Lévy-based Tempo-Spatial Modelling; with Applications to Turbulence

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Abstract

This paper discusses certain types of tempo-spatial models constructed from Lévy bases. The dynamics are described by a field of stochastic processes $X = \{X_t(\sigma)\}$, on a set $\mathcal{S}$ of sites σ, defined as integrals

$$X_t(\sigma) = \int_{-\infty}^{t} \int_{\mathcal{S}} f_t(\rho, s; \sigma) Z(\mathrm{d}\rho \times \mathrm{d}s)$$

where Z denotes a Lévy basis. The integrands f are deterministic functions of the form $f_t(\rho, s; \sigma) = h_t(\rho, s; \sigma) \mathbf{1}_{A_t(\sigma)}(\rho, \sigma)$ where $h_t(\rho, s; \sigma)$ is of a special kind and $A_t(\sigma)$ is a subset of $\mathcal{S} \times \mathbb{R}_{\leq t}$.

We first consider OU (Ornstein-Uhlenbeck) fields $X_t(\sigma)$ representing several extensions of the concept of OU processes (processes of Ornstein-Uhlenbeck type), with the main focus on the potential of $X_t(\sigma)$ for dynamic modelling. Applications to dynamical spatial processes of Cox type are briefly indicated.

The second part of the paper discusses the modelling of tempo-spatial correlations of SI (stochastic intermittency) fields of the form

$$Y_t(\sigma) = \exp\{X_t(\sigma)\}.$$

This form allows for explicit calculation of expectations $\mathrm{E}\{Y_{t_1}(\sigma_1) \cdots Y_{t_n}(\sigma_n)\}$, which we use to characterise correlations. SI fields can be thought of as a dynamical, continuous and homogeneous generalisation of turbulent cascades. In this respect we construct an SI field with tempo-spatial scaling behaviour that accords with the energy dissipation observed in turbulent flows. Some parallels of this construction are also briefly sketched.

Keywords: Cox processes; Fully developed turbulence; Lévy basis; Ornstein-Uhlenbeck type processes; Scaling; Stochastic intermittency; Stochastic volatility; Spatial-temporal modelling.

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Contents

1 Introduction

The concept of Lévy bases constitutes a rich source for tempo-spatial modelling, such bases providing the innovations in, for instance, Ornstein-Uhlenbeck related models and models for scaling behaviour in turbulence. The present paper indicates some of the potential in this. In particular, relations to recent major advances in the modelling of turbulence from the viewpoint of multiplicative cascades will be discussed.

We shall be thinking in terms of a set of sites $\mathcal{S}$, with points σ, where things develop in time t. The dynamic developments will be described by a field of stochastic processes $X = \{X_t(\sigma)\}$ on $\mathcal{S}$, which will be defined by integrals

$$X_t(\sigma) = \int_{-\infty}^{t} \int_{\mathcal{S}} f_t(\rho, s; \sigma) Z(\mathrm{d}\rho \times \mathrm{d}s) \tag{1}$$

where Z denotes a Lévy basis, that is an infinitely divisible and independently scattered random measure on $\mathcal{R} = \mathcal{S} \times \mathbb{R}$. For concreteness, in the present paper the integrands f are taken to be deterministic functions and $\mathcal{S}$ will be a Borel subset of $\mathbb{R}^d$. To each point $\xi = (\sigma, t)$ in $\mathcal{R}$ will be associated a set $A_t(\sigma)$ in $\mathcal{B}(\mathcal{R})$, the Borel σ-algebra of $\mathcal{R}$, and we consider functions f of the form

$$f_t(\rho, s; \sigma) = h_t(\rho, s; \sigma) \mathbf{1}_{A_t(\sigma)}(\rho, s)$$

with h being subject to various restrictions. For instance $h_t(\rho, s; \sigma) = \exp(-\lambda(\sigma)(t - s))$ or $h_t(\rho, s; \sigma) = h(|\rho - \sigma|)$ where $||$ denotes distance in $\mathbb{R}^d$. We shall refer to the $A_t(\sigma)$ as *ambit sets*.

The focus will be on two cases, that we call OU fields and SI fields, respectively. In both cases the sets $A_t(\sigma)$ relate to past events in the sense that $A_t(\sigma) \cap (\mathbb{R}^d \times (t, \infty)) = \emptyset$. For OU fields a prototypical example of $A_t(\sigma)$ is shown in Figure 1, while Figure 2 indicates a prototype $A_t(\sigma)$ for SI fields.

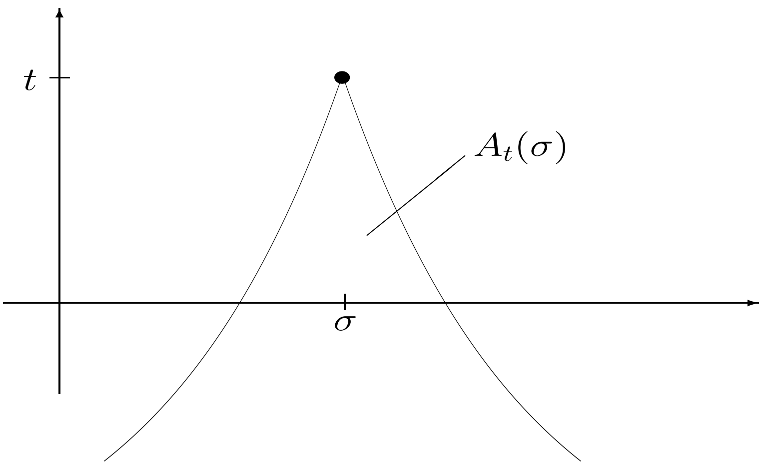

Figure 1: Illustration of the ambit set $A_t(\sigma)$ associated to $X_t(\sigma)$.

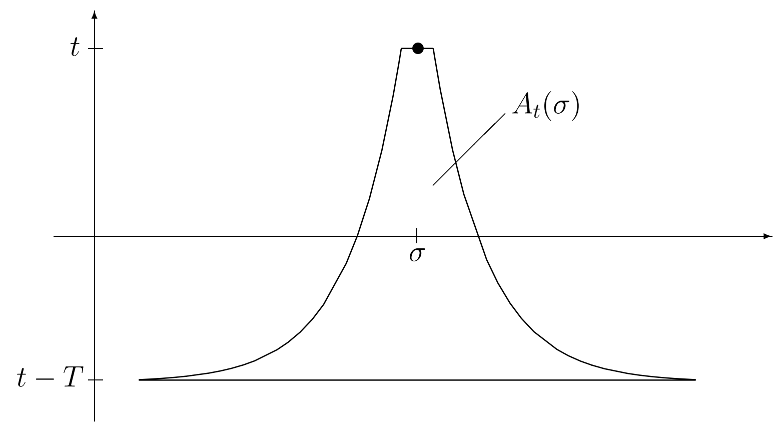

Figure 2: Illustration of the ambit set $A_t(\sigma)$ associated to $Y_t(\sigma)$.

The OU fields constitute extensions of the concept of OU processes (processes of Ornstein-Uhlenbeck type). For a general discussion of such processes see Sato (1999). In a number of recent papers in mathematical finance (Barndorff-Nielsen and Shephard (2001a, 2001b), Nicolato and Venardos (2002); see also Schoutens (2003) and Barndorff-Nielsen and Shephard (2004)), positive one-dimensional OU processes have figured prominently as models of the stochastic volatility of financial assets (for some related work on the multivariate case, see Barndorff-Nielsen, Pedersen and Sato (2001)). The substantial similarities between the dynamics of financial markets and of turbulent fluids (cf. for instance, Barndorff-Nielsen (1998a, 2002), Ghashgaie et al. (1996)), motivates an interest in exploring the possibilities for realistic modelling of intermittency, the turbulence analogue of stochastic volatility, using ideas related to OU processes. These ideas may also, as we shall indicate, be useful in other areas of spatial-temporal modelling, such as the modelling of intensity processes for use in dynamical spatial processes of Cox type.

The motivation for the introduction of SI (Stochastic Intermittency) fields comes from recent advances in the modelling of turbulent cascades from the viewpoint of multiplicative processes (Schmiegel, Eggers and Greiner (2001)). (In this connection see also Greiner (2002).)

The paper presents, in outline, a number of results on OU and SI fields, leaving many mathematical details to fuller expositions elsewhere. Section 2 provides various background material for Lévy based modelling and for turbulence. Sections 3 and 4 consider OU and SI fields, respectively. The OU fields are of the linear type $X_t(\sigma)$ (as in (1)) while the SI fields are of multiplicative type, i.e. exponentiated versions $Y_t(\sigma) = \exp\{X_t(\sigma)\}$. Integrals of $X_t(\sigma)$ and of $Y_t(\sigma)$ with respect to σ are of some particular interest.[1] The main focus of the paper is on scaling behaviour of the energy dissipation ($\frac{4}{5}$ law and modifications), leaving aside the more commonly considered scaling features of the velocity field ($\frac{n}{3}$ laws and their modifications) and the power spectrum ($\frac{5}{3}$ law and modifications) for discussion

[1]The latter type of integrals brings us into the realm of what could be termed Lévy-based exponential functionals. In this connection, see Carmona, Petit and Yor (2001).

in a more extensive study (Barndorff-Nielsen and Schmiegel (2003))[2]. In Section 4.3 we consider dynamic models for fields of energy dissipation with special focus on the temporal bivariate correlators. Sections 4.4 and 4.5 briefly discuss modelling of further aspects of SI fields. Again the focus is on temporal modelling, but the results are, in many cases, translatable to spatial relations. Section 5 concludes.

2 Background

The present section summarises some well-known results from the theory of infinite divisibility and Lévy processes and from turbulence theory. This serves as background material for the discussion in the rest of the paper.

We shall use the following notation for the log characteristic function and the log Laplace transform of a random variable X

$$\mathrm{C}\{\zeta \ddagger X\} = \log \mathrm{E}\{e^{i\zeta X}\} \quad \text{and} \quad \mathrm{K}\{\theta \ddagger X\} = \log \mathrm{E}\{e^{\theta X}\}$$

and, to distinguish, we will refer to these as the cumulant function and the kumulant function, respectively. Similar notation applies for vector variates and conditional laws. Thus, for instance, $\mathrm{C}\{\zeta \ddagger X|Y\}$ is the conditional cumulant function of X given Y.

