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Preface

The first MaphySto Conference on 'Lévy Processes: Theory and Applications', held 18-22 January 1999 at the Department of Mathematical Sciences, University of Aarhus, made it clear that the field of infinite divisibility, Lévy processes, etc. was in a very active state of wide ranging research developments, in regard to applications as well as theory. It was therefore natural, soon after the event, to think of organising a second conference with the same title, in the framework of MaPhySto. The present volume, which consists of extended abstracts of the papers presented at this second conference, testifies to the fact that interest in the field and the diversity of aspects studied have further increased in the intervening three years.

A Mini-proceedings volume, like the present, was also produced after the first conference and is available in the MaPhySto Miscellanea Series as No. 11 (1999). Furthermore, a volume of state-of-the-art articles, consisting to a large extent of fully developed accounts of papers presented at that conference, was published in 2001 in collaboration with Birkhauser Verlag under the title "Lévy Processes. Theory and Applications" (Eds.: Ole E. Barndorff-Nielsen, Thomas Mikosch and Sidney I. Resnick).

As for the first Miniproceedings, it is a pleasure to thank the participants for their willingness to contribute to the present volume, which hopefully will be found of interest to a large group of readers.

A special warm thanks to Ken-iti Sato and Thomas Mikosch for their generous help in organising both conferences.

Ole E. Barndorff-Nielsen

Stochastic integrals and Lévy-Ito decomposition on separable Banach spaces

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Abstract

The results obtained in [1] and reported at the Aarhus conference are reported here. A direct definition of stochastic integrals for deterministic Banach valued functions on separable Banach spaces is given with respect to compensated Poisson random measures and applied to provide a direct proof of the Lévy -Ito decomposition of a càdlàg process with stationary, independent increments into a jump and Brownian component on Banach spaces of type 2.

1 Introduction

In this paper we present the results obtained in [1] and reported at the Aarhus conference by the second named author. In the first part of this article we give a direct definition of stochastic integrals for deterministic functions with respect to compensated Poisson random measures of Lévy processes $(X_t)_{t\geq 0}$ on separable Banach spaces. (In [2] the results of this article are extended for the case of compensated Poisson random measure of additive processes $(X_t)_{t\geq 0}$. In [32] the whole approach is extended to the case of random functions.) In the second part of the present paper this approach is used to provide a direct proof of the corresponding Lévy-Ito decomposition theorem on separable Banach spaces of type 2. To the best of our knowledge the only existing proofs of this decomposition of càdlàg processes with stationary, independent increments (Lévy processes) into a jump and a Brownian part on infinite dimensional state spaces are given in [13], [35], where the decomposition is proven for the case where the state space is a (co -) nuclear space, and in [8] for the case of Banach spaces ([8] was pointed out to us during the conference by J. Rosinski and W. A. Woyczynski). In the preceding works however no direct expression of the Lévy part as connected with stochastic integrals is given. (As the proof in [8] is rather sketchy we describe it in Remark 5.3, trying to complete it as much as possible with precise references.) For our proof of the decomposition theorem for the Lévy-processes $(X_t)_{t\geq 0}$ on separable Banach spaces $(E, \mathcal{B}(E))$ of type 2, where the above mentioned stochastic integral is defined, we need that the corresponding Lévy measure ν (see Definition 2.10) satisfies the condition

$$\int_{E\setminus\{0\}} \min(1, \|x\|^2)\nu(dx) < \infty \tag{1.1}$$

It is well known that on $(I\!R \setminus 0, \mathcal{B}(I\!R \setminus 0))$ the condition (1.1) is satisfied by any Lévy measure. Viceversa, any σ-finite measure on $(I\!R \setminus \{0\}, \mathcal{B}(I\!R \setminus \{0\}))$, satisfying (1.1), is a Lévy measure. The same correspondence between Lévy measures and the above condition holds on any separable Hilbert space ([25]), while on general separable Banach spaces $(E, \mathcal{B}(E))$ (1.1) is neither necessary nor sufficient for a σ-finite measure on $(E \setminus 0, \mathcal{B}(E \setminus 0))$ to be a Lévy measure. However separable Banach spaces of cotype 2 are characterized by the condition that any Lévy measure satisfies condition (1.1), while separable Banach spaces of type 2 are characterized by the condition that any σ-finite measure on $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\}))$, which satisfies (1.1), is a Lévy measure (see, e.g., [3], [4],[10], [22]). We recall the definition of type -p, resp. cotype-p Banach spaces (see e.g. [3]).

Definition 1.1. *A separable Banach space B is of type p, $1 \le p \le 2$, if there is a constant K_p, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent B-valued random variables, such that $E[\|X_i\|^p] < \infty$, then*

$$E[\|\sum_{i=1}^n X_i\|^p] \le K_p \sum_{i=1}^n E[\|X_i\|^p] \tag{1.2}$$

Definition 1.2. *A separable Banach space B is of cotype p, $p \ge 2$, if there is a constant C_p, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent B-valued random variables, such that $E[\|X_i\|^p] < \infty$, then*

$$E[\|\sum_{i=1}^n X_i\|^p] \ge C_p \sum_{i=1}^n E[\|X_i\|^p] \tag{1.3}$$

A Banach space is type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space [17]. Typical examples of separable Banach spaces of cotype 2 (resp. type 2) are the spaces $L_p(\Omega, P)$,$p \in [1, 2]$ (resp. $p \in [2, \infty)$), where (Ω, P) is any measure space.

The structure of the paper is as follows. In Sect.2 we define Lévy measures and processes on separable Banach spaces, as well as the associated Poisson random measures. Section 3 recalls the Lévy Ito decomposition theorem on $(I\!R, \mathcal{B}(I\!R))$. Section 4 gives our construction of deterministic stochastic integrals on separable Banach spaces. Section 5 gives our Lévy-Ito decomposition theorem (Theorem 5.1).

2 Poisson and Lévy measures of Lévy processes on separable Banach spaces

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le +\infty}, P)$, satisfying the "usual hypothesis", is given:

i) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$.

ii) $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t} \mathcal{F}_t$ for all t such that $0 \le t < +\infty$, i.e. the filtration is right continuous.

We shall study Lévy processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le +\infty}, P)$ with values in $(E, \mathcal{B}(E))$, where in the whole paper we assume that E is a separable Banach space with norm $\| \cdot \|$ and $\mathcal{B}(E)$ is the corresponding σ-algebra. We start by recalling the well known definition of Lévy process.

Definition 2.1 (Lévy process). *A process $(X_t)_{t \ge 0}$ with state space $(E, \mathcal{B}(E))$, is an $\mathcal{F}_t$-Lévy process on $(\Omega, \mathcal{F}, P)$ if*

i) $(X_t)_{t \ge 0}$ *is adapted (to* $(\mathcal{F}_t)_{t \ge 0}$*)*

ii) $X_0 = 0$ *a.s.*

iii) $(X_t)_{t \ge 0}$ *has increments independent of the past, i.e.* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *if* $0 \le s < t$.

iv) $(X_t)_{t \ge 0}$ *has stationary increments, that is* $X_t - X_s$ *has the same distribution as* X_{t-s}, $0 \le s < t$.

v) $(X_t)_{t \ge 0}$ *is stochastically continuous.*

vi) $(X_t)_{t \ge 0}$ *is càdlàg.*

Remark 2.2. The class satisfying i)–v) is given canonically once an infinitely divisible probability measure E is fixed (as easily seen e.g. from [21] and [7]). That any such process has a càdlàg version follows from its being, after compensation, a martingale (see e.g. [9], [15],[26]).

Let $(X_t)_{t\geq 0}$ be a Lévy process on $(E, \mathcal{B}(E))$ (in the sense of Definition 2.1). Set $X_{s^-} := \lim_{s\uparrow t} X_s$ and $\Delta X_s := X_s - X_s^-$. Following the same lines as for the case where the state space is $(I\!R, \mathcal{B}(I\!R))$ (see e.g. [28], [33]) we prove in [1] the following results (Theorems 2.3, 2.4, 2.7, 2.9 and Corollaries 2.5, 2.8 below).

Theorem 2.3. *Let $\Lambda \in \mathcal{B}(E)$, $0 \in (\overline{\Lambda})^c$ (where as usual $\overline{\Lambda}$ denotes the closure of the set Λ and N^c denotes the complement of a set N),*

$$N_t^\Lambda := \sum_{0<s\leq t} \mathbf{1}_\Lambda(\Delta X_s) = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n^\Lambda} \tag{2.1}$$

where

$$T_1^\Lambda := inf\{s > 0 : \Delta X_s \in \Lambda\} \tag{2.2}$$

$$T_{n+1}^\Lambda := inf\{s > T_\Lambda^n : \Delta X_s \in \Lambda\}, \quad n \in I\!N. \tag{2.3}$$

N_t^Λ is an adapted counting process without explosion. Moreover it is a Poisson process.

Theorem 2.4. *Let $\mathcal{B}(E \setminus \{0\})$ be the trace σ-algebra on $E \setminus \{0\}$ of the Borel σ-algebra $\mathcal{B}(E)$ on E , and $\mathcal{F}((E \setminus \{0\}) := \{\Lambda \in \mathcal{B}(E \setminus \{0\}) : 0 \in (\overline{\Lambda})^c\}$, then $\mathcal{F}((E \setminus \{0\})$ is a ring and for all $\omega \in \Omega$ the set function*

$$N_t^\cdot := N_t(\omega, \cdot)\colon \mathcal{F}(E \setminus \{0\}) \to I\!R_+ \tag{2.4}$$

$$\Lambda \to N_t^\Lambda(\omega) \tag{2.5}$$

is a σ-finite pre-measure.

Corollary 2.5. *For any $\omega \in \Omega$ there is a unique σ-finite measure on $\mathcal{B}(E \setminus \{0\})$*

$$N_t(\omega, \cdot) : \quad \mathcal{B}(E \setminus \{0\}) \to I\!R_+ \tag{2.6}$$

$$A \to N_t^A(\omega) \tag{2.7}$$

which is the continuation of the σ-finite pre-measure on $\mathcal{F}(E \setminus \{0\})$ given by Theorem 2.4.

From Theorem 2.4, Corollary 2.5 it follows that $N_t : \Lambda \to N_t^\Lambda$ is a random measure on $(E, \mathcal{B}(E)$.

Definition 2.6. *$N_t : \Lambda \to N_t^\Lambda$ is called the Poisson random measure of the Lévy process $(X_t)_{t\geq 0}$.*

Theorem 2.7. *The set function $\nu(\Lambda) := E[N_1^\Lambda(\omega)] \in I\!R$, $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $\omega \in \Omega$ satisfies:*

$$\nu : \quad \mathcal{F}(E \setminus \{0\}) \to I\!R_+ \tag{2.8}$$

$$\Lambda \to E[N_1^\Lambda(\omega)] \tag{2.9}$$

and is a σ-finite pre-measure on $\big((E \setminus \{0\}), \mathcal{F}(E \setminus \{0\})\big)$.

Corollary 2.8. *There is a unique σ-finite measure on the σ-algebra $\mathcal{B}(E \setminus \{0\})$*

$$\nu : \quad \mathcal{B}(E \setminus \{0\}) \to I\!R_+ \tag{2.10}$$

$$A \to E[N_1^A(\omega)] \tag{2.11}$$

which is the continuation to $\mathcal{B}(E \setminus \{0\})$ of the σ-finite pre-measure ν on the ring $((E \setminus \{0\}), \mathcal{F}(E \setminus \{0\}))$, given by Theorem 2.7.

Theorem 2.9. *The σ-finite measure ν of Corollary 2.8 is a Lévy measure.*

We recall the definition of Lévy measures on separable Banach spaces.

Definition 2.10. *A σ-finite positive measure ν on $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\}))$ is a "Lévy measure", if there is a probability measure μ on $(E, \mathcal{B}(E))$ such that the Fourier transform $\hat{\mu}(F)$, $F \in E'$ satisfies*

$$\hat{\mu}(F) = \exp \int_{E\setminus\{0\}} \exp(iF(x) - 1 - iF(x)\mathbf{1}_{\|x\|\leq 1})\nu(dx) \tag{2.12}$$

We call μ the "Poisson type measure" associated with the "Lévy measure" ν.

Definition 2.11. *We call the measure ν of Theorem 2.9 "the Lévy measure of the Lévy process $(X_t)_{t\geq 0}$".*

Definition 2.12. *We call the random measure $q_t(\omega,\cdot) := N_t(\omega,\cdot)-t\nu(\cdot)$ "the compensated Poisson random measure of the Lévy process $(X_t)_{t\geq 0}$".*

Integrals of Banach valued bounded functions w.r.t. the random measure $N_t(\omega, dx)$ are naturally defined on $\mathcal{F}(E\setminus\{0\})$ (i.e. "when excluding small jumps") as follows.

Definition 2.13. *Let $\Lambda \in \mathcal{F}(E\setminus\{0\})$, $f: E \to F$ be $\mathcal{F}(E\setminus\{0\})/\mathcal{B}(F)$ - measurable and bounded on $E\setminus\{0\}\cap\Lambda$, where $(F,\mathcal{B}(F)$ is a separable Banach space. Then:*

$$\int_\Lambda f(x)\, N_t(\omega, dx) = \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_\Lambda(\Delta X_s) \tag{2.13}$$

Definition 2.14. *Let $\Lambda \in \mathcal{F}(E\setminus\{0\})$, $f: E \to F$ be $\mathcal{F}(E\setminus\{0\})/\mathcal{B}(F)$ - measurable and bounded on $E\setminus\{0\}\cap\Lambda$, then the natural integral w.r.t. the compensated Poisson random measure $N_t(dx)-t\nu(dx)$ is*

$$\int_\Lambda f(x)\,(N_t(\omega,dx)-t\nu(dx)) = \sum_{0<s\leq t} f((\Delta X_s)(\omega))\,\mathbf{1}_\Lambda(\Delta X_s) - t\int_\Lambda f(x)\nu(dx) \tag{2.14}$$

where the last term is a Bochner integral.

3 The Lévy-Ito decomposition theorem on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and related stochastic integration

In [12], Ito proved the following well known decomposition for Lévy processes on the real line (and the corresponding decomposition for more general real valued additive processes):

Theorem 3.1. *Let $(X_t)_{t\geq 0}$ be a Lévy-process on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, and ν the corresponding Lévy measure (according to Definition 2.11).*
Then for all $K>0$, there is a constant α_K such that $\forall t\geq 0$

$$X_t = B_t + \int_{\|x\|<K} x(N_t(dx)-t\nu(dx)) + \alpha_K t + \int_{\|x\|\geq K} xN_t(dx) \quad P-a.s. \tag{3.1}$$

where $N_t(\omega,dx)$ is the Poisson random measure of the Lévy process $(X_t)_{t\geq 0}$. $(B_t)_{t\geq 0}$ is a Brownian motion with 0-mean. For all $\Lambda\in\mathcal{F}(\mathbb{R}\setminus\{0\})$, $(B_t)_{t\geq 0}$ is independent of $(N_t^\Lambda)_{t\geq 0}$, (with the notation $N_t^\Lambda(\omega) := N_t(\omega,\Lambda)$).

(as usual we omit for simplicity to write the dependence on ω in (3.1).)

K.I. Sato gives in his book [33] a short description of the history of the above theorem, in the case of additive processes known under "Lévy-Ito decomposition theorem", starting with the following sentence: "the decomposition was conceived by Lévy [19], [20], and formulated and proved by Ito [12] using many pages".
In fact, Theorem 3.1 can be proven by showing first the results of Section 1, i.e. Theorems 2.3, 2.4, 2.7, 2.9 and Corollaries 2.5, 2.8, and then proving that the centered Lévy process $X_t - J_t - E[X_t - J_t]$, with $J_t := \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\|\Delta X_s\|\geq K}$, $E[X_1 - J_1] = \alpha_K$ decomposes into the sum $B_t + \int_{\|x\|<K} x(N_t(dx)-t\nu(dx))$, one of the main difficulties being the definition of the integral $\int_{\|x\|<K} x(N_t(dx)-t\nu(dx))$, which is shown to be well defined as the limit in $L^2(\Omega,P)$ of $\int_{\frac{1}{n}\|x\|<K} x(N_t(dx)-t\nu(dx))$.
If we are however interested in studying the stochastic differential equations with non Gaussian white noise, given by the random measure $q_t(dx) := N_t(\omega,dx)-t\nu(dx)$, the above integral should also be expressed as a special case of a (deterministic) stochastic integral like it is done e.g. in [34] (for the case of stable laws).
In the present paper we only treat the extension of this theorem for Lévy processes to the case of Banach spaces of type 2, an extension to the case of Banach space valued additive processes is presented in [2].

4 Stochastic (deterministic) integrals w.r.t. the compensated Poisson random measure on separable Banach spaces

In this Section we present our results concerning the stochastic integration with respect to the compensated Poisson random measure $q_t(\omega, \cdot) := N_t(\omega, \cdot) - t\nu(\cdot)$, $\forall t \geq 0$, where N_t (resp. ν) is the Poisson random measure (resp. Lévy measure) of the Lévy process $(X_t)_{t\geq 0}$ on the separable Banach space $(E, \mathcal{B}(E))$. For the proof of these results we refer to [1]. (In [32] the whole approach is extended to the case of random functions.) We shall consider here integration of deterministic functions $f : E \to F$, f being $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ -measurable, where $(F, \mathcal{B}(F))$ is a separable Banach space with norm $\|\cdot\|_F$. We will define different kinds of integrability conditions , strong p-integrability and simple p-integrability, $p \geq 1$.
The simple p-integrability condition is satisfied when the "natural" integral of f on a set $\Lambda_n := \{\delta_n < \|x\| \leq 1\}$ (definition 2.14) converges, when δ_n goes to zero, in $L^p(\Omega, \mathcal{F}, P)$.
The strong p-integral is defined by approximation in $L^p(\Omega, \mathcal{F}, P)$ of the "natural" integrals of simple functions (Definition 4.3). This concept generalizes the known definition of stochastic integration of real valued functions with respect to martingales measures [14], [36]), to Banach space valued functions, for the case where the martingale measures are given by compensated Poisson random measures.

We prove that functions which are Bochner integrable w.r.t. the Lévy measure ν of $(X_t)_{t\geq 0}$ are strong 1-integrable, and functions with values on separable Banach spaces F of type 2 (Definition 1.1 below), which satisfy the condition $\int \|f(x)\|^2 \nu(dx) < \infty$, are strong 2-integrable, the strong p-integrability being equivalent to the simple p-integrability, under the above conditions. Moreover we introduce the notion of simple integral (Definition 4.16) and prove that under the above conditions a function which is simply integrable is (simply or strong) p-integrable, $p = 1, 2$.

To this purpose we give the following definition:

Definition 4.1. *Let $p \geq 1$. $L_p^F(\Omega, \mathcal{F}, P)$ is the space of F-valued random variables, such that $E\|Y\|^p = \int \|Y\|^p dP < \infty$. We denote by $\|\cdot\|_p^F$ (or simply $\|\cdot\|_p$ when $E = F$) the norm given by $\|Y\|_p^F = (E\|Y\|_F^p)^{1/p}$. Given $(Y_n)_{n\in\mathbb{N}}, Y \in L_p^F(\Omega, \mathcal{F}, P)$, we write $\lim^p_{n\to\infty} Y_n = Y$ if $\lim_{n\to\infty} \|Y_n - Y\|_p^F = 0$*

We are interested in the following set of functions. Let $p \geq 1$,

$$M_\nu^p(E/F) := \{f : E \to F \quad \mathcal{B}(E \setminus \{0\})/\mathcal{B}(F) \quad -\, measurable, \quad \int \|f(x)\|^p \,\nu(dx) < \infty\} \tag{4.1}$$

Remark 4.2. A function $f : E \to F$ $\quad \mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ $\quad -$ *measurable* is Bochner integrable w.r.t. ν on $E \setminus \{0\}$ if and only if $f \in M_\nu^1(E/F)$

Let us define the set $\mathcal{S}(E/F)$ of "simple functions".

Definition 4.3. *A function f belongs to the sets $\mathcal{S}(E/F)$ of simple functions , if $f : E \setminus \{0\} \to F$ is such that*

$$f(x) = \sum_{k=1}^{N} a_k \mathbf{1}_{A_k}, \quad A_k \in \mathcal{F}(E \setminus \{0\}), \tag{4.2}$$

with $N \in \mathbb{N}$, $a_k \in F$, $k \in (1, ...N)$. If $E = F$, we write $\mathcal{S}$ instead of $:= \mathcal{S}(E/E)$

Similar to the proof of the "if -part" of a Theorem (S.Bochner) in [39], Chapt. V, §5 the following can be proved

Proposition 4.4. *For any σ-finite measure ν on $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\})$ and for any $f \in M_\nu^p(E/F)$, there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions converging ν -a.s. to f, such that*

$$\lim_{n\to\infty} \int \|f_n - f\|_F^p \, d\nu = 0. \tag{4.3}$$

Definition 4.5. *Let $p \geq 1$ We say that $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$-measurable, is strong p-integrable on $\Lambda \in \mathcal{B}(E \setminus \{0\})$ w.r.t. the random measure q_t if the limit*

$$\int_\Lambda f(x) q_t(dx) := \lim_{n\to\infty}^{p} \int_\Lambda f_n(x) q_t(dx) \tag{4.4}$$

exists for any sequence $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{S}$ which satisfies the condition in Proposition 4.4, and does not depend on the choice of the sequence $\{f_n\}_{n\in\mathbb{N}}$.

Remark 4.6. Let f be strong p-integrable. Then $\forall \Lambda \in \mathcal{B}(E \setminus \{0\})$

$$E[\int_\Lambda f(x) q_t(dx)] = 0 \tag{4.5}$$

In fact, by definition of Bochner integral of random variables from $(\Omega, \mathcal{F}_\infty, P)$ to $(F, \mathcal{B}(F))$, one has

$$E[\int_\Lambda f(x) q_t(dx)] = \lim_{n\to\infty} E[\int_\Lambda f_n(x) q_t(dx)] = 0 \tag{4.6}$$

as

$$\lim_{n\to\infty} \| \int_\Lambda f(x) q_t(dx) - \int_\Lambda f_n(x) q_t(dx) \| = 0 \quad P-a.s. \tag{4.7}$$

and $\int_\Lambda f_n(x) q_t(dx) \in \mathcal{S}(E/F)$.

Remark 4.7. Let f, g be strong p-integrable. For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is strong p-integrable and we have that $\forall \Lambda \in \mathcal{B}(E \setminus \{0\})$

$$\alpha \int_\Lambda f(x) q_t(dx) + \beta \int_\Lambda g(x) q_t(dx) = \int_\Lambda (\alpha f(x) + \beta g(x)) q_t(dx) \tag{4.8}$$

Theorem 4.8. *Let $f \in M^1_\nu(E/F)$, then f is strong 1-integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|] \leq 2t \int_\Lambda \|f(x)\| \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{4.9}$$

Theorem 4.9. *Suppose $(F, \mathcal{B}(F))$ is a separable Banach space of type 2. Let $f \in M^2_\nu(E/F)$, then f is strong 2-integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|^2] \leq 4K_2 t \int_\Lambda \|f(x)\|^2 \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{4.10}$$

where K_2 is the constant K_p, $p = 2$ in the Definition 1.1 (of type p Banach spaces).

