of r can be shown to determine $\mathcal{D}_H$, see [3]. This method was applied to triangulated planar surfaces (or pure two-dimensional quantum gravity) in [5], where it was argued that $\mathcal{D}_H = 4$ (see also [9]).

A different method was recently applied in [11] to quadrangulated planar maps, making it possible to exploit a clever bijective correspondence between such maps and so-called *well labelled trees*. The main result of this work is to establish the existence of the limit in law of the random variables $n^{-\frac{1}{4}} r_n$ as $n \to \infty$, where r_n is the radius of a random quadrangulation with n faces, identifying the limit as the width of the so-called one-dimensional Integrated SuperBrownian Excursion (or ISE) up to a constant multiple. In view of the information carried by this result about the radius of a generic surface of given volume, it seems reasonable to make the identification $\mathcal{D}_H = 4$.

Yet another approach was recently proposed in [6, 7]. In [6] a *uniform probability measure* is first constructed on *infinite planar triangulations* as a limit of uniform measures on finite planar triangulations. In [7] it is then proven that the ball of radius r around a marked triangle in a sample triangulation contains of order r^4 triangles up to logaritmic corrections, which can be seen as yet another manifestation of the Hausdorff dimension being equal to 4.

In this article we adapt the technique of [6] to the case of well labelled trees and thereby construct by simple combinatorial arguments a uniform probability measure μ on infinite well labelled trees. We show how to identify well labelled trees in the support of this measure with infinite quadrangulated planar surfaces through a mapping that shares the basic properties of the mapping used in [11] for finite trees and quadrangulations. In particular, there is a one-to-one correspondence between vertices in a tree and the vertices in the corresponding surface, except for a certain marked vertex in the surface, and the label $r \in \mathbb{N}$ of a vertex in a tree equals the (graph) distance between the corresponding vertex in the surface and the marked vertex.

Viewing, via this identification, μ as a measure on quadrangulated planar surfaces, we prove the relation (1) with $\mathcal{D}_H = 4$.

The article is organized as follows. In Section 2, we define the space of rooted (labelled) trees, it is endowed with a topology, and its topological and combinatorial properties are studied. In Section 3 we construct a so-called uniform measure on the set of infinite rooted trees, whose vertices are labelled by positive integers, such that the labels of neighbouring vertices deviate by at most 1, and such that the root has a fixed label. In case the root has label 1 such trees are called *well labelled* in [11]. In Section 4 we show that almost surely the trees have exactly one infinite branch, allowing a definition of the spine of a sample well labelled tree as the unique infinite non-self-intersecting path starting at the root. A sample tree can be obtained by attaching (finite) branches along the spine, independently distributed according to a common measure $\hat{\rho}$. In particular, we show in Section 4.2, that the labels along the spine are described by a certain birth & death process, and that the labels in the branches are described in terms of a multitype Galton-Watson process. In Section 5 these two processes are investigated in more detail. In particular, the birth & death process is shown to be transient, and we determine, as a main result, the asymptotic behaviour of the average number $\langle N_r \rangle$ of vertices with a fixed label r to be

$$\langle N_r \rangle = \Theta(r^3) \tag{2}$$

for r large. In Section 6 we show how to extend the mapping of well labelled trees onto planar quadrangulations to infinite ones. In combination with (2) this yields (1) and the value 4 for the statistical Hausdorff dimension of planar quadrangulated surfaces.

2 Labelled trees

2.1 Definitions and notations

By $\overline{\mathcal{T}}$ we shall denote the set of rooted planar trees, where *rooted* means that one oriented edge (i_0, i_1) is distinguished, called the *root*, and i_0 and i_1 are called the first and second *root-vertex*, respectively. Here, trees are allowed to be infinite, but vertices are of finite degree. The adjective planar means that trees are assumed embedded into the plane $\mathbb{R}^2$ such that no two edges intersect except at common vertices, and we identify trees that can be mapped onto each other by an orientation preserving homeomorphism of the plane, that maps root onto root. In addition, certain regularity requirements on the embeddings are needed, the discussion of which we postpone until Section 6. A more precise combinatorial definition is as follows. Once a fixed orientation of $\mathbb{R}^2$ is chosen, the set of vertices at distance r from the first root-vertex in a given rooted planar tree τ has a natural ordering. This can be obtained e.g. by choosing a righthanded coordinate system for $\mathbb{R}^2$ and mapping the tree into $\mathbb{R}^2$ such that the vertices at distance r from the root i_0 are mapped into the the vertical line through $(r, 0)$ and then ordering according to their second coordinate, in such a way that, for $r = 1$, the second root i_1 is smallest. We call this ordered set $\Delta_r = (i_{r1}, \dots, i_{rn_r})$. The edges in the tree are specified by mappings $\phi_r : \Delta_r \to \Delta_{r-1}$, $r \geq 1$, preserving the ordering, i.e. the edges in τ are given by $(i_{rk}, i_{r-1\phi_r(k)})$, $1 \leq k \leq n_r$. It is clear that any (finite or infinite) sequence $(\Delta_0, \Delta_1, \Delta_2, \dots)$ of finite pairwise disjoint ordered sets, where $\Delta_0 = i_0$ is a one-point set, together with orientation preserving maps $(\phi_1, \phi_2, \dots)$ as above, uniquely specifies a rooted planar tree τ, in which case we write $\tau = (\Delta_r, \phi_r)_{r\in\mathbb{N}}$. We then have $(\Delta_r, \phi_r)_{r\in\mathbb{N}} = (\Delta'_r, \phi'_r)_{r\in\mathbb{N}}$ if and only if there exist order preserving bijective maps $\psi_r : \Delta_r \to \Delta'_r$ such that $\phi'_r = \psi_{r-1} \circ \phi_r \circ \psi_r^{-1}$ for all r.

We have

$$\overline{\mathcal{T}} = \left(\bigcup_{N=1}^{\infty} \mathcal{T}_N \right) \cup \mathcal{T}_\infty \ ,$$

where $\mathcal{T}_N$ consists of trees with maximal vertex distance from the first root equal to N, i.e. $\Delta_r = \emptyset$ for $r > N$ but $\Delta_N \neq \emptyset$, and $\mathcal{T}_\infty$ consists of infinite trees. We say that $\tau \in \mathcal{T}_N$ has *height* (or radius) $\rho(\tau) = N$. The set $\bigcup_{N=1}^{\infty} \mathcal{T}_N$ of finite trees will be denoted by $\mathcal{T}$ and the *size* (or volume) $|\tau|$ of a finite tree is defined to be the number of edges in τ. For $\tau \in \mathcal{T}_\infty$ we set $\rho(\tau) = |\tau| = \infty$.

By a labelled tree we mean a pair (τ, ℓ), where $\ell : i \to \ell_i$ is a mapping from the vertices of τ into the integers $\mathbb{Z}$, such that

$$|\ell_i - \ell_j| \leq 1 \quad \text{if } (i, j) \text{ is an edge in } \tau \ .$$

If, furthermore,

$$\ell_{i_0} = k \quad \text{and } \ell_i \geq 1 \text{ for all vertices } i \text{ in } \tau \ ,$$

we say that (τ, ℓ) is a *k-labelled* tree. A 1-labelled tree is also called a *well labelled tree* [11, 12]. We call the set of k-labelled trees $\overline{\mathcal{W}}^{(k)}$ and, for $k = 1$, we set $\overline{\mathcal{W}}^{(1)} = \overline{\mathcal{W}}$. The corresponding sets of finite labelled trees are denoted similarly without overlining. Obviously, $\overline{\mathcal{W}}^{(k)}$ and $\overline{\mathcal{W}}$ inherit from $\overline{\mathcal{T}}$ the natural decompositions

$$\begin{aligned}\overline{\mathcal{W}}^{(k)} &= \mathcal{W}^{(k)} \cup \mathcal{W}^{(k)}_\infty = \left(\bigcup_{N=1}^{\infty} \mathcal{W}^{(k)}_N\right) \cup \mathcal{W}^{(k)}_\infty \ , \\ \overline{\mathcal{W}} &= \mathcal{W} \cup \mathcal{W}_\infty = \left(\bigcup_{N=1}^{\infty} \mathcal{W}_N\right) \cup \mathcal{W}_\infty \ ,\end{aligned}$$

into finite and infinite k-labelled trees. If $\omega = (\tau, \ell)$ is a labelled tree we set $|\omega| = |\tau|$ and $\rho(\omega) = \rho(\tau)$. Moreover, if $\tau = (\Delta_r, \phi_r)_{r\in\mathbb{N}}$ and $\Delta_r = (i_{r1}, \dots, i_{rn_r})$, we set

$$\Xi_r = ((i_{r1}, \ell_{i_{r1}}), \dots, (i_{rn_r}, \ell_{i_{rn_r}})) \ ,$$

in which case we have

$$|\ell_{rk} - \ell_{r-1\phi_i(k)}| \leq 1 \quad \text{for all } r \geq 1,\ 1 \leq k \leq n_r \ . \tag{3}$$

Clearly, any sequence (Ξ_r, ϕ_r), $r \in \mathbb{N}$, obtained from a tree $\tau = (\Delta_r, \phi_r)_{r\in\mathbb{N}}$ by adding labels as above to the vertices of each Δ_r fulfilling (3) defines a unique labelled tree ω, in which case we write $\omega = (\Xi_r, \phi_r)_{r\in\mathbb{N}}$. Furthermore, we have $(\Xi_r, \phi_r)_{r\in\mathbb{N}} = (\Xi'_r, \phi'_r)_{r\in\mathbb{N}}$ if and only if there exist maps ψ_r identifying the underlying unlabelled trees and such that the labels of a vertex i and its image $\psi_r(i)$ are identical.

2.2 Topology on labelled trees

For $r \in \mathbb{N}_0$ and $\omega \in \mathcal{W}^{(k)}$ with distance classes $\Xi_s(\omega)$, $s \in \mathbb{N}_0$, we define the ball $B_r(\omega)$ of radius r in ω to be the labelled subtree of ω generated by $\Xi_0(\tau), \dots, \Xi_r(\tau)$, if $r < \rho(\omega)$, and equal to ω otherwise. In other words,

$$B_r(\omega) = ((\Xi_1, \phi_1), ..., (\Xi_r, \phi_r)) \quad \text{if } \omega = ((\Xi_1, \phi_1), (\Xi_2, \phi_2), \dots) \ .$$

Next we define, for $\omega, \omega' \in \overline{\mathcal{W}}^{(k)}$ and k fixed,

$$d(\omega, \omega') = \inf \left\{ \frac{1}{r+1} \ \middle| \ B_r(\omega) = B_r(\omega') \ , \ r \in \mathbb{N}_0 \right\} \ .$$

It is trivially verified that d defines a metric on $\overline{\mathcal{W}}^{(k)}$. The corresponding open balls in $\overline{\mathcal{W}}^{(k)}$ are given by

$$\mathcal{B}_s(\omega_0) = \{\omega \in \overline{\mathcal{W}}^{(k)} \mid d(\omega, \omega_0) < s\} \ , \quad \text{for } s > 0.$$

Remark 2.1. The following facts are easy to verify:

- The set $\mathcal{W}^{(k)}$ of finite k-labelled trees is a countable dense subset of $\overline{\mathcal{W}}^{(k)}$, and its boundary $\partial\mathcal{W}^{(k)}$ in $\overline{\mathcal{W}}^{(k)}$ equals $\mathcal{W}^{(k)}_\infty$.

- For $s > 0$ and $\omega \in \overline{\mathcal{W}}^{(k)}$ the ball $\mathcal{B}_s(\omega)$ is both open and closed and

$$\{\omega \in \mathcal{B}_s(\omega_0)\} \Leftrightarrow \{\mathcal{B}_s(\omega) = \mathcal{B}_s(\omega_0)\} .$$

  As a consequence, either two balls are disjoint or one is contained in the other.