2.1 Lévy bases

By $\mathcal{B} = \mathcal{B}(\mathcal{R})$ we denote the Borel σ-algebra of $\mathcal{R} = \mathcal{S} \times \mathbb{R}$, and $\mathcal{B}_b$ will stand for the class of bounded elements in $\mathcal{B}$. Let $Z = \{Z(A); A \in \mathcal{C}\}$, where $\mathcal{C} = \mathcal{B}$ or $\mathcal{B}_b$, be an independently scattered random measure on $\mathcal{R}$.

Suppose that Z is infinitely divisible in the sense that for each $A \in \mathcal{C}$, $Z(A)$ is an infinitely divisible random variable whose cumulant function can be written as

$$\mathrm{C}\{\zeta \ddagger Z(A)\} = i\zeta a(A) - \frac{1}{2}\zeta^2 b(A) + \int_{\mathbb{R}} \{e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x)\}\nu(\mathrm{d}x, A) \tag{2}$$

where a is a signed measure on $\mathcal{B}$, b is a positive measure on $\mathcal{B}$, and $\nu(\mathrm{d}x, A)$ is (for fixed A) a Lévy measure on $\mathbb{R}$ and a measure on $\mathcal{B}$ for fixed $\mathrm{d}x$. Then Z is called a *Lévy basis* with characteristics (a, b, ν). We shall refer to ν as a *generalized Lévy measure*. In this paper our interest is only in the cases where there is no Gaussian part so b will be 0 throughout. Then the *control measure* of Z is

$$\omega(\mathrm{d}\xi) = |a|(\mathrm{d}\xi) + \int_{\mathbb{R}} \left(x^2 \wedge 1\right) \nu(\mathrm{d}x, \mathrm{d}\xi)$$

If the Lévy basis Z on $\mathcal{R}$ is such that $Z(A)$ is Poisson distributed for all A then we call Z a *Poisson basis*. In this case the generalised Lévy measure is of the form $\nu(\mathrm{d}x, A) =$

[2]It is, however, appropriate here to make reference to the work of Cağlar and Çinlar (2001), that introduces and studies velocity field processes of shot noise type, with particular regard to the modelling of medium scale structures in oceanic flows.

$\Lambda(A)\delta_1(\mathrm{d}x)$ for some measure Λ on $\mathcal{B}$ and where δ_1 is the Dirac measure at 1. We shall, however, mostly be interested in cases where ν is a diffuse measure on $\mathbb{R}\times\mathcal{R}$.

Heuristically it is useful to reexpress (2) in infinitesimal form, at $\xi \in \mathcal{R}$, as

$$\mathrm{C}\{\zeta \ddagger Z(\mathrm{d}\xi)\} = ia(\mathrm{d}\xi) - \frac{1}{2}\zeta^2 b(\mathrm{d}\xi) + \int_{\mathbb{R}} \left\{e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x)\right\} \nu(\mathrm{d}x;\mathrm{d}\xi). \tag{3}$$

There is no essential loss of generality in assuming that the measure ν in (3) factorizes as

$$\nu(\mathrm{d}x;\mathrm{d}\xi) = U(\mathrm{d}x;\xi)\mu(\mathrm{d}\xi) \tag{4}$$

for some measure μ on $\mathcal{R}$ and with $U(\mathrm{d}x;\xi)$ being a Lévy measure for each fixed ξ.

In most of the cases to be considered, the Lévy measure $U(\cdot;\xi)$ is, for each ξ, absolutely continuous with respect to Lebesgue measure on $\mathcal{R}$, with a density $u(\mathrm{d}x;\xi)$.

We may think of

$$\kappa(\zeta;\xi) = \int_{\mathbb{R}} \{e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x)\} U(\mathrm{d}x;\xi)$$

as the cumulant function of a random variable $Z'(\xi)$, say, having cumulant function

$$\kappa(\zeta;\xi) = \mathrm{C}\{\zeta \ddagger Z'(\xi)\};$$

and if $a(\mathrm{d}\xi) \equiv 0$ then

$$\mathrm{C}\{\zeta \ddagger Z(\mathrm{d}\xi)\} = \mathrm{C}\{\zeta \ddagger Z'(\xi)\}\mu(\mathrm{d}\xi) = \kappa(\zeta;\xi)\mu(\mathrm{d}\xi).$$

In many cases of applied interest, $U(\cdot;\xi)$ does not depend on ξ. Then we say that the Lévy basis is *factorisable* and we write $U(\mathrm{d}x)$, etc. If, moreover, μ is (proportional to) Lebesgue measure then Z is called *homogeneous.* When this is the case we take Z' to mean a random variable with Lévy measure U.

Extension of the above to multivariate Lévy bases $Z = (Z_1, ..., Z_m)$ is immediate, the infinitesimal Lévy-Khintchine representation taking the form

$$\mathrm{C}\{\zeta \ddagger Z(\mathrm{d}\xi)\} = i\langle a(\mathrm{d}\xi), \zeta\rangle - \frac{1}{2}\langle \zeta b(\mathrm{d}\xi), \zeta\rangle + \int \left\{e^{i\langle\zeta,x\rangle} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x)\right\} \nu(\mathrm{d}x;\mathrm{d}\xi) \tag{5}$$

where now $\zeta = (\zeta_1, ..., \zeta_m)$, a is an m-dimensional measure and b is an $m \times m$ matrix valued measure.

2.2 Lévy-Ito representation of Lévy bases

Let Z be an independently scattered random measure on $\mathbb{R}^k$ with characteristic triplet $(a, 0, \nu)$ and nonatomic control measure ω. Then (for a proof see Pedersen (2003)) there exists a Poisson basis N on $\mathbb{R} \times \mathbb{R}^k$ with intensity measure ν such that Z is representable as

$$Z(\mathrm{d}\xi) = a(\mathrm{d}\xi) + \int_{|x|>1} xN(\mathrm{d}x;\mathrm{d}\xi) + \int_{|x|\le 1} x(N-\nu)(\mathrm{d}x;\mathrm{d}\xi). \tag{6}$$

For Z a nonnegative Lévy basis this may be reexpressed as

$$Z(\mathrm{d}\xi) = a_0(\mathrm{d}\xi) + \int_{\mathbb{R}_+} xN(\mathrm{d}x; \mathrm{d}\xi) \tag{7}$$

for some measure a_0 on $\mathbb{R}^k$.

2.3 Integrals with respect to Lévy bases

The integral of a measurable function f on $\mathcal{R}$ with respect to Z will be denoted by $f \bullet Z$. (For the theory of integration with respect to independently scattered random measures see Kallenberg (1989) or Kwapien and Woyczynski (1992). Rüdiger (2003) discusses the general question of stochastic integration with respect to compensated Poisson random measures.)

A key result for many calculations is embodied in the formula

$$\mathrm{C}\{\zeta \ddagger f \bullet Z\} = f \bullet a + \int \kappa(\zeta f(\xi); \xi)\mu(\mathrm{d}\xi). \tag{8}$$

The essential condition for this to hold is that the integral on the right hand side should exist (cf. Barndorff-Nielsen and Thorbjørnsen (2003) and Barndorff-Nielsen and Schmiegel (2003)).

Integrals of nonnegative functions f with respect to a nonnegative Lévy basis with generalised Lévy measure $\nu(dx; d\xi)$ are representable as

$$f \bullet Z = f \bullet a_0 + \int_{\mathbb{R}_+} \int_{\mathcal{S}} f(\xi) xN(\mathrm{d}x; \mathrm{d}\xi) \tag{9}$$

and the kumulant functional of the basis Z is of the form

$$\mathrm{K}\{\theta f \ddagger Z\} = \mathrm{K}\{\theta \ddagger f \bullet Z\} = \theta f \bullet a_0 + \int_{\mathbb{R}_+} \int_{\mathcal{S}} (e^{\theta f(\xi) x} - 1)\nu(\mathrm{d}x; \mathrm{d}\xi) \tag{10}$$

Furthermore, the Lévy measure of $f \bullet Z$ is $U_f = \bar{f} \circ \nu$, the lifting of ν by $\bar{f}$ where $\bar{f}(x, \xi) = xf(\xi)$.

2.4 Examples of Lévy bases

Example 2.1 *TS Lévy basis* When $U(\mathrm{d}x; \xi)$ in (4) is concentrated on $\mathbb{R}_{>0}$ and of the form

$$U(\mathrm{d}x; \xi) = x^{-1-\alpha/2} e^{-\frac{1}{2}\gamma^2(\xi)x} \mathrm{d}x$$

with $0 < \alpha < 2$ and $\gamma(\xi) > 0$, we say that Z constitutes a *tempered stable Lévy basis*. The 'local' random variable $Z'(\xi)$ then has an $\alpha/2$ tempered stable distribution, the inverse Gaussian law occurring for $\alpha = 1$. (For a discussion of tempered stable (*TS*) laws and

Lévy processes, see Barndorff-Nielsen and Shephard (2001c).) In the limiting case $\alpha = 0$ the law of $Z'(\xi)$ is the Gamma distribution $\Gamma(1, \frac{1}{2}\gamma^2(\xi))$.

In case γ does not depend on ξ, the basis Z is factorisable and $Z(A)$ follows a TS law (a Gamma law in case $\alpha = 0$) for all A.

