# ZERO ENERGY ASYMPTOTICS OF THE RESOLVENT IN THE LONG RANGE CASE

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ABSTRACT. We present a limiting absorption principle at zero energy for two-body Schrödinger operators with a long-range potential having a positive virial at infinity. Furthermore, we prove existence of limits (in weighted spaces), as the spectral parameter tends to zero, of all powers of the resolvent. The principal tools of proof are absence of eigenvalue at zero, singular Mourre theory and microlocal estimates. Some elements of the proof will be explained.

## 1. STATEMENT OF MAIN RESULTS

We give an account of some recent results on asymptotic expansion at zero of the resolvent $R(\zeta) = (H - \zeta)^{-1}$ of a two-body Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$; see [5] for details. It is well-known, see [18], [12] and the more recent work [13] in which further references can be found, that if $V(x) = O(|x|^{-(2+\epsilon)})$ with $\epsilon > 0$ then such an asymptotic expansion exists. For the 'long-range' case, $V(x) = O(|x|^{-\mu})$ with $\mu < 2$, much less is known. To our knowledge, the only results on limiting absorption principles for such potentials are [22] and [17] (and [2] for the purely Coulombic case). In [22] only radially symmetric potentials are treated, and though radial symmetry is not imposed in [17] some of the assumptions of that paper appear unnecessarily restrictive. Here we present a complete asymptotic expansion of the resolvent at zero energy, for a much wider class of potentials. Our basic assumption is a sign condition at infinity,

$$V(x) \leq -\epsilon|x|^{-\mu};\ |x| > R, \tag{1.1}$$

and a similar positive virial condition.

For such potentials we prove complete asymptotic expansions (in weighted spaces)

$$R(\lambda + (-)i0) \asymp \sum_{j=0}^{\infty} R_j^{+(-)} \lambda^j \text{ for } \lambda \to 0^+; \tag{1.2}$$

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here $R_0^+ \neq R_0^-$. We also show that zero is not an eigenvalue. (This is implicit in (1.2).) We notice that there is no explicit dimension-dependence or fractional/inverse powers in $\lambda$.

It is well-known that for 'long-range' potentials that are positive at infinity, zero can indeed be an eigenvalue. This explains one aspect of the condition (1.1). Probably the best intuitive explanation of the result (1.2) is given in terms of the WKB-ansatz for stationary solutions to the Schrödinger equation $-\psi'' + V\psi = E\psi$ in dimension $d = 1$

$$\psi \approx C_+(E-V)^{-\frac{1}{4}} e^{i\int (E-V)^{\frac{1}{2}} dx} + C_-(E-V)^{-\frac{1}{4}} e^{-i\int (E-V)^{\frac{1}{2}} dx}. \quad (1.3)$$

Under the condition (1.1) the oscillatory behaviour survives for $E \approx 0$, with $E > 0$ (since $\int (-V)^{\frac{1}{2}} dx \sim |x|^{1-\frac{\mu}{2}} \to \infty$). Moreover, (1.3) suggests that zero is not an eigenvalue, and also indicates which weights one needs in (1.2). We remark that indeed (1.2) can be proved for $d = 1$ by WKB-methods, see [22], which can also be used to prove optimality of the weights in the results below.

Let us state our main results precisely. Let $0 < \theta < \pi$ and define

$$\Gamma_\theta = \{z \in \mathbb{C} \setminus \{0\} \,\big|\, |z| \leq 1, \arg z \in (0, \theta)\}. \quad (1.4)$$

We have the following *limiting absorption principle* at zero energy for a Schrödinger operator $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R}^d)$ recalling the notation $R(\zeta) = (H - \zeta)^{-1}$. Although we shall not elaborate here, it is enough to impose the conditions (1) and (3) near infinity.