Theorem 4.10. *Suppose $(F, \mathcal{B}(F)) := (H, \mathcal{B}(H))$ is a separable Hilbert space. Let $f \in M^2_\nu(E/H)$, then f is strong 2-integrable w.r.t. q_t. Moreover*

$$E[\| \int_\Lambda f(x) q_t(dx) \|^2] = t \int_\Lambda \|f(x)\|^2 \nu(dx) \quad \forall \Lambda \in \mathcal{B}(E \setminus \{0\}) \tag{4.11}$$

Proposition 4.11. *Let $p \geq 1$, f be p-strong integrable and $f \in M^p_\nu$. For all $\Lambda \in \mathcal{F}(E \setminus \{0\})$ the strong p-integral of f coincides with the natural integral of f, i.e.*

$$\int_\Lambda f(x) q_t(dx) = \sum_{0<s<t} f(\Delta X_s) \mathbf{1}_{\Delta X_s \in \Lambda} - t \int_\Lambda f(x) \nu(dx) \quad P-a.s. \tag{4.12}$$

Definition 4.12. *Let $p \geq 1$. We say that $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable, is simply p-integrable w.r.t. the random measure q_t if for any sequence $\delta_n > 0$, which converges to zero when $n \to \infty$, the limit*

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x) - t\nu(dx)) := \lim_{n\to\infty}^{p} \sum_{0<s\leq t} f((\Delta X_s)(\omega)) \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t \int_{\Lambda_{\delta_n}} f(x) \nu(dx) \tag{4.13}$$

with

$$\Lambda_{\delta_n} := \{x \in E \setminus \{0\} : \delta_n < \|x\| \leq 1\} \tag{4.14}$$

exists.
The simple p-integral on a set $\Lambda \in \mathcal{F}(E \setminus \{0\}$ coincides with the natural integral in Definition 2.14. While on a set $\Lambda \in \mathcal{B}(E \setminus \{0\}$ it is defined as

$$\int_\Lambda f(x)(N_t(x) - t\nu(dx)) := \lim_{n\to\infty}^{p} \sum_{0<s\leq t} f((\Delta X_s)(\omega)) \mathbf{1}_{\Lambda_{\delta_n}\cap\Lambda}(\Delta X_s) - t \int_{\Lambda_{\delta_n}\cap\Lambda} f(x) \nu(dx) \tag{4.15}$$

(and does henceforth not depend on the choice of the sequence δ_n satisfying the above hypothesis).

Proposition 4.13. *Let $f \in M^1_\nu(E/F)$, then f is simply 1-integrable and the simple 1-integral coincides with the strong 1-integral.*

Proposition 4.14. *Let $f \in M^2_\nu(E/F)$, F a separable Banach space of type 2, then f is simply 2-integrable. The simple 2-integral coincides with the strong 2-integral.*

Corollary 4.15. *Let $f \in M^2_\nu(E/H)$, and H a separable Hilbert space, then f is simply 2-integrable. The simple 2-integral coincides with the strong 2-integral.*

Definition 4.16. *We say that $f : E \to F$, which is $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ - measurable, is simply integrable w.r.t. the random measure q_t if for any sequence $\delta_n > 0$, which converges to zero when $n \to \infty$, the limit*

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x) - t\nu(dx)) := \lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t\int_{\Lambda_{\delta_n}} f(x)\nu(dx) \quad (4.16)$$

exists a.s., (Λ_{δ_n} is defined in (4.14)).
The simple integral on a set $\Lambda \in \mathcal{F}(E \setminus \{0\}$ coincides with the natural integral in Definition 2.14. While on a set $\Lambda \in \mathcal{B}(E \setminus \{0\}$ it is defined as

$$\int_\Lambda f(x)(N_t(x) - t\nu(dx)) := \lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}\cap\Lambda}(\Delta X_s) - t\int_{\Lambda_{\delta_n}\cap\Lambda} f(x)\nu(dx) \quad (4.17)$$

where the sequence converges a.s..

Remark 4.17. We remark that the convergence in this definition is a.s., whereas in Definition 4.12 it is in the L^F_p-sense.

Remark 4.18. Let f be simply integrable. Then for all $\Lambda \in \mathcal{B}(E \setminus \{0\})$

$$E[\int_\Lambda f(x)q_t(dx)] = 0 \quad (4.18)$$

Let f, g be simply integrable. For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is simply integrable and we have that $\forall \Lambda \in \mathcal{B}(E \setminus \{0\})$

$$\alpha \int_\Lambda f(x)q_t(dx) + \beta\int_\Lambda g(x)q_t(dx) = \int_\Lambda (\alpha f(x) + \beta g(x))q_t(dx) \quad (4.19)$$

Proposition 4.19. *If f is simply p-integrable for some $p \geq 1$, then it is simply integrable, and the simple p-integral coincides with the simple integral.*

Proof of Proposition 4.19:
As f is simply p-integrable for some $p \geq 1$, it follows that for any sequence $\{\delta_n\}$, such that $\delta_n \to 0$ when $n \to \infty$

$$\int_{0<\|x\|\leq 1} f(x)(N_t(x) - t\nu(dx)) := \lim_{n\to\infty} \sum_{0<s\leq t} f((\Delta X_s)(\omega))\, \mathbf{1}_{\Lambda_{\delta_n}}(\Delta X_s) - t\int_{\Lambda_{\delta_n}} f(x)\nu(dx)\,, \quad (4.20)$$

the convergence being in probability. From a theorem of Ito-Nisio (Theorem 3.1, [?])(see also [?]) it follows that the limit exists a.s. and coincides with the p-simple integral of f w.r.t. q_t. ■

Corollary 4.20. *Let $f : E \to F$ be $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ -measurable, $f \in M^1_\nu(E/F)$. f is simply integrable. The simple integral coincides with the simple 1-integral and strong 1-integral.*

Corollary 4.21. *Let $f : E \to F$ be $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ -measurable, $f \in M^2_\nu(E/F)$, F a separable Banach space of type 2. f is simply integrable. The simple integral coincides with the simple 2-integral and the strong 2-integral.*

Corollary 4.22. *Let $f : E \to F$ be $\mathcal{B}(E \setminus \{0\})/\mathcal{B}(F)$ -measurable, $f \in M^2_\nu(E/F)$, F a separable Hilbert space. The simple integral coincides with the simple 2-integral and the strong 2-integral.*

Proof of the Corollaries 4.20, 4.21, 4.22:
The proof follows from Proposition 4.19 and Proposition 4.13, resp. Proposition 4.14, Corollary 4.15. ■

Proposition 4.23. *If f is simply integrable and for some $p \geq 1$ fixed*

$$sup_{n \geq N} E[\| \sum_{0<s\leq t} f((\Delta X_s))\, \mathbf{1}_{\Lambda_n}(\Delta X_s) - t \int_{\Lambda_n} f(x)\nu(dx)\|^p] < \infty \tag{4.21}$$

then f is simply p-integrable.

Proof of Proposition 4.23:
The statement of Proposition 4.23 follows from a theorem of Hoffmann-Joergensen ([11] Theorem 5.5 Chap. II). ■

Remark 4.24. The simple integrability of a function f does not imply in general that the function f is Bochner integrable on $E \setminus \{0\}$ w.r.t. ν (hence simply integrability does not imply strong 1-integrability). In fact, the function $f(x) = x$ is simply integrable w.r.t. any compensated Poisson Random measure $N_t - t\nu$ as results from the proof of the Lévy - Khinchine formula on separable Banach spaces (see e.g.[3], [21])) but it is not true that f is Bochner integrable on $E \setminus \{0\}$ for all Lévy measures ν (i.e. there exist Lévy measures ν such that $f(x) = x \notin M^1_\nu$), see e.g. [3].

Remark 4.25. For previous definitions of stochastic integrals with respect to general "abstract" martingales on Banach spaces, see e.g. [5],[8], [18], [23],[24], [26],[29],[30], [31], [37], [38] and, for the case of Hilbert spaces, e.g. [16], [27]. For previous definitions of stochastic integrals of real valued functions with respect to general "abstract" martingale measures on Banach spaces, see e.g. [14], [36]. The main point of the present paper, in this context, is to define stochastic integrals of Banach valued functions with respect to a "concretely constructed" Banach valued Lévy noise.

5 The Lévy-Ito decomposition theorem on separable Banach spaces of type 2

In [1] we prove the following

Theorem 5.1 (Lévy-Ito decomposition theorem on separable Banach spaces). *Let $(X_t)_{t\geq 0}$ be a Lévy-process on a separable Banach space $(E, \mathcal{B}(E))$, and ν the corresponding Lévy measure (according to Definition 2.11). Suppose $N_t(\omega, dx)$ is the Poisson random measure and respectively $q_t(\omega, dx) := N_t(\omega, dx) - t\nu(dx)$ the compensated Poisson random measure associated to the Lévy process $(X_t)_{t\geq 0}$. Suppose the following condition holds*

c) E is a separable Banach space of type 2, and

$$\int_{\{E\setminus 0\}} min(1, \|x\|^2)\, \nu(dx) < \infty\,. \tag{5.1}$$

Then for all $K > 0$, there is $\alpha_K \in E$ such that $\forall t \geq 0$

$$X_t = B_t + \int_{\|x\|<K} x(N_t(dx) - t\nu(dx)) + \alpha_K t + \int_{\|x\|\geq K} xN_t(dx) \quad P-a.s. \tag{5.2}$$

(we omit here for simplicity to write the dependence on $\omega \in \Omega$), where $(B_t)_{t\geq 0}$ is an E-valued Brownian motion with 0-mean. For all $\Lambda \in \mathcal{F}(E \setminus \{0\})$, $(B_t)_{t\geq 0}$ is independent of $(N^\Lambda_t)_{t\geq 0}$, (with the notation $N^\Lambda_t(\omega) := N_t(\omega, \Lambda)$).
The integral $\int_{\|x\|\geq K} x(N_t(dx) - t\nu(dx))$ is the strong 2 (or equivalently simple 2) -integral of the function $f(x) = x$ w.r.t. q_t.

Remark 5.2. Let μ be such that $\mu(A) = P(X_1 \in A)$, $\forall A \in \mathcal{B}(E)$. Let us take $K = 1$ in the decomposition (5.2). From Theorem 2.9 and the Lévy-Khinchine representation theorem for infinitely divisible laws on separable Banach spaces (see e.g. [21] Theorem 5.7.3 or [3]) it follows that $\mu = \mathcal{G} \star \delta_{\alpha_1} \star \mathcal{L}$, where $\mathcal{G}$ is the distribution of the Brownian motion $(B_t)_{t\geq 0}$, $\mathcal{L}$ is the Poisson type probability measure associated with the Lévy measure ν, and $\alpha_1 \in E$.

Remark 5.3. As the main work in this paper was already finished, we learned that the Lévy-Ito decomposition theorem, without however a direct expression of the "Lévy part" $\int_{0<\|x\|\leq 1}\int x\,[N_t(dx)-t\nu(dx)]$ as a special case of a (deterministic) stochastic integral, in the sense of Definition 4.5, is also stated in [8], however with a rather sketchy proof. ($\int_{0<\|x\|\leq 1}\int x\,[N_t(dx)-t\nu(dx)]$ is in [8] the limit in $L_p^E(\Omega,\mathcal{F},P)$, $p\geq 1$ of $\int_{\frac{1}{n}<\|x\|\leq 1}\int x\,[N_t(dx)-t\nu(dx)]$, when $n\to\infty$). Let us describe a proof of the Lévy-Ito decomposition theorem along the lines of [8], however completing it as much as possible with precise references. Let ν be a Lévy measure on the separable Banach space $(E,\mathcal{B}(E))$. By [21] (Prop. 5.4.5, iii, p. 76), given any sequence $\delta_n\downarrow 0$, the sequence of probability measures $\rho_n(\cdot):=e^{-\nu(E)}\sum_{k=0}^{\infty}\frac{\nu^k(\cdot\cap\|x\|>\delta_n)}{k!}$ is such that there exist points $x_n\in E$, such that $(\rho_n\star\delta_{x_n})(\cdot)$ contains a subsequence which converges weakly as $\delta_n\downarrow 0$. From Corollary 5.4.6 [21] it follows that $x_n:=\int_{\delta_n<\|z\|\leq 1}zt\nu(dz)$. Moreover the Fourier transform $\widehat{(\rho_n\star\delta_{x_n})}(k)$ of $(\rho_n\star\delta_{x_n})(\cdot)$ converges itself for $n\to\infty$ point wise to $\exp(\int_{E\setminus 0}e^{i<k,x>}-1-i<x,k>)\nu(dx)$ ([21], Theorem 5.4.8, ii), p. 78). This identifies the weak limit of $\rho_n\star\delta_{x_n}(\cdot)$, as $n\to\infty$. Moreover by [21] (Theorem 5.3.6., p. 70) $\nu(\cdot\cap\|x\|>\delta_n)$ has a weakly convergent subsequence, the weak limit being $\nu(\cdot)$. Suppose now that ν is the Lévy measure of the Lévy process X_t (according to definition 2.11). From the proof of the Theorem 2.9 it follows that $\tilde{\nu}(\cdot):=\nu(\cdot\cap\|x\|\leq 1)$ is the Lévy measure of the Lévy process $X_t-J_t-E[X_t-J_t]$, with $J_t=\sum_{0<s\leq t}\Delta X_s\mathbf{1}_{\|\Delta X_s\|\geq 1}$. The distribution of $S_t^n:=\int_{1/n<\|x\|\leq 1}x(N_t(dx)-t\nu(dx))$ is given by $\rho_n(\cdot\cap\|x\|\leq 1)\star\delta_{\int_{\frac{1}{n}<\|z\|\leq 1}zt\nu(dz)}(dx)$ and by the above arguments converges weakly , as $n\to\infty$, to a random variable Y_t with Fourier transform $\exp(\int_{0<\|x\|\leq 1}(e^{i<k,x>}-1-i<x,k>)\nu(dx)$. From the Ito -Nisio theorem it follows that S_t^n converges a.s., for $n\to\infty$. It follows (see e.g. [6], pag. 72)

$$\sup_{n\in\mathbb{N}}E[\|S_t^n\|^p\leq E[\|X_t-J_t-E[X_t-J_t]\|^p]\quad\forall p\geq 1\tag{5.3}$$

From this one can deduce by a theorem of [11] (theorem II.5.5) that S_t^n converges in $L_p^E(\Omega,P)$ to Y_t, and

$$P[sup_{0\leq r\leq t}\|S_r^n-Y_r\|>a]\leq a^{-p}E[\|S_t^n-Y_t\|^p]\quad\forall p>1\tag{5.4}$$

so that there is a subsequence n_k for which $X_t-J_t-E[X_t-J_t]-S_r^{n_k}$ converges a.s. uniformly in $r\in[0,t]$, when $k\to\infty$. From this one can deduce the Lévy -Ito decomposition, with $X_t-J_t-Y_t$ the Gaussian part and $Y_t+\int_{\|x\|>1}xN_t(dx)$ the Lévy part, like it was done in our Proof of Theorem 5.1.

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On the Wold decomposition for cocycle perturbations of a quantum Levy process

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Abstract

We define a class of Markovian cocycle perturbations for a quantum Levy process j. Every Markovian cocycle perturbation in the sense of our definition defines a new quantum Levy process $\tilde{j}$ which is isomorphic to the initial. It gives that for the E_0-semigroup associated with the group of automorphisms translating orbits of $\tilde{j}$ in time we can choose the restriction being a semiflow of Powers shifts which determines a nondeterministic part of the perturbed process. It can be considered as some analogue of the Wold decomposition for a classical stationary stochastic process which allows to obtain a nondeterministic part of the process.

The quantum stochastic process with stationary increments is an one-parameter family $j = (j_t)_{t\in\mathbb{R}},\ j_0 = 0$, consisting of *-homomorphisms j_t embedding the involutive algebra $\mathcal{A}$ into the algebra of linear (non bounded in general) operators $\mathcal{L}$ in a Hilbert space such that there exists a one-parameter w-continuous group $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ of *-automorphisms $\mathcal{L}$ translating the increments of j in time, $\alpha_t(j_s(x) - j_r(x)) = j_{s+t}(x) - j_{r+t}(x),\ x \in \mathcal{A}$. We also suppose that there exists a state ω on $\mathcal{L}$ determining the expectation $\mathbb{E}$ on quantum random variables associated with the process such that $\mathbb{E}(x) = \omega(x)$. Under the quantum Levy process we mean the quantum stochastic process with stationary increments satisfying the additional property that the increments $y_i = j_{t_i}(x_i) - j_{s_i}(x_i),\ x_i \in \mathcal{A}$, are independent for $(s_i, t_i) \cap (s_j, t_j) = \emptyset,\ i \neq j,\ 1 \leq i \leq n$,

(i) in the classical sense which is $\omega(\eta_1(y_{i_1})\eta_2(y_{i_2})\dots\eta_k(y_{i_k})) = \omega(\eta_1(y_{i_1}))\dots\omega(\eta_k(y_{i_k}))$ for an arbtrary choise of functions $\eta_s \in L^\infty$,

(ii) the increments are commutative, $[y_i, y_j] = y_iy_j - y_jy_i = 0,\ i \neq j$.

Let $\mathcal{M}_{t]}$ and $\mathcal{M}_{[t}$ be the von Neumann algebras generated by the past before the time t and the future after the time t of the process j correspondingly, i.e. $\mathcal{M}_{t]} = \{j_t(x) - j_s(x),\ s \leq t,\ x \in \mathcal{A}\}''$ and $\mathcal{M}_{[t} = \{j_s(x) - j_t(x),\ s \geq t,\ x \in \mathcal{A}\}''$. Then $\mathcal{M}_{t]} = \alpha_t(\mathcal{M}_{0]}),\ t \in \mathbb{R}$. In the following we also need the von Neumann algebra $\mathcal{M} = \vee_t\mathcal{M}_{t]} = \vee_t\mathcal{M}_{[t}$ associated with the whole process. The group of automorphisms α is called a Kolmogorov flow if $\cap_{t\in\mathbb{R}}\mathcal{M}_{t]} = \{\mathbb{C}\mathbf{1}\}$. In the case when j is the quantum Levy process, the group α is the Kolmogorov flow. Notice that one doesn't need to claim j to be a quantum Levy process to obtain the Kolmogorov flow α. An one-paramenter family of *-automorphisms $w = (w_t)_{t\in\mathbb{R}}$ is called a multiplicative α-cocycle if

$$w_{t+s} = w_t \circ \alpha_t \circ w_s \circ \alpha_{-t},\ s, t \in \mathbb{R}.$$

We call the multiplicative α-cocycle w Markovian (see [1, 2, 3]) if

$$w_t(x) = x,\ x \in \mathcal{M}_{[t},\ t \geq 0.$$

Every orbit $y_t = j_t(x),\ x \in \mathcal{A}$, of the quantum stochastic process with the stationary increments satisfyies the property of additive $1 - \alpha$-cocycle which is $y_{t+s} = y_t + \alpha_t(y_s),\ s, t \in \mathbb{R}$. Let $H^* =$

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$\oplus_{i=1}^{+\infty}H^i$ be the group of all cohomologies for the group α with the coefficients in $\mathcal{M}$ associated with the bar resolvent. The group H^* is a ring with respect to the cohomological multiplication $\cup: H^i\times H^j\to H^{i+j}$ defined by the formula $(y\cup z)_{t_1,\dots,t_{i+j}}=y_{t_1,\dots,t_i}\alpha_{t_1+\dots+t_i}(z_{t_{i+1},\dots,t_{i+j}}),\ y\in H^i,\ z\in H^j$.

Proposition. *Given a Markovian multiplicative cocycle w, there exists a limit $\lim\limits_{t\to+\infty}w_{-t}(y)=w_{-\infty}(y)$ in the sense that $\eta(w_{-t}(y)-w_{-\infty}(y))\to 0,\ t\to+\infty$, for all $y\in\mathcal{M},\ \eta\in\mathcal{M}_*$. The map $w_{-\infty}$ is a *-endomorphism on $\mathcal{M}$. The image of $w_{-\infty}$ is certain von Neuman subalgebra $\tilde{\mathcal{M}}\subset\mathcal{M}$.*

Corrolary 1. *Let $\mathbf{h}$ be a subring of H^* generated by additive $1-\alpha$-cocycles $j_t(x),\ x\in\mathcal{A}$. Then the Markovian multiplicative cocycle w correctly determines a homomorphism of $\mathbf{h}$ to its image.*

Corrolary 2. *Given a Markovian cocycle w, the formula $\tilde{j}_t(x)=w_{-\infty}\circ j_t(x),\ x\in\mathcal{A},\ t\in\mathbb{R}$, defines a quantum Levy process $\tilde{j}$ which is isomorphic to j.*

The restriction $\beta_t=\alpha_{-t}|_{\mathcal{M}_{0]}},\ t\ge 0$, for the Kolmogorov flow α is a semiflow of Powers shifts. It means that every β_t is a shift in the sense of Powers (see [4]), i.e. $\cap_{n=1}^{+\infty}\beta_{tn}(\mathcal{M}_{0]})=\{\mathbb{R}\mathbf{1}\},\ t>0$. Put $\tilde{\alpha}_t=w_t\circ\alpha_t,\ t\in\mathbb{R}$. Then $\tilde{\alpha}=(\tilde{\alpha}_t)_{t\in\mathbb{R}}$ is a group of *-automorphisms on $\mathcal{M}$, which is a cocycle perturbation of α. The Markov property for w allows to define the restriction $\tilde{\beta}_t=\tilde{\alpha}_{-t}|_{\mathcal{M}_{0]}},\ t\ge 0$. The semigroup $\tilde{\beta}=(\tilde{\beta}_t)_{t\ge 0}$ is a E_0-semigroup in the sense of [4]. One can ask is it possible to find a restriction $\tilde{\beta}|_{\mathcal{N}}$ which is a semiflow of Powers shifts isomorphic to β. In this way we can extract a nondeterministic part of the quantum stochastic process and, therefore, define an analogue of the Wold decomposition for the quantum stochastic process which is obtained by a cocycle perturbation of the quantum Levy process.

Theorem. *The restriction $\tilde{\beta}|_{\mathcal{N}}$, where $\mathcal{N}=w_{-\infty}(\mathcal{M}_{0]})$, is isomorphic to the semiflow of Powers shifts β such that the endomorphism $w_{-\infty}$ defines a nondeterministic part of the perturabation of the quantum Levy process by the Markovian cocycle w.*

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Dynamic models of long-memory processes driven by Lévy noise*

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Abstract

A class of long-memory models with continuous time is developed for the purpose of modelling heavy-tailed data. These models are based on the Green function solutions of fractional differential equations driven by Lévy noise. Exact results on the second-order and higher-order characteristics of the equations are obtained. Some applications in finance and macroeconomics are discussed.

