# ZERO ENERGY ASYMPTOTICS OF THE RESOLVENT FOR A CLASS OF SLOWLY DECAYING POTENTIALS

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Abstract. We prove a limiting absorption principle at zero energy for two-body Schrödinger operators with long-range potentials having a positive virial at infinity. More precisely, we establish a complete asymptotic expansion of the resolvent in weighted spaces when the spectral parameter varies in cones; one of the two branches of boundary for the cones being given by the positive real axis. The principal tools are absence of eigenvalue at zero, singular Mourre theory and microlocal estimates.

## Contents



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## 1. Introduction and statement of the main results

In this paper we derive an asymptotic expansion at zero of the resolvent $R(\zeta) = (H-\zeta)^{-1}$ of a two-body Schrödinger operator $H = -\Delta+V$ on $L^2(\mathbb{R}^d)$. It is well-known, see [Rau78], [JenKat79] and the more recent work [JenNen01] in which further references can be found, that if $V(x) = O(|x|^{-(2+\epsilon)})$ (for $\epsilon > 0$) then such an asymptotic expansion exists. The result is rather complicated; it depends on the dimension $d$ and on the possible existence of zero-energy eigenstates and/or a resonance state. For the 'long-range' case, $V(x) = O(|x|^{-\mu})$ with $\mu < 2$, the literature is more sparse. In fact, the only papers on asymptotics of the resolvent for such potentials, we are aware of, are [Yaf82] and [Nak94] (and [BSS85] for the purely Coulombic case). In [Yaf82] only radially symmetric potentials are treated, and some of the assumptions of [Nak94] (although not requiring radial symmetry) appear unnecessarily strong. The purpose of the present paper is to prove a limiting absorption principle at zero, and in fact a complete asymptotic expansion of the resolvent, for a much wider class of potentials. Our basic assumption is a sign condition at infinity,

$$V(x) \leq -\epsilon|x|^{-\mu};\ |x| > R, \tag{1.1}$$

and a similar positive virial condition.

For such potentials we prove complete asymptotic expansions (in weighted spaces)

$$R(\lambda + (-)i0) \asymp \sum_{j=0}^{\infty} R_j^{+(-)}\lambda^j \text{ for } \lambda \to 0^+; \tag{1.2}$$

here $R_0^+ \neq R_0^-$. We also show that zero is not an eigenvalue. (This is implicit in (1.2).) We notice that there is no explicit dimension-dependence or fractional/inverse powers in $\lambda$.

It is well-known that for 'long-range' potentials that are positive at infinity, zero can indeed be an eigenvalue. This explains one aspect of the condition (1.1). Probably the best intuitive explanation of the result (1.2) is given in terms of the WKB-ansatz for stationary solutions to the Schrödinger equation $-\psi'' + V\psi = E\psi$ in dimension $d = 1$

$$\psi \approx C_+(E-V)^{-\frac{1}{4}}e^{i\int(E-V)^{\frac{1}{2}}dx} + C_-(E-V)^{-\frac{1}{4}}e^{-i\int(E-V)^{\frac{1}{2}}dx}.$$

Under the condition (1.1) there is a trace of scattering theory even for $E \approx 0$ (but $\geq 0$) in the sense that the oscillatory behaviour survives in this regime (since $\int(-V)^{\frac{1}{2}}dx \sim |x|^{1-\frac{\mu}{2}} \to \infty$). Moreover, clearly the ansatz suggests that zero is not an eigenvalue, and it is also suggestive for which weights one needs in (1.2). We remark that indeed (1.2) can be proved for $d = 1$ by WKB-methods, see [Yaf82].

Let us state our main results precisely. Let $0 < \theta < \pi$ and define

$$\Gamma_\theta = \{z \in \mathbb{C} \setminus \{0\} \,\big|\, |z| \le 1, \arg z \in (0, \theta)\}. \tag{1.3}$$

For a Hilbert space $\mathcal{H}$ (which in our case mostly will be $L^2(\mathbb{R}^d)$) we denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$. A $\mathcal{B}(\mathcal{H})$-valued function $B(\cdot)$ on $\Gamma_\theta$ is said to be uniformly Hölder continuous in $\Gamma_\theta$ if there exist $C, \gamma > 0$ such that

$$\|B(z_1) - B(z_2)\| \le C|z_1 - z_2|^\gamma \text{ for all } z_1, z_2 \in \Gamma_\theta.$$

A main result of the present paper is the following *limiting absorption principle* at zero energy for a Schrödinger operator $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R}^d)$. We recall the notation $R(\zeta) = (H - \zeta)^{-1}$. Notice that it is enough to impose the conditions (1) and (3) near infinity. We will give arguments to this effect in Section 3.

**Theorem 1.1** (Limiting absorption principle). *Let $V(x) = V_1(x) + V_2(x)$, $x \in \mathbb{R}^d$, be a real-valued potential. Suppose there exists $0 < \mu < 2$ such that $V$ satisfies the conditions* (1)–(6) *below.*

(1) *There exists $\epsilon_1 > 0$ such that $V_1(x) \le -\epsilon_1 \langle x\rangle^{-\mu}$; $\langle x\rangle = \sqrt{1 + x^2}$.*
(2) *For all $\alpha \in (\mathbb{N} \cup \{0\})^d$ there exists $C_\alpha > 0$ such that*
$$\langle x\rangle^{\mu+|\alpha|}|\partial^\alpha V_1(x)| \le C_\alpha.$$
(3) *There exists $\epsilon_2 > 0$ such that $-|x|^{-2}\left(x \cdot \nabla(|x|^2 V_1)\right) \ge -\epsilon_2 V_1$.*
(4) *$V_2(-\Delta + i)^{-1}$ is a compact operator on $L^2(\mathbb{R}^d)$.*
(5) *There exists $\delta, C, R > 0$ such that*
$$|V_2(x)| \le C|x|^{-1-\mu/2-\delta},$$
*for $|x| > R$.*
(6) *$V$ satisfies unique continuation at infinity (see Assumption 2.1 in Section 2).*

*Then for all $s \in (\frac{1}{2} + \frac{\mu}{4}, \frac{1}{2} + \frac{\mu}{4} + \delta)$ and all $0 < \theta < \pi$ the family of operators $B(\zeta) = \langle x\rangle^{-s}R(\zeta)\langle x\rangle^{-s}$ is uniformly Hölder continuous in $\Gamma_\theta$. For $s > 3/2(1 + \mu/2)$ (in the case $\delta > (1 + \mu/2)$) the Hölder exponent may be chosen to be $\gamma = 1$, and for $s \le 3/2(1 + \mu/2)$ to be any $\gamma < s(1 + \mu/2)^{-1} - 1/2$. In particular there exists $C_{s,\theta} > 0$ such that*

$$\sup_{\zeta\in\Gamma_\theta} \left\|\langle x\rangle^{-s}R(\zeta)\langle x\rangle^{-s}\right\| \le C_{s,\theta}, \tag{1.4}$$

*and the limits*

$$\langle x\rangle^{-s}R(0+i0)\langle x\rangle^{-s}\equiv\lim_{\zeta\to 0,\zeta\in\Gamma_\theta}\langle x\rangle^{-s}R(\zeta)\langle x\rangle^{-s},$$

$$\langle x\rangle^{-s}R(0-i0)\ \langle x\rangle^{-s}\equiv\lim_{\zeta\to 0,\zeta\in\Gamma_\theta}\langle x\rangle^{-s}R\left(\bar{\zeta}\right)\langle x\rangle^{-s}$$

*exist in* $\mathcal{B}(L^2(\mathbb{R}^d))$.

From the facts that $V$ is negative at infinity and decays slower than $r^{-2}$ it follows, cf. [ReSi78, Theorem XIII.6 (Vol. IV, p. 87)], that $H$ has infinitely many negative eigenvalues accumulating at zero. Clearly by (1.4), zero is not an eigenvalue of $H$. (See Section 2 for a more general result.)

We also get existence of limits for powers of the resolvent. The asymptotic expansion (1.2) is an easy consequence of Theorem 1.2 below.

**Theorem 1.2** (Iterated resolvents)**.** *Let* $V=V_1+V_2$ *satisfy the conditions in Theorem 1.1 with* (5) *replaced by*

(5') $\operatorname{supp}(V_2)$ *is compact.*

*Let* $H=-\Delta+V$ *and* $R(\zeta)=(H-\zeta)^{-1}$ *be given as before and* $k(x)=\langle x\rangle^{1+\mu/2}$, $x\in\mathbb{R}^d$. *Let* $m\in\mathbb{N}$, $\theta\in(0,\pi)$ *and* $\epsilon>0$. *Then there exists* $C>0$ *such that*

$$\left\|k^{-(m-1/2)-\epsilon}R(\zeta)^m k^{-(m-1/2)-\epsilon}\right\|\leq C, \tag{1.5}$$

*for all* $\zeta\in\Gamma_\theta$.
*Furthermore, the function*

$$\zeta\mapsto k^{-(m-1/2)-\epsilon}R(\zeta)^m k^{-(m-1/2)-\epsilon},$$

*extends to a continuous function on* $\overline{\Gamma}_\theta$ *(the closure of* $\Gamma_\theta$*).*

**Remarks 1.3.** 1) For fixed $m\in\mathbb{N}$ the bound $V_2=O(k^{-m-\epsilon})$ suffices for (1.5). (This follows readily from our proof by a little more bookkeeping.)

2) In dimension $d=1$ one may show by WKB-analysis, see for example [Yaf82], that (1.5) is optimal in the following sense: There exists $\phi\in L^d(\mathbb{R}^d)$ such that

$$\sup_{\zeta\in\Gamma_\theta}\left\|k^{-(m-1/2)}R(\zeta)^m\phi\right\|=\infty.$$

(This may be done using $R(\zeta)^m=((m-1)!)^{-1}\frac{d^{m-1}}{d\zeta^{m-1}}R(\zeta)$ and the standard formula for $R(\zeta)$ in terms of outgoing/incoming generalized eigenfunctions; analysing the large $x$-asymptotics of the $(m-1)$'th derivative w.r.t. $\zeta$ of those functions yields the result.) We would expect that the same result is true in any dimension, also for potentials that are not radially symmetric.

Due to Theorem 1.1 we may define

$$E'(+0) = (2\pi i)^{-1}\,(R(0+i0) - R(0-i0)) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2);$$
$$\mathcal{H}_1 = k^{-1/2-\epsilon} L^2(\mathbb{R}^d),\ \mathcal{H}_2 = k^{1/2+\epsilon} L^2(\mathbb{R}^d).$$

Motivated by the next result we shall prove (in Section 5) that

$$E'(+0) \neq 0. \tag{1.6}$$

Let $F(|x| < C)$ denote the multiplication operator by the characteristic function of $\{x|\ |x| < C\}$.

**Theorem 1.4** (Local decay estimates)**.** *Under the conditions of Theorem 1.2:*

(i) *For all* $s > \frac{5}{2}(1 + \frac{\mu}{2})$ *and* $f \in C_0^\infty(\mathbb{R})$

$$\|\langle x\rangle^{-s}\Big(e^{-itH}(f 1_{[0,\infty)})(H) + it^{-1} f(0) E'(+0)\Big)\langle x\rangle^{-s}\| = O(t^{-2}).$$

(ii) *For all* $0 \le \epsilon' < \epsilon \le 1$*, all real* $s$ *such that for some integer* $m$

$$s(1+\mu/2)^{-1} + \frac{1}{2} > m > \frac{1}{2} + \epsilon^{-1}(1 + \frac{1}{2}\epsilon'),$$

*and for all* $f \in C_0^\infty(\mathbb{R})$

$$\|F(|x| < t^\kappa) e^{-itH}(f 1_{[0,\infty)})(H)\langle x\rangle^{-s}\| = O(t^{-(1+\epsilon')\frac{1}{2}}); \tag{1.7}$$
$$\kappa = (1-\epsilon)(1+\mu/2)^{-1}.$$

**Remarks 1.5.** 1) By time reversal invariance there are similar bounds for $t \to -\infty$.

2) Due to (1.6) and Theorem 1.4 (i) the best one could hope for to the right in (1.7) would be the bound $O(t^{-1})$ (for $f(0) \neq 0$). Moreover we would expect that $\kappa = (1 + \mu/2)^{-1}$ is indeed the borderline for this kind of low energy, minimal velocity estimate; see Theorem 4.7 for an analogous bound in classical mechanics.

3) If $V_2 \in C_0^\infty(\mathbb{R}^d)$ one may take $f = 1$ in Theorem 1.4 (i) and (ii). This follows readily from the given statements and well-established high energy estimates, see [Kit84, Theorem 1.1], [CyPe84, Theorem 1] or [Jen89, Theorem 1.2 (ii)]. Some local singularities may be included (see for example [Jen89, Section 6] for a specific treatment of the purely Coulombic case of the Hydrogen atom).

There are other (main) results that are in fact important in the proof of the above Theorems 1.1 and 1.2. They are however too complicated to be stated in this introduction. They concern *microlocal estimates* of the resolvent.

One perspective of our results is zero-energy asymptotics of the scattering matrix (or related quantities) for the class of potentials considered in this paper. It is well-known that microlocal estimates are important in the study of the scattering matrix for positive energies (and in the high energy regime), cf. the seminal work [IsKi85]. We would hope that our estimates would open up for results on asymptotics of scattering quantities; this needs elaboration elsewhere.

Let us also mention the perspective of asymptotics at thresholds for the many-body problem. This appears, although very interesting, far more ambitious. In the time-dependent approach to the asymptotic completeness problem control of the dynamical behaviour at thresholds is important. We were motivated to look at the above two-body problems by the papers [Gér93] and [Sk03] on three-body asymptotic completeness with long-range potentials; some estimates, similar in nature to (1.7), play an important role in these papers.

To minimize confusion, let us mention a number of (standard) notations and conventions that will be used in the paper.

We will write $\Re z = \Re(z), \Im z = \Im(z)$ for the real and imaginary part of $z$, both in the case where $z$ is a complex number and, more generally, when it is an operator.

We define for any open $U \subseteq \mathbb{R}^d$

$$\mathcal{B}^\infty(U) = \left\{ f \in C^\infty(U) \,\middle|\, \partial^\alpha f \in L^\infty(U) \text{ for all } \alpha \in (\mathbb{N} \cup \{0\})^d \right\}.$$

Apart from the notation $\langle x \rangle = \sqrt{1+x^2}$, used above, we will also need the notation $p = -i\nabla$ and $A = (x \cdot p + p \cdot x)/2$.

The virial $W$ of the potential $V$ will play an important role throughout the paper. It is defined by

$$W = -2V - x \cdot \nabla V. \tag{1.8}$$

We recall the (formal) identity $i[H, A] = 2H + W$. The assumptions (1) and (3) of Theorem 1.1 in the case $V_2 = 0$ yield

$$W(x) \geq \epsilon_1 \epsilon_2 \langle x \rangle^{-\mu}, \tag{1.9}$$

so in particular, $W$ is positive in this case.

In Section 2 we will prove a result of independent interest (Theorem 2.4), namely the non-existence of eigenvalue at zero under slightly weaker assumptions than those of Theorem 1.1. This is a basic result which will allow us to prove Theorem 1.1 by a perturbative method: First we shall prove (1.4) in the case $V_2 = 0$, and then later on in general by perturbation theory and Theorem 2.4. We prove Theorem 2.4 by extending the method of proof of [ReSi78, Theorem XIII.58] dealing with absence of positive eigenvalues to incorporate the case $E = 0$.

The bound (1.4) of Theorem 1.1 in the case $V_2 = 0$ is shown in Section 3—this is done using a non-standard Mourre theory.

In Section 4 we prove Theorem 1.2 in the case $V_2 = 0$. The proof is accomplished using the strategy from [GIS96] and an energy-dependent pseudodifferential calculus. Some technical verifications concerning this calculus will be given in Appendix B. In Appendix A a certain 'algebraic verification' is given. Using Theorems 1.2 (with $V_2 = 0$) and 2.4 we finally prove Theorem 1.1 in the general case in Section 5. That section also contains the proof of Theorem 1.2, (1.6) and Theorem 1.4.

## 2. Absence of eigenvalue at zero

In this section we will prove a basic result, namely that for long-range potentials $V$ that are *negative* at infinity, there is no $L^2$-eigenfunction with energy zero. That is the result of Theorem 2.4. In Subsection 5.1 we will use this insight to obtain the limiting absorption principle at zero energy, Theorem 1.1.

We intend to generalize the 'Kato-Agmon-Simon Theorem' [ReSi78, Theorem XIII.58] by using a modification of its proof. In particular, we shall apply unique continuation. The proof uses ODE techniques in the radial coordinate, so let us specify the notation $x = r\omega \in \mathbb{R}^d$, with $\omega \in \mathbb{S}^{d-1}$.

The conditions which exclude zero-energy eigenfunctions are given in Assumptions 2.1 and 2.3.