**Theorem 1.1.** *Let $V(x) = V_1(x) + V_2(x)$, $x \in \mathbb{R}^d$, be a real-valued potential. Suppose there exists $0 < \mu < 2$ such that $V$ satisfies the conditions (1)–(6) below.*

(1) *There exists $\epsilon_1 > 0$ such that $V_1(x) \leq -\epsilon_1 \langle x \rangle^{-\mu}$; $\langle x \rangle = \sqrt{1 + x^2}$.*
(2) *For all $\alpha \in (\mathbb{N} \cup \{0\})^d$ there exists $C_\alpha > 0$ such that*

$$\langle x \rangle^{\mu + |\alpha|} |\partial^\alpha V_1(x)| \leq C_\alpha.$$

(3) *There exists $\epsilon_2 > 0$ such that $-|x|^{-2}\left(x \cdot \nabla(|x|^2 V_1)\right) \geq -\epsilon_2 V_1$.*
(4) *$V_2(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$.*
(5) *There exists $\delta, C, R > 0$ such that*

$$|V_2(x)| \leq C|x|^{-1-\mu/2-\delta},$$

*for $|x| > R$.*
(6) *$V$ satisfies unique continuation at infinity (see Assumption 2.1 in Section 2).*

*Then for all $s \in (\frac{1}{2} + \frac{\mu}{4}, \frac{1}{2} + \frac{\mu}{4} + \delta)$ and all $0 < \theta < \pi$ the family of operators $B(\zeta) = \langle x \rangle^{-s} R(\zeta) \langle x \rangle^{-s}$ is uniformly Hölder continuous in*

$\Gamma_\theta$. *In particular there exists* $C_{s,\theta} > 0$ *such that*

$$\sup_{\zeta\in\Gamma_\theta} \left\| \langle x\rangle^{-s} R(\zeta) \langle x\rangle^{-s} \right\| \le C_{s,\theta}, \tag{1.5}$$

*and the limits*

$$\begin{aligned} \langle x\rangle^{-s} R(0+i0)\langle x\rangle^{-s} &\equiv \lim_{\zeta\to 0, \zeta\in\Gamma_\theta} \langle x\rangle^{-s} R(\zeta)\langle x\rangle^{-s}, \\ \langle x\rangle^{-s} R(0-i0)\ \langle x\rangle^{-s} &\equiv \lim_{\zeta\to 0, \zeta\in\Gamma_\theta} \langle x\rangle^{-s} R\left(\bar\zeta\right) \langle x\rangle^{-s} \end{aligned}$$

*exist in* $\mathcal{B}(L^2(\mathbb{R}^d))$.

Next, we have existence of limits for powers of the resolvent. The asymptotic expansion (1.2) is an easy consequence of Theorem 1.2 below. Notice also that Theorem 1.1 is a particular case of Theorem 1.2.

**Theorem 1.2.** *Let* $V = V_1 + V_2$ *satisfy the conditions in Theorem 1.1 with* (5) *replaced by: For some* $m_0 \in \mathbb{N}$

*(5') $V_2 = O(k^{-m_0-\epsilon})$; $k = k(x) = \langle x\rangle^{1+\mu/2}$.*

*Let* $m \le m_0$, $\theta \in (0,\pi)$ *and* $\epsilon > 0$. *Then there exists* $C > 0$ *such that*

$$\left\| k^{-(m-1/2)-\epsilon} R(\zeta)^m k^{-(m-1/2)-\epsilon} \right\| \le C, \tag{1.6}$$

*for all* $\zeta \in \Gamma_\theta$.

*Furthermore, the function*

$$\zeta \mapsto k^{-(m-1/2)-\epsilon} R(\zeta)^m k^{-(m-1/2)-\epsilon}$$

*is uniformly Hölder continuous in* $\Gamma_\theta$.

Using Theorem 1.1 one may define (with $\mathcal{H}_1 = k^{-1/2-\epsilon} L^2(\mathbb{R}^d)$, $\mathcal{H}_2 = k^{1/2+\epsilon} L^2(\mathbb{R}^d)$):

$$E'(+0) = (2\pi i)^{-1} \left\{ R(0+i0) - R(0-i0) \right\} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2).$$

One can prove that indeed

$$E'(+0) \ne 0. \tag{1.7}$$

Let $F(|x| < C)$ denote the multiplication operator by the characteristic function of $\{x|\ |x| < C\}$ and let $\kappa = (1+\mu/2)^{-1}$.