1 Introduction

It is well documented that many macroeconomic and financial time series such as real output growth, consumption prices, asset returns and interest rates may exhibit long-range dependence (LRD) Moreover, the distributions of these time series typically have heavier tails than the normal distribution. These distributions may be well fitted by hyperbolic distributions (see Barndorff-Nielsen (1998, 2001), Barndorff-Nielsen and Shephard (2001)). Alternatively, Heyde (1999) proposed to use a t-distribution with degree of freedom ν typically in the range 3-5. This, of course, implies an infinite k-th moment for $k \geq \nu$. Another issue in modelling economic and financial time series is that their sample autocorrelation functions (acf) may decay quickly, but their absolute increments or squares may have acfs with non-negligible values for large lags (see Heyde (1999), Barndorff-Nielsen (1998, 2001), and the references therein). These ubiquitous phenomena call for an effort to develop more reasonable models which can be integrated into the economic and financial theories.

An approach is to develop a theory of stochastic differential equations driven by fractional Brownian motion (FBM), which is a classical example of a non-stationary Gaussian process with LRD. In this approach, the effect of LRD can be obtained from the noise term. However, such models have inherent difficulties because FBM is not a semimartingale and the resulting Black-Scholes market contain arbitrage opportunities. A generalization of FBM is fractional Riesz-Bessel motion (FRBM)(see Anh, Leonenko and McVinish (2001)and the references therein).

Recently, Heyde (1999) proposed a risky asset model with LRD through fractal activity time. The idea is to replace Brownian time in geometrical Brownian motion by some process with stationary LRD increments and heavy tails.

Barndorff-Nielsen (2001) proposed to use discrete or continuous-type superposition of Ornstein-Uhlenbeck processes with Lévy motion input to obtain a class of random processes with LRD and infinitely divisible marginal distributions as their marginal law. These processes have been used to represent stochastic volatility in models of log prices (see Barndorff-Nielsen and Shephard (2001)).

In a continuous-parameter framework, it is known that LRD can be obtained by replacing ordinary derivative or ordinary differential operators by fractional derivative or fractional differential operators in differential or partial differential equations driven by white noise or via random initial conditions(see Woyczynski (1998), Leonenko and Woyczynski (1998), Leonenko (1999), Anh and Leonenko (1999) and their references). Following this approach, we introduce in this paper a class of fractional differential equations driven by Lévy noise, whose solutions are obtained as convolutions of the Green functions of the corresponding deterministic fractional differential equations with Lévy noise or stochastic path integrals with respect to Lévy processes. The main advantage of this approach is that LRD can be effected via the Green function of the fractional operator involved, hence freeing up the noise term to represent the effects of non-Gaussianity or multifractality. We will obtain exact results on the Green functions, correlation functions, spectra and

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higher-order spectra of particular forms of these fractional differential equations. These dynamic models for LRD processes with possible heavy-tail distributions provide useful tools for prediction and simulation purposes. These processes can be used to model the stochastic volatility of log price processes and macroeconomic processes with long memory (see Barndorff-Nielsen and Shephard (2001)).

2 Lévy processes

As standard notation, we will write $C\{\zeta; y\}$ for the cumulant function of a random vector $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$, i.e.,

$$C\{\zeta; y\} = \log E \exp\left\{ i \sum_{j=1}^{p} \zeta_j y_j \right\}, \ \zeta \in \mathbb{R}^p.$$

Let $L = \{L(t), t \geq 0\}$ be a Lévy process and the Lévy triple of $L(1)$ is (a, b, Q) where $a \in \mathbb{R}$, $b \geq 0$ and Q is the Lévy measure(see Barndorff-Nielsen (1998, 2001))

It is known that the law of L is infinitely divisible with

$$C\{\zeta; L(t)\} = tC\{\zeta; L(1)\}.$$

where

$$\Psi(\zeta) = C\{\zeta; L(1)\}$$

We will assume that

(A) The Lévy measure of $L(1)$ satisfies for some $\varepsilon > 0$ and $\lambda > 0$

$$\int_{(-\varepsilon,\varepsilon)^c} \exp\{\lambda |u|\} Q(du) < \infty.$$

This implies that

$$\int_{\mathbb{R}} |u|^k Q(du) < \infty, \qquad k \geq 2,$$

and that the characteristic function $E \exp\{i\zeta L(t)\}, \ \zeta \in \mathbb{R}$, is analytic in a neighborhood of 0. As a consequence, $L(t)$ has moments of all orders.

3 Finite-memory processes and long-memory processes

We consider the finite-memory process

$$X(t) = \int_0^t G(t-s)\, dL(s), \ EL^2(1) < \infty, \tag{3.1}$$

where L is a Lévy process and G is the memory function such that

$$\int_0^t G^2(s)\, ds < \infty. \tag{3.2}$$

Under the condition

$$\int_0^\infty G^2(s)\, ds < \infty, \tag{3.3}$$

the process (3.1) is asymptotically equivalent to the stationary process

$$\tilde{X}(t) = m + \int_{-\infty}^t G(t-s)\, dL(s) \tag{3.4}$$

in the sense that

$$\lim_{t\to\infty} E\left(X(t) - m - \tilde{X}(t) \right)^2 = 0,$$

where m is a constant.

The stochastic integral (3.1) or (3.4) can be interpreted in the $L_2(\Omega)$-sense if (3.2) or (3.3) holds. On the other hand the stochastic integral (3.1) exists *a.s.* as path-by-path integral on $[0, t]$ in the following cases (see Carmona et.al. (1998), Mikosch and Narvaiša (2000)):

a) the function G has bounded p-variation and L *a.s.* has paths of finite q-variation with $p^{-1}+q^{-1}>1$

b) the function G is Hölder continuous of index α and L *a.s.* has paths being Hölder continuous of index β with $\alpha+\beta>1$.

Moreover, in the case a), the integral (3.1) exists

1. in the Riemann-Stieltjes sense whenever G and paths of L have no discontinuities at the same points;
2. in the Moore-Pollard-Stieltjes sense whenever G and paths of L have no one-sided discontinuities at the same points;
3. always in the sense defined by Young.

Remark 3.1. *Suppose that there exists the Laplace transform of a memory function G in (3.1) or (3.4):*

$$g(p)=\int_0^\infty e^{-pt}G(t)\,dt.$$

Then the process (3.4) is stationary, at least in the second order, with spectral density

$$f_2(\omega)=\frac{1}{2\pi}\left|g(i\omega)\right|^2,\ \omega\in\mathbb{R} \tag{3.5}$$

if $f(\omega)\in L_1(\mathbb{R})$. Note that if

$$G_0(t)=e^{-\lambda t}\mathbf{1}_{(0,\infty)}(t),\ \lambda>0, \tag{3.6}$$

then the process (3.4) is called Ornstein-Uhlenbeck process (see Barndorff-Nielsen (1998, 2001)). Chambers (1996) investigated stationary processes with LRD of type (3.4) with

$$G_1(t)=\left[t^{\alpha-1}/\Gamma(\alpha)\right]\mathbf{1}_{(0,\infty)}(t),\ \alpha\in\left(\frac{1}{2},\frac{3}{2}\right) \tag{3.7}$$

in the case when $L(t)$ is the Wiener process $L(t)=W(t)$. In fact LRD occurs in the range $\alpha\in\left(1,\frac{3}{2}\right)$, while for $\alpha\in\left(\frac{1}{2},1\right)$ the spectral density of the increments has a zero in the spectrum.

Remark 3.2. *The process (3.1) is the finite-memory part of FBM if $L(t)$ is the Wiener process . Marinucci and Robinson (1999, 2001) investigated the process (3.1) in the case $L(t)=W(t)$. They called the process (3.1) "Type II fractional Brownian motion" in this special case. In particular, Marinucci and Robinson (1999, 2001) pointed out that the process (3.1) with memory function (3.7), $\alpha\in\left(0,\frac{1}{2}\right)$ and $L(t)=W(t)$ asymptotically behaves as FBM. That is why the finite-memory process (3.1) can be considered as a model of processes with LRD in an asymptotic sense. Alternatively we may define LRD of (3.1) via its asymptotic equivalence to the stationary process (3.4).*

The following key result presents the exact form for the finite-dimensional distributions of a process (3.1) which is defined constructively for a large class of functions G and Lévy processes L in terms of their characteristic functions.

Theorem 3.1. *Suppose that condition (A) holds and the process (3.1) is well-defined for a function*

$$G\in L_1([0,t])\cap L_2([0,t]). \tag{3.8}$$

Then for all $t_j\in[0,t]$, $j=1,\ldots,p$

$$C\big\{\zeta_1,\ldots,\zeta_p;X(t_1),\ldots,X(t_p)\big\}=\int_0^\infty\Psi\left(\sum_{j=1}^p\zeta_j\mathbf{1}_{[0,\infty)}(t_j-s)\,G(t_j-s)\right)ds, \tag{3.9}$$

where $\Psi(\zeta)=C\{\zeta;L(1)\}$ and

$$cum\left(X(t_1),\ldots,X(t_p)\right)=i^{-p}\Psi^{(p)}(0)\int_0^{\min(t_1,\ldots,t_p)}\left[\prod_{j=1}^pG(t_j-\tau)\right]d\tau \tag{3.10}$$

if the last integral is finite.

The proof of (3.9) follows that of Proposition 2.1 of Barndorff-Nielsen (2001) .

Remark 3.3. *Provided that corresponding integrals exist for a function G, we may obtain the formulae which are analogous to (3.9) and (3.10) for the stationary version (3.4) of a process (3.1), that is, for every $t_j \in [0,t]$, $j = 1,\dots,p$*

$$C\left\{\zeta_1,\dots,\zeta_p;\tilde{X}(t_1),\dots,\tilde{X}(t_p)\right\} = \int_{\mathbb{R}} \Psi\left(\sum_{j=1}^{p} \zeta_j \mathbf{1}_{[0,\infty)}(t_j - s)\, G(t_j - s)\right) ds \tag{3.11}$$

and

$$cum\left(\tilde{X}(t_1),\dots,\tilde{X}(t_p)\right) = i^{-p}\Psi^{(p)}(0) \int_0^{\min(t_1,\dots,t_p)} \left[\prod_{j=1}^{p} G(t_j - \tau)\right] d\tau. \tag{3.12}$$

The higher-order spectral densities of a stationary process (3.4) with $E\left|\tilde{X}(t)\right|^q < \infty$, $2 \le q \le p$ can be obtained as inverse Fourier transforms of (3.12) if they exist. In the stationary case the spectral density of q-th order $(2 \le q \le p)$ depends on $q-1$ variables and can be defined as

$$\begin{gathered} f_q(\omega_1,\dots,\omega_{q-1}) = (2\pi)^{-q+1} \frac{\Psi^{(q)}(0)}{i^q} g(i\omega_1)\dots g(i\omega_{q-1})\, g(-i(\omega_1+\dots+\omega_{q-1})), \\ \omega_j \in \mathbb{R},\ 1 \le j \le q-1, \end{gathered} \tag{3.13}$$

at least if this complex-valued function belongs to $L_1\left(\mathbb{R}^{q-1}\right)$, $q \ge 2$. For $q = 2$ (3.13) reduces to (3.6). In fact both formulae (3.6) and (3.13) can be used without integrability conditions if the processes are interpreted in the generalized sense. For example, the spectral density of "Type II fractional Brownian motion" is of the form

$$const\,|\omega|^{-2\alpha},\ \omega \in \mathbb{R},\ \alpha \in \left(\frac{1}{2},\frac{3}{2}\right),$$

which does not belong to $L_1(\mathbb{R})$, but its singular properties are clearly seen from the expression. Such a situation is typical for non-stationary processes.

4 Fractional differential equations with Lévy noise

We consider the processes of type (3.1) or (3.4) in which the memory functions G are the Green functions of some fractional differential equations. Let us recall some definitions of fractional derivatives and integrals (see Podlubny(1999), for example).

Assuming reasonable behavior for $f(t)$, the Riemann-Liouville fractional derivative is defined as

$$\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, \tag{4.1}$$

$\alpha \in [n-1,n)$, $n = 1,2,\dots,$ and the Riemann-Liouville fractional integral is defined as

$$\mathcal{J}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0 \tag{4.2}$$

We will widely use the notion of fractional Green function of a deterministic fractional differential equation

$$\mathcal{L}y(t) = f(t), \tag{4.3}$$

where the linear differential operator $\mathcal{L}$ with constant coefficients is given by

$$\mathcal{L}y(t) = A_n \mathcal{D}_t^{\beta_n} y(t) + \dots + A_1 \mathcal{D}_1^{\beta_1} y(t) + A_0 \mathcal{D}_t^{\beta_0} y(t) \tag{4.4}$$

and $\mathcal{D}_t^{\beta_j}$, $1 \le j \le n$, are defined in (4.1),

$$\beta_n > \beta_1 > \dots > \beta_1 > \beta_0,\ n \ge 1. \tag{4.5}$$

In this paper, we consider fractional differential equations of the following form:

$$\mathcal{L}X(t) = A_n \mathcal{D}_t^{\beta_n} X(t) + \cdots + A_1 \mathcal{D}_t^{\beta_1} X(t) + A_0 \mathcal{D}_t^{\beta_0} X(t) = \dot{L}(t), \tag{4.6}$$

where condition (4.5) is satisfied and $\dot{L}$ is Lévy noise. Note that Lévy noise $\dot{L}$ has the following properties: (1) it is infinitely divisible; (2) its probability distribution is translation invariant and (3) $\dot{L}(t)$ and $\dot{L}(s)$ are independent if $t \neq s$.

Suppose that a Green function G and its Laplace transform are known for the deterministic fractional differential equation (4.4) and the stochastic integral

$$X(t) = \int_0^t G(t-s)\, dL(s) \tag{4.7}$$

exists *a.s.* as path-by-path integral in $[0,t)$ or in $L_2(\Omega)$-sense (see Section 3 for details). Eq. (4.7) gives the Green function solution of the fractional differential equation (4.6) with Lévy noise $\dot{L}$, where formally $L(t) = \int_0^t \dot{L}(s)\, ds$. As an alternative to Lévy noise, we may consider a random measure with infinitely divisible distribution (Barndorff-Nielsen (2001)).

5 One-term equations

Consider now the following one-term fractional differential equation

$$A\mathcal{D}_t^{\alpha} X(t) = \dot{L}(t), \ \alpha > 0, \ A > 0, \tag{5.1}$$

where $\mathcal{D}_t^{\alpha}$ is defined in (4.1) and $\dot{L}$ is Lévy noise. Then the Green function solution of (5.1) is of the form (if exists)

$$X_1(t) = \int_0^t \frac{1}{A} \left[(t-s)^{\alpha-1} / \Gamma(\alpha) \right] dL(s), \tag{5.2}$$

where the Green function

$$G_1(t) = \frac{1}{A} \left[t^{\alpha-1} / \Gamma(\alpha) \right] \mathbf{1}_{(0,\infty)}(t) \tag{5.3}$$

of one-term equation (4.3) (with $A_n = A$, $\beta_n = \alpha$, $A_j = 0$, $0 \le j \le n-1$) has the Laplace transform

$$g_1(p) = \frac{1}{Ap^{\alpha}}, \ \mathrm{Re}(p) > 0.$$

If the Lévy process $L(t) = W(t)$, where $W(t)$, $t \geq 0$ is Brownian motion, the process (5.2) has been studied by Comte (1996) and Marinucci and Robinson (1999, 2000). In particular, Marinucci and Robinson (1999) called the process (5.2) with $L(t) = W(t)$ and $\alpha \in \left(1, \frac{3}{2}\right)$ "Type II fractional Brownian motion", because its asymptotic properties are similar to the (non-asymptotic) properties of FBM.

The process (5.2) exists in the $L_2(\Omega)$-sense if $\alpha > 1/2$ and $EL^2(1) < \infty$. It follows from Section 3 that (5.2) exists *a.s.* path by path if the condition (3.4) is satisfied. This condition holds for same hyperbolic Lévy processes if $\alpha > 1/2$ (see Barndorff-Nielsen and Shephard (2001)).

From (3.9), (3.10) and (5.2) we obtain

$$EX_1(t) = EL(1)\, t^{\alpha} / \left[\alpha \Gamma(\alpha) A \right], \ \alpha \in \left(\frac{1}{2}, \frac{3}{2} \right) \tag{5.4}$$

if $EL(1) < \infty$, and

$$cov(X_1(t), X_1(s)) = \frac{-\Psi''(0)}{A^2 \Gamma^2(\alpha)} \int_0^{\min(t,s)} (t-\tau)^{\alpha-1} (s-\tau)^{\alpha-1} d\tau, \ \alpha \in \left(\frac{1}{2}, \frac{3}{2} \right) \tag{5.5}$$

if $EL^2(1) < \infty$. In particular

$$varX_1(t) = t^{2\alpha-1} \left[-\Psi''(0) / \left(\Gamma^2(\alpha)(2\alpha-1) A^2 \right) \right] \tag{5.6}$$

and for $0 < t < s$

$$E\left(X_1(s) - X_1(t)\right)^2 \sim K_1 (s-t)^{2\alpha-1}, \quad \frac{t}{s-t} \to \infty, \tag{5.7}$$

$$E\left(X_1(s) - X_1(t)\right)^2 \sim K_2 (s-t)^{2\alpha-1}, \quad \frac{t}{s-t} \to 0, \tag{5.8}$$

where K_1 and K_2 are positive constants.

Moreover, if $EL^p(1) < \infty$ we obtain from (3.10) the following expression for the higher-order cumulant function:

$$cum\left(X(t_1), \ldots, X(t_k)\right) = i^{-k}\Psi^{(k)}(0)\, A^{-k}\Gamma^{-k}(\alpha)$$
$$\times \int_0^{\min(t_1,\ldots,t_k)} \left[\prod_{j=1}^{k} (t_j - \tau)^{\alpha-1}\right] d\tau, \; 2 \le k \le p, \; \alpha > 1 - \frac{1}{k}. \tag{5.9}$$

For $\alpha \in \left(1, \frac{3}{2}\right)$ the above results (5.5)-(5.9) indicate LRD in the solution process.

Alternatively we may obtain LRD in the solution by using the process (3.4) with memory function (5.4), which is asymptotically equivalent to the process (5.2) (see Section 3). The process

$$\tilde{X}_1(t) = \frac{1}{A\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1}\, dL(s) \tag{5.10}$$

can be interpreted also as Weyl's fractional integral of Lévy noise $\dot{L}$. In fact, (5.10) is the formal solution of the fractional differential equation

$$\mathcal{D}_W^{\alpha} \tilde{X}(t) = \dot{L}(t), \; \alpha > 0,$$

where

$$\mathcal{D}_W^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} (t-\tau)^{-\alpha} f(\tau)\, d\tau$$

is the Weyl fractional derivative of a function f. The process (5.10) is not stationary because the function (5.3) does not satisfy condition (3.3). The integral 5.10 cannot generally be defined. This is the reason why the fractional Brownin motion is generally defined by its increments. Formal calculations from (3.5), (3.13) and (5.3) give us the spectral density of the second order as

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi A^2} |\omega|^{-2\alpha}, \; \omega \in \mathbb{R} \tag{5.11}$$

and the spectral densities of higher-order as

$$f_p(\omega_1, \ldots, \omega_{p-1}) = \frac{\Psi^{(p)}(0)}{i^p (2\pi)^{p-1} A^p} (i\omega_1)^{-\alpha} (i\omega_2)^{-\alpha} \ldots (i\omega_{p-1})^{-\alpha} \left(-i\left(\omega_1 + \cdots + \omega_{p-1}\right)\right)^{-\alpha}, \tag{5.12}$$
$$\omega_j \in \mathbb{R}, \; 1 \le j \le p-1, \; p \ge 2,$$

which are typical spectral densities of nonstationary processes.

We will, however, be concerned with the stationary process obtained as the first integer differences of the process (5.10), namely,

$$X_s(t) = \tilde{X}_1(t) - \tilde{X}_1(t-1) = \frac{1}{A\Gamma(\alpha)} \left\{ \int_{-\infty}^{t} (t-s)^{\alpha-1}\, dL(s) - \int_{-\infty}^{t} (t-1-s)^{\alpha-1}\, dL(s) \right\}. \tag{5.13}$$

From (5.13) we obtain by direct calculations that

$$R_s(\tau) = cov\left(X_s(t), X_s(t-\tau)\right) = c_1(\alpha) \left[|\tau+1|^{2\alpha-1} - 2|\tau|^{2\alpha-1} + |\tau-1|^{2\alpha-1} \right], \; \tau \in \mathbb{R},$$

where

$$c_1(\alpha) = \left[\Psi''(0) / \left(2\Gamma(2\alpha)\cos(\alpha\pi)\right)\right] > 0$$

and the corresponding second-order spectral density is

$$f_2(\omega) = \left[-2\Psi''(0)/\pi\right](1-\cos\omega)|\omega|^{-2\alpha}, \ \omega \in \mathbb{R}. \tag{5.14}$$

From (5.14) we obtain that

$$\lim_{\tau\to\infty} \tau^{3-2\alpha} R_s(\tau)/R_s(0) = (\alpha-1)(2\alpha-1), \ \alpha \in \left(\frac{1}{2}, \frac{3}{2}\right), \ \tau > 0. \tag{5.15}$$

The property (5.15) shows second-order LRD for $\alpha \in \left(1, \frac{3}{2}\right)$, while for $\alpha \in \left(\frac{1}{2}, 1\right)$ the correlation function $R_s(\tau)$ is negative (or spectral density has zeroes).

We can now summarize our results in the following

Theorem 5.1. *Suppose that condition (A) is satisfied and the process (5.2) exists a.s. as path-by-path stochastic integral or in the* $L_2(\Omega)$*-sense. Then the process (5.2) has second-order properties (5.5)-(5.8) and higher-order property (5.9). The (generalized) process (5.13) has covariance function* $R_s(\tau)$ *and spectral density (5.14). This process displays LRD if* $\alpha \in \left(1, \frac{3}{2}\right)$.

6 Two-term equations

The two-term equations will take the form

$$A\mathcal{D}_t^{\alpha} X(t) + BX(t) = \dot{L}(t), \ \alpha > 0, \tag{6.1}$$

where $\mathcal{D}_t^{\alpha}$ is defined in (4.1) and $\dot{L}$ is the Lévy noise. For their analysis, we will widely use the two-parameter Mittag-Leffler function which can be defined by the series expansion

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ z \in \mathbb{C}, \ \alpha > 0, \ \beta > 0. \tag{6.2}$$

The Green function solution (see Section 4) of the fractional differential equation (6.1) is of the form

$$X_2(t) = \int_0^t G_2(t-s)\,dL(s) = \frac{1}{A}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{B}{A}(t-s)^{\alpha}\right) dL(s), \tag{6.3}$$
$$\alpha > 0, \ B/A \geq 0,$$

if the stochastic integral (6.3) exists *a.s.* as path-by-path integral on $[0,t]$ or in the $L_2(\Omega)$-sense (see Section 3), where $E_{\alpha,\beta}$ is Mittag-Leffler function (6.2) of negative real argument.