For relevance below we note that there is a natural extension of the family of TS laws to the so-called generalised Gamma (or G) family, see Brix (1999) and references there. □

Example 2.2 *NIG Lévy basis* When $U(\mathrm{d}x;\xi)$ in (4) is of the form

$$U(\mathrm{d}x;\xi) = \pi^{-1}\delta\alpha(\xi)\,|x|^{-1}\,K_1(\alpha(\xi)\,|x|)e^{\beta(\xi)x}$$

(with $\alpha(\xi) > |\beta(\xi)| \geq 0$) we say that Z constitutes a *normal inverse Gaussian Lévy basis.* In case α and β do not depend on ξ, Z is factorisable and $Z(A)$ follows an NIG law for all A. □

2.5 Selfdecomposability and OU processes

A stationary stochastic process $\{X\}_{t\in\mathbb{R}}$ is said to be an OU process if it is representable in the form

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)}\mathrm{d}Z_{\lambda s} \tag{11}$$

where λ is a positive parameter–the rate parameter of the OU process–and Z_t is a Lévy process. In this case the law of X_t is necessarily selfdecomposable and, on the other hand, for any selfdecomposable law D on $\mathbb{R}$ there exists a Lévy process Z–called the background driving Lévy process (BDLP)–such that (11) determines an OU process with X_t being distributed according to D. For the general theory of selfdecomposability and OU processes see Sato (1999). Applications of these concepts to finance and to turbulence are discussed in Barndorff-Nielsen(1998a,b, 2001, 2002) and Barndorff-Nielsen and Shephard (2001a,b); see also Schoutens (2003) and Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen, Jensen and Sørensen (1990, 1993).

2.6 Turbulence

One of the most striking features of turbulent flow is the intermittent behaviour of the velocity field and related to that the intermittency of the energy dissipation (Frisch (1995), Monin and Yaglom (1971)). Intermittency here means that fluctuations around the mean occur in bursts with amplitudes that are clearly non-Gaussian. Furthermore, these intermittent bursts are clustered, indicating their temporal dependencies. These characteristic features of a turbulent flow can also be described in a multifractal formalism where moments of velocity-differences over a given lag and moments of the coarse grained energy dissipation with coarse graining domain of given size display scaling behaviour over a certain range, which is called the inertial range (Meneveau and Sreenivasan (1991), Sreenivasan and Antonia (1997)). A first systematic description of inertial range scaling is due to

Kolmogorov (Kolmogorov (1941a,b)). He used two basic premises about the universality of a turbulent flow for high Reynolds numbers. The first is that when the fluid viscosity ν is small, the average energy-dissipation rate ϵ is independent of ν. The second premise is that small-scale turbulence is homogeneous and isotropic. With that he concluded that velocity increments $\Delta v_r \equiv v(\sigma + r) - v(\sigma)$ for distances r within the inertial range obey the scaling relation

$$\mathrm{E}\{\Delta v_r^n\} \propto (r\epsilon)^{n/3}.$$

Much effort has been devoted to verifying this equation, especially to the spectral equivalent of the case with $n = 2$

$$\Phi(\kappa) \propto \epsilon^{2/3}\kappa^{-5/3}$$

where $\Phi(\kappa)$ is the one-dimensional spectrum of energy. It turned out that the monofractal scaling of velocity increments is violated by experimental observations for orders $n > 3$ (see Frisch (1995) and Sreenivasan and Antonia (1997) for an overview and further references). One reason for this failure is the strong variability of the energy dissipation rate. Already 1962, Obukhov (Obukhov (1962)) and Kolmogorov (Kolmogorov(1962)) suggested to replace ϵ by the locally coarse-grained energy dissipation rate

$$\epsilon_r = \frac{1}{|V|}\int_V \epsilon(\sigma)\mathrm{d}\sigma$$

where $|V| = O(r^3)$ is a volume of linear dimension r. According to this proposition the moments of the locally averaged energy dissipation itself displays a scaling behaviour with non-linear, i.e. multifractal scaling exponents $\tau(n)$

$$\mathrm{E}\{\epsilon_r^n\} \propto |V|^{-\tau(n)}.$$

This scaling relation is assumed to hold within the inertial range (a certain interval for r) for very high Reynolds number flows. Especially for higher order n (and necessarily finite Reynolds numbers) it is difficult to extract the scaling exponents $\tau(n)$.

However in Schmiegel et al. (2001,2002) it is shown that n-point correlators (defined below) also display a scaling behaviour which compared to the coarse grained energy dissipation is much more accurate, even for lower Reynolds numbers. A simple calculation for the one-dimensional case helps to illustrate this. Assume we have $\mathrm{E}\{\epsilon_r^2\} \propto r^{-\tau(2)}$ to hold exactly for some range of r. Then we can differentiate this relation twice and get $\mathrm{E}\{\epsilon(\sigma)\epsilon(\sigma + r)\} \propto r^{-\tau(2)}$ since we have

$$\mathrm{E}\{\epsilon_r^2\} = 2\int_0^r (r - \sigma)\mathrm{E}\{\epsilon(0)\epsilon(\sigma)\}\mathrm{d}\sigma \tag{12}$$

if we assume translational invariance. Thus the scaling of two-point correlations is more fundamental, in the sense that exact scaling of the locally averaged energy dissipation implies scaling of the correlators. But the converse need not be true. This is a very important point and is the reason why we focus on scaling relation for n-point correlators in Section 4.3.

The scaling description of a turbulent flow (in terms of velocity increments or in terms of the energy dissipation) is, as yet, not understood from the basics of the Navier-Stokes dynamics - with one exception, the famous 4/5-law of Kolmogorov (Kolmogorov (1941a,b)) (see also Shiryaev (1999) for some historical background). It is for this reason, that phenomenological models play such an important role in the description of turbulent behaviour (Frisch (1995) and Bohr et al. (1998)). One of the simplest and at the same time most successful types of such models are cascade models which are able to describe the intermittent scaling behaviour of the energy dissipation in a transparent and analytically tractable way. They are also able to reproduce observed multiplier distributions, their correlation effects, cumulants and Markov properties (Schmiegel (2002), Jouault, Greiner and Lipa (2000), Cleve and Greiner (2000), Eggers, Dziekan and Greiner (2001), Schmiegel et al. (2002)). Despite this remarkable success they have two major drawbacks. If they are defined according to a cascade tree-structure they are not translational invariant and result in a discrete spatial resolution. On the other hand if they are defined without referring to a tree-like structure, their correlation structure has to be introduced by hand. In both cases it is hard to give a simple and complete description of a cascade-process. The second drawback consists in their static character. Turbulence is a dynamical phenomenon and clearly asks for a simultaneous temporal and spatial description. In the context of cascading processes it is so far not clear, how dynamics should be incorporated in a parsimonious way. Thus there is clearly need for a dynamical generalisation of the cascade processes while keeping their property of a nested structure of interweaving scales. Along this line SI fields, as discussed in Section 4 below, seem to be a natural way to introduce dynamics in a multiplicative process of interacting scales. As an example we discuss in Section 4.3 scaling relations for n-point correlators. We do not discuss the marginal distribution of the energy dissipation, since some of its features are flow-specific. In view of universality, the scaling exponents $\tau(n)$ of the coarse grained energy dissipation and the scaling exponents of the n-point correlators are the appropriate observables. However, in more detailed applications the general framework of SI fields allows the additional modelling of marginal characteristics.

3 Tempo-spatial processes of OU type

In the present Section we consider extensions of the idea of OU processes to tempo-spatial contexts. We start by discussing the simplest, basic type of such processes. For this the region $\mathcal{S}$ equals the full d-dimensional Euclidean space $\mathbb{R}^d$ and the field of ambit sets is translation invariant, i.e. $A_t(\sigma) = (\sigma, t) + A_0(0)$. We also, for simplicity, assume that the Lévy basis has Lévy measure of the form

$$\nu(\mathrm{d}x, \mathrm{d}(\sigma, t)) = U(\mathrm{d}x)\mathrm{d}\sigma\mathrm{d}t$$

so that Z is homogeneous.

3.1 OU$_\wedge$ processes

Consider a two-index family of subsets of $\mathbb{R}^d$:

$$\{C_s(\sigma) : -\infty < s \leq 0, \sigma \in \mathbb{R}^d\}$$

with $C_0(\sigma) = \{\sigma\}$ and

$$C_s(\sigma) \subset C_{s'}(\sigma) \quad \text{for } 0 > s > s'. \tag{13}$$

In terms of this family we may define translation invariant ambit sets $A_t(\sigma)$ by

$$A_t(\sigma) = \{(\rho, s) : -\infty < s \leq t, \rho \in C_{s-t}(\sigma)\}.$$

For brevity we will write C_s for $C_s(0)$. Thus $C_s(\sigma) = (\sigma, t) + C_s$. Also, for $s > 0$, we let $C_s = C_{-s}$.

We may now, under some mild conditions on the sets C_s, define a random field process X on $\mathcal{S} = \mathbb{R}^d$ by

$$X_t(\sigma) = \int_{-\infty}^{t} e^{-t+s} Z(C_{s-t}(\sigma) \times \mathrm{d}s). \tag{14}$$

We call $X_t(\sigma)$ an *OU$_\wedge$ process*. The expression for $X_t(\sigma)$ may be rewritten as

$$X_t(\sigma) = \int_{-\infty}^{0} e^{u} Z(C_u(\sigma) \times \mathrm{d}_t u)$$

where $\mathrm{d}_t u$ denotes the infinitesimal element $\mathrm{d}u$ placed at point t on the time axis. (We shall similarly use the notation $\mathrm{d}_\sigma \rho$.)

Note also that for fixed t we may think of $Z(C_{s-t}(\sigma) \times \mathrm{d}s)$ as $\mathrm{d}Z^{(\sigma,t)}(s)$, where $Z^{(\sigma,t)}$ is an additive process on $(-\infty, t]$, and then

$$X_t(\sigma) = \int_{-\infty}^{t} e^{-t+s} \mathrm{d}Z^{(\sigma,t)}(s). \tag{15}$$

Theorem 3.1 The OU$_\wedge$ process $\{X_t(\sigma)\}_{t\in\mathbb{R}}$ is stationary and Markovian. For $t \geq 0$, let

$$U_t(\sigma) = e^{-t} \int_{-\infty}^{0} e^{s} Z(C_{s-t}(\sigma) \backslash C_s(\sigma) \times \mathrm{d}s) \tag{16}$$

and

$$V_t(\sigma) = e^{-t} \int_{0}^{t} e^{s} Z(C_{s-t}(\sigma) \times \mathrm{d}s). \tag{17}$$

Then $\{X_t(\sigma)\}_{t\geq 0}$ is decomposable as

$$X_t(\sigma) = e^{-t} X_0(\sigma) + U_t(\sigma) + V_t(\sigma) \tag{18}$$

where $X_0(\sigma)$, $\{U_t(\sigma)\}_{t\geq 0}$ and $\{V_t(\sigma)\}_{t\geq 0}$ are independent. (See Figure 3 for an illustration.)

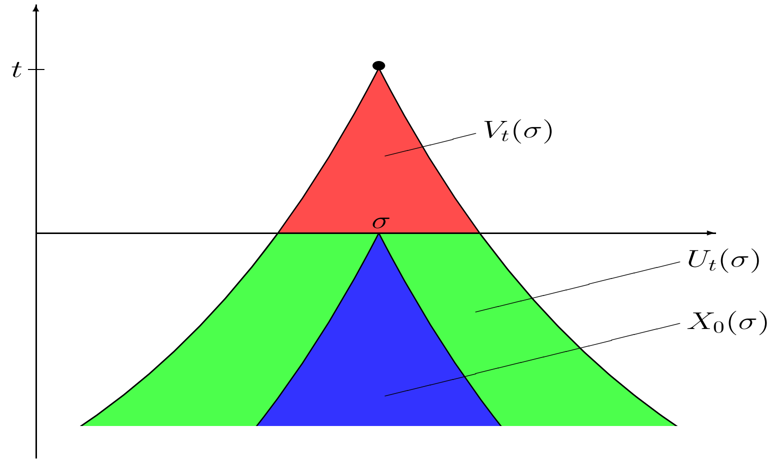

Figure 3: Illustration of the contributions $V_t(\sigma)$ and $U_t(\sigma)$.

□

Remark From the decomposition (18) we find, in particular, that

$$\mathrm{C}\{\zeta \ddagger X_t(\sigma)|X_0(\sigma)\} = e^{-t}X_0(\sigma) + \mathrm{C}\{\zeta \ddagger U_t(\sigma)\} + \mathrm{C}\{\zeta \ddagger V_t(\sigma)\}$$

This shows that the process $\{X_t(\sigma)\}_{t\in\mathbb{R}}$ is affine in the sense that the conditional cumulant function is an affine function of the conditioning state. Markov processes having this type of property have recently been studied in great generality by Duffie, Filipović and Schachermayer (2002). □

Proof By definition,

$$X_0(\sigma) = \int_{-\infty}^{0} e^s Z(C_s(\sigma) \times \mathrm{d}s).$$

Furthermore, using (13) we find, for $t > 0$,

$$\begin{aligned} X_t(\sigma) &= e^{-t}\int_{-\infty}^{0} e^s Z(C_{s-t}(\sigma) \times \mathrm{d}s) + e^{-t}\int_{0}^{t} e^s Z(C_{s-t}(\sigma) \times \mathrm{d}s) \\ &= e^{-t}\int_{-\infty}^{0} e^s Z(C_s(\sigma) \times \mathrm{d}s) + e^{-t}\int_{-\infty}^{0} e^s Z(C_{s-t}(\sigma)\backslash C_s(\sigma) \times \mathrm{d}s) \\ &\quad + e^{-t}\int_{0}^{t} e^s Z(C_{s-t}(\sigma) \times \mathrm{d}s) \end{aligned}$$

i.e.

$$X_t(\sigma) = e^{-t}X_0(\sigma) + U_t(\sigma) + V_t(\sigma).$$

Since Z is independently scattered, $\{Z(C_s(\sigma) \times \mathrm{d}s) : s \leq 0\}$ is independent of

$$\{Z(C_{s-t}(\sigma)\backslash C_s(\sigma) \times \mathrm{d}s) : s \leq 0\}.$$

This implies that $X_0(\sigma)$, $\{U_t(\sigma)\}_{t\in\mathbb{R}}$ and $\{V_t(\sigma)\}_{t\in\mathbb{R}}$ are independent and that $X_t(\sigma)$ is Markov. The stationarity of $\{X_t(\sigma)\}_{t\in\mathbb{R}}$ follows immediately from the homogeneity of Z and the translation invariance of the ambit sets. $\square$

Since, by convention, $C_s(\sigma) = C_{-s}(\sigma)$ we have that

$$U_t(\sigma) \stackrel{law}{=} \int_t^\infty e^{-u} Z(C_u(\sigma)\backslash C_{u-t}(\sigma) \times \mathrm{d}u)$$

or, equivalently,

$$U_t(\sigma) \stackrel{law}{=} e^{-t} Q_t(\sigma)$$

where

$$Q_t(\sigma) = \int_0^\infty e^{-v} Z(C_{v+t}(\sigma)\backslash C_v(\sigma) \times \mathrm{d}v)$$

which is an additive process [3]. Furthermore,

$$V_t(\sigma) \stackrel{law}{=} V_t^{\uparrow}(\sigma)$$

where

$$V_t^{\uparrow}(\sigma) = \int_0^t e^{-v} Z(C_v(\sigma) \times \mathrm{d}v)$$

which is also an additive process.

Theorem 3.2 The process $\{X_t^{\updownarrow}(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}$ defined by

$$X_t^{\updownarrow}(\sigma) = e^{-t} X_0(\sigma) + e^{-t} Q_t(\sigma) + V_t^{\uparrow}(\sigma) \tag{19}$$

behaves in law exactly as $\{X_t(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}$. Here $Q_t(\sigma)$ and $V_t^{\uparrow}(\sigma)$ are additive processes and

$$\mathcal{L}\left(X_0(\sigma), \{U_t(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}, \{V_t(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}\right) = \mathcal{L}\left(X_0(\sigma), \{e^{-t}Q_t(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}, \{V_t^{\uparrow}(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}\right). \tag{20}$$

$\square$

From (19) it follows that $X_t^{\updownarrow}(\sigma)$ solves the stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t^{\updownarrow}(\sigma) &= -X_t^{\updownarrow}(\sigma)\mathrm{d}t + \mathrm{d}V_t^{\uparrow}(\sigma) + e^{-t}V_t^{\uparrow}(\sigma)\mathrm{d}t \\ &= -X_t^{\updownarrow}(\sigma)\mathrm{d}t + e^{-t}\left(V_t^{\uparrow}(\sigma)\mathrm{d}t + Z(C_t(\sigma) \times \mathrm{d}t)\right) \end{aligned} \tag{21}$$

and the quadratic variation process of $\{X_t^{\updownarrow}(\sigma)\}_{t\in\mathbb{R}_{\geq 0}}$ is consequently given by

$$\mathrm{d}[X^{\updownarrow}(\sigma)]_t = e^{-2t} Z^2(C_t(\sigma) \times \mathrm{d}t). \tag{22}$$

[3]We recall that an additive process is a process with independent increments. For an extension of the concept of additivity to higher dimensions see Pedersen (2003).

As a step towards modelling various types of timewise behaviour of fields we next consider time dilations of (14) thereby obtaining the more general form

$$X_t(\sigma) = \int_{-\infty}^{t} e^{-\lambda(\sigma)(t-s)} Z(C_{s-t}(\sigma) \times \lambda(\sigma)\mathrm{d}s). \tag{23}$$

Example 3.1 A particularly simple case occurs when $\mathcal{S} = \mathbb{R}$, $\lambda = \lambda(\sigma)$ does not depend on σ and $C_s(\sigma) = \sigma + [-g(s), g(s)]$ for some nonnegative decreasing function g on $(-\infty, 0]$. Then

$$X_t(\sigma) = e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} Z([-g(s-t)+\sigma, g(s-t)+\sigma] \times \lambda \mathrm{d}s). \tag{24}$$

Expressions for joint cumulants of $X_t(\sigma)$ considered as a random field on $\mathbb{R}\times\mathbb{R}$ are easily derived from formula (24).

In partial extension of this setting, it is some interest to take $g(s) = c|s|$, for some constant $c > 0$, and define $X_t(\sigma)$ by

$$X_t(\sigma) = \int_{-\infty}^{t} \int_{-c|s-t|+\sigma}^{c|s-t|+\sigma} e^{-\lambda|(\sigma,t),(\rho,s)|} Z(\mathrm{d}\rho \times \lambda \mathrm{d}s)$$

where $||$ indicates Euclidean distance. □

3.2 intOU$_\wedge$ and supOU$_\wedge$ processes

For any OU$_\wedge$ process $\{X_t(\sigma)\}_{t\in\mathbb{R}}$, we let

$$X_t^*(\sigma) = \int_0^t X_s(\sigma)\mathrm{d}s$$

and refer to this as an *intOU$_\wedge$ process*.

In mathematical finance the analogous concept of intOU processes occurs centrally in descriptions of the key concept of integrated volatility. See, for instance, Barndorff-Nielsen and Shephard (2003,2004). Another type of application is indicated in the following subsection.

To throw light on the nature of $X_t^*(\sigma)$ we shall use the representation (19) of the process $X_t^{\updownarrow}(\sigma)$ that is equivalent to $X_t(\sigma)$. Having in mind that $Q_t(\sigma)$ and $V_t^{\uparrow}(\sigma)$ are additive processes we find (in obvious notation)

$$\begin{aligned} X_t^{\updownarrow *}(\sigma) = \int_0^t X_s^{\updownarrow}(\sigma)\mathrm{d}s &= \left(1-e^{-t}\right) X_0(\sigma) + \int_0^t e^{-s} Q_s(\sigma)\mathrm{d}s + \int_0^t V_s^{\uparrow}(\sigma)\mathrm{d}s \\ &= \left(1-e^{-t}\right) X_0(\sigma) - e^{-t} Q_t(\sigma) + \int_0^t e^{-s}\mathrm{d}Q_s(\sigma) + V_t^{\uparrow *}(\sigma) \\ &= X_0^{\updownarrow}(\sigma) - X_t^{\updownarrow}(\sigma) + \int_0^t e^{-s}\mathrm{d}Q_s(\sigma) + V_t^{\uparrow}(\sigma) + V_t^{\uparrow *}(\sigma). \end{aligned} \tag{25}$$

From this rewrite we note that the joint process $(X_t^{\updownarrow}(\sigma), X_t^{\updownarrow *}(\sigma))$ is affine, and thus the same holds for $(X_t(\sigma), X_t^*(\sigma))$.