**Assumption 2.1** (Unique continuation at infinity)**.** The function $V : \mathbb{R}^d \to \mathbb{R}$ is measurable, and if $u \in H^2(\mathbb{R}^d)$, $u = 0$ in a neighbourhood of $\infty$, the product $V\psi \in L^2(\mathbb{R}^d)$ and $u$ is a distributional solution to

$$-\Delta u + Vu = 0,$$

then $u = 0$.

**Remark 2.2.** Different results exist in this direction. The condition $V \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$ suffices for $d \geq 3$, see [JerKen85].

**Assumption 2.3.** The function $V$ can be written as $V = V_1 + V_2$, such that: For some $s \in [0, 1)$, some $R, C > 0$ and a positive differentiable function $h = h(r)$ defined on $[R, \infty)$ we have

(1) $V_1$ and $V_2$ are bounded on $|x| > R$, and $V_1$ is negative on $|x| > R$.
(2) $\sup_{\omega \in S^{d-1}} \frac{d}{dr}(r^{s+1}V_1(r\omega)) \leq -r^s h^2(r)$ when $r > R$.
(3) $r^{-1} + r \sup_{\omega \in S^{d-1}} |V_2(r\omega)| = o(h)$ as $r \to \infty$.
(4) $h'(r) \leq Ch^2(r)$ on $|x| > R$.

With the above assumptions we can prove the absence of zero-energy eigenstates.

**Theorem 2.4.** *Suppose $V = V_1 + V_2$ satisfies Assumptions 2.1 and 2.3. Suppose furthermore that $\psi \in H^2_{\rm loc}(\mathbb{R}^d)$ satisfies* (1)–(3) *below.*

(1) $\int_{|x|>R} h^2(r)|\psi(x)|^2\,dx < \infty$ *and* $\int_{|x|>R} |V_1(x)||\psi(x)|^2\,dx < \infty$.

(2) $p_j\psi \in L^2(\mathbb{R}^d)$; $j = 1, \cdots, d$.

(3) *The product $V\psi \in L^2_{\rm loc}(\mathbb{R}^d)$, and $(-\Delta + V)\psi = 0$ in the sense of distributions.*

*Then $\psi = 0$.*

**Remarks 2.5.** 1) The assumptions in Theorem 1.1 are stronger than Assumption 2.3: Take $h = \epsilon r^{-\mu/2}$ for a small $\epsilon > 0$ and $s$ close to 1.

2) Suppose Assumption 2.3 and that $\psi \in H^2_{\rm loc}(\mathbb{R}^d)$ obeys (1), (2) and (3) except that the condition $(-\Delta + V)\psi = 0$ is only required to be fulfilled outside some sufficiently large ball $B_\rho = \{x \in \mathbb{R}^d |\ |x| \le \rho\}$, then our proof yields the conclusion that $\psi = 0$ outside $B_\rho$. (This remark will be used in Subsection 5.3.)

3) Using the negativity of $V_1$ we see that

$$\sup_{\omega \in S^{d-1}} \frac{d}{dr} r^2 V_1(r\omega) \le -rh^2(r); \text{ for } r > R. \tag{2.1}$$

4) By integrating (2.1) we get the bound

$$\limsup_{\rho\to\infty} \rho^{-2} \int_R^\rho rh^2(r)dr \le \sup_{|x|>R} |V_1|,$$

which essentially is a boundedness condition on $h$.

5) A slightly more general result may be obtained in terms of an $h$ that depends on the angle $\omega$ as well.

**Example 2.6.** Suppose that for some $\mu \in [0, 2)$, $V_1(x) = v(\omega)r^{-\mu} + R(x)$ where $\sup v(\omega) < 0$, $R(x) = o(r^{-\mu})$ and $\sup_{\omega\in S^{d-1}} \frac{d}{dr}R(r\omega) = o(r^{-\mu-1})$, and $V_2 = o(r^{-1-\mu/2})$. Then Assumption 2.3 holds, cf. Remarks 2.5 1. The particular case given by putting $\mu = 0$ and $v$ equal to a negative constant (in that case we may take $s = 0$ in Assumption 2.3) yields the Kato-Agmon-Simon theorem on absence of positive eigenvalues for Schrödinger operators [ReSi78, Theorem XIII.58]. (Notice that any positive energy may be absorbed into $V_1$.)

*Proof of Theorem 2.4.* The proof is based on unique continuation, so we aim at proving that $\psi$ vanishes identically on $\mathbb{R}^d \setminus \{|x| \le R_1\}$ for some (sufficiently big) $R_1$. Let $B$ be the (negative) Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ such that

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}B.$$

We will need the inner product $\langle \cdot,\cdot\rangle_{L^2(\mathbb{S}^{d-1})}$ and the associated norm $\|\cdot\|_{L^2(\mathbb{S}^{d-1})}$ repeatedly. For shortness we will therefore just write $\langle \cdot,\cdot\rangle$ and $\|\cdot\|$. Unless otherwise stated, all inner products and norms in the proof will refer to these—this is contrary to the convention in the later sections of the paper where inner products and norms are on $L^2(\mathbb{R}^d)$ unless otherwise stated.

Let $\psi$ be the function from the statement of the theorem. We may assume that $\psi$ is real-valued. We define, for $r \in (0,\infty)$, the function $w_\psi(r,\cdot) = w(r,\cdot) \in L^2(\mathbb{S}^{d-1})$ by

$$w(r,\omega) = r^{(d-1)/2}\psi(r\omega).$$

Notice, that if $\phi \in C_0^\infty(\mathbb{R}^d)$, then

$$w_{H\phi} = -w_\phi'' - r^{-2}Bw_\phi + \tfrac{1}{4}(d-1)(d-3)r^{-2}w_\phi + Vw_\phi. \tag{2.2}$$

We define furthermore,

$$F(r) = \|w'\|^2 + r^{-2}\langle w, Bw\rangle - \langle w, V_1 w\rangle - sr^{-1}\langle w, w'\rangle, \tag{2.3}$$

where $w'(r,\omega) = \frac{\partial}{\partial r}w(r,\omega)$.

One may note that since $V$ is locally bounded outside a sufficiently large ball centered at the origin, one gets by elliptic regularity (see for instance [GiTr01]) that $\psi \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^d \setminus \{|x| \le R\})$ (for $R$ given as in Assumption 2.3 and for all $\alpha \in [0,1)$). Thus $F$ is a Hölder continuous function.

The first important step in the proof is to establish that $F$ is integrable at infinity, i.e. $F \in L^1((R,\infty); dr)$. In order to see this, notice that for $\phi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_0^\infty \left\{ \|r^{(d-1)/2}\frac{\partial}{\partial r}\phi(r\omega)\|^2 - r^{-2}\langle w_\phi, Bw_\phi\rangle \right\} dr = \||p|\phi\|^2_{L^2(\mathbb{R}^d)}.$$

This implies, since $-B \ge 0$,

$$\begin{aligned}\int_R^\infty \|w_\phi'\|^2\,dr &\le C\int_R^\infty \left( \|r^{(d-1)/2}\frac{\partial}{\partial r}\phi(r\omega)\|^2 + r^{-2}\|w_\phi\|^2 \right) dr \\ &\le C'\left( \||p|\phi\|^2_{L^2(\mathbb{R}^d)} + \int_{|x|\ge R} h(r)^2|\phi(x)|^2\,dx \right).\end{aligned}$$

This, and an approximation argument, proves that the first term in (2.3) is integrable. Similar considerations apply to the other terms. Thus $F \in L^1((R,\infty); dr)$.

We will next compute $(rF)'$ using that formally

$$0 = w_{H\psi} = -w'' - r^{-2}Bw + \tfrac{1}{4}(d-1)(d-3)r^{-2}w + Vw. \tag{2.4}$$

Using (2.4), a formal computation gives

$$\begin{aligned}(rF(r))' = 2\langle w', [rV_2 + \tfrac{1}{4}(d-1)(d-3)r^{-1}]w\rangle + (1-s)\|w'\|^2 \\ - \langle w, (rV_1)'w\rangle - (1-s)r^{-2}\langle w, Bw\rangle \\ - s\langle w, [\tfrac{1}{4}(d-1)(d-3)r^{-2} + V]w\rangle. \qquad (2.5)\end{aligned}$$

Let $\delta = 1 - s > 0$. Using Cauchy-Schwarz, we estimate the first term in (2.5) below by

$$-\delta\|w'\|^2 - \langle w, o(h^2)w\rangle. \qquad (2.6)$$

For the third term we estimate

$$-(rV_1)' = -r^{-s}\frac{d}{dr}(r^{s+1}V_1) + sV_1 \geq h^2 + sV_1. \qquad (2.7)$$

Notice that the contribution from $sV_1$ to the right and the contribution from the term $V_1$ in the fifth term of (2.5) in fact cancel. Clearly the fourth term is non-negative. Hence we conclude from (2.5), (2.6) and (2.7) that for a sufficiently large $R_1 \geq R$

$$\frac{d}{dr}(rF(r)) \geq \langle w, (h^2 + o(h^2))w\rangle \geq 0 \text{ when } r > R_1. \qquad (2.8)$$

Therefore we have, for all $r > r_1 > R_1$,

$$F(r) \geq \frac{r_1 F(r_1)}{r}. \qquad (2.9)$$

The deduction of (2.9) above was based on formal calculations. However, it is standard to use approximation arguments (see the proof of [ReSi78, Thm. XIII.58]) to justify the inequality rigorously. Below we will do a similar formal calculation without comment—again it can easily be justified using approximations by smooth functions.

Since $F \in L^1(dr)$ it follows from (2.9) that

$$F(r) \leq 0 \text{ for all } r > R_1. \qquad (2.10)$$

Next we define, with $w_m = r^m w$ for $m > 0$,

$$\begin{aligned}G(m,r) = \|w_m'\|^2 + r^{-2}\langle w_m, Bw_m\rangle \\ + \langle w_m, (m(m+1)r^{-2} - g - V_1)w_m\rangle, \qquad (2.11)\end{aligned}$$

where $g = \epsilon r^{-1}h(r)$ with $\epsilon = (2C)^{-1}$. Here $C$ is the constant from Assumption 2.3. The function $w_m$ satisfies

$$w_m'' - 2mr^{-1}w_m' + r^{-2}\left(m(m+1) - \tfrac{1}{4}(d-1)(d-3) + B\right)w_m - Vw_m \\ = 0.$$

We now compute, using (2.1) in the first inequality, for $r > \tilde{R}_1$ and $m > M$, for some large positive $\tilde{R}_1$ and $M$:

$$\begin{aligned}
&(2r)^{-1}\frac{d}{dr}(r^2G(m,r)) \\
&= \langle w_m, (rV_2 - rg + \tfrac{1}{4}(d-1)(d-3)r^{-1})w'_m\rangle \\
&\quad + (2m+1)\|w'_m\|^2 - \langle w_m, (2r)^{-1}(r^2(g+V_1))'w_m\rangle \\
&\geq (2m+1)\|w'_m\|^2 + (\tfrac{1}{2}h^2 - (2r)^{-1}(r^2g)')\|w_m\|^2 \\
&\quad + \langle w_m, (o(h) - rg)w'_m\rangle \\
&= (2m+1)\|w'_m\|^2 + (\tfrac{1}{4}h^2 + o(h^2))\|w_m\|^2 + \langle w_m, O(h)w'_m\rangle \\
&\geq 0;\ \text{for } r > \tilde{R}_1, m > M.
\end{aligned} \tag{2.12}$$

For convenience we assume henceforth that $\tilde{R}_1 = R_1$. We learn from (2.12) that $r^2G(m,r)$ is non-decreasing in $r > R_1$, provided that $m > M$.

We will now combine (2.10) and (2.12) to prove that $w$ (and therefore $\psi$) vanishes in a neighbourhood of infinity. Clearly

$$\begin{aligned}
&r^{-2m}G(m,r) = \\
&\|w' + mr^{-1}w\|^2 + r^{-2}\langle w, Bw\rangle + \langle w, (m(m+1)r^{-2} - g - V_1)w\rangle.
\end{aligned} \tag{2.13}$$

Suppose $w(r_1) \neq 0$ for some $r_1 > R_1$. Then we have for some large enough $m_1 > M$,

$$r_1^{-2}\langle w(r_1), Bw(r_1)\rangle + \langle w(r_1), (m_1(m_1+1)r_1^{-2} - g(r_1))w(r_1)\rangle > 0. \tag{2.14}$$

Clearly (2.12), (2.13) and (2.14) yield

$$G(m_1, r) > 0 \text{ for all } r \geq r_1. \tag{2.15}$$

Next, we get from Assumption 2.3 (3) that $\lim_{r\to\infty} r^2g(r) = \infty$. Therefore, for some $R_2 > r_1$, we have

$$\left[\frac{2m_1 + s}{2} + m_1(2m_1+1)\right] r^{-2} - g(r) \leq 0 \text{ for all } r > R_2. \tag{2.16}$$

Since by assumption

$$\int_{r\geq R_2} r^{-2}\|w(r)\|^2\, dr < \infty,$$

the function $r \mapsto r^{-1}\|w(r)\|^2$ cannot be strictly increasing near infinity. We pick $r_2 > R_2$, such that $(r^{-1}\|w(r)\|^2)'\,\big|_{r=r_2} \leq 0$ and estimate using (2.15) and (2.16),

$$0 < r_2^{-2m_1}G(m_1, r_2) \leq F(r_2),$$

which by (2.10) is impossible.

We conclude that $w(r) = 0$ for all $r > R_1$. Since therefore $\psi = 0$ on a neighbourhood of infinity, $\psi \in H^2(\mathbb{R}^d)$ and $V\psi \in L^2(\mathbb{R}^d)$. From Assumption 2.1 we conclude that $\psi = 0$. □

## 3. Limiting absorption principle for $V_1$

In this section we will prove the limiting absorption principle bound (1.4) in the case $V_2 = 0$ in the set of conditions of Theorem 1.1. A somewhat more general result will be stated in Corollary 3.5 below. Explicitly, we study the operator $H_1 = p^2 + V_1$, where $V = V_1$ satisfies the assumptions of Theorem 1.1. In Section 5 we will make a perturbative argument to include $V_2$ (the short-range, non sign-definite part of the potential). For simplicity we will skip the indices in this section and write $H$ and $V$ instead of $H_1$ and $V_1$, respectively. Our approach is very different from [Nak94], where a similar (but much weaker) result is proved. There an auxiliary operator (the inverse of the Birman-Schwinger kernel) is studied, whereas we do Mourre theory directly on $H$. Since we do not have a positive commutator in the sense of Mourre [Mou81] at zero-energy, this will be a non-standard Mourre theory. (For another non-standard Mourre theory, used in a very different setting, we refer to [Her91].)

To motivate our strategy of proof of Theorem 1.1 we notice that if $V = V_1 + V_2$ only satisfies the bounds of the assumptions (1) and (3) for $|x| > R$ (and in addition (2), (4) and (5)) then there exists another decomposition $V = \tilde{V}_1 + \tilde{V}_2$ for which (1)–(5) hold. To prove this we define $\tilde{V}_1 = V_1\chi_+(r/n) - \epsilon_1\langle x\rangle^{-\mu}\chi_-(r/n)$. Here $1 = \chi_+ + \chi_-$ is a standard, smooth partition of unity on $\mathbb{R}_+$: $\operatorname{supp}\chi_+ = [1,\infty)$, $\chi_+(t) = 1$ for $t \geq 2$. Clearly we may assume that $\epsilon_2 < 2-\mu$. We claim that for any $n > R$

$$-x \cdot \nabla\tilde{V}_1 \geq (2-\epsilon_2)\tilde{V}_1. \tag{3.1}$$

Using that (3.1) is satisfied for $V_1$ on $|x| > R$, we find

$$\begin{aligned}
-x \cdot \nabla\tilde{V}_1 \geq{}& (2-\epsilon_2)\tilde{V}_1 \\
&+ \epsilon_1\langle x\rangle^{-\mu}\chi_-(r/n)\left((2-\epsilon_2) + \langle x\rangle^{\mu}(x\cdot\nabla\langle x\rangle^{-\mu})\right) \\
&- \frac{r}{n}\chi_-'(r/n)\langle x\rangle^{-\mu}\left(-\langle x\rangle^{\mu}V_1 - \epsilon_1\right) \\
\geq{}& (2-\epsilon_2)\tilde{V}_1,
\end{aligned}$$

proving (3.1).

We define the operator $\langle A\rangle = (C + A^2)^{1/2}$ with $C \gg 1$ fixed (yielding better commutation properties than with $C = 1$, cf. [MøSk02, Lemma

2.5]). For $E \geq 0$, let $f_E$ be the function

$$f_E = \left(E + \langle x\rangle^{-\mu}\right)^{1/2}. \tag{3.2}$$

For the purpose of proving (1.4) one could replace $f_E$ by $\langle x\rangle^{-\mu/2}$ everywhere in this section. But we include it here since the estimates we establish (with $f_E$) will be important in Section 4.

It is easy to see that $p^2 + f_E^2$ defines a positive, self-adjoint operator, so we can use the spectral theorem to define, for $\zeta \in \mathbb{C}$, the operator

$$\gamma = \gamma_{|\zeta|} = (p^2 + f_{|\zeta|}^2)^{1/2}, \tag{3.3}$$

Recall from (1.8) and (1.9) that

$$W = -2V - x \cdot \nabla V \geq c\langle x\rangle^{-\mu}.$$

Consider

$$R_\zeta(\epsilon) = (H - i\epsilon i[H, A] - \zeta)^{-1}, \tag{3.4}$$

in a range of the form

$$0 < \Im\zeta,\ |\zeta| \leq 1,\ -2\epsilon_0\Re\zeta \leq \Im\zeta;\ 0 < \epsilon \leq \epsilon_0' \leq \epsilon_0. \tag{3.5}$$

(The existence of the inverse in the definition of $R_\zeta(\epsilon)$ follows from the calculation (3.7) and the theory of numerical range [Kat95, Section V.3.1].) The results given below have completely analogous versions with the signs of $\Im\zeta$ and $\epsilon$ both negative; for convenience we shall only explicitly state the results for the case of positive signs.

We first formulate and prove a version of the quadratic estimate of [Mou81].

**Lemma 3.1.** *For all $\epsilon_0 > 0$ there exists $\epsilon_0' > 0$ such that for all positive $\epsilon \leq \epsilon_0'$ and all $\zeta$ as in* (3.5)*, we have with $\gamma = \gamma_{|\zeta|}$ given by* (3.3) *and for any bounded operator $B$*

$$\|\gamma R_\zeta(\epsilon)B\|^2 \leq C\epsilon^{-1}\|B^* R_\zeta(\epsilon)B\|. \tag{3.6}$$

**Remark 3.2.** Note that $\zeta$ appears twice in (3.6): In the definition of $\gamma = \gamma_{|\zeta|}$ and in $R_\zeta(\epsilon)$.

*Proof.* Let $T = \gamma R_\zeta(\epsilon)B$. Then by the definition of $\gamma$

$$T^*T = (R_\zeta(\epsilon)B)^*(p^2 + |\zeta| + \langle x\rangle^{-\mu})R_\zeta(\epsilon)B.$$

Notice that with $C_1 = 2C_2 - 1$ we have, using $i[H, A] = 2H + W$,

$$\begin{aligned} &- C_1\Re\left(H - i\epsilon i[H, A] - \zeta\right) - \epsilon^{-1}C_2\Im\left(H - i\epsilon i[H, A] - \zeta\right) \\ &= p^2 + C_2W + V + g_\zeta(\epsilon);\ g_\zeta(\epsilon) = C_1\Re\zeta + \epsilon^{-1}C_2\Im\zeta. \end{aligned} \tag{3.7}$$

From the hypothesis on $W$ and $V$, we have $C_2W + V \geq \langle x\rangle^{-\mu}$ for $C_2$ sufficiently big. Fix such $C_2$. Next, notice that there exists a positive

$\epsilon_0'$ such that for all $\zeta$ obeying (3.5) and for all $\epsilon \le \epsilon_0'$ indeed $|\zeta| \le |\Re\zeta| + |\Im\zeta| \le g_\zeta(\epsilon)$. Therefore, we can estimate

$$T^*T \le -C_1 \Re(B^* R_\zeta(\epsilon)^* B) - C_2 \epsilon^{-1} \Im(B^* R_\zeta(\epsilon)^* B),$$

yielding (3.6). □

Notice that Lemma 3.1 also holds with $\gamma$ replaced by $f_{|\zeta|}, \langle x \rangle^{-\mu/2}, p_j$ or $|p|$. This follows from the proof since, for instance $\langle x \rangle^{-\mu} \le \gamma^2$.

**Lemma 3.3.** *The operator $R_\zeta(\epsilon)$ (defined in (3.4)) has the following derivative,*

$$\frac{d}{d\epsilon} R_\zeta(\epsilon) = (1 - 2i\epsilon)^{-1} \left\{ R_\zeta(\epsilon) A - A R_\zeta(\epsilon) + i\epsilon R_\zeta(\epsilon)(x \cdot \nabla W) R_\zeta(\epsilon) \right\}.$$

*Proof.* We compute

$$\begin{aligned}
&\frac{d}{d\epsilon} R_\zeta(\epsilon) = -R_\zeta(\epsilon)[H, A] R_\zeta(\epsilon) \\
&= R_\zeta(\epsilon) A - A R_\zeta(\epsilon) + \epsilon R_\zeta(\epsilon)[[H, A], A] R_\zeta(\epsilon) \\
&= R_\zeta(\epsilon) A - A R_\zeta(\epsilon) - 2i\epsilon R_\zeta(\epsilon)[H, A] R_\zeta(\epsilon) + i\epsilon R_\zeta(\epsilon)(x \cdot \nabla W) R_\zeta(\epsilon).
\end{aligned}$$

We now insert the first line of the above calculation in the last and isolate $\frac{d}{d\epsilon} R_\zeta(\epsilon)$ in the resulting equation. □

Using Mourre's technique of differential inequalities, we can (with some work) prove limiting absorption principle estimates.

**Lemma 3.4.** *For all $\epsilon_0 > 0$ there exists $\epsilon_0' > 0$ such that for all positive $\epsilon \le \epsilon_0'$, all positive $\delta < 1/2$ and all $\zeta$ as in (3.5) we have the following estimate uniformly in $\zeta$ and $\epsilon$, with $f = f_{|\zeta|}$ given by (3.2),*

$$\left\| \langle \epsilon A \rangle^{-1} f_{|\zeta|}^{1/2} \langle x \rangle^{-(1/2+\delta)} R_\zeta(\epsilon) \langle x \rangle^{-(1/2+\delta)} f_{|\zeta|}^{1/2} \langle \epsilon A \rangle^{-1} \right\| \le C_\delta. \tag{3.8}$$

*Proof.* We study the function

$$F_\zeta(\epsilon) = \langle \epsilon A \rangle^{-1} f^{1/2} \langle x \rangle^{-s} R_\zeta(\epsilon) \langle x \rangle^{-s} f^{1/2} \langle \epsilon A \rangle^{-1};\ s = 1/2 + \delta.$$

From Lemma 3.1 we get

$$\|F_\zeta(\epsilon)\| \le C\epsilon^{-1}. \tag{3.9}$$

We will prove

$$\|\frac{d}{d\epsilon} F_\zeta(\epsilon)\| \le C \left\{ \|F_\zeta(\epsilon)\| + \epsilon^{-1+\delta'} \|F_\zeta(\epsilon)\|^{1/2} \right\}, \tag{3.10}$$

for any positive $\delta' \le \delta$. Combining (3.9) and (3.10) we clearly get (3.8) by repeated integration.

To prove (3.10) we calculate $\frac{d}{d\epsilon}F_\zeta(\epsilon)$ using Lemma 3.3. After commutation we need to estimate the following terms (and some 'adjoint' expressions):

$$\big\|\langle \epsilon A\rangle^{-1} A f^{1/2}\langle x\rangle^{-s} R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\big\|\,, \tag{3.11}$$

$$\big\|\langle \epsilon A\rangle^{-3} \epsilon A^2 f^{1/2}\langle x\rangle^{-s} R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\big\|\,, \tag{3.12}$$

$$\epsilon\Big\|\langle \epsilon A\rangle^{-1} f^{1/2}\langle x\rangle^{-s} R_\zeta(\epsilon) \times (x\cdot\nabla W) R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\Big\|. \tag{3.13}$$

Here and below it is useful to note the uniform bound (4.9) which is also valid with the present definition of $f_E$.

The term $x\cdot\nabla W$ of (3.13) satisfies $|x\cdot\nabla W|\leq C\langle x\rangle^{-\mu}$. Whence, using Lemma 3.1, the expression (3.13) may be estimated in agreement with (3.10).

So it suffices to prove that

$$\|AF_\zeta(\epsilon)\|\leq C\epsilon^{-1+\delta'}\|F_\zeta(\epsilon)\|^{1/2}. \tag{3.14}$$

Using that

$$\|\langle \epsilon A\rangle^{-1}\langle A\rangle^{1/2-\delta'}\|\leq C\epsilon^{-1/2+\delta'},$$

(3.14) will follow from the bound

$$\|\langle A\rangle^{1/2+\delta'} f^{1/2}\langle x\rangle^{-s} R_\zeta(\epsilon)\psi\|^2\leq C\epsilon^{-1}\|F_\zeta(\epsilon)\|, \tag{3.15}$$

where $\psi = f^{1/2}\langle x\rangle^{-s}\langle \epsilon A\rangle^{-1}\phi$; $\|\phi\|=1$. For this bound we estimate

$$\begin{aligned}
&\left(\langle A\rangle^{1/2+\delta'} f^{1/2}\langle x\rangle^{-s}\right)^{*}\langle A\rangle^{1/2+\delta'}\langle x\rangle^{-s} f^{1/2}\\
&\leq C_1 f\langle x\rangle^{-2s} + f^{1/2}\langle x\rangle^{-s}A^2\langle A\rangle^{2\delta'-1}\langle x\rangle^{-s}f^{1/2}\\
&\leq \sum_{j=1}^{d}\Re\left(p_j B_j\langle A\rangle^{2\delta'} f\langle x\rangle^{-2\delta}\right) + C_2 f^{1/2}\langle x\rangle^{-s}\langle A\rangle^{2\delta'}\langle x\rangle^{-s}f^{1/2};\\
&B_j = f^{1/2}\langle x\rangle^{-s}x_j\langle A\rangle^{2\delta'-1}Af^{-1/2}\langle x\rangle^{s-1}\langle A\rangle^{-2\delta'}.
\end{aligned}$$

The operators $B_j$ are bounded uniformly in $\zeta$. This may readily be proved by the techniques of the proof of [MøSk02, Lemma 2.5] and will not be done here. (We move the factor $\langle A\rangle^{2\delta'-1}$ to the right by commutation using the representation formula [MøSk02, (2.5)].)

The two terms on the right hand side are estimated in the same fashion in the expectation in the state $R_\zeta(\epsilon)\psi$, so let us only consider the first one in details. We get with $g = \langle x\rangle^{-2\delta} f$

$$\begin{aligned}&\left|\langle p_j R_\zeta(\epsilon)\psi, B_j \langle A\rangle^{2\delta'} g R_\zeta(\epsilon)\psi\rangle\right|\\ &\le C \left\|p_j R_\zeta(\epsilon) f^{1/2}\langle x\rangle^{-s}\langle \epsilon A\rangle^{-1}\right\| \cdot \left\|\langle A\rangle^{2\delta'} g R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\right\|.\end{aligned}$$

An application of Lemma 3.1 to the first factor gives

$$\left\|p_j R_\zeta(\epsilon) f^{1/2}\langle x\rangle^{-s}\langle \epsilon A\rangle^{-1}\right\| \le C\epsilon^{-1/2}\|F_\zeta(\epsilon)\|^{1/2}.$$

To estimate the second factor we use [MøSk02, Lemma 7.1], which says that for all $s' > 0$ there exists a constant $C > 0$ such that

$$\|\langle A\rangle^{s'}\langle x\rangle^{-s'}\langle p\rangle^{-s'}\| \le C,$$

to write

$$\begin{aligned}&\left\|\langle A\rangle^{2\delta'} g_j R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\right\|\\ &\le C \left\|\langle p\rangle\langle x\rangle^{2\delta'} g R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\right\|.\end{aligned}$$

By Lemma 3.1

$$\left\|\langle x\rangle^{2\delta'} g R_\zeta(\epsilon) f^{1/2}\langle x\rangle^{-s}\langle \epsilon A\rangle^{-1}\right\| \le C\epsilon^{-1/2}\|F_\zeta(\epsilon)\|^{1/2}.$$

Moreover

$$\begin{aligned}&(p_i\langle x\rangle^{2\delta'} g)^* p_i\langle x\rangle^{2\delta'} g = p_i g^2\langle x\rangle^{4\delta'} p_i + O(\langle x\rangle^{-2+4\delta'-4\delta} f^2)\\ &= p_i O(1) p_i + O(f^2),\end{aligned}$$

from which we obtain the final estimate

$$\left\|\langle A\rangle^{2\delta'} g R_\zeta(\epsilon)\langle x\rangle^{-s} f^{1/2}\langle \epsilon A\rangle^{-1}\right\| \le C\epsilon^{-1/2}\|F_\zeta(\epsilon)\|^{1/2}$$

by another application of Lemma 3.1. Therefore we have proved that the contribution from the first term agrees with the bound (3.15). Since the other term can be treated similarly the proof of (3.8) is done. □

By taking $\epsilon \to 0$ we obtain

**Corollary 3.5.** *For any $\delta > 0$ and $\theta \in (0, \pi)$ the operators*

$$\zeta \mapsto \langle x\rangle^{-(1/2+\mu/4+\delta)}(H-\zeta)^{-1}\langle x\rangle^{-(1/2+\mu/4+\delta)},$$

*and*

$$\zeta \mapsto \langle x\rangle^{-(1/2+\delta)} f_{|\zeta|}^{1/2}(H-\zeta)^{-1} f_{|\zeta|}^{1/2}\langle x\rangle^{-(1/2+\delta)}$$

*are bounded uniformly in the sector $\Gamma_\theta$.*

**Remark 3.6.** One may show uniform Hölder continuity in $\Gamma_\theta$ of the first function in Corollary 3.5 by a certain modification of the Mourre type method of [PSS81] using bounds from the present section. The proof we give in Section 5 is more efficient and yields a better Hölder exponent although it requires more smoothness of the potential.

## 4. Iterated resolvents for $V_1$

4.1. **Discussion and statement of results.** In this section we will use a strategy similar to the one from [GIS96] to prove weighted estimates for iterated resolvents. In [GIS96] this strategy was developed to prove weighted estimates for powers of resolvents at energies for which a Mourre estimate holds. Although the different (multiple commutator) approach of [JMP84] can be extended to give some results in the present context, it seems impossible to derive *optimal* weights by some [JMP84]-type technique. Only the case $V = V_1$ is considered, as in Section 3.

The following symbols will play a prominent role:

$$a_0(x,\xi) = \frac{\xi^2}{f_E(x)^2}, \quad b(x,\xi) = \frac{x}{\langle x\rangle} \cdot \frac{\xi}{f_E(x)}. \tag{4.1}$$

Here and in the rest of the paper

$$f = f_E = \sqrt{\kappa_0^{-2}E + (1-\mu/2)^{-1}\langle x\rangle^{-\mu}} \tag{4.2}$$

is essentially the function from (3.2)—we have only introduced a dependence on the constants $\mu, \kappa_0 > 0$ that appear in Theorem 4.1 below. These constants are imposed on us, due to the need for (4.35) and (4.36) (and their quantum analogues) to hold. In a part of this section the simpler definition (3.2) of $f$ would be sufficient, but in the critical Subsection 4.5 the definition (4.2) above will be important. These symbols will be studied together with the resolvent $R(\zeta)$. We always take $E = |\zeta|$ with $\zeta \in \Gamma_\theta$ in (4.1). Thus $a_0$ and $b$ depend on $\zeta$, even though we do not include this explicitly in the notation. We also define a useful auxiliary weight function $w$ and the symbol $h$ of the operator $H$ by

$$w = w_E(x) = \langle x\rangle f_E(x), \quad h = h(x,\xi) = \xi^2 + V(x). \tag{4.3}$$

We will let $\mathrm{Op}^{\mathrm{w}}(a)$ denote the Weyl quantization of a symbol $a$. Explicitly $\mathrm{Op}^{\mathrm{w}}(a)$ acts as follows

$$(\mathrm{Op}^{\mathrm{w}}(a)\phi)(x) = (2\pi)^{-d} \iint e^{i(x-y)\xi} a((x+y)/2,\xi)\phi(y)\, dy d\xi,$$

with mapping properties depending, of course, on the symbol $a$.

We will be concerned with proving the following result:

**Theorem 4.1.** *Let $V(x)$ satisfy the conditions of Theorem 1.1 with $V_2 = 0$. We reformulate the assumption (3) as: For some $\kappa_0 > 0$ and $2 > \mu > 0$,*

$$W(x) = -2V(x) - x \cdot \nabla V(x) \geq 2\kappa_0^2 \langle x \rangle^{-\mu}. \tag{4.4}$$

*Let $\theta \in (0, \pi)$ and $\Gamma_\theta$ be as defined in* (1.3)*. Let $a_0$ and $b$ be as defined in* (4.1) *with $E = |\zeta|$. Define $k = k(x) = \langle x \rangle^{1+\mu/2}$. Then the following conclusions,* (i) - (iv)*, hold for $H = p^2 + V$ with all bounds being uniform in $\zeta \in \Gamma_\theta$:*

(i) *For all $\epsilon > 0$ there exists $C > 0$ such that*

$$\|k^{-1/2-\epsilon} R(\zeta) k^{-1/2-\epsilon}\| \leq C. \tag{4.5a}$$

(ii) *There exists $C_0 > 0$, depending only on $V$, such that if $F_+ \in \mathcal{B}^\infty(\mathbb{R})$,* $\operatorname{supp}(F_+) \subset (C_0, \infty)$ *and $F_+' \in C_0^\infty(\mathbb{R})$, then for all $\epsilon > 0$ and all $t > 0$ there exists $C > 0$ such that*

$$\|k^{t-1/2-\epsilon} \mathrm{Op^w}(F_+(a_0)) R(\zeta) k^{-t-1/2-\epsilon}\| \leq C, \tag{4.5b}$$

$$\|k^{-t-1/2-\epsilon} R(\zeta) \mathrm{Op^w}(F_+(a_0)) k^{t-1/2-\epsilon}\| \leq C. \tag{4.5c}$$

(iii) *Let $\tilde{F}_+, \tilde{F}_- \in \mathcal{B}^\infty(\mathbb{R})$ satisfy (with $\kappa_0$ from* (4.4)*) for some $\kappa > 0$,*
- $\inf \operatorname{supp}(\tilde{F}_+) > -\kappa > -\kappa_0$, $\sup \operatorname{supp}(\tilde{F}_-) < \kappa < \kappa_0$.
- $\tilde{F}_-', \tilde{F}_+' \in C_0^\infty(\mathbb{R})$.

*Let $F_- \in C_0^\infty(\mathbb{R})$. Then for all $\epsilon, t > 0$ there exists $C > 0$ such that*

$$\|k^{t-1/2-\epsilon} \mathrm{Op^w}(F_-(a_0)\tilde{F}_-(b)) R(\zeta) k^{-t-1/2-\epsilon}\| \leq C, \tag{4.5d}$$

$$\|k^{-t-1/2-\epsilon} R(\zeta) \mathrm{Op^w}(F_-(a_0)\tilde{F}_+(b)) k^{t-1/2-\epsilon}\| \leq C. \tag{4.5e}$$

(iv) *Suppose $\tilde{F}_+$ and $\tilde{F}_-$ satisfy the assumptions from* (iii)*, $F_-^1, F_-^2 \in C_0^\infty(\mathbb{R})$ and*

$$\operatorname{dist}(\operatorname{supp}(\tilde{F}_+), \operatorname{supp}(\tilde{F}_-)) > 0.$$

*Then for all $t > 0$ there exists $C > 0$ such that*

$$\|k^t \mathrm{Op^w}(F_-^1(a_0)\tilde{F}_-(b)) R(\zeta) \mathrm{Op^w}(F_-^2(a_0)\tilde{F}_+(b)) k^t\| \leq C. \tag{4.5f}$$

*Suppose $F_+$ is given as in* (ii)*, some functions $\tilde{F}_+, \tilde{F}_-, F_-$ are given as in* (iii) *and suppose*

$$\operatorname{dist}(\operatorname{supp}(F_-), \operatorname{supp}(F_+)) > 0.$$

*Then for all $t > 0$ there exists $C > 0$ such that*

$$\|k^t \mathrm{Op^w}(F_+(a_0))R(\zeta)\mathrm{Op^w}(F_-(a_0)\tilde{F}_+(b))k^t\| \leq C, \tag{4.5g}$$

$$\|k^t \mathrm{Op^w}(F_-(a_0)\tilde{F}_-(b))R(\zeta)\mathrm{Op^w}(F_+(a_0))k^t\| \leq C. \tag{4.5h}$$

Using purely algebraic arguments, cf. [Jen85] or [Iso85], one can get the following result from Theorem 4.1. An elaboration is given in Appendix A. (See (5.8) for another application of the partition of unity needed in our case.)

**Theorem 4.2.** *Let the assumptions and notations be as in Theorem 4.1. Then the following conclusions hold for $H = p^2 + V$ (uniformly in $\zeta \in \Gamma_\theta$):*

(i) *Let $m \in \mathbb{N}$ and let $\epsilon > 0$ be arbitrary. Then there exists $C > 0$ such that*

$$\|k^{-(m-1/2)-\epsilon}R(\zeta)^m k^{-(m-1/2)-\epsilon}\| \leq C. \tag{4.6a}$$

(ii) *Let $C_0$ be the number from* (ii) *in Theorem 4.1 and suppose $F_+ \in \mathcal{B}^\infty(\mathbb{R})$, $\mathrm{supp}(F_+) \subset (C_0, \infty)$ and $F_+' \in C_0^\infty(\mathbb{R})$. Then for all $m \in \mathbb{N}$ and all $\epsilon, t > 0$ there exists $C > 0$ such that*

$$\|k^{t-1/2-\epsilon}\mathrm{Op^w}(F_+(a_0))R(\zeta)^m k^{-t-m+1/2-\epsilon}\| \leq C, \tag{4.6b}$$

$$\|k^{-t-m+1/2-\epsilon}R(\zeta)^m \mathrm{Op^w}(F_+(a_0))k^{t-1/2-\epsilon}\| \leq C. \tag{4.6c}$$

(iii) *Let $\tilde{F}_+, \tilde{F}_-, F_-$ satisfy the assumptions from* (iii) *in Theorem 4.1. Then for all $m \in \mathbb{N}$ and all $\epsilon, t > 0$ there exists $C > 0$ such that*

$$\|k^{t-1/2-\epsilon}\mathrm{Op^w}(F_-(a_0)\tilde{F}_-(b))R(\zeta)^m k^{-t-m+1/2-\epsilon}\| \leq C, \tag{4.6d}$$

$$\|k^{-t-m+1/2-\epsilon}R(\zeta)^m \mathrm{Op^w}(F_-(a_0)\tilde{F}_+(b))k^{t-1/2-\epsilon}\| \leq C. \tag{4.6e}$$

(iv) *Suppose $\tilde{F}_+$ and $\tilde{F}_-$ satisfy the assumptions from* (iii)*, $F_-^1, F_-^2 \in C_0^\infty(\mathbb{R})$ and*

$$\mathrm{dist}(\mathrm{supp}(\tilde{F}_+), \mathrm{supp}(\tilde{F}_-)) > 0.$$

*Then for all $m \in \mathbb{N}$ and all $t > 0$ there exists $C > 0$ such that*

$$\|k^t \mathrm{Op^w}(F_-^1(a_0)\tilde{F}_-(b))R(\zeta)^m \mathrm{Op^w}(F_-^2(a_0)\tilde{F}_+(b))k^t\| \leq C. \tag{4.6f}$$

*Suppose $F_+$ is given as in* (ii)*, some functions $\tilde{F}_+, \tilde{F}_-, F_-$ are given as in* (iii) *and suppose*

$$\mathrm{dist}(\mathrm{supp}(F_-), \mathrm{supp}(F_+)) > 0.$$

*Then for all $m \in \mathbb{N}$ and all $t > 0$ there exists $C > 0$ such that*

$$\|k^t \mathrm{Op^w}(F_+(a_0))R(\zeta)^m \mathrm{Op^w}(F_-(a_0)\tilde{F}_+(b))k^t\| \leq C, \tag{4.6g}$$

$$\|k^t \mathrm{Op^w}(F_-(a_0)\tilde{F}_-(b))R(\zeta)^m \mathrm{Op^w}(F_+(a_0))k^t\| \leq C. \tag{4.6h}$$

The remaining part of the present section is devoted to proving Theorem 4.1.

The first statement (4.5a) is a restatement of (1.4) (in the case $V_2 = 0$) which was proved in Section 3, cf. Corollary 3.5.

In Subsection 4.2 we shall discuss a pseudodifferential calculus given in terms of the metric $\frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{f_E^2}$, where $f_E$ is the function from (4.2).

The localization $F_+$ in the second equation (4.5b) can be thought of as an energy localization (uniform in energy). In Subsection 4.3 we use the pseudodifferential calculus to deal with (4.5b), and with (4.5c), (4.5g) and (4.5h).

The symbol $F_-(a_0)\tilde{F}_-(b)$ lies in a good symbol class; this would not have been the case without the factor $F_-(a_0)$. Thus to prove (4.5d) we can use the pseudodifferential calculus. A positive commutator will play a major role in the analysis, which is carried out in Subsection 4.5. The remaining estimates, (4.5e) and (4.5f), can be proved similarly.

4.2. **Pseudodifferential calculus.** We will use pseudodifferential operators in the Weyl calculus associated with the metric $g_E = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{f^2}$ (cf. [Hör94, Chapt. XVIII]), where $f = f_E$ is the energy-dependent function given in (4.2). For a part of the argument one could instead use the (energy-independent) metric $\frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle x\rangle^{-\mu}}$. However the crucial positivity arguments in Subsections 4.3 and 4.5 (applications of the Fefferman-Phong inequality) rely indeed on the more precise energy-dependent estimates. It is clear that for $E = 0$, $f_E = C\langle x\rangle^{-\mu/2}$ and the two metrics are essentially equal.

Since $\mu < 2$ we have a 'Planck's constant' of size

$$w^{-1} = \langle x\rangle^{-1} f^{-1} \leq (1 - \frac{\mu}{2})^{1/2} \langle x\rangle^{-1+\mu/2}.$$

We will prove that $\frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{f^2}$ satisfies the definition of a Hörmander metric with estimates that are uniform in $E \in (0, 1]$. Therefore we get uniform (in $E$) control of the constants appearing in the pseudodifferential calculus. In particular the boundedness (on $L^2(\mathbb{R}^d)$) of pseudodifferential operators ([Hör94, Theorem 18.6.3]) and the positivity (to highest order) of pseudodifferential operators with positive symbols (the Fefferman-Phong inequality [Hör94, Theorem 18.6.8]) hold uniformly in $E$.

**Lemma 4.3.**

(i) *(A uniform Hörmander metric $g_E$.)*
*For points $v \in \mathbb{R}^d \times \mathbb{R}^d$ write $v = (v_x, v_\xi)$. Define for $E \in (0, 1]$*

*and* $(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d$ *the metric* $g(v)=g_E(v)=g_{(x,\xi)}(v)$ *by*

$$g_{(x,\xi)}(v)=\frac{v_x^2}{\langle x\rangle^2}+\frac{v_\xi^2}{f_E^2(x)}.$$

*Then* $g$ *is a Hörmander metric uniformly in* $E\in(0,1]$*, i.e. there exist constants* $C_1,C_2,N>0$ *independent of* $E\in(0,1]$ *such that* $g$ *satisfies*

- (slow variation) *If* $g_{(x,\xi)}((y,\eta))\le 1/C_1$ *then*

$$g_{(x,\xi)+(y,\eta)}(v)\le C_1 g_{(x,\xi)}(v) \text{ for all } v\in\mathbb{R}^d\times\mathbb{R}^d.$$

- (uncertainty principle) $g_E\le g_E^\sigma$*, where* $g_E^\sigma$ *denotes the dual metric of* $g_E$ *with respect to the standard symplectic form* $\sigma=dx\wedge d\xi$*.*
- (temperateness) *For all* $v\in\mathbb{R}^d\times\mathbb{R}^d\backslash\{0\}$ *and all* $(x,\xi),(y,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$ *we have*

$$g_{(x,\xi)}(v)\le C_2 g_{(y,\eta)}(v)\left(1+g^\sigma_{(x,\xi)}((y,\eta)-(x,\xi))\right)^N.$$

(ii) *(Uniform weight functions.)*
*A positive function* $m=m_E=m(x,\xi)$ *is said to be a uniformly temperate weight (w.r.t.* $\sigma,g$*) if the following two conditions are satisfied with constants independent of* $E\in(0,1]$

- *There exists* $c,C>0$ *such that for all* $v,v_1\in\mathbb{R}^{2d}$*:*

$$g_v(v_1)\le c\Rightarrow m(v)/C\le m(v+v_1)\le Cm(v).$$

- *There exists* $C,N>0$ *such that for all* $v,v_1\in\mathbb{R}^{2d}$*:*

$$m(v_1)\le Cm(v)\left(1+g^\sigma_{v_1}(v-v_1)\right)^N.$$

*With this definition, any of the functions* $\langle x\rangle$*,* $f_E$*,* $\langle\xi\rangle$*, or* $\langle\frac{\xi}{f_E}\rangle$*, as well as any combination of products of real powers of these functions, is a uniformly temperate weight function for the metric* $g_E$*.*

The proof of Lemma 4.3 is given in Appendix B.

For a uniformly temperate weight function $m=m_E$ we denote by $S_{unif}(m,g_E)$ the space of $C^\infty$ functions ('symbols') $a=a_\zeta$ satisfying

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}m(x,\xi)\langle x\rangle^{-|\alpha|}f_E^{-|\beta|}, \tag{4.7}$$

for all $\alpha,\beta\in(\mathbb{N}\cup\{0\})^d$ with constants $C_{\alpha,\beta}$ independent of $\zeta=Ee^{i\phi}\in\Gamma_\theta$. We let $\Psi_{unif}(m,g_E)$ denote the space of operators given as the

Weyl quantization of symbols from $S_{unif}(m, g_E)$. Thus, for instance, a short verification gives (with $h$ from (4.3))

$$h, h - \zeta \in S_{unif}\left(f_{|\zeta|}^2 \langle \frac{\xi}{f_{|\zeta|}} \rangle^2, g_{|\zeta|}\right). \tag{4.8}$$

Notice that for any $C_0^\infty$–function $G$ we have with $b$ from (4.1) for $E = 0$,

$$\partial_x^\alpha \partial_\xi^\beta G(b) \approx (\langle x \rangle^{(\mu/2-1)} |\xi|)^{|\alpha|} \langle x \rangle^{\mu|\beta|/2} + \cdots .$$

Consequently $G(b)$ is not a good symbol (not even with $\xi$ considered as bounded since $\mu$ can be greater than 1). The remedy for this has already been introduced in (4.5): Let $F_- \in C_0^\infty(\mathbb{R})$ and study $F_-(a_0)G(b)$ for any $C^\infty$–function $G$. Notice that $|b|^2 \leq a_0$, so $b$ is bounded on $\mathrm{supp}(F_-(a_0))$. Using the elementary bound, valid for any $s \in \mathbb{R}$,

$$|\partial_x^\alpha f_E^s| \leq C_\alpha f_E^s \langle x \rangle^{-|\alpha|}, \tag{4.9}$$

with $C_\alpha$ independent of $E$ (or in short $f_E \in S_{unif}(f_E, g_E)$ recalling the convention $E = |\zeta|$), we readily infer that indeed

$$F_-(a_0)G(b) \in S_{unif}(1, g_E).$$

Once Lemma 4.3 is established we have from [Hör94, Sections 18.4-6] (cf. in particular [Hör94, Theorems 18.6.3 and 18.6.8]):

**Theorem 4.4.**

(i) *Let $a \in S_{unif}(1, g_E)$. Then there exists a constant $C > 0$, independent of $\zeta \in \Gamma_\theta$, such that*

$$\|\mathrm{Op}^{\mathrm{w}}(a)\| \leq C.$$

(ii) *Let $a \in S_{unif}(w_E^2, g_E)$ and suppose $a \geq 0$. Then there exists a constant $C > 0$ independent of $\zeta \in \Gamma_\theta$ such that*

$$\mathrm{Op}^{\mathrm{w}}(a) \geq -C,$$

*as a quadratic form on $\mathcal{S}(\mathbb{R}^d)$.*

Another useful tool is the composition rule for pseudodifferential operators (see [Hör94, Theorem 18.5.4]): Suppose $a_1 \in S_{unif}(m_1, g_E)$ and $a_2 \in S_{unif}(m_2, g_E)$. Then $\mathrm{Op}^{\mathrm{w}}(a_1)\mathrm{Op}^{\mathrm{w}}(a_2) = \mathrm{Op}^{\mathrm{w}}(s)$ with $s \in S_{unif}(m_1 m_2, g_E)$, and asymptotically $s = \sum_{j=0}^\infty s_j(x, \xi)$ with

$$s_j(x, \xi) \in S_{unif}\left(w_E^{-j} m_1 m_2, g_E\right)$$

given by

$$s_j(x, \xi) = 2^{-j} i^j \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\alpha|}}{\alpha! \beta!} (\partial_\xi^\alpha \partial_x^\beta a_1)(\partial_\xi^\beta \partial_x^\alpha a_2). \tag{4.10}$$

Here 'asymptotically' means that for all $N \in \mathbb{N}$ we have

$$s - \sum_{j=0}^{N} s_j \in S_{unif}\left(w_E^{-(N+1)}, g_E\right).$$

Finally we shall discuss some further results for the class $S_{unif}(m, g_E)$. Although they will not be used in this paper, we feel that including them might be clarifying for the reader.

Let us topologize $S_{unif}(m, g_E)$ by the seminorms $\|a\|_{\alpha,\beta}$, each defined as the smallest constant $C_{\alpha,\beta}$ independent of $\zeta = Ee^{i\phi} \in \Gamma_\theta$ such that (4.7) holds. From [Hör94, Theorem 18.5.10] (and its proof) we learn that

$$\begin{aligned} &e^{i\kappa\langle p_x, p_\xi\rangle} : S_{unif}(m, g_E) \to S_{unif}(m, g_E) \\ &\text{is a topological isomorphism; } \kappa \in \mathbb{R}. \end{aligned} \tag{4.11}$$

Of particular interest are the values $\kappa = -1, -\frac{1}{2}, \frac{1}{2}, 1$, cf. [Hör94, (18.5.20)], linking Weyl quantization with left and right Kohn-Nirenberg quantization.

Yet another basic result is a uniform (partial) version of the Beals criterion [Bea81, Theorem 4.4] (see also [BoCh94]). It is a useful tool for linking pseudodifferential and functional calculi, cf. [DeGé97, Appendix D]. We introduce $\mathrm{ad_B(A)} = \mathrm{AB} - \mathrm{BA}$ and similarly for vector-valued operators, $\mathrm{ad}^\beta_{\mathrm{B}}(\mathrm{A})$, defined as $|\beta|$ compositions of operations of the previous type. The Beals criterion in our case is the characterisation of the space $\Psi_{unif}(1, g_E)$ as the set of $\mathcal{B}(L^2)$-valued functions $A = A_\zeta$ on $\Gamma_\theta$ for which for all multiindices $\alpha, \beta \in \mathbb{N}^d$

$$\|\langle x\rangle^{|\alpha|} f_{|\zeta|}^{|\beta|} \mathrm{ad}_p^\alpha \mathrm{ad}_x^\beta (A)\| \le D_{\alpha,\beta},$$

where the constants $D_{\alpha,\beta}$ are independent of $\zeta \in \Gamma_\theta$. This characterisation may be proved using (4.9), (4.11), the proof of [Bea77, Theorem 1.4] and conjugation by the Fourier transform, see the proof of [DeGé97, Theorem D.8.2].

4.3. **Energy estimate.** In this subsection we will prove (4.5b). The proof of (4.5c) is very similar (we may proceed in the same fashion for the 'adjoint' expression), and the statements (4.5g) and (4.5h) may be proved along the same pattern using in addition (4.5d) and (4.5e) (the latter estimates will be proved independently in Subsection 4.5).

We will start by proving the following lemma in which $f = f_{|\zeta|}$ and $w = w_{|\zeta|}$ are given by (4.2) and (4.3), respectively.

**Lemma 4.5.** *For all $\epsilon > 0$, $s \geq 0$ and all functions $F_+$ as in* (4.5b) *the operator*

$$\langle x\rangle^{-1/2-\epsilon} f^{1/2} w^s \mathrm{Op}^{\mathrm{w}}(F_+(a_0))R(\zeta)w^{-s} f^{1/2}\langle x\rangle^{-1/2-\epsilon}, \tag{4.12}$$

*is bounded uniformly in $\zeta \in \Gamma_\theta$.*

*Proof.* We will use the inequality (with $h$ from (4.3)),

$$\frac{\xi^2}{f^2} = \frac{\Re(h-\zeta)}{f^2} - \frac{\Re(V-\zeta)}{f^2} \leq \frac{\Re(h-\zeta)}{f^2} + C, \tag{4.13}$$

where the constant $C$ only depends on $V$.

By (4.13)

$$F_+(a_0)^2 \leq F_+(a_0)^2 \frac{\xi^2}{f^2 C_0} \leq F_+(a_0)^2 \frac{f^{-2}\Re(h-\zeta) + C}{C_0}.$$

Thus, if $C/C_0 \leq 1/2$, i.e. if $C_0$ has been chosen sufficiently large,

$$F_+(a_0)^2 \leq \frac{2}{C_0} F_+(a_0)^2 \frac{\Re(h-\zeta)}{f^2}. \tag{4.14}$$

By (4.8) and (4.9) this is an inequality for symbols in $S_{unif}(\langle \frac{\xi}{f}\rangle^2, g_{|\zeta|})$. The constant $C_0 > 0$ only depends on $V$ as demanded in Theorem 4.1 (ii).

For $s = 0$, the bounds of (4.12) follow from Corollary 3.5 and the pseudodifferential calculus.

Suppose we have proved uniform boundedness (in $\zeta$) of the operator (4.12) for all $\epsilon > 0$, functions $F_+$ as in (4.5b), and $s \leq s_0$ for some $s_0 \geq 0$. We will then prove that the operator

$$\mathrm{Op}^{\mathrm{w}}(f w^{s-1/2} F_+(a_0))R(\zeta) f w^{-s-1/2-\epsilon}$$

is also uniformly bounded for all $\epsilon > 0$, such functions $F_+$, and $s \in (s_0, s_0 + 1)$. Since

$$f w^{s-1/2} = (\langle x\rangle^{-1/2} w^{-\delta}) f^{1/2} w^{s+\delta},$$

and $w = \langle x\rangle f \geq \langle x\rangle^{1-\mu/2}$, this is equivalent to uniform boundedness of (4.12) for $s$ is the same range.

Thus our goal will be to prove a uniform bound on

$$\|\mathrm{Op}^{\mathrm{w}}(f w^{s-1/2} F_+(a_0))RT^{-1}\|,$$

where $R = R(\zeta)$ and $T^{-1} = f w^{-s-1/2-\epsilon}$.

Using Theorem 4.4, (4.14) and the symbolic calculus we may estimate

$$\begin{aligned}
&\mathrm{Op^w}(a)^* P \mathrm{Op^w}(a) \geq -C; \qquad (4.15)\\
&a = \langle \frac{\xi}{f} \rangle^{-1} f^{-1} w^{\frac{3}{2}-s} \ (\in S_{unif}(a, g_{|\zeta|})),\\
&P = \mathrm{Op^w}\left( w^{2s-1} F_+(a_0)^2 (\frac{2}{C_0} \Re(h-\zeta) - f^2) \right).
\end{aligned}$$

To write (4.15) in a more convenient form we first use the standard parametrix construction to find a symbol $a^{(m)} \in S_{unif}(a^{-1}, g_{|\zeta|})$ such that

$$\mathrm{Op^w}(a)\mathrm{Op^w}(a^{(m)}) - I = \mathrm{Op^w}(r^{(m)}) \in \Psi_{unif}(\langle x \rangle^{-m}, g_{|\zeta|}); \ m > 0.$$

Pick a function $F^1_+$ satisfying the same assumptions as $F_+$, and furthermore, $F^1_+ = 1$ on a neighbourhood of $\operatorname{supp} F_+$. We readily show, that for any $m > 0$

$$\begin{aligned}
&P \geq -D^* D - C_1 \mathrm{Op^w}(r^1_m); \qquad (4.16)\\
&D = \mathrm{Op^w}(\langle \frac{\xi}{f} \rangle) B_1, \ B_1 = \mathrm{Op^w}(b_1 F^1_+(a_0)), \ b_1 \in S_{unif}(f w^{s-\frac{3}{2}}, g_{|\zeta|}),\\
&r^1_m \in S_{unif}(\langle \frac{\xi}{f} \rangle^2 \langle x \rangle^{-m}, g_{|\zeta|}).
\end{aligned}$$

Using (4.16) we have to bound

$$T^{-1} R^* \Big( \frac{2}{C_0} \mathrm{Op^w}\big( w^{2s-1} F_+(a_0)^2 \Re(h-\zeta) \big) + D^* D + C_1 \mathrm{Op^w}(r^1_m) \Big) R T^{-1}. \quad (4.17)$$

For the contribution from the first term in (4.17), we write

$$\begin{aligned}
\mathrm{Op^w}\big( w^{2s-1} F_+(a_0)^2 \Re(h-\zeta) \big) &\leq \Re \left( \mathrm{Op^w}\left( w^{2s-1} F_+(a_0)^2 \right) (H-\zeta) \right)\\
&\quad + \hat{B}_1^* \mathrm{Op^w}(\langle \frac{\xi}{f} \rangle)^2 \hat{B}_1 + C_2 \mathrm{Op^w}(\hat{r}^1_m), \quad (4.18)
\end{aligned}$$

where $\hat{B}_1$ and $\hat{r}^1_m$ have the same form as $B_1$ and $r^1_m$, respectively. Clearly we may estimate the contribution from the first term on the right hand side of (4.18) by the induction hypothesis since

$$\begin{aligned}&T^{-1}R^*\Re\left(\mathrm{Op^w}\left(w^{2s-1}F_+(a_0)^2\right)(H-\zeta)\right)RT^{-1}\\&=\Re\left(T^{-1}R^*\mathrm{Op^w}(w^{2s-1}F_+(a_0)^2T^{-1}\right)\\&=\Re\left((\{\mathrm{Op^w}(a)fw^{s-3/2-\epsilon}\mathrm{Op^w}(F_+^1(a_0))+\mathrm{Op^w}(r_m)\}Rfw^{-s-1/2-\epsilon})^*\right),\end{aligned}$$

where $a\in S_{unif}(1,g_{|\zeta|})$ and $r_m\in S_{unif}(\langle x\rangle^{-m},g_{|\zeta|})$.

So in order to finish the proof we only have to take care of the second and third terms from (4.17). We write for the second one

$$D^*D=\Re\left(B_1^*\check{B}\mathrm{Op^w}(\langle\frac{\xi}{f}\rangle^2)\right)+\mathrm{Op^w}(\check{r}_m),\tag{4.19}$$

where $\check{B}=\mathrm{Op^w}(\check{b}\check{F}_+^1(a_0))$ with $\check{b}\in S_{unif}(fw^{s-\frac{3}{2}},g_{|\zeta|})$ and $\check{F}_+^1$ is given as $F_+^1$ but such that $\check{F}_+^1=1$ on a neighbourhood of $\operatorname{supp}F_+^1$, and $\check{r}_m\in S_{unif}(\langle\frac{\xi}{f}\rangle^2\langle x\rangle^{-m},g_{|\zeta|})$.

We write

$$\begin{aligned}&\mathrm{Op^w}(\langle\frac{\xi}{f}\rangle^2)=f^{-2}p^2+\mathrm{Op^w}(a);\\&a=1-i\xi\cdot\nabla f^{-2}+4^{-1}\Delta f^{-2}\in S_{unif}(\langle\frac{\xi}{f}\rangle,g_{|\zeta|}).\end{aligned}$$

Substituted into (4.19) this yields

$$D^*D=\Re(B_1^*\check{B}f^{-2}p^2)+\Re(D^*\tilde{B})+\mathrm{Op^w}(\tilde{r}_m),\tag{4.20}$$

where $\tilde{B}$ and $\tilde{r}_m$ are defined similarly. Next we substitute $p^2=(H-\zeta)+(\zeta-V)$ and apply Cauchy-Schwarz to the second term to the right in (4.20). After a subtraction we conclude that

$$D^*D\le 2\Re\left(B_1^*\check{B}f^{-2}(H-\zeta)\right)+\bar{B}^*\bar{B}+\mathrm{Op^w}(\bar{r}_m),\tag{4.21}$$

for yet another couple of similar operators $\bar{B}$ and $\mathrm{Op^w}(\bar{r}_m)$.

We may treat the contribution from the first term on the right hand side of (4.21) as above. The second term is handled by the induction hypothesis. The third term is similar to the third term from (4.17). For these terms we use the resolvent identity $R(z)=R(i)+(z-i)R(i)R(z)$ and the fact that $\langle x\rangle^{-m}p^2R(i)\langle x\rangle^m$ is bounded; we need $m$ sufficiently large. □

The estimate (4.5b) is an easy consequence of Lemma 4.5 and Lemma 4.6 below. Lemma 4.6 will also be useful in Subsection 4.5.

**Lemma 4.6.** *Suppose $A=A_\zeta$ is a $\mathcal{B}(L^2)$-valued function in $\zeta\in\Gamma_\theta$ such that for all $\epsilon>0$, $s\ge 0$ the operator*

$$\langle x\rangle^{-\epsilon}fw^{s-1/2}AR(\zeta)w^{-s-1/2}f\langle x\rangle^{-\epsilon}\tag{4.22}$$

*is uniformly bounded. Then also*

$$k^{t-1/2-\epsilon}AR(\zeta)k^{-t-1/2-\epsilon},$$

*is bounded uniformly in $\zeta \in \Gamma_\theta$ for all $\epsilon > 0$, $t \geq 0$.*

*Proof of Lemma 4.6.* Let $1 = F_1 + F_2$ be a sharp partition of unity on $\mathbb{R}$, $\mathrm{supp}(F_1) = (-\infty, 0]$, $\mathrm{supp}(F_2) = [0, +\infty)$ and let us consider the two terms

$$\langle x\rangle^{-\epsilon}k^{t-1/2}AR(\zeta)k^{-t-1/2}\langle x\rangle^{-\epsilon}F_1(|\zeta| - \langle x\rangle^{-\mu}) \tag{4.23}$$

$$\langle x\rangle^{-\epsilon}k^{t-1/2}AR(\zeta)k^{-t-1/2}\langle x\rangle^{-\epsilon}F_2(|\zeta| - \langle x\rangle^{-\mu}). \tag{4.24}$$

First we study (4.23). On $\mathrm{supp}(F_1(|\zeta| - \langle x\rangle^{-\mu}))$ we have $|\zeta| \leq \langle x\rangle^{-\mu}$ and therefore

$$\langle x\rangle^{-\mu/2} \leq f_{|\zeta|}(x) \leq C\langle x\rangle^{-\mu/2}. \tag{4.25}$$

Let us choose $s$ such that

$$(-t - 1/2)(1 + \mu/2) = -s - 1/2 - (-s + 1/2)\mu/2, \tag{4.26}$$

that is $s = t(1 + \mu/2)/(1 - \mu/2)$ (notice that $s \geq 0$), and therefore

$$(t - 1/2)(1 + \mu/2) = s - 1/2 + (-s - 1/2)\mu/2. \tag{4.27}$$

So (using (4.27)) to the left of the resolvent in (4.23) we can write

$$k^{t-1/2} = w^{s-1/2}f\left(\frac{\langle x\rangle^{-\mu/2}}{f}\right)^{s+1/2}.$$

We know, since $f \geq \langle x\rangle^{-\mu/2}$, that the $(\cdot)^{s+1/2}$-term is bounded.

To the right in (4.23) we can write using (4.26)

$$k^{-t-1/2} = w^{-s-1/2}f\left(\frac{\langle x\rangle^{-\mu/2}}{f}\right)^{-s+1/2}.$$

Now we infer uniform boundedness of (4.23) from (4.22) and (4.25).

The boundedness of (4.24) is more subtle. Here we cannot convert all $f$'s to $\langle x\rangle^{-\mu/2}$'s. Instead we have to compare some $f$'s to $|\zeta|^{1/2}$ and others to $\langle x\rangle^{-\mu/2}$. Notice that on $\mathrm{supp}(F_2(|\zeta| - \langle x\rangle^{-\mu}))$ we have

$$\langle x\rangle^{-\mu/2} \leq f_{|\zeta|}(x) \leq C|\zeta|^{1/2}. \tag{4.28}$$

For $s = (1 + \mu/2)t$ we have

$$(t - 1/2)(1 + \mu/2) = -\mu/4 + s - 1/2, \tag{4.29}$$

$$(t + 1/2)(1 + \mu/2) = s + 1/2 + \mu/4. \tag{4.30}$$

To the left in (4.24) we write, using (4.29),

$$k^{t-1/2} = \left(\frac{|\zeta|^{1/2}}{f}\right)^s \left(\frac{\langle x\rangle^{-\mu/2}}{f}\right)^{1/2} |\zeta|^{-s/2} f w^{s-1/2}.$$

To the right in (4.24) we write

$$k^{-t-1/2} = \left(\frac{f}{|\zeta|^{1/2}}\right)^s \left(\frac{\langle x\rangle^{-\mu/2}}{f}\right)^{1/2} |\zeta|^{s/2} f w^{-s-1/2}.$$

Now we infer uniform boundedness of (4.24) from (4.22) and (4.28). □

4.4. **Classical mechanics.** The purpose of this subsection is twofold: We present various notation and computations needed in the next subsection. Secondly we show how one can construct a classical 'propagation observable' which yields a classical analogue of Theorem 1.4. The more technical material of Subsection 4.5 may be viewed as being based on the classical proof presented here.

Recall the definitions of $a_0$, $b$, $f$, $h$, $w$ and $W$ from (4.1), (4.2), (4.3) and (4.4). For any $E \geq 0$ we define

$$v = v_E(x) = \kappa_0^{-2} E + \langle x\rangle^{-\mu},$$

and compute

$$\nabla f(x) = \frac{-\mu}{2-\mu} f^{-1} \langle x\rangle^{-\mu-1} \frac{x}{\langle x\rangle},$$
$$\nabla w(x) = (\kappa_0^{-2} E + \langle x\rangle^{-\mu}) \frac{x}{w} = v\frac{x}{w}. \tag{4.31}$$

Recall also the definition of the Poisson bracket:

$$\{a,b\}_P = \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b.$$

With our definition (4.3) of $h(x,\xi)$ we get for any symbol $a$, $\{h,a\}_P = 2\xi \cdot \nabla_x a - \nabla V \cdot \nabla_\xi a$. Thus, with $b$ as in (4.1), an elementary calculation yields

$$\{h,b\}_P = w^{-1}\left(2h + W(x) - 2b^2 v(x)\right). \tag{4.32}$$

**Theorem 4.7.** *Let $\kappa_0 > 0$ be given as in* (4.4)*. Then for any classical orbit $x(t)$ with energy $E \geq 0$*

$$\liminf_{|t|\to\infty} |tC|^{-(1+\frac{\mu}{2})^{-1}} |x(t)| \geq 1;\ C = \kappa_0 \frac{2+\mu}{(1-\frac{\mu}{2})^{\frac{1}{2}}}. \tag{4.33}$$

*Proof.* We shall only prove the bound for $t \to +\infty$. Let us fix $0 < \tilde{\kappa} < \kappa < \kappa' \le \kappa_0$. We pick a real-valued, decreasing, smooth function $\tilde{F}_-$ such that

$$\tilde{F}_-(\tilde{\kappa}) = 1,\ \tilde{F}_-(\kappa) = 0 \text{ and } \operatorname{supp}(\tilde{F}_-') \subset (\tilde{\kappa}, \kappa),$$

and consider the observable

$$q = q(x,\xi) = w(\kappa' - b)\tilde{F}_-(b). \tag{4.34}$$

We claim that $q$ has a non-positive derivative: First we compute using (4.32):

$$\begin{aligned} \frac{d}{dt}\tilde{F}_-(b) &= \left(2E + W - 2b^2 v\right) w^{-1}\tilde{F}_-'(b) \\ &\le 2(\kappa_0^2 - \kappa^2) v w^{-1}\tilde{F}_-'(b) \le 0. \end{aligned} \tag{4.35}$$

The contribution from the derivative of the other factors on the right hand side of (4.34) is computed and estimated using (4.31) and (4.32) as

$$\begin{aligned} \left(\frac{d}{dt}\left(w(\kappa' - b)\right)\right)\tilde{F}_-(b) &= \left(2vb(\kappa' - b) - 2E - W + 2vb^2\right)\tilde{F}_-(b) \\ &\le -\left(2E + W - 2v\kappa\kappa'\right)\tilde{F}_-(b) \\ &\le -2(\kappa_0^2 - \kappa\kappa')v\tilde{F}_-(b) \le 0. \end{aligned} \tag{4.36}$$

In particular, we infer that

$$\frac{d}{dt}q \le -2(\kappa_0^2 - \kappa\kappa')\langle x\rangle^{-\mu}\tilde{F}_-(b) \le 0. \tag{4.37}$$

By integrating (4.37) we obtain the uniform bound

$$q(T) + \int_0^T \langle x(t)\rangle^{-\mu}\tilde{F}_-(b(t))dt \le C,$$

by which we will now prove that

$$\tilde{F}_-(b(t)) \to 0 \text{ for } t \to \infty. \tag{4.38}$$

Notice that by (4.35), $\tilde{F}_-(b(t)) \to c$; so we need only to show that $c = 0$. There are two cases: 1) $x$ is bounded, or 2) $|x(t)| \to \infty$ along some sequence $t = t_n \to \infty$. In Case 1) we have $\tilde{F}_-(b(t_n)) \to 0$ for some sequence $t_n \to \infty$ (by the boundedness of the integral) yielding $c = 0$. In Case 2) we learn from the boundedness of $q$ that $\tilde{F}_-(b(t_n)) \to 0$ yielding $c = 0$ in this case too.

Finally, define

$$F(r) = \frac{1}{2}\int_1^r (\kappa_0^{-2}E + (1 - \frac{\mu}{2})^{-1}s^{-\mu})^{-\frac{1}{2}}ds,$$

and compute

$$\frac{d}{dt}F(\langle x\rangle) = b.$$

Combined with (4.38) this yields

$$\frac{d}{dt}F(\langle x\rangle) \geq \tilde{\kappa} \text{ for } t \geq t_{\tilde{\kappa}}.$$

Whence, by integrating,

$$\frac{1}{2}(1-\mu/2)^{\frac{1}{2}}(1+\mu/2)^{-1}\langle x(t)\rangle^{1+\frac{\mu}{2}} \geq \tilde{\kappa}t - C.$$

Since $\tilde{\kappa}$ can be taken arbitrarily close to $\kappa_0$, we are done. □

**Remarks 4.8.** 1) The explicit bounding constant of Theorem 4.7 is optimal. This may readily be seen by examining an almost-bounded orbit for the potential $V(x) = -C|x|^{-\mu}$, cf. [DeGé97, Example 2.2.4], [Gér93] and [Sk03].

2) The zero-energy orbits can behave somewhat unexpected like logarithmic spirals. (We encountered first such example in a preliminary version of the book [DeGé97].)

4.5. **Phase space localization.** In this subsection we will prove (4.5d). This is the main difficulty in proving Theorem 4.1. The proofs of (4.5e) and (4.5f) are essentially identical to the present proof of (4.5d) and will be omitted.

Let us fix a real $\kappa'$ with $\kappa < \kappa' < \kappa_0$, and consider the observable $Q_s = \mathrm{Op}^{\mathrm{w}}(q_s)$, where

$$q_s = q_s(x,\xi) = (w(\kappa' - b))^s \tilde{F}_-(b)F_-(a_0); s \in \mathbb{R}. \tag{4.39}$$

We recall from (4.3) that $w = \langle x\rangle f$; here and henceforth $f = f_{|\zeta|}$ with $\zeta \in \Gamma_\theta$ (as for $a_0$ and $b$). On $\mathrm{supp}(F_-(a_0))$ the symbol $b$ is bounded, and therefore $q_s \in S_{unif}\left(w^s, g_{|\zeta|}\right)$. We will prove the following result.

**Lemma 4.9.** *For all $\epsilon > 0$, $s \geq 0$ the operator*

$$\langle x\rangle^{-1/2-\epsilon} f^{1/2} Q_s R(\zeta) w^{-s} f^{1/2} \langle x\rangle^{-1/2-\epsilon}$$

*is bounded uniformly in $\zeta \in \Gamma_\theta$.*

*Proof.* Notice first that the statement for $s = 0$ follows from Corollary 3.5. We will prove the lemma by induction in $s$ using some ideas from the proof of [GIS96, Lemma 2.6].

Suppose we have proved Lemma 4.9 for all $s \leq s_0$ for some $s_0 \geq 0$. We will then prove that for all $\epsilon > 0$ and $s \in (s_0, s_0 + 1/2)$ the operator

$$fQ_{s-1/2}R(\zeta)fw^{-s-1/2-\epsilon} \text{ is uniformly bounded.} \tag{4.40}$$

By the calculus of pseudodifferential operators and Corollary 3.5 this is equivalent to the statement of the lemma for $s$ in the same range, cf. the proof of Lemma 4.5.

We may without loss of generality assume that $F_-$ and $\tilde{F}_-$ are real-valued, and (by the ΨDO calculus) that

$$F_- = 1 \text{ on a neighbourhood of } [0, C_0], \tag{4.41}$$

$$\tilde{F}_- \text{ is decreasing and } \operatorname{supp}(\tilde{F}_-') \subset (-\kappa, \kappa), \tag{4.42}$$

where $C_0$ is the constant from Theorem 4.1 (ii).

We will place us in a situation, where we can apply the Fefferman-Phong inequality (i.e. Theorem 4.4 (ii)) as follows: Clearly $q_s^2 \in S_{unif}\left(w^{2s}, g_{|\zeta|}\right)$, and therefore, since (by (4.8)) $h \in S_{unif}(f^2\langle \frac{\xi}{f}\rangle^2, g_{|\zeta|})$ and $\langle \frac{\xi}{f}\rangle$ is bounded on $\operatorname{supp}(F_-(a_0))$,

$$\{h, q_s^2\}_P \in S_{unif}\left(f^2 w^{2s-1}, g_{|\zeta|}\right).$$

We will estimate the bracket from above by a $\sigma \in S_{unif}(f^2 w^{2s-1}, g_{|\zeta|})$. We then get as an operator inequality on $L^2(\mathbb{R}^d)$

$$w^{-(s-3/2)} f^{-1} \mathrm{Op}^{\mathrm{w}}(\sigma - \{h, q_s^2\}_P) f^{-1} w^{-(s-3/2)} \geq -C.$$

The effective form of this estimate suited for implementing the induction hypothesis will be: For any $m > 0$

$$\mathrm{Op}^{\mathrm{w}}(\sigma - \{h, q_s^2\}_P) \geq -CB^*B - C_m\langle x\rangle^{-m}, \tag{4.43}$$

where $B = \langle x\rangle^{-1/2} w^{s-s_o-1} f^{1/2} Q_{s_0}^1$ with $Q_{s_0}^1$ given as the quantization of a symbol $q_{s_0}^1$ of the form (4.39) with the functions $F_-$ and $\tilde{F}_-$ replaced by say $F_-^1$ and $\tilde{F}_-^1$; these functions obey (4.41) and (4.42) but they are 'larger' than the previous ones.

Let us first calculate the principal symbol $\{h, q_{2s}\}_P$ of $i[H, Q_s^* Q_s]$. Here and henceforth we put, with a slight abuse of notation, $q_{2s} = q_s^2$. We decompose

$$\{h, q_{2s}\}_P = T_1 + T_2 + T_3; \tag{4.44}$$

$$T_1 = 2s\left(w(\kappa' - b)\right)^{2s-1} \tilde{F}_-^2(b) F_-^2(a_0)\{h, w(\kappa' - b)\}_P,$$

$$T_2 = 2\left(w(\kappa' - b)\right)^{2s} \tilde{F}_-(b)\tilde{F}_-'(b) F_-^2(a_0)\{h, b\}_P,$$

$$T_3 = 2\left(w(\kappa' - b)\right)^{2s} \tilde{F}_-^2(b) F_-(a_0) F_-'(a_0)\{h, a_0\}_P.$$

Defining $q_{2s-1} = (w(\kappa' - b))^{2s-1}\tilde{F}_-^2(b)F_-^2(a_0)$ we get from the computation in (4.36)

$$T_1 = 2s q_{2s-1}(2\kappa' v b - 2h - W). \tag{4.45}$$

By (4.32) the second term in (4.44) becomes, cf. (4.35),

$$T_2 = 2\hat{q}_{2s-1}\tilde{F}'_-(b)(2h + W - 2b^2 v), \tag{4.46}$$

where $\hat{q}_{2s-1} = w^{-1}\left(w(\kappa' - b)\right)^{2s}\tilde{F}_-(b)F_-^2(a_0)$.

Finally for the third term in (4.44) we may write with some symbol $\check{q}_{2s-1} \in S_{unif}\left(w^{2s-1}, g_{|\zeta|}\right)$

$$T_3 = f^2\check{q}_{2s-1}F'_-(a_0). \tag{4.47}$$

The rest of the calculation is split in two depending on the relative size of $\Re\zeta$ and $|\zeta|$.

**Case 1.** $\Re\zeta \geq \frac{(\kappa')^2}{\kappa_0^2}|\zeta|$. Using this, the fact that $b < \kappa$ on $\mathrm{supp}(q_{2s-1})$ and (4.4) we may estimate the right hand side of (4.45)

$$\begin{aligned}
&2sq_{2s-1}(2\kappa' vb - 2h - W)\\
&\leq 2sq_{2s-1}\left\{2\kappa\kappa'(|\zeta|/\kappa_0^2 + \langle x\rangle^{-\mu}) - 2\Re\zeta - 2\Re(h-\zeta) - 2\kappa_0^2\langle x\rangle^{-\mu}\right\}\\
&\leq -\delta q_{s-1/2}f^2 q_{s-1/2} - 2s\left\{(h-\overline{\zeta})q_{2s-1} + q_{2s-1}(h-\zeta)\right\},
\end{aligned} \tag{4.48}$$

here with $\delta = \delta(\kappa, \kappa', \kappa_0, s, \mu) > 0$ and

$$q_{s-1/2}(x,\xi) = (w(\kappa' - b))^{s-1/2}\tilde{F}_-(b)F_-(a_0).$$

Since $f^{-2}h$ is bounded on $\mathrm{supp}(q_{2s-1})$ the right hand side of (4.48) clearly lies in $S_{unif}\left(f^2 w^{2s-1}, g_{|\zeta|}\right)$.

To estimate the right hand side of (4.46) we use the property (4.42). Thus $\tilde{F}'_- \leq 0$ and we have $b^2 \leq \kappa^2$ on $\mathrm{supp}(\tilde{F}'_-(b))$. So we see as above that

$$2\hat{q}_{2s-1}\tilde{F}'_-(b)(2h + W - 2b^2 v) \leq 4\hat{q}_{2s-1}\tilde{F}'_-(b)\Re(h-\zeta). \tag{4.49}$$

Clearly the right hand side of (4.49) is a symbol in $S_{unif}\left(f^2 w^{2s-1}, g_{|\zeta|}\right)$.

The input to our application of the Fefferman-Phong inequality is therefore the estimate (combining (4.44) - (4.49))

$$\begin{aligned}
&\delta(fq_{s-1/2})^2 \leq -\{h, q_{2s}\}_P - 2s\left\{(h-\overline{\zeta})q_{2s-1} + q_{2s-1}(h-\zeta)\right\}\\
&\quad + 2\left\{(h-\overline{\zeta})\hat{q}_{2s-1}\tilde{F}'_-(b) + \hat{q}_{2s-1}\tilde{F}'_-(b)(h-\zeta)\right\} + f^2\check{q}_{2s-1}F'_-(a_0)\\
&\quad = -\{h, q_{2s}\}_P\\
&\qquad\qquad + \left\{(h-\overline{\zeta})q_{2s-1}^{\text{final}} + q_{2s-1}^{\text{final}}(h-\zeta)\right\} + f^2\check{q}_{2s-1}F'_-(a_0),
\end{aligned} \tag{4.50}$$

with

$$q_{2s-1}^{\text{final}} = 2\hat{q}_{2s-1}\tilde{F}'_-(b) - 2sq_{2s-1}.$$

To show (4.40) we apply (4.43) and (4.50). We introduce $R = R(\zeta)$, $T^{-1} = fw^{-s-1/2-\epsilon}$, $\phi \in L^2$, $\|\phi\| = 1$ and $\psi = RT^{-1}\phi$, and use the induction hypothesis and Corollary 3.5 to obtain

$$\begin{aligned}\delta\|fQ_{s-1/2}\psi\|^2 - C_1 &\le \delta\left\langle \mathrm{Op}^{\mathrm{w}}((q_{s-1/2}f)^2)\right\rangle_\psi \\ &\le -\left\langle \mathrm{Op}^{\mathrm{w}}(\{h, q_{2s}\}_P)\right\rangle_\psi \\ &+ 2\left\langle \mathrm{Op}^{\mathrm{w}}\left((h-\overline{\zeta})q^{\mathrm{final}}_{2s-1} + q^{\mathrm{final}}_{2s-1}(h-\zeta)\right)\right\rangle_\psi \\ &+ \left\langle \mathrm{Op}^{\mathrm{w}}(f^2\check{q}_{2s-1}F'_-(a_0))\right\rangle_\psi + C_2.\end{aligned} \tag{4.51}$$

On the right hand side of (4.51) we consider each term separately. For the first term we know that $\{h, q_{2s}\}_P$ is the principal symbol of the pseudodifferential operator $i[H, Q_{2s}]$; $Q_{2s} = Q_s^* Q_s$. Thus, the pseudodifferential calculus and the induction hypothesis yield

$$-\langle \mathrm{Op}^{\mathrm{w}}(\{h, q_{2s}\}_P)\rangle_\psi \le -\langle i[H, Q_{2s}]\rangle_\psi + C_3. \tag{4.52}$$

Clearly

$$-\langle i[H, Q_{2s}]\rangle_\psi = 2\Im\langle T^{-1}\phi, Q_{2s}\psi\rangle - 2\Im(\zeta)\|Q_s\psi\|^2.$$

Since $\Im\zeta > 0$ we may drop the second term. Thus we conclude from (4.52) (after an application of the pseudodifferential calculus, the induction hypothesis and Cauchy-Schwarz) that

$$\begin{aligned}&-\langle \mathrm{Op}^{\mathrm{w}}(\{h, q_{2s}\}_P)\rangle_\psi \\ &\le \eta\|fQ_{s-1/2}\psi\|^2 + \eta^{-1}\|f^{-1}Q_{s+1/2}T^{-1}\phi\|^2 + C_4;\ \eta = \delta/2.\end{aligned} \tag{4.53}$$

Notice that with $T^{-1} = fw^{-s-1/2-\epsilon}$ the second term on the right hand side is clearly bounded since it does not contain a resolvent.

The second term in (4.51) is similar but easier. We calculate as above,

$$\begin{aligned}&\left\langle \mathrm{Op}^{\mathrm{w}}\left((h-\overline{\zeta})q^{\mathrm{final}}_{2s-1} + q^{\mathrm{final}}_{2s-1}(h-\zeta)\right)\right\rangle_\psi \\ &\le 2\Re\langle (H-\zeta)\psi, \mathrm{Op}^{\mathrm{w}}(q^{\mathrm{final}}_{2s-1})\psi\rangle + C_5 \\ &\le \|f^{-1}w^{s+1/2+\epsilon}T^{-1}\phi\| \times \|fw^{-s-1/2-\epsilon}\mathrm{Op}^{\mathrm{w}}(q^{\mathrm{final}}_{2s-1})\psi\| + C_5.\end{aligned} \tag{4.54}$$

The final expression can clearly be estimated by using the induction hypothesis.

The third term in (4.51) is easily seen to be bounded using the property (4.41) and Lemma 4.5. Therefore, inserting (4.53) and (4.54) in (4.51) proves boundedness of $\|fQ_{s-1/2}\psi\|$, which is what we aimed at.

**Case 2.** $\Re\zeta < \frac{(\kappa')^2}{\kappa_0^2}|\zeta|$**.** In this case $2\Re\zeta - \frac{2\kappa\kappa'}{\kappa_0^2}|\zeta|$ can be negative. So instead of writing $2h + W = 2\Re(h-\zeta) + (2\Re\zeta + W)$ as in (4.48), we write

$$2h + W = [2\Re(h-\zeta) + C\Im(h-\zeta)] + (2\Re\zeta + C\Im\zeta + W),$$

for $C > 0$. Using that $\Re\zeta < \frac{(\kappa')^2}{\kappa_0^2}|\zeta|$, we choose $C$ so big that

$$2\Re\zeta + C\Im\zeta - \frac{2\kappa\kappa'}{\kappa_0^2}|\zeta| \geq \Im\zeta \geq \delta'|\zeta|.$$

Thus, instead of (4.48), we find, for some $\delta'' > 0$,

$$\begin{aligned}
&2sq_{2s-1}(2\kappa' vb - 2h - W) \\
&\leq -\delta'' q_{s-1/2} f^2 q_{s-1/2} - 2s\left\{(h-\overline{\zeta})q_{2s-1} + q_{2s-1}(h-\zeta)\right\} \\
&- iCs\left\{(h-\overline{\zeta})q_{2s-1} - q_{2s-1}(h-\zeta)\right\}.
\end{aligned}$$

The same thing is done in (4.49). The rest of the proof is now similar to Case 1.

□

We now remove the extra $b$'s appearing in the statement of Lemma 4.9 compared to (4.5d). Notice that this follows readily from the pseudodifferential calculus since $(\kappa' - b)^{-s}$ is bounded on $\mathrm{supp}(F_-(b)F_-(a_0))$ (this was in fact also used in the proof of Lemma 4.9).

**Lemma 4.10.** *Let $s \geq 0$ be arbitrary. Let $A = A_\zeta = \mathrm{Op^w}(a)$ be the quantization of the symbol $a = \tilde{F}_-(b)F_-(a_0)$. Then for all $\epsilon > 0$ the operator*

$$\langle x\rangle^{-\epsilon} f w^{s-1/2} A R(\zeta) w^{-s-1/2} f \langle x\rangle^{-\epsilon},$$

*is bounded uniformly in $\zeta \in \Gamma_\theta$.*

We can now finish the proof of (4.5d):

*Proof of* (4.5d). The final step in proving (4.5d) consists of replacing the weights $f$ and $w$ in Lemma 4.10 by their limits as $|\zeta|$ goes to zero. That is the content of Lemma 4.6. □

## 5. Proof of main results

5.1. **Proof of Theorem 1.1.** We will only consider the case $\zeta \in \Gamma_\theta$—the other case, $\overline{\zeta} \in \Gamma_\theta$, can be proved analogously.

Let us write for $\zeta \in \mathbb{C} \setminus \mathbb{R}$

$$R(\zeta) = (H - \zeta)^{-1},\ R_1(\zeta) = (H_1 - \zeta)^{-1}, \tag{5.1}$$

with $H_1 = p^2 + V_1 = H - V_2$. We shall proceed perturbatively using

$$R(\zeta)(I + V_2 R_1(\zeta)) = R_1(\zeta). \tag{5.2}$$

First we show that $R_1(\zeta)$ is uniformly Hölder continuous in $\Gamma_\theta$. For that we interpolate (4.6a) for $m = 1$ and $m = 2$. We consider the family of bounded operators

$$B(z) = k^{-z-\epsilon}\{R_1(\zeta_1) - R_1(\zeta_2)\}k^{-z-\epsilon};\ \Re(z) \in [1/2, 3/2]. \tag{5.3}$$

For $\Re(z) = 3/2$ (using $R_1(\zeta_1) - R_1(\zeta_2) = \int_0^1 \frac{d}{dt} R_1(\zeta_2 + t(\zeta_1 - \zeta_2))\, dt$) we have the bound $\|B(z)\| \le C|\zeta_1 - \zeta_2|$, and for $\Re(z) = 1/2$ the bound $\|B(z)\| \le C$, yielding

$$\|B(z)\| \le C|\zeta_1 - \zeta_2|^{\Re(z)-1/2}. \tag{5.4}$$

For $s \le 3/2(1 + \mu/2)$ we choose $z = \Re(z) = s(1 + \mu/2)^{-1}$ in (5.4); otherwise we take $z = 3/2$. This proves the Hölder continuity statement of Theorem 1.1 in the case $V_2 = 0$. In particular $R_1^+ = R_1(0 + i0) = \lim_{\zeta\to 0, \zeta\in\Gamma_\theta} R_1(\zeta)$ and $R_1^- = R_1(0 - i0) = \lim_{\zeta\to 0, \zeta\in\Gamma_\theta} R_1(\overline{\zeta})$ are well-defined (in weighted spaces).

To show (1.4) (in the general case) it suffices to show that $\langle x\rangle^s(I + V_2R_1^+)\langle x\rangle^{-s}$ is invertible as an operator on $L^2(\mathbb{R}^d)$. (Here we use (5.2), as well as the standard limiting absorption principle for positive energies and absence of positive eigenvalues, cf. [Mou81], [Tam89] and [DeGé97, Section 6.5].)

Notice that $\langle x\rangle^s V_2 R_1^+ \langle x\rangle^{-s}$ is compact (being the norm-limit of a compact operator-valued function). Whence by Fredholm theory, it suffices to show that the equation

$$\phi = -V_2R_1^+\phi, \tag{5.5}$$

has no nonzero solution $\phi \in \langle x\rangle^{-s}L^2(\mathbb{R}^d)$. Let $\psi = R_1^+\phi$ ($\in \langle x\rangle^s L^2(\mathbb{R}^d)$). Then we have in the sense of distributions

$$H\psi = 0 \text{ and } \mathrm{V}_2\psi = -\phi. \tag{5.6}$$

We can calculate

$$\begin{aligned} 0 &= \Im\langle\psi, V_2\psi\rangle = -\Im\langle\psi, \phi\rangle \\ &= -\Im\langle R_1^+\phi, \phi\rangle = (2i)^{-1}\langle\phi, (R_1^+ - R_1^-)\phi\rangle. \end{aligned}$$

Since $\frac{R_1^+ - R_1^-}{2i} \ge 0$ we get that

$$\psi = R_1^+\phi = R_1^-\phi. \tag{5.7}$$

**Lemma 5.1.** $\psi \in L^2(\mathbb{R}^d)$.

*Proof.* A priori $\phi \in \langle x\rangle^{-s}L^2$ and $\psi \in \langle x\rangle^{s'}L^2$; $s' > \frac{1}{2} + \frac{\mu}{4}$. From (5.6) we learn that if $\psi \in \langle x\rangle^{s'}L^2$ for some real $s'$ then $\phi \in \langle x\rangle^{s'-1-\mu/2-\delta}L^2$. In particular we have $\phi \in \langle x\rangle^{-\tilde{s}}L^2$; $\tilde{s} < \frac{1}{2} + \frac{\mu}{4} + \delta$. The idea of the proof is to show by a bootstrap argument that we may take $s'$ arbitrary. A bootstrap argument for a similar problem for the free Laplacian was given by Agmon in his proof of [Ag75, Theorem 3.3]. Our analysis is based on Theorem 4.1.

We pick a real-valued function $F_+$ as in Theorem 4.1 (ii) such that $F_+(x) = 1$ for $|x| > 2C_0$. Let $F_- = 1 - F_+$. Pick real-valued functions

$\tilde{F}_-$ and $\tilde{F}_+$ as in Theorem 4.1 (iii) such that $\tilde{F}_- + \tilde{F}_+ = 1$. Then we decompose with the symbols $a_0$ and $b$ being defined as in (4.1) with $E = 0$ in the expression (4.2) for $f$

$$\psi = \mathrm{Op}^{\mathrm{w}}(F_+(a_0))\psi + \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_-(b))\psi \\ + \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b))\psi. \tag{5.8}$$

By (4.5b) and (4.5d) the first two terms on the right hand side of (5.8) belong to $\langle x\rangle^{s'}L^2$ where (assuming here $\phi \in \langle x\rangle^{-s}L^2$)

$$s' = (1+\tfrac{\mu}{2})(-t+\tfrac{1}{2}+\epsilon);\; t = \frac{s}{1+\frac{\mu}{2}} - \tfrac{1}{2} - \epsilon. \tag{5.9}$$

We notice that the bound (4.5e) is equivalent to

$$\|k^{t-1/2-\epsilon}\mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b))R_1(\zeta)^*k^{-t-1/2-\epsilon}\| \le C. \tag{5.10}$$

Taking $\zeta \to 0$ in the sector $\Gamma_\theta$, (5.10) leads to

$$\|k^{t-1/2-\epsilon}\mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b))R_1^- k^{-t-1/2-\epsilon}\| \le C, \tag{5.11}$$

with the same convention for $a_0$ and $b$ as above. We use the representation $\psi = R_1^-\phi$ of (5.7) and apply (5.11), and conclude that also the third term on the right hand side of (5.8) belongs to $\langle x\rangle^{s'}L^2$ with $s'$ given by (5.9); so $\psi \in \langle x\rangle^{s'}L^2$.

From this we learn that

$$\phi \in \langle x\rangle^{s'-1-\frac{\mu}{2}-\delta}L^2 = \langle x\rangle^{-s-\delta+(2+\mu)\epsilon}L^2;$$

so by taking $\epsilon < (2+\mu)^{-1}\delta$ we improve the decay of $\phi$. Iterating this argument (gaining at each iteration almost a factor $\langle x\rangle^{-\delta}$) leads to $s' \le 0$ eventually. □

Combining Theorem 2.4 and Lemma 5.1 yields $\psi = \phi = 0$, completing the proof of (1.4) in the general case. It remains to show the Hölder continuity statement of Theorem 1.1 in the general case. This may easily be done using (5.2) and the known result for $R_1(\zeta)$; we omit the details.

**Remark 5.2.** There exists another approach to proving (1.4) based on a virial type argument and Theorem 2.4: Formally, on one hand

$$\langle \psi, i[H_1, A]\psi\rangle = \langle \psi, W\psi\rangle,$$

while on the other hand

$$\langle \psi, i[H_1, A]\psi\rangle = -2\Im(\langle \phi, A\psi\rangle).$$

This leads to the conclusion (rigorously, after some work using again (5.7)) that $p_j\psi, \langle x\rangle^{-\mu/2}\psi \in L^2(\mathbb{R}^d)$. Therefore, we can apply Theorem 2.4 to conclude that $\psi$, and therefore $\phi$, vanish identically. To make this work one needs a stronger decay assumption than Theorem 1.1 (5). As an advantage, being independent of Theorem 4.1, this method only requires a few derivatives of $V_1$, cf. Remark 3.6.

5.2. **Proof of Theorem 1.2.** We will only prove (1.5), since the Hölder continuity follows from (1.5) using interpolation as in Subsection 5.1.

Let $V = V_1 + V_2$ be the decomposition of $V$ given in the statement of Theorem 1.2. Two successive applications of the resolvent identity (5.2) give

$$R(\zeta) = R_1(\zeta) - R_1(\zeta)V_2R_1(\zeta) + R_1(\zeta)V_2R(\zeta)V_2R_1(\zeta). \tag{5.12}$$

We will prove by induction in $m$ that

$$\left\|k^{-(m-1/2)-\epsilon}R_1(\zeta)^s R(\zeta)^t k^{-(m-1/2)-\epsilon}\right\| \le C, \tag{5.13}$$

for $s, t \in \mathbb{N} \cup \{0\}$ with $1 \le s + t \le m$. It is clear from Theorem 1.1 that (5.13) holds for $m = 1$. So let us assume that (5.13) holds for all $m \le m_0$ and prove that it holds for $m = m_0 + 1$.

The case $s = m_0 + 1$, $t = 0$ is statement (4.6a) of Theorem 4.2. So suppose

$$\|k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^s R(\zeta)^t k^{-(m_0+1/2)-\epsilon}\| \le C, \tag{5.14}$$

where $s \ge \sigma$ and $s + t = m_0 + 1$. Then need to bound

$$k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^{\sigma-1}R(\zeta)^{\tau+1}k^{-(m_0+1/2)-\epsilon};\ \tau = m_0 - \sigma + 1. \tag{5.15}$$

Upon substitution of (5.12) the expression (5.15) becomes

$$\begin{aligned} &k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^\sigma R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon} \\ -&k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^\sigma V_2 R_1(\zeta)R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon} \\ +&k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^\sigma V_2 R(\zeta)V_2R_1(\zeta)R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon} \\ =&E_1 + E_2 + E_3. \end{aligned}$$

By the hypothesis (5.14), $E_1$ is uniformly bounded.

To estimate $E_2$ we write (with $N$ sufficiently big)

$$E_2 = \left(k^{-(m_0+1/2)-\epsilon}R_1(\zeta)^\sigma \langle x\rangle^{-N}\right)\left(\langle x\rangle^N V_2 R_1(\zeta)R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon}\right).$$

We only need to estimate the last factor. By using the resolvent equation it may written as

$$\begin{aligned} &BB_1 + (\zeta - i)BB_2;\ B = \langle x\rangle^N V_2 R_1(i)\langle x\rangle^N, \\ &B_1 = \langle x\rangle^{-N}R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon},\ B_2 = \langle x\rangle^{-N}R_1(\zeta)R(\zeta)^\tau k^{-(m_0+1/2)-\epsilon}. \end{aligned}$$

Using the relative boundedness of $V_2$ and the property of compact support, we see that $B$ is bounded. The two other factors $B_1$ and $B_2$ are bounded by the induction hypotheses.

The argument for the term $E_3$ is similar; it is omitted.

Thus we have a uniform bound of (5.15), and (5.13) follows. Clearly the bound (5.13) with $s = 0$, $m = t$ and (1.5) coincide, so the proof is finished.

5.3. **Proof of** (1.6)**.** Suppose on the contrary that $E'(+0) = 0$. Let $R^+ = R(0 + i0)$ and $R^- = R(0 - i0)$, so that $R^+ = R^-$ as an identity in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We also use the notation $R_1^+$ and $R_1^-$ of the proof of Theorem 1.1 in Subsection 5.1.

We compute

$$\begin{aligned}
&(2i)^{-1}\langle \phi, (R_1^+ - R_1^-)\phi\rangle \\
&= \lim_{\kappa\to 0^+} \kappa \|R_1(i\kappa)\phi\|^2 \\
&= \lim_{\kappa\to 0^+} \kappa \|R(i\kappa)\tilde{\phi}(\kappa)\|^2;\ \tilde{\phi}(\kappa) = (I + V_2 R_1(i\kappa))\phi, \\
&= (2i)^{-1}\langle \tilde{\phi}(+0), (R^+ - R^-)\tilde{\phi}(+0)\rangle = 0,
\end{aligned}$$

from which we conclude that also

$$R_1^+ = R_1^-. \tag{5.16}$$

Using the proof of Theorem 1.1 we may infer from (5.16) that for all $t > 0$

$$k^{t-1/2-\epsilon} R_1^+ k^{-t-1/2-\epsilon} \in \mathcal{B}(L^2(\mathbb{R}^d)). \tag{5.17}$$

Let us consider vectors $\phi \in L^2(B_\rho)$ where $B_\rho = \{x \in \mathbb{R}^d |\ |x| \leq \rho\}$ with $\rho$ large. By taking $t = 1/2 + \epsilon$ in (5.17) we learn that $\psi := R_1^+\phi \in L^2(\mathbb{R}^d)$. Since $H_1\psi = \phi$ we see that $H_1\psi = 0$ outside $B_\rho$. Using this and Remarks 2.5 2) we conclude that indeed $\psi = 0$ outside $B_\rho$. We conclude that $R_1^+ \in \mathcal{B}(L^2(B_\rho))$ is the inverse of the Dirichlet Laplacian $H_{1,\rho} = -\Delta + V_1$ for the region $B_\rho$. We obtain a contradiction from this by choosing a large $\rho$ such that zero is an eigenvalue of $H_{1,\rho}$. Notice that all branches of eigenvalues will cross zero as $\rho \to \infty$, cf. [ReSi78, Theorem XIII.6 (Vol. IV, p. 87)].

5.4. **Proof of Theorem 1.4.**

**Proof of (i).** Due to Theorem 1.2 we may define

$$\begin{aligned}
E^{(m)}(\lambda) = \frac{d^{m-1}}{d\lambda^{m-1}} E'(\lambda) = (2\pi i)^{-1} \frac{d^{m-1}}{d\lambda^{m-1}} \left(R(\lambda + i0) - R(\lambda - i0)\right) \\
\in \mathcal{B}(k^{-(m-1/2)-\tilde{\epsilon}} L^2, k^{(m-1/2)+\tilde{\epsilon}} L^2),
\end{aligned}$$

and

$$E^{(m)}(+0) = \lim_{\lambda \to 0^+} E^{(m)}(\lambda).$$

Using similar notation for $(fE')(\lambda)$ we represent as an operator in this space, cf. [JMP84],

$$\begin{aligned} e^{-itH}(f1_{[0,\infty)})(H) &= \int_0^\infty e^{-it\lambda} f(\lambda)E'(\lambda)d\lambda \qquad (5.18) \\ &= \sum_{n=1}^{m-1} (it)^{-n}(fE')^{(n-1)}(+0) + \int_0^\infty e^{-it\lambda}(it)^{-m+1}(fE')^{(m-1)}(\lambda)d\lambda. \end{aligned}$$

Take $m = 3$.

**Proof of (ii).** We multiply (5.18) by $F(\langle x \rangle < 2t^\kappa)$ (where $\kappa = (1-\epsilon)(1+\mu/2)^{-1}$) from the left and $k^{-(m-1/2)-\tilde{\epsilon}}$ from the right. Using again Theorem 1.2 we estimate

$$\begin{aligned} &\|F(\langle x \rangle < 2t^\kappa) \int_0^\infty e^{-it\lambda}(it)^{-m+1}(fE')^{(m-1)}(\lambda)d\lambda k^{-(m-1/2)-\tilde{\epsilon}}\| \\ &\leq C_1 \int_0^\infty \|(\frac{t^{1-\epsilon}}{k})^{(m-1/2)+\tilde{\epsilon}} e^{-it\lambda}(it)^{-m+1}(fE')^{(m-1)}(\lambda)k^{-(m-1/2)-\tilde{\epsilon}}\|d\lambda \\ &\leq C_2 t^{\frac{1}{2}+\tilde{\epsilon}(1-\epsilon)-\epsilon(m-\frac{1}{2})}. \end{aligned}$$

Suppose

$$\frac{1}{2} - \epsilon(m - \frac{1}{2}) < -(1+\epsilon')\frac{1}{2}.$$

Then for all sufficiently small $\tilde{\epsilon} > 0$

$$\frac{1}{2} + \tilde{\epsilon}(1-\epsilon) - \epsilon(m - \frac{1}{2}) \leq -(1+\epsilon')\frac{1}{2}.$$

This is the argument for the contribution from the last term on the right hand side of (5.18).

We deal with the other terms in a similar way:

$$\begin{aligned} &\|F(\langle x \rangle < 2t^\kappa)(it)^{-n}(fE')^{(n-1)}(+0)k^{-(m-1/2)-\tilde{\epsilon}}\| \\ &\leq C_1\|(\frac{t^{1-\epsilon}}{k})^{(n-1/2)+\tilde{\epsilon}}(it)^{-n}(fE')^{(n-1)}(+0)k^{-(m-1/2)-\tilde{\epsilon}}\| \\ &= C_2 t^{(1-\epsilon)((n-1/2)+\tilde{\epsilon})}t^{-n} \\ &= C_2 t^{\tilde{\epsilon}(1-\epsilon)+(\frac{1}{2}-n)\epsilon-\frac{1}{2}}. \end{aligned}$$

The worst bound is for $n = 1$. Clearly we get the result.

## Appendix A. Algebraic verification of Theorem 4.2

We prove the equations (4.6) by induction in $m$. For $m = 1$ the equations (4.6) become (4.5). So assume that (4.6) are true for all $m \leq m_0$ for some $m_0 \geq 1$. We will prove (4.6a), (4.6b), (4.6d), (4.6f), and (4.6g) for $m = m_0+1$. By symmetry (look at the 'adjoint' expressions) (4.6c), (4.6e) and (4.6h) will follow from the previous proofs, so these cases will not need further elaboration. For shortness we write $R = R(\zeta)$.

**Proof of** (4.6a). To prove (4.6a) for $m = m_0 + 1$ we choose a real-valued function $F_+$ as in Theorem 4.1 (ii) such that $F_+ \equiv 1$ on a neighbourhood of $+\infty$. Let $F_- = 1-F_+$. Pick real-valued functions $\tilde{F}_-$ and $\tilde{F}_+$ as in Theorem 4.1 (iii) such that $\tilde{F}_- + \tilde{F}_+ = 1$. Then obviously $1 = F_+(a_0) + F_-(a_0)\tilde{F}_+(b) + F_-(a_0)\tilde{F}_-(b)$. We decompose

$$\begin{aligned}
&k^{-(m_0+1/2)-\epsilon} R^{m_0+1} k^{-(m_0+1/2)-\epsilon} \\
&= \left(k^{-(m_0+1/2)-\epsilon} R k^{-s}\right) \left(k^s \mathrm{Op}^{\mathrm{w}}(F_+(a_0)) R^{m_0} k^{-(m_0+1/2)-\epsilon}\right) \qquad \text{(A.1)}\\
&+ \left(k^{-(m_0+1/2)-\epsilon} R k^{-s}\right) \left(k^s \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_-(b)) R^{m_0} k^{-(m_0+1/2)-\epsilon}\right)\\
&+ \left(k^{-(m_0+1/2)-\epsilon} R \mathrm{Op}^{\mathrm{w}}(F_-(a_0)\tilde{F}_+(b)) k^s\right) \left(k^{-s} R^{m_0} k^{-(m_0+1/2)-\epsilon}\right),
\end{aligned}$$

cf. (5.8). Choosing $s = 1/2 + \epsilon/3$, the first term in (A.1) is seen to be uniformly bounded by using (4.5a) and (4.6b) with $m = m_0$ and $t = 1 + \epsilon 2/3$ and $\epsilon \to \epsilon/3$. The second term is treated similarly using (4.6d) instead of (4.6b). For the third term we choose $s = m_0+\epsilon/3-1/2$ and apply (4.5e) and (4.6a) to get the conclusion.

In the rest of the proof we will not explicitly introduce factors $k^s$ and $k^{-s}$ as we did above.

**Proof of** (4.6b). For this part we introduce functions $G_+, \tilde{G}_+, \tilde{G}_-$ analogous to the $F$'s in the argument above and satisfying the same conditions. We assume that $G_+ \equiv 1$ on a neighbourhood of $\operatorname{supp} F_+$. Let $G_- = 1 - G_+$. Then we write with $B = \mathrm{Op}^{\mathrm{w}}(F_+(a_0))$ and $\tau = t + m_0 + 1/2 + \epsilon$:

$$\begin{aligned}
k^{t-1/2-\epsilon} B R^{m_0+1} k^{-\tau} =& k^{t-1/2-\epsilon} B R \mathrm{Op}^{\mathrm{w}}(G_+(a_0)) R^{m_0} k^{-\tau} \qquad \text{(A.2)}\\
&+ k^{t-1/2-\epsilon} B R \mathrm{Op}^{\mathrm{w}}(G_-(a_0)\tilde{G}_-(b)) R^{m_0} k^{-\tau}\\
&+ k^{t-1/2-\epsilon} B R \mathrm{Op}^{\mathrm{w}}(G_-(a_0)\tilde{G}_+(b)) R^{m_0} k^{-\tau}.
\end{aligned}$$

The first term in (A.2) is easily estimated using (4.5b) and (4.6b) with $m = m_0$. The second term is estimated by combining (4.5b) and (4.6d) (with $m = m_0$). Finally, the third term is estimated, using the support properties of $G_-$, by combining (4.5g) and (4.6a) (with $m = m_0$).

**Proof of** (4.6d). We shall use a set of functions $G$'s as above such that $\tilde{G}_- \equiv 1$ on a neighbourhood of $\operatorname{supp} \tilde{F}_-$. We consider (A.2) now with $B = \mathrm{Op^w}(F_-(a_0)\tilde{F}_-(b))$. The first term is bounded using (4.5d) and (4.6b). For the second term we apply (4.5d) and (4.6d). Furthermore, using the support properties of $\tilde{G}_-(b)$, we can apply (4.5f) and (4.6a) to bound the third term.

**Proof of** (4.6f). In order to prove (4.6f) we choose a set of functions $G$'s as above such that $G_- \equiv 1$ on a neighbourhood of $\operatorname{supp} F_-^2$, $\operatorname{dist}(\operatorname{supp} \tilde{G}_-, \operatorname{supp} \tilde{F}_+) > 0$ and $\operatorname{dist}(\operatorname{supp} \tilde{G}_+, \operatorname{supp} \tilde{F}_-) > 0$. Now we write with $B_1 = \mathrm{Op^w}(F_-^1(a_0)\tilde{F}_-(b))$ and $B_2 = \mathrm{Op^w}(F_-^2(a_0)\tilde{F}_+(b))$

$$\begin{aligned} k^t B_1 R^{m_0+1} B_2 k^t = & k^t B_1 R \mathrm{Op^w}(G_+(a_0)) R^{m_0} B_2 k^t \\ & + k^t B_1 R \mathrm{Op^w}(G_-(a_0)\tilde{G}_-(b)) R^{m_0} B_2 k^t \\ & + k^t B_1 R \mathrm{Op^w}(G_-(a_0)\tilde{G}_+(b)) R^{m_0} B_2 k^t. \end{aligned} \tag{A.3}$$

To bound the first term in (A.3), we use the support property of $G_+$, (4.5d) and (4.6g). The second term is bounded using the separation of the supports of $\tilde{G}_-$ and $\tilde{F}_+$, (4.5d) and (4.6f). Finally for the third term we combine (4.5f), (4.6e) and the fact that the supports of $\tilde{F}_-$ and $\tilde{G}_+$ are separated.

**Proof of** (4.6g). We finally consider (4.6g) with $m = m_0 + 1$. Here we choose $G$'s as before, but this time satisfying the conditions $\operatorname{dist}(\operatorname{supp} G_+, \operatorname{supp} F_-) > 0$, $\operatorname{dist}(\operatorname{supp} G_-, \operatorname{supp} F_+) > 0$ and $\operatorname{dist}(\operatorname{supp} \tilde{G}_-, \tilde{F}_+) > 0$. Introducing the corresponding partition of unity, we have to bound the terms in (A.3) with $B_1 = \mathrm{Op^w}(F_+(a_0))$ and $B_2 = \mathrm{Op^w}(F_-(a_0)\tilde{F}_+(b))$. Since $\operatorname{dist}(\operatorname{supp} G_+, \operatorname{supp} F_-) > 0$, we can use (4.5b) and (4.6g) to bound the first term. For the second term we use (4.5b), $\operatorname{dist}(\operatorname{supp} \tilde{G}_-, \tilde{F}_+) > 0$ and (4.6f). For the third term we apply the support condition on $G_-$, (4.5g), and (4.6e).

## Appendix B. Uniformity of Hörmander metric

In this appendix we verify that the metric $g_E = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{f_E(x)^2}$ satisfies the estimates in the definition of a Hörmander metric *uniformly* in the parameter $E \in (0, 1]$, cf. Lemma 4.3.

*Proof.* First we prove that for some $C > 0$ independent of $E, x$ and $y$

$$\frac{f(x)}{f(y)} \leq C \left( 1 + \frac{\langle y \rangle}{\langle x \rangle} \right)^{\mu/2}. \tag{B.1}$$

Suppose $\langle x\rangle^{-\mu} \leq E$. Then

$$\frac{f(x)}{f(y)} \leq \sqrt{\frac{\kappa_0^{-2}E + (1-\mu/2)^{-1}E}{\kappa_0^{-2}E}} = \sqrt{\frac{\kappa_0^{-2} + (1-\mu/2)^{-1}}{\kappa_0^{-2}}}.$$

On the other hand if $\langle x\rangle^{-\mu} \geq E$, then

$$\frac{f(x)}{f(y)} \leq \sqrt{\frac{\kappa_0^{-2}\langle x\rangle^{-\mu} + (1-\mu/2)^{-1}\langle x\rangle^{-\mu}}{(1-\mu/2)^{-1}\langle y\rangle^{-\mu}}} = C\left(\frac{\langle y\rangle}{\langle x\rangle}\right)^{\mu/2}.$$

To prove the slow variation, let us assume that $1/C_1 < 1/4$. The inequality $g_{(x,\xi)}((y,\eta)) \leq 1/C_1$ implies $|y|^2 \leq \langle x\rangle^2/C_1$ and therefore

$$\frac{2}{3} \leq \frac{\langle x\rangle}{\langle x+y\rangle} \leq 2. \tag{B.2}$$

Now using (B.1) and (B.2),

$$\begin{aligned}
g_{(x,\xi)+(y,\eta)}(v) &= \frac{v_x^2}{\langle x+y\rangle^2} + \frac{v_\xi^2}{f(x+y)^2} \\
&\leq \sup\left\{\left(\frac{\langle x\rangle}{\langle x+y\rangle}\right)^2, \left(\frac{f(x)}{f(x+y)}\right)^2\right\} g_{(x,\xi)}(v) \\
&\leq \sup\left\{4, C\left(\frac{5}{2}\right)^{\mu}\right\} g_{(x,\xi)}(v).
\end{aligned}$$

The dual metric is given by

$$g^\sigma = f^2 dx^2 + \langle x\rangle^2 d\xi^2.$$

Since clearly

$$\langle x\rangle^{-2} \leq \langle x\rangle^{-\mu} \leq f^2,$$

we see that the uncertainty principle is satisfied.

Finally, we prove that the metric is temperate. Let us first consider the case $|x| \geq |y|$. Then

$$\frac{g_{(x,\xi)}(v)}{g_{(y,\eta)}(v)} = \frac{\frac{v_x^2}{\langle x\rangle^2} + \frac{v_\xi^2}{f(x)^2}}{\frac{v_x^2}{\langle y\rangle^2} + \frac{v_\xi}{f(y)^2}} = \frac{a^2 v_x^2 + b^2 v_\xi^2}{c^2 v_x^2 + d^2 v_\xi^2},$$

where $c^2 \geq a^2$, $b^2 \geq d^2$. One now sees that the function $s \mapsto \frac{a^2+b^2 s}{c^2+d^2 s}$ is increasing on $[0,\infty)$. Therefore, using also (B.1), we infer that for all $v \neq 0$

$$\frac{g_{(x,\xi)}(v)}{g_{(y,\eta)}(v)} \leq \frac{b^2}{d^2} = \frac{f(y)^2}{f(x)^2} \leq C^2\left(1 + \frac{\langle x\rangle}{\langle y\rangle}\right)^{\mu} \leq C^2\left(1 + \frac{\langle x\rangle}{\langle y\rangle}\right)^2.$$

A similar argument shows that in the case $|x| < |y|$

$$\frac{g_{(x,\xi)}(v)}{g_{(y,\eta)}(v)} \le \frac{a^2}{c^2} = \frac{\langle y\rangle^2}{\langle x\rangle^2}.$$

We need to find $\tilde{C}, N > 0$ such that

$$\max\left\{\frac{\langle x\rangle^2}{\langle y\rangle^2}, \frac{\langle y\rangle^2}{\langle x\rangle^2}\right\} \le \tilde{C}\Big(1 + g^\sigma_{(x,\xi)}((y,\eta)-(x,\xi))\Big)^N. \tag{B.3}$$

Clearly

$$g^\sigma_{(x,\xi)}((y,\eta)-(x,\xi)) \ge f(x)^2|y-x|^2 \ge c\langle x\rangle^{-\mu}(|x|-|y|)^2.$$

If $|x| \ge 2|y|$, then

$$\langle x\rangle^{-\mu}(|x|-|y|)^2 \ge 4^{-1}\langle x\rangle^{-\mu}|x|^2.$$

Using the trivial bound $\frac{\langle x\rangle^2}{\langle y\rangle^2} \le \langle x\rangle^2$ we conclude that in this case any $N \ge 2/(2-\mu)$ suffices. If $|x| \le 2^{-1}|y|$, then

$$\langle x\rangle^{-\mu}(|x|-|y|)^2 \ge 4^{-1}\langle y\rangle^{-\mu}|y|^2,$$

and again $N \ge 2/(2-\mu)$ suffices. If, on the other hand, $2^{-1}|y| < |x| < 2|y|$, then $\max\{\frac{\langle x\rangle^2}{\langle y\rangle^2}, \frac{\langle y\rangle^2}{\langle x\rangle^2}\} \le 4$.

We have proved (B.3) and hence that the metric is temperate uniformly in $E$. That finishes the proof of the first part of Lemma 4.3.

The second part of Lemma 4.3 for $m = \langle x\rangle$ or $m = f_E$ follows readily from (B.1), (B.2) and (B.3). The statement for $m = \langle\xi\rangle$ or $m = \langle\frac{\xi}{f_E}\rangle$ may be verified along the same line; we omit the proof. The statement for products of powers is a general property for uniformly temperate weight functions.

□

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