The Green function of the two-term equation (4.3) is of the form

$$G_2(t) = \frac{1}{A} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{B}{A}t^{\alpha}\right) \mathbf{1}_{(0,\infty)}(t), \ \alpha > 0 \tag{6.4}$$

and its Laplace transform has the form

$$g_2(p) = \frac{1}{Ap^{\alpha}+B}, \ \mathrm{Re}(p) > |A|^{1/\alpha}. \tag{6.5}$$

The process (6.3) exists in the $L_2(\Omega)$ sense if $\alpha > 1/2$ and $EL^2(1) < \infty$. It follows from the results of Section 3 that (6.3) exists *a.s.* path by path if condition (3.4) is satisfied. This condition holds for many Lévy processes given hyperbolic if $\alpha > 1/2$.

We obtain from (6.3) the following expression

$$EX_2(t) = \frac{EL(1)}{A} t^{\alpha} E_{\alpha,\alpha+1}\left(-\frac{B}{A}t^{\alpha}\right), \ A > 0$$

if $EL(1) < \infty$.

Note that

$$EX_2(t) \sim \frac{EL(1)}{A\Gamma(\alpha+1)} t^{\alpha} \left(1 - \frac{B}{A\Gamma(2\alpha+1)} t^{\alpha}\right) \tag{6.6}$$

as $t \to 0$, and we obtain

$$\lim_{t\to\infty} EX_2(t) = EL(1)/B. \tag{6.7}$$

The formulae (6.6) and (6.7) are in contrast to (5.4).

If $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $EL^2(1) < \infty$ we obtain from (3.10) and (6.4) the following expression for the covariance function:

$$\begin{aligned} cov(X_2(t), X_2(s)) &= \frac{-\Psi''(0)}{A^2} \int_0^{\min(t,s)} (t-\tau)^{\alpha-1} (s-\tau)^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}\left(-\frac{B}{A}(t-\tau)^{\alpha}\right) E_{\alpha,\alpha}\left(-\frac{B}{A}(s-\tau)^{\alpha}\right) d\tau. \end{aligned} \tag{6.8}$$

In particular,

$$varX_2(t) = \frac{-\Psi''(0)}{A^2} \int_0^t (t-\tau)^{2\alpha-2} E^2_{\alpha,\alpha}\left(-\frac{B}{A}(t-s)^{\alpha}\right) d\tau. \tag{6.9}$$

From (6.9) we obtain

$$varX_2(t) \sim \frac{-\Psi''(0)}{A^2\Gamma^2(\alpha)(2\alpha-1)} t^{2\alpha-1}$$

as $t \to 0$, which is asymptotically similar to (5.6).

Consider now a stationary version of the process (6.3), that is, a process of the form (3.4) with memory function (6.4):

$$\tilde{X}_2(t) = \frac{1}{A} \int_{-\infty}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{B}{A}(t-s)^{\alpha}\right) dL(s), \ \alpha > 1/2. \tag{6.10}$$

In contrast to (5.10) the process (6.10) is second-order stationary if $\alpha > 1/2$ and $EL^2(1) < \infty$.

From (3.5) and (6.5) we obtain the second-order spectral density of the stationary process (6.10), which is given by

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi} \frac{1}{|A(i\omega)^{\alpha} + B|^2}, \ \omega \in \mathbb{R}, \tag{6.11}$$

and from (3.13) and (6.5) we have the spectral density of k-th order (if exists) as

$$\begin{aligned} &f_k(\omega_1, \ldots, \omega_{k-1}) = \\ &\frac{\Psi^{(k)}(0)}{(2\pi)^{k-1} i^k} \frac{1}{((i\omega_1)^{\alpha} A + B) \ldots ((i\omega_{k-1})^{a} A + B)((-i(\omega_1 + \cdots + \omega_{k-1}))^{\alpha} A + B)}, \\ &\qquad \omega_j \in \mathbb{R}, \ 1 \le j \le k. \end{aligned} \tag{6.12}$$

The spectral density (6.12) exists if $EL^k(1) < \infty$ and $\alpha > 1/2$. Note that

$$A(i\omega)^{\alpha} + B = A|\omega|^{\alpha} e^{\frac{i\pi\alpha}{2}} + B \tag{6.13}$$

and by direct calculation we obtain from (6.11) and (6.13) that

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi} \frac{1}{\left(B^2 + 2AB|\omega|^{\alpha}\cos\frac{\alpha\pi}{2} + A^2|\omega|^{2\alpha}\right)}, \ \omega \in \mathbb{R}. \tag{6.14}$$

If $\alpha > 1/2$ and $\omega \to \infty$

$$f_2(\omega) = O\left(\frac{1}{|\omega|^{2\alpha}}\right). \tag{6.15}$$

We can conclude that for $\alpha > 1/2$ the spectral density (6.14) belongs to $L_1(\mathbb{R})$ and has no singularity at zero, but this spectral density displays intermittency (6.15) as $\omega \to \infty$.

We therefore arrive at

Theorem 6.1. *Suppose that condition (A) is satisfied and the process (6.3) has second-order characteristics (6.8)-(6.9), while the stationary process (6.10) with $\alpha > 1/2$ has second-order spectral density (6.14) and higher-order spectral densities (6.12). The second-order spectral density (6.14) does not display LRD if $B \neq 0$, but indicates second-order intermittency (6.15).*

From (6.12) we can, in principle, obtain higher-order intermittency by direct calculations with the help of (6.13).

7 Three-term equations

The three-term equations take the form

$$A\mathcal{D}_t^\beta X(t) + B\mathcal{D}_t^\alpha X(t) + CX(t) = \dot{L}(t), \ \beta > \alpha > 0, \ A, C > 0, \ B \geq 0, \tag{7.1}$$

where $\mathcal{D}_t^\alpha$ is as defined in (4.1) and $\dot{L}$ is Lévy noise. We will need the k-th derivative of Mittag-Leffler function (6.2) of the form

$$E_{\alpha,\beta}^{(k)}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(j+k)! z^j}{j! \Gamma(\alpha j + \alpha k + \beta)}. \tag{7.2}$$

Suppose now that the following stochastic integral

$$X_3(t) = \int_0^t G_3(t-s) \, dL(s) \tag{7.3}$$

exists *a.s.* as path-by-path integral on $[0,t]$ or in the $L_2(\Omega)$-sense, where the Green function of the three-term equation is

$$G_3(t) = \frac{1}{A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{C}{A}\right)^k t^{\beta(k+1)-1} E_{\beta-\alpha,\beta+\alpha k}^{(k)} \left(-\frac{B}{A} t^{\beta-\alpha}\right) \mathbf{1}_{(0,\infty)}(t). \tag{7.4}$$

Its Laplace transform is given by

$$g_3(p) = \frac{1}{Ap^\beta + Bp^a + C}, \ \beta > \alpha > 0. \tag{7.5}$$

The formula (7.5) follows from (7.2) and (7.4).

In our definition (see Section 4), the process (7.3) represents the Green function solution of the three-term equation (7.1) with Lévy noise.

Note that formula (7.4) is rather elegant. In fact, if $C = 0$ we obtain

$$G_3(t) = \frac{1}{A} t^{\beta-1} E_{\beta-\alpha,\beta} \left(-\frac{B}{A} t^{\beta-\alpha}\right) \mathbf{1}_{(0,\infty)}(t), \ \beta > \alpha > 0. \tag{7.6}$$

The last formula reduces to (6.4) if $\alpha = 0$ (formally, the role of α in (6.4) is played by β in (7.6)). Moreover, if $B = 0$ and $\beta = 1$ the Green function (7.4) reduces to

$$\begin{aligned} G_3(t) &= \frac{1}{A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{C}{A}\right)^k t^k \mathbf{1}_{(0,\infty)}(t) \\ &= \frac{1}{A} e^{-\frac{C}{A}t} \mathbf{1}_{(0,\infty)}(t), \ \frac{C}{A} > 0, \end{aligned} \tag{7.7}$$

which is, of course, the Green function (3.6) (up to constants) which corresponds to Ornstein-Uhlenbeck- type processes. It means that the stationary process (3.4) with memory function (7.7) is an Ornstein-Uhlenbeck- type process driven by Lévy process (see Barndorff-Nielsen (2001). In this case, we can interpret the Langevin type equation (7.1) with $B = 0$, $\beta = 1$ in the Itô sense and the Itô solution coincides with the Green function solution.

Let us now study in frequency domain the stationary process

$$\tilde{X}_3(t) = \int_{-\infty}^{t} G_3(t-s)\, dL(s), \tag{7.8}$$

where G_3 is as defined in (7.4) or (7.6).

From (3.5) and (7.5) we obtain the second-order spectral density of the process (7.8) (if exists) as

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi} \frac{1}{\left|A(i\omega)^{\beta} + B(i\omega)^{\alpha} + C\right|^2},\ \omega \in \mathbb{R}, \tag{7.9}$$

which can be written with the help of (6.13) in the following form:

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi} \frac{1}{P(\omega)},\ \omega \in \mathbb{R}, \tag{7.10}$$

where

$$\begin{aligned} P(\omega) \quad = \quad & C^2 + B^2 |\omega|^{2\alpha} + A^2 |\omega|^{2\beta} + 2AB |\omega|^{\alpha+\beta} \cos\frac{\beta-\alpha}{2}\pi \\ & +2AC |\omega|^{\beta} \cos\frac{\beta\pi}{2} + 2BC |\omega|^{\alpha} \cos\frac{\alpha\pi}{2}. \end{aligned} \tag{7.11}$$

It follows from (7.10) and (7.11) that the spectral density (7.10) belongs to $L_1(\mathbb{R})$ if $\beta > \alpha > 0$, and $\beta > 1/2$ or $\alpha + \beta > 1$. Thus, under these conditions, the process (7.8) is second-order stationary if $EL^2(1) < \infty$.

Another interesting observation is that the spectral density (7.10) with $C = 0$ reduces to

$$f_2(\omega) = \frac{-\Psi''(0)}{2\pi} \frac{1}{|\omega|^{2\alpha}\left(B^2 + A^2|\omega|^{2(\beta-\alpha)} + 2AB|\omega|^{\beta-\alpha}\cos\frac{\beta-\alpha}{2}\pi\right)},\ \omega \in \mathbb{R}. \tag{7.12}$$

If $EL^2(1) < \infty$, $\beta > \alpha > 0$ and $\beta > 1/2$ or $\alpha + \beta > 1$, the spectral density (7.12) represents a stationary process (7.8) in which the memory function G_3 is as defined in (7.6). The spectral density (7.12) displays LRD with fractional parameter $\alpha \in \left(0, \frac{1}{2}\right)$, because it behaves as $O\left(|\omega|^{-2\alpha}\right)$ as $\omega \to 0$. Moreover, the spectral density (7.12) displays second-order intermittency, because it behaves as $O\left(|\omega|^{-2\beta}\right)$ as $\omega \to \infty$. Thus the second fractional parameter $\beta > 1/2$ indicates intermittency. Such effects can also be expressed in terms of higher-order spectral densities which can be obtained from (3.13) with the function g_3 given by (7.5). In particular LRD effects can be described by the singularities of higher-order spectral densities at zero and also on the diagonal. We can now summarize the results in

Theorem 7.1. *Suppose that condition (A) is satisfied and the process (7.3) exists as path-by-path integral or in the $L_2(\Omega)$-sense. If $\beta > \alpha > 0$ and $\beta > 1/2$ or $\alpha + \beta > 1$, the process (7.8) is second-order stationary with spectral density (7.10) which displays intermittency of the form $O\left(|\omega|^{-2\beta}\right)$ as $\omega \to \infty$. Moreover, if $C = 0$, the spectral density (7.12) displays intermittency and LRD simultaneously and the LRD effect can be described by the behavior $O\left(|\omega|^{-2\alpha}\right)$ as $\omega \to 0$ of the spectral density (7.12) with $\alpha \in \left(0, \frac{1}{2}\right)$.*

In the same spirit we consider the four-term and n-term equations. Exact results on the second-order and higher-order characteristics of the equations are obtained. Some applications in finance and macroeconomics are discussed.

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Lévy Processes in the Heisenberg Group - the Schrödinger Representation and Dirichlet Forms

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Abstract

We want to find out what we can learn about stochastic processes in groups through studying their representations. In the case of Lévy processes in the Heisenberg group, the Schrödinger representation of the generator contains interesting probabilistic information. Under certain conditions, we are able to construct a Dirichlet form and so associate a Hunt process in Euclidean space to the group-valued process,

1 Motivation

Let $\rho = (\rho(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a group G. Let π be a unitary representation of G in a complex Hilbert space $\mathcal{H}$. Form the *unitary operator-valued process* $U = (U(t), t \geq 0)$ where, for each $\omega \in \Omega$,

$$U(t)(\omega) = \pi(\rho(t)(\omega)).$$

Question 1: What does U tell us about ρ ?

Now suppose $\mathcal{H} = L^2(S, m)$ for some nice space S equipped with a σ-finite measure m.

Question 2: When do we have

$$U(t)f = f \circ \phi(t),$$

for some random mappings $\phi(t) : S \to S$? What does $\phi = (\phi(t), t \geq 0)$ tell us about ρ ?

We'll study the case where ρ is a *Lévy process* in G.

2 Lévy Processes in Lie Groups

Let G be a Lie group with Lie algebra $\mathbf{g}$. A *Lévy process* in G is a G-valued stochastic process $\rho = (\rho(t), t \geq 0)$ which satisfies the following,

1. ρ has stationary and independent left increments, where the increment between s and t with $s \leq t$ is $\rho(s)^{-1}\rho(t)$.

2. $\rho(0) = e$ (a.s.).

3. ρ is stochastically continuous i.e.

$$\lim_{s \to t} P(\rho(s)^{-1}\rho(t) \in A) = 0,$$

for all $A \in \mathcal{B}(G)$ with $e \notin \bar{A}$.

Let p_t denote the law of $\rho(t)$, so $p_t(A) = P(X(t) \in A)$ for each $t \geq 0, A \in \mathcal{B}(G)$, then $(p_t, t \geq 0)$ is a weakly continuous, convolution semigroup of probability measures, where the convolution is defined by

$$(p_s * p_t)(A) = \int_G p_s(\tau^{-1}A)p_t(d\tau),$$

for each $s, t \geq 0$.

Let $C_0(G)$ be the Banach space (with respect to the supremum norm) of functions on G which vanish at infinity. We obtain a Feller semigroup $(T(t), t \geq 0)$ on $C_0(G)$ by the prescription,

$$\begin{aligned} T(t)f(\tau) &= \mathbf{E}(f(\tau\rho(t))) \\ &= \int_G f(\tau\sigma)p_t(d\sigma), \end{aligned}$$

for each $t \geq 0, \tau \in G, f \in C_0(G)$. The infinitesimal generator will be denoted as $\mathcal{L}$. The starting point of probabilistic investigations of Lévy processes in Lie groups is the "Lévy-Khinchine"-type structure of $\mathcal{L}$, which we now describe.

We fix a basis $\{Z_1, \ldots, Z_n\}$ for $\mathbf{g}$ and define a dense subspace $C_2(G)$ of $C_0(G)$ as follows:-

$$C_2(G) = \{f \in C_0(G); Z_i^L(f) \in C_0(G) \text{ and } Z_i^L Z_j^L(f) \in C_0(G) \text{ for all } 1 \leq i, j \leq n\}$$

where Z^L denotes the left invariant vector field associated to $Z \in \mathbf{g}$ by differential left translation.

In [9], Hunt proved that there exist functions $y_i \in C_2(G), 1 \leq i \leq n$ so that each

$$y_i(e) = 0 \text{ and } Z_i^L y_j(e) = \delta_{ij}$$

and a map $h \in \text{Dom}(\mathcal{L})$ which is such that

1. $h > 0$ on $G - \{e\}$,
2. There exists a compact neighborhood of the identity V such that for all $\tau \in V$,

$$h(\tau) = \sum_{i=1}^{n} y_i(\tau)^2.$$

Any such function is called a *Hunt function* in G.

A positive measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever

$$\int_{G-\{e\}} h(\sigma)\nu(d\sigma) < \infty$$

for some Hunt function h.

We are now ready to state the main result of [9].

Theorem 1 (Hunt's theorem). *Let ρ be a Lévy process in G with infinitesimal generator $\mathcal{L}$ then*

1. *$C_2(G) \subseteq Dom(\mathcal{L})$*
2. *For each $\tau \in G, f \in C_2(G)$*

$$(\mathcal{L}f)(\tau) = b^i Z_i^L f(\tau) + c^{ij} Z_i^L Z_j^L f(\tau) + \int_{G-\{e\}} (f(\tau\sigma) - f(\tau) - y^i(\sigma)Z_i^L f(\tau))\nu(d\sigma) \quad (2.1)$$

where $b = (b^1, \ldots b^n) \in \mathbf{R}^n, c = (c^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Furthermore, any linear operator with a representation as in (2.1) is the restriction to $C_2(G)$ of the infinitesimal generator of a unique (up to modification) Lévy process.

Several obscure features of Hunt's paper were later clarified by Ramaswami in [11] and then incorporated into Heyer's seminal treatise [8]. For a survey of these and related ideas see [2].

3 Group Representations

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{U}(\mathcal{H})$ be the group of all unitary operators in $\mathcal{H}$. A *representation* of G in $\mathcal{H}$ is a strongly continuous homomorphism π from G into $\mathcal{U}(\mathcal{H})$, so that

- For each $g \in G, \pi(g)$ is a unitary operator in $\mathcal{H}$.
- For each $g, h \in \mathcal{H}$,
$$\pi(gh) = \pi(g)\pi(h).$$
- For each $g \in G$,
$$\pi(e) = I, \quad \pi(g^{-1}) = \pi(g)^*.$$
- For each $\psi \in \mathcal{H}$, the mapping from G to $\mathcal{H}$ given by $g \to \pi(g)\psi$ is continuous.

A closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ is *invariant* for π if $\pi(\mathcal{H}_1) \subseteq \mathcal{H}_1$. A representation is *irreducible* if the only invariant subspaces are $\{0\}$ and $\mathcal{H}$. We denote as $\mathrm{Irr}(G)$ the set of all irreducible representations of G.

Now let π be an arbitrary representation of G in some $\mathcal{H}$.

We will have need of $C^\infty(\pi) = \{\psi \in h; g \to \pi(g)\psi \text{ is } C^\infty\}$ which is the dense linear space of *smooth vectors* for π in $\mathcal{H}$.

Let $X \in \mathbf{g}$, then $d\pi(X)$ is an essentially skew-adjoint operator in $\mathcal{H}$ where

$$d\pi(X)\psi = \frac{d}{da}\pi(\exp(aX))\psi\bigg|_{a=0},$$

for all $\psi \in C^\infty(\pi)$.

Now define a *unitary operator valued process* $U = (U(t), t \geq 0)$ in $\mathcal{H}$ by

$$U(t) = \pi(\rho(t)),$$

for each $t \geq 0$.

The main object of our investigations are the linear operators $(\mathcal{T}_t^\pi, t \geq 0)$ in $\mathcal{H}$ defined by

$$\mathcal{T}_t^\pi = \mathbf{E}(\pi(\rho(t))),$$

for each $t \geq 0$, so that for each $\psi \in \mathcal{H}$,

$$\mathcal{T}_t^\pi \psi = \int_G (\pi(\sigma)\psi) p_t(d\sigma).$$

We have the following important result.

Theorem 2. *$(\mathcal{T}_t^\pi, t \geq 0)$ is a strongly continuous, contraction semigroup on $\mathcal{H}$.*

We denote the infinitesimal generator of $(\mathcal{T}_t^\pi, t \geq 0)$ as $\mathcal{L}^\pi$.

It follows from the arguments of [3], that $C^\infty(\pi) \subseteq \mathrm{Dom}(\mathcal{L}^\pi)$ and for all $\psi \in C^\infty(\pi)$ we have

$$\begin{aligned}\mathcal{L}^\pi \psi = {} & b^i d\pi(Z_i)\psi + c^{ij} d\pi(Z_i) d\pi(Z_j)\psi + \\ & + \int_{G-\{e\}} (\pi(\sigma) - I - y^i(\sigma) d\pi(Z_i))\psi \nu(d\sigma). \end{aligned} \tag{3.1}$$

The following is a useful tool for computations as it gives a relation with the Markov semigroup induced by the process. Fix $\psi_1, \psi_2 \in \mathcal{H}$ and define $f \in C_b(G)$ by $f(\sigma) = <\psi_1, \pi(\sigma)\psi_2>$ where $\sigma \in G$, then we have

$$(T(t)f)(e) = <\psi_1, \mathcal{T}_t^\pi \psi_2>.$$

If $\psi_2 \in C^\infty(\pi)$, we have $f \in \mathrm{Dom}(\mathcal{L})$ and

$$(\mathcal{L}f)(e) = <\psi_1, \mathcal{L}^\pi \psi_2>.$$

In [3], it is shown that U satisfies an operator-valued stochastic differential equation.

$$U(t) = I + \int_0^t U(s-)d\pi(X_i)dB^i(s) + \int_0^t U(s-)\mathcal{L}^\pi ds$$
$$\int_0^{t+} \int_{G-\{e\}} U(s-)(\pi(\sigma) - I)\hat{N}(ds, d\sigma),$$

where $B = (B(t), t \geq 0)$ is a Brownian motion on $\mathbf{R}^n$ with $\mathrm{Cov}(B^i(t)B^j(t)) = 2c^{ij}t$ for $1 \leq i, j \leq n, t \geq 0$ and $\hat{N}$ is a (compensated) Poisson random measure on $\mathbf{R}^+ \times (G - \{e\})$, which is independent of B, and whose intensity measure is the Lévy measure ν.

A key result which to some extent answers Question 1 is due to E.Siebert, [12].

Theorem 3 (Siebert). *The collection of linear operators* $\{\mathcal{L}^\pi, \pi \in Irr(G)\}$ *determines* $(p_t, t \geq 0)$.

Note. In general, if μ is a bounded measure on G and $\pi : G \to \mathcal{H}$ is a representation, we can define the *generalised Fourier transform* to be the bounded linear operator $\hat{\mu}(\pi)$ on $\mathcal{H}$, defined by

$$< \psi_1, \hat{\mu}(\pi)\psi_2 >= \int_G < \psi_1, \pi(\sigma)\psi_2 > \mu(d\sigma),$$

for each $\psi_1, \psi_2 \in \mathcal{H}$ (see [7]). From this point of view we have

$$\mathcal{T}_t^\pi = \widehat{p_t}(\pi),$$

for each $t \geq 0$.