- $\overline{\mathcal{W}}^{(k)}$ is not compact: Let ω_n be the (unique) k-labelled tree of height 1 with $n+1$ vertices and all labels equal to k. Then $d(\omega_n, \omega_m) = 1$ for $n \neq m$ and hence ω_n, $n \in \mathbb{N}$, has no convergent subsequence.

As a substitute for compactness we shall make use of the following result:

Proposition 2.2. *Let K_r, $r \in \mathbb{N}$, be a sequence of positive numbers. Then the subset*

$$C = \bigcap_{r=1}^{\infty} \{\omega \in \overline{\mathcal{W}}^{(k)} \mid |B_r(\omega)| \leq K_r\}$$

of $\overline{\mathcal{W}}^{(k)}$ is compact.

Proof. Let ω_n, $n \in \mathbb{N}$, be any sequence in C. For each $r \in \mathbb{N}$ the set

$$\left\{ \omega \in \mathcal{W}_r^{(k)} \middle| \, |\omega| \leq K_r \right\}$$

is finite. Hence there exists a subsequence ω_{n_i}, $i \in \mathbb{N}$, such that $B_r(\omega_{n_i})$ is constant as a function of i. Applying a diagonal argument we may choose this subsequence such that $B_i(\omega_{n_j}) = B_i(\omega_{n_i})$ for all $i \leq j$. It follows that this subsequence determines a unique tree $\omega \in C$ such that $B_i(\omega) = B_i(\omega_{n_i})$ for all $i \in \mathbb{N}$. In particular, $\omega_{n_i} \to \omega$ as $i \to \infty$, which completes the proof. □

2.3 Combinatorics of finite labelled trees

In Section 3 we shall consider the sequence μ_N, $N \in \mathbb{N}$, of measures on $\overline{\mathcal{W}}$, where μ_N is defined as the uniform probability measure concentrated on

$$\mathcal{W}'_N = \{\omega \in \mathcal{W} \mid |\omega| = N\} , \quad N \in \mathbb{N}_0 ,$$

that is

$$\mu_N(\omega) = D_N^{-1} \text{ for } \omega \in \mathcal{W}'_N , \quad \mu_N(\mathcal{W} \setminus \mathcal{W}'_N) = 0 ,$$

where $D_N = \sharp \mathcal{W}'_N$ is the number of well labelled trees of size N.

In order to establish weak convergence of μ_N, $N \in \mathbb{N}$, we need some basic facts about the sequence D_N, $N \in \mathbb{N}$, and, more generally, about the sequence $D_N^{(k)}$, $N \in \mathbb{N}$, where $D_N^{(k)}$ is the number of k-labelled trees of size N.

As shown in [11, 12], D_N equals the number of quadrangulated planar maps with N faces, see also Section 6 below. The corresponding generating function

$W(x)$ has been computed in [25] and is given by

$$\begin{aligned} W(x) &= \sum_{N=0}^{\infty} D_N x^N \\ &= 1 + 2x + 9x^2 + \dots \\ &= \frac{18x - 1 + (1-12x)^{\frac{3}{2}}}{54x^2}, \quad \text{for } x \leq \tfrac{1}{12}, \end{aligned} \tag{4}$$

yielding

$$D_N = 2 \cdot 3^N \frac{(2N)!}{N!(N+2)!}. \tag{5}$$

Note that we have included in $W(x)$ the contribution $D_0 = 1$ from the tree with only one vertex. As consequences of (5) we have

$$D_N \simeq \frac{2}{\sqrt{\pi}} N^{-\frac{5}{2}} 12^N \tag{6}$$

and, for each $k \in \mathbb{N}$,

$$D_N^{(k)} = \Theta(N^{-\frac{5}{2}} 12^N). \tag{7}$$

We shall give a more precise estimate for $D_N^{(k)}$ below.

By decomposing a tree into trees with first root i_0 of degree 1 as in Figure 1 we obtain

$$W^{(k)}(x) = \frac{1}{1 - Z^{(k)}(x)}, \tag{8}$$

where

$$Z^{(k)}(x) = \sum_{N=1}^{\infty} E_N^{(k)} x^N$$

is the generating function for the number $E_N^{(k)}$ of k-labelled trees with N edges and first root i_0 of degree 1. In the following we shall only need the values

$$z_k = Z^{(k)}\left(\frac{1}{12}\right) \quad \text{and} \quad w_k = W^{(k)}\left(\frac{1}{12}\right).$$

Proposition 2.3. *For $k \in \mathbb{N}$ we have*

$$z_k = \frac{1}{2} - \frac{1}{k(k+3)} \tag{9}$$

and

$$w_k = 2\frac{k(k+3)}{(k+1)(k+2)}. \tag{10}$$

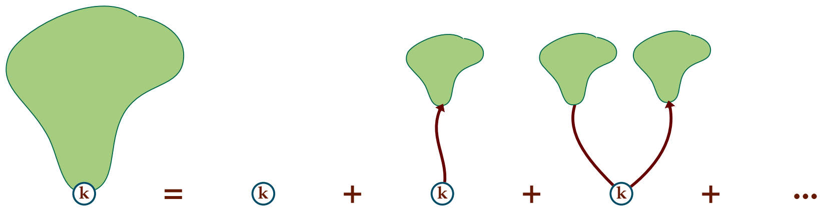

Figure 1: $W^{(k)} = \frac{1}{1-Z^{(k)}}$.

Proof. By (8) it suffices to prove (9). For this purpose we note the relations

$$Z^{(1)}(x) = \frac{x}{1-Z^{(1)}(x)} + \frac{x}{1-Z^{(2)}(x)} \tag{11}$$

$$Z^{(k)}(x) = \frac{x}{1-Z^{(k-1)}(x)} + \frac{x}{1-Z^{(k)}(x)} + \frac{x}{1-Z^{(k+1)}(x)}\,, \quad k \geq 2\,, \tag{12}$$

which are obtained by decomposing the sum over trees defining $Z^{(k)}$ according to the degree and the label of their second root i_1.

Clearly, these relations determine $Z^{(k)}$ in terms of $Z^{(1)}$. Inserting $x = \frac{1}{12}$ and the value $z_1 = \frac{1}{4}$ obtained from (4), one finds that (9) solves (11) and (12). □

A precise estimate for $D_N^{(k)}$ is established in the subsequent lemma. Set

$$d_{k,N} = \frac{D_N^{(k)}}{D_N}. \tag{13}$$

Lemma 2.4. *For each $k \in \mathbb{N}$, the sequence $d_{k,N}$, $N \in \mathbb{N}$, converges to a limit*

$$d_k = \frac{3}{280}\,\frac{k(k+3)}{(k+1)(k+2)}\,\left(5k^4 + 30k^3 + 59k^2 + 42k + 4\right). \tag{14}$$

This limit fulfills

$$d_1 = 1 \tag{15}$$

$$d_1 + d_2 = 12 d_1 (w_1)^{-2} \tag{16}$$

$$d_{k-1} + d_k + d_{k+1} = 12 d_k (w_k)^{-2}\,, \quad k \geq 2\,. \tag{17}$$

Corollary 2.5.

$$D_N^{(k)} \simeq \frac{2\,d_k}{\sqrt{\pi}}\, N^{-\frac{5}{2}} 12^N.$$

Proof. By decomposing a tree $\omega \in \mathcal{W}_N^{(k)}$ into the two branches attached to the two root vertices, and the root, we obtain

$$D_N^{(k)} = \sum_{N_1+N_2=N-1} D_{N_1}^{(k)} (D_{N_2}^{(k-1)} + D_{N_2}^{(k)} + D_{N_2}^{(k+1)})\,, \tag{18}$$

where by convention $D_N^{(0)} = 0$. Thus $d_{k,N}$ is a sum of terms of the form

$$D_N^{-1} \sum_{N_1+N_2=N-1} D_{N_1}^{(k)} D_{N_2}^{(k')} \ . \tag{19}$$

The contribution $f_{k,k',N}$ from $0 \le N_1 \le N/2$ to this sum can be written as

$$\begin{aligned} f_{k,k',N} &= D_N^{-1} \sum_{0 \le N_1 \le N/2} D_{N_1}^{(k)} D_{N-N_1-1}^{(k')} \\ &= (D_N^{-1} 12^{1-N} N^{-5/2}) \sum_{0 \le N_1 \le N/2} (D_{N_1}^{(k)} 12^{-N_1})(D_{N-N_1-1}^{(k')} 12^{N-N_1-1} N^{-5/2}) \ . \end{aligned}$$

Considering the factors in parentheses we see that the first one tends to $12\frac{\sqrt{\pi}}{2}$ by (6), the sum $\sum_{N_1=0}^{\infty} D_{N_1}^{(k)} 12^{-N_1}$ is convergent with sum w_k, and the last factor is bounded and converges to $d_{k'} \frac{2}{\sqrt{\pi}}$ as $N \to \infty$ for each N_1, provided the limit $d_{k'}$ exists. It follows that

$$\lim_{N \to \infty} f_{k,k',N} = 12 w_k d_{k'}$$

if the limit $d_{k'}$ exists.

Similarly, the contribution from $N/2 < N_1 \le N-1$ to the sum (19) converges to $12 d_k w_{k'}$ provided the limit d_k exists. Together with (18), (8) and (12) this yields

$$\begin{aligned} d_k &= 12 d_k (w_{k-1} + w_k + w_{k+1}) + 12 w_k (d_{k-1} + d_k + d_{k+1}) && (20) \\ &= d_k z_k + 12 w_k (d_{k-1} + d_k + d_{k+1}) && (21) \end{aligned}$$

provided the limits $d_k, d_{k\pm1}$ exist, and $d_0 = w_0 = 0$ by convention. Using (8) once more one obtains (16) and (17). But since $D_N = D_N^{(1)}$, the limit d_1 trivially exists and equals 1. Using (18) recursively we obtain the existence of the limit d_k for all $k \in \mathbb{N}$. Finally, it turns out (tediously checking identities between polynomials of order 8) that the expression given in (14) satisfies (15), (16) and (17). Clearly, this last set of equations determines d_k uniquely. □

Another proof of Lemma 2.4, perhaps less transparent, could be obtained as follows. First, rewrite relations (8), (11) and (12) in the form

$$W^{(k+1)}(x) = \frac{1}{x}\left(1 - \frac{1}{W^{(k)}(x)}\right) - W^{(k-1)}(x) - W^{(k)}(x) \,, \quad k \ge 2 \,, \tag{22}$$

and note that an expansion of the form

$$W^{(k)}(x) = w_k + a_k(1-12x) + b_k(1-12x)^{3/2} + o\left((1-12x)^{3/2}\right),$$

which holds for $k = 1$ due to (4), also holds for general k as a consequence of (22). Under suitable analyticity conditions around $1/12$, transfer theorems (cf. [16] or [17, Ch. 5.4]) imply that

$$D_N^{(k)} \simeq \frac{3\, b_k}{4\sqrt{\pi}}\, N^{-5/2} 12^N.$$

It follows that d_k exists and

$$d_k = \frac{b_k}{b_1} = \frac{3b_k}{8}.$$

Relations (16) and (17) are then immediate consequences of (22).

Remark 2.6. In [9] the solution of equations (11) and (12) with $Z^{(1)}(x)$ given by (4) and (8) is obtained for all $x \leq \frac{1}{12}$, and a closed form for d_k (formula (4.19)) is then deduced from the closed form for $W^{(k)}(x)$.

3 Uniform measure on infinite labelled trees

We are now ready to prove one of the main results of this paper.