Note also that

$$\int_0^t e^{-s} Q_s(\sigma) \mathrm{d}s = \int_0^t (e^{-s} - e^{-t}) \mathrm{d}Q_s(\sigma)$$

and

$$V_t^{\uparrow *}(\sigma) = \int_0^t (t-s) \mathrm{d}V_s^{\uparrow}(\sigma)$$

so that

$$X_\sigma^{\updownarrow *}(\sigma) = (1 - e^{-t}) X_0(\sigma) + \int_0^t (e^{-s} - e^{-t}) \mathrm{d}Q_s(\sigma) + \int_0^t (t-s) \mathrm{d}V_s(\sigma).$$

It follows, in particular, that the conditional cumulant transform of $X_t^*(\sigma)$ given $X_0(\sigma)$, which is of importance for prediction, can be expressed as

$$\begin{aligned} C\left\{\zeta \ddagger X_t^*(\sigma) | X_0(\sigma)\right\} &= (1 - e^{-t}) X_0(\sigma) + \int_0^t C\left\{(e^{-s} - e^{-t})\zeta \ddagger Z'\right\} q(s) ds \\ &\quad + \int_0^t C\left\{(t-s)\zeta \ddagger Z'\right\} |C_s(\sigma)| ds \end{aligned}$$

where $q(s) = \int_0^\infty e^{-v} \frac{\mathrm{d}|C_{s+v}(\sigma)|}{\mathrm{d}s} \mathrm{d}v$ (assuming differentiability of $|C_{s+v}|$).

Furthermore, supposing for simplicity that the Lévy basis Z is nonnegative, it follows from (25) that for large t the value of $X_t^*(\sigma)$ is largely determined by $V_t^{\star}(\sigma)$.

By superposition of independent $\mathrm{OU}_\wedge$ processes with different rate functions $\lambda(\sigma)$ a considerable range of dependence structures can be introduced. These processes are called $\mathrm{supOU}_\wedge$ processes. We have in mind particularly time-wise dependencies. A similar approach has been followed in finance, see Barndorff-Nielsen and Shephard (2001b,2004). The construction is via an additional Lévy basis, cf. Barndorff-Nielsen (2001).

3.3 Applications to Cox processes

Applications of particular types of Lévy bases (Gamma, inverse Gaussian, tempered stable) have, in purely spatial contexts, been used to model intensity measures for Cox processes by Brix (1998,1999) and Wolpert and Ickstadt (1998). Spatial dependencies are introduced by mixing of the form $m(A) = \int k(A, \xi) Z(\mathrm{d}\xi)$ with suitably chosen kernels k. The resulting random measures m are again infinitely divisible and thus give rise to independent increment processes m_t of random measures.This has been used, by Brix and Chadœuf (2002), to model weed growth using Cox processes where the intensity surfaces develop in time as a suitably chosen such process m_t, the underlying Lévy basis being of the G type (cf. Example 2.1). See also Møller (2002).

Another approach has been to model the spatial intensity λ as $\lambda(\xi) = \exp\{G(\xi)\}$ where the $G(\xi)$ constitute a Gaussian random field, see Møller, Syversveen and Waagepetersen

(1998). A spatio-temporal extension of this approach, using a Gaussian random field process $G_t(\xi)$ which develops in time according to an Ornstein-Uhlenbeck stochastic differential equation, is discussed in Brix and Diggle (2001).

Alternatively, any nonnegative supOU$_\wedge$ field process $\{X_t(\cdot)\}_{t\in\mathbb{R}}$ may be used as intensity process of a dynamic Cox process on $\mathbb{S}$, while the integrated field process $\{X_t^*(\cdot)\}_{t\in\mathbb{R}}$ may serve as a spatial time-change. This has the advantage over Gaussian log-intensity specification of having the innovations appear linearly.

3.4 Extension

In generalisation of (14), let $X_t(\sigma)$ be defined by

$$X_t(\sigma) = e^{-\lambda(\sigma)t}\int_{-\infty}^{t} e^{\lambda(\sigma)s}\int_{\mathcal{R}} f_{s-t}(\rho;\sigma)Z(\mathrm{d}\rho\times\lambda(\sigma)\mathrm{d}s)$$

where $\lambda(\sigma) > 0$ is a time dilation parameter while the function $f_s(\rho;\sigma)$ is nonnegative, defined for $s \le 0$ and $(\rho,\sigma)\in\mathbb{R}^d\times\mathbb{R}^d$, and such that for $0 \ge s > s'$

$$f_{s'}(\rho;\sigma) \ge f_s(\rho;\sigma). \tag{26}$$

This condition holds, in particular, if $f_s(\rho;\sigma) = h(\rho;\sigma)\mathbf{1}_{C_s(\sigma)}(\rho)$ for some (nonnegative) function $h(\rho;\sigma)$ and with the sets $C_s(\sigma)$ specified as in Section 3.1.

In this case we again have a decomposition

$$X_t(\sigma) = e^{-\lambda(\sigma)t}X_0(\sigma) + U_t(\sigma) + V_t(\sigma)$$

where now

$$U_t(\sigma) = e^{-\lambda(\sigma)t}\int_{-\infty}^{0} e^{\lambda(\sigma)s}\int_{\mathcal{R}}\{f_{s-t}(\rho;\sigma) - f_s(\rho;\sigma)\}Z(\mathrm{d}\rho\times\lambda(\sigma)\mathrm{d}s)$$

$$V_t(\sigma) = e^{-\lambda(\sigma)t}\int_{0}^{t} e^{\lambda(\sigma)s}\int_{\mathcal{R}} f_{s-t}(\rho;\sigma)Z(\mathrm{d}\rho\times\lambda(\sigma)\mathrm{d}s)$$

which together with (26) shows that $X_t(\sigma)$ is Markovian. Moreover, $X_t(\sigma)$ can also be written as

$$X_t(\sigma) = \int_{-\infty}^{0} e^{\lambda(\sigma)s} f_s(\rho;\sigma)Z(\mathrm{d}\rho\times\lambda(\sigma)\mathrm{d}_t s)$$

and it follows that $X_t(\sigma)$ is a stationary process.

3.5 On the spatial structure of OU$_\wedge$ processes

Above we have discussed properties of OU$_\wedge$ processes, i.e. the timewise behaviour of $X_t(\sigma)$ where $X_t(\sigma)$ is of the form (14). Here we shall briefly consider the spatial behaviour in the sense of looking at $X_t(\sigma)$ as a stochastic process with σ, taken one-dimensional, playing the role of time.

From the construction it is clear that for any fixed t, $\{X_t(\sigma)\}_{\sigma\in\mathbb{R}}$ is a stationary process. To simplify notation we take $t=0$. Assuming for simplicity that $C_s(0) = [-c|s|, c|s|]$ for some $c > 0$, $\{X_0(\sigma)\}_{\sigma\geq 0}$ may be decomposed in terms of additive processes as follows (see Figure 4 for an illustration)

$$X_0(\sigma) = X_0(0) - X_0^{\searrow}(\sigma) + X_0^{\rightarrow}(\sigma) - X_0^{\downarrow}(\sigma) \tag{27}$$

where $X_0^{\searrow}(\sigma)$, $X_0^{\rightarrow}$ and $X_0^{\downarrow}$ are all additive and defined by

$$X_0^{\searrow}(\sigma) = X_0(0) - \int_{-\infty}^{-\frac{1}{2}c^{-1}\sigma} e^s Z([\sigma - c|s|, c|s|] \times \mathrm{d}s \tag{28}$$

$$X_0^{\rightarrow}(\sigma) = \int_{-\infty}^{0} e^s Z([c|s|, \sigma + c|s|] \times \mathrm{d}s \tag{29}$$

$$X_0^{\downarrow}(\sigma) = \int_{-\frac{1}{2}c^{-1}\sigma}^{0} e^s Z([c|s|, \sigma - c|s|] \times \mathrm{d}s. \tag{30}$$

Furthermore $\{X_0^{\searrow}(\sigma)\}_{\sigma\geq 0}$ is independent of $\{X_0^{\rightarrow}(\sigma), X_0^{\downarrow}(\sigma)\}_{\sigma\geq 0}$.

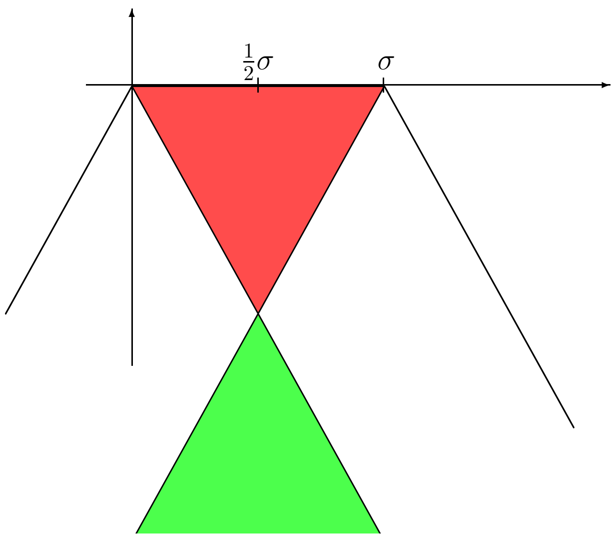

Figure 4: Illustration of the decomposition (27).

Note that the dynamics in σ is rather different from that in t (compare (18) or (19), with $C_s(0) = [-c|s|, c|s|]$, to (27)). In particular, in contrast to $\{X_t(\sigma)\}_{t\in\mathbb{R}}$, the process $\{X_t(\sigma)\}_{\sigma\in\mathbb{R}}$ is not Markovian. However, to an important extent, this difference hinges on the assumption (13). In fact, as we shall show in Section 4.3, there is at least one choice of the ambit sets $A_t(\sigma)$ for which key aspects of the dynamics are the same for the two processes, and this is related to Taylor's Frozen Flow Hypothesis for turbulence.