4 The Heisenberg Group

This section is based on Chapter 2 of [13] (see also the monograph [5]). The *Heisenberg group* $\mathbf{H}^n$ is a Lie group with underlying manifold $\mathbf{R}^{2n+1}$ equipped with the composition law

$$(a_1, q_1, p_1)(a_2, q_2, p_2) = (a_1 + a_2 + \frac{1}{2}(p_1 \cdot q_2 - q_1 \cdot p_2), q_1 + q_2, p_1 + p_2)$$

where each $a_i \in \mathbf{R}, q_i, p_i \in \mathbf{R}^n (i = 1, 2)$ and $\cdot$ is the usual scalar product in $\mathbf{R}^n$.

A basis for the Lie algebra of left-invariant vector fields is $\{T, L_1, \ldots, L_n, M_1, \ldots, M_n\}$ where for $1 \leq j \leq n$,

$$T = \frac{\partial}{\partial t}, L_j = \frac{\partial}{\partial q_j} + \frac{1}{2}p_j \frac{\partial}{\partial t}, M_j = \frac{\partial}{\partial p_j} - \frac{1}{2}q_j \frac{\partial}{\partial t}$$

and we have the commutation relations

$$[L_j, L_k] = [M_j, M_k] = [M_j, T] = [L_j, T] = 0, [M_j, L_k] = \delta_{jk}T$$

for $1 \leq j, k \leq n$ so that $\mathbf{H}^n$ is step-2 nilpotent.

By the Stone-von Neumann uniqueness theorem, $\mathrm{Irr}(\mathbf{H}^n) = (\mathbf{R} - \{0\}) \cup \mathbf{R}^{2n}$, where

- $(x, y) \in \mathbf{R}^{2n}, \mathcal{H} = \mathbf{C}$,
$$\pi_{x,y}(a, q, p) = e^{i(x.q+y.p)},$$
for each $(a, q, p) \in \mathbf{H}^n$.

- (*The Schrödinger Representation*) $\lambda \in \mathbf{R} - \{0\}, \mathcal{H} = L^2(\mathbf{R}^n)$.
We have $C^\infty(\pi_\lambda) = S(\mathbf{R}^n)$ where $S(\mathbf{R}^n)$ is the Schwartz space of rapidly decreasing functions.
For $\lambda > 0$,
$$\pi_{\pm\lambda}(a, q, p) = e^{i(\pm\lambda aI \pm \lambda^{\frac{1}{2}}q.X + \lambda^{\frac{1}{2}}p.D)}$$
where $X = (X_1, \ldots, X_n)$ and each $X_i u(x) = x_i u(x)$ for $u \in S(\mathbf{R}^n)$ and $D = (D_1, \ldots, D_n)$ where each $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$. Hence for all $f \in S(\mathbf{R}^n), x \in \mathbf{R}^n$,
$$(\pi_{\pm\lambda}(a, q, p)f)(x) = e^{\pm i(\lambda a + \lambda^{\frac{1}{2}}q.x + \frac{\lambda}{2}q.p)} f(x + \lambda^{\frac{1}{2}}p).$$

In fact the linear operator $p.D + q.X$ is essentially self-adjoint on $S(\mathbf{R}^n)$. A basis for the representation of the Lie algebra is

$$d\pi_{\pm\lambda}(T) = \pm i\lambda I, d\pi_{\pm\lambda}(L_j) = \pm i\lambda^{\frac{1}{2}} X_j, d\pi_{\pm\lambda}(M_j) = i\lambda^{\frac{1}{2}} D_j$$

for $1 \leq j \leq n$, where I is the identity operator.

The remainder of this article is based on joint work with Serge Cohen (Toulouse University) [1].

5 Lévy Processes in the Heisenberg Group

Let $\rho = (\rho_1, \rho_2, \rho_3)$ be a Lévy process in $\mathbf{H}^n$, where ρ_1 is a real-valued process and ρ_2 and ρ_3 are both $\mathbf{R}^n$-valued. In the light of Siebert's theorem, we should examine the unitary operator valued processes $\pi(\rho)$ where π is an irreducible representation of $\mathbf{H}^n$. Clearly, for each $x, y \in \mathbf{R}^n, t \geq 0$,

$$\pi_{x,y}(\rho(t)) = e^{i(x.\rho_2(t) + y.\rho_3(t))}.$$

It then follows from the structure of $\mathcal{L}^{\pi_{x,y}}$ that (ρ_2, ρ_3) is a Lévy process in $\mathbf{R}^{2n}$.

It is more interesting to examine the Schrödinger representation, and from now on we write $\pi = \pi_1$, for convenience.

We compute the form of the generator (3.1) on the domain $C^\infty(\pi) = S(\mathbf{R}^n)$. We will find it simplifies matters if we write the vector $b = (b_0, b^1, b^2)$, where $b_0 \in \mathbf{R}$ and $b^i \in \mathbf{R}^n, i = 1, 2$. We also write the non-negative definite matrix $c = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{01} & C_1 & E \\ c_{02} & E^T & C_2 \end{pmatrix}$ where for $i = 1, 2, c_{00} \geq 0, c_{0i} \in \mathbf{R}^n$ and C_i, E are $n \times n$ matrices with each C_i symmetric ($(\cdot)^T$ denotes the transpose matrix).

Proposition 1.

$$\begin{aligned} \mathcal{L}^\pi = {} & i\left(b_0 + \sum_{j=1}^n E_{jj}\right) I + (ib_j^1 - 2c_j^{01})X^j + (ib_j^2 - 2c_j^{02})D^j \qquad (5.1) \\ & - c_{00}I - c_{jk}^1 X^j X^k - c_{jk}^2 D^j D^k - 2E_{jk}X^j D^k \\ & + \int_{\mathbf{R}^{2n+1}-\{0\}} \left(e^{i(aI+q.X+p.D)} - I - \frac{i(aI + q.X + p.D)}{1 + |a|^2 + |q|^2 + |p|^2}\right) \nu(da, dq, dp) \end{aligned}$$

It is interesting that $\mathcal{L}^\pi$ can be exhibited as a pseudo-differential operator using the *Weyl calculus*. Details of this, which lead to a probabilistic derivation of *Mehler's formula* for the symbol of each $\mathcal{T}_t^\pi$, can be found in [1].

$C_c^\infty(\mathbf{R}^n)$ is a core for $\mathcal{L}^\pi$, and we will use $\mathcal{L}_0^\pi$ to denote the restriction of $\mathcal{L}^\pi$ to $C_c^\infty(\mathbf{R}^n)$.

Our aim here is to try to answer Question 2, by constructing Dirichlet forms. To do this, we need to place some constraints on $\mathcal{L}^\pi$.

- $c_{0i} = b_0 = b_j^i = 0 (i = 1, 2, j = 1, \ldots n), E = 0$
- ν is a symmetric measure i.e. $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(\mathbf{R}^{2n+1})$.

We then have that $-\mathcal{L}_0^\pi$ is a positive symmetric operator and

$$\begin{aligned} -\mathcal{L}_0^\pi = {} & c_{00}I + c_{jk}^1 X^j X^k + c_{jk}^2 D^j D^k \qquad (5.2) \\ & + \int_{\mathbf{R}^{2n+1}-\{0\}} (I - \cos(aI + q.X + p.D))\nu(da, dq, dp). \end{aligned}$$

Now since $-\mathcal{L}_0^\pi$ is positive symmetric, we can define a positive quadratic form $\mathcal{E}_\pi$ with domain $C_c^\infty(\mathbf{R}^n)$ by the prescription

$$\mathcal{E}_\pi(f) = - < f, \mathcal{L}_0^\pi f >$$

for each $f \in C_c^\infty(\mathbf{R}^n)$, then $\mathcal{E}_\pi$ is closable with closure $\overline{\mathcal{E}_\pi}$. Moreover $-\mathcal{L}^\pi$ is the Fredholm extension of $-\mathcal{L}_0^\pi$, so that $(\mathcal{T}_t^\pi, t \geq 0)$ is a self-adjoint semigroup in $L^2(\mathbf{R}^n)$ and $\overline{\mathcal{E}_\pi}(f) = - < f, \mathcal{L}^\pi f >$ for all $f \in \mathrm{Dom}(\mathcal{L}^\pi)$.

In order to construct Dirichlet forms, we need to know when the operator $\mathcal{L}^\pi$ preserves real-valued functions.

Proposition 2. $-\mathcal{L}_0^\pi$ *maps real-valued functions to real-valued functions if and only if*

$$\int_{\mathbf{R}^{2n+1}-\{0\}} \sin(a + q.x + \frac{1}{2}p.q) f(x+p)\nu(da, dq, dp) = 0. \tag{5.3}$$

for all $f \in C_c^\infty(\mathbf{R}^n), x \in \mathbf{R}^n$.

Under these conditions, we then find that for all $f \in C_c^\infty(\mathbf{R}^n)$,

$$\begin{aligned}\mathcal{E}_\pi(f) = & \sum_{j,k=1}^n c_{jk}^2 \int_{\mathbf{R}^n} \partial_j f(x)\partial_k f(x)dx + \Bigg(\int_{\mathbf{R}^n}\Big(c_{00} + c_{jk}^1 x^j x^k \\ & + \int_{\mathbf{R}^{2n+1}-\{0\}} \sin^2(\frac{1}{2}(a+q.x+\frac{1}{2}q.p)) + \sin^2(\frac{1}{2}(a+q.x-\frac{1}{2}q.p))\nu(da,dq,dp)\Big) f(x)^2 dx \\ & + \frac{1}{2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^{2n+1}-\{0\}} \cos(a+q.x+\frac{1}{2}p.q)(f(x)-f(x+p))^2\nu(da,dq,dp)dx\Bigg).\end{aligned}$$

This formula is very closely related to the famous Beuring-Deny formula for regular symmetric Dirichlet forms (see e.g. [6], p.108-11), and we would like to make the following interpretation:-

$$\begin{aligned}\mathcal{E}(f) = & \mathcal{E}_c(f) + \int_{\mathbf{R}^n} f(x)^2 k(dx) \\ & + \int_{(\mathbf{R}^{2n+1}-\{0\})\times\mathbf{R}^n} (f(x) - f(x+p))^2 J(da, dq, dp, dx)\end{aligned} \tag{5.4}$$

where the local part of the form is given by $\mathcal{E}_c(f) = \sum_{j,k=1}^n c_{jk}^2 \int_{\mathbf{R}^n} \partial_j f(x)\partial_k f(x)dx$, the killing measure is $k(dx) = (c_{00}+c_{jk}^1 x^j x^k + \int_{\mathbf{R}^{2n+1}-\{0\}} \sin^2(\frac{1}{2}(a+q.x+\frac{1}{2}q.p)) + \sin^2(\frac{1}{2}(a+q.x-\frac{1}{2}q.p))\nu(da,dq,dp))dx$ and the jump measure is $J(da,dq,dp,dx) = \frac{1}{2}\cos(a+q.x+\frac{1}{2}p.q)\nu(da,dq,dp)dx$.

In general, however J will not be a positive measure. It will be in some cases though, e.g. when $\text{supp}(\nu) = \{(0,0,p), p \in \mathbf{R}^n\}$.

When $\mathcal{E}$ is a bona fide Dirichlet form, we can assert the existence of a Hunt process $(Y(t), t \geq 0)$ on $\mathbf{R}^n \cup \{\Delta\}$ (where Δ is the cemetery point) which is unique up to exceptional sets and whose transition semigroup is a quasi-continuous version of $(\mathcal{T}_t^\pi, t \geq 0)$. This gives a partial answer to Question 2. However this raises more questions than answers -

- What further information about ρ can we obtain from the Dirichlet form $\mathcal{E}$ and/or the Hunt process Y ?
- Does $\mathcal{E}$ have a probabilistic interpretation when the jump measure J fails to be positive ?
- Can this procedure work with other groups, e.g. the Lorentz group seems to be an obvious candidate.

Notes. The Heisenberg group is a nice setting for both harmonic analysis and probability theory. [5] is a good reference for the former. For the latter, with a particular emphasis on limit theorems, see the monograph by Neuenschwander [10].

The volume by Diaconis [4] contains a fascinating general account of applications of group representations in probability theory.

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Martingale calculations for Lévy–driven Russian options and many–server queues

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Let X be a Lévy process with Lévy exponent $\kappa(\alpha)$ and $W(t) = e^{\alpha X(t) - t\kappa(\alpha)}$ the Wald martingale. Kella & Whitt (1992) considered martingales of the form $\int_0^t e^{\alpha Y(t)+t\kappa(\alpha)} W(dt)$ where Y is an adapted process of locally bounded variation (depending on properties of Y, the integral takes different forms after integration by parts). The present talk deals with applications of these martingales to the calculation of $Ee^{-a\tau}$ where $\tau = \inf\{t : Z(t) \geq k\}$ and $Z(t) = X(t) + L(t)$ for a suitable adapted process L.

The first example is Russian options. In the setup of Shepp & Shiryaev 1993, 1994), the main problem in evaluation of the price is precisely to determine $Ee^{-a\tau}$ for the case where X is a Brownian motion and L the local time at 0 (Z is then the reflected version). Later work on Lévy models (Kou, 2000, Mordecki & Moreira, 2001, Avram *et al*, 2001) makes specific model restrictions. We work here in the dense class of Lévy processes which may have jumps in both directions but where the jumps are in the dense class of phase-type distributions (Neuts, 1981). The Markovian interpretation of such distributions allows to deal with the problem of controlling the overshoot, which is one of the main obstacles in generalizing beyond Brownian motion. The approach uses an imbedding in a continuous Markov additive process and the extension of the Kella–Whitt martingale derived in Asmussen & Kella (2000); $Ee^{-a\tau}$ then comes out by solving a certain set of linear equations.

Characteristics of τ are also of interest in queueing theory where one often identifies τ with the time of first buffer overflow say in a data buffer. The martingale technique has earlied been exploited by Asmussen *et al.* (2002/03) for Markov–modulated M/M/1 queues which are just a special case of reflected Markov additive processes ($L =$ the local time). For many–server queues, L is a more complicated boundary modification, which appears to give more unknowns than equations in the martingale approach, and we show how to overcome this problem using specific properties of the many–server queueing model.

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Exit problems for (reflected) spectrally negative Lévy processes and applications to exotic option pricing.

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1 Spectrally negative Lévy processes

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. In the sequel we do not allow for the case of subordinators and further, if X is a process of bounded variation, then we assume that its Lévy measure is absolutely continuous with respect to Lebesgue measure. Denote $\psi(\theta)$ the cumulant of X, which is convex and finite at least for $\theta \geq 0$. Further define the change of measure

$$\left.\frac{d\mathbb{P}^c}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\{cX_t - \psi(c)t\}$$

for all c such that $\psi(c) < \infty$. Note that it is not difficult to prove that under such a change of measure, the process $(X, \mathbb{P}^c)$ remains within the class of spectrally negative Lévy processes.

In order to discuss exit problems for this class of Lévy processes we need to introduce two functions $Z^{(q)}$ and $W^{(q)}$ which are called scale functions. To this end, denote for $q \geq 0$ the largest root of $\psi(\theta) = q$ by $\Phi(q)$. For this same range of q, define $W^{(q)} : \mathbb{R} \to [0, \infty)$ as the unique function which is identically zero for all $x \leq 0$ and for $x > 0$ is continuous and determined by the inverse transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = (\psi(\theta) - q)^{-1} \text{ when } \theta \geq \Phi(q).$$

Further, define

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\, dy.$$

From Bertoin (1997) and Lambert (2000) we have that for the afore mentioned class of spectrally negative Lévy processes the function $W^{(q)}$ restricted to $(0, \infty)$ is C^1 with the consequence that $Z^{(q)}$, when restricted to $(0, \infty)$ is C^2. Further, both $W^{(q)}$ and $Z^{(q)}$ as functions of q can be analytically extended to entire functions in $\mathbb{C}$.

2 Exit problems

It turns out that the two functions $W^{(q)}$ and $Z^{(q)}$ are all one needs to describe an ensemble of exit problems for the process X and the reflected process

$$Y = \{Y_t = \overline{X}_t - X_t : t \geq 0\}$$

where $\overline{X}_t = \sup_{u\in[0,t]} X_t$. Denote the passage times

$$T_a = \inf\{t \geq 0 : X_t \geq a\} \text{ and } T_0 = \inf\{t \geq 0 : X_t \leq 0\}$$

and let $\mathbb{P}_x$ be the translation of $\mathbb{P}$ under which $X_0 = x$. The following result has a long history in the literature, starting with Takács (1966), running through Suprun (1976), Bingham (1975) Rogers (1990) and ending with the formulation that we have adopted here given in Bertoin (1997).

[*]Speaker at Aarhus 25.01.02

Proposition 2.1 (two sided exit of X). *For $q \geq 0$*

$$\mathbb{E}_x \left[e^{-qT_a} \mathbf{1}_{(T_a<T_0)} \right] = W^{(q)}(x) / W^{(q)}(a) \tag{2.1}$$

and

$$\mathbb{E}_x \left[e^{-qT_0} \mathbf{1}_{(T_a>T_0)} \right] = Z^{(q)}(x) - Z^{(q)}(a) W^{(q)}(x) / W^{(q)}(a).$$

Note that with the given definitions of $W^{(q)}$ and $Z^{(q)}$ on $(-\infty, 0]$ these equalities hold for all $x \in (-\infty, a]$.

Denote $\Phi_c, W_c^{(q)}$ and $Z_c^{(q)}$ the versions of $\Phi, W^{(q)}$ and $Z^{(q)}$ associated with the process $(X, \mathbb{P}^c)$. It can be checked that by taking Laplace transforms and using analytical extention that for all real u and v such that $\psi(v) < \infty$

$$W^{(q)}(x) = e^{vx} W_v^{(u-\psi(v))}(x).$$

By exponentially tilting and taking the upper boundary a to infinity one can reach the following conclusion (also due to Emery (1975)).

Proposition 2.2 (one sided exit of X). *For $u \geq 0$ and v such that $\psi(v) < \infty$ we have for all $x \in \mathbb{R}$*

$$\mathbb{E}_x \left[e^{-uT_0 + vX_{T_0}} \right] = e^{vx} \left(Z_v^{(p)}(x) - W_v^{(p)}(x) p / \Phi_v(p) \right)$$

where $p = u - \psi(v)$ and $p/\Phi_v(p)$ is understood in the limiting sense when $p = 0$.

In recent work, Avram *et al.* (2002) have shown that these scale functions also serve a purpose for exit problems of the process Y. Let

$$\tau_k = \inf\{t \geq 0 : Y_t \geq k\}$$

and we shall alter the definition of $\mathbb{P}_x$ to $\mathbb{P}_{s,x}$ meaning that both $X_0 = x$ and $\overline{X}_0 = s$.

Theorem 2.3 (one sided exit of Y). *For $u \geq 0$ and v such that $\psi(v) < \infty$,*

$$\mathbb{E}_{s,x} \left[e^{-u\tau_k - vY_{\tau_k}} \right] = h(s - x)$$

where

$$h(z) = e^{-vz} \left(Z_v^{(p)}(k-z) - W_v^{(p)}(k-z) \frac{pW_v^{(p)}(k) + vZ_v^{(p)}(k)}{W_v^{(p)\prime}(k) + vW_v^{(p)}(k)} \right) \tag{2.2}$$

where $p = u - \psi(v)$ and $x \in \mathbb{R}$.

The functions $W^{(q)}$ and $Z^{(q)}$ can be considered positive-harmonic with respect to the process killed at rate q which is further killed on exiting $(0, a)$. In the case of diffusions harmonic would mean that they are in the kernel of the operator $L - q$ restricted to $(0, a)$, where L is the generator of the diffusion. Unfortunately we cannot phrase harmonicity in this way for these scale functions on account of the fact that they are not necessarily in the domain of the generator of X killed on exiting $(0, a)$. Harmonicity can be expressed however in the more general sense of a martingale. Note that $\mathbf{1}_{(T_a<T_0)} = W^{(q)}(X_{T_a \wedge T_0}) / W^{(q)}(a)$. By placing this expression back into (2.1) and applying the Strong Markov property, one can deduce that for $x \in (0, a)$

$$\left\{ e^{-q(t \wedge T_a \wedge T_0)} W^{(q)}(X_{t \wedge T_a \wedge T_0}) / W^{(q)}(a) : t \geq 0 \right\}$$

is a $\mathbb{P}_x$-martingale with respect to $\mathbf{F}$. In fact it is quite straightforward to prove that $W^{(q)}(x)/W^{(q)}(a)$ restricted to $(-\infty, a]$ is the unique function which is zero at $x \leq 0$ and one at $x = a$ and that is harmonic for $x \in (0, a)$ in the above sense. The exit problem thus keeps its analogy with diffusions. Similar representations of the results in the other exit problems can be written down.

3 Applications to exotic option pricing

A genuine motivation for studying these types of exit problems comes from the theory of pricing of perpetual options. Consider the standard Black-Scholes market, where the bank rate is fixed at $r > 0$. In place of the underlying Brownian motion, we can now work with the spectrally negative Lévy process X. In particular then, the value of the risky asset is the process $\{\exp(X_t) : t \geq 0\}$.

Let us give one example of option pricing which uses the exit problems above, in particular Theorem 2.3. The perpetual Russian option is an American-type option with no expiry offering the holder the possibility to exercise at any $\mathbf{F}$-stopping time in order to claim

$$e^{-\alpha t} \max\left\{e^s, e^{\overline{X}_t}\right\}$$

at time t. Given that our market is now incomplete, the issue of option 'price' is a big discussion which we do not wish to indulge in. Instead we will assume that a risk-neutral measure has been selected and this will now inherit the symbol $\mathbb{P}$. That is to say, we make the extra assumption that under $\mathbb{P}$ the process $\{\exp\{X_t - rt\} : t \geq 0\}$ is a martingale. Referring back to the original paper of Shepp and Shiryayev (1994) who give the price of a Russian option in the traditional Black-Scholes market one finds that by changing measure using the afore mentioned martingale the analogy of the price of this option reduces to the solution to the optimal stopping problem

$$\sup_{\tau} \mathbb{E}^1_{s,x}\left(e^{-\alpha\tau + Y_\tau}\right)$$

where the supremum is taken over all almost surely finite $\mathbf{F}$-stopping times and we modify $\mathbb{P}^1_x$ to $\mathbb{P}^1_{s,x}$ in the obvious way as before. It turns out that the expression in Theorem 2.3 makes for an easy solution to this optimal stopping problem.