Theorem 3.1. *The sequence μ_N, $N \in \mathbb{N}$, converges weakly to a Borel probability measure, μ, concentrated on $\mathcal{W}_\infty$. We call μ the uniform probability measure on $\mathcal{W}_\infty$.*

Proof. By Remark 2.1 the denumerable family of balls

$$\mathcal{V} = \{\mathcal{B}_{\frac{1}{r}}(\omega) \mid r \in \mathbb{N},\ |\omega| < +\infty\}$$

consists of open and closed sets and

i) any finite non empty intersection of sets in $\mathcal{V}$ belongs to $\mathcal{V}$,

ii) any open set in $\overline{\mathcal{W}}$ can be written as a union of sets in $\mathcal{V}$.

By Theorems 2.1, 2.2 and 6.1 in [8] it will suffice to prove that the sequence μ_N, $N \in \mathbb{N}$, is tight, and that $\mu_N(A)$ converges as $N \to \infty$ for all $A \in \mathcal{V}$.

We first prove tightness by showing that for any given $\varepsilon > 0$ and $r \in \mathbb{N}$ there exists $K_r > 0$ such that

$$\mu_N\left(\{\omega \in \mathcal{W} \mid |B_r(\omega)| > K_r\}\right) < \varepsilon \tag{23}$$

for all N. Replacing ε by $\varepsilon/2^r$ in (23) and choosing K_r correspondingly, Proposition 2.2 gives the desired compact set C fulfilling $\mu_N(C) > 1 - \varepsilon$ for all N.

We proceed to show (23) by induction on r.

If $r = 1$, then $|B_r(\omega)|$ equals the degree of the root-vertex $i_0(\omega)$. By the argument leading to (8) there is a one-to-one correspondence between trees ω in $\mathcal{W}_N$ with $i_0(\omega)$ of degree K and K-tuples $(\omega_1, \ldots, \omega_K)$ of trees, such that the first root-vertex of each ω_a has degree 1 and

$$|\omega_1| + \cdots + |\omega_K| = N\ .$$

This gives

$$\mu_N(\{\omega \in \mathcal{W} \mid |B_1(\omega)| = K\}) \ = \ D_N^{-1} \sum_{N_1+\cdots+N_K=N} \ \prod_{a=1}^{K} E^{(1)}_{N_a}\ .$$

In this sum $N_a \geq N/K$ for at least one value of $a = 1, \ldots, K$. Combining this with (7) and

$$E_N^{(k)} \leq D_N^{(k)}$$

we obtain

$$\begin{aligned}
&\mu_N(\{\omega \in \mathcal{W} \mid |B_1(\omega)| = K\}) \\
&\leq \quad K \sum_{\substack{N_1+\cdots+N_K=N \\ N_1 \geq N/K}} c' \cdot K^{\frac{5}{2}} \prod_{a=2}^{K} E_{N_a}^{(1)} \cdot 12^{-N_a} \\
&\leq \quad c' \cdot K^{\frac{7}{2}} z_1^{K-1} \; = \; c' K^{\frac{7}{2}} 4^{1-K} \; ,
\end{aligned}$$

where $c' > 0$ is a constant independent of N. This proves (23) for $r = 1$ and for sufficiently large K_1.

Now assume (23) holds for a given $r \geq 1$. For any $K > 0$, we then have

$$\begin{aligned}
&\mu_N(\{\omega \in \mathcal{W} \quad | \quad |B_{r+1}(\omega)| > K\}) \\
\leq \varepsilon + &\mu_N(\{\omega \in \mathcal{W} \quad | \quad |B_{r+1}(\omega)| > K, \; |B_r(\omega)| \leq K_r\}) \; .
\end{aligned}$$

Since there are only finitely many different balls $B_r(\omega)$ with $|B_r(\omega)| \leq K_r$, it suffices to show that

$$\mu_N(\{\omega \in \mathcal{W} \mid |B_{r+1}(\omega)| > K, \; B_r(\omega) = \hat{\omega}\}) \to 0 \tag{24}$$

as $K \to \infty$, uniformly in N for any fixed $\hat{\omega} \in \mathcal{W}_r$. This is obtained in a similar fashion as for $r = 1$:

Set $\Xi_r(\hat{\omega}) = ((j_1, k_1), \ldots, (j_R, k_R))$ and $K' = K - |\hat{\omega}|$. For $\omega \in \mathcal{W}$ with $B_r(\omega) = \hat{\omega}$ and $|B_{r+1}(\omega)| = K$ let $(\omega_1, \cdots, \omega_{K'})$ be the ordered set of branches with roots of degree 1 attached to the vertices of $\Xi_r(\hat{\omega})$, such that the first $L_1 \geq 0$ are attached to j_1, the next $L_2 \geq 0$ are attached to $j_2, \ldots$, and the last L_R are attached to j_R. Then

$$L_1 + \cdots + L_R = K' \; , \tag{25}$$

and the labels $\ell_1, \ldots, \ell_{K'}$ of the first root vertices of $\omega_1, \ldots, \omega_{K'}$ are determined by $k_1, \ldots, k_R$. Hence, the subset of trees ω with fixed values of $L_1, \ldots, L_R \geq 0$ fulfilling (25) has μ_N-measure given by

$$\begin{aligned}
&D_N^{-1} \sum_{N_1+\cdots+N_{K'}=N-|\hat{\omega}|} \prod_{s=1}^{K'} E_{N_s}^{(\ell_s)} \\
&\leq \quad D_N^{-1} \sum_{t=1}^{K'} \sum_{\substack{N_1+\cdots+N_{K'}=N-|\hat{\omega}| \\ N_t \geq (N-|\hat{\omega}|)/K'}} \prod_{s=1}^{K'} E_{N_s}^{(\ell_s)} \\
&\leq \quad c'' \sum_{t=1}^{K'} K'^{\frac{5}{2}} \prod_{\substack{s=1 \\ s \neq t}}^{K'} z_{\ell_s} \\
&\leq \quad c'' K^{\frac{7}{2}} \, 2^{|\hat{\omega}|+1-K} \; ,
\end{aligned}$$

where the first two inequalities follow by the same arguments as for $r = 1$, and where c'' is a constant depending only on $\hat{\omega}$. Since there are

$$\binom{K' + R - 1}{R - 1} \leq \frac{K^{R-1}}{(R-1)!}$$

different ways of choosing $L_1, \ldots, L_R \geq 0$ fulfilling (25) we have

$$\mu_N(\{\omega \in \mathcal{W} \mid |B_{r+1}(\omega)| = K \ , \ B_r(\omega) = \hat{\omega}\}) \leq c''' \, K^{R+\frac{5}{2}} \, 2^{-K} \ ,$$

for some positive constant c''' depending only on $\hat{\omega}$. Clearly, this proves (24) for $r+1$, if K_{r+1} is chosen large enough. This completes the proof of tightness of the sequence μ_N, $N \in \mathbb{N}$.

It remains to establish convergence of $\mu_N(\mathcal{B}_{\frac{1}{r}}(\hat{\omega}))$ as $N \to \infty$ for all $r \in \mathbb{N}$ and finite $\hat{\omega} \in \mathcal{W}$. When $\rho(\hat{\omega}) \leq r - 1$, we have $\mathcal{B}_{\frac{1}{r}}(\hat{\omega}) = \{\hat{\omega}\}$, and

$$\lim_N \mu_N(\mathcal{B}_{\frac{1}{r}}(\hat{\omega})) = 0.$$

Since $\mathcal{B}_{\frac{1}{r}}(\hat{\omega}) = \mathcal{B}_{\frac{1}{r}}(B_r(\hat{\omega}))$, we assume from now on that $\hat{\omega} \in \mathcal{W}_r$, and we set $\hat{N} = |\hat{\omega}|$. Then

$$\mathcal{B}_{\frac{1}{r}}(\hat{\omega}) = \{\omega \in \mathcal{W} \mid B_r(\omega) = \hat{\omega}\} \ .$$

With the notation $\Delta_r(\hat{\omega}) = (j_1, \ldots, j_R)$, any $\omega \in \mathcal{B}_{\frac{1}{r}}(\hat{\omega})$ is obtained by grafting a sequence $(\omega_1, \ldots, \omega_R)$ of R trees in $\mathcal{W}$ on $\hat{\omega}$ such that the root-vertex $i_0(\omega_s)$ is identified with j_s and has same label k_s as that of j_s in $\hat{\omega}$. This gives

$$\mu_N\left(\mathcal{B}_{\frac{1}{r}}(\hat{\omega})\right) = D_N^{-1} \sum_{N_1+\cdots+N_R=N-\hat{N}} \prod_{s=1}^{R} D_{N_s}^{(k_s)} \ , \tag{26}$$

where $N_s = |\omega_s|$. For a given $t = 1, \ldots, R$, let $N_s, \ s \neq t$, be fixed, while N_t is determined as a function of N by $N_1 + \cdots + N_R = N - \hat{N}$. For the corresponding term in the sum we obtain from Corollary 2.5 as $N \to \infty$

$$\begin{aligned} D_N^{-1} \prod_{s=1}^{R} D_{N_s}^{(k_s)} &= 12^{-\hat{N}} \frac{D_{N_t}^{(k_t)}}{D_{N_t}} \, \frac{12^{-N_t} D_{N_t}}{12^{-N} D_N} \prod_{s \neq t} D_{N_s}^{(k_s)} 12^{-N_s} \\ &\to 12^{-\hat{N}} d_{k_t} \prod_{s \neq t} D_{N_s}^{(k_s)} 12^{-N_s} \ . \end{aligned}$$

From this we conclude, for any fixed $A > 0$, that

$$\begin{aligned} &\lim_N \ \mu_N\left(\{\omega \in \mathcal{B}_{\frac{1}{r}}(\hat{\omega}) \mid N_s \leq A \text{ for all } s \text{ but one}\}\right) \\ &= 12^{-\hat{N}} \sum_{t=1}^{R} d_{k_t} \prod_{s \neq t} \sum_{N_s=0}^{A} D_{N_s}^{(k_s)} 12^{-N_s} \ . \end{aligned} \tag{27}$$

On the other hand, for fixed $1 \le t, u \le R,\ t \neq u$ we have

$$
\begin{aligned}
& D_N^{-1} \sum_{\substack{N_1+\cdots+N_R=N-\hat{N} \\ N_t \ge (N-\hat{N})/R,\ N_u \ge A}} \prod_{s=1}^{R} D_{N_s}^{(k_s)} \\
\le\ & cst \cdot \sum_{\substack{N_s,\ s\neq t,u \\ N_u \ge A}} 12^{-\hat{N}} \left(\frac{NR}{N-\hat{N}}\right)^{5/2} N_u^{-5/2} \prod_{s\neq t,u} D_{N_s}^{(k_s)} 12^{-N_s} \\
\le\ & cst \cdot A^{-3/2} \prod_{s\neq t,u} w_{k_s} \\
=\ & cst \cdot A^{-3/2}\ , \qquad (28)
\end{aligned}
$$

where the constants depend on $\hat{\omega}$ only, and we have used (7). This shows that

$$\mu_N \Big(\{\omega \in \mathcal{B}_{\frac{1}{r}}(\hat{\omega}) \mid \exists u \neq t \text{ s.t. } N_u \ge A, N_t \ge A\} \Big) \le cst \cdot A^{-3/2}\ , \qquad (29)$$

where the constant depends on $\hat{\omega}$ only. Letting $A \to \infty$ we finally conclude from (26), (27) and (29) that

$$\mu_N(\mathcal{B}_{\frac{1}{r}}(\hat{\omega})) \overset{N\to\infty}{\longrightarrow} 12^{-|\hat{\omega}|} \sum_{t=1}^{R} d_{k_t} \prod_{s\neq t} w_{k_s}\ . \qquad (30)$$

This concludes the proof of Theorem 3.1, since it is clear from the definition that the (countable) set of finite well labelled trees has vanishing μ-measure. $\square$

For later use we note that the proof extends immediately to the corresponding situation for k-labelled trees. Defining $\mu_N^{(k)}$ as the uniform probability measure concentrated on the set ${\mathcal{W}'}_N^{(k)}$ of k-labelled trees of size n, that is

$$\mu_N^{(k)}(\omega) = (D_N^{(k)})^{-1} \text{ for } \omega \in {\mathcal{W}'}_N^{(k)}\ , \quad \mu_N^{(k)}\left(\overline{\mathcal{W}}^{(k)} \setminus {\mathcal{W}'}_N^{(k)}\right) = 0\ ,$$

we thus have the following corollary of the preceding proof.