4 Tempo-spatial processes of SI type

In this Section we will use the results from Sections 2 and 3 to build processes of SI type defined as the exponentials of processes $X_t(\sigma)$. In the context of SI type processes we are mainly interested in the modelling of n-point correlations and n-point correlators, as defined below. These quantities are all expressible in terms of the kumulant function of the process $X_t(\sigma)$, due to the multiplicative set-up, and therefore allow for explicit calculations by use of the fundamental equation (8).

As to modelling, in the context of the present paper we have in mind primarily the problem of describing the stochastic behaviour of turbulent energy-dissipation fields $\{\varepsilon_t(\sigma) : \sigma \in \mathcal{S}\}_{t \in \mathbb{R}}$. As an application, we show how the SI fields can be understood as a continuous analogue of multiplicative cascade-processes in the description of the turbulent energy-dissipation field.

Throughout the Section, for simplicity we restrict discussion to ambit sets of the form

$$A_t(\sigma) = \{(\rho, s) : -\infty < s < t, \rho \in C_{s-t}(\sigma)\} \tag{31}$$

where

$$C_s(\sigma) = [\sigma - g(s), \sigma + g(s)] \tag{32}$$

and the function $g(s)$, defined on $(-\infty, 0]$, is nonnegative and decreasing on $[-T, 0]$ while $C_s(\sigma) = \emptyset$ for $s \in (-\infty, -T)$. As in Section 3 we define $g(s)$ for $s > 0$ by $g(s) = g(-s)$.

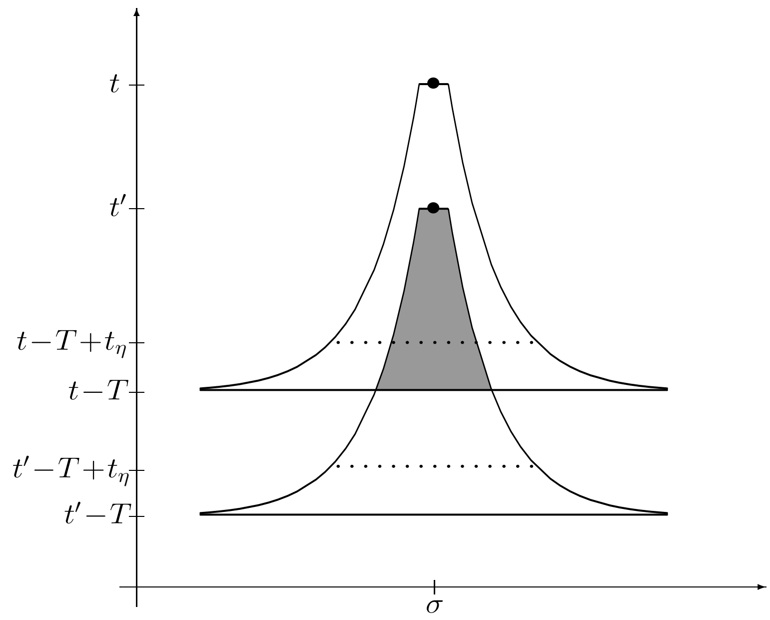

Figure 5: Illustration of the overlap $A_t(\sigma) \cap A_{t'}(\sigma)$.

Note that then (see Figure 5)

$$|A_t(\sigma) \cap A_{t'}(\sigma)| = G((T - |t - t'|)_+)$$

where

$$G(u) = 2\int_0^u g(s)\mathrm{d}s.$$

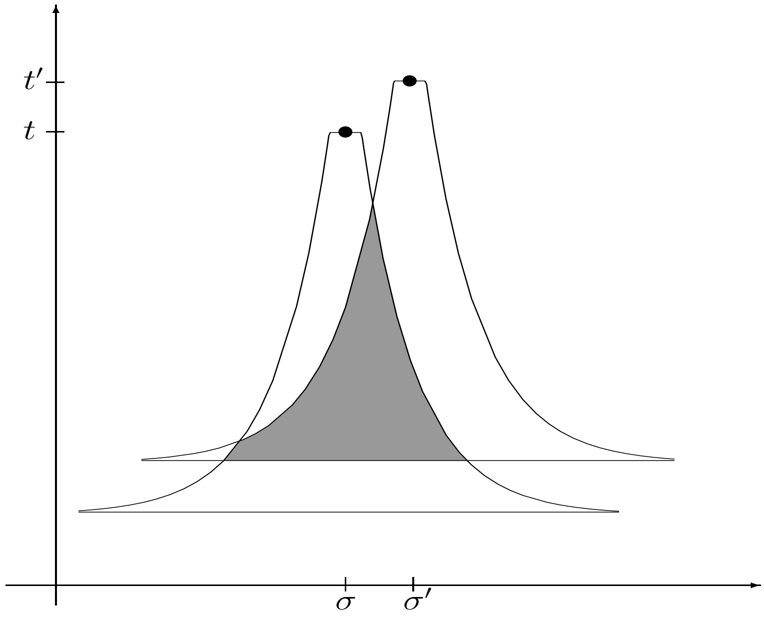

Figure 6: Illustration of the overlap $A_t(\sigma) \cap A_{t'}(\sigma')$.

For comparison, Figure 6 depicts the geometry of the overlap $A_t(\sigma) \cap A_{t'}(\sigma')$ for $\sigma \neq \sigma'$.

4.1 SI fields

The processes we are interested in here are build with the help of the additive processes $X_t(\sigma)$ defined, as in the previous Sections, by

$$X_t(\sigma) = \int_{-\infty}^t \int_{\mathcal{S}} f_t(\rho, s; \sigma) Z(\mathrm{d}\rho \times \mathrm{d}s)$$

where $\sigma \in \mathcal{S}$, Z denotes a Lévy basis on $\mathcal{R} = \mathcal{S} \times \mathbb{R}$ and $\mathcal{S} = \mathbb{R}^d$. The functions f are of the form

$$f_t(\rho, s; \sigma) = h_t(\rho, s; \sigma)\mathbf{1}_{A_t(\sigma)}(\rho, \sigma) \tag{33}$$

where h will be of some particular type and where to each point $a = (\sigma, t)$ in $\mathcal{R}$ will be associated an ambit set $A_t(\sigma)$ as in (31).

With that, the SI field $Y_t(\sigma)$ is defined as

$$Y_t(\sigma) = \exp\{X_t(\sigma)\}. \tag{34}$$

This kind of set-up constitutes a multiplicative process of independent multiplicative weights since the Lévy basis Z is an independently scattered random measure. Accordingly these fields are here called Stochastic Intermittency fields (SI), as the multiplicativity allows for adequate modelling of intermittent behaviour. In this respect, the SI fields can be viewed as a dynamic, continuous and homogeneous generalisation of multiplicative cascade processes. Heuristically this can be seen from the following considerations. In the most simplest formulation a multiplicative cascade for a quantity $\epsilon(\sigma)$ can be described by

$$\epsilon(\sigma) = \prod_j q_j(\sigma)$$

where the multiplicative weights q_j are independent random variates. They act on a hierarchy of scales l_j characterised by the scale-index j, i.e. for every j there is a family of independent multiplicative weights $\{q_j^i : i \in \mathbb{N}\}$ and an associated family of sets $\{A_j^i \in \mathcal{B}(\mathcal{S}) : i \in \mathbb{N}\}$ constituting a partition of $\mathcal{S}$ and with Euclidean volume $l_j < l_{j'}$ for $j' < j$ and $q_j(\sigma) = q_j^i$ if $\sigma \in A_j^i$. Now, if we assume that q_j is positive for all j we can rewrite the cascade process as

$$\epsilon(\sigma) = \exp\left\{\sum_j \ln q_j(\sigma)\right\}.$$

A natural and dynamical generalisation now consists in a densification of scales, i.e. the scale-index j becomes continuous and is identified with the variable t, thus resulting in structures that are within the set-up of (34). The usual identification of t with time now establishes the dynamical aspect of this generalisation.

4.2 Correlators

To describe the correlation structure of the SI field we will use two closely related quantities that are useful for multiplicative processes. *n-point correlations* are defined as

$$m(a_1, m_1; \ldots; a_n, m_n) = \mathrm{E}\left\{Y_{t_1}(\sigma_1)^{m_1} \cdot \ldots \cdot Y_{t_n}(\sigma_n)^{m_n}\right\} \tag{35}$$

and *n-point correlators* as

$$c(a_1, m_1; \ldots; a_n, m_n) = \frac{m(a_1, m_1; \ldots; a_n, m_n)}{m(a_1, m_1) \cdots m(a_n, m_n)} \tag{36}$$

provided the respective moments exist and are finite. Here $a_i = (\sigma_i, t_i)$ and $m_i \in \mathbb{R}$ for $i = 1, \ldots n$. This definition of correlators is naturally adapted to the multiplicativity of the SI field since it allows for a cancellation of independent factors in the nominator and denominator. In the following we always assume that the cited correlations and correlators exist. In this case we can use (8) and get the important result

$$m(a_1, m_1; \ldots; a_n, m_n) = \exp\left\{\int_{\mathcal{R}} \mathrm{K}\left(\sum_{j=1}^{n} m_j f_{t_j}(a; \sigma_j); a\right) \mu(\mathrm{d}a)\right\} \tag{37}$$

where $\mathrm{K}(\theta;\xi) = \kappa(-i\theta;\xi)$ is the kumulant function (assumed to exist) of the random variable $Z'(\xi)$ defined in Section 2.1. When Z is homogeneous we write $\mathrm{K}(\theta)$ for $\mathrm{K}(\theta;\xi)$.

4.3 Scaling of correlators

Scaling of some function $\Psi(x)$ here means that $\Psi(x)$ displays a power-law behaviour $cx^{-\tau}$ where c is a constant and the so-called scaling exponent τ is independent of x. The term scaling arises because a change of the scale by $x' = kx$ only results in a change of the constant c. Thus the power-law behaviour (and τ) is independent of the scale.