To this end, note that for any $k > 0$ we can manipulate the expression (2.2) to deduce that

$$\mathbb{E}^1_{s,x}\left[e^{-\alpha\tau_k + Y_{\tau_k}}\right] = e^{(s-x)}\left(Z^{(q)}(k-s+x) - W^{(q)}(k-s+x)\frac{Z^{(q)}(k) - qW^{(q)}(k)}{W^{(q)\prime}(k) - W^{(q)}(k)}\right)$$

where $q = \alpha + r$. Now let us assume that there exists a $k^* \in (0,\infty)$ such that $Z^{(q)}(k^*) - qW^{(q)}(k^*)$. [In fact such a k^* does exist when X contains a Gaussian component; for a general discussion around this issue, see Avram *et al.* (2002)]. It follows that the expression above simplifies somewhat to

$$\mathbb{E}^1_{s,x}\left[e^{-\alpha\tau_{k^*} + Y_{\tau_{k^*}}}\right] = e^{(s-x)} Z^{(q)}(k^* - s + x).$$

Rather conveniently the function $e^z Z^{(q)}(k^* - z)$ is a C^2 function in $\mathbb{R}\backslash\{k^*\}$. At this point it can be checked that with the presence of the Gaussian component in X the function $W^{(q)}$ becomes continuous at zero implying that $e^z Z^{(q)}(k^* - z)$ is C^1 on $\mathbb{R}$. This is sufficient smoothness to use $Z^{(q)}$ in the context of Itô's formula. Indeed the fact that

$$\left\{e^{-\alpha(t\wedge\tau_{k^*}) + Y_{t\wedge\tau_{k^*}}} Z^{(q)}(k^* - Y_{t\wedge\tau_{k^*}}) : t \geq 0\right\}$$

is a $\mathbb{P}^1_{s,x}$-martingale (see earlier remarks) implies that

$$\left(\widehat{\Gamma}_1 - \alpha\right)\left[e^z Z^{(q)}(k^* - z)\right] = 0 \text{ for } z \in [0, k^*)$$

where $\widehat{\Gamma}_1$ is the generator of the process $(-X, \mathbb{P}^1)$. Now recall that for $z \geq k^*$ we have that $e^z Z^{(q)}(k^* - z) = e^z$. Since $\{\exp\{rt - X_t\} : t \geq 0\}$ is a $\mathbb{P}^1$-martingale, it follows again from Itô's formula that

$$0 = \left(\widehat{\Gamma}_1 + r\right)\left[e^z Z^{(q)}(k^* - z)\right] \geq \left(\widehat{\Gamma}_1 - \alpha\right)\left[e^z Z^{(q)}(k^* - z)\right] \text{ for } z \in (k^*, \infty).$$

Now by a final application of Itô's formula together with the above variational inequatlities, it follows that

$$\left\{e^{-\alpha t + Y_t} Z^{(q)}(k^* - Y_t) : t \geq 0\right\}$$

is a supermartingale.

The solution to the optimal stopping problem can now be solved quite easily. Doob's Optimal Stopping Theorem together with the fact that $e^z Z^{(q)}(k^* - z) \geq e^z$ for all z implies that for any almost surely finite **F**-stopping time, τ,

$$\mathbb{E}^1_{s,x}\left(e^{-\alpha\tau+Y_\tau}\right) \leq \mathbb{E}^1_{s,x}\left(e^{-\alpha\tau+Y_\tau}Z^{(q)}(k^* - Y_\tau)\right) \leq e^{(s-x)}Z^{(q)}(k^* - s + x)$$

giving an upper bound for $\sup_\tau \mathbb{E}^1_{s,x}\left(e^{-\alpha\tau+Y_\tau}\right)$. Since this sequence of inequalities becomes a sequence equalities with the choice $\tau = \tau_{k^*}$ we have our solution to the optimal stopping problem:

$$\sup_\tau \mathbb{E}^1_{s,x}\left(e^{-\alpha\tau+Y_\tau}\right) = e^{(s-x)}Z^{(q)}(k^* - s + x)$$

with optimal strategy $\tau = \tau_{k^*}$.

Although we have assumed a Gaussian component is present, it is not strictly necessary in order to produce this latter conclusion. See Avram *et al.* (2002) for further details.

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Introduction to Fractional Lévy Motions.

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Abstract

In this article the class of Moving Average Fractional Lévy Motions is introduced. This class is built from the Fractional Brownian Motion and illustrates what are fractional processes beyond FBM. The asymptotic self-similarity and the smothness of the paths of these fields are studied, and they are compared to Real Harmonizable Fractional Lévy Motions . Some MAFLM's are locally self-similar with an index $\widetilde{H}$, have $H-d/2$ Hölder continous sample paths and the L^2 norm of the increments is H Hölder continous. This shows that in a non-Gaussian setting these indexes may be different. Moreover we can establish a multiscale behavior of some of these fields. Eventually identification of all the indexes of such MAFLM's is performed.

Key-words: Identification, Local Asymptotic Self Similarity, Second order fields, Stable fields.
AMS classification (2000): 60G12,60G17,62G05.

The concept of self-similarity is often used to give a mathematical meaning to the heuristical concept of roughness. In this domain the Fractional Brownian Motion (in short FBM) $B_H(t)$ of fractional index $0 < H < 1$, introduced in [8, 10], is certainly the most celebrated model. Let us recall that the FBM is the only centered Gaussian self-similar process with stationary increments and with index H. However it is a well known fact that in some fields of applications the data do not fit Gaussian models. See for instance [9, 13, 15] for image modeling. Moreover some phenomena seem to have different regime depending on the scale they are considered (see [14] in Computer Science, or [2] in Mathematical Finance). The need for so-called multiscale models that are nearly Gaussian for some scales and very far from Gaussian at other scale is hence driven by the applications. From a mathematical point of view dropping the Gaussian assumption leads to wonder which properties of the FBM shall be kept for fractional non Gaussian models, and if new features appear for these generalized models. For instance roughness is described by H for the FBM, is it still true for other models ?

Let us be more precise and give mathematical statements. The FBM is self-similar in distribution :

$$(B_H(\varepsilon t))_{t\in\mathbb{R}} \overset{(d)}{=} \varepsilon^H \left(B_H(t)\right)_{t\in\mathbb{R}}$$

for every $\varepsilon > 0$, the Hölder exponent of the sample paths is almost surely H; the increments are centered Gaussian variables with variance:

$$\mathbb{E}\left(B_H(t) - B_H(s)\right)^2 = |t-s|^{2H} .$$

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However the index H of the FBM is the same in every point $t \in \mathbb{R}$ which is not desirable to model roughness. This index H can be localized and replaced by a function $h(t)$ ([5, 11]). The resulting processes X_h are centered and Gaussian and satisfy the following three properties.

- Local self-similarity.

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_h(t+\varepsilon u) - X_h(t)}{\varepsilon^{h(t)}} \right)_{u \in \mathbb{R}} \stackrel{(d)}{=} \left(B_{h(t)}(u) \right)_{u \in \mathbb{R}} .$$

  One then says that the FBM is the tangent process of X at point t.

- Hölder continuity. For every t the pointwise Hölder exponent at point t is almost surely $h(t)$.

- Variance of the increments. The increments are centered Gaussian variables and the variances satisfy:

$$\lim_{t \to s} \frac{\mathbb{E}\left(X_h(t) - X_h(s) \right)^2}{|t-s|^{2h(s)}} = 1 .$$

Models that share these three properties will be called fractional fields. It is known that fractional fields are not necessarily Gaussian since [3] and the so-called Real Harmonizable Fractional Lévy Motion (in short RHFLM) which is a large class of fractional fields including the FBM and other non-Gaussian fields. In these class at every point t the tangent field in the lass property is a FBM with fractional index H. For the class of RHFLM's, one index H governs the three properties that define fractional fields. Then a new question arises : Is the roughness concept reducible to only one index or is it a weakness of the previous models ?

The first aim of this paper is to prove that the local self-similarity, the Hölder continuity and the variance of the increments can be governed by three different indexes. Let us now shortly describe our model. The well-balanced Moving Average representation of FBM in d-dimension:

$$B_H(t) = \int_{\mathbb{R}^d} \left(||t-s||^{H-d/2} - ||s||^{H-d/2} \right) W(ds) ,$$

where W is the Wiener measure is the starting point of our construction. Then the measure W is replaced by a Lévy measure M with moments of every orders and without Brownian component:

$$X_H(t) = \int_{\mathbb{R}^d} \left(||t-s||^{H-d/2} - ||s||^{H-d/2} \right) M(ds) .$$

The process X is called a Moving Average Fractional Lévy Motion (in short MAFLM). In this paper, we prove the following.

- Local self-similarity. For some control measures of the Lévy measure called truncated stable measure which are vanishing at infinity and such that $\nu(du) \sim \dfrac{du}{|u|^{1+\alpha}}$ when u is close to 0.

$$\lim_{\varepsilon \to 0^+} \left(\frac{X(t+\varepsilon u) - X(t)}{\varepsilon^{\widetilde{H}}} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} \left(Y_{\widetilde{H}}(u) \right)_{u \in \mathbb{R}^d} ,$$

  where $\widetilde{H} - d/\alpha = H - d/2$ as soon as $0 < \widetilde{H} < 1$, and where $Y_{\widetilde{H}}$ is a Moving Average Fractional Stable Motion cf [12].

- Hölder continuity.

  MAFLM are almost surely continuous if and only if $H > d/2$. Then, the sample paths belong almost surely to Hölder spaces $C^{H-d/2-\varepsilon}$ for all $\varepsilon > 0$ and they do not belong $C^{H-d/2+\varepsilon}$ with probability one.

- Variance of the increments.

  By construction, MAFLM's have exactly the same second-order structure than FBM. Therefore:

$$\mathbb{E}\left(X(t) - X(s) \right)^2 = ||t-s||^{2H} .$$

It is now clear that three indexes of roughness can be defined for the MAFLM: $\widetilde{H}$, $H-d/2$ and H.

Moreover an asymptotic self-similar property is proved for all MAFLM's :

$$\lim_{R\to+\infty}\left(\frac{X_H(Rt)}{R^H}\right)_{t\in\mathbb{R}^d} \stackrel{(d)}{=} (B_H(t))_{t\in\mathbb{R}^d}\,.$$

In particular this means that the MAFLM with a truncated stable measure denoted by $X_{H,\alpha}$ exhibits different behaviors through the scales. For low scales, there are locally self-similar with a Moving Average Stable Motion as tangent process. At large scale, there are asymptotically self-similar with a FBM as tangent process. This phenomenon of different asymptotic self similarities at low and large scales has already been encountered in [4, 3] and it has been already detected in some applications cf [7, 2], it is called a multiscale behavior. Since $\widetilde{H}>H$ for MAFLM's whereas $\widetilde{H}<H$ for RHFLM's these two models are complementary for modeling multiscale data.

The previous indexes of MAFLM's are then identified from the observation of a single sample path on a bounded interval. The local self-similarity for $X_{H,\alpha}$ suggests to use log-variations to identify $\widetilde{H}$, as it was done by [1, 6] for stable processes in a wavelet setting. β-variations are used for the identification of H. Actually it is shown that the β-variations behave differently if $\beta<\alpha$ or if $\beta>\alpha$. This fact is reminiscent of the multiscale behavior of $X_{H,\alpha}$.

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Merton's portfolio optimization problem and non-Gaussian stochastic volatility

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1 Introduction

We will in this extended abstract review some existing results and announce some forthcoming work on portfolio optimization problems based on the stochastic models for financial assets recently introduced by Barndorff-Nielsen and Shephard [3]. The control problems discussed here are considered in detail in the papers Benth, Karlsen, and Reikvam [9, 10] and in the forthcoming paper [11].

Consider an investor who at time t wants place money in a risky asset and a bond in such a way that terminal expected utility at time T is optimized. If the value of the risky asset follows a stochastic process $S(s), s \geq t$ and the bond has (deterministic) value $R(s)$, the dynamics of the investor's total wealth becomes

$$dW(s) = \frac{\pi(s)W(s)}{S(s)}\,dS(s) + \frac{(1-\pi(s))W(s)}{R(s)}\,dR(s)\,.$$

Here, $\pi(s)$ is the *fraction* of the total wealth $W(s)$ invested in the risky asset at time s. The goal of the investor is, in mathematical terms, to find an investment strategy $\pi^*(s)$ which optimizes the expected terminal utility

$$\mathbb{E}\big[U(W(T))\big]\,,$$

where U is the investor's utility function. This stochastic portfolio optimization problem is known in finance as *Merton's problem.* The value function of the problem is

$$V(t,w) = \sup_{\pi} \mathbb{E}^{t,w}\big[U(W(T))\big]\,,$$

where w is the initial wealth at time t and the supremum is taken over all admissible controls (to be clarified later). $\mathrm{E}^{t,w}$ means the expectation conditioned on $W(t) = w$.

To study such *stochastic control problems* we need a dynamics on $S(s)$. Of course, this dynamics should reflect the statistical stylized facts of the underlying risky asset, but at the same time it is desirable to use models which are tractable within the framework of stochastic control. From the *dynamic programming (or Hamilton-Jacobi-Bellman)* point of view, this means models having the Markov property.

The simplest, and traditional, choice of asset price dynamics is the geometric Brownian motion (also known as the Samuelson-Black-Scholes model). We will briefly review Merton's problem for this case in the next section. In the recent years Barndorff-Nielsen and others have introduced different classes of models which fit observed stock price dynamics extremely well (see [1, 2, 3, 4, 13, 18, 19]). These models have the desirable Markov property, being stochastic processes driven by Lévy processes. In the subsequent sections, the Ornstein-Uhlenbeck stochastic volatility model of Barndorff-Nielsen and Shephard will be used to model the dynamics of the risky asset. The resulting portfolio problem will be considered from a dynamic programming point of view. We will also discuss the problem of pricing derivatives in a market where the underlying risky asset follows the Barndorff-Nielsen and Shephard model. We suggest to use a utility optimization technique based on the idea of Hodges and Neuberger [14] to find prices of options in this incomplete market.

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2 Merton's problem in Black & Scholes markets

Let us review the classical Merton problem: The stock price follows a geometric Brownian motion on the form

$$S(s) = S(t)\exp\big(\mu\cdot(s-t)+\sigma(B(s)-B(t))\big),$$

σ being the volatility, μ the expected logreturn and $B(s)$ a standard Brownian motion. Using a bond price dynamics (which will be the bond price in the rest of this abstract)

$$dR(s) = rR(s)\,ds, \quad R(t)=1$$

we obtain the following wealth process from the self-financing hypothesis:

$$dW(s) = \big(r+(\mu+\frac{1}{2}\sigma^2-r)\pi(s)\big)W(s)\,ds+\sigma\pi(s)W(s)\,dB(s)\,.$$

The initial wealth is $W(t)=w$. Restricting our attention to feedback controls (e.g. Markov controls, $\pi(s)\equiv\pi(s,W(s))$), the associated Hamilton-Jacobi-Bellman equation (from now on the HJB-equation for short) for the value function $V(t,w)$ is given by

$$V_t+\max_{\pi}\Big\{\big(r+(\mu+\frac{1}{2}\sigma^2-r)\pi\big)wV_w+\frac{1}{2}\sigma^2\pi^2w^2V_{ww}\Big\}=0$$
$$v(T,w)=U(w)$$

Considering a utility function U of HARA-type, i.e. $U(w)=w^\gamma/\gamma$ for $\gamma\in(0,1)$, this control problem has an explicit solution:

$$\pi^*(s)=\frac{\mu+\frac{1}{2}\sigma^2-r}{\sigma^2(1-\gamma)}$$

and with a value function equal to

$$V(t,w)=\frac{w^\gamma}{\gamma}\mathrm{e}^{\gamma k(T-t)}$$

where

$$k=\big(r+\frac{(\mu+\frac{1}{2}\sigma^2-r)^2}{2\sigma^2(1-\gamma)}\big).$$

Using a standard verification theorem we easily prove these results, see Merton [15, 16].

3 Merton's problem and non-Gaussian stochastic volatility

Barndorff-Nielsen and Shepard [3] (see also [4] and [2]) have recently suggested a class of stochastic volatility models where the risky asset follows the dynamics

$$d\ln S(s)=\big(\mu+\beta\sigma(s)\big)\,ds+\sqrt{\sigma(s)}\,dB(s) \tag{3.1}$$

where

$$\sigma(s)=\sum_{j=1}^{m}\omega_jY_j(s)$$

and

$$dY_j(s)=-\lambda_jY_j(s)\,ds+dZ_j(\lambda_js)\,,$$

ω_j are positive weigths summing to one and $Z_j(s)$ are subordinators independent of B. β is a constant modelling skewness in the logreturns, while the parameters λ_j model the autocorrelation. In fact, the autocorrelation of the logreturns will in this stochastic volatility model be a sum of functions decreasing to zero exponentially at the speed given by λ_j. The unusual timing of the subordinators assures that the invariant distribution of $\sigma(s)$ is independent of λ_j, thus separating the modelling of logreturn and autocorrelation. For more on the features of this class of stochastic volatility models, we refer to the extensive study in Barndorff-Nielsen and Shephard [3].

Using (3.1), we are led to the wealth dynamics

$$dW(s) = \Big(r + \big(\mu + (\beta + \frac{1}{2})\sigma(s) - r\big)\pi(s)\Big)W(s)\,ds + \pi(s)\sqrt{\sigma(s)}W(s)\,dB(s)$$

Initial wealth at time t is $W(t) = w$. Progressivly measurable controls $\pi(s) \in [0,1]$ are called *admissible* if a unique solution to the wealth dynamics exists. The set of admissible controls are denoted $\mathcal{A}_t$. Note that we assume no short selling or borrowing of money (the controls are in the interval $[0,1]$). We can relax this assumption to become $\pi \in [a,b]$ for two arbitrary constants $a < b$. The constraint $[0,1]$ is just for convenience. The value function of the control problem looks like

$$V(t,w,y) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}^{t,w,y}\big[U(W(T))\big]$$

where we start the processes $W(s)$ in w and $Y_j(s)$ in y_j at time t.

From the *dynamic programming principle* we may associate a Hamilton-Jacobi-Bellman equation for the value function:

$$\begin{aligned} &V_t + \max_{\pi\in[0,1]}\Big\{\pi\left(\mu + \left(\tfrac{1}{2}+\beta\right)\sigma - r\right)wV_w + \tfrac{1}{2}\pi^2\sigma w^2 V_{ww}\Big\} + rwV_w \\ &\quad - \sum_{j=1}^m \lambda_j y_j V_{y_j} + \sum_{j=1}^m \lambda_j \int_0^\infty \Big(V(t,w,y+z\cdot e_j) - V(t,w,y)\Big)\,\ell_j(dz) = 0, \end{aligned} \tag{3.2}$$

with terminal condtion $V(T,w,y) = U(w)$. Here, $\sigma = \sum_{j=1}^m \omega_j y_j$, $\ell_j(dz)$ are the Lévy measures of the subordinators Z_j and the domain is $(t,w,y) \in [0,T) \times R_+^{m+1}$. It turns out that we can find a more or less explicit solution to (3.2) in the case of HARA-utility, $U(w) = w^\gamma/\gamma$. We now refer the main result of [9] which holds under some explicit exponential integrability conditions of the Lévy measures $\ell_j(dz)$.

Theorem 3.1.

$$V(t,w,y) = \gamma^{-1}w^\gamma\,\mathbb{E}^{t,y}\left[e^{\int_t^T \gamma\Pi(\sigma(s))\,ds}\right]$$

where

$$\Pi(\sigma) = \max_{\pi\in[0,1]}\Big\{\pi\left(\mu + \left(\tfrac{1}{2}+\beta\right)\sigma - r\right) - \tfrac{1}{2}\pi^2\sigma(1-\gamma)\Big\} + r.$$

Furthermore, the optimal investment strategy is $\pi^*(\sigma) = 1$ *when* $\frac{1}{2} + \beta/1-\gamma \geq 1$,

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0,\hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1,\infty) \end{cases}$$

when $\frac{1}{2} + \beta/1-\gamma \in (0,1)$, *and*

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0,\hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1,\hat{\sigma}_0], \\ 0, & \sigma \in (\hat{\sigma}_0,\infty). \end{cases}$$

when $\frac{1}{2} + \beta/1-\gamma < 0$. *The function* $\bar{\pi}(\sigma)$ *is defined as*

$$\bar{\pi}(\sigma) = \frac{1}{1-\gamma}\left(\frac{\mu-r}{\sigma} + \frac{1}{2} + \beta\right)$$

and

$$\hat{\sigma}_1 = \frac{\mu - r}{(1-\gamma) - \left(\frac{1}{2}+\beta\right)}, \qquad \hat{\sigma}_0 = -\frac{\mu - r}{\frac{1}{2}+\beta}.$$

We refer to [9] for the proof of this result. As expected, we change the fraction of wealth invested in the risky asset inversely proportional to the level of volatility, contrary to the classical Merton case where a fixed fraction is placed in the stock for the whole investment period. Notice that when $\beta = 0$ and $\lambda_1 = \ldots = \lambda_m = 0$, $\bar{\pi}$ is the classical Merton solution, and our optimal π^* is the Merton investment strategy under the constraint that short-selling and borrowing of money is prohibited. We remark that in [10] the control problem is studied for general utility functions where we characterize the value function as the unique viscosity solution of (3.2).

4 Pricing options using utility optimization techniques

Let us consider the problem of pricing a European claim with payoff $f(S(T))$ at time T in the *incomplete* market defined by the dynamics (3.1). Motivated from Hodges and Neuberger's [14] utility optimization approach, we may use the control problem from the section above to find prices of the claim. We consider the pricing problem from the *issuer's* point of view. In passing, we remark that the following results are preliminary and will be presented in a rigourous fashion in the forthcoming paper [11].

The issuer has two opportunities as an investor: Either he do not issue the claim, but optimize his wealth. The optimization problem is

$$V^0(t,w,y) = \sup_\pi \mathbb{E}^{t,w,y}\Big[U\Big(W(T)\Big)\Big]$$

Alternativly, he can issue the claim, and then optimize the wealth, which leads to the portfolio optimization problem

$$V(t,w,y,x) = \sup_\pi \mathbb{E}^{t,w,y,x}\Big[U\Big(W(T) - f(S(T))\Big)\Big]$$

Here, x is the stock price at time t, $S(t) = x$ The *price* of the claim is defined as the premium Λ for which the issuer is *indifferent* between the two investment alternatives.

$$V^0(t,w,y) = V(t,w+\Lambda,y,x)$$

If we choose $U(w) = w^\gamma/\gamma$, Λ will be dependent on initial wealth w. The price of a claim should not depend on the wealth of the issuer, so the HARA-utility seems to be undesirable in this context. Let us choose the exponential utility function $U(w) = 1 - \exp\big(-\gamma w\big)$ instead (motivated from Hodges and Neuberger [14]). γ is the index of risk since $-U''(w)/U'(w) = \gamma$. We furthermore assume that the investment strategies π admissible to the issuer are unconstrained, that is, any position in stocks is admissible. To ease exposition, assume for the moment that $r = 0$.