**Corollary 1.3.** *Under the conditions of Theorem 1.2 with (5') valid for all* $m_0 \in \mathbb{N}$:

(i) *For all* $s > \frac{5}{2}(1+\frac{\mu}{2})$ *and* $f \in C_0^\infty(\mathbb{R})$

$$\|\langle x\rangle^{-s} \Big( e^{-itH} (f 1_{[0,\infty)})(H) + it^{-1} f(0) E'(+0) \Big) \langle x\rangle^{-s}\| = O(t^{-2}).$$

(ii) *For all* $0 \leq \epsilon' < \epsilon \leq 1$ *there exists* $s > 1$ *(depending on* $\epsilon$*) such that for all* $f \in C_0^\infty(\mathbb{R})$

$$\|F\left(|x| < t^{(1-\epsilon)\kappa}\right) e^{-itH}(f 1_{[0,\infty)})(H)\langle x\rangle^{-s}\| = O(t^{-(1+\epsilon')\frac{1}{2}}). \qquad (1.8)$$

**Remark 1.4.** 1) By time reversal invariance there are similar bounds for $t \to -\infty$.

2) Due to (1.7) and Corollary 1.3 (i) the best one could hope for to the right in (1.8) would be the bound $O(t^{-1})$ (for $f(0) \neq 0$). Moreover we would expect that $t^\kappa$ is indeed the borderline for this kind of low energy, minimal velocity estimate. In fact there is a sharp analogous bound in classical mechanics, cf. [6] and [20].

3) If $V_2 \in C_0^\infty(\mathbb{R}^d)$ one may take $f = 1$ in Corollary 1.3 (i) and (ii). This follows readily from the given statements and well-established high energy estimates, see [15, Theorem 1.1], [3, Theorem 1] or [11, Theorem 1.2 (ii)], in fact some local singularities may be included.

We shall outline the proof of Theorems 1.1 and 1.2 in the following sections. Apart from the notation $\langle x\rangle = \sqrt{1+x^2}$, used above, we will also need the notation $p = -i\nabla$ and $A = (x\cdot p + p\cdot x)/2$.

The virial $W$ of the potential $V$ is defined by $W = -2V - x\cdot\nabla V$. We recall the (formal) identity $i[H, A] = 2H + W$. By design of the splitting of $V$, the assumptions (1) and (3) of Theorem 1.1 yield

$$W_1(x) = -2V_1(x) - x\cdot\nabla V_1(x) \geq \epsilon_1\epsilon_2\langle x\rangle^{-\mu},$$

so in particular, the virial $W$ is positive in the case $V_2 = 0$.

## 2. Absence of bound states for Schrödinger operators

In this section we present a basic result of independent interest, namely the absence of zero-energy bound states for long range potentials negative at infinity. The proof is a variation of the technique applied in the proof of [19, Theorem XIII.58] of which the present result is a generalization.

The conditions which exclude zero-energy eigenfunctions are given in Assumptions 2.1 and 2.2 below. Notice that the assumptions in Theorem 1.1 are stronger than Assumption 2.2: Take $h = \epsilon r^{-\mu/2}$ for a small $\epsilon > 0$ and $s$ close to 1. Let us specify the notation $x = r\omega \in \mathbb{R}^d$, with $\omega \in \mathbb{S}^{d-1}$.

**Assumption 2.1.** The function $V : \mathbb{R}^d \to \mathbb{R}$ is measurable, and if $u \in H^2(\mathbb{R}^d)$, $u = 0$ in a neighbourhood of $\infty$, the product $V\psi \in L^2(\mathbb{R}^d)$ and $u$ is a distributional solution to

$$-\Delta u + Vu = 0,$$

then $u = 0$.

We remark that for $d \geq 3$ the condition $V \in L^{d/2}_{\text{loc}}(\mathbb{R}^d)$ suffices, see [14].

**Assumption 2.2.** The function $V$ can be written as $V = V_1 + V_2$, such that: For some $s \in [0, 1)$, some $R, C > 0$ and a positive differentiable function $h = h(r)$ defined on $[R, \infty)$ we have

(1) $V_1$ and $V_2$ are bounded on $|x| > R$, and $V_1$ is negative on $|x| > R$.
(2) $\sup_{\omega \in S^{d-1}} \frac{d}{dr}(r^{s+1} V_1(r\omega)) \leq -r^s h^2(r)$ when $r > R$.
(3) $r^{-1} + r \sup_{\omega \in S^{d-1}} |V_2(r\omega)| = o(h)$ as $r \to \infty$.
(4) $h'(r) \leq Ch^2(r)$ on $|x| > R$.

With the above assumptions we can prove the absence of zero-energy eigenstates.