As an example of the usefulness of (37) we construct an SI field which displays scaling relations for n-point correlators. This kind of correlation structure is observed for the energy dissipation in a turbulent flow and is closely related to the multifractal and intermittent nature of the coarse grained energy dissipation field (see Schmiegel (2002), Schmiegel, Eggers and Greiner (2001,2003)). For brevity we restrict ourselves to the case $\mathcal{S} = \mathbb{R}$ and assume the Lévy basis Z to be homogeneous with finite kumulant function $\mathrm{K}(\theta;\xi) = \mathrm{K}(\theta)$ for all $\xi \in \mathcal{S}$ and $\theta \in D(\mathrm{K})$, the set of θ where K is defined. The function h in (33) is set to be constant $h = 1$.

For the function $g(s)$, which determines the ambit sets via (32), we take

$$g(s) = \frac{1}{2(T+s)} \quad \text{for } s \in [t_\eta - T, 0] \tag{38}$$

while on $[-T, t_\eta - T]$ the function is left arbitrary, subject to the restrictions that it is decreasing on $[-T, 0]$ and that $|A_t(\sigma)|$ is finite, cf. Figure 7.

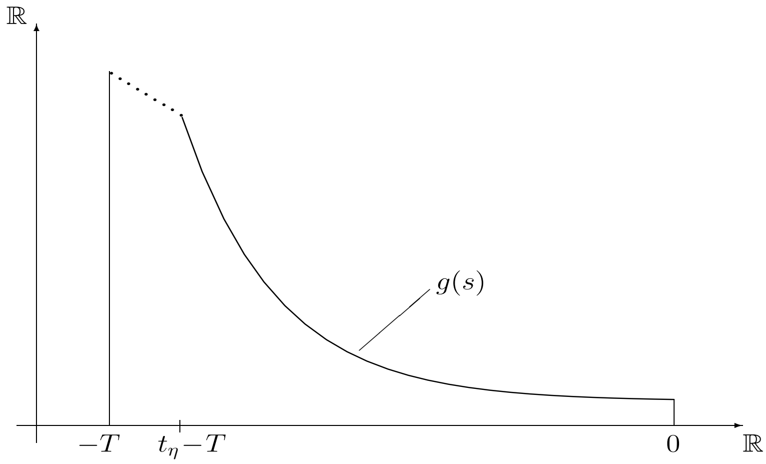

Figure 7: Illustration of the function $g(t)$.

With this set-up it is easy to see, using (36), (37) and Figure 7, that, provided $m_1, m_2, m_1 + m_2 \in D(\mathrm{K})$,

$$c((\sigma,t), m_1; (\sigma,t'), m_2) = \left(\frac{T}{|t-t'|}\right)^{\mathrm{K}(m_1+m_2)-\mathrm{K}(m_1)-\mathrm{K}(m_2)}, \quad |t-t'| \in [t_\eta, T],$$

which is scaling in $|t-t'|$,and

$$c((\sigma,t), m_1; (\sigma',t), m_2) \propto |\sigma-\sigma'|^{-(\mathrm{K}(m_1+m_2)-\mathrm{K}(m_1)-\mathrm{K}(m_2))}, \quad |\sigma-\sigma'| \in [T^{-1}, t_\eta^{-1}],$$

which is scaling in $|\sigma - \sigma'|$. For positive m_1 and m_2, the scaling exponent $\tau = \mathrm{K}(m_1+m_2) - \mathrm{K}(m_1) - \mathrm{K}(m_2)$ is positive, as follows from the Minkowski Inequality. In the last expression, the constant of proportionality depends on g. Similar relations also hold for higher order correlators, they can all be expressed as functions of the overlaps of all contributing ambit sets, which in turn gives rise to scaling relations. For a detailed characterisation of higher order correlators and generalisation to the case $S = \mathbb{R}^d$ we refer to Schmiegel (2002), Schmiegel, Eggers and Greiner (2001,2003) and Barndorff-Nielsen and Schmiegel (2003). Here we state only the main results. For an increasing sequence $\{t_j, 1 \leq j \leq n\}$ with $a_i = (\sigma, t_i)$ we get for temporal correlators

$$c(a_1, m_1; \ldots; a_n, m_n) \propto \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (t_k - t_{k-j})^{-\tau[m_{k-j}, \ldots, m_k]}, \tag{39}$$

where the exponents $\tau[m_{k-j}, \ldots, m_k]$ are defined as

$$\tau[m_1, \ldots, m_j] = \mathrm{K}\left(\sum_{i=1}^{j-2} m_i\right) + \mathrm{K}\left(\sum_{i=1}^{j} m_i\right) - 2\mathrm{K}\left(\sum_{i=1}^{j-1} m_i\right). \tag{40}$$

These exponents can take positive and negative sign, depending on j and the orders m. However for $j = 2$ they are strictly positive for positive m.

A next step in the modelling is to choose the kumulant function K so that the implied values of the exponents $\tau[m_1, \ldots, m_j]$, as given by (40), correspond as closely as possible to empirically determined values of these exponents. We will discuss this further in Barndorff-Nielsen and Schmiegel (2003) (see also Schmiegel, Eggers and Greiner (2003)).

For the spatial correlators we get, for an increasing sequence $\{\sigma_j, 1 \leq j \leq n\}$ with $a_i = (\sigma_i, t)$, the similar relation

$$c(a_1, m_1; \ldots; a_n, m_n) \propto \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (\sigma_k - \sigma_{k-j})^{-\tau[m_{k-j}, \ldots, m_k]}. \tag{41}$$

Note that the scaling exponents τ are the same in both cases. However in Schmiegel (2002) it is shown how to contruct an SI field where these exponents are different.

To further illustrate these scaling relations we give the expression for the temporal correlator with $n = 3$

$$c(a_1, m_1; a_2, m_2; a_3, m_3) \propto (t_2 - t_1)^{-\tau[m_1, m_2]} (t_3 - t_2)^{-\tau[m_2, m_3]} (t_3 - t_1)^{-\tau[m_1, m_2, m_3]}. \tag{42}$$

The arguments of the scaling exponents τ reflect the nested structure of the various overlaps. Here the points a_1, a_2 and a_2, a_3 are immediate neighboring points while a_1, a_3 are only next-to-neighbors.

It is also possible to motivate the form (38) of the function $g(t)$ from the Taylor Frozen Flow Hypothesis (for a discussion of this hypothesis, see Frisch (1995)). It states that the spatial variation of the energy dissipation can be expressed in terms of the temporal variation by means of the mean velocity v_0 of the flow if the relative fluctuations around v_0 are small. Using this assumption and the definition of the ambit set $A_t(\sigma)$ as being bounded by a function g as in (32), it is possible to derive (38). More specifically, (38) follows from requiring that $\{Y_t(\sigma)\}_{t\in\mathbb{R}}$ and $\{Y_t(\sigma)\}_{\sigma\in\mathbb{R}}$ have the same two-point correlators. (For more details see Schmiegel (2003) and Barndorff-Nielsen and Schmiegel (2003)).

4.4 Further modelling of two-point correlators

In the last Subsection we used the SI field to model scaling two point correlators by a special choice of the function $g(s)$ in (38) and, with that, of the ambit sets $A_t(\sigma)$. In the following we give some more examples of what can be modelled by exploiting more of the degrees of freedom that are inherent in the definition of the SI field $Y_t(\sigma)$.

To give the ideas in a transparent way we use the set-up of Section 4.3 and specify the function $g(s)$ in order to model other forms than scaling of two-point correlators. The ideas presented here may also serve as a starting point to model higher order tempo-spatial correlators by use of all degrees of freedom in the general definition of the SI field. Here we consider two-point correlators only and we focus on the degree of freedom in the choice of the ambit sets $A_t(\sigma)$ in the case of $S = \mathbb{R}$ and a homogeneous Lévy basis Z. We also assume that the SI field itself is homogeneous which is achieved by shape-invariance of the ambit sets, i.e. $A_t(\sigma) = (\sigma, t) + A$. The starting point is equation (37) which in this set-up simplifies to

$$c((\sigma,t),1;(\sigma,t'),1) = c(|t-t'|) = e^{|A_t(\sigma)\cap A_{t'}(\sigma)|(\mathrm{K}(2)-2\mathrm{K}(1))} \tag{43}$$

where $\mathrm{K}(2)-2\mathrm{K}(1)$ is positive and $c(|t-t'|)$ is a shorthand expression. Otherwise expressed,

$$\log c(u) = (\mathrm{K}(2)-2\mathrm{K}(1))G(T-u)$$

for $u \in (0,T]$. Thus, if g is continuous on $[-T,0]$ we have

$$g(u) = -\frac{1}{\mathrm{K}(2)-2\mathrm{K}(1)}\frac{c'(T-u)}{c(T-u)} \tag{44}$$

for $u \in (0,T]$. This tells us how to choose g in order to obtain a desired behaviour of the two-point correlator function c.

Example 4.1 *Exponential decay* Assume we want to model a process with correlator function $c(t) = 1 + e^{-\lambda t}$ for $0 < t < T$ and $c(t) = 1$ for $t > T$, with $\lambda > 0$. Then (44) gives

$$g(u) = -\frac{1}{\mathrm{K}(2)-2\mathrm{K}(1)}\frac{\lambda e^{-\lambda(T-u)}}{1+e^{-\lambda(T-u)}},$$

defining the ambit set $A_t(\sigma)$ by (31) and (32). In physics, exponential decay is a common model for short range correlations in equlibrium systems. □

Example 4.2 *Power-law decay* Assume we want to model a process with $c(t) = 1 + (t + t_0)^{-\alpha}$ where $\alpha > 0$ and $t_0 > 0$. Then, in this case,

$$g(u) = -\frac{1}{\mathrm{K}(2) - 2\mathrm{K}(1)} \frac{\alpha(T - u + t_0)^{-\alpha-1}}{(1 + (T - u + t_0)^{-\alpha})}.$$

This example can be understood as an approximation to scaling two-point correlators for $t \gg t_0$. $\square$

4.5 Dynamic mean-variance modelling

General remarks

In this Subsection, the main focus is on non-stationary SI fields. In particular, we are interested in building SI fields with given temporal evolution of the mean-process and the variance-process, leaving aside a discussion of possible correlations. Thus the aim of this subsection is to model the temporal evolution of the first and second moments

$$\mathrm{E}\{Y_t(\sigma)\} = \exp\left\{\int_{\mathcal{R}} \mathrm{K}(h_t(a;\sigma)1_{A_t(\sigma)};a)\mu(\mathrm{d}a)\right\} \equiv \mu_1(t) \tag{45}$$

$$\mathrm{E}\left\{(Y_t(\sigma))^2\right\} = \exp\left\{\int_{\mathcal{R}} \mathrm{K}(2h_t(a;\sigma)1_{A_t(\sigma)};a)\mu(\mathrm{d}a)\right\} \equiv \mu_2(t). \tag{46}$$

which are assumed to be independent of σ. The basic assumption again is a factorisable Lévy basis Z implying $\mathrm{K}(\zeta;a)$ to be independent of $a \in \mathcal{R}$. But we will allow $A_t(\sigma)$ to explicitly depend on t and furthermore use non-constant functions $h_t(a;\sigma)$. This gives enough freedom to model the moments $\mu_1(t)$ and $\mu_2(t)$ independently.

Variance-process with constant mean

The simplest situation is given by a constant mean $\mu_1(t) = 1$ and a varying second moment $\mu_2(t)$. The first condition is achieved simply by setting $h_t(a,\sigma) = c_h = \text{const.}$ in such a way that $\mathrm{K}(c_h) = 0$ and $\mathrm{K}(2c_h) \neq 0$, assuming such a constant exists. This requires that the kumulant function K exists in a neighbourhood of 0. Then the mean gets independent of $A_t(\sigma)$ while

$$\log \mu_2(t) = \mathrm{K}(2c_h)\mu(A_t(\sigma)) \tag{47}$$

and we generally are able to choose $A_t(\sigma)$ and the measure μ to model $\mu_2(t)$.

Example 4.3 *Increasing second moment (I)* For simplicity we set $\mathcal{R} = \mathbb{R}^4$ and take μ to be the Lebesgue measure. Suppose we wish to have $\mu_2(t) \to 1$ for $t \to -\infty$, and $\mu_2(t)$

differentiable with $\mu_2'(t) > 0$. Let $B(r)$ denote a ball of radius r and suppose $A_t(\sigma)$ is of the form

$$A_t(\sigma) = \{(\sigma, s) + B(r(s)) : s \leq t\}$$

for some function $r(s) \geq 0$.

Then

$$\mu(A_t(\sigma)) = \int_{-\infty}^{t} \mathrm{d}s \int_{(\sigma,s)+B(r(s))} \mathrm{d}\rho \tag{48}$$

and hence, choosing

$$r(t) = \left(\frac{3\mu_2'(t)}{4\pi \mathrm{K}(2c_h)\mu_2(t)} \right)^{1/3}, \tag{49}$$

we obtain that (46) is satisfied.

An example is given by $\mu_2(t) = 1 + e^t$ and

$$Y_t(\sigma) = \exp\left\{ \int_{-\infty}^{t} \int_{(\sigma,t')+B(r(t'))} Z(\mathrm{d}a') \right\}, \tag{50}$$

the Lévy basis Z defined as being homogeneous with Z' distributed according to a stable distribution $S_\alpha(\sigma, \beta, \mu)$ [4] with $\alpha \neq 1$, $\beta = -1$ and

$$\mu = \frac{\sigma^\alpha}{\cos\left(\frac{\pi\alpha}{2}\right)}.$$

With that the kumulant function of Z' is given by

$$\mathrm{K}(\theta) = \mu\left(\theta - \theta^\alpha\right)$$

with two zeros $\theta = 0$ and $\theta = 1$. Thus we set $c_H = 1$. According to (49) we choose

$$r(t) = \frac{3e^t}{4\pi \mathrm{K}(2)\left(1 + e^t\right)}$$

and obtain $\mu_1(t) = 1$ and $\mu_2(t) = 1 + e^t$. □

Example 4.4 *Increasing second moment (II)* Here we use a homogeneous NIG Lévy basis instead of the stable Lévy basis in the previous example. The kumulant function is then given by

$$\mathrm{K}(\theta) = \delta\left\{ \sqrt{(\alpha^2 - \beta^2)} - \sqrt{(\alpha^2 - (\beta + \theta)^2} \right\} + \mu\theta$$

[4] We are using here the notation established in Samorodnitsky and Taqqu (1994) according to which $S_\alpha(\sigma, \beta, \mu)$ is the infinitely divisible law with cumulant function

$$\mathrm{C}\{\theta \ddagger X\} = \begin{cases} \exp\left\{-\sigma^\alpha |\theta|^\alpha \left(1 - i\beta(\mathrm{sgn}(\theta)\tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu\theta\right\}, & (\alpha \neq 1) \\ \\ \exp\left\{-\sigma|\theta| \left(1 + i\beta\frac{2}{\pi}(\mathrm{sgn}(\theta)\ln(|\theta|)\right) + i\mu\theta\right\}, & (\alpha = 1). \end{cases}$$

where $\mu \in \mathbb{R}$, $\delta \in \mathbb{R}_+$ and $0 \leq |\beta| < \alpha$. If we define the constant c_h by $(\beta + c_h)^2 = \alpha^2$ and set

$$\delta = \frac{-\mu c_h}{\sqrt{\alpha^2 - \beta^2}},$$

we have $\mathrm{K}(c_h) = 0$, and $Y_t(\sigma)$ defined according to (50) with

$$r(t) = \frac{3e^t}{4\pi \mathrm{K}(2c_h)(1+e^t)}$$

yields $\mu_1(t) = 1$ and $\mu_2(t) = 1 + e^t$. $\square$

Example 4.5 *Arbitrarily varying second moment* Define

$$Y_t(\sigma) = \exp\left\{ \int_{(\sigma,t)+B(r(t))} Z(\mathrm{d}a') \right\}$$

where Z is again homogeneous. If we set

$$r(t) = \left(\frac{3 \ln \mu_2(t)}{4\pi \mathrm{K}(2)} \right)^{1/3}$$

then $\mathrm{E}\{Y_t(\sigma)^2\} = \mu_2(t)$ and $\mathrm{E}\{Y_t(\sigma)\} = 1$ where $\mu_2(t) \geq 1$ can be choosen arbitrarily. $\square$

Mean and variance process

Now we focus on SI fields, where both moments $\mu_1(t)$ and $\mu_2(t)$ vary with t. In contrast to above, we now allow the function $h_t(a;\sigma)$ to vary with the variable t, but not with a and σ. Accordingly we write it as h_t. Then (45) and (46) translate into

$$\mathrm{E}\{Y_t(\sigma)\} = \exp\left\{\mu(A_t(\sigma))\mathrm{K}(h_t 1_{A_t(\sigma)})\right\} \equiv \mu_1(t) \tag{51}$$

$$\mathrm{E}\left\{(Y_t(\sigma))^2\right\} = \exp\left\{\mu(A_t(\sigma))\mathrm{K}(2h_t 1_{A_t(\sigma)})\right\} \equiv \mu_2(t). \tag{52}$$

To solve these two equations we have at our disposal three quantities, namely K, h and $\mu(A_t(\sigma))$. Thus we can restrict ourself to choose the kumulant-function K suitably in advance and solve (51) and (52) for h_t and $\mu(A_t(\sigma))$. It is to note here, that $A_t(\sigma)$ is still arbitrary, only the measure $\mu(A_t(\sigma))$ is determined. Thus here is still some freedom in modelling the correlation-structure of the SI field $Y_t(\sigma)$.

Example 4.6 A particularly simple example is given by the homogeneous stable Lévy basis on $\mathcal{R} = \mathbb{R}^4$ defined as in Example 4.3. Equations (51) and (52) yield

$$q_t \equiv \frac{\ln \mu_2(t)}{\ln \mu_1(t)} = \frac{\mathrm{K}(2h_t)}{\mathrm{K}(h_t)} = \frac{2h_t - (2h_t)^\alpha}{h_t - h_t^\alpha}.$$

Now, provided $q_t > 2^\alpha$ for all t (which trivially holds for $\alpha < 1$) we get

$$h_t = \left(\frac{q_t - 2}{q_t - 2^\alpha} \right)^{1/(\alpha-1)} .$$

Finally we set

$$r(t) = \left(\frac{3 \ln \mu_1(t)}{4\pi \mathrm{K}(h_t)} \right)^{1/3}$$

and define the ambit sets $A_t(\sigma) = (\sigma, t) + B_{r(t)}$ and the SI field $Y_t(\sigma)$ as in Example 4.3. With this set-up equations (51) and (52) hold for arbitrary functions $\mu_1(t)$ and $\mu_2(t)$ provided $q_t > 2^\alpha$. A particular example is given by $\mu_2(t) = \mu_1(t)^\tau + \mu_1(t)^2$ with τ arbitrary. This type of relation between the first and second moment is a defining feature of the exponential family of Tweedie models (cf., for instance, Jørgensen (1997)). □

5 Conclusion

We have discussed some tempo-spatial models $X_t(\sigma)$ and $Y_t(\sigma) = \exp X_t(\sigma)$ where the $X_t(\sigma)$ are defined in terms of integrals with respect to Lévy bases. The main aim has been to construct realistic dynamic models, continuous in time and space, for the energy dissipation fields $\epsilon_t(\sigma)$ of high Reynolds number turbulence, in line with recent developments, in particular random cascade formulations, of the original phenomenological theory of homogeneous and isotropic turbulence due to Kolmogorov. (The applicability of the results obtained for tempo-spatial modelling in, for instance, image analysis, is also briefly indicated.) In contrast to the discrete random cascade models discussed by Greiner (2002), Cleve and Greiner (2000), Eggers, Dziekan and Greiner (2001) and Jouault, Greiner and Lipa (2000), the models proposed here are continuous and homogeneous; and we have, in particular, specified how one of the latter models is in full consistency with Taylor's Frozen Flow Hypothesis.

We hope in future work to tap much more of the potential of the present framework for a more detailed, comprehensive and integrated modelling of turbulent fields, with close attention to recent experimental evidence.

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