Following the same program as for Merton's problem above we find from the dynamic programming principle equations for V^0 and V. If we separate the solution like in the HARA-case,

$$V^0(t,w,y) = 1 - \mathrm{e}^{-\gamma w} H^0(t,y), \quad V(t,w,y,x) = 1 - \mathrm{e}^{-\gamma w} H(t,y,x)$$

we can show that H^0 and H solve integro-differential equations with terminal conditions:

$$H^0_t - g(\sigma)H^0 + \mathcal{B}H^0 = 0, \qquad H^0(T,y) = 1$$

where, $\sigma^2 = \sum_{j=1}^m \omega_j y_j$, $g(\sigma) = \big(\mu + (\frac{1}{2}+\beta)\sigma^2 - r\big)^2/2\sigma^2$, and

$$\mathcal{B}h = \sum_{j=1}^m \lambda_j \int_0^\infty h(y + z\cdot e_j) - h(y)\ell_j(dz) - \sum_{j=1}^m \lambda_j y_j h_{y_j}$$

Equation for H

$$H_t - g(\sigma)H - \frac{1}{2}\sigma^2 x^2 \frac{H_x^2}{H} + \frac{1}{2}\sigma^2 x^2 H_{xx} + \mathcal{B}H = 0, \qquad H(T,y,x) = \mathrm{e}^{\gamma f(x)}$$

From direct calculations (introducing r again) we find the price to be

$$\Lambda \equiv \Lambda(t,y,x) = \mathrm{e}^{-r(T-t)} \cdot \gamma^{-1} \ln\Big(\frac{H(t,y,x)}{H^0(t,y)}\Big) \tag{4.1}$$

Note that this price is not dependent on the initial wealth w. To actually calculate the price Λ, we need to solve two integro-differential equations (numerically) It is easy, however, to prove that when $\lambda_1 = \ldots = \lambda_m = 0$ (i.e. σ^2 =const), then $\Lambda \equiv \Lambda(t,x)$ solves the Black & Scholes equation

$$\Lambda_t - r\Lambda + rx\Lambda_x + \frac{1}{2}\sigma^2 x^2 \Lambda_{xx} = 0, \qquad \Lambda(T,x) = f(x)$$

and therefore coincides with the arbitrage-free price in the complete case.

A more thorough study of the price fomula (4.1) from both an analytical and numerical point of view will be presented in the forthcoming paper [11]. But what can we expect from such a pricing methodology compared to arbitrage-free pricing in incomplete markets? In order to give some idea we consider a one-period market which is incomplete, and characterize both the arbitrage-free pricing interval, and prices obtained from utility optimization.

4.1 An example from one-period markets

Assume the stock (the risky investment) has initial value $S_0 = s$ and

$$S_1 = \begin{cases} s(1+u), & \text{prob. } p \\ s, & \text{prob. } q \\ s(1-d), & \text{prob. } r \end{cases}$$

where $p, q, r \in (0,1)$ and $p + q + r = 1$. Furthermore, we assume $u, d > 0$. Let the risk-free investment opportunity be a bank account with zero interest rate. A portfolio in the market will consist of α \$ in the bank and π number of shares in the stock. If we start with initial wealth w, we must choose α and π such that

$$\alpha + \pi s = w$$

The total value of such a portfolio in the next period will be,

$$W_1 = \alpha + \pi S_1 = w + \pi(S_1 - s)$$

Assume the market also consists of a claim X with payoff

$$X = \begin{cases} x, & \text{prob. } p \\ y, & \text{prob. } q \\ z, & \text{prob. } r \end{cases}$$

where x, y, z all are positive. Furthermore, for reasons to be clear below, we assume

$$\frac{dx}{u+d} + \frac{dz}{u+d} > y \tag{4.2}$$

Our market is incomplete since the number of possible states of the stock are larger than 2.

The following lemma characterizes all equivalent martingale measures in this market. Let $Q(\omega_1) = \tilde{p}, Q(\omega_2) = \tilde{q}$ and $Q(\omega_3) = \tilde{r}$. Then we have,

Lemma 4.2. *A probability Q will be an equivalent martingale measure if and only if*

$$\tilde{p} = \tilde{r}\frac{d}{u}, \quad 0 < \tilde{r} < \frac{1}{1 + \frac{d}{u}}$$

Proof. Q is an equivalent martingale measure if and only if $\tilde{p}, \tilde{r} \in (0,1)$, $\tilde{p} + \tilde{r} < 1$ (which implies $\tilde{q} = 1 - \tilde{p} - \tilde{r} \in (0,1)$), and $\mathbb{E}_Q\big[S_1\big] = s$. The last condition is equivalent with

$$\tilde{p}s(1+u) + (1 - \tilde{p} - \tilde{r})s + \tilde{r}(1-d) = s$$

which again is equivalent with $\tilde{p}u = \tilde{r}d$. Together with $\tilde{p} + \tilde{r} < 1$ we have proven the Lemma. $\square$

We can now find the interval of arbitrage-free prices of the claim X:

Lemma 4.3. *For any equivalent martingale measure Q the arbitrage-free price of X is,*

$$\Lambda_Q = \tilde{r}\frac{d}{u}x + (1 - \tilde{r}(1 + \frac{d}{u}))y + \tilde{r}z$$

All the arbitrage-free prices are in the price interval

$$\Lambda_Q \in \big(y, \frac{dx}{u+d} + \frac{uz}{u+d}\big)$$

Proof. Straightforward calculation, using the characterization of equivalent martingale measures and the bound of $\tilde{r}$. ☐

We now consider the superreplicating strategies in this market, that is, we look for initial wealth w and a trading strategy in stocks π such that $W_1 = w + \pi(S_1 - s) \geq X$. An equivalent formulation is

$$\begin{aligned} w + \pi s u &\geq x \\ w &\geq y \\ w - \pi s d &\geq z \end{aligned}$$

Multiplying the first inequality with d and the third with u, and then adding the two, yields, $w(u+d) \geq xd + zu$, or $w \geq \frac{ux}{u+d} + \frac{dz}{u+d}$. By the assumption (4.2), we have that with this w, the inequality $w \geq y$ is satisfied. Inserting the constraint on w into the first inequality above, gives, $\pi \geq \frac{x-z}{s(u+d)}$. We conclude,

Lemma 4.4. *The cheapest superreplicating strategy is*

$$\pi_{super} = \frac{x-z}{s(u+d)}$$

which costs

$$\Lambda_{super} = \frac{ux}{u+d} + \frac{dz}{u+d}$$

Observe that Λ_{super} coincides with the upper bound of Λ_Q, not unexpectedly.

We now move on to the utility optimization approach to price claims. Assume the investor has a utility function $1 - \exp(-\gamma w)$. For an initial wealth w, he will seek to optimize

$$V_0(w) = 1 - \inf_{\pi} \mathbb{E}\big[\exp(-\gamma W_1)\big] \tag{4.3}$$

where $W_1 = w + \pi(S_1 - s)$. If the investor issues the claim X, his portfolio optimization problem takes the form

$$V^i(w + \Lambda^i_\gamma) = 1 - \inf_{\pi} \mathbb{E}\big[\exp(-\gamma W_1 + \gamma X)\big] \tag{4.4}$$

where $W_1 = w + \Lambda^i_\gamma + \pi(S_1 - s)$. The price Λ^i_γ is defined as the solution of the equation

$$V_0(w) = V(w + \Lambda^i_\gamma)$$

For convenience, define

$$\rho = p\big(\frac{dr}{up}\big)^{u/u+d}(1 + \frac{u}{d}) \tag{4.5}$$

We find the following results for the two portfolio optimization problems:

Lemma 4.5. *The optimal strategy for the portfolio problem* (4.3) *is*

$$\pi^*_0 = \frac{\ln(up) - \ln(dq)}{\gamma s(u+d)}$$

with the optimal expected utility

$$V_0(w) = 1 - (q + \rho)\exp(-\gamma w)$$

where ρ is given in (4.5).

Proof. Let

$$J(\pi) = p\exp(-\gamma w - \gamma\pi s u) + q\exp(-\gamma w) + r\exp(-\gamma w + \gamma \pi s d)$$

A straightforward differentiation shows that $J'(\pi) = 0$ when $\pi = \frac{\ln(up) - \ln(dq)}{\gamma s(u+d)}$. Inserting this into the optimal expected utility function yields $V_0(w)$. ☐

Lemma 4.6. *The optimal strategy for the portfolio problem* (4.4) *is*

$$\pi^* = \frac{\ln(up) - \ln(dq)}{\gamma s(u+d)} + \frac{x-z}{s(u+d)}$$

with the optimal expected utility

$$V^i(w + \Lambda^i_\gamma) = 1 - (qe^{\gamma y} + \rho e^{\gamma \Lambda_{super}}) \exp(-\gamma(w + \Lambda^i_\gamma))$$

where ρ is given in (4.5).

Proof. Let

$$J(\pi) = p \exp(-\gamma w - \gamma \pi s u + \gamma x) + q \exp(-\gamma w + \gamma y) + r \exp(-\gamma w + \gamma \pi s d + \gamma z)$$

A straightforward differentiation shows that $J'(\pi) = 0$ when $\pi = \frac{\ln(up)-\ln(dq)}{\gamma s(u+d)} + \frac{x-z}{s(u+d)}$. Inserting this into the optimal expected utility function yields $V^i(w + \Lambda^i_\gamma)$. ☐

Observe that π^* is the sum of π^*_0 and the cheapest superreplicating strategy! We find the utility price of the claim as

Proposition 4.7. *The utility optimization price of the claim X from the issuer's point of view is*

$$\Lambda^i_\gamma = \frac{dx}{u+d} + \frac{uz}{u+d} - \frac{1}{\gamma} \ln\left(\frac{q+\rho}{q \exp(-\gamma(\Lambda_{super} - y)) + \rho} \right)$$

where ρ is given in (4.5).

Proof. Letting $V_0(w) = V(w + \Lambda^i_\gamma)$ yields the equation

$$(q+\rho)\exp(-\gamma w) = (q e^{\gamma y} + \rho e^{\gamma(\frac{dx}{u+d} + \frac{uz}{u+d})}) \exp(-\gamma(w + \Lambda^i_\gamma))$$

which gives,

$$\Lambda^i_\gamma = \frac{1}{\gamma} \ln\left(\frac{q e^{\gamma y} + \rho e^{\gamma(\frac{dx}{u+d} + \frac{uz}{u+d})}}{p+\rho} \right)$$

Reorganizing gives the result. ☐

The price of the claim is given as the difference between Λ_{super}, the price of the cheapest superreplicating portfolio, and some price dependent on the the risk aversion factor γ. The last term may be interpreted as a reduction in the price coming from the willingness of the investor to take some risk on his own account since Λ_{super} puts him in the position of superreplicating the claim, that is, cover all the risk associated with the claim.

We find a price interval by letting $\gamma \to \infty$ and $\gamma \downarrow 0$:

Lemma 4.8.

$$\lim_{\gamma \to \infty} \Lambda^i_\gamma = \Lambda_{super}$$

Proof. Observe that

$$\lim_{\gamma \to \infty} \ln\left(\frac{q+\rho}{q \exp(-\gamma(\frac{dx}{u+d} + \frac{uz}{u+d} - y)) + \rho} \right) = \ln\left(\frac{q+\rho}{\rho} \right)$$

Hence, the Lemma follows. ☐

When the investor's risk aversion becomes infinite, he will charge the same price as is requested for superreplicating the claim. The optimal portfolio strategy also becomes $\pi^* = \frac{x-z}{s(u+d)}$, the superreplicating strategy, when $\gamma \to \infty$. What is interesting, is that when the risk aversion goes to zero, the price Λ does *not* tend to the lower bound y of the arbitrage-free pricing interval, but somewhere inbetween y and the upper bound Λ_{super}!

Lemma 4.9.

$$\lim_{\gamma \downarrow 0} \Lambda^i_\gamma = \frac{\rho \Lambda_{super}}{q+\rho} + \frac{qy}{q+\rho} \in \big(y, \Lambda_{super}\big)$$

where ρ is given in (4.5).

Proof. Note that $\ln\Big(\frac{q+\rho}{q\exp(-\gamma(\frac{dx}{u+d}+\frac{uz}{u+d}-y))+\rho}\Big)$ tends to zero when $\gamma \downarrow 0$. Hence, by L'Hopital's rule we find the limit given in the Lemma.

Since $\Lambda_{\text{super}} > y$ by (4.2), we get

$$\frac{\rho \Lambda_{\text{super}}}{q+\rho} + \frac{qy}{q+\rho} > \frac{\rho y}{q+\rho} + \frac{qy}{q+\rho} = y$$

Conversely, since $y < \Lambda_{\text{super}}$,

$$\frac{\rho \Lambda_{\text{super}}}{q+\rho} + \frac{qy}{q+\rho} < \frac{\rho \Lambda_{\text{super}}}{q+\rho} + \frac{q \Lambda_{\text{super}}}{q+\rho} = \Lambda_{\text{super}}$$

□

We conclude that the pricing interval for the *issuer* is

$$\Lambda^i_\gamma \in \Big(\frac{\rho \Lambda_{\text{super}}}{q+\rho} + \frac{qy}{q+\rho}, \Lambda_{\text{super}}\Big) \subset \Big(y, \Lambda_{\text{super}}\Big)$$

Let us now consider the pricing problem from the *buyer's* point of view. The buyer of the claim will pay a premium Λ^b_γ and receive the claim payoff X in the next period. His optimization problem becomes

$$V^b(w - \Lambda^b_\gamma) = 1 - \inf_\pi \mathbb{E}\big[\exp(-\gamma W_1 + \gamma X)\big] \tag{4.6}$$

Lemma 4.10. *The optimal strategy for the portfolio problem* (4.6) *is*

$$\pi^* = \frac{\ln(up) - \ln(dq)}{\gamma s(u+d)} - \frac{x-z}{s(u+d)}$$

with the optimal expected utility

$$V^b(w - \Lambda^b_\gamma) = 1 - (qe^{-\gamma y} + \rho e^{-\gamma(\frac{dx}{u+d}+\frac{uz}{u+d})}) \exp(-\gamma(w - \Lambda^b_\gamma))$$

where ρ is given in (4.5).

Proof. Let

$$J(\pi) = p\exp(-\gamma w - \gamma \pi s u - \gamma x) + q\exp(-\gamma w - \gamma y) + r\exp(-\gamma w + \gamma \pi s d - \gamma z)$$

A straightforward differentiation shows that $J'(\pi) = 0$ when $\pi = \frac{\ln(up)-\ln(dq)}{\gamma s(u+d)} - \frac{x-z}{s(u+d)}$. Inserting this into the optimal expected utility function yields $V^b(w - \Lambda^b_\gamma)$. □

Note that the buyer goes short in the superreplicating strategy. The price of the claim from the buyer's point of view will now be the solution of $V_0(w) = V^b(w - \Lambda^b_\gamma)$:

Proposition 4.11. *The utility optimization price of the claim X from the buyer's point of view is*

$$\Lambda^b_\gamma = \Lambda_{super} - \frac{1}{\gamma} \ln\Big(\frac{q\exp(\gamma(\Lambda_{super} - y)) + \rho}{q+\rho}\Big)$$

where ρ is given in (4.5).

Now taking limits when $\gamma \to \infty$ and $\gamma \downarrow 0$ yields the following:

Lemma 4.12.

$$\lim_{\gamma\to\infty}\Lambda^b_\gamma = y$$
$$\lim_{\gamma\downarrow 0}\Lambda^b_\gamma = \frac{\rho\Lambda_{super}}{q+\rho}+\frac{qy}{q+\rho}$$

where ρ is given in (4.5).

Proof. Note that $\ln\left(\frac{q\exp(\gamma(\Lambda_{\text{super}}-y)+\rho}{q+\rho}\right)$ tends to infinity when $\gamma\to\infty$. L'Hopital's rule therefore gives the first limit. When $\gamma\downarrow 0$ $\ln\left(\frac{q\exp(\gamma(\Lambda_{\text{super}}-y)+\rho}{q+\rho}\right)$ tends to 0. L'Hopital again yields the limit. □

We conclude that the pricing interval for the *buyer* is

$$\Lambda^b_\gamma \in \Big(y, \frac{\rho\Lambda_{\text{super}}}{q+\rho}+\frac{qy}{q+\rho}\Big)\subset\Big(y,\Lambda_{\text{super}}\Big)$$

As we see, the issuer and buyer are dividing the arbitrage-free pricing interval into two parts with a separating price

$$\Lambda^{i,b}_0 = \frac{\rho\Lambda_{\text{super}}}{q+\rho}+\frac{qy}{q+\rho} \tag{4.7}$$

To the left of this point we find the buyer's indifferent prices, while to the right we have the indifferent prices of the issuer.

4.2 Discussion

Nicolato and Venardos [17] find arbitrage-free pricing intervals for European claims based on an underlying following the dynamics (3.1). We expect that the utility approach to pricing will yield similar intervals, at least when we consider the pricing problem from both the issuer's and the buyer's point of view. We conjecture that the arbitrage-free pricing interval will be separated into two parts, where the left part is the prices acceptable for the buyer of the claim, while the right is the prices that the issuer will charge depending on his utility. The optimal strategy for the issuer in the case when he sells a claim can be separated into the cheapest superreplicating strategy and the optimal strategy when no claim is issued. These conjectures will be analyzed in more detail in the forthcoming paper [11].

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Smoluchowski's coagulation equation and Lévy processes *

Jean Bertoin

Roughly, Smoluchowski's coagulation equation is meant to describe the evolution in the hydrodynamic limit of a particle system in which particles coagulate pairwise as time passes. Typically, we are given a symmetric function $K :]0,\infty[\times]0,\infty[\to [0,\infty[$, called the coagulation kernel, where $K(x, y)$ specifies the rate at which two particles with respective masses x and y coagulate (that is they merge as a single particle with mass $x+y$). If we represent the density of particles with mass dx at time t by a measure $\mu_t(dx)$ on $]0,\infty[$, the dynamics of the system are thus described by the equation

$$\partial_t\langle\mu_t, f\rangle \;=\; \frac{1}{2}\int_{]0,\infty[\times]0,\infty[} (f(x+y)-f(x)-f(y))\,K(x,y)\mu_t(dx)\mu_t(dy)\,, \qquad (1)$$

where f is a generic test function, and the notation $\langle\mu_t, f\rangle$ stands for the integral of f with respect to $\mu_t(dx)$. More precisely, we may assume for simplicity unit total mass density at some given time t_0, i.e. $\int_{]0,\infty[} x\mu_{t_0}(dx) = 1$, and then by taking $f(x) = x$ in (1), we see that

$$\int_{]0,\infty[} x\mu_t(dx) = 1 \qquad \text{for all } t. \qquad (2)$$

Definition 1. *Let $\mu = (\mu_t(dx), t \in \mathbb{R})$ be a one-parameter family of measures on $]0,\infty[$ such that (2) holds. We call μ an* eternal solution *to Smoluchowski's coagulation equation with additive kernel if $K(x, y) = x + y$ and (1) is fulfilled for every continuous function $f :]0,\infty[\to\mathbb{R}$ with $\sup_{0<x<\infty}|f(x)|/x < \infty$. In that case, we simply write $\mu \in \mathcal{S}_{\text{eternal}}$.*

Our analysis of eternal solutions will heavily rely on the following.

Definition 2. *We call (σ^2, Λ) a* Lévy pair *and write $(\sigma^2, \Lambda) \in \mathcal{L}$ if $\sigma^2 \geq 0$ and $\Lambda(dx)$ is a measure on $]0,\infty[$ with*

$$\int_{]0,\infty[} (x \wedge x^2)\Lambda(dx) < \infty\,,$$

such that

$$\sigma^2 > 0 \quad \textit{or} \quad \int_{]0,\infty[} x\Lambda(dx) = \infty\,.$$

For every $(\sigma^2, \Lambda) \in \mathcal{L}$, we define for $q \geq 0$

$$\Psi_{\sigma^2,\Lambda}(q) \;=\; \frac{1}{2}\sigma^2 q^2 + \int_{]0,\infty[} \left(\mathrm{e}^{-qx} - 1 + qx\right)\Lambda(dx)\,.$$

The central result of this work is the following characterization of eternal solutions.

Theorem 1.

(i) *Let $\mu \in \mathcal{S}_{\text{eternal}}$. Then there exists a unique $(\sigma^2, \Lambda) \in \mathcal{L}$, which we call the* Lévy pair *of μ, such that*

$$\lim_{t\to-\infty} \mu_t(]\mathrm{e}^t x, \infty[) \;=\; \Lambda(]x, \infty[)$$

for every $x > 0$ which is not an atom of Λ, and

$$\lim_{x\to 0}\lim_{t\to-\infty} \int_0^x y\mu_t(]\mathrm{e}^t y, \infty[)dy \;=\; \frac{1}{2}\sigma^2\,.$$

*Based on a paper to appear in *Annals of Applied Probability*

(ii) *Conversely, take any* $(\sigma^2,\Lambda)\in\mathcal{L}$. *There exists a unique* $\mu\in\mathcal{S}_{\text{eternal}}$ *with Lévy pair* (σ^2,Λ). *More precisely, for every* $s>0$, *let* $\Phi(\cdot,s):[0,\infty[\to[0,\infty[$ *be the inverse of the bijection* $q\to\Psi_{\sigma^2,\Lambda}(sq)+q$. *Then* μ_t *is specified by the identity*

$$\Phi(q,\mathrm{e}^t)\;=\;\int_{]0,\infty[}(1-\mathrm{e}^{-qx})\mu_t(dx)\,,\qquad q\ge 0\,.$$

The general form of eternal solutions given in Theorem 1 invites a probabilistic interpretation. Indeed, the formula that defines $\Psi_{\sigma^2,\Lambda}$ is of the Lévy-Khintchine type. Specifically, $\Psi_{\sigma^2,\Lambda}$ can be viewed as the Laplace exponent of a centered Lévy processes with no positive jumps, where σ^2 is the so-called Brownian coefficient and $\Lambda(dx)$ the image of the Lévy measure by the map $x\to -x$.