**Theorem 2.3.** *Suppose $V = V_1 + V_2$ satisfies Assumptions 2.1 and 2.2. Suppose furthermore that $\psi \in H^2_{\text{loc}}(\mathbb{R}^d)$ satisfies (1)–(3) below.*

(1) $\int_{|x|>R} h^2(r)|\psi(x)|^2\, dx < \infty$ *and* $\int_{|x|>R} |V_1(x)||\psi(x)|^2\, dx < \infty$.
(2) $p_j \psi \in L^2(\mathbb{R}^d)$; $j = 1, \cdots, d$.
(3) *The product $V\psi \in L^2_{\text{loc}}(\mathbb{R}^d)$, and $(-\Delta + V)\psi = 0$ in the sense of distributions.*

*Then $\psi = 0$.*

## 3. Extended limiting absorption principles

We introduce that following symbols:

$$a_0(x, \xi) = f_E(x)^{-2}\xi^2, \quad b(x, \xi) = \frac{x}{\langle x \rangle} \cdot \frac{\xi}{f_E(x)}, \tag{3.1}$$

$$\text{with} \qquad f = f_E = \sqrt{\kappa_0^{-2} E + (1 - \mu/2)^{-1} \langle x \rangle^{-\mu}}; \ \kappa_0, E > 0, \tag{3.2}$$

and where the parameters will be specified below.

Let us denote by $\mathrm{Op}^{\mathrm{w}}(a)$ the Weyl quantization of a symbol $a$. Explicitly $\mathrm{Op}^{\mathrm{w}}(a)$ acts as follows

$$(\mathrm{Op}^{\mathrm{w}}(a)\phi)(x) = (2\pi)^{-d} \iint e^{i(x-y)\xi} a((x+y)/2, \xi)\phi(y)\, dy d\xi.$$

**Theorem 3.1.** *Let $V(x)$ satisfy the conditions of Theorem 1.1 with $V_2 = 0$. We reformulate the assumption (3) as: For some $\kappa_0 > 0$ and $2 > \mu > 0$,*

$$W(x) = -2V(x) - x \cdot \nabla V(x) \geq 2\kappa_0^2 \langle x \rangle^{-\mu}. \tag{3.3}$$

*Let $\theta \in (0, \pi)$ and $\Gamma_\theta$ be as defined in (1.4). Let $a_0$ and $b$ be as defined in (3.1) with $E = |\zeta|$. Define $k = k(x) = \langle x \rangle^{1+\mu/2}$. Then the following* *conclusions,* (i) - (iv)*, hold for* $H = p^2 + V$ *with all bounds being uniform in* $\zeta \in \Gamma_\theta$*:*

(i) *Let* $m \in \mathbb{N}$ *and let* $\epsilon > 0$ *be arbitrary. Then there exists* $C > 0$ *such that*

$$\|k^{-(m-1/2)-\epsilon} R(\zeta)^m k^{-(m-1/2)-\epsilon}\| \le C. \tag{3.4a}$$

(ii) *There exists* $C_0 > 0$*, depending only on* $V$*, such that if* $\operatorname{supp}(F_+) \subset (C_0, \infty)$ *and* $F_+' \in C_0^\infty(\mathbb{R})$*, then for all* $m \in \mathbb{N}$ *and all* $\epsilon, t > 0$ *there exists* $C > 0$ *such that*

$$\|k^{t-1/2-\epsilon} \operatorname{Op}^{\mathrm{w}}(F_+(a_0)) R(\zeta)^m k^{-t-m+1/2-\epsilon}\| \le C, \tag{3.4b}$$

$$\|k^{-t-m+1/2-\epsilon} R(\zeta)^m \operatorname{Op}^{\mathrm{w}}(F_+(a_0)) k^{t-1/2-\epsilon}\| \le C. \tag{3.4c}$$

(iii) *Let* $\tilde{F}_+, \tilde{F}_-$ *satisfy (with* $\kappa_0$ *from* (3.3)*) for some* $\kappa > 0$*,*
- $\inf\operatorname{supp}(\tilde{F}_+) > -\kappa > -\kappa_0$, $\sup\operatorname{supp}(\tilde{F}_-) < \kappa < \kappa_0$.
- $\tilde{F}_-', \tilde{F}_+' \in C_0^\infty(\mathbb{R})$.

*Let* $F_- \in C_0^\infty(\mathbb{R})$*. Then for all* $m \in \mathbb{N}$ *and all* $\epsilon, t > 0$ *there exists* $C > 0$ *such that*

$$\|k^{t-1/2-\epsilon} \operatorname{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_-(b)) R(\zeta)^m k^{-t-m+1/2-\epsilon}\| \le C, \tag{3.4d}$$

$$\|k^{-t-m+1/2-\epsilon} R(\zeta)^m \operatorname{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b)) k^{t-1/2-\epsilon}\| \le C. \tag{3.4e}$$

(iv) *Suppose* $\tilde{F}_+$ *and* $\tilde{F}_-$ *satisfy the assumptions from* (iii)*,* $F_-^1, F_-^2 \in C_0^\infty(\mathbb{R})$ *and*

$$\operatorname{dist}(\operatorname{supp}(\tilde{F}_+), \operatorname{supp}(\tilde{F}_-)) > 0.$$

*Then for all* $m \in \mathbb{N}$ *and all* $t > 0$ *there exists* $C > 0$ *such that*

$$\|k^t \operatorname{Op}^{\mathrm{w}}(F_-^1(a_0)\tilde{F}_-(b)) R(\zeta)^m \operatorname{Op}^{\mathrm{w}}(F_-^2(a_0)\tilde{F}_+(b)) k^t\| \le C. \tag{3.4f}$$

*Suppose* $F_+$ *is given as in* (ii)*, some functions* $\tilde{F}_+, \tilde{F}_-, F_-$ *are given as in* (iii) *and suppose* $\operatorname{dist}(\operatorname{supp}(F_-), \operatorname{supp}(F_+)) > 0$*. Then for all* $m \in \mathbb{N}$ *and all* $t > 0$ *there exists* $C > 0$ *such that*

$$\|k^t \operatorname{Op}^{\mathrm{w}}(F_+(a_0)) R(\zeta)^m \operatorname{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b)) k^t\| \le C, \tag{3.4g}$$

$$\|k^t \operatorname{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_-(b)) R(\zeta)^m \operatorname{Op}^{\mathrm{w}}(F_+(a_0)) k^t\| \le C. \tag{3.4h}$$

The proof of Theorem 3.1 reduces by elementary algebra to the case of bounds with only one resolvent i.e. $m = 1$, cf. [10] or [9]. The partition of unity needed for this reduction is indicated in (4.7). In the case $m = 1$, (3.4a) follows by a singular Mourre theory while (3.4d) and (3.4e) follow by a certain modification of a method of [7]. The bounds (3.4b) and (3.4c) may be thought of as energy localizations. Certain energy-dependent positivity bounds, given by the Fefferman-Phong inequality in a certain Hörmander-Weyl calculus, play an important role in the proof of the bounds in (ii), (iii) and (iv) for $m = 1$.

## 4. Perturbative argument

Using Theorems 2.3 and 3.1 one may easily prove Theorems 1.1 and 1.2 by perturbative arguments. In this section we will show Theorem 1.1. Let us write for $\zeta \in \Gamma_\theta$

$$R(\zeta) = (H - \zeta)^{-1},\ R_1(\zeta) = (H_1 - \zeta)^{-1};\ H_1 = p^2 + V_1. \tag{4.1}$$

We shall proceed perturbatively using

$$R(\zeta)(I + V_2 R_1(\zeta)) = R_1(\zeta). \tag{4.2}$$

First we notice that $R_1(\zeta)$ is uniformly Hölder continuous in $\Gamma_\theta$. If $s > 3/2(1 + \mu/2)$ this follows from (3.4a) with $m = 2$ (showing in fact Lipschitz continuity in this case). If $s \leq 3/2(1 + \mu/2)$ we may interpolate the bounds of (3.4a) with $m = 1$ and $m = 2$.

In particular $R_1^+ = R_1(0 + i0) = \lim_{\zeta \to 0, \zeta \in \Gamma_\theta} R_1(\zeta)$ and $R_1^- = R_1(0 - i0) = \lim_{\zeta \to 0, \zeta \in \Gamma_\theta} R_1(\overline{\zeta})$ are well-defined (in weighted spaces).

To show (1.5) (in the general case) it suffices to show that $\langle x \rangle^s (I + V_2 R_1^+) \langle x \rangle^{-s}$ is invertible as an operator on $L^2(\mathbb{R}^d)$. This follows from (4.2), the standard limiting absorption principle for positive energies and absence of positive eigenvalues, cf. [16], [21] and [4, Section 6.5]. Since $\langle x \rangle^s V_2 R_1^+ \langle x \rangle^{-s}$ is compact it suffices to show that the equation

$$\phi = -V_2 R_1^+ \phi, \tag{4.3}$$

has no nonzero solution $\phi \in \langle x \rangle^{-s} L^2(\mathbb{R}^d)$. Let $\psi = R_1^+ \phi$ $(\in \langle x \rangle^s L^2(\mathbb{R}^d))$. Then we have in the sense of distributions

$$H\psi = 0 \text{ and } \mathrm{V}_2 \psi = -\phi. \tag{4.4}$$

Using that $\frac{R_1^+ - R_1^-}{2i} \geq 0$ we obtain from the calculation

$$0 = \Im\langle \psi, V_2 \psi \rangle = -\Im\langle \psi, \phi \rangle = -\Im\langle R_1^+ \phi, \phi \rangle = (2i)^{-1} \langle \phi, (R_1^+ - R_1^-) \phi \rangle,$$

that

$$\psi = R_1^+ \phi = R_1^- \phi. \tag{4.5}$$

We claim that

$$\psi \in L^2(\mathbb{R}^d). \tag{4.6}$$

We shall prove (4.6) using Theorem 3.1 in a bootstrap argument. (For a similar problem for the free Laplacian see the proof of [1, Theorem 3.3].) We pick a real-valued function $F_+$ as in Theorem 3.1 (ii) such that $F_+(x) = 1$ for $|x| > 2C_0$. Let $F_- = 1 - F_+$. Pick real-valued functions $\tilde F_-$ and $\tilde F_+$ as in Theorem 3.1 (iii) such that $\tilde F_- + \tilde F_+ = 1$. Then we decompose with the symbols $a_0$ and $b$ being defined as in (3.1) with $E = 0$ in the expression (3.2) for $f$

$$\psi = \mathrm{Op}^{\mathrm{w}}(F_+(a_0))\psi + \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde F_-(b))\psi + \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde F_+(b))\psi. \tag{4.7}$$

By (3.4b) and (3.4d) the first two terms on the right hand side of (4.7) belong to $\langle x\rangle^{s'}L^2$ where (assuming here $\phi \in \langle x\rangle^{-s}L^2$)

$$s' = (1+\tfrac{\mu}{2})(-t+\tfrac12+\epsilon);\ t = \frac{s}{1+\frac{\mu}{2}} - \tfrac12 - \epsilon. \tag{4.8}$$

We notice that the bound (3.4e) for $m = 1$ is equivalent to

$$\|k^{t-1/2-\epsilon}\mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde F_+(b))R_1(\zeta)^*k^{-t-1/2-\epsilon}\| \le C. \tag{4.9}$$

Taking $\zeta \to 0$ in the sector $\Gamma_\theta$, (4.9) leads to

$$\|k^{t-1/2-\epsilon}\mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde F_+(b))R_1^- k^{-t-1/2-\epsilon}\| \le C, \tag{4.10}$$

with the same convention for $a_0$ and $b$ as above. We use the representation $\psi = R_1^-\phi$ of (4.5) and apply (4.10), and conclude that also the third term on the right hand side of (4.7) belongs to $\langle x\rangle^{s'}L^2$ with $s'$ given by (4.8); so $\psi \in \langle x\rangle^{s'}L^2$.

From this and (4.4) we learn that $\phi \in \langle x\rangle^{s'-1-\frac{\mu}{2}-\delta}L^2 = \langle x\rangle^{-s-\delta+(2+\mu)\epsilon}L^2$; so by taking $\epsilon << (2+\mu)^{-1}\delta$ we improve the decay of $\phi$ by almost a factor $\langle x\rangle^{-\delta}$. Iterating this argument leads to $s' \le 0$ eventually. We have proved (4.6).

Combining Theorem 2.3 and (4.6) yields $\psi = \phi = 0$, completing the proof of (1.5) in the general case. The Hölder continuity statement of Theorem 1.1 in the general case follows readily by using (4.2) and the known result for $R_1(\zeta)$.

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