Corollary 3.2. *The sequence $\mu_N^{(k)}$, $N \in \mathbb{N}$, converges to a Borel probability measure, $\mu^{(k)}$, concentrated on $\mathcal{W}_\infty^{(k)}$. We call $\mu^{(k)}$ the uniform probability measure on $\mathcal{W}_\infty^{(k)}$.*

Having proven the existence of the measure μ we obtain by a slight modification of the last part of the proof of Theorem 3.1 the following result on the measure $d\mu(\omega_1, \ldots, \omega_R \mid A(\hat{\omega}))$ obtained by conditioning μ on the event

$$A(\hat{\omega}) = \mathcal{B}_{\frac{1}{r}}(\hat{\omega}) = \{\omega \in \overline{\mathcal{W}} \mid B_r(\omega) = \hat{\omega}\}\ ,$$

where $\hat{\omega} \in \mathcal{W}_r$ is a finite tree of height r and with R vertices at maximal distance r from the first root i_0, and we identify $A(\hat{\omega})$ (homeomorphically) with $\overline{\mathcal{W}}^{(k_1)} \times \cdots \times \overline{\mathcal{W}}^{(k_R)}$ as previously, where $k_1, \ldots, k_R$ are the labels of those R vertices.

Corollary 3.3. *For $\hat{\omega} \in \mathcal{W}_r$ we have*

$$\mu(A(\hat{\omega})) = 12^{-|\hat{\omega}|} \sum_{t=1}^{R} d_{k_t} \prod_{s \neq t} w_{k_s}$$

and

$$d\mu(\omega_1, \ldots, \omega_R \mid A(\hat{\omega})) = \mu(A(\hat{\omega}))^{-1} \sum_{t=1}^{R} d\mu^{(k_t)}(\omega_t) 12^{-|\hat{\omega}|} \prod_{s \neq t} d\rho^{(k_s)}(\omega_s) \ , \quad (31)$$

where the measure $\rho^{(k)}$ is the supported on $\mathcal{W}^{(k)}$ and defined by

$$\rho^{(k)}(\omega) = 12^{-|\omega|} \quad \text{for } \omega \in \mathcal{W}^{(k)}.$$

Remark 3.4. For $\hat{\omega} \in \mathcal{W}_r^{(k)}$ and

$$A(\hat{\omega}) = \{\omega \in \overline{\mathcal{W}}^{(k)} \mid B_r(\omega) = \hat{\omega}\} \ ,$$

we have similarly with same notation that

$$\mu^{(k)}(A(\hat{\omega})) = 12^{-|\hat{\omega}|} \sum_{t=1}^{R} \frac{d_{k_t}}{d_k} \prod_{s \neq t} w_{k_s}$$

and

$$d\mu^{(k)}(\omega_1, \ldots, \omega_R \mid A(\hat{\omega})) = \mu^{(k)}(A(\hat{\omega}))^{-1} \sum_{t=1}^{R} d\mu^{(k_t)}(\omega_t) 12^{-|\hat{\omega}|} \prod_{s \neq t} d\rho^{(k_s)}(\omega_s) \ .$$

Remark 3.5. It is worth noting that the proof of the existence of the limit (30) and of the factorized form (31) of the conditional probability measure depends crucially on the fact that the exponent $-\frac{5}{2}$ of the asymptotic power dependence of $D_N^{(k)}$ on N has value less than -1. For unlabelled planar trees this exponent is $-\frac{3}{2}$ such that similar, but simpler, arguments apply, see [14, 15].

4 Description of the uniform probability measure on $\mathcal{W}_\infty$

4.1 More on the topology on labelled trees

Consider a labelled tree $\omega \in \mathcal{W}^{(k)}$ with a marked leaf λ. Let $h(\lambda)$ denote the height of λ, i.e, the graph distance in ω of λ from the first root, and let $\ell(\lambda)$ be the label of λ. Let $gr(\omega, \lambda, \omega')$ denote the labelled tree formed, starting from ω, by grafting a labelled tree $\omega' \in \overline{\mathcal{W}}^{(\ell(\lambda))}$ at the leaf λ of ω (see Figure 2). Set

$$Gr(\omega, \lambda) = \left\{ gr(\omega, \lambda, \omega') \mid \omega' \in \overline{\mathcal{W}}^{(\ell(\lambda))} \right\} .$$

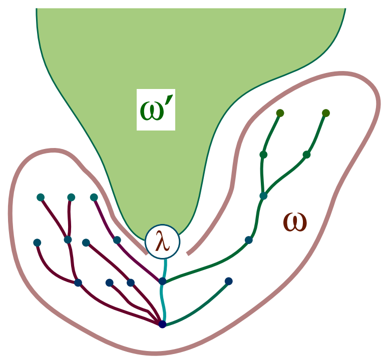

Figure 2: Grafting a tree: $gr(\omega, \lambda, \omega')$.

Proposition 4.1. *i) Each set $Gr(\omega, \lambda)$ is both open and closed and can be written as a union of sets in $\mathcal{V}$.*

ii)

$$\mu\left(Gr(\omega, \lambda)\right) = 12^{-|\omega|} d_{\ell(\lambda)},$$

iii) If $\hat{\omega}$ is finite *with radius r, then $\mathcal{B}_{1/r}(\hat{\omega}) = A(\hat{\omega})$ can be written, up to a set of μ–measure 0, as a union of sets $Gr(\omega, \lambda)$, all satisfying $h(\lambda) = r$.*

Proof. i) Setting

$$a = \rho(\omega) - h(\lambda) + 1 \quad \text{and} \quad k = \ell(\lambda)\ ,$$

we have

$$Gr(\omega, \lambda) = \bigcup_{\substack{\omega' \in \mathcal{W}^{(k)} \\ \rho(\omega') \leq a}} \mathcal{B}_{\frac{1}{\rho(\omega)+1}}\left(gr(\omega, \lambda, \omega')\right).$$

Since the family of all open balls with a given radius form a partition of $\overline{\mathcal{W}}$, both $Gr(\omega, \lambda)$ and its complement are open, being unions of open balls with radius $\frac{1}{\rho(\omega)+1}$.

ii) By i) we have

$$\lim_N \mu_N\left(Gr(\omega, \lambda)\right) = \mu\left(Gr(\omega, \lambda)\right).$$

Here

$$\mu_N\left(Gr(\omega, \lambda)\right) = \frac{D^{(k)}_{N-|\omega|}}{D_N} = \frac{D^{(k)}_{N-|\omega|}}{D^{(k)}_N}\ \frac{D^{(k)}_N}{D_N},$$

for $N > |\omega|$, and ii) follows from Corollary 2.5.

iii) Given a tree $\hat{\omega}$, the grafting operation can be generalized in an obvious way to a finite sequence of leaves, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_R)$, in $\hat{\omega}$ and a finite sequence of trees, $\underline{\omega} = (\omega_1, \ldots, \omega_R)$ with $\omega_i \in \overline{\mathcal{W}}^{(\ell(\lambda_i))}$. We denote the resulting tree by $gr\,(\hat{\omega}, \underline{\lambda}, \underline{\omega})$.

Assume that the tree $\hat{\omega}$ has height r, and, as in Section 3, let $\Xi_r(\hat{\omega}) = ((j_1, k_1), \ldots, (j_R, k_R))$. Furthermore, let Λ_t be the sequence obtained by erasing j_t from $(j_1, \ldots, j_R)$. Setting

$$\overline{\Omega}_t = \prod_{s \neq t} \overline{\mathcal{W}}^{(k_s)}, \qquad \Omega_t = \prod_{s \neq t} \mathcal{W}^{(k_s)},$$

we then have

$$\begin{aligned} A(\hat{\omega}) &= \bigcup_{t=1}^{R} \bigcup_{\underline{\omega} \in \overline{\Omega}_t} Gr\,(gr(\hat{\omega}, \Lambda_t, \underline{\omega}), j_t) \\ &\supset \bigcup_{t=1}^{R} \bigcup_{\underline{\omega} \in \Omega_t} Gr\,(gr(\hat{\omega}, \Lambda_t, \underline{\omega}), j_t) \doteq A'(\hat{\omega}). \end{aligned}$$

Thus, the set $A'(\hat{\omega})$ consists of the trees in $A(\hat{\omega})$ such that only one among the R grafted trees is infinite, the others being finite. From ii) and Corollary 3.3 we conclude that

$$\begin{aligned} \mu(A'(\hat{\omega})) &= \sum_{t=1}^{R} \sum_{(\omega_1, \cdots, \omega_{R-1}) \in \Omega_t} d_{k_t} 12^{-|\hat{\omega}| - |\omega_1| - \cdots - |\omega_{R-1}|} \\ &= \mu(A(\hat{\omega})). \end{aligned}$$

□

Remark 4.2. Note that statement ii) in Proposition 4.1 has a straightforward generalization to a family of leaves, possibly at different heights.

4.2 The spine

We define a *spine* of a labelled tree ω to be any infinite sequence of labelled vertices (λ_r) starting at the first root–vertex of ω, and such that λ_r is a son of λ_{r-1}. Equivalently, a spine of ω is an infinite labelled linear subtree with the same first root–vertex as ω. Similarly, for $r \in \mathbb{N}$ an r-spine of ω is a labelled linear subtree of height r with same first root–vertex as ω.

The following result will be important for the subsequent developments.

Theorem 4.3. *With μ–probability 1, a tree contains exactly one spine.*

In other words, the probability measure μ is supported on the set $\mathcal{S}$ of trees with exactly one spine. This is not unexpected, since the limit law in the simpler case of unlabelled trees has a similar description [1, 14, 15, 18, 22]. Note that trees in $\mathcal{S}$ are obtained by grafting, at each vertex of its spine, a pair of *finite* well labelled trees, one on the right and one on the left.

Note that by convention we draw the root at the bottom of the tree so the right of the spine means the right, when looking at the spine *from the root.*

Proof. Set

$$\mathcal{S}_r = \bigcup_{\substack{\omega\in\mathcal{W}\\ h(\lambda)=r}} Gr\,(\omega,\lambda) \quad \text{and} \quad \mathcal{S} = \bigcap_{r\geq 1} \mathcal{S}_r \ ,$$

and assume that η belongs to $\mathcal{S}$. Then η has infinite size, for it contains a vertex at each height r. Being an element in $\mathcal{S}_r \cap \mathcal{W}_\infty$ it follows that η has a unique decomposition of the form $gr(\omega_r, \lambda_r, \omega'_r)$, where $\omega_r \in \mathcal{W}$ and $h(\lambda_r) = r$. Necessarily, in such a decomposition, ω'_r belongs to $\mathcal{W}_\infty^{(\ell(\lambda_r))}$. Hence, for each r, there is a unique pair (ω_r, λ_r) such that $\eta \in Gr\,(\omega_r, \lambda_r)$. It follows that all spines in η, if any, coincide up to height r, and λ_{r+1} is necessarily the son of λ_r. This shows that $(\lambda_r)_{r\geq 0}$ is the unique spine of η.

Finally, due to Proposition 4.1 iii), $\mu(\mathcal{S}_r) = 1$, and thus $\mu\,(\mathcal{S}) = 1$. □

For a random μ–distributed element ω of $\mathcal{S} \subset \mathcal{W}_\infty$ we introduce the following notation. By e_n we denote the vertex at height n on its spine, and the label of e_n is denoted by $X_n(\omega)$. Furthermore, we let $L_n(\omega)$ (resp. $R_n(\omega)$) be the *finite* subtree of ω attached to e_n on the left (resp. on the right) of its spine. Finally, we let $\hat{\rho}^{(k)}$ denote the measure obtained by normalizing $\rho^{(k)}$, that is

$$\hat{\rho}^{(k)}(\omega) \doteq \frac{12^{-|\omega|}}{w_k} \quad \text{for} \quad \omega \in \mathcal{W}^{(k)} \ .$$

Theorem 4.4. *The measure μ has the following probabilistic description.*

i) $X = (X_n)_{n\geq 0}$ is a Markov chain with values in $\mathbb{N}$, starting at $X_0 = 1$ and with transition probabilities

$$\mathbb{P}\,(X_{n+1} = \ell \mid X_n = k) = \frac{(w_k)^2}{12\, d_k}\ d_\ell \qquad (32)$$

for $|k-\ell| \leq 1,\ k,\ell \geq 1$, and zero otherwise. In other words, X is a discrete birth & death process with parameters

$$q_k \doteq \frac{(w_k)^2}{12\, d_k}\ d_{k-1}, \quad r_k \doteq \frac{(w_k)^2}{12}, \quad p_k \doteq \frac{(w_k)^2}{12\, d_k}\ d_{k+1},$$

for $k \in \mathbb{N}$, where by convention $d_0 = 0$.

ii) Conditionally, given that $X = (s_n)_{n\geq 0}$, the L_n's and R_n's are independent, and distributed according to the measure $\hat{\rho}^{(s_n)}$ on $\mathcal{W}^{(s_n)}$.

Proof. Given $r \in \mathbb{N}_0$ and $(s_0, s_1, \ldots, s_r) \in \mathbb{N}^r$ fulfilling

$$s_0 = 1 \quad \text{and} \quad |s_n - s_{n-1}| \leq 1 \text{ for } n = 1, \ldots, r\ ,$$

we observe that the set

$$\{\omega \in \mathcal{S} \ \mid (L_n, R_n) = (\omega'_n, \omega''_n),\ 0 \leq k \leq r-1, \text{ and } X_n = s_n,\ 0 \leq n \leq r\}$$

equals $Gr(\omega,\lambda)\cap\mathcal{S}$, where ω is formed by grafting the pairs (ω'_n,ω''_n) of *finite* well labelled trees in $\mathcal{W}^{(s_n)}$ on both sides of the r-spine $((e_0,s_0),(e_1,s_1),\ldots,(e_r,s_r))$, and where $\lambda=(e_r,s_r)$.

Thus, by Proposition 4.1 ii), the "finite–dimensional distributions" are given by

$$\begin{aligned}\mu\left(\{(L_n,R_k,X_n)=(\omega'_n,\omega''_n,s_n),\ 0\le k\le r-1,\ \text{and } X_r=s_r\}\right)\\ &= d_{s_r}12^{-|\omega|}\\ &=\prod_{i=0}^{r-1}\frac{1}{12}\frac{d_{s_{i+1}}}{d_{s_i}}(w_{s_i})^2\prod_{i=0}^{r-1}\frac{12^{-|\omega'_i|}}{w_{s_i}}\frac{12^{-|\omega''_i|}}{w_{s_i}}.\end{aligned}$$

Upon realizing that relations (15)-(17) imply $p_k+q_k+r_k=1$, the statements in the theorem can be read off from this formula. □

Remark 4.5. By a slight extension of the argument, it follows that, upon conditioning on a fixed r-spine, all $2r+1$ branches, including the infinite one attached to the end of the r-spine, are independently distributed, and the latter is distributed according to the measure μ_{s_r}, where s_r is the end-label (cf. also Corollary 3.3).

4.3 The branches

The following theorem describes in more detail the probabilistic structure of finite subtrees grafted on the left and on the right of the spine: the labels of the nodes can be seen as types of *multitype* Galton–Watson processes. In this setting we have

Theorem 4.6. *Conditionally, given that $X_n=k$, R_n and L_n are independent multitype Galton-Watson trees, in which the ancestor has type k, and a (type ℓ)-individual can only have progeny of type $\ell+\varepsilon$, $\varepsilon\in\{0,\pm1\}$. In such multitype Galton-Watson trees, the progeny of a (type ℓ)-individual is determined by a sequence of independent trials with 4 possible outcomes, $\ell+1$, $\ell-1$, ℓ and e (for "extinction"), with respective probabilities $w_{\ell+1}/12$, $w_{\ell-1}/12$, $w_\ell/12$ and $\frac{1}{w_\ell}$ (that add up to 1), sequence stopped just before the first occurence of e. So a (type ℓ)-individual has as many children of type $\ell+1$, $\ell-1$ or ℓ as there are occurences of $\ell+1$, $\ell-1$, ℓ in the sequence, before the first occurence of e.*

Proof. Note that, owing to (22),

$$\left(w_{\ell+1}+w_{\ell-1}+w_\ell\right)\frac{1}{12}+\frac{1}{w_\ell}=1. \tag{33}$$

With respect to $\hat\rho^{(k)}$, the probability that the ancestor has m sons with respective labels $(k_1,\cdots,k_m)$ and with associated subtrees $(\omega_1,\cdots,\omega_m)$ is

$$\frac{12^{-m-\sum|\omega_i|}}{w_k},$$

provided that $|k-k_i| \leq 1$ and $\omega_i \in \mathcal{W}_{k_i}$. This probability can be written:

$$\frac{1}{w_k} \prod_{i=1}^{m} \frac{w_{k_i}}{12} \prod_{i=1}^{m} \hat{\rho}^{(k_i)}(\omega_i).$$

The theorem follows by induction. □

Remark 4.7. Thus, for the special case of multitype Galton Watson trees we consider here, the picture is quite alike the picture given in [1, 14, 15, 18, 22], for monotype Galton Watson trees. However, there are some differences: the spine is not pasted uniformly on the available leaves but with a bias introduced by the different types, and the finite branching trees are multitype GW tree, critical in the sense that the average progeny of a (type ℓ)-individual,

$$w_\ell - 1 = 1 - \frac{4}{(k+1)(k+2)} \ ,$$

has supremum in ℓ equal to 1, but, as we see from the expansion of $W^{(k)}(x)$ at 1/12, with finite expected size. As an additional feature, the succession of types on the spine is a birth & death process.

5 Label occurrences in a uniform labelled tree

As will be seen in Section 6, the volume of the ball with radius k in a random uniform quadrangulation with N faces has the same distribution as the number of nodes with label *smaller than* $k+1$ in a well labelled tree with N edges. In this section we study the number of nodes with label *exactly* k in the uniform infinite well labelled tree, i.e. the number N_k of occurrences of label k in a well labelled tree with respect to the measure μ. In particular, we determine the asymptotic behavior of the average value $\langle N_k \rangle_\mu = \mathbb{E}_\mu [N_k]$. For this purpose we need to investigate, in subsections 5.1 and 5.2, two types of random walks associated to μ.

5.1 The random walk along the spine

We wish to determine the asymptotic behaviour of the number S_k of occurrences of label k along the spine of the uniform infinite well labelled tree. This behaviour eventually depends on the asymptotic behaviour of q_k, r_k, p_k as $k \to \infty$. We shall prove

Proposition 5.1.

$$\lim_{k\to\infty} \frac{\langle S_k \rangle_\mu}{k} = \lim_{k\to\infty} \frac{\mathbb{E}_1 [S_k]}{k} = \frac{3}{7}.$$

In probabilistic terms, S_k can be seen as the sojourn time of the process X at level k, i.e.

$$S_k(\omega) = \sharp\{n \in \mathbb{N} \mid X_n(\omega) = k\} \ .$$

Proposition 5.1 will be obtained as a consequence of a more general property of sojourn times S_k of general birth & death processes, stated in the next lemma.

Lemma 5.2. *Assume*

i) all q_k and p_k are positive with the exception of $q_1 = 0$,

ii) there exist constants $a > 1$ and c such that, for all k,

$$k^2 \left| \frac{q_k}{p_k} - 1 + \frac{a}{k} \right| \leq c$$

iii) the sequence r_k, $k \in \mathbb{N}$, has a limit $b < 1$.

Then

$$\lim_k \frac{\mathbb{E}_i\left[S_k\right]}{k} = \frac{2}{(1-b)(a-1)}$$

Remark 5.3. In ii), the assumption $a > 1$ ensures that X is transient, and that a.s. $\lim X_n = +\infty$.

Remark 5.4. As is usual for Markov chains, the index i in $\mathbb{E}_i\left[S_k\right]$ stresses the assumption $\mathbb{P}\left(X_0 = i\right) = 1$. For instance, in the case of the random uniform infinite labelled tree, $\mathbb{E}_i\left[S_k\right] = \left\langle S_k \right\rangle_{\mu^{(i)}}$. Due to transience, $\mathbb{E}_i\left[S_k\right]$ does not depend on i if $i \leq k$.

Proof. Set

$$m_{i,k} = \frac{q_i q_{i+1} \cdots q_{i+k-1}}{p_i p_{i+1} \cdots p_{i+k-1}} ,$$

which is positive for $i \geq 2$ by assumption. By definition, set $m_{i,0} = 1$, and let T_i denote the first hitting time of level i. As is well known [19, Chapter 3], X is transient if and only if $m_{1,k}$ is the general term of a converging series. Also

$$\mathbb{P}_{i+k}\left(T_i = +\infty\right) = \frac{\sum_{j=0}^{k-1} m_{i+1,j}}{\sum_{j \geq 0} m_{i+1,j}},$$

with the special case

$$\mathbb{P}_{i+1}\left(T_i = +\infty\right) = \frac{1}{\sum_{j \geq 0} m_{i+1,j}} \doteq \rho_i.$$

Let D_k denote the number of downcrossings $k+1 \downarrow k$ and let Y_ℓ be the sojourn time of X at level k after the ℓ-th and before the $\ell+1$-th (if it occurs) downcrossing. Also, let Y_0 be the sojourn time of X at level k before the first (if any) downcrossing. Then, for $i \leq k$ and $n \geq 0$,

$$\mathbb{P}_i\left(D_k \geq n\right) = \left(1 - \rho_k\right)^n.$$

Furthermore, conditionally, given that $D_k = n$, $(Y_0, Y_1, \ldots, Y_n)$ is a sequence of i.i.d. random variables satisfying, for $m \geq 1$,

$$\mathbb{P}_i\left(Y_j \geq m\right) = \left(1 - p_k\right)^{m-1}.$$

As a consequence, by Wald's identity,

$$\mathbb{E}_i\left[S_k\right] = \mathbb{E}_i\left[1 + D_k\right]\mathbb{E}_i\left[Y_0\right] = \frac{1}{\rho_k p_k}. \tag{34}$$

To determine the asymptotics of ρ_k we use that, due to assumption ii), the expression

$$k^2\left(\frac{q_k(k+1)^a}{p_k\, k^a} - 1\right)$$

is bounded as a function of k. Thus

$$\limsup_k \sup_{\ell\geq 0}\left|m_{k,\ell}\ \left(\frac{k+\ell}{k}\right)^a - 1\right| = 0\,,$$

and, as a consequence,

$$\lim_k \frac{1}{k}\ \sum_{\ell\geq 0} m_{k,\ell} = \lim_k \frac{1}{k}\ \sum_{\ell\geq 0}\frac{1}{\left(1+\frac{\ell}{k}\right)^a} = \int_1^{+\infty}\frac{dx}{x^a} = \frac{1}{a-1}.$$

Combining this with the definition of ρ_k and

$$\lim_k p_k = \frac{1-b}{2}\,,$$

the claimed limit follows from (34). ☐

In the next lemma we collect the large-k behaviours of q_k, r_k, p_k that follow immediately from (10), (14) and (32).

Lemma 5.5.

$$\begin{aligned} q_k &= \frac{1}{3} - \frac{4}{3k} + \mathcal{O}\left(k^{-2}\right),\\ r_k &= \frac{1}{3}\left(1 - \frac{4}{k^2}\right) + \mathcal{O}\left(k^{-3}\right),\\ p_k &= \frac{1}{3} + \frac{4}{3k} + \mathcal{O}\left(k^{-2}\right). \end{aligned}$$

Proposition 5.1 now follows from Lemma 5.2 and Lemma 5.5, with $a = 8$ and $b = 1/3$.

Remark 5.6. By discretisation of the d-dimensional Bessel process, one obtains a birth & death process that satisfies the assumptions of Lemma 5.2 for $(a, b) = (d-1, 0)$. So the birth & death process we meet here is, in a sense, close to a 9-dimensional Bessel process. Similar birth & death processes also appear, in connection with random non-labelled trees, in the study of the 3-dimensional Bessel process, and lead to an elegant proof of the decomposition theorem of Williams [21].

Proposition 5.1 ensures that S_k is finite almost surely, in other words that each label k occurs almost surely only finitely many times on the spine. It will turn out essential for the interpretation of well labelled trees as quadrangulated surfaces in the next section that the same result holds for the total number N_k of occurences of a label k on the whole tree, i.e. that N_k is finite almost surely for each $k \in \mathbb{N}$. We shall do this by proving that $\langle N_k \rangle_\mu$ is finite, and also determine its asymptotic behaviour as $k \to \infty$ in subsection 5.3 below. For this purpose a study of the average value of N_k in a generic branch attached to the spine is first needed.

5.2 A random walk associated to branches

Consider a branch ω_n, left or right, grafted at the n-th site e_n of the spine of a well labelled tree $\omega \in \mathcal{S}$. According to Theorem 4.4, conditionally, given that k is the label of e_n, ω_n is is distributed according to $\hat{\rho}^{(k)}$. Letting $N_j(\omega_n)$ denote the number of occurrences of the label j in ω_n we denote by $G(k,j)$ the (normalized) average value of N_j, i.e.

$$\begin{aligned} G(k,j) &= \mathbb{E}_{\hat{\rho}^{(k)}}[N_j] \\ &= \frac{1}{w_k} \sum_{\omega \in \mathcal{W}^{(k)}} N_j(\omega) 12^{-|\omega|} . \end{aligned}$$

Consider the set $\mathcal{W}_\star^{(k)}$ of *marked* finite trees in $\mathcal{W}^{(k)}$, i.e. couples (ω, e) with $\omega \in \mathcal{W}^{(k)}$ and e a marked vertex of ω, endowed with the measure $\frac{12^{-|\omega|}}{w_k}$ for each element (ω, e) of $\mathcal{W}_\star^{(k)}$. Then $G(k,j)$ can be seen as the measure of the set of marked trees whose marked vertex has label j.

Given an element (ω, e) of $\mathcal{W}_\star^{(k)}$, it has a distinguished (finite) spine

$$((f_0, \theta_0), (f_1, \theta_1), \ldots, (f_L, \theta_L)) ,$$

namely the path connecting the root $i_0(\omega) = f_0$ to the marked vertex $e = f_L$. We let

$$(\omega'_1, \ldots, \omega'_{L-1}) \quad \text{and} \quad (\omega''_1, \ldots, \omega''_{L-1}) ,$$

respectively, with ω'_t and ω''_t in $\mathcal{W}^{(\theta_t)}$, denote the sequences of subtrees (branches) in ω grafted on this spine on the left and right, and by $\omega_L \in \mathcal{W}^{(j)}$ the subtree attached to the marked vertex. Since this correspondence between marked trees on one side and spines together with branches on the other is bijective we obtain, by summing over branches first, the representation

$$G(k,j) = \sum_{\theta : k \to j} 12^{-|\theta|} \prod_{t=0}^{|\theta|-1} w_{\theta_t} w_{\theta_{t+1}} , \tag{35}$$

where we have set

$$\theta = (\theta_0, \theta_1, \ldots, \theta_L) \text{ and } |\theta| = L ,$$

and the sum is over spine-label sequences θ with initial label k and final label j. Evidently, θ can also be viewed as a walk in $\mathbb{N}$ from k to j whose steps $\theta_{t+1} - \theta_t$ belong to $\{0, \pm 1\}$ and have "probability"

$$\frac{w_{\theta_t} w_{\theta_{t+1}}}{12}.$$

Alternatively, by Theorem 4.6, the average number of children of an individual of type k is $w_k - 1$, and due to Wald's identity, for $\varepsilon \in \{0, \pm 1\}$, the average number $p_{k,k+\varepsilon}$ of children of type $k + \varepsilon$ of an individual of type k is given by

$$\begin{aligned} p_{k,k+\varepsilon} &= (w_k - 1) \frac{w_{k+\varepsilon}}{w_{k-1} + w_k + w_{k+1}} \\ &= \frac{w_k w_{k+\varepsilon}}{12}, \end{aligned}$$

the second equality owing to (33). Let $\Sigma_\theta(\omega_n)$ denote the number of spines of ω_n whose spine-label sequence is θ. Let us consider θ as a word in

$$\mathbb{N}^\star = \{\emptyset\} \cup \left(\bigcup_{k \geq 1} \mathbb{N}^k \right).$$

Then, for any a in $\mathbb{N}$, conditionally given $\Sigma_\theta = k$, $\Sigma_{\theta a}$ is distributed as

$$\sum_{i=1}^{k} Y_i,$$

in which Y_i stands for the number of children of type a of the last vertex of the i–th spine with spine-label sequence θ. Thus the Y_i's are i.i.d. with expectation $p_{\theta_L, a}$, and, according to Wald's identity,

$$\mathbb{E}\left[\Sigma_{\theta a}\right] = \mathbb{E}\left[\Sigma_\theta\right] \mathbb{E}\left[Y_1\right] = \mathbb{E}\left[\Sigma_\theta\right] p_{\theta_L, a}.$$

By induction, the average number of paths θ in a tree distributed according to $\hat{\rho}^{(k)}$ is given by

$$\prod_{t=0}^{L-1} p_{\theta_t, \theta_{t+1}} = 12^{-L} \prod_{t=0}^{L-1} w_{\theta_t} w_{\theta_{t+1}}.$$

Summing over all spines with initial label k and final label j, we obtain again the formula for $G(k, j)$.

Theorem 5.7. *The function G has the following properties.*

$$\begin{aligned} &i) && G(k,j) = G(j,k) \quad \textit{for } k, j \in \mathbb{N} \\ &ii) && \phi'_+(k)\phi'_-(j) \leq G(k,j) \leq \phi_+(k)\phi_-(j) \quad \textit{for } j > k \geq 1 \\ &iii) && G(k,k) = d \cdot k + O(1) \end{aligned}$$

where

$$\phi'_+(k) = \Theta(k^4), \qquad \phi_+(k) = \Theta(k^4), \tag{36}$$

$$\phi'_-(k)) = \Theta(k^{-3}), \qquad \phi_-(k)) = \Theta(k^{-3}). \tag{37}$$

Proof. The symmetry property i) follows immediately from (35). From (35) we also deduce the difference equation

$$G(k,j) = \delta_{kj} + \frac{1}{12} w_j \sum_{\varepsilon=0,\pm1} w_{j+\varepsilon} G(k, j+\varepsilon)$$

where we use the convention $w_0 = 0$. Setting

$$H(k,j) = \frac{1}{12} w_k G(k,j) w_j \,, \tag{38}$$

this equation can be rewritten as

$$\Delta_j H(k,j) = 3\left(4(w_j)^{-2} - 1\right) H(k,j) - \delta_{kj} \,,$$

where Δ_j is the discrete Laplace operator with respect to j, whose action on functions $\phi : \mathbb{N} \to \mathbb{R}$ is given by

$$\Delta_j \phi(j) = \phi(j+1) + \phi(j-1) - 2\phi(j) \,,$$

where $\phi(0) = 0$ by convention.

We now proceed to establish the upper bound in ii). Since $w_k \to 2$ as $k \to \infty$ we may replace G by H. For any positive sequence u_k, $k \in \mathbb{N}$, let us define $H^u(k,j)$ in analogy with H by replacing w by u in formulas (35) and (38), such that $H = H^w$. Using

$$3\left(4w_k^{-2} - 1\right) \geq \frac{12}{k(k+3)}$$

and defining $u_k \in \mathbb{R}$ by

$$3\left(4u_k^{-2} - 1\right) = \frac{12}{k(k+3)}$$

we have

$$w_k < u_k < 2 \,. \tag{39}$$

It follows that

$$H(k,j) \leq H^u(k,j)$$

and H^u fulfills

$$\Delta_j H^u(k,j) = \frac{12}{j(j+3)} H^u(k,j) - \delta_{kj} \,. \tag{40}$$

Two linearly independent solutions to

$$\Delta_j \phi(j) = \frac{12}{j(j+3)} \phi(j) \,, \quad j \geq 2 \,,$$

are

$$\phi_+(j) = \Theta(j^4) \quad \text{and} \quad \phi_-(j) = \Theta(j^{-3}) \,. \tag{41}$$

These can, in fact, be found explicitly: one first verifies that

$$\phi_+(j) = (j+3)(j+2)(j+1)j$$

is a solution. Applying the method of reduction of order, one then finds

$$\phi_-(j) = (j+3)(j+2)(j+1)j \sum_{k=j}^{\infty} \frac{1}{k(k+1)^2(k+2)^2(k+3)^2(k+4)} ,$$

leading to (41).

We conclude that

$$H^u(k,j) = c_k\phi_-(j) + c_k^+\phi_+(j) \quad \text{for} \quad j > k \geq 1 ,$$

where c_k and c_k^+ are constants depending only on k. Here $c_k^+ = 0$, because, according to (39), $H^u(k,j)$ is bounded above by the function $H^2(k,j)$, defined analogously to $H^u(k,j)$ by replacing u_k by 2. This case corresponds to the simple random walk where $q_k = r_k = p_k = \frac{1}{3}$ for $k \geq 2$ with reflecting boundary condition at $k = 1$, and the function H^2 fulfills

$$\Delta_j H^2(k,j) = 0 ,$$

i.e. it is a linear function of j. But, since $\phi_+(j) = \Theta(j^4)$, it follows that $c_k^+ = 0$ and so

$$H^u(k,j) = c_k\phi_-(j) \quad \text{for } j > k \geq 1 .$$

Using the symmetry of $H^u(k,j)$, we get that c_k is a linear combination of $\phi_+(k)$ and $\phi_-(k)$ for $k \geq 2$, say

$$c_k = c \cdot \phi_-(k) + d \cdot \phi_+(k) .$$

Since $H^u > 0$ we obviously have $d \geq 0$. We claim that $d > 0$. Otherwise, we would have

$$H^u(k,j) = c \cdot \phi_-(k)\phi_-(j) \quad \text{for } k,j \geq 2 ,\ k \neq j ,$$

and, in particular, $H^u(k, k\pm 1) \to 0$ as $k \to \infty$. Using this in (40) for $k = j$, i.e.

$$\left(2 + \frac{12}{k(k+3)}\right) H^u(k,k) = 1 + H^u(k,k-1) + H^u(k,k+1) ,$$

we conclude that $H^u(k,k) \to \frac{1}{2}$ as $k \to \infty$. But this contradicts the inequality

$$H^u(k,k+1) \geq (u_{k+1})^2 H^u(k,k)/12$$

which is an easy consequence of the definition of H^u. Hence we must have $d > 0$ and (41) implies

$$H^u(k,k) = d \cdot k + O(1) . \tag{42}$$

Redefining $\phi_+(k)$ to be equal to c_k for $k \in \mathbb{N}$, we have established the upper bound in ii).

Similarly, we obtain a lower bound $H^v(k,j)$ for $H(k,j)$ of the same type by using

$$3\left(4(w_j)^{-2} - 1\right) \leq \frac{12}{(k-1)(k+2)}\,, \quad k \geq 2\,,$$

and defining $v_k \leq w_k$ by

$$v_k = \begin{cases} w_1 & \text{for } k = 1 \\ u_{k-1} & \text{for } k \geq 2\,. \end{cases}$$

This finishes the proof of ii).

Finally, iii) follows from (42) and the corresponding relation for H^v. □

5.3 Label occurences.

We are now ready to calculate the asymptotic behaviour of the average number $\langle N_j \rangle_\mu = \mathbb{E}_\mu [N_j]$ of occurences of the label j in the full uniformly distributed well labelled tree.

Theorem 5.8.

$$\mathbb{E}_\mu [N_j] = \Theta(j^3)\,.$$

Proof. For a random uniform well labelled infinite tree $\omega \in \mathcal{S}$, we have

$$N_j(\omega) = \sum_{n=0}^{\infty} (N_j(R_n) + N_j(L_n)) - S_j\,.$$

By Corollary 3.3 or Theorem 4.4 ii) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{E}_\mu [N_j(R_n) + N_j(L_n)] &= 2\mathbb{E}_1 \left[\sum_{n=0}^{\infty} G(X_n, j)\right] \\ &= 2\mathbb{E}_1 \left[\sum_{k=1}^{\infty} S_k\, G(k,j)\right] \end{aligned}$$

so that

$$\begin{aligned} \mathbb{E}_\mu [N_j] &= 2\sum_{k=1}^{\infty} \mathbb{E}_1 [S_k]\, G(k,j) - \mathbb{E}_1 [S_j] \\ &\leq 2\sum_{k=1}^{j-1} \mathbb{E}_1 [S_k]\, \phi_+(k)\phi_-(j) \\ &+ 2\sum_{k=j+1}^{\infty} \mathbb{E}_1 [S_k]\, \phi_-(k)\phi_+(j) \\ &+ \mathbb{E}_1 [S_j]\, (2G(j,j) - 1)\,. \end{aligned}$$

Inserting the asymptotic behaviours of $\phi_\pm$ and $\mathbb{E}_1[S_k]$ from (36), (37), and Proposition 5.1 one finds that the first two terms in the last expression equal $\Theta(j^3)$ while the last term equals $\Theta(j^2)$. Similarly one obtains a lower bound of the same type and the theorem is proven. $\square$

Corollary 5.9. *For each $j \in \mathbb{N}$ the number of vertices with label j is $\mu-$almost surely finite.*

6 The uniform infinite random quadrangulation

In this Section, we show how to draw an infinite planar map, starting from an infinite labelled tree in the set

$$\mathcal{C} = \{\omega \in \mathcal{S} \,|\, \forall j \geq 1, \quad N_j(\omega) < +\infty\} ,$$

which, as we saw in the previous section, has μ-measure 1. The result will be an infinite rooted quadrangulation of a domain in the plane, a notion defined more precisely below. Our construction follows the same steps as the finite analog given in [11, Section 3.4]. In particular, the vertices of the tree can be identified with the vertices of the corresponding quadrangulation, with the exception of a distinguished root-vertex in the latter. We stress, however, that we do not establish a one-to-one correspondence between infinite well labelled trees and infinite quadrangulations in general, since it appears that the reverse construction of a finite well labelled tree from a finite quadrangulation does not apply in the infinite case. What is important for our purposes, is that the constructed mapping $\mathcal{Q}$ allows us to transport the measure μ on $\mathcal{C}$ to the image set $\mathcal{Q}(\mathcal{C})$ of quadrangulations, and, furthermore, that $\mathcal{Q}$ possesses the property that the label of a vertex of a tree $\omega \in \mathcal{C}$ equals the distance to the root of the corresponding vertex in the quadrangulation $\mathcal{Q}(\omega)$ (see Property 6.3 below).

6.1 Regular infinite planar maps

By an *infinite planar map* we mean an embedding $\mathcal{M} = E(G)$ of an infinite graph G, which we assume is connected and all of whose vertices are of finite degree, into the 2-sphere S^2, such that the edges are represented by smooth curve-segments that do not intersect each other except at common vertices. In order to be a useful concept, some regularity properties of the embedding are necessary, in addition. For instance, it is evidently possible to embed the infinite linear tree into S^2, thought of as the plane $\mathbb{R}^2$ with the point ∞ added, such that it is mapped onto a circle, which we shall not accept as a valid embedding. To avoid this we make the following assumption (see also [6]):

α) *If p_i, $i \in \mathbb{N}$, is a sequence of points in a planar map $\mathcal{M}$, considered as the union of its edges in S^2, such that, for $i \neq j$, p_i and p_j are contained in different edges, then the sequence has no condensation point in $\mathcal{M}$.*

Consider now a closed continuous curve C in an infinite planar map $\mathcal{M}$ composed of a sequence of edges. By asumption α), the set of different edges must be finite and so the complement of C in $\mathbb{R}^2$ decomposes into a finite number of connected components. If one of the components contains only a finite number of vertices of $\mathcal{M}$ then the part of $\mathcal{M}$ inside or on the boundary of this component is a finite planar map. The faces of this finite planar map inside the connected component are then also called faces of $\mathcal{M}$. Thus the faces of $\mathcal{M}$ are those obtained in this way for some closed curve C. In particular, each face is bounded by a polygonal loop composed of a *finite* number of edges. This leads us to make another assumption:

β) *either an edge of a planar map $\mathcal{M}$ is shared by exactly two faces, or it occurs twice in the boundary of one face of $\mathcal{M}$.*

For instance, such a regularity assumption does not hold for infinite trees, that have only one "face" with infinite degree. Let $\mathcal{D}(\mathcal{M})$ denote the union of the closed faces of $\mathcal{M}$: due to β), $\mathcal{D}(\mathcal{M})$ is an open connected subset of S^2. We identify embeddings $\mathcal{M}$ and $\mathcal{M}'$ that are related by an orientation preserving homeomorphism between $\mathcal{D}(\mathcal{M})$ and $\mathcal{D}(\mathcal{M}')$.

The planar map is called a *quadrangulation* if all of its faces are quadranngles, i.e. bounded by polygons with four edges. By a *rooted planar map* we mean a (finite or infinite) planar map with a distinguished oriented edge (i_0, i_1), called the *root* of the planar map. As in the case of trees, we call i_0 the first root-vertex.

6.2 The mapping $\mathcal{Q}$

In order to define $\mathcal{Q}$, we need regularity and uniqueness conditions on the embedding of labelled trees in S^2: for instance, even with restriction α), the linear tree can be embedded in many non-homeomorphic ways into S^2. However, requiring the embedding to be such that sequences of type α) have exactly one and the same condensation point in S^2, then the embedding is unique up to homeomorphisms of S^2. More generally, this also holds for arbitrary infinite trees:

The combinatorial definition of a tree given at the beginning of Section 2 determines a unique embedding of the corresponding graph into S^2, up to homeomorphisms, such that all sequences as in α) have exactly one and the same condensation point.

Indeed, such an embedding has already been indicated in Section 2, where vertices at distance r from the first root-vertex are mapped into vertical lines through $(r, 0)$, the only possible condensation point in question being ∞. We leave it to the reader to verify uniqueness. Below, we shall consider rooted trees (finite or infinite) as planar maps via this correspondence.

We are now ready to describe the mapping $\mathcal{Q}$, following closely [11, Section 3.4]. Let ω be a tree in $\mathcal{C}$, considered as a planar map with condensation point for sequences as in α) equal to c, which we assume is in $\mathbb{R}^2$ (as also indicated in the figures of this section). By F_0 we denote the complement of ω in $\mathbb{R}^2$. A *corner* of F_0 is a sector between two consecutive edges around a vertex. A

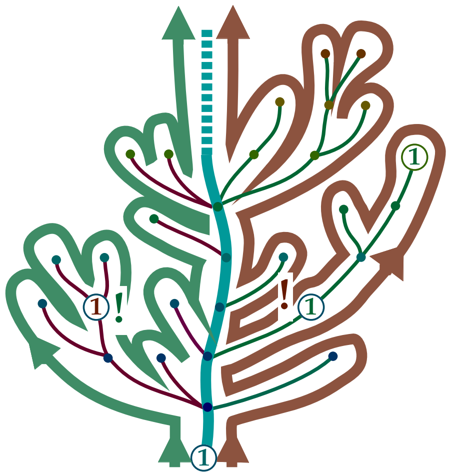

Figure 3: Contour traversal a) of the left side, b) of the right side of the spine. The last occurence of label 1 is signaled by an exclamation mark.

vertex of degree k defines k corners. The label of a corner is by definition the label of the corresponding vertex. A labelled tree $\omega \in \mathcal{C}$ has a finite number $C_k(\omega) \geq N_k(\omega)$ of corners with label k.

The image $\mathcal{Q}(\omega)$ is defined in three steps.

(1) A vertex v_0 with label 0 is placed in $F_0 \setminus \{c\}$ and one edge is added between this vertex and each of the $C_1(\omega) < +\infty$ corners with label 1. Notice that this is possible because ω has only one spine. The new root is taken to be the edge arriving from v_0 at the corner before the root of ω.

After Step (1) a uniquely defined rooted planar map $\mathcal{M}_0$ with $C_1(\omega) - 1$ faces has been obtained (see Figure 4, with $C_1(\omega) = 7$). It is natural to consider the complement of $\mathcal{M}_0$ and its faces as an additional face, which we shall call the *infinite face*: as $C_1(\omega) < +\infty$, there is a last occurence of the label 1 when one does a contour traversal, as indicated on Figure 3, of the left (resp. right) side of the spine, and this corresponds to some corner c_ℓ (resp. c_r) with label 1. The infinite face is the one with corner c_ℓ–v_0–c_r. The other faces are bounded by edges joining v_0 to two corners on the same side of the spine, and are thus finite.

The next steps take place independently in each of those faces and will be described for a generic[1] face F of $\mathcal{M}_0$. Let k be the degree of F (k can be infinite, and by construction $k \geq 3$). Among the corners of F only one belongs to v_0 and has label 0. If F is finite, let the corners be numbered from 0 to $k-1$ in clockwise order along the border, starting with v_0. Otherwise, F contains infinitely many corners both on the left and right side of the spine. Let the corners on the right

[1]The infinite face does not require special treatment.