This incites us to introduce a Lévy process with no positive jumps, $X=(X_r,r\ge 0)$, such that

$$E\left(\exp\left(qX_r\right)\right)\;=\;\exp\left(r\Psi_{\sigma^2,\Lambda}(q)\right)\,,\qquad q\ge 0\,.$$

The condition $(\sigma^2,\Lambda)\in\mathcal{L}$ is equivalent to assuming that the sample paths of X have unbounded variation and oscillate (i.e. $\sup_{r\ge 0}X_r=\infty$ and $\inf_{r\ge 0}X_r=-\infty$ a.s.). Similarly, the function $\Psi(\cdot,s)$ which plays a key role in the preceding section, can be viewed as another Laplace exponent; more precisely introduce for every $s>0$

$$X_r^{(s)}\;:=\;sX_r+r\,,\qquad r\ge 0$$

which is again Lévy process with no positive jumps, whose Laplace exponent given by

$$\Psi(q,s)\;=\;\Psi_{\sigma^2,\Lambda}(sq)+q\,,\qquad q\ge 0\,.$$

It is then well-known that the first passage process

$$T_x^{(s)}\;:=\;\inf\left\{r\ge 0:X_r^{(s)}>x\right\}\,,\qquad x\ge 0$$

is a subordinator, i.e. an increasing process with independent and stationary increments. More precisely, the inverse $\Phi(\cdot,s)$ of the bijection $\Psi(\cdot,s)$ is the Laplace exponent of $T^{(s)}$, in the sense that

$$E\left(\exp\left(-qT_x^{(s)}\right)\right)\;=\;\exp\left(-x\Phi(q,s)\right)\,,\qquad q,x\ge 0\,.$$

By the Lévy-Khintchine formula for $\Phi(\cdot,s)$, we conclude that the eternal solution $\mu_t(dx)$ associated with the Lévy pair (σ^2,Λ) coincides with the so-called Lévy measure of the subordinator $T^{(s)}$ for $s=\mathrm{e}^t$.

We may use each subordinator $T^{(s)}$ to construct an interesting random partition of $[0,\infty[$. To that end, introduce the closed range

$$\mathcal{T}^{(s)}\;:=\;\left\{T_x^{(s)},x\ge 0\right\}^{\mathrm{cl}}\,.$$

Because the subordinator $T^{(s)}$ has zero drift, $\mathcal{T}^{(s)}$ is a closed random set with zero Lebesgue measure. The complementary set $[0,\infty[\backslash\mathcal{T}^{(s)}$ can be expressed as the union of disjoint open intervals, which we may view as a random partition of $[0,\infty[$. We now make the key observation that

$$\mathcal{T}^{(s)}\subseteq\mathcal{T}^{(s')}\qquad\text{for } 0<s'<s, \tag{3}$$

because an instant at which $X^{(s)}$ reaches a new maximum is necessarily also an instant at which $X^{(s')}$ reaches a new maximum. Roughly, (3) means that the random partitions get coarser as the parameter s increases; and therefore they induce a process in which intervals aggregate.

As a first application, we point out that eternal solutions can be obtained from a hydrodynamic limit of this stochastic aggregation model. To give a precise statement, denote for every $r,s>0$ by $\lambda_{r,1}^{(s)}\ge\lambda_{r,2}^{(s)}\ge\cdots$ the ranked sequence of the lengths of the intervals components of $[0,r[\backslash\mathcal{T}^{(s)}$. We may think of the latter as massive particles that aggregate when time s increases. Let us now introduce the associated (re-weighted) empirical measure

$$\mu_{r,\ln s}(dx)\;=\;\frac{1}{r}\sum_{k=1}^{\infty}\delta_{\lambda_{r,k}^{(s)}}(dx)\,,$$

where $\delta_y(dx)$ stands for the Dirac point mass at y; note that the normalization has been chosen such that $\int_{]0,\infty[} x\mu_{r,t}(dx) = 1$ for all $r > 0$ and $t \in \mathbb{R}$.

Corollary 1. *With probability one,*

$$\lim_{r\to\infty} \mu_{r,t}(dx) \;=\; \mu_t(dx)$$

in the sense of vague convergence of measures on $]0,\infty[$*, where* $(\mu_t(dx), t \in \mathbb{R}) = \mu \in \mathcal{S}_{\mathrm{eternal}}$ *is the eternal solution with Lévy pair* (σ^2, Λ)*.*

As another application, we present a simple criterion that ensures the existence of a smooth density for eternal solutions. Specifically, suppose that (σ^2, Λ) is a Lévy pair such that

$$\sigma^2 > 0 \quad \text{or} \quad \liminf_{\varepsilon\to 0+} \varepsilon^{-\alpha} \int_{]0,\varepsilon[} x^2 \Lambda(dx) \;>\; 0 \qquad \text{for some } \alpha < 2. \tag{4}$$

We stress that this is a very mild assumption as it always holds that $\int_{]0,\varepsilon[} x\Lambda(dx) = \infty$. Then it is known that the one-dimensional distributions of the Lévy process X are absolutely continuous with a $\mathcal{C}^\infty$ density; more precisely, there exists a $\mathcal{C}^\infty$ function

$$\begin{aligned} p: \quad]0,\infty[\times\mathbb{R} &\longrightarrow [0,\infty[\\ (r,x) &\longrightarrow p_r(x) \end{aligned}$$

such that

$$P(X_r \in dy) \;=\; p_r(y)dy \qquad \text{for every } r > 0\,.$$

Corollary 2. *Let* $\mu \in \mathcal{S}_{\mathrm{eternal}}$ *be an eternal solution with Lévy pair* (σ^2, Λ)*; and assume that (4) is fulfilled. Then for every* $t \in \mathbb{R}$*,* $\mu_t(dx)$ *is absolutely continuous and there is a version of its density,* $n_t(x) = \mu_t(dx)/dx$ *such that* $n : (t,x) \to n_t(x)$ *is a* $\mathcal{C}^\infty$ *function. More precisely, in the notation introduced above, we have*

$$n_t(x) \;=\; \mathrm{e}^{-t} x^{-1} p_x(-x\mathrm{e}^{-t})\,, \qquad x > 0\,.$$

For instance, in the Brownian case ($\sigma^2 = 1$ and $\Lambda = 0$), we get the eternal solution

$$\mu_t(dx) \;=\; \frac{\mathrm{e}^{-t}}{\sqrt{2\pi x^3}} \exp\left(-\frac{x\mathrm{e}^{-2t}}{2}\right) dx\,, \qquad t \in \mathbb{R},\, x > 0\,.$$

In the α-stable case $\sigma^2 = 0$ and $\Lambda(dx) = cx^{-\alpha-1}dx$, where $1 < \alpha < 2$, we obtain a different eternal solution in terms of the completely asymmetric α-stable density, say

$$\rho_\alpha(y) \;=\; P(X_1 \in dy)/dy\,.$$

More precisely, the scaling property of stable processes yields the identity

$$p_r(y) \;=\; r^{-1/\alpha} \rho_\alpha(yr^{-1/\alpha})\,,$$

and hence

$$n_t(x) \;=\; \mathrm{e}^{-t} x^{-1-1/\alpha} \rho_\alpha\left(-x^{1-1/\alpha}\mathrm{e}^{-t}\right)\,.$$

It does not seem easy to check directly that indeed this provides a solution of Smoluchowski's coagulation equation.

We also observe that the formula in Corollary 2 shows that the density of the solution decays exponentially as a function of time t, and more precisely

$$n_t(x) \;\sim\; x^{-1} p_x(0)\mathrm{e}^{-t} \qquad \text{as } t \to \infty\,.$$

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Fokker–Planck equations and conservation laws involving Lévy diffusion generators

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Abstract

In this paper we report on our results concerning linear and nonlinear evolution equations in which standard Laplacian diffusion operator is replaced by an infinitesimal generator of a Lévy process. In particular, we have studied the following two cases:

We consider extensions of the classical linear Fokker–Planck equation

$$u_t + \mathcal{L}u = \nabla \cdot (u \nabla V(x)) \tag{1}$$

on $\mathbb{R}^d$ with $\mathcal{L} = -\Delta$ and $V(x) = \frac{1}{2}|x|^2$, where $\mathcal{L}$ is a general operator describing the diffusion and V is a suitable potential.

Let $-\mathcal{L}$ be the generator of a Lévy semigroup on $L^1(\mathbb{R}^n)$ and $f : \mathbb{R} \to \mathbb{R}^n$ be a nonlinearity. Nonlinear equations with nonlocal diffusive terms like

$$u_t + \mathcal{L}u + \nabla \cdot f(u) = 0 \tag{2}$$

appear as models with anomalous diffusion in continuum mechanics. We study the asymptotic behavior of solutions of these nonlocal equations as time t tends to infinity, analyzing the L^p-decay and two terms of the asymptotics of solutions. In the critical case when the diffusion and nonlinear terms are balanced, i.e. $\mathcal{L} \sim (-\Delta)^{\alpha/2}$, $1 < \alpha < 2$, $f(s) \sim s|s|^{(\alpha-1)/n}$, the solutions feature genuinely nonlinear self-similar asymptotics.

1 Generalized Fokker–Planck equations

Main results of the paper [7] include the following theorems which extend exponential convergence of solution of (1) to the corresponding steady states, discussed recently for second order elliptic operators $\mathcal{L}$ in [1] where entropy methods and logarithmic Sobolev inequalities are employed.

Define the entropy functional W

$$W[u(t)|u_\infty] \equiv W(t) = \int \Psi\left(\frac{u(x,t)}{u_\infty(x)}\right) u_\infty(x)\, dx, \tag{1.1}$$

where $u(t) = u(x,t) \ge 0$ is a solution of (1), u_∞ is a stationary solution of (1), the function Ψ is a C^2 convex function on $\mathbb{R}^+$, $\Psi(s) \ge 0$ for $s \ge 0$, and $\Psi(1) = 0$. Typical (and the most important for our purposes) examples include the *logarithmic* (or *physical*) *entropy* with $\Psi_1(s) = s\log s - s + 1$, and the *quadratic entropy* $\Psi_2(s) = (s-1)^2$.

Theorem 1.1. *Suppose that $u = u(x,t) \ge 0$ is a sufficiently regular solution of the Fokker–Planck equation* (1), *Ψ generates an entropy functional as in* (1.1), *and $u_0 \in L^1(\mathbb{R}^d)$ satisfies moreover $W[u_0|u_\infty] < \infty$ for the unique steady state u_∞ with $\int u_\infty = \int u_0$, $u_\infty \in L^1(\mathbb{R}^d)$. Then the entropy W decays monotonically to* 0*: $\lim_{t\to\infty} W(t) = 0$ and, as a consequence,*

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{L^1} = 0. \tag{1.2}$$

The equations of the form

$$u_t + \mathcal{L}u = \nabla \cdot (ux), \tag{1.3}$$

called here the *Lévy–Fokker–Planck equations*, are particular examples of (1). Moreover, they appear as the rescaled versions of equations

$$z_t + \mathcal{L}z = 0 \tag{1.4}$$

when $\mathcal{L}$ has suitable scaling properties. Indeed, if $\mathcal{L} = (-\Delta)^{\alpha/2}$, defined by the Fourier multiplier $a(\xi) = |\xi|^\alpha$, $0 < \alpha \leq 2$, corresponds to the α-stable law, then the space-time rescaling of (1.4)

$$z(x,t) = (\alpha t + 1)^{-d/\alpha} u\left(x(\alpha t + 1)^{-1/\alpha}, \alpha^{-1}\log(\alpha t + 1)\right)$$

leads exactly to (1.3).

Let $p_t^\alpha(x-y) = K_t(x,y)$ be the integral kernel of the semigroup $e^{-t(-\Delta)^{\alpha/2}}$, $\|p_t^\alpha\|_{L^q} = ct^{-d(1-1/q)/\alpha}$. Thus, the results on the *intermediate asymptotics* of $z(t)$:

$$t^{d(1-1/q)/\alpha} \left\| z(t) - \left(\int z_0\right) p_t^\alpha \right\|_{L^q} \to 0 \quad \text{as } t \to \infty$$

are equivalent to the results on the convergence of $u(t)$ to u_∞.

Remark. Our assumptions on the symbol a guarantee not only the hyper- and ultracontractivity of the semigroup $e^{-t\mathcal{L}}$, but also the decay estimates

$$\left\|e^{-t\mathcal{L}}v\right\|_{L^p} \leq C \min\left(t^{-d(1-1/p)/2}, t^{-d(1-1/p)/\alpha}\right) \|v\|_{L^1}, \tag{1.5}$$

for the semigroup $e^{-t\mathcal{L}}$ are discussed, but also the boundedness of the solutions $u \mapsto u(t)$ of (1.3) from $L^1(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.

Theorem 1.2. *The solutions of the equation* (1.3) *converge to the corresponding steady states at an exponential rate*

$$\|u(t) - u_\infty\|_{L^p} \leq Ce^{-\varepsilon t} \tag{1.6}$$

for all $1 \leq p \leq \infty$*, some* $\varepsilon = \varepsilon(p) > 0$*, and* C *which depend on* $\mathcal{L}$*,* p*, and* u_0*, resp. If* $1 \leq p \leq 2$*, some more restrictive regularity and decay assumptions on* u_0 *are needed.*

We expect that the exponential convergence in Theorem 2 can also be established in the more general context of Theorem 1, thus improving (1.2).

A study of relations between Dirichlet forms involving differences of functions (that define Lévy operators without Brownian part) and the entropy production formula (which might lead to above mentioned results) is under way.

2 Multifractal and Lévy conservation laws

Our aim is to present some results in the spirit of papers [8], [10], [11] and [9] on the asymptotics of solutions of the Cauchy problem (2) with the Lévy operator $\mathcal{L}$ defined by the symbol a represented by the Lévy–Khintchine formula, so that $e^{-t\mathcal{L}}$ is a symmetric, positivity-preserving semigroup on $L^1(\mathbb{R}^n)$. We suppose (w.l.o.g.) that the drift term in the Lévy–Khintchine formula is $b = 0$. The function $q(\xi)$ describing the diffusion part is a positive-definite quadratic form (in the wide sense) and the Lévy measure Π satisfies the usual integrability conditions.

The equation (2) can be viewed as a far reaching generalization of the Burgers equation $u_t - u_{xx} + (u^2)_x = 0$. Particular cases of (2) have been studied in our papers, see the references below. An overview of recent applications of the Burgers equation to turbulence models is in [19].

The present work is motivated by physical applications of nonlinear equations with nonlocal integro-differential or pseudodifferential diffusive terms, which include, e.g., anomalous growth models of molecular interfaces involving hopping and trapping phenomena, and hydrodynamic models with modified diffusivity. Various *linear* differential equations involving fractional derivatives, and their applications to statistical physics, hydrodynamics, molecular biology etc., have been also discussed in physical literature. We studied also connections with probability theory, stochastic differential equations and Monte Carlo-type approximations of their solutions via finite systems of interacting particles ("propagation of chaos"). The work on interpretations of the asymptotic behavior of solutions of those equations as Central Limit Theorems is in progress.

The list of the hypotheses imposed on the semigroup $e^{-t\mathcal{L}}$ include the following conditions satisfied for all $t > 0$, $1 \le p \le \infty$ and some $0 < \alpha,\ \widetilde{\alpha} < 2$

$$\|e^{-t\mathcal{L}}\|_{1,p} \le \min(c_1 t^{-n(1-1/p)/2}, c_2 t^{-n(1-1/p)/\alpha}), \tag{2.1}$$

$$\|\nabla e^{-t\mathcal{L}}\|_{1,p} \le \min(c_1 t^{-n(1-1/p)/2-1/2}, c_2 t^{-n(1-1/p)/\alpha-1/\alpha}), \tag{2.2}$$

All these assumptions are verified by, e.g., *multifractal diffusion operators*

$$\mathcal{L} = -a_0\Delta + \sum_{j=1}^{k} a_j(-\Delta)^{\alpha_j/2},\ 0 < \alpha_j < 2,\ \alpha = \min_{1\le j\le k} \alpha_j,\quad a_j > 0, \tag{2.3}$$

with $a_0 > 0$.

Our main results can be summarized as follows.

Theorem 2.3. *Assume that $f \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{L}$ is of the Lévy–Khintchine form and $e^{-t\mathcal{L}}$ satisfies estimates (2.1), (2.2) as above. Given $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, there exists a unique solution $u \in \mathcal{C}([0,\infty);\ L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of the equation (2) with $u(x,0) = u_0(x)$. This solution is regular, $u \in C((0,\infty); W^{2,2}(\mathbb{R}^n)) \cap C^1((0,\infty); L^2(\mathbb{R}^n))$, satisfies the conservation of integral property $\int u(x,t)\,dx = \int u_0(x)\,dx$, and the contraction property*

$$\|u(t)\|_{L^p} \le \|u_0\|_{L^p}$$

for each $p \in [1,\infty]$ and all $t > 0$. Moreover, the maximum and minimum principles hold: $\operatorname{ess\,inf} u_0 \le u(x,t) \le \operatorname{ess\,sup} u_0$, *a.e. x, t, as well as the comparison principle for $u_0 \le v_0 \in L^1(\mathbb{R}^n)$:*

$$u(x,t) \le v(x,t) \quad \text{a.e. } x,\ t, \text{ and } \|u(t) - v(t)\|_{L^1} \le \|u_0 - v_0\|_{L^1}.$$

The estimates of solutions of the nonlinear equation (2), which turn out to be the same as for the linear semigroup, can be proved under quite general assumptions on the decay of the semigroup, much weaker than (2.1).

Theorem 2.4.

(i) If the semigroup $e^{-t\mathcal{L}}$ verifies the estimate $\|e^{-t\mathcal{L}}\|_{1,\infty} \le m(t)$ for some decreasing C^1 function m, then positive solutions of the Cauchy problem for (2) satisfy the bound

$$\|u(t)\|_{L^2} \le m(t)^{1/2}\|u_0\|_{L^1}.$$

Moreover, if $m(t) = ct^{-\varepsilon}$ (as it is whenever (2.1) holds), then the same estimate is valid for solutions of arbitrary sign.

(ii) If $\|e^{-t\mathcal{L}}\|_{2,\infty} \le M(t)$, then $\|u(t)\|_{L^\infty} \le M(t)\|u_0\|_{L^2}$ for u_0 of arbitrary sign.

(iii) Under assumption (2.1) on $e^{-t\mathcal{L}}$ the bound

$$\|u(t)\|_{L^p} \le C_p \min(t^{-n(1-1/p)/2}, t^{-n(1-1/p)/\alpha})\|u_0\|_{L^1}$$

holds for all $1 \le p \le \infty$. Moreover, if $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$\|u(t)\|_{L^p} \le C(1+t)^{-n(1-1/p)/\alpha}$$

with a constant C which depends on $\|u_0\|_{L^1}$ and $\|u_0\|_{L^p}$.

Two consecutive terms of asymptotics of solutions of (2) are described in next two theorems.

Theorem 2.5. *Assume that u is a solution of the Cauchy problem* (2) *with $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $e^{-t\mathcal{L}}$ satisfies* (2.1)–(2.2) *with some $0 < \alpha < 2$. Furthermore, suppose that $f \in C^1$, $\limsup_{s\to 0} |f(s)|/|s|^r < \infty$ for some $r > \max((\alpha-1)/n+1, 1)$. Then, for every $1 \le p \le \infty$ the relation*

$$t^{n(1-1/p)/\alpha}\|u(t) - e^{-t\mathcal{L}}u_0\|_{L^p} \to 0$$

as $t \to \infty$ holds.

Theorem 2.6. *Let the semigroup $e^{-t\mathcal{L}}$ satisfy* (2.1)–(2.2)*, the symbol a of $\mathcal{L}$ satisfy some regularity assumptions off the origin, and $f \in C^2$, $f'(0) = 0$. If $n = 1$ and $\alpha \ge 1$, suppose moreover that $f \in C^3$, $f''(0) = 0$. Then for each $1 < p \le \infty$ the limit relation*

$$t^{n(1-1/p)/\alpha+1/\alpha}\|u(t) - e^{-t\mathcal{L}}u_0 + F\cdot(\nabla e^{-t\mathcal{L}}\delta_0)\|_{L^p} \to 0$$

as $t \to \infty$ holds with $F = \int_0^\infty \int_{\mathbb{R}^n} f(u(y,\tau))\,dy\,d\tau$.

The solutions of (2) with a multifractal operator $\mathcal{L}$ and f satisfying the condition

$$\lim_{s\to 0} f(s)/(s|s|^{(\alpha-1)/n}) \in \mathbb{R}$$

behave asymptotically like self-similar solutions U of the fractal Burgers equation

$$u_t + (-\Delta)^{\alpha/2}u + c\cdot\nabla(u|u|^{r-1}) = 0, \quad c \in \mathbb{R}^n, \tag{2.4}$$

with singular initial data $M\delta_0$, i.e., the, so-called, *source solutions*. Note that here u_0 is not necessarily positive, while positivity of U is a subtle consequence of (2.4) and $M > 0$. More generally, we have

Theorem 2.7. *Let u be a solution of the Cauchy problem* (2) *with the operator $\mathcal{L} = (-\Delta)^{\alpha/2} + \mathcal{K}$ for some $1 < \alpha < 2$, and another Lévy operator $\mathcal{K}$ whose symbol k fulfills $\lim_{\xi\to 0} k(\xi)/|\xi|^\alpha = 0$ (in particular, $\mathcal{L}$ can be a multifractal operator of the form* (2.3) *with $a_0 \ge 0$, $1 < \alpha_j < 2$), and $u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u_0(x)\,dx = M > 0$. Then, for each $1 \le p \le \infty$,*

$$t^{n(1-1/p)/\alpha}\|u(t) - U(t)\|_{L^p} \to 0$$

as $t \to \infty$, where $U = U_M$ is the unique solution of the problem (2.4) *with the initial data $M\delta_0$. Moreover, U is of self-similar form $U(x,t) = t^{-n/\alpha}U(xt^{-1/\alpha}, 1)$, $\int_{\mathbb{R}^n} U(x,1)\,dx = M$, and $U \ge 0$.*

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Tail probabilities of subadditive functionals acting on Lévy processes

Michael Braverman

1 Introduction

This talk is based on the joint work with T. Mikosch and G. Samorodnitsky (see References).

Both in the theory and in applications of stochastic processes one is often interested in two types of questions: *When does the process* $\mathbf{X} = \{X(t),\, t \geq 0\}$ *lie above a certain deterministic function (curve)* $\boldsymbol{\mu} = \{\mu(t),\, t \geq 0\}$*, and given the process exceeds this curve, what are its values?* For example, what can be said about the distribution of the biggest excess of the process over the curve and, if both the process and the function are measurable, what is the distribution of the time the process spends above the curve?

Here we outline a general approach to the asymptotic tail behavior of the distributions of these and other subadditive functionals acting on Lévy processes with "not too light" tails. We consider the distributional tails of various subadditive functionals of their paths. These examples will show in detail how successfully this method works and how general it is.

Let $\mathbf{X}$ be an infinitely divisible process without Gaussian component and Lévy measure ν. Following [8], the distribution of $\mathbf{X}$ is characterized as follows:

$$Ee^{i\langle \boldsymbol{\beta},\mathbf{X}\rangle} = \exp\left\{\int_{\mathbb{R}^{[0,\infty)}} \left(e^{i\langle\boldsymbol{\beta},\boldsymbol{\alpha}\rangle} - 1 - i\langle\boldsymbol{\beta},\boldsymbol{\tau}(\boldsymbol{\alpha})\rangle\right)\nu(d\boldsymbol{\alpha})\right\}\,, \quad \boldsymbol{\beta} \in \mathbb{R}^{([0,\infty))}\,.$$

Here ν is the projective limit of the Lévy measures corresponding to the finite dimensional distributions of $\mathbf{X}$. The symbol $\mