On the relative distribution of eigenvalues of exceptional Hecke operators and automorphic Laplacians

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Contents

Introduction

In [1] we considered character perturbations of the automorphic Laplacian $A = A(\Gamma_0, \chi)$ for the Hecke group $\Gamma_0(N)$ with primitive character χ. We assume that $N = 4N_2$ or $N = 4N_3$, where N_2 and N_3 are products of distinct primes and $N_2 \equiv 2 \mod 4$, $N_3 \equiv 3 \mod 4$. In these cases we are dealing with regular perturbations of A, which allows for a rigorous analysis of the problem of stability of embedded eigenvalues. We introduce a character perturbation of the form $\alpha M + \alpha^2 N$, where the first order term M is a first order differential operator and N a multiplication operator. In order to prove instability of an embedded eigenvalue λ we prove that the Phillips-Sarnak integral $I(\Phi, \lambda) = \langle M\Phi, E\rangle \neq 0$ for a common eigenfunction Φ of A with eigenvalue λ and the Hecke operators, where E is a generalized eigenfunction with eigenvalue λ. We consider only odd eigenfunctions, since $\langle M\Phi, E\rangle = 0$ for Φ even. We proved in [1] that $I(\Phi, \lambda) \neq 0$ for each such eigenfunction under the following condition on the eigenvalue $\lambda = \frac{1}{4} + r^2$ of A and the eigenvalues $\rho(q)$ of the exceptional Hecke operators $U(q)$, $q \mid N$, with the common eigenfunction Φ. The operators $U(q)$ are unitary ([1] Theorem 4.1) so the eigenvalues $\rho(q)$ lie on the unit circle. The main result about the Phillips-Sarnak integral follows from [1](7.23), (7.24). It states that $I(\Phi, \lambda) \neq 0$ if and only if $\rho(2) \neq 2^{ir}$ and $\rho(q) \neq q^{ir}/\varepsilon_q$, where the ε_q are real parameters of the perturbation, $\varepsilon_q \neq 0$. In [1] Lemma 4.3 it is stated that $\rho(q) = \pm 1$, which gives rise only to the exceptional sequences $r_{n2} = \frac{\pi}{\log 2} n$ and $r_{nq} = \frac{\pi}{\log q} n$, $n \in \mathbb{Z}$, $q \mid N$, $q > 2$ if $\varepsilon_q = \pm 1$. This Lemma, however, is not correct. The eigenvalues of $U(q)$ may lie anywhere on the unit circle. For $q > 2$ we can obtain $\rho(q) \neq q^{ir}/\varepsilon_q$ by choosing $\varepsilon_q \neq \pm 1$. For $q = 2$ we might a priori have $e^{2ir_n} = \rho_n(2)$ for all eigenvalues $\lambda_n = \frac{1}{4} + r_n^2$ or for no such λ_n. It is a delicate problem to establish that $e^{2ir_n} \neq \rho_n(2)$ at least for a certain proportion of the eigenvalues λ_n. That is the subject of the present paper. We shall prove that asymptotically at least $\frac{1}{4}$ of all eigenvalues of A_{odd}, counted with multiplicity, satisfy the condition $e^{2ir_n} \neq \rho_n(2)$, where $\lambda_n = \frac{1}{4} + r_n^2$ and $\rho_n(2)$ are eigenvalues of $U(2)$ with the same eigenvector Φ (Theorem 6).

Assume further that the dimensions of odd eigenspaces are bounded, $\dim(N(A_{\text{odd}} - \lambda_n)) \leq m$ for all λ_n. Then asymptotically at least $\frac{1}{4m}$ of all odd eigenfunctions turn into resonance functions under this perturbation, and in that sense the Weyl law is violated under the perturbation $\alpha M + \alpha^2 N$ at $\alpha = 0$.

The proof that $e^{2ir_n} \neq \rho_n(2) = e^{i\eta_n}$ for at least $\frac{1}{4}$ of all eigenvalues is based on the Weyl law (Theorem 5) for a certain operator T introduced in §4, combining A, $U(2)$ and $U(2)^*$ and measuring the average distance of 2^{ir_n}

from $e^{i\eta_n}$. In order to prove this Weyl law we first establish the Weyl law for A_{odd} (Theorem 4). The main term coming from the identity is $\varepsilon^{-1}\frac{|F|}{8\pi}$. The remaining terms are proved to be of smaller order as $\varepsilon \downarrow 0$. Then the Weyl law follows from a Tauberian theorem.

To prove these results we derive a version of the Selberg trace formula involving the exceptional Hecke operators $U(2), U(2)^*$ on the Hilbert space of (Γ, χ)-automorphic odd functions. Because of this specific restriction, this version of the Selberg trace formula was not considered before (as far as we know). But some terms in this formula were calculated before (see [2, 3, 4, 5, 6]). Anyway we make the derivation of the trace formula self-contained.

We would like to thank Fredrik Strömberg for pointing out to us the mistake in the proof of Lemma 4.3 in [1].

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1 Hecke exceptional operators $U(2), U(2)^*$ and their squares

We have

$$\begin{aligned} U(2)f(z) &= \frac{1}{\sqrt{2}}[f(\tfrac{z}{2}) + f(\tfrac{z+1}{2})] \\ U(2)^*f(z) &= \frac{1}{\sqrt{2}}[f(2z) + f(\tfrac{2z}{-Nz+1})]. \end{aligned} \tag{$*$}$$

Recall our discrete group $\Gamma = \Gamma_0(N)$ where $N = 4N_2$ or $N = 4N_3$ (see [1]). We can see that $U(2)^*U(2)f = U(2)U(2)^*f = f$. We identify linear-fractional maps with corresponding elements of $\mathrm{PSL}(2,\mathbb{R})$. We denote $U = U(2)$, $U^* = U(2)^*$ and we have the following correspondence, where all matrices are taken $\mod \pm 1$:

For U:

$$\begin{cases} z \longrightarrow \dfrac{z}{2} & \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \\ z \longrightarrow \dfrac{z+1}{2} & \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} \end{cases} \tag{1.1}$$

For U^*:

$$\begin{cases} z \longrightarrow 2z & \leftrightarrow \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ z \longrightarrow \dfrac{2z}{-Nz+1} & \leftrightarrow \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \end{cases} \tag{1.2}$$

For U^2:

$$\begin{cases} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 2 \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{pmatrix} \end{cases} \tag{1.3}$$

For U^{*2}:

$$\begin{cases} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{-N}{2} & \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ \frac{-3N}{2} & \frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-N}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -N & \frac{1}{2} \end{pmatrix} \end{cases} \tag{1.4}$$

We have 4 sets of elements of $\mathrm{PSL}(2,\mathbb{R})$ on the right hand sides of (1.1)–(1.4) which we denote by P_1, P_2, P_3, P_4.

Lemma 1. *For $j = 1, 2, 3, 4$, $\Gamma P_j = \Gamma P_j \Gamma$.*

Proof. This follows from the definition of Hecke operators. But since we start with the definition of U, U^* by $(*)$ we can recall the argument. Let φ be a continuous function of compact support in H, then for

$$f_\varphi(z) = \sum_{\gamma\in\Gamma} \chi(\gamma)\varphi(\gamma z)$$

we have $f_\varphi(\gamma_0 z) = \chi(\gamma_0) f_\varphi(z)$, where $\gamma_0 \in \Gamma$ and χ is a real primitive character on $\Gamma_0(N)$, $\chi(\gamma_0) = \chi(\gamma_0^{-1})$, (see [1]). If we denote by T_j one of the operators U, U^*, U^2, U^{*2}, then

$$T_j f_\varphi(\gamma_0 z) = \chi(\gamma_0) T_j f_\varphi(z) \qquad \text{and} \qquad T_j f_\varphi(z) = \sum_{\gamma\in\Gamma} \chi(\gamma) \sum_{p\in P_j} \varphi(\gamma p z).$$

From that follows

$$\begin{aligned} \sum_{\gamma\in\Gamma} \chi(\gamma) \sum_{p\in P_j} \varphi(\gamma p \gamma_0 z) &= \sum_{\gamma\in\Gamma} \chi(\gamma_0^{-1}\gamma) \sum_{p\in P_j} \varphi(\gamma p z) \\ &= \sum_{\gamma\in\Gamma} \chi(\gamma) \sum_{p\in P_j} \varphi(\gamma_0 \gamma p z) \end{aligned} \tag{1.5}$$

Since (1.5) holds for all functions φ of this type, the statement of Lemma 1 follows. □

We will study now the sets ΓP_j and their conjugacy classes by conjugation from Γ like $\{\gamma p\}_\Gamma = \{\gamma_1 \gamma p \gamma_1^{-1} \mid \gamma_1 \in \Gamma\}$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$.

Lemma 2. *There are no parabolic classes* $\{\gamma p\}_\Gamma$ *in each* ΓP_j, $j = 1, 2, 3, 4$.

Proof. We have to check that $\mathrm{tr}(\gamma p) \neq 2 \pmod{\pm 1}$. Take

$$\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma,$$

$ad - Nbc = 1$. Since $N = 4N_2$ or $4N_3$, a and d are odd integers, and the lemma follows. $\square$

The next result is well known.

Lemma 3.

(1) There are at most finitely many elliptic classes $\{\gamma p\}_\Gamma$ *in* ΓP_j, $j = 1, 2, 3, 4$.

(2) There are infinitely many hyperbolic classes in ΓP_j, $j = 1, 2, 3, 4$.

It is obvious that the unity $e \in \Gamma$ is not in $\Gamma P = \bigcup_{j=1}^{4} \Gamma P_j$. We will study now the centralizers $\Gamma_{\gamma p}$ of γp in Γ. By definition

$$\Gamma_{\gamma p} = \{\gamma_1 \in \Gamma \mid \gamma_1 \gamma p = \gamma p \gamma_1\}$$

for $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$.

It is clear that $\Gamma_{\gamma p}$ is a subgroup of Γ and it is known that each $\Gamma_{\gamma p}$ is a cyclic group and possibly trivial, $\Gamma_{\gamma p} = \{e\}$ for some γp.

Lemma 4. *A hyperbolic element of* $\mathrm{PSL}(2, \mathbb{R})$ *commutes only with the identity and with hyperbolic elements.*

Proof. We check this using the language of linear fractional transformations. Any hyperbolic transformation is conjugated in $\mathrm{PSL}(2, \mathbb{R})$ to a transformation of the type

$$z \longrightarrow \lambda^2 z, \quad \lambda > 1, \quad z \in H$$

Assume that

$$z \longrightarrow \frac{az + b}{cz + d}$$

corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R})$ commutes with the above transformation, so that

$$\frac{a\lambda^2 z + b}{c\lambda^2 z + d} = \lambda^2 \frac{az + b}{cz + d} \qquad \text{for all } z \in H$$

This implies that $a \neq 0$, $b = c = 0$, $d = a^{-1}$, so that the transformation defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is hyperbolic or the identity. $\square$

Similarly we have

Lemma 5. *A parabolic element of* $\mathrm{PSL}(2,\mathbb{R})$ *commutes only with the identity and with parabolic elements.*

Also a simple check shows

Lemma 6. *There are no elliptic classes in* Γ.

From Lemmas 3–5 follows

Theorem 1.

(1) Any elliptic class $\{\gamma p\}_\Gamma$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$, *has only trivial centralizer* $\Gamma_{\gamma p} = \{e\}$.

(2) For a hyperbolic class $\{\gamma p\}$, $\gamma p \in \Gamma P_j$, *we have the alternatives*

(a) $\Gamma_{\gamma p} = \{e\}$

(b) $\Gamma_{\gamma p}$ *is generated by a hyperbolic element in* Γ.

We will study in more detail hyperbolic classes in ΓP_j and characterize their centralizers.

From the proof of Lemma 4 follows

Lemma 7. *The hyperbolic elements of* $\mathrm{PSL}(2,\mathbb{R})$ *commute with each other if and only if they have the same fixed points as linear fractional transformations.*

Let

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} \in \Gamma$$

be a hyperbolic element. Then we have $c_1 \neq 0$. The equation $\frac{a_1 z + b_1}{c_1 z + d_1} = z$ has two solutions

$$z_{1,2} = \frac{a_1 - d_1 \pm \sqrt{(a_1 + d_1)^2 - 4}}{2Nc_1} \tag{1.6}$$

Since $N = 4N_2$ or $N = 4N_3$, then a_1 and d_1 are odd integers (recall $a_1 d_1 - Nb_1 c_1 = 1$), $a_1 + d_1$ is an even integer and $(a_1 + d_1)^2$ is divisible by 4. From this follows that for any hyperbolic element γ of Γ the integer $(\operatorname{tr}\gamma)^2 - 4$ can not be the square of an integer.

We consider now a hyperbolic element

$$g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma P.$$

It has two fixed points as transformation of H

$$t_{1,2} = \frac{a_2 - d_2 \pm \sqrt{(a_2 + d_2)^2 - 4}}{2c_2} \tag{1.7}$$

The problem is how for given g_2 to find g_1 with the same fixed points $z_{1,2} = t_{1,2}$. Since $\sqrt{(a_1 + d_1)^2 - 4}$ is always irrational we have

$$\begin{aligned} \frac{a_1 - d_1}{2Nc_1} &= \frac{a_2 - d_2}{2c_2} \\ \frac{(a_1 + d_1)^2 - 4}{4N^2 c_1^2} &= \frac{(a_2 + d_2)^2 - 4}{4c_2^2} \end{aligned} \tag{1.8}$$

It will work, of course, if $(a_2 + d_2)^2 - 4$ is an integer and it is not a square, which is not necessarily true for all hyperbolic elements of ΓP. We will try to solve the system of equations (1.8). From (1.8) follows

$$\begin{aligned} \frac{a_1 d_1 - 1}{N^2 c_1^2} &= \frac{a_2 d_2 - 1}{c_2^2} \\ \frac{b_1}{Nc_1} &= \frac{b_2}{c_2} \end{aligned} \tag{1.9}$$

From (1.8), (1.9) follows

$$\begin{aligned} a_1 &= d_1 + Nc_1 \frac{a_2 - d_2}{c_2} \\ b_1 &= Nc_1 \frac{b_2}{c_2} \end{aligned} \tag{1.10}$$

Let

$$\begin{aligned} g &= \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma \\ p &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in P_i \end{aligned}$$

then

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ Nc\alpha + d\gamma & Nc\beta + d\delta \end{pmatrix} \tag{1.11}$$

Since in (1.10) we have the ratios $\frac{a_2-d_2}{c_2}$, $\frac{b_2}{c_2}$ we can multiply the matrix (1.11) by $\sqrt{2}$ in the case $i=1,2$ or by 2 in the case $i=3,4$, not changing these ratios but getting ratios of integers.

We have in the first case $i=1,2$

$$\begin{aligned} a_1 &= d_1 + Nc_1 \frac{\sqrt{2}(a_2-d_2)}{\sqrt{2}c_2} \\ b_1 &= Nc_1 \frac{\sqrt{2}b_2}{\sqrt{2}c_2} \end{aligned} \tag{1.12}$$

and $\sqrt{2}(a_2-d_2), \sqrt{2}b_2, \sqrt{2}c_2 \in \mathbb{Z}$. In the second case $i=3,4$

$$\begin{aligned} a_1 &= d_1 + Nc_1 \frac{2(a_2-d_2)}{2c_2} \\ b_1 &= Nc_1 \frac{2b_2}{2c_2} \end{aligned} \tag{1.13}$$

and $2(a_2-d_2), 2b_2, 2c_2 \in \mathbb{Z}$. In the first case we assume that

$$Nc_1 = 2k_1\sqrt{2}c_2, \quad k_1 \in \mathbb{Z} \tag{1.14}$$

and we will have that

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} = \begin{pmatrix} d_1 + 2k_1\sqrt{2}(a_2-d_2) & 2k_1\sqrt{2}b_2 \\ 2k_1\sqrt{2}c_2 & d_1 \end{pmatrix} \tag{1.15}$$

is an integer matrix with the left lower matrix element divisible by N. We have to prove now that there exist $d_1, k_1 \in \mathbb{Z}$ with the property that $\det g_1 = 1$. That means we have to prove the existence of integer solutions of the equation

$$d_1^2 + 2k_1 d_1 \sqrt{2}(a_2-d_2) - 4k_1^2 \cdot 2b_2c_2 = 1 \tag{1.16}$$

or the equation

$$m_1^2 - k_1^2 \cdot 2[(a_2+d_2)^2 - 4] = 1 \tag{1.17}$$

where $m_1 = d_1 + k_1\sqrt{2}(a_2-d_2)$. This is Pell's equation which has infinitely many integer solutions in $m_1, k_1 \in \mathbb{Z}$ for given a_2, d_2 if $2[(a_2+d_2)^2-4]$ is not the square of an integer (recall that $\sqrt{2}a_2, \sqrt{2}d_2$ are integers). Notice that this is a square if and only if $|\sqrt{2}(a_2+d_2)| = 3$. If $|\sqrt{2}(a_2+d_2)| \neq 3$, we can always find a matrix g_1 with $z_{1,2} = t_{1,2}$, which belongs to the centralizer $\Gamma_{\gamma p}$ of a given hyperbolic γp. If on the contrary $|\sqrt{2}(a_2+d_2)| = 3$ so that

$2[(a_2+d_2)^2-4]=1$, there is no such matrix g_1, so the centralizer of γp is $\{e\}$.

In the case $i=3,4$ we obtain a similar result, using (1.13). This leads to the equation

$$m_2^2-4k_2^2[(a_2+d_2)^2-4]=1$$

in the integers m_2,k_2. Recall that $2a_2,2d_2$ are given integers in that case. This equation has integer solutions if and only if $4[(a_2+d_2)^2-4]$ is not the square of an integer, that is if $2|a_2+d_2|\neq 5$. In this case the centralizer $\Gamma_{\gamma p}$ of γp is non-trivial and generated by a hyperbolic element of Γ. If $2|a_2+d_2|=5$, $\Gamma_{\gamma p}=\{e\}$.

We have proved

Theorem 2. *For a hyperbolic class $\{\gamma p\}_\Gamma$ in ΓP from (2) of Theorem 1 the alternative (a) occurs when $\sqrt{2}|\mathrm{tr}\,\gamma p|=3$ for $p\in P_i$, $i=1,2$ and when $2|\mathrm{tr}\,\gamma p|=5$ for $p\in P_i$, $i=3,4$. In those cases the fixed points of γp in $\overline{H}$ are rational points. The norms of the classes $\{\gamma p\}_\Gamma$ are 2 in case 1 and 4 in case 2. For other values of $\mathrm{tr}(\gamma p)$ alternative (b) holds.*

We shall see later that there are only finitely many classes $\{\gamma p\}_\Gamma$ from Theorem 2.

2 The involution $J\colon z \mapsto -\bar{z}$ and the exceptional Hecke operators

Let $g \in \mathrm{PSL}(2, \mathbb{R})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{\pm 1}.$$

Then it is easy to see that JgJ acts on H as the matrix

$$\tilde{g} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \pmod{\pm 1}.$$

It is convenient now to introduce an isomorphic model of H. This model is well known. We consider the set of positive definite symmetric matrices

$$z(x, y) = \begin{pmatrix} y + x^2 y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{pmatrix} \tag{2.1}$$

where $y > 0$, $x \in \mathbb{R}$. We define the action of $g \in \mathrm{PSL}(2, \mathbb{R})$ on such matrices by

$$gz(x, y) = g[z(x, y)]g^t \tag{2.2}$$

where g^t is the transpose of g and the product on the right of (2.2) is the usual product of matrices. It is easy to see that the set $\hat{H} = \{z(x, y)\}$ with the action (2.2) has the structure of a symmetric space and is isomorphic to H. The isomorphism is given by the map

$$z(x, y) \to z = x + iy.$$

This model $\hat{H}$ of the hyperbolic plane has the following useful property, which can not be seen in the case of H. The reflection J in the model $\hat{H}$ is given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{\pm 1} \tag{2.3}$$

i.e. it is an element in $\mathrm{GL}(2, \mathbb{R})/(\pm E)$, where E is the identity matrix. It therefore makes sense to consider the products Jg, gJ in $\mathrm{PGL}(2, \mathbb{R})$ for $g \in \mathrm{PSL}(2, \mathbb{R})$.

We shall study now the relative conjugacy classes $\{gJ\}_G$ under conjugation by elements from $\mathrm{PSL}(2, \mathbb{R}) = G$. We have obviously $GJ = GJG$, so we can consider the conjugation $g_1 g J g_1^{-1}$, $g_1 \in G$, for fixed $g \in G$. From the point of view of the trace formula there is an important alternative for the relative conjugacy classes:

(1) $\operatorname{tr}(gJ) \neq 0$

(2) $\operatorname{tr}(gJ) = 0$ (see [2] §6.5)

The fixed points of $gJ\colon H \to H$ are determined by the equation

$$\frac{-a\bar{z}+b}{-c\bar{z}+d} = z$$
$$b - a\bar{z} = -c|z|^2 + dz.$$

For $z = x + iy$ we have the system

$$b - ax = dx - c|z|^2$$
$$dy = ay.$$

In the case (1) $\operatorname{tr}(gJ) = d - a \pmod{\pm 1} \neq 0$ we have

$$y = 0$$
$$b - ax = dx - cx^2$$

Therefore we obtain for $c = 0$, $x = \frac{b}{a+d}$.

For $c \neq 0$ the fixed points are

$$\begin{aligned} z_{1,2} &= t_{1,2} \\ &= \frac{a+d}{2c} \pm \sqrt{\frac{(a+d)^2}{4c^2} - \frac{b}{c}} \\ &= \frac{a+d}{2c} \pm \frac{\sqrt{(a-d)^2+4}}{2c} \end{aligned} \tag{2.4}$$

where we use $\det g = 1$.

In the case (2) $\operatorname{tr}(gJ) = d - a = 0$ we have a one-parameter family of fixed points z given by the equation

$$c|z|^2 - a(z+\bar{z}) + b = 0 \tag{2.5}$$

where $z = x + iy$, $\bar{z} = x - iy$, $y > 0$, $x \in \mathbb{R}$.

Consider now $gJ, g \in G$ where J is given by (2.3) with the property $\operatorname{tr}(gJ) \neq 0$. There exists $g_1 \in G$ such that

$$g_1(gJ)g_1^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \pmod{\pm 1} \quad , \quad \lambda > 1 \tag{2.6}$$

By definition the norm $N(gJ) = \lambda^2$.

Similar to Lemma 4 we have

Lemma 8. *An element gJ from* (2.6) *with* $\operatorname{tr}(gJ) \neq 0$ *commutes in* $\mathrm{PSL}(2, \mathbb{R})$ *only with the identity and with hyperbolic elements.*

We will specify later exactly which hyperbolic elements commute with gJ from Lemma 8. From Lemma 1 follows

Lemma 9. *For $j = 1, 2, 3, 4$, $\Gamma P_j \Gamma = \Gamma P_j J \Gamma$.*

Lemma 10. *For any $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$ we have* $\operatorname{tr}(\gamma p J) \neq 0$, *where J is given by* (2.3).

Proof. This follows directly from (1.1)–(1.4) and the fact that for any

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma$$

a and d are odd integers, $N = 4N_2$ or $N = 4N_3$. □

Similar to Lemma 7 we have

Lemma 11. *An element gJ from Lemma 8 commutes with a hyperbolic element g_1 if and only if they have the same fixed points.*

We will modify now the proof of Theorem 2 for the case

$$g_2 = \begin{pmatrix} a_2 & -b_2 \\ c_2 & -d_2 \end{pmatrix} \in \Gamma P J \tag{2.7}$$

The fixed points of g_2 are (see (2.4))

$$t_{1,2} = \frac{a_2 + d_2}{2c_2} \pm \frac{\sqrt{(a_2 - d_2)^2 + 4}}{2c_2} \tag{2.8}$$

Now, for given g_2 we have to find $g_1 \in \Gamma$, hyperbolic, with the same fixed points $x_{1,2} = t_{1,2}$. Since $\sqrt{(a_1 + d_1)^2 - 4}$ is always irrational we have (see (1.8))

$$\begin{aligned} \frac{a_1 - d_1}{2Nc_1} &= \frac{a_2 + d_2}{2c_2} \\ \frac{(a_1 + d_1)^2 - 4}{4N^2 c_1^2} &= \frac{(a_2 - d_2)^2 + 4}{4c_2^2} \end{aligned} \tag{2.9}$$

and similarly to (1.9)

$$\begin{aligned} \frac{a_1 d_1 - 1}{N^2 c_1^2} &= \frac{1 - a_2 d_2}{c_2^2} \\ \frac{b_1}{Nc_1} &= -\frac{b_2}{c_2} \end{aligned} \tag{2.10}$$

That gives the matrix

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ Nc_1 & d_1 \end{pmatrix} = \begin{pmatrix} d_1 + Nc_1\frac{a_2+d_2}{c_2} & \frac{-Nc_1b_2}{c_2} \\ Nc_1 & d_1 \end{pmatrix} \tag{2.11}$$

We take now $g_2 \in \Gamma P_j J$ where $j = 1, 2$. In that case we have similarly to (1.12)

$$\begin{aligned} a_1 &= d_1 + Nc_1\frac{\sqrt{2}(a_2+d_2)}{\sqrt{2}c_2} \\ b_1 &= Nc_1\frac{\sqrt{2}b_2}{\sqrt{2}c_2}(-1) \end{aligned} \tag{2.12}$$

and $\sqrt{2}(a_2+d_2), \sqrt{2}c_2, \sqrt{2}b_2 \in \mathbb{Z}$. Similar to (1.14) we have

$$Nc_1 = 2\kappa_1\sqrt{2}c_2 \quad , \quad \kappa_1 \in \mathbb{Z} \tag{2.13}$$

where

$$g_1 = \begin{pmatrix} d_1 + 2\kappa_1\sqrt{2}(a_2+d_2) & -2\kappa_1\sqrt{2}b_2 \\ 2\kappa_1\sqrt{2}c_2 & d_1 \end{pmatrix} \tag{2.14}$$

is an integer matrix with $2\kappa_1\sqrt{2}c_2$ divisible by N. Similar to (1.16), (1.17) we have

$$d_1^2 + 2\kappa_1 d_1\sqrt{2}(a_2+d_2) + 4\kappa_1^2 \cdot 2b_2c_2 = 1 \tag{2.15}$$

or

$$m_1^2 - 2\kappa_1^2((a_2-d_2)^2+4) = 1 \tag{2.16}$$

where $m_1 = d_1 + \kappa_1\sqrt{2}(a_2+d_2)$.

If $g_2 \in \Gamma P_j J$ and $j = 3, 4$, we have

$$\begin{aligned} a_1 &= d_1 + Nc_1\frac{2(a_2+d_2)}{2c_2} \\ b_1 &= -Nc_1\frac{2b_2}{2c_2} \end{aligned} \tag{2.17}$$

$$Nc_1 = 2\kappa_2 \cdot 2c_2 \quad , \quad \kappa_2 \in \mathbb{Z} \tag{2.18}$$

where

$$g_1 = \begin{pmatrix} d_1 + 2\kappa_2 \cdot 2(a_2+d_2) & -2\kappa_2 \cdot 2b_2 \\ 2\kappa_2 \cdot 2c_2 & d_1 \end{pmatrix} \tag{2.19}$$

is an integer matrix with $2\kappa_2 \cdot 2c_2$ divisible by N. Since $\det g_1 = 1$, we have

$$d_1^2 + 2\kappa_2 d_1 \cdot 2(a_2 + d_2) + 4\kappa_1^2 \cdot 4b_2c_2 = 1 \tag{2.20}$$

or

$$m_2^2 - 4\kappa_2^2((a_2 - d_2)^2 + 4) = 1 \tag{2.21}$$

where

$$m_2 = d_1 + \kappa_2 \cdot 2(a_2 + d_2)$$

We can always solve the equations (2.16), (2.21) if $2((a_2 - d_2)^2 + 4)$ (or $4((a_2 - d_2)^2 + 4)$ in the second case) is not an integer squared. This holds if and only if $\sqrt{2}\,\mathrm{tr}(\gamma p) \neq \pm 1$ for $p \in P_j$, $j = 1, 2$ and $2\,\mathrm{tr}(\gamma p) \neq \pm 3$, $p \in P_j$, $j = 3, 4$.

We have proved

Theorem 3. *For a relative conjugacy class $\{\gamma p J\}_\Gamma$, $\gamma \in \Gamma$, we have*

(a) If $\sqrt{2}|\mathrm{tr}(\gamma p)| = 1$ for $j = 1, 2$ and $2|\mathrm{tr}(\gamma p)| = 3$ for $j = 3, 4$, then the centralizer $\Gamma_{\gamma p J}$ of $\gamma p J$ in Γ is $\{e\}$.

(b) If $\sqrt{2}|\mathrm{tr}(\gamma p)| \neq 1$ for $j = 1, 2$ and $2|\mathrm{tr}(\gamma p)| \neq 3$ for $j = 3, 4$, then $\Gamma_{\gamma p J}$ is a cyclic group generated by a hyperbolic element.

3 The trace formula for odd functions and the Weyl law

We recall now the definition of the Eisenstein series (non-holomorphic) for $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$, which correspond to open cusps for the real primitive character χ (see [1] (2.1)). For the definitions we introduced elements g_j in $\mathrm{PSL}(2,\mathbb{R})$. We now parametrize these elements by the divisors $d \mid N$, $d > 0$. We have

$$g_d = \begin{pmatrix} \sqrt{m_d} & 0 \\ d\sqrt{m_d} & \sqrt{m_d}^{-1} \end{pmatrix} \qquad\qquad g_d S_\infty g_d^{-1} = S_d,$$

where S_∞ is the stabilizer of the cusp at ∞.

For each open cusp $\frac{1}{d}$ we define the Eisenstein series

$$E_d(z,s) = E_d(z,s,\Gamma,\chi) = \sum_{\gamma\in\Gamma_d\backslash\Gamma} y^s(g_d^{-1}\gamma z)\chi(\gamma), \tag{3.1}$$

where $\operatorname{Re} s > 1$, $\chi(\gamma) = \overline{\chi(\gamma)}$. Let us calculate $y^s(g_d^{-1}\gamma z)$. Let

$$\gamma = \begin{pmatrix} a & b \\ Nc & h \end{pmatrix} \in \Gamma,$$

and we have

$$g_d^{-1} = \begin{pmatrix} \sqrt{m_d}^{-1} & 0 \\ -d\sqrt{m_d} & \sqrt{m_d} \end{pmatrix},$$

$z = x + iy \in H$. We have with $N = dm_d$

$$\begin{aligned} g_d^{-1}\gamma &= \begin{pmatrix} * & * \\ -d\sqrt{m_d}a + \sqrt{m_d}Nc & -d\sqrt{m_d}b + \sqrt{m_d}h \end{pmatrix} \\ y^s(g_d^{-1}\gamma z) &= y^s[\{(-d\sqrt{m_d}a + \sqrt{m_d}Nc)x - d\sqrt{m_d}b + \sqrt{m_d}h\}^2 \\ &\quad + (-d\sqrt{m_d}a + \sqrt{m_d}Nc)^2y^2]^{-s} \\ &= y^s|m_d|^{-s}[\{(Nc-ad)x - db + h\}^2 + (Nc-ad)^2]^{-s} \end{aligned} \tag{3.2}$$

We notice that (3.2) is unchanged when (d,x,b,c) is replaced by $(-d,-x,-b,-c)$. It follows that

$$E_{-d}(-\bar{z},s) = E_d(z,s) \tag{3.3}$$

Lemma 12. *The cusps $\frac{1}{d}$ and $\frac{-1}{d}$ are equivalent, $d \mid N$,*

$$E_d(z,s) = E_{-d}(z,s) = E_d(Jz,s). \tag{3.4}$$

Proof. We want to find a matrix

$$\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \in \Gamma_0(N),$$

where $N = 4N_2$ or $N = 4N_3$ with $\alpha\delta - N\beta\gamma = 1$, such that

$$\begin{pmatrix} \alpha & \beta \\ N\gamma & \delta \end{pmatrix} \left(\frac{-1}{d} \right) = \frac{1}{d}.$$

This yields the equation

$$\delta^2 - (\beta d + \gamma m_d)\delta + \beta\gamma d m_d + 1 = 0 \tag{3.5}$$

with the solutions

$$\delta = \frac{\beta d + \gamma m_d}{2} \pm \frac{\sqrt{(\beta d - \gamma m_d)^2 - 4}}{2} \tag{3.6}$$

Here $(\beta d - \gamma m_d)^2 - 4$ is the square of an integer if and only if

$$\beta d - \gamma m_d = \pm 2 \tag{3.7}$$

From that follows $\alpha = \delta$, since α is a solution of the same equation (3.5). Since $N = dm_d = 4p_1 \cdots p_k$, where the p_j are distinct primes, we have the following cases:

(1) $d = 2d'$, $m_d = 2m_d'$, $(d', m_d') = 1$. Then (3.7) yields

$$\beta d' - \gamma m_d' = \pm 1 \tag{3.8}$$

The equation (3.8) has integer solutions β, γ and by (3.6) $\delta = \beta d' + \gamma m_d'$.

(2) $d = 2d'$, $2 \mid d'$, $(d', m_d) = 1$. Then by (3.7), γ must be even, $\gamma = 2\gamma'$, and

$$\beta d' - \gamma' m_d = \pm 1. \tag{3.9}$$

The equation (3.9) has integer solutions β, γ', and by (3.6) $\delta = \beta d' + \gamma' m_d$.

(3) $m_d = 2m_d'$, $2 \mid m_d'$, $(d, m_d') = 1$. This is similar to (2), exchanging d with m_d and β with γ.

Hence $\frac{1}{d}$ and $\frac{-1}{d}$ are equivalent, and it follows that $E_d(z,s) = E_{-d}(z,s)$. Together with (3.3) this proves the Lemma. $\square$

We recall that the involution $J\colon z \to -\bar{z}$ acts on the space of all continuous (Γ,χ)-automorphic functions and splits this space into the sum of even and odd functions given by

$$f(Jz) = f(z) \qquad\qquad f(Jz) = -f(z) \tag{3.10}$$

The operator J commutes with the automorphic Laplacian $A(\Gamma,\chi)$ and with all Hecke operators. The Hilbert space $\mathcal{H} = \mathcal{H}(\Gamma,\chi)$ according to (3.10) decomposes into an orthogonal sum of two subspaces $\mathcal{H} = \mathcal{H}_{\text{odd}} \oplus \mathcal{H}_{\text{even}}$. From Lemma 12 follows

Lemma 13. *Let $\mathcal{D}(A)$ be the domain of definition of $A(\Gamma,\chi)$ in $\mathcal{H}$ and $A_{\text{odd}} = A(\Gamma,\chi)|_{\mathcal{D}(A)\cap\mathcal{H}_{\text{odd}}}$. Then the operator A_{odd} has discrete spectrum as a selfadjoint operator in $\mathcal{H}_{\text{odd}}$.*

Let $N_{\text{odd}}(\lambda)$ be the distribution function of eigenvalues of A_{odd}. We will prove now the Weyl law

$$N_{\text{odd}}(\lambda) \sim \frac{\mu(F)}{8\pi}\lambda \quad , \quad \lambda \to \infty \tag{3.11}$$

where $\mu(F)$ is the area ($d\mu$-area) of the fundamental domain F of Γ. The proof is an extension of the proof from [2] and [7]. We have a preliminary trace formula on the space of odd functions

$$\sum_j h_\varepsilon(\lambda_j) = \lim_{Y\to\infty} \frac{1}{2}\int_{F_Y} \sum_{\gamma\in\Gamma} \chi(\gamma)(k_\varepsilon(u(z,\gamma z)) - k_\varepsilon(u(z,\gamma Jz)))d\mu(z) \tag{3.12}$$

where $\{\lambda_j\}$ is the set of all eigenvalues of A_{odd} and F_Y is the cut-off fundamental domain of Γ in H (see (3.32)), $F_Y \to F$ as $Y \to \infty$. Here u is the distance function

$$u(z,z') = \frac{|z-z'|^2}{yy'}.$$

The test function $h_\varepsilon(\lambda)$ is given by

$$h_\varepsilon(\lambda) = h_\varepsilon(\tfrac{1}{4} + r^2) = e^{-\varepsilon r^2} \quad , \quad \varepsilon > 0 \tag{3.13}$$

and $k_\varepsilon(u)$ is the corresponding Selberg transform of $h_\varepsilon(\lambda)$, given by

$$
\begin{aligned}
g_\varepsilon(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iru} h_\varepsilon(\tfrac{1}{4}+r^2)dr \\
h_\varepsilon(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda u} g_\varepsilon(u)du \quad , \quad \lambda = \tfrac{1}{4}+r^2 \\
Q_\varepsilon(e^u+e^{-u}-2) &= g_\varepsilon(u) \\
k_\varepsilon(t) &= -\frac{1}{\pi}\int_t^{\infty} \frac{dQ_\varepsilon(\omega)}{\sqrt{\omega-t}} \\
Q_\varepsilon(\omega) &= \int_\omega^{\infty} \frac{k_\varepsilon(t)}{\sqrt{t-\omega}}dt
\end{aligned}
\tag{3.14}
$$

We have

$$\int_{F_Y} \sum_{\gamma\in\Gamma} \chi(\gamma) k_\varepsilon(u(z,\gamma z))d\mu(z) = \sum_{\{\gamma\}_\Gamma} \chi(\gamma)\int_{F_Y^\gamma} k_\varepsilon(u(z,\gamma z))d\mu(z) \tag{3.15}$$

where $\{\gamma\}_\Gamma$ is the conjugacy class in Γ with representative γ,

$$F_Y^\gamma = \bigcup_{\gamma'\in\Gamma_\gamma\backslash\Gamma} \gamma' F_Y \tag{3.16}$$

Γ_γ is the centralizer of γ in Γ, and $\Gamma_\gamma\setminus\Gamma$ is the left co-set. We have $F_Y^\gamma \xrightarrow[Y\to\infty]{} F^\gamma$, where F^γ is a fundamental domain of Γ_γ in H.

In analogy we have

$$\int_{F_Y} \sum_{\gamma\in\Gamma} \chi(\gamma) k_\varepsilon(u(z,\gamma J z))d\mu(z) = \sum_{\{\gamma J\}_\Gamma} \chi(\gamma)\int_{F_Y^{\gamma J}} k_\varepsilon(u(z,\gamma J z))d\mu(z) \tag{3.17}$$

where $\{\gamma J\}_\Gamma$ is the relative conjugacy class in ΓJ by conjugation of Γ,

$$F_Y^{\gamma J} = \bigcup_{\gamma'\in\Gamma_{\gamma J}\backslash\Gamma} \gamma' F_Y \xrightarrow[Y\to\infty]{} F^{\gamma J} \tag{3.18}$$

$\Gamma_{\gamma J}$ is the centralizer of γJ in Γ, $F^{\gamma J}$ is a fundamental domain of $\Gamma_{\gamma J}$ in H. We consider first the sum given by (3.15). It is well known that the sum over all conjugacy classes $\{\gamma\}_\Gamma$ in (3.15) splits into $\{e\}_\Gamma$, $\{h\}_\Gamma$, $\{p\}_\Gamma$, identity, hyperbolic, parabolic classes (Γ has no elliptic classes). Also the sum over all parabolic classes splits into two sums according to the character χ ($\chi=1$, $\chi=-1$) (see [1]). The contribution from $\{e\}_\Gamma$ is equal to

$$\int_{F_Y} k_\varepsilon(u(z,z))d\mu(z) = \frac{\mu(F_Y)}{4\pi}\int_{-\infty}^{\infty} r\cdot(\tanh\pi r)h_\varepsilon(\tfrac{1}{4}+r^2)dr \tag{3.19}$$

$$\mu(F_Y) \xrightarrow[Y\to\infty]{} \mu(F) \tag{3.20}$$

The contribution from all hyperbolic classes to (3.15) is equal to

$$\sum_{\{h\}_\Gamma} \sum_{k=1}^{\infty} \frac{\chi^k(h) \log N(h)}{N(h)^{k/2} - N(h)^{-k/2}} g_\varepsilon(k \log N(h)) + \underset{Y\to\infty}{o(1)} \tag{3.21}$$

where $\gamma = h^k$ is a positive integer power of a primitive hyperbolic element h, $N(h)$ is the norm of h,

$$g_\varepsilon(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-riu} h_\varepsilon(\tfrac{1}{4} + r^2) dr \tag{3.22}$$

Let $a(\Gamma, \chi)$ be the number of open cusps for F relative to χ, and $b(\Gamma, \chi)$ the number of closed cusps of F. Then the contribution from all of the parabolic conjugacy classes to (3.15) is equal to

$$\begin{aligned} a(\Gamma, \chi) \Big[& g_\varepsilon(0) \log Y - g_\varepsilon(0) \log 2 + \frac{h_\varepsilon(\frac{1}{4})}{4} \\ & - \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\varepsilon(\tfrac{1}{4} + r^2) \frac{\Gamma'}{\Gamma}(1 + ir) dr \Big] \\ & - b(\Gamma, \chi) g_\varepsilon(0) \log 2 + \underset{Y\to\infty}{o(1)} \end{aligned} \tag{3.23}$$

where $\Gamma(s)$ the Euler function and $\underset{Y\to\infty}{o(1)}$ means $o(1) \xrightarrow[Y\to\infty]{} 0$.

Now we consider the sum given by (3.17). In order to calculate the right hand side of (3.17) we have to seperate two different cases for the conjugacy classes $\{\gamma J\}_\Gamma$, $\gamma \in \Gamma$ (as in § 2)

$$\sum_{\{\gamma J\}_\Gamma} = \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) \neq 0}} + \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) = 0}} \tag{3.24}$$

There is a significant difference between the classes $\{\gamma J\}_\Gamma$ and $\{\gamma p J\}_\Gamma$, $\gamma \in \Gamma$, $p \in P_j$, $j = 1, 2, 3, 4$. We shall make use of the following results about these classes.

Lemma 14.

(1) For any $\{\gamma J\}_\Gamma$ with the property $\operatorname{tr}(\gamma J) \neq 0$ the centralizer $\Gamma_{\gamma J}$ in Γ of γJ is generated by a hyperbolic element $h = h(\gamma J)$.

(2) There are classes $\{\gamma J\}_\Gamma$, $\gamma \in \Gamma$, $\operatorname{tr}(\gamma J) = 0$.

Proof. (1) It is sufficient to show that $\Gamma_{\gamma J}$ contains a hyperbolic element. Then from the discreteness of Γ follows that $\Gamma_{\gamma J}$ is a cyclic group, generated by a hyperbolic element. We have $\gamma J \gamma J \in \Gamma$ if $\gamma \in \Gamma$. Clearly $(\gamma J \gamma J)\gamma J = \gamma J(\gamma J \gamma J)$. Similar to (2.6) we have

$$g_1(\gamma J)g_1^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \pmod{\pm 1} \quad , \quad \lambda > 1$$

Then

$$g_1(\gamma J \gamma J)g_1^{-1} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} \pmod{\pm 1}$$

is a hyperbolic element.

(2) This follows from the definition of Γ.

□

Remark. We shall see later that there are at most finitely many classes $\{\gamma J\}_\Gamma$ from (2) of Lemma 14.

From the proof of (1) of Lemma 14 follows

Lemma 15. *For any* $\{\gamma J\}_\Gamma$ *with* $\operatorname{tr}(\gamma J) \neq 0$ *we have*

$$N(h(\gamma J)) \leq N^2(\gamma J).$$

From the definition of the group $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$, and (2.6) follows

Lemma 16. *Let* $\gamma \in \Gamma$ *and* $\{\gamma J\}_\Gamma$ *be such that* $\operatorname{tr}(\gamma J) \neq 0$. *Then* γ *is a hyperbolic element and* $N(\gamma J) = N(\gamma)$.

The following is well known.

Lemma 17. *The series*

$$\sum_{k=1}^{\infty} \sum_{\{\gamma\}_\Gamma} \frac{1}{N(\gamma)^{ks}} \quad , \gamma \text{ hyperbolic primitive}$$

is absolutely convergent for $\operatorname{Re} s > 1$, *where the sum is taken over all hyperbolic conjugacy classes in* Γ.

We will now calculate the contribution to (3.24) from classes with $\operatorname{tr}(\gamma J) = 0$.

Lemma 18. *Let $\gamma \in \Gamma$ and $\operatorname{tr}(\gamma J) = 0$, then the centralizer $\Gamma_{\gamma J}$ of γJ in Γ is either trivial $\Gamma_{\gamma J} = \{e\}$ or $\Gamma_{\gamma J}$ is a cyclic hyperbolic group.*

Proof. The element γJ is conjugated by $g \in G$ to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (mod ± 1). Let $g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We consider the commutation condition

$$\frac{-a\bar{z} + b}{-c\bar{z} + d} = -\frac{a\bar{z} + b}{c\bar{z} + d} \tag{3.25}$$

which is supposed to be valid for all $z \in H$. From this follows that g_1 is a hyperbolic element or $g_1 = e$ or g_1 is elliptic with $\operatorname{tr}(g_1) = 0$. Then the result follows from Lemma 6 and the discreteness of Γ. □

We will calculate now in a little more detail the sum in (3.17) since it is not so much known as the sum (3.15). We start from the sum in (3.24) with $\operatorname{tr}(\gamma J) \neq 0$. Any $\gamma' J$ in ΓJ is an odd positive integer power of a primitive element γJ. Let us denote by $h(\gamma J) \in \Gamma$ the generator of $\Gamma_{\gamma J}$. Let $F_{\gamma J}$ be a fundamental domain of $\Gamma_{\gamma J}$ in H. If $g(\gamma J) \in G$ brings $h(\gamma J)$ to the diagonal form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ similar to (2.6), then we take as $F_{\gamma J}$ the following domain

$$g(\gamma J) F_{\gamma J} = \{ z = re^{i\varphi} \in H \mid 1 \le r < N(h(\gamma J)), 0 < \varphi < \pi \} \tag{3.26}$$

Then the part of (3.17) with the condition $\operatorname{tr}(\gamma J) \neq 0$ is equal to

$$\sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) \neq 0}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z)$$

$$= \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) \neq 0}}{}' \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \int_{F_{\gamma J}} k_\varepsilon(u(z, (\gamma J)^{2k-1} z)) d\mu(z) + \underset{Y \to \infty}{o(1)}$$

$$= \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) \neq 0}}{}' \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \int_1^{N(h)} \frac{dr}{r} \int_0^{\pi} \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon \Big(\frac{|z + N(\gamma J)^{2k-1} \bar{z}|^2}{y^2 N(\gamma J)^{2k-1}} \Big) + \underset{Y \to \infty}{o(1)}$$

$$= \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) \neq 0}}{}' \sum_{k=1}^{\infty} \Bigg[\chi(\gamma)^{2k-1} \log N(h(\gamma J))$$

$$2 \int_0^{\pi/2} \frac{d\varphi}{\sin^2 \varphi} k_\varepsilon \Big(\frac{|e^{i\varphi} + N(\gamma J)^{2k-1} e^{-i\varphi}|^2}{\sin^2 \varphi \cdot N(\gamma J)^{2k-1}} \Big) \Bigg] + \underset{Y \to \infty}{o(1)}$$

$$
\begin{aligned}
&= \sideset{}{'}\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)\neq 0}} \sum_{k=1}^{\infty} \Bigg[\chi(\gamma)^{2k-1} \log N(h(\gamma J)) \\
&\qquad \int_0^\infty \frac{dt}{\sqrt{t}} k_\varepsilon\Big(t\big(N(\gamma J)^{k-\frac{1}{2}} + N(\gamma J)^{\frac{1}{2}-k}\big)^2 + \\
&\qquad\qquad \big(N(\gamma J)^{k-\frac{1}{2}} - N(\gamma J)^{\frac{1}{2}-k}\big)^2\Big)\Bigg] + \underset{Y\to\infty}{o(1)} \\
&= \sideset{}{'}\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)\neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \log N(h(\gamma J)) \frac{Q_\varepsilon\big(N(\gamma J)^{2k-1} + N(\gamma J)^{1-2k} - 2\big)}{N(\gamma J)^{k-1/2} + N(\gamma J)^{1/2-k}} + \underset{Y\to\infty}{o(1)} \\
&= \sideset{}{'}\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)\neq 0}} \sum_{k=1}^{\infty} \chi(\gamma)^{2k-1} \log N(h(\gamma J)) \frac{g_\varepsilon((2k-1)\log N(\gamma J))}{N(\gamma J)^{k-1/2} + N(\gamma J)^{1/2-k}} + \underset{Y\to\infty}{o(1)}
\end{aligned}
\tag{3.27}
$$

where the summation in $\sum'$ is only taken over primitive relative classes and N is the norm.

We consider now the situation when $\mathrm{tr}(\gamma J) = 0$ and $\Gamma_{\gamma J}$ is a hyperbolic cyclic group. Denote by $h(\gamma J)$ the generator of $\Gamma_{\gamma J}$. Then we have

$$
\begin{aligned}
&\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}\ \text{nontrivial}}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z,\gamma J z))d\mu(z) \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}\ \text{nontrivial}}} \chi(\gamma) \int_{F_{\gamma J}} k_\varepsilon(u(z,\gamma J z))d\mu(z) + \underset{Y\to\infty}{o(1)} \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}\ \text{nontrivial}}} \chi(\gamma) \int_1^{N(h(\gamma J))} \frac{dr}{r} \int_0^\pi \frac{d\varphi}{\sin^2\varphi} k_\varepsilon(u(z,Jz)) + \underset{Y\to\infty}{o(1)} \\
&= \sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}\ \text{nontrivial}}} \chi(\gamma) \log N(h(\gamma J)) \int_0^\pi \frac{d\varphi}{\sin^2\varphi} k_\varepsilon\Big(\frac{4\cos^2\varphi}{\sin^2\varphi}\Big) + \underset{Y\to\infty}{o(1)}
\end{aligned}
$$

$$= \sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \chi(\gamma) \frac{\log N(h(\gamma J))}{2} g_\varepsilon(0) + \underset{Y\to\infty}{o(1)} \tag{3.28}$$

The calculation is similar to (3.27). We shall see later that the sum in (3.28) contains only finitely many terms, at most.

Finally we have to calculate the contribution to (3.24) from classes with $\mathrm{tr}(\gamma J) = 0$ and trivial centralizer $\Gamma_{\gamma J} = \{e\}$. We have to find the asymptotics of

$$\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}=\{e\}}} \chi(\gamma) \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z) \quad , \quad Y \to \infty \tag{3.29}$$

where

$$F_Y^{\gamma J} = \bigcup_{\gamma' \in \Gamma_{\gamma J} \backslash \Gamma} \gamma' F_Y = \bigcup_{\gamma' \in \Gamma} \gamma' F_Y \tag{3.30}$$

We recall that $\Gamma = \Gamma_0(N)$, $N = 4N_2$ or $N = 4N_3$ and $F = F_0(N)$, a fundamental domain of Γ in H. We introduce $\Gamma(1)$ to be the modular group and $F(1)$ to be a fundamental domain of $\Gamma(1)$ in H. For the purpose of calculation let us take

$$F(1) = \{z \in H, z = x + iy \mid x^2 + y^2 > 1, 0 \le x \le \tfrac{1}{2} \text{ or } x^2 + y^2 \ge 1, \tfrac{-1}{2} < x < 0\}$$

Then we take

$$F_Y(1) = \{z \in F \mid y \le Y\} \quad , \quad Y > 1 \tag{3.31}$$

We have

$$F_0(N) = \bigcup_{\gamma \in \Gamma(1)/\Gamma_0(N)} \gamma F(1)$$

and we now define F_Y by

$$F_Y = F_Y^0(N) = \bigcup_{\gamma \in \Gamma(1)/\Gamma_0(N)} \gamma F_Y(1) \tag{3.32}$$

From that follows the continuation of (3.30)

$$\bigcup_{\gamma' \in \Gamma} \gamma' F_Y = \bigcup_{\gamma \in \Gamma(1)} \gamma F_Y(1) \tag{3.33}$$

In the sum (3.29) we will first consider the term with $\gamma = e$

$$\int_{F_Y^J} k_\varepsilon(u(z, Jz))d\mu(z) \tag{3.34}$$

We define two sets $\Omega_j \subset H$, $j = 1, 2$. By definition

$$\begin{aligned} \Omega_1 = H(Y) &= \{z \in H, z = x + iy \mid y > Y\} \\ \Omega_2 = \Omega_2(Y) &= \{z = \tfrac{-1}{z'} \mid z' \in \Omega_1(Y)\} \\ &= \{x + iy \mid x^2 + (y - \tfrac{1}{2Y})^2 < \frac{1}{4Y^2}\} \end{aligned}$$

From (3.30)–(3.33) follows

$$F_Y^J \subset H \setminus \Omega_1 \cup \Omega_2 = \Omega_3 \tag{3.35}$$

We will not calculate here explicitly the integral (3.34), but we will calculate the divergent term and we will estimate the remainder term for the purpose of proving the Weyl law.

We define now

$$\Omega_4 = \Omega_4(Y) = \{z \in H, z = x + iy \mid \frac{1}{Y} \le y \le Y\} \tag{3.36}$$

We can see that

$$\Omega_4(Y) \subset F_Y^J \tag{3.37}$$

This follows from (3.30)–(3.33) and the fact that

$$\begin{aligned} \max \operatorname{Im}(\gamma z) \le \frac{1}{y} \quad &, \quad z = x + iy \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in \Gamma(1) \\ c \ne 0 & \end{aligned} \tag{3.38}$$

We will prove now that the only divergent part of (3.34) for $Y \to \infty$ is given by

$$\int_{\Omega_4(Y)} k_\varepsilon(u(z, Jz))d\mu(z) \tag{3.39}$$

which is equal to

$$2\int_0^\infty dx \int_{1/Y}^Y \frac{dy}{y^2} k_\varepsilon(\tfrac{4x^2}{y^2}) = \frac{1}{2}\int_{1/Y}^Y \frac{dy}{y}\int_0^\infty \frac{k_\varepsilon(t)}{\sqrt{t}}dt$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot 2 \log Y \cdot g_\varepsilon(0) \\
&= g_\varepsilon(0) \log Y
\end{aligned} \tag{3.40}
$$

We will estimate now the rest of the integral (3.34) given by

$$
\int_{F_Y^J \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) \tag{3.41}
$$

From (3.14), (3.22), (3.13) follows

$$
\begin{aligned}
g_\varepsilon(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iru} e^{-\varepsilon(r^2 + \frac{1}{4})} dr = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-u^2/4\varepsilon} \cdot e^{-\varepsilon/4} \\
k_\varepsilon(e^v + e^{-v} - 2) &= -\frac{1}{\pi} \int_v^{\infty} \frac{g'_\varepsilon(u) du}{\sqrt{e^u + e^{-u} - e^v - e^{-v}}} \\
g'_\varepsilon(u) &= \frac{d}{du} g_\varepsilon(u)
\end{aligned} \tag{3.42}
$$

From (3.42) follows that $k_\varepsilon(t) \geq 0$, $t \geq 0$. Therefore

$$
\int_{F_Y^J \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) \leq \int_{\Omega_3(Y) \setminus \Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) = T_1 \tag{3.43}
$$

for all $Y > 1$, $\varepsilon > 0$ by (3.35). We will calculate now the right hand side of (3.43). We have

$$
\begin{aligned}
T_1 &= 2 \int_0^{1/Y} \frac{dy}{y^2} \int_{y\sqrt{1/yY - 1}}^{\infty} k_\varepsilon(\tfrac{4x^2}{y^2}) dx \\
&= \frac{1}{2} \int_0^{1/Y} \frac{dy}{y} \int_{4(1/yY - 1)}^{\infty} \frac{k_\varepsilon(t)}{\sqrt{t}} dt \\
&= \frac{1}{2} \int_1^{\infty} \frac{d\tau}{\tau} \int_{4(\tau - 1)}^{\infty} \frac{k_\varepsilon(t)}{\sqrt{t}} dt \\
&= -g_\varepsilon(0) \log 2 + \frac{1}{2} \int_0^{\infty} \log(t + 4) \frac{k_\varepsilon(t)}{\sqrt{t}} dt
\end{aligned} \tag{3.44}
$$

For the purpose of estimations of integrals (3.44) is good enough, but we can also transform (3.44) to integrals with the h_ε function. It was done in [2] §6.5. We have

$$
\int_0^{\infty} \log(t + 4) \frac{k_\varepsilon(t)}{\sqrt{t}} dt = -\frac{1}{\pi} \int_0^{\infty} dQ_\varepsilon(\omega) \int_0^{\omega} \frac{\log(t + 4)}{\sqrt{t}\sqrt{\omega - t}} dt
$$

$$
\begin{aligned}
&= \frac{1}{\pi} g_\varepsilon(0) \log 2 \int_0^1 \frac{1}{\sqrt{t}\sqrt{1-t}} dt \\
&\quad + \frac{1}{\pi} \int_0^\infty Q_\varepsilon(\omega) d\omega \int_0^1 \frac{d\tau}{(\omega + 4/\tau)\sqrt{\tau}\sqrt{1-\tau}} \\
&= 2 \log 2 g_\varepsilon(0) + \frac{1}{\pi} \int_0^\infty Q_\varepsilon(\omega) \cdot \pi \cdot \frac{1}{\omega + 4 + 2\sqrt{\omega + 4}} d\omega \\
&= 2 \log 2 g_\varepsilon(0) + \int_0^\infty g_\varepsilon(u) \tanh(u/4) du \qquad (3.45)
\end{aligned}
$$

Similarly we obtain (cf. Appendix Lemma A.3) as the main term of the asymptotics of (3.29)

$$
\int_{\Omega_4(Y)} k_\varepsilon(u(z, Jz)) d\mu(z) = g_\varepsilon(0) \log y
$$

and the remaining terms given by

$$
\int_{c^{-1}(\Omega_3(Y) \setminus \Omega_4(Y))} k_\varepsilon(u(z, Jz)) d\mu(z) = T_1
$$

and

$$
\int_{c^{-1}\frac{1}{Y}}^{1/Y} k_\varepsilon(u(z, Jz)) d\mu(z)
$$

see (4.34), (4.36).

Lemma 19. *The number of classes* $\{\gamma J\}_\Gamma$, $\operatorname{tr}(\gamma J) = 0$, $\Gamma_{\gamma J}$ *nontrivial, is finite.*

Proof. We can derive a formula similar to (3.12) for a trivial character χ. Then we can repeat the calculation of the contributions from all classes to (3.12). Instead of (3.28) we get

$$
\frac{g_\varepsilon(0)}{2} \sum_{\substack{\{\gamma J\}_\Gamma \\ \operatorname{tr}(\gamma J) = 0 \\ \Gamma_{\gamma J} \text{ nontrivial}}} \log N(h(\gamma J)) + \underset{Y \to \infty}{o(1)} \qquad (3.46)
$$

For any fixed $Y > 1$ and trivial character χ the integral in (3.12) is finite. From that follows that the sum in (3.46) is finite, which can happen only if we have finitely many terms in the sum. $\square$

We denote by $a(\Gamma, 1) = a(\Gamma)$ the number of all pairwise inequivalent open cusps of F relative to χ and set $m(\Gamma) = \#\{\gamma J\}_\Gamma$, $\operatorname{tr}(\gamma J) = 0$, $\Gamma_{\gamma J} = \{e\}$.

Lemma 20.

(1) $a(\Gamma) = m(\Gamma)$

(2) $a(\Gamma, \chi) = \sum\limits_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)=0 \\ \Gamma_{\gamma J}=\{e\}}} \chi(\gamma)$

Proof. Similar to the proof of Lemma 19. We compare the coefficients of the $\log Y$ terms in (3.12) (1) for trivial χ (2) for the primitive nontrivial character coming from the $\{\gamma\}_\Gamma$ and $\{\gamma J\}_\Gamma$ classes. $\square$

Theorem 4. *The Weyl law (see* (3.11)*) is valid.*

Proof. To prove the theorem we have to see the asymptotics of each term in (3.12) when $\varepsilon \to +0$ and then apply a Tauberian theorem. Similar to [2] we have

$$\begin{aligned}\frac{\mu(F)}{4\pi}\int_{-\infty}^{\infty} r(\tanh \pi r) h_\varepsilon(\tfrac{1}{4}+r^2)dr &= \frac{\mu(F)}{4\pi}\cdot\frac{1}{\varepsilon} + \underset{\varepsilon\to+0}{O(1)} \\ \sum_{\{h\}_\Gamma}\sum_{k=1}^{\infty}\frac{\chi^k(h)\log N(h)}{N(h)^{k/2}-N(h)^{-k/2}} g_\varepsilon(k\log N(h)) &= \underset{\varepsilon\to+0}{o(1)} \\ g_\varepsilon(0) &= \underset{\varepsilon\to+0}{O(\tfrac{1}{\sqrt{\varepsilon}})} \\ \int_{-\infty}^{\infty} h_\varepsilon(\tfrac{1}{4}+r^2)\frac{\Gamma'}{\Gamma}(1+ir)dr &= \underset{\varepsilon\to+0}{O(\tfrac{|\log\varepsilon|}{\sqrt{\varepsilon}})}\end{aligned} \tag{3.47}$$

We have also

$$\int_0^{\infty} g_\varepsilon(u)\tanh(\tfrac{u}{4})du = \underset{\varepsilon\to0}{o(1)} \tag{3.48}$$

Applying Lemmas 15, 16, 17 we have (see (3.27))

$$\sideset{}{'}\sum_{\substack{\{\gamma J\}_\Gamma \\ \mathrm{tr}(\gamma J)\neq 0}}\sum_{k=1}^{\infty}\chi(\gamma)^{2k-1}\log N(h(\gamma J))\frac{g_\varepsilon((2k-1)\log N(\gamma J))}{N(\gamma J)^{k-1/2}+N(\gamma J)^{1/2-k}} = \underset{\varepsilon\to+0}{o(1)} \tag{3.49}$$

Finally, using (3.43) and Lemmas 19, 20, we obtain from (3.12)

$$\begin{aligned}\sum_j h_\varepsilon(\lambda_j) &= \int_0^{\infty} e^{-\varepsilon\lambda}dN_{\mathrm{odd}}(\lambda) \\ &= \frac{\mu(F)}{8\pi}\cdot\frac{1}{\varepsilon} + \underset{\varepsilon\to+0}{O(\tfrac{|\log\varepsilon|}{\sqrt{\varepsilon}})}\end{aligned} \tag{3.50}$$

and from that follows (3.11) and Theorem 4. $\square$

4 More advanced trace formula

We introduce three functions

$$\begin{aligned} h_1(\lambda) &= h_1(\tfrac{1}{4} + r^2) = e^{-\varepsilon(\frac{1}{4}+r^2)} \\ h_2(\lambda) &= h_2(\tfrac{1}{4} + r^2) = 4\cos(r\log 2)e^{-\varepsilon(\frac{1}{4}+r^2)} \\ h_3(\lambda) &= h_3(\tfrac{1}{4} + r^2) = 2\cos(2r\log 2)e^{-\varepsilon(\frac{1}{4}+r^2)} \end{aligned} \tag{4.1}$$

All these functions depend on a parameter $\varepsilon > 0$ similar to (3.13). For each $h_j(\lambda)$ we denote by k_j the corresponding Selberg transform from (3.14), $j = 1, 2, 3$. We introduce also

$$K_j(z, z', \Gamma, \chi) = \sum_{\gamma\in\Gamma} \chi(\gamma)k_j(u(z, \gamma z')) \tag{4.2}$$

where γ runs over the whole group $\Gamma = \Gamma_0(N)$, $N = 4N_2$, $N = 4N_3$ and χ is our real primitive character and $u(z, z') = \frac{|z-z'|^2}{yy'}$.

We denote also by $K_j(\Gamma, \chi) = K_j$ the corresponding integral operator in the Hilbert space $\mathcal{H}(\Gamma) = L_2(F, d\mu)$ with the kernel given by (4.2). We will study now the operator

$$T = (4 + U^2(2) + U^{*2}(2))K_1 - (U(2) + U^*(2))K_2 + K_3 \tag{4.3}$$

on the space of odd functions $\mathcal{H}_{\text{odd}}$ (see the above Lemma 13). We have $K_j = h_j(A(\Gamma, \chi))$, $j = 1, 2, 3$. $U(2)K_j = K_jU(2)$, $U^*(2)K_j = K_jU^*(2)$, $K_j\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$, $U(2)\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$, $U^*(2)\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}$ and hence

$$T\mathcal{H}_{\text{odd}} \subset \mathcal{H}_{\text{odd}}. \tag{4.4}$$

From Lemma 12 follows that $A(\Gamma, \chi)$ has only discrete spectrum in $\mathcal{H}_{\text{odd}} \cap \mathcal{D}(A(\Gamma, \chi))$. From Theorem 4.2 [1] follows that there exists a common basis of eigenfunctions of $A(\Gamma, \chi)$, $U(2)$, $U^*(2)$ in $\mathcal{H}_{\text{odd}}$. Let us denote by $\{v_j(z, \Gamma, \chi)\}_{j=1}^{\infty}$ the orthonormal basis of common eigenfunctions in $\mathcal{H}_{\text{odd}}$. If

$$A(\Gamma, \chi)v_j(z) = \lambda_j v_j(z) \qquad \lambda_j = \lambda_j(\Gamma, \chi) \qquad v_j(z) = v_j(z, \Gamma, \chi)$$

and

$$U(2)v_j(z) = \nu_j v_j(z) \qquad \nu_j = \nu_j(\Gamma, \chi)$$

$$\lambda_j \in \mathbb{R} \qquad \nu_j \in \mathbb{C} \qquad |\nu_j| = 1 \qquad \nu_j = e^{i\eta_j} \qquad \eta_j \in \mathbb{R}$$

then $U^*(2)v_j = e^{-i\eta_j} v_j$ and we have

$$\begin{aligned} Tv_j &= \big[(4 + e^{2i\eta_j} + e^{-2i\eta_j})h_1(\lambda_j) - (e^{i\eta_j} + e^{-i\eta_j})h_2(\lambda_j) + h_3(\lambda_j)\big]v_j \\ &= \big[(4 + e^{2i\eta_j} + e^{-2i\eta_j}) - (e^{i\eta_j} + e^{-i\eta_j}) \cdot 4\cos(r_j \log 2) + 2\cos(2r_j \log 2)\big]e^{-\varepsilon\lambda_j} v_j \end{aligned} \tag{4.5}$$

where $\lambda_j = \frac{1}{4} + r_j^2$. We can continue (4.5)

$$\begin{aligned} Tv_j &= (e^{i\eta_j} + e^{-i\eta_j} - 2^{ir_j} - 2^{-ir_j})^2 e^{-\varepsilon\lambda_j} v_j \\ &= (2\cos\eta_j - 2\cos(r_j \log 2))^2 e^{-\varepsilon\lambda_j} v_j \end{aligned} \tag{4.6}$$

We denote $\omega_j = \cos\eta_j - \cos(r_j \log 2)$. It is not difficult to see that the operator T is of trace class on $\mathcal{H}_{\mathrm{odd}}$ and its spectral trace is equal to

$$\operatorname{tr} T = 4\sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon\lambda_j} \tag{4.7}$$

Using the trace formula we can calculate the matrix trace of T and obtain the asymptotics as $\varepsilon \to +0$. Then we apply the Tauberian theorem to get information on a bound for ω_j.

From (1.4), (4.2), (4.3) it follows that the kernel $\hat{T}(z, z')$ of the operator T (as an integral operator) on the space of odd functions $\mathcal{H}_{\mathrm{odd}}$ is given by

$$\begin{aligned} \hat{T}(z, z') &= \frac{1}{2}(T(z, z') - T(z, Jz')) \\ &= 2\sum_{\gamma\in\Gamma} \chi(\gamma)[k_1(u(z, \gamma z')) - k_1(u(z, \gamma Jz'))] \\ &\quad + \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{p\in P_3} \chi(\gamma)[k_1(u(z, \gamma pz')) - k_1(u(z, \gamma pJz'))] \\ &\quad + \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{p\in P_4} \chi(\gamma)[k_1(u(z, \gamma pz')) - k_1(u(z, \gamma pJz'))] \\ &\quad - \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{p\in P_1} \chi(\gamma)[k_2(u(z, \gamma pz')) - k_2(u(z, \gamma pJz'))] \\ &\quad - \frac{1}{2}\sum_{\gamma\in\Gamma}\sum_{p\in P_2} \chi(\gamma)[k_2(u(z, \gamma pz')) - k_2(u(z, \gamma pJz'))] \\ &\quad + \frac{1}{2}\sum_{\gamma\in\Gamma} \chi(\gamma)[k_3(u(z, \gamma z')) - k_3(u(z, \gamma Jz'))] \end{aligned} \tag{4.8}$$

From (4.7), (4.8) we can construct a trace formula similar to (3.12),

$$4\sum_{j=1}^{\infty}\omega_j^2 e^{-\varepsilon\lambda_j} = \lim_{Y\to\infty}\int_{F_Y}\hat{T}(z,z)d\mu(z) \tag{4.9}$$

where the λ_j were explained above (4.5). The right hand side of (4.8) is a sum of six automorphic kernels according to the decomposition (4.3). We denote them as $\hat{T}_j(z,z')$, $j=1,2,\dots,6$, starting from

$$\hat{T}_1(z,z') = 2\sum_{\gamma\in\Gamma}\chi(\gamma)[k_1(u(z,\gamma z')) - k_1(u(z,\gamma Jz'))]$$

and so on finishing by

$$\hat{T}_6(z,z') = \frac{1}{2}\sum_{\gamma\in\Gamma}\chi(\gamma)[k_3(u(z,\gamma z')) - k_3(u(z,\gamma Jz'))].$$

It is not difficult to see that for each $j=1,\dots,6$ there exists a finite limit

$$\lim_{Y\to\infty}\int_{F_Y}\hat{T}_j(z,z)d\mu(z) = \int_F\hat{T}_j(z,z)d\mu(z) = I_j(\varepsilon) \tag{4.10}$$

for any fixed $\varepsilon>0$. We have to find an asymptotics (or bound) for all $I_j(\varepsilon)$, $\varepsilon\to+0$. We did that in § 3 for I_1 (see (3.50)). We have

$$I_1(\varepsilon) = \frac{\mu(F)}{2\pi}\cdot\frac{1}{\varepsilon} + \underset{\varepsilon\to+0}{O}\left(\tfrac{|\log\varepsilon|}{\sqrt{\varepsilon}}\right) \tag{4.11}$$

The next integral we will consider is $I_6(\varepsilon)$ since it is close to the previous case. The contribution from the identity similar to (3.12), (3.13) is equal to

$$\frac{\mu(F)}{8\pi}\int_{-\infty}^{\infty}r(\tanh\pi r)h_3(\tfrac{1}{4}+r^2)dr \tag{4.12}$$

We will estimate this integral when $\varepsilon\to+0$. We have

$$\int_{-\infty}^{\infty}r(\tanh\pi r)h_3(\tfrac{1}{4}+r^2)dr = 4\int_0^{\infty}r(\tanh\pi r)(e^{2ir\log 2}+e^{-2ir\log 2})e^{-(\frac{1}{4}+r^2)\varepsilon}dr \tag{4.13}$$

Since

$$\int_0^{\infty}r(1-\tanh\pi r)h_3(\tfrac{1}{4}+r^2)dr = \underset{\varepsilon\to+0}{O(1)}$$

we have to evaluate

$$\int_0^\infty r(2\cos(2r\log 2))e^{-r^2\varepsilon}dr = \frac{1}{\varepsilon}\int_0^\infty t(2\cos(2\tfrac{t}{\sqrt{\varepsilon}}\log 2))e^{-t^2}dt$$
$$= \frac{2}{\varepsilon}\int_0^\infty t\cdot e^{-t^2}d(\sin(\tfrac{t}{\sqrt{\varepsilon}}2\log 2))\frac{\sqrt{\varepsilon}}{2\log 2}$$
$$= -\frac{1}{\sqrt{\varepsilon}\log 2}\int_0^\infty \frac{d}{dt}(t\cdot e^{-t^2})\sin(\tfrac{t}{\sqrt{\varepsilon}}2\log 2)dt$$
$$= \underset{\varepsilon\to+0}{O}(\tfrac{1}{\sqrt{\varepsilon}}) \tag{4.14}$$

From (4.14) follows that (4.12) is $\underset{\varepsilon\to+0}{O}(\frac{1}{\sqrt{\varepsilon}})$, which is smaller than the leading term in (4.11). To see the contribution from hyperbolic elements similar to (3.21), (3.47) we have to find

$$g_3(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iru}h_3(\tfrac{1}{4}+r^2)dr$$
$$= \frac{1}{2\sqrt{\pi\varepsilon}}[e^{-\frac{(u-2\log 2)^2}{4\varepsilon}} + e^{-\frac{(u+2\log 2)^2}{4\varepsilon}}]e^{-\frac{\varepsilon}{4}} \tag{4.15}$$

The worst that could happen is if there exists $\{h\}_\Gamma$ (see the second line in (3.47)) with the property

$$\kappa\log N(h) = 2\log 2 \tag{4.16}$$

which gives us instead of $\underset{\varepsilon\to+0}{o}(1)$ the estimate $\underset{\varepsilon\to+0}{O}(\frac{1}{\sqrt{\varepsilon}})$ coming from the $g_3(2\log 2)$ term (we know that at most finitely many $\{h\}_\Gamma$ have the same norm). We have then that the contribution from hyperbolic classes to the integral I_6 is bounded by $\underset{\varepsilon\to+0}{O}(\frac{1}{\sqrt{\varepsilon}})$ which again is smaller than the leading term in (4.11). The contribution from parabolic classes to I_6 is estimated in complete analogy to the previous case of I_1 and is estimated by $\underset{\varepsilon\to+0}{O}(\frac{1}{\sqrt{\varepsilon}})$.

The estimations of contributions from the $\{\gamma J\}_\Gamma$ classes also proceed in analogy to the previous case with obvious change. For example in (3.43) we estimate in numerical value

$$\int_{F_Y^J\backslash\Omega_4(Y)}|k_3(u(z,Jz))|d\mu(z) \leq \int_{\Omega_3(Y)\backslash\Omega_4(Y)}|k_3(u(z,Jz))|d\mu(z) \tag{4.17}$$

and in (3.49) we will get instead of $\underset{\varepsilon\to+0}{o}(1)$ the estimate $\underset{\varepsilon\to+0}{O}(\frac{1}{\sqrt{\varepsilon}})$ using the argument similar to the one in the proof of (4.15). Finally we obtain

$$I_6(\varepsilon) = \underset{\varepsilon\to+0}{O}(\tfrac{|\log\varepsilon|}{\sqrt{\varepsilon}}) \tag{4.18}$$

We are going to evaluate now the remaining four integrals I_2, I_3, I_4, I_5. From the point of view of the trace formula there is no substantial difference between these cases. We will consider in more detail the case I_2 and then explain the differences with other cases. We have for a fixed $\varepsilon > 0$

$$\begin{aligned}
I_2 &= I_2(\varepsilon) \\
&= \lim_{Y\to\infty} \int_{F_Y} \frac{1}{2} \sum_{\gamma\in\Gamma} \sum_{p\in P_3} \chi(\gamma) \big[k_1(u(z,\gamma p z)) - k_1(u(z,\gamma p J z))\big] d\mu(z) \\
&= \tfrac{1}{2} \lim_{Y\to\infty} \Big\{ \sum_{\gamma\in\Gamma} \chi(\gamma) \sum_{p\in P_3} \int_{F_Y} k_1(u(z,\gamma p z)) d\mu(z) \\
&\qquad - \sum_{\gamma\in\Gamma} \chi(\gamma) \sum_{p\in P_3} \int_{F_Y} k_1(u(z,\gamma p J z)) d\mu(z) \Big\} \\
&= \tfrac{1}{2} \lim_{Y\to\infty} \Big\{ \sum_{\{\gamma p\}_\Gamma} \chi(\gamma) \int_{F_Y^{\gamma p}} k_1(u(z,\gamma p z)) d\mu(z) \\
&\qquad - \sum_{\{\gamma p J\}_\Gamma} \int_{F_Y^{\gamma p J}} k_1(u(z,\gamma p J z)) d\mu(z) \Big\}
\end{aligned} \tag{4.19}$$

where $\{\gamma p\}_\Gamma$, $\{\gamma p J\}_\Gamma$ are relative conjugacy classes with Γ conjugation (see § 1, § 2). Similar to (3.16), (3.18) we have

$$F_Y^{\gamma p} = \bigcup_{\gamma'\in\Gamma_{\gamma p}\backslash\Gamma} \gamma' F_Y \qquad\qquad F_Y^{\gamma p J} = \bigcup_{\gamma'\in\Gamma_{\gamma p J}\backslash\Gamma} \gamma' F_Y \tag{4.20}$$

where $\Gamma_{\gamma p}$, $\Gamma_{\gamma p J}$ are centralizers of γp and $\gamma p J$ in Γ. We have $F_Y^{\gamma p} \xrightarrow[Y\to\infty]{} F^{\gamma p}$, $F_Y^{\gamma p J} \xrightarrow[Y\to\infty]{} F^{\gamma p J}$, where $F^{\gamma p}$, $F^{\gamma p J}$ are fundamental domains of $\Gamma_{\gamma p}$, $\Gamma_{\gamma p J}$ in H. We have to see now the contribution to (4.19) from different conjugacy classes. From Lemma 2 we know there is no parabolic classes $\{\gamma p\}$ in each ΓP_j, $j = 1, 2, 3, 4$. From Lemma 3 follows that there are at most finitely many elliptic classes $\{\gamma p\}_\Gamma$ in ΓP_j, $j = 1, 2, 3, 4$. Let the elliptic class $\{\gamma p\}_\Gamma$ have order d, then it is not difficult to see that the contribution to the trace is

$$\frac{\chi(\gamma)}{2\sin\frac{\pi}{d}} \int_{-\infty}^{\infty} \frac{\exp(-2\pi r/d)}{1+\exp(-2\pi r)} h(\tfrac{1}{4} + r^2) dr \tag{4.21}$$

where for I_2, I_3 we have $h = h_1$ and for I_4, I_5 we have $h = h_2$. From (4.1) follows that in all cases the contribution from all elliptic classes is given by $\underset{\varepsilon\to+0}{O(1)}$. From the first sum in (4.19) we have to evalute now only contributions from hyperbolic classes since there is no contribution from the identity in all these cases. According to Theorem 2 we have hyperbolic classes of two different types. We will consider first the case when $\Gamma_{\gamma p}$ is non-trivial cyclic group generated by a hyperbolic element $h(\gamma p) \in \Gamma$. We have infinitely many such classes and the total contribution can be calculated very similar to (3.21)

$$\sum_{\substack{\{\gamma p\}_\Gamma \\ \{\gamma p\} \text{ hyperbolic} \\ \Gamma^{\gamma p} \text{ non-trivial}}} \chi(\gamma) \log N(h(\gamma p)) \frac{1}{N(\gamma p)^{1/2} - N(\gamma p)^{-1/2}} \cdot g(\log N(\gamma p)) + \underset{Y\to\infty}{o(1)} \tag{4.22}$$

where again $g = g_1$ for I_2, I_3 and $g = g_2$ for I_4, I_5. It is known that the series (4.22) is absolutely convergent. We can take the limit $Y \to \infty$ and then evaluate (4.22) by $\underset{\varepsilon\to+0}{o(1)}$ for I_2, I_3 and $\underset{\varepsilon\to+0}{O(\frac{1}{\sqrt{\varepsilon}})}$ for I_4, I_5. To complete the study of the first sum in (4.19) we have to consider hyperbolic classes with trivial $\Gamma_{\gamma p}$. We have $F_Y^{\gamma p} = \Gamma_Y^0$ (see Appendix) and we will consider the more general situation assuming $p \in P_j$, $j = 1, 2, 3, 4$. Then

$$\int_{\Gamma_Y^0} k(u(z, \gamma p z)) d\mu(z) = \int_{g\Gamma_Y^0} k(u(z, \alpha z)) d\mu(z) \tag{4.23}$$

where $g \in \mathrm{PSL}(2, \mathbb{R})$, $\alpha = g\gamma p g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$, $u(z, \alpha z) = \frac{|z-\lambda^2 z|}{y^2\lambda^2}$. Similar to (3.40) the main term of asymptotics of (4.23) for $Y \to \infty$ is given by

$$\int_{-\infty}^{\infty} dx \int_{1/Y}^{Y} k(\tfrac{|z-\lambda^2 z|^2}{y^2\lambda^2}) \frac{dy}{y^2} = A \tag{4.24}$$

(see Appendix Lemma A.4). We have

$$\begin{aligned} A &= 2\int_0^\infty dx \int_{1/Y}^{Y} k(a(1 + \tfrac{x^2}{y^2})) \frac{dy}{y^2} \\ &= 2(\log(Y) - \log(1/Y)) \int_0^\infty dt k(a(1 + t^2)) \\ &= 4 \log Y \int_0^\infty k(a(1 + t^2)) dt \\ &= \frac{2\log Y}{\lambda - 1/\lambda} g(2 \log \lambda) \end{aligned} \tag{4.25}$$

where $a = (\lambda - \frac{1}{\lambda})^2$ and $g(u)$ is the Selberg transform of $k(t)$ (see (3.14)). For $k = k_1$ or k_2 we have

$$g(2\log\lambda) = \underset{\varepsilon\to 0}{O}(\tfrac{1}{\sqrt{\varepsilon}}) \tag{4.26}$$

If $h_0^{(j)}(\Gamma)$ is the number of classes $\{\gamma p\}_\Gamma$, $p \in P_j$, with trivial centralizer $\Gamma_{\gamma p} = \{e\}$, we can prove similar to Lemma 20 that each $h_0^{(j)} < \infty$, and the total divergent term from the first sum of (4.19) is equal to

$$\Big(\sum_{\substack{\{\gamma p\}_\Gamma \\ \Gamma_{\gamma p}=\{e\} \\ p\in P_j}} \chi(\gamma)\frac{1}{\lambda - 1/\lambda}\Big) 2\log Y g(2\log\lambda) \tag{4.27}$$

where the sum in (4.27) consists of $h_0^{(j)}$ terms, $\lambda = \lambda(\gamma p)$. We have from Theorem 2

(a) $\mathrm{tr}(\gamma p) = \lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$ if $p \in P_1$ or P_2

(b) $\mathrm{tr}(\gamma p) = \lambda + \lambda^{-1} = \frac{5}{2}$ if $p \in P_3$ or P_4.

From that follows in case (a) $\lambda = \sqrt{2}$ and in case (b) $\lambda = 2$, so it is independent of γ. We can rewrite (4.27)

$$(2\log Y) g(2\log\lambda)\frac{1}{\lambda - 1/\lambda} \sum_{\substack{\{\gamma p\}_\Gamma \\ \Gamma_{\gamma p}=\{e\} \\ p\in P_j}} \chi(\gamma) \quad , \quad \lambda = \sqrt{2} \text{ or } \lambda = 2 \tag{4.28}$$

Now we have to see the divergent terms in the second sum of (4.19). From Lemma 10 follows that there is no term in this sum with $\mathrm{tr}(\gamma pJ) = 0$. We have to split this sum according to

$$\sum_{\substack{\{\gamma pJ\}_\Gamma \\ \Gamma_{\gamma pJ} \text{ non-trivial}}} + \sum_{\substack{\{\gamma pJ\}_\Gamma \\ \Gamma_{\gamma pJ} \text{ trivial}}} \tag{4.29}$$

The first sum in (4.29) is transforming similar to (3.27) with $g = g(\varepsilon)$ where $g = g_1$ or $g = g_2$. It is absolutely convergent, has a finite limit as $Y \to \infty$ and is estimated by $\underset{\varepsilon\to 0}{O}(\frac{1}{\sqrt{\varepsilon}})$. And we have to calculate the asymptotics as $Y \to \infty$ of the second sum in (4.29). We have $F_Y^{\gamma pJ} = \Gamma_Y^0$ and

$$\int_{\Gamma_Y^0} k(u(z, \gamma pJz))d\mu(z) = \int_{g\Gamma_Y^0} k(u(z, \beta z))d\mu(z) \tag{4.30}$$

where $g^* \in \mathrm{PSL}(2, \mathbb{R})$,

$$\beta = g^* \gamma p J g^{*-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix},$$

$\lambda > 1$, $p \in P_j$.

The main term of asymptotics for $Y \to \infty$ of (4.30) is given by

$$\int_{-\infty}^{\infty} dx \int_{1/Y}^{Y} k(\tfrac{|z+\lambda^2 \bar{z}|^2}{y^2 \lambda^2}) \frac{dy}{y^2} = 2(\log(Y) - \log(1/Y)) \int_0^{\infty} k(b^2 x^2 + d^2) dx \tag{4.31}$$

$b^2 = (\lambda + \frac{1}{\lambda})^2$, $d^2 = (\lambda - \frac{1}{\lambda})^2$. From Theorem 3 follows that $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ in cases $j = 1, 2$ and $\lambda - \lambda^{-1} = \frac{3}{2}$ if $j = 3, 4$. That means $\lambda = \sqrt{2}$ in cases (1), (2) and $\lambda = 2$ in cases (3), (4). The integral (4.31) is equal to

$$4 \log Y \int_0^{\infty} k(b^2 x^2 + d^2) dx = \frac{\lambda}{1 + \lambda^2} \cdot 2 \log Y g(2 \log \lambda) \tag{4.32}$$

where g is the Selberg transform of k from (3.14). Again for $k = k_1$ or k_2 we have (4.26). If $h_1^{(j)}(\Gamma)$ is the number of classes $\{\gamma p J\}_\Gamma$, $p \in P_j$ with trivial centralizer $\Gamma_{\gamma p J} = \{e\}$, similar to Lemma 20 we can prove that each $h_1^{(j)} < \infty$ and the total divergent term from the second sum of (4.19) is equal to (more general $p \in P_j$)

$$(2 \log Y) g(2 \log \lambda) \frac{\lambda}{1 + \lambda^2} \sum_{\substack{\{\gamma p J\}_\Gamma \\ \Gamma_{\gamma p J} = \{e\} \\ p \in P_j}} \chi(\gamma) \qquad \lambda = \sqrt{2} \text{ or } \lambda = 2 \tag{4.33}$$

where the sum in (4.33) consists of $h_1^{(j)}$ terms. From the existence of a finite limit (4.10) we obtain that the divergent terms (4.28) and (4.33) coincide (not only for primitive character χ but also for $\chi = 1$). We have proved the following

Lemma 21. *We have (the notation explained in* (4.27), (4.33)*)*

(1) $h_1^{(j)}(\Gamma) = 3 h_0^{(j)}(\Gamma)$

(2) $\displaystyle\sum_{\substack{\{\gamma p J\}_\Gamma \\ p \in P_j}} \chi(\gamma) = 3 \sum_{\substack{\{\gamma p\}_\Gamma \\ p \in P_j}} \chi(\gamma)$

(3) $3 h_1^{(j)}(\Gamma) = 5 h_0^{(j)}(\Gamma)$

(4) $3 \sum_{\substack{\{\gamma p J\}_\Gamma \\ p \in P_j}} \chi(\gamma) = 5 \sum_{\substack{\{\gamma p\}_\Gamma \\ p \in P_j}} \chi(\gamma)$

The cases (1), (2) apply when $j = 1, 2$ *and the cases (3), (4) apply when* $j = 3, 4$.

To complete now the evaluation of terms in (4.19) (more general $p \in P_j$) we first have to evaluate the second terms in the right hand sides of (4.25), (4.32) for $\varepsilon \to 0$, and this is done by (4.26). Secondly we have to evaluate the differences between (4.23), (4.24), and between (4.30), (4.31). In both of these last cases we have for the domain of integration (see Appendix Lemmas A.4 and A.5) for some $c \geq 1$

$$g\Gamma_Y^0 \subset \left\{ z \in H, z = x + iy \mid \frac{1}{cY} < y < Y \right\} \cup \left\{ 0 < y < \frac{1}{cY}, |x| > y\sqrt{\frac{1}{yYc} - 1} \right\} \tag{4.34}$$

It follows then, that we have to evaluate the following integrals

$$\int_{1/cY}^{1/Y} \frac{dy}{y^2} \int_{-\infty}^{\infty} dx k(a(1 + \tfrac{x^2}{y^2})) \tag{4.35}$$

$$\int_0^{1/cY} \frac{dy}{y^2} \int_{y\sqrt{1/yY - 1}}^{\infty} dx k(a(1 + \tfrac{x^2}{y^2})) \tag{4.36}$$

where $c \geq 1$ is a constant (independent of Y, ε, but generally different for γp, $\gamma p J$), $a = (\lambda - \lambda^{-1})^2$ (see (4.25)), and the integrals

$$\int_{1/cY}^{1/Y} \frac{dy}{y^2} \int_{-\infty}^{\infty} dx k(d^2 + \tfrac{x^2}{y^2} b^2)) \tag{4.37}$$

$$\int_0^{1/cY} \frac{dy}{y^2} \int_{y\sqrt{1/yY - 1}}^{\infty} dx k(d^2 + \tfrac{x^2}{y^2} b^2)) \tag{4.38}$$

$b = \lambda + \lambda^{-1}$, $d = \lambda - \lambda^{-1}$ (see (4.31)).

It is easy to see (by calculation, similar to (4.25), (4.32)) that the integrals (4.35), (4.37) are independent of Y and up to a multiplicative constant they are equal to $g(2 \log \lambda)$, which is estimated in (4.26). We consider now (4.36). By obvious change of variables we reduce it to (up to a multiplicative constant)

$$\int_a^{\infty} \frac{dy}{y} \int_y^{\infty} \frac{k(\tau)}{\sqrt{\tau - a}} d\tau = \int_a^{\infty} d(\log y) \int_y^{\infty} \frac{k(\tau)}{\sqrt{\tau - a}}$$

$$= -\log(a)g(2\log\lambda) + \int_a^\infty \log y \frac{k(y)}{\sqrt{y-a}}dy \qquad (4.39)$$

We only have to estimate

$$\begin{aligned}
&\int_a^\infty \log y \frac{k(y)}{\sqrt{y-a}}dy \\
&\quad = \int_a^\infty \frac{\log y}{\sqrt{y-a}}\Big(\frac{-1}{\pi}\int_y^\infty \frac{dQ(\omega)}{\sqrt{\omega-y}}\Big) \\
&\quad = \frac{-1}{\pi}\int_a^\infty d\omega \int_a^\omega dy \frac{Q'(\omega)\log y}{\sqrt{y-a}\sqrt{\omega-y}} \\
&\quad = \frac{-1}{\pi}\int_a^\infty dQ(\omega)\int_a^\infty \frac{\log y}{\sqrt{y-a}\sqrt{\omega-y}}dy \\
&\quad = \frac{-1}{\pi}\int_a^\infty dQ(\omega)\int_0^1 \frac{\log((\omega-a)y+a)}{\sqrt{y}\sqrt{1-y}}dy \\
&\quad = (\log a)g(2\log\lambda) + \frac{1}{\pi}\int_a^\infty Q(\omega)\int_0^1 \frac{\sqrt{y}dy}{\sqrt{1-y}((\omega-a)y+a)}d\omega \\
&\quad = (\log a)g(2\log\lambda) + \frac{1}{\pi}\int_a^\infty Q(\omega)\frac{\pi}{\omega+\sqrt{a\omega}}d\omega \\
&\quad = (\log a)g(2\log\lambda) + \int_{2\log\lambda}^\infty g(u)\frac{e^u-e^{-u}}{e^u+e^{-u}-2+(\lambda-1/\lambda)(e^{u/2}-e^{-u/2})}du \\
&\quad = (\log a)g(2\log\lambda) + \int_{2\log\lambda}^\infty g(u)\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}+\lambda-1/\lambda}du \qquad (4.40)
\end{aligned}$$

We have $g = g_1$ or g_2 and

$$\begin{aligned}
g_1(\varepsilon) &= \frac{1}{2\sqrt{\pi}}\cdot\frac{1}{\sqrt{\varepsilon}}e^{-u^2/4\varepsilon}e^{-\varepsilon/4} \\
g_2(\varepsilon) &= e^{-\varepsilon/4}\frac{1}{\sqrt{\pi\varepsilon}}\big(e^{-(u-\log 2)^2/4\varepsilon} + e^{-(u+\log 2)^2/4\varepsilon}\big)
\end{aligned} \qquad (4.41)$$

From (4.40), (4.41) follows that the integral (4.39) is estimated by $O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \to +0$.

The last integral (4.38) can be easily reduced to the sum of $g(2\log\lambda)$ and $\int_{d^2}^\infty \log(t+4)\frac{k(t)}{\sqrt{t-d^2}}dt$ (up to multiple constant coefficients) which is estimated as before by $O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \to +0$.

We have proved the following

Lemma 22. *There are estimates $I_j = I_j(\varepsilon) = O(\frac{1}{\sqrt{\varepsilon}})$, $\varepsilon \to +0$, $j = 2, 3, 4, 5$ (see (4.8), (4.10)).*

From (4.11), (4.18), Lemma 22 follows

Theorem 5. *For the trace of the operator T (see (4.7)) we have the following asymptotics, $\varepsilon \to +0$*

$$\operatorname{tr} T = 4\sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon\lambda_j} = \frac{\mu(F)}{2\pi} \cdot \frac{1}{\varepsilon} + O\Big(\frac{|\log\varepsilon|}{\sqrt{\varepsilon}}\Big)$$

We have by (3.50) and Theorem 5 with $\omega_j = \cos(\eta_j) - \cos(r_j \log 2)$,

$$\sum_{j=1}^{\infty} e^{-\varepsilon\lambda_j} = \frac{\mu(F)}{8\pi} \cdot \frac{1}{\varepsilon} + O\Big(\frac{|\log\varepsilon|}{\sqrt{\varepsilon}}\Big) \tag{4.42}$$

$$\sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon\lambda_j} = \frac{\mu(F)}{8\pi} \cdot \frac{1}{\varepsilon} + O\Big(\frac{|\log\varepsilon|}{\sqrt{\varepsilon}}\Big) \tag{4.43}$$

The smallest number of terms with $\omega_j \neq 0$ is obtained if $|\omega_j| = 2$ for all j with $\omega_j \neq 0$. Assume that this holds and that

$$\frac{\#\{j \mid 1 \leq j \leq X, \omega_j^2 = 4\}}{X} \xrightarrow[X\to\infty]{} \frac{1}{4} \tag{4.44}$$

Then we shall see that (4.43) holds.

By a Tauberian theorem (4.42) implies (Theorem 4)

$$\#\{\lambda_j \leq \lambda\} \approx \frac{\mu(F)}{8\pi}\lambda \text{ as } \lambda \to \infty \tag{4.45}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and λ_j is repeated according to multiplicity. Let $j_{k_1} < j_{k_2} < \cdots < j_{k_n} < \cdots$ be the values of j such that $\omega_{j_k}^2 = 4$. Then (4.44) implies

$$\frac{\#\{\lambda_{j_k} \leq \lambda\}}{\#\{\lambda_j \leq \lambda\}} \xrightarrow[\lambda\to\infty]{} \frac{1}{4}$$

and hence by (4.45)

$$\#\{\lambda_{j_k} \leq \lambda\} \approx \frac{1}{4} \cdot \frac{\mu(F)}{8\pi}\lambda$$

Then

$$\sum_{j=1}^{\infty} \omega_j^2 e^{-\varepsilon\lambda_j} = 4\sum_{k=1}^{\infty} e^{-\varepsilon\lambda_{j_k}} \underset{\varepsilon\downarrow 0}{\sim} 4\frac{1}{4}\frac{\mu(F)}{8\pi}\frac{1}{\varepsilon} = \frac{\mu(F)}{8\pi}\frac{1}{\varepsilon}$$

in agreement with (4.42). The same proof shows that we cannot have

$$\liminf_{X\to\infty} \frac{\#\{j \mid 1 \le j \le X, \omega_j^2 = 4\}}{X} < \frac{1}{4}$$

so the minimal number of j with $\omega_j \neq 0$ is given by (4.44).

We have proved

Theorem 6.

$$\liminf_{\lambda\to\infty} \frac{\#\{\lambda_{j_k} \le \lambda \mid \omega_{j_k} \neq 0\}}{\#\{\lambda_j \le \lambda\}} \ge \frac{1}{4} \tag{4.46}$$

where λ_j is repeated according to multiplicity.

Thus, for at least $\frac{1}{4}$ of the eigenvalues $\lambda_j = \frac{1}{4} + r_j^2$ of the automorphic Laplacian with odd eigenfunction Φ_j the corresponding eigenvalue $e^{i\eta_j}$ of the exceptional Hecke operator $U(2)$ satisfies $\omega_j \neq 0$ or $e^{i\eta_j} \neq 2^{ir_j}$.

We now consider the other exceptional Hecke operators $U(q)$ for $q > 2$, $q \mid N$, $N = 4N_2$ or $N = 4N_3$. Here we have to establish the condition $\rho_j(q) \neq \frac{q^{ir_j}}{\varepsilon_q}$, where $\rho_j(q)$ is the eigenvalue of $U(q)$ corresponding to the eigenvector Φ_j. The parameters ε_q are any real numbers. We now assume that $\varepsilon_q \neq \pm 1$ for all $q \mid N$, $q > 2$. Then we obtain from [1] (7.23), (7.24) and Theorem 6

Theorem 7. *Let the forms $\omega(z)$ be defined as in [1] Theorem 6.2 and assume that $\varepsilon_l \neq \pm 1$ for $l = 2, \dots, k$. Then for at least $\frac{1}{4}$ of the eigenvalues λ_j of $A_{\mathrm{odd}}(\Gamma, \chi)$ (in the sense of Theorem 6) with eigenfunctions Φ_j the Philips-Sarnak integral $I_j(\Phi_j) \neq 0$, where $I_j(\Phi_j)$ is given by [1] (7.2).*

As a consequence, for each λ_j in the sequence of Theorem 7 at least one eigenvector Φ in the eigenspace $N(A_{\mathrm{odd}}(\Gamma, \chi)) - \lambda_j$ turns into a resonance function under perturbation by the form $\omega(z)$, and the total dimension of the eigenspace is reduced by at least one (cf. [1] Theorem 5.8).

For each eigenvalue λ_{j_k} with $\omega_{j_k} \neq 0$ there is at least one eigenfunction $\tilde{\Phi}$ with eigenvalue λ_{j_k} such that $\tilde{\Phi}$ turns into a resonance function under perturbation. Let $\{\tilde{\Phi}_i\}_{i=1}^{\infty}$ be an orthonormal sequence of all such eigenfunctions with increasing eigenvalues λ_i.

Theorem 8. *Assume that* $\dim\{N(A_{\mathrm{odd}} - \lambda_j)\} \le m$ *for all eigenvalues λ_j. Then*

$$\liminf_{\lambda\to\infty} \frac{\#\{\tilde{\Phi}_i \mid \lambda_i \le \lambda, \omega_i \neq 0\}}{\#\{\Phi_j \mid \lambda_j \le \lambda\}} \ge \frac{1}{4m}$$

Proof. The minimal number of eigenfunctions $\tilde{\Phi}_i$ with eigenvalues $\lambda_i \leq \lambda$ occurs if all eigenfunctions Φ_{j_k} with $\omega_{j_k} \neq 0$ are distributed with m in each m-dimensional eigenspace of $A_{\text{odd}}(\Gamma, \chi)$. Then each such eigenspace contributes at least one $\tilde{\Phi}_i$, and the result follows from Theorem 7. $\square$

A Transformation of the domain of integration of $k_\varepsilon(u(z, Jz))$ by $g \in \mathrm{PSL}(2, \mathbb{Q})$

In (3.29)–(3.34) it was shown that $F_Y^J = F_Y^0 = \bigcup_{\gamma \in \Gamma_1} \gamma F_Y(1)$, where Γ_1 is the modular group with fundamental domain $F(1)$ and $F_Y(1) = \{z \in F(1) \mid y \leq Y\}$. Then

$$\int_{F_Y^J} k_\varepsilon(u(z, Jz)) d\mu(z) = \int_{F_Y^0} k_\varepsilon(u(z, Jz)) d\mu(z) \tag{A.1}$$

Let $H(Y) = \{z \in H \mid y > Y\}$, $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, and for $c_1 \neq 0$ let $C(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ be the circle with center $(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ and radius $\frac{1}{2Yc_1^2}$, touching $\mathbb{R}$ at $\frac{a_1}{c_1}$, with interior $C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$.

If $c_1 = 0$, $\gamma(H(Y)) = H(Y)$. We now prove that for $c_1 \neq 0$, $\gamma(H(Y)) = C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$. We have

$$\gamma(x + iY) = \frac{(a_1 x + b_1)(c_1 x + d_1) + a_1 c_1 Y^2 + iY}{(c_1 x + d_1)^2 + c_1^2 Y^2} = x' + iy'$$

Since $\gamma(\infty) = \frac{a_1}{c_1}$, $\gamma(H(Y))$ is a circle $C^0(\frac{a_1}{c_1}, R)$ with radius R to be determined by the equation

$$(x' - \frac{a_1}{c_1})^2 + y'^2 - 2Ry' = 0$$

implying

$$(-x - \frac{d_1}{c_1})^2 + Y^2 = 2RYc_1^2[(x + \frac{d_1}{c_1})^2 + Y^2]$$

which gives

$$R = \frac{1}{2Yc_1^2}$$

From this we obtain

Lemma A.1.

$$H \setminus F_Y^0 = H(Y) \cup \bigcup_{\substack{\gamma \in \Gamma \\ c_1 \neq 0}} C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$$

Proof. No point in $F_Y(1)$ is mapped by $\gamma \in \Gamma_1$ to a point of $F(1) \setminus F_Y(1)$, and hence $H(Y) \subset H \setminus \Gamma_Y^0$. Therefore, $\gamma(H_Y) = C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}) \subset H \setminus \Gamma_Y^0$ for $\gamma \in \Gamma_1$, $c_1 \neq 0$, and the Lemma follows. $\square$

In order to estimate the integrals in (3.29) we also need to calculate for $\gamma \in \Gamma_0(N)$

$$\int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z) \tag{A.2}$$

in the case where $\mathrm{tr}(\gamma J) = 0$ and the centralizer $\Gamma_{\gamma J} = \{e\}$. Again we have $F_Y^{\gamma J} = F_Y^0$, and if $g \in \mathrm{PSL}(2, \mathbb{Q})$ such that

$$g(\gamma J)g^{-1} = J,$$

we have

$$\begin{aligned} \int_{F_Y^{\gamma J}} k_\varepsilon(u(z, \gamma J z)) d\mu(z) &= \int_{F_Y^0} k_\varepsilon(u(gz, Jgz)) d\mu(z) \\ &= \int_{g\Gamma_Y^0} k_\varepsilon(u(z, Jz)) d\mu(z) \end{aligned} \tag{A.3}$$

By Lemma A.1

$$\begin{aligned} H \setminus g\Gamma_Y^0 &= g(H \setminus \Gamma_Y^0) \\ &= g(H(Y)) \cup \bigcup_{\substack{\gamma \in \Gamma_1 \\ c_1 \neq 0}} g(C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})) \end{aligned}$$

In order to estimate the integral (A.2) we therefore need to calculate $g(H(Y))$ and $g(C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}))$ for $c_1 \neq 0$. The set $g(H(Y))$ is given by the proof of Lemma A.1 to be $C^0(\frac{a}{c}, \frac{1}{2Yc^2})$ if $c \neq 0$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and it is $H(Ya^2)$ if $c = 0$.

Let $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_1$, $c_1 \neq 0$ and set $r_1 = \frac{a_1}{c_1}$. The equation of $C(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})$ is

$$(x - r_1)^2 + (y - \frac{1}{2Yc_1})^2 = (\frac{1}{2Yc_1})^2$$

or

$$x^2 + y^2 = 2r_1 x + \frac{1}{Yc_1^2} y - r_1^2 \tag{A.4}$$

We have two cases:

(1) $\frac{a_1}{c_1} = -\frac{d}{c}$. Then $g(\frac{a_1}{c_1}) = \infty$, so $g(C^0(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2})) = H(Y_0)$, Y_0 to be determined.

(2) $\frac{a_1}{c_1} \neq -\frac{d}{c}$. Then $g(\frac{a_1}{c_1}) = g(r_1) = \frac{ar_1+b}{cr_1+d}$, so

$$g\Big(C^0\Big(\frac{a_1}{c_1}, \frac{1}{2Yc_1^2}\Big)\Big) = C^0\Big(\frac{ar_1+b}{cr_1+d}, R\Big),$$

R to be determined.

We have

$$g(x+iy) = \frac{(ax+b)(cx+d) + acy^2 + iy}{(cx+d)^2 + c^2y^2} = x' + iy' \tag{A.5}$$

(1) $\frac{a_1}{c_1} = -\frac{d}{c}$. Then by (A.5)

$$\frac{y}{(cx+d)^2 + c^2y^2} = Y_0$$

or

$$(x - r_1)^2 + y^2 = (x + \tfrac{d}{c})^2 = \frac{1}{Y_0c^2},$$

so by (A.4)

$$Y_0 = Y\frac{c_1^2}{c^2}.$$

(2) $\frac{a_1}{c_1} \neq -\frac{d}{c}$. We determine the radius R of the circle

$$C^0(\tfrac{ar_1+b}{cr_1+d}, R) = g(C(r_1, \tfrac{1}{2Yc_1^2})),$$

where $C(r_1, \frac{1}{2Yc_1^2})$ is given by equation (A.4). The equation of $C^0(\frac{ar_1+b}{cr_1+d}, R)$ with x' and y' given by (A.5) is

$$(x' - \tfrac{ar_1+b}{cr_1+d})^2 + y'^2 = 2Ry'$$

or

$$\Big\{(ax+b)(cx+d) + acy^2 - \frac{ar_1+b}{cr_1+d}\big[(cx+d)^2 + c^2y^2\big]\Big\}^2 + y^2$$
$$= 2Ry\big\{(cx+d)^2 + c^2y^2\big\}$$

Using (A.4) this reduces to

$$\{x+\frac{ac}{Yc_1^2}y+\frac{a}{b}\}^2+y^2=2Ry\{2\frac{c}{a}x+\frac{c^2}{Yc_1^2}y+\frac{ad+bc}{a^2}\}.$$

Squaring and using (A.4) again, we get

$$\frac{c^2}{(Yc_1^2)^2(cr_1+d)^2}y^2+\frac{2c}{Yc_1^2(cr_1+d)}xy+\frac{1}{Yc_1^2}\frac{d-cr_1}{cr_1+d}y= \\ 2Ry\Big\{\frac{c^2}{Yc_1^2}y+2c(cr_1+d)x+d^2-c^2r_1^2\Big\} \tag{A.6}$$

Equating coefficients of y^2, xy and y, we get that (A.6) holds if and only if

$$R=\frac{1}{2Yc_1^2}\frac{1}{(cr_1+d)^2}=\frac{1}{2Y}\frac{1}{(ca_1+dc_1)^2}.$$

We have proved

Lemma A.2. *Let*

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{PSL}(2,\mathbb{Q}),$$

$c\neq 0$. *Then with*

$$\gamma=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\in \Gamma_1,$$

$r_1=\frac{a_1}{c_1}$,

$$H\setminus g\Gamma_Y^0=C^0\Big(\frac{a}{c},\frac{1}{2Yc^2}\Big)\cup\bigcup_{\substack{\gamma\in\Gamma_1\\ c_1\neq 0\\ r_1\neq\frac{-d}{c}}}C^0\Big(\frac{ar_1+b}{cr_1+d},\frac{1}{2Y}\frac{1}{(ca_1+dc_1)^2}\Big)\cup H\Big(Y\frac{c_2^2}{c^2}\Big)$$

where

$$\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\in\Gamma_1\quad,\quad \frac{a_2}{c_2}=-\frac{d}{c}.$$

In particular,

$$g(C^0(-\tfrac{b}{a},\tfrac{1}{2Yc_1^2}))=C^0(0,\tfrac{1}{2Y(ca_1+dc_1)^2}))$$

for $a \neq 0$, $\frac{a_1}{c_1} = -\frac{b}{a}$.

$$g(H(Y)) = C^0(0, \tfrac{1}{2Yc^2})$$

for $a = 0$.

In order to estimate (A.3) we apply Lemma A.2 to

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q}),$$

where $g(\gamma_0 J)g^{-1} = J$,

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma_0(N) \\ \mathrm{tr}(\gamma_0 J) &= 0 \\ J &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

We solve the equation $g(\gamma_0 J)g^{-1} = J$ or

$$\begin{pmatrix} a_0 a + N c_0 b & -b_0 a - a_0 b \\ a_0 c + N c_0 d & -b_0 c - a_0 d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$$

or the two pairs of dependent equations

$$\begin{matrix} (a_0 - 1)a + N c_0 b = 0 & -b_0 a - (a_0 + 1)b = 0 \\ (a_0 + 1)c + N c_0 d = 0 & -b_0 c - (a_0 - 1)d = 0 \end{matrix} \tag{A.7}$$

Case 1: $b_0 \neq 0$, $c_0 \neq 0$, so $a_0 \neq \pm 1$. We find

$$g = \begin{pmatrix} \frac{1-a_0^2}{2b_0} c^{-1} & \frac{a_0 - 1}{2} c^{-1} \\ c & -\frac{b_0}{a_0 - 1} c \end{pmatrix} \quad , \quad c \in \mathbb{Q} \setminus \{0\}.$$

Let $e_0 = (1 - a_0, b_0)$ and choose $c = c^* = \frac{1-a_0}{e_0}$. Then $d = d^* = \frac{b_0}{c_0}$, so c^* and d^* are integers with $(c^*, d^*) = 1$.

Case 2: $b_0 = 0$, $c_0 \neq 0$, $a_0 = -1$. We find $c^* = 1$, $d^* = 0$.

Case 3: $b_0 \neq 0$, $c_0 = 0$, $a_0 = -1$. We find

$c^* = 2$, $d^* = b_0$ if b_0 is odd, $a^* = 0$.

$c^* = 1$, $d^* = \frac{b_0}{2}$ if b_0 is even, $a^* = 0$.

Case 4: $b_0 = 0$, $c_0 \neq 0$, $a_0 = 1$. We find $c^* = -\frac{Nc_0}{2}$, $d^* = 1$, $b = 0$.

Case 5: $b_0 \neq 0$, $c_0 = 0$, $a_0 = 1$. We find $c = 0$, and choose $d^* = 1$,

$$g^* = \begin{pmatrix} 1 & \frac{-b_0}{2} \\ 0 & 1 \end{pmatrix}.$$

Applying Lemma A.2 to $g^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$, we obtain

Lemma A.3. *Let*

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ Nc_0 & d_0 \end{pmatrix} \in \Gamma_0(N)$$

and $g^*(\gamma_0 J)g^{*-1} = J$, *where*

$$g^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q})$$

with c^* *and* d^* *integers,* $(c^*, d^*) = 1$. *Then*

$$\int_{F^{\gamma_0 J}} k_\varepsilon(u(z, \gamma_0 Jz))d\mu(z) = \int_{g^*\Gamma_Y^0} k_\varepsilon(u(z, Jz))d\mu(z)$$

where

$$\{y \mid \tfrac{1}{Y} < y < Y\} \subset g^*\Gamma_Y^0 \subset \{y \mid 0 < y < Y\} \setminus C^0(0, \frac{1}{2Yc}), c \geq 1,$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2, \\ \frac{a_1}{c_1} &= -\frac{b^*}{a^*}, \\ (a_1, c_1) &= 1 \textit{ for } a^* \neq 0, \\ c &= 1 \textit{ for } b_0 \textit{ even and } c = 4 \textit{ for } b_0 \textit{ odd, if } a^* = 0. \end{aligned}$$

Proof. Let $\gamma \in \Gamma_1$, $c_1 \neq 0$, $r_1 = \frac{a_1}{c_1} \neq -\frac{d^*}{c^*}$. Since $(c^*, d^*) = 1$, $\min(c^* a_1 + d^* c_1)^2 = 1$, $\max \frac{1}{2Y} \frac{1}{(c^* a_1 + d^* c_1)^2} = \frac{1}{2Y}$. Alsom $\frac{a_2}{c_2} = -\frac{d^*}{c^*}$ in the cases 1,3,4, so $c_2 = \pm c^*$ and $H(Y\frac{c_2^2}{c^{*2}}) = H(Y)$. Clearly, $c = (c^* a_1 + d^* c_1)^2$ with $\frac{a_1}{c_1} = -\frac{b^*}{a^*}$ for $a^* \neq 0$. In case 3 $a^* = 0$, $c = c^{*2} = 1, 4$.

In case 2 $c^* = 1$, $d^* = 0$. If $\frac{a_1}{d_1} \neq -\frac{d^*}{c^*} = 0$, then $a_1 \neq 0$. For $a_1 = 0$ we have $c = c^* a_1 + d^* c_1 = 1$. Now c_2 is determined by $\frac{a_2}{c_2} = -\frac{d^*}{c^*} = 0$, so $a_2 = 0$. Then $c_2 = 1$, $b_2 = -1$, so $c_2 = c^* = 1$, and $H(Y\frac{c_2^2}{c^{*2}}) = H(Y)$. In case 5 g^* is translation by $\frac{-b_0}{2}$, and the Lemma is proved. $\square$

In order to estimate the analogous integrals from section 4 we consider now $\gamma_0 p$, $\gamma_0 \in \Gamma_0(N)$, $p = p_1, p_2, p_3, p_4$, $\Gamma_{\gamma_0 p} = \{e\}$. Let

$$\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ Nc_0 & d_0 \end{pmatrix} \in \Gamma_0(N),$$

we shall find

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q})$$

such that

$$g(\gamma_0 p)g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(I) $\gamma_0 p_1$ with

$$p_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & q\frac{1}{sqrt2} \\ 0 & \sqrt{2} \end{pmatrix} \quad , \quad q = 0, 1$$

By Theorem 2, $\Gamma_{\gamma_0 p_1} = \{e\}$ iff $\sqrt{2}\,\mathrm{tr}(\gamma_0 p_1) = 3$. Then $\lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$, $\lambda = \sqrt{2}$ or $\frac{1}{\sqrt{2}}$. We have

$$\gamma_0 p_1 = \begin{pmatrix} \frac{a_0}{\sqrt{2}} & a_0\frac{q}{\sqrt{2}} + b_0\sqrt{2} \\ \frac{Nc_0}{\sqrt{2}} & Nc_0\frac{q}{\sqrt{2}} + d_0\sqrt{2} \end{pmatrix} \tag{A.8}$$

With $\lambda = \sqrt{2}$ we get the dependent equations in c and d

$$\begin{aligned} (a_0 - 1)c + Nc_0 d &= 0 \\ (a_0 q + 2b_0)c + (Nc_0 q + 2d_0 - 1)d &= 0 \\ a_0 + Nc_0 q + 2d_0 &= 3 \end{aligned} \tag{A.9}$$

(II)

$$\gamma_0 p_2 = \begin{pmatrix} \sqrt{2}a_0 + \frac{1}{\sqrt{2}}qb_0 & \frac{1}{\sqrt{2}}b_0 \\ \sqrt{2}Nc_0 + \frac{1}{\sqrt{2}}qd_0 & \frac{1}{\sqrt{2}}d_0 \end{pmatrix} \quad , \quad q = 0, -N$$

By Theorem 2, $\Gamma_{\gamma_0 p_2} = \{e\}$ iff $\mathrm{tr}(\gamma_0 p_2) = \frac{3}{\sqrt{2}}$. With $\lambda = \sqrt{2}$ solution of $\lambda + \lambda^{-1} = \frac{3}{\sqrt{2}}$ we get the dependent equations in c and d

$$\begin{aligned} b_0 c + (d_0 - 1)d &= 0 \\ (2 - d_0)c + (2Nc_0 + qd_0)d &= 0 \quad , \quad q = 0, -N \\ 2a_0 + qb_0 + d_0 &= 3 \end{aligned} \tag{A.10}$$

(III)

$$\gamma_0 p_3 = \begin{pmatrix} \frac{1}{2}a_0 & \frac{1}{2}qa_0 + 2b_0 \\ \frac{1}{2}Nc_0 & \frac{1}{2}qNc_0 + 2d_0 \end{pmatrix} \quad , \quad q = 0,1,2,3$$

By Theorem 2, $\mathrm{tr}(\gamma_0 p_3) = \frac{5}{2}$ iff $\Gamma_{\gamma_0 p_3} = \{e\}$. With $\lambda = 2$ solution of $\lambda + \lambda^{-1} = \frac{5}{2}$ we obtain the dependent equations for c and d

$$\begin{aligned} (a_0 - 1)c + Nc_0 d &= 0 \\ (qa_0 + 4b_0)c + (4 - a_0)d &= 0 \quad , \quad q = 0,1,2,3 \\ a_0 + qNc_0 + 4d_0 &= 5 \end{aligned} \tag{A.11}$$

(IV)

$$\gamma_0 p_4 = \begin{pmatrix} 2a_0 + qb_0 & \frac{1}{2}b_0 \\ 2Nc_0 + qd_0 & \frac{1}{2}d_0 \end{pmatrix} \quad , \quad q = 0, -\frac{N}{2}, -N, -\frac{3N}{2}$$

By Theorem 2, $\mathrm{tr}(\gamma_0 p_4) = \frac{5}{2}$ iff $\Gamma_{\gamma_0 p_4} = \{e\}$. With $\lambda = 2$ solution of $\lambda + \lambda^{-1} = \frac{5}{2}$ we obtain the dependent equations

$$\begin{aligned} b_0 c + (d_0 - 1)d &= 0 \\ (2a_0 + qb_0 - \tfrac{1}{2})c + (2Nc_0 + qd_0)d &= 0 \\ 4a_0 + 2qb_0 + d_0 &= 5 \end{aligned} \tag{A.12}$$

In all the cases I–IV we solve the equations (A.9)–(A.12) in the same way as we solved the second set of equations (A.7). Similarly to Lemma A.3 we obtain from Lemma A.2

Lemma A.4. *Let $\gamma_0 \in \Gamma_0(N)$ and $\sqrt{2}\,\mathrm{tr}(\gamma_0 p_i) = 3$ for $i = 1,2$, $2\,\mathrm{tr}(\gamma_0 p_i) = 5$ for $i = 3,4$. Then $\Gamma_{\gamma p_i} = \{e\}$ and $F_Y^{\gamma_0 p_i} = F_Y^0$. Let g^* be defined as in I–IV such that*

$$g^*(\gamma_0 p_i) g^{*-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix},$$

where $\lambda_i = \sqrt{2}$ for $i = 1,2$ and $\lambda_i = 2$ for $i = 3,4$. Then

$$\int_{F_Y^{\gamma_0 p_i}} k_\varepsilon(u(z, \gamma_0 p_i z)) = \int_{g^* F_Y^0} k_\varepsilon(u(z, \lambda_i^2 z)) d\mu(z)$$

where

$$\{z \in H \mid \tfrac{1}{Y} < \mathrm{Im}\, z < Y\} \subset g^* F_Y^0 \subset \{z \in H \mid 0 < y < Y\} \setminus C^0(0, \tfrac{1}{2Yc}) \quad , \quad c \geq 1$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2 \\ \frac{a_1}{c_1} &= -\frac{b^*}{a^*} \text{ for } a^* \neq 0 \\ c &\geq 1 \text{ for } a^* = 0 \end{aligned}$$

We finally consider $\gamma_0 p_i J,\ \gamma_0 \in \Gamma_0(N),\ p = p_1, p_2, p_3, p_4,\ J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $\Gamma_{\gamma_0 p_i J} = \{e\}$. By Theorem 3 this holds iff $\sqrt{2}\,\mathrm{tr}(\gamma_0 p_i J) = 1$ for $i = 1, 2$ and $2\,\mathrm{tr}(\gamma_0 p_i J) = 3$ for $i = 3, 4$. We find

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that

$$g(\gamma_0 p_i J) g^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

(I)

$$\gamma_0 p_1 J = \begin{pmatrix} \frac{a_0}{\sqrt{2}} & -(a_0 \frac{q}{\sqrt{2}} + b_0 q) \\ \frac{N c_0}{\sqrt{2}} & -(-N c_0 \frac{q}{\sqrt{2}} + d_0 \sqrt{2}) \end{pmatrix} \quad , \quad q = 0, 1.$$

With the solution $\lambda = \sqrt{2}$ of $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ we get the dependent equations fpr c and d

$$\begin{aligned} (a_0 + 1)c + N c_0 d &= 0 \\ -(a_0 q + 2b_0 + 1)c - a_0 d &= 0 \\ a_0 - (N c_0 q + 2d_0) &= 1 \end{aligned} \tag{A.13}$$

(II)

$$\gamma_0 p_2 J = \begin{pmatrix} \sqrt{2} a_0 + \frac{1}{\sqrt{2}} \gamma_0 b_0 & -\frac{1}{\sqrt{2}} b_0 \\ \sqrt{2} N c_0 + \frac{1}{\sqrt{2}} q d_0 & -\frac{1}{\sqrt{2}} d_0 \end{pmatrix} \quad , \quad q = 0, -N \quad , \quad \mathrm{tr}(\gamma_0 p_2 J) = \frac{1}{\sqrt{2}}$$

With $\lambda = \sqrt{2}$ solution of $\lambda - \lambda^{-1} = \frac{1}{\sqrt{2}}$ we get the two equations in c and d

$$\begin{aligned} -b_0 c - (d_0 + 1)d &= 0 \\ (d_0 + 2)c + (2N c_0 + q d_0)d &= 0 \\ 2a_0 + \gamma_0 b - d_0 &= 1 \end{aligned} \tag{A.14}$$

(III)

$$\gamma_0 p_3 J = \begin{pmatrix} \frac{1}{2}a_0 & -(\frac{1}{2}qa_0 + 2b_0) \\ \frac{1}{2}Nc_0 & -(\frac{1}{2}qNc_0 + 2d_0) \end{pmatrix} \quad , \quad q = 0,1,2,3 \quad , \quad \mathrm{tr}(\gamma_0 p_3 J) = \frac{3}{2}$$

With $\lambda = 2$ solution of $\lambda - \lambda^{-1} = \frac{3}{2}$ we get the equations in c and d

$$\begin{aligned} (a_0+1)c + Nc_0 d &= 0 \\ -(qa_0+4b_0)c - (a_0-2)d &= 0 \\ a_0 - (qNc_0 + 4d_0) &= 3 \end{aligned} \tag{A.15}$$

(IV)

$$\gamma_0 p_4 J = \begin{pmatrix} 2a_0 + qb_0 & -\frac{1}{2}b_0 \\ 2Nc_0 + qd_0 & -\frac{1}{2}d_0 \end{pmatrix} \quad , \quad q = 0, -\frac{N}{2}, -N, -\frac{3N}{2} \quad , \quad \mathrm{tr}(\gamma_0 p_4 J) = \frac{3}{2}$$

With $\lambda = 2$ solution of $\lambda - \lambda^{-1} = \frac{3}{2}$ we get the equations in c and d

$$\begin{aligned} b_0 c + (-d_0+1)d &= 0 \\ (d_0+4)c + (4Nc_0 + 2qd_0)d &= 0 \\ 4a_0 + 2qb_0 - d_0 &= 3 \end{aligned} \tag{A.16}$$

In all cases I–IV we solve the equations (A.13)–(A.16) in the same way as the equations (A.7) and (A.9)–(A.12). Then we obtain from Lemma A.2

Lemma A.5. *Let* $\gamma_0 \in \Gamma_0(N)$ *and* $\sqrt{2}\,\mathrm{tr}(\gamma_0 p_i J) = 1$ *for* $i = 1,2$, $2\,\mathrm{tr}(\gamma_0 p_i J) = 3$ *for* $i = 3,4$. *Then* $\Gamma_{\gamma_0 p_i J} = \{e\}$, *and* $F_Y^{\gamma_0 p_i J} = F_Y^0$. *Let* $g^* \in \mathrm{PSL}(2,\mathbb{Q})$ *be defined as in I–IV such that*

$$g^*(\gamma_0 p_i J) g^{*-1} = \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i^{-1} \end{pmatrix},$$

where $\lambda_i = \sqrt{2}$ *for* $i = 1,2$ *and* $\lambda_i = 2$ *for* $i = 3,4$. *Then*

$$\int_{F_Y^{\gamma_0 p_i J}} k_\varepsilon(u(z, \gamma_0 p_i J z))d\mu(z) = \int_{g^* F_Y^0} k_\varepsilon(u(z, \lambda_i^2 z))d\mu(z)$$

where

$$\{z \mid \tfrac{1}{Y} < y < Y\} \subset g^* F_Y^0 \subset \{z \mid 0 < y < Y\} \setminus C^0(0, \tfrac{1}{2Yc}) \quad , \quad c \geq 1$$

and

$$\begin{aligned} c &= (c^* a_1 + d^* c_1)^2 \\ \frac{a_1}{c_1} &= -\frac{b^*}{a^*} \text{ for } a^* \neq 0 \\ c &\geq 1 \text{ for } a^* = 0 \end{aligned}$$

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