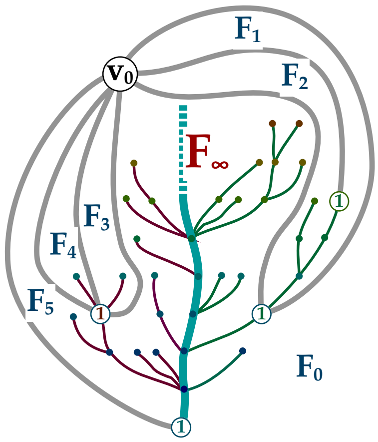

Figure 4: Step (1), leading to finite faces
F_i, $0 \leq i \leq 5$, and the infinite face F_∞.

of the spine be numbered by nonnegative integers, in clockwise order, starting with v_0, and let the corners on the left of the spine be numbered by negative integers, in counterclockwise order, starting right after v_0. Let moreover $\ell(i)$ be the label of corner i (so that $\ell(0) = 0$ and in the finite case $\ell(1) = \ell(k-1) = 1$, while in the infinite case $\ell(1) = \ell(-1) = 1$). In Figure 5 the corners are explicitly represented with their numbering for the infinite face.

(2) In each face, the function successor s is defined for all corners, but the corner at v_0, by

$$s(i) = \min\{j \triangleright i \mid \ell(j) = \ell(i) - 1\},$$

in which $j \triangleright i$ has the same meaning as $j > i$ for couples of positive integers, and also for couples of negative integers, but we assume that a negative integer is larger than a positive integer. More precisely, if $j \leq 0$ and $i > 0$, then $j \triangleright i$.

(3) For each corner $i \geq 2$ such that $s(i) \neq i + 1$, a chord $(i, s(i))$ is added inside the face. This can be done in such a way that the various chords do not intersect (Property 6.1 below).

Once this construction has been carried out in each face, a planar map $\mathcal{M}'$ is obtained.

(4) All edges of $\mathcal{M}'$ with the same label at both ends are deleted. The resulting map is a quadrangulation $\mathcal{M} = \mathcal{Q}(\omega)$ satisfying β) (Property 6.2 below).

Note that a chord $(i, s(i))$ of the infinite face can very well join both sides of the spine ($is(i) < 0$) if, in the contour traversal of the the right side of the spine,

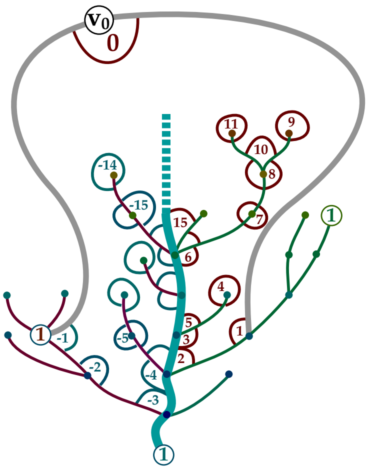

Figure 5: Numbering the corners of the infinite face.

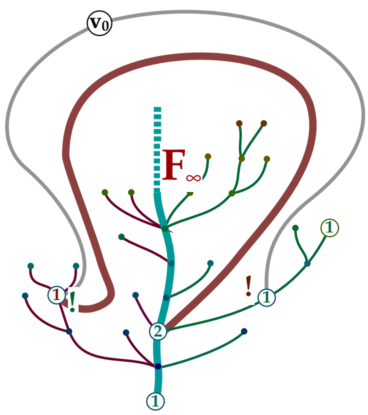

Figure 6: Inside F_∞, a chord joining two sides of the spine, with $\ell(2) = 2$, $\ell(-1) = 1$, and $s(2) = -1 \triangleright 2$

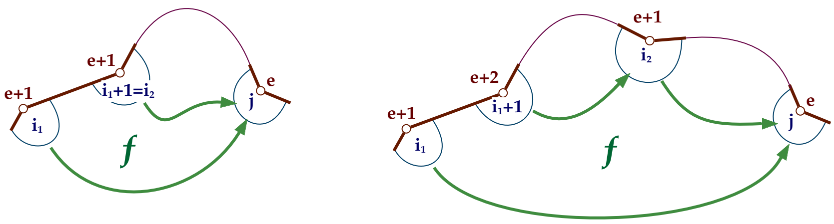

Figure 7: Two possible sizes for f: triangular or quadrangular.

the corner i appears after the last occurence of label $\ell(i) - 1$ (see Figure 6). Let us first prove the two properties that validate the preceding construction.

Property 6.1. *The chords $(i, s(i))$ do not intersect.*

Proof. Suppose that two chords $(i, s(i))$ and $(j, s(j))$ cross each other. Perhaps upon exchanging i and j, one has $i \lhd j \lhd s(i) \lhd s(j)$. The first two inequalities imply, together with the definition of s, that $\ell(j) > \ell(s(i))$, while the two last inequalities imply $\ell(s(i)) \geq \ell(j)$. This is a contradiction. □

Property 6.2. *The faces of $\mathcal{M}'$ are of one of the two types in Figure 7: either triangular with labels $e, e+1, e+1$, or quadrangular with labels $e, e+1, e+2, e+1$.*

Proof. Let f be a face of $\mathcal{M}'$. Then f is included in a face F of $\mathcal{M}_0$ so that its corners inherit the numbering and labelling of those of F. Let j be the corner with largest number (w.r.t. $\lhd$) in f, let $e = \ell(j)$ and let $i_1 \lhd i_2 \lhd j$ be the two neighbours of j in f (cf. Figure 7). The two latter corners both have label $e+1$, because the edge (i_1, j) has to be a chord so that $j = s(i_1)$, and, as $i_1 \lhd i_2 \lhd j$, this implies $\ell(i_2) \geq \ell(i_1)$ and hence $\ell(i_1) = \ell(i_2) = \ell(j) + 1$.

By construction, no other chord leaves i_1 so that the face is bordered by the edge $(i_1, i_1 + 1)$ of F. Two cases may occur:

- Either $i_1 + 1 = i_2$, and the face is triangular,
- Or the face is quadrangular: indeed the corner $i_1 + 1$ has label $e + 2$ (otherwise chord $(i_1 + 1, j)$ would exclude i_2 from the face) and the chord leaving $i_1 + 1$ goes to i_2 (otherwise $s(i_1 + 1) \neq i_2$, so that $\ell(s(i_1 + 1)) = \ell(i_1 + 1) - 1 = \ell(i_2)$ and chord $(s(i_1 + 1), j)$ would exclude i_2).

Observe finally that the deletion of edges with same labels at both ends will join triangular faces pairwise to form quadrangular faces. □

Property 6.3. *The labels of the vertices in the tree ω are the distances to the root of the corresponding vertices in $\mathcal{M} = \mathcal{Q}(\omega)$.*

Proof. By construction, the variation along the edges of the quadrangulation $\mathcal{Q}(\omega)$ is 0 or ± 1, thus any path from a vertex v with label k to v_0 has length at least k. Choose a face F of $\mathcal{M}_0$, such that v is a corner of F and let s be the successor function associated with F. Then $v, s(v), s^2(v), \ldots, s^k(v)$ is a length–k–path in $\mathcal{Q}(\omega)$, that ends at v_0. ☐

6.3 Properties of $\mathcal{Q}(\omega)$

In this Section, for sake of brevity, $\mathcal{D}(\omega)$ will denote $\mathcal{D}(\mathcal{Q}(\omega))$. As remarked earlier, since, by construction, $\mathcal{Q}(\omega)$ satisfies β), $\mathcal{D}(\omega)$ is an open connected subset of S^2. It should be noted that $\mathcal{D}(\omega)$ depends on the way the chords and edges in ω are drawn in the plane. However, it is easy to see that, as a planar map, $\mathcal{Q}(\omega)$ is unique by construction, and that $\mathcal{Q}$ is an injective mapping. In particular, $\mathcal{D}(\omega)$ is unique up to homeomorphisms for any given $\omega \in \mathcal{C}$.

Let now $\bar{\mu}$ be the measure on $\mathcal{Q}(\mathcal{C})$ obtained by transporting μ from $\mathcal{C}$, that is

$$\bar{\mu}(A) = \mu(\mathcal{Q}^{-1}(A))$$

for subsets $A \subseteq \mathcal{Q}(\mathcal{C})$, such that $\mathcal{Q}^{-1}(A)$ is μ-measurable in $\mathcal{C}$. Also, let $B_r(\mathcal{M})$ denote the ball of radius r in a quadrangulation $\mathcal{M}$, that is the finite planar map whose vertices are those of $\mathcal{M}$ with (graph) distance from the first root-vertex less than or equal to r, together with the edges connecting them (see Section 1). By Property 6.2 we can then reformulate Theorem 5.8 as follows.

Theorem 6.4. *Let $|B_r(\mathcal{M})|$ denote the number of vertices in $B_r(\mathcal{M})$. Then*

$$\mathbb{E}_{\bar{\mu}}\left[|B_r|\right] = \Theta(r^4) .$$

Our final results concern the shape of the domain $\mathcal{D}(\omega)$, which should be compared with Theorem 1.10 in [6].

Theorem 6.5. *For any well labelled tree $\omega \in \mathcal{C}$, the complement of $\mathcal{D}(\omega)$ in S^2 is connected. As such, the domain $\mathcal{D}(\omega)$ is homeomorphic to a disc,and, in particular, it has exactly one boundary component.*

Proof. According to [23], or [24, Chap. 13], only the first assertion of the Theorem needs a proof: assume on the contrary that the complement of $\mathcal{D}(\omega)$ is not connected, such that we can write

$$S^2 \setminus \mathcal{D}(\omega) = K_1 \cup K_2 ,$$

where K_1 and K_2 are non-empty compact subsets of S^2 contained in two disjoint open sets O_1 and O_2, respectively. From property α) it follows that there is only a finite number of edges in $\mathcal{Q}(\omega)$ that are not contained in $O_1 \cup O_2$, and clearly each of the sets O_1 and O_2 contains an infinite number of edges. Let now $R_0 > 0$ be large enough such that $B_{R_0}(\mathcal{Q}(\omega))$ contains all edges that are not contained in $O_1 \cup O_2$. It follows that if we remove from $\mathcal{D}(\omega)$ any finite set of faces containing all vertices in $B_{R_0}(\omega)$, then the remaining part of $\mathcal{D}(\omega)$ is not connected. We shall now obtain a contradiction with this statement.

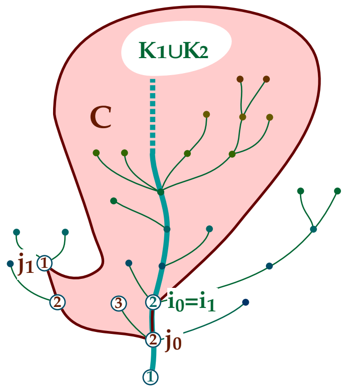

Figure 8: The closed curve C, for $R = 2$.

Let $R \geq 2$ be a fixed integer. Let i_1 be the vertex with the last occurrence of label R by (clockwise) contour traversal on the right hand side of ω and, similarly, let j_1 be the vertex with last occurrence of label $R-1$ by (counter-clockwise) contour traversal on the left hand side of ω. Furthermore, let i_0 (resp. j_0) be the vertex on the spine of ω at which the branch containing i_1 (resp. j_1) is attached. We then obtain a closed curve C in $\mathcal{M}'$ made up of the shortest paths connecting j_1 to j_0, j_0 to i_0 and i_0 to i_1 together with the chord from i_1 to j_1 (see Figure 8). By construction all vertices enclosed by this curve have labels larger than or equal to $R-1$ and only finitely many vertices are outside C. Obviously, the faces of $\mathcal{M}'$ enclosed by C form a connected set. By adding to this the triangles corresponding to cutting edges in C with equal labels, we obtain a connected set of faces in $\mathcal{M} = \mathcal{Q}(\omega)$ all of whose vertices have labels larger than or equal to $R-2$ and such that only finitely many vertices in $\mathcal{Q}(\omega)$ are not in this set. Since this holds for arbitrary $R \geq 2$ we have established the claimed contradiction, thus finishing the proof. $\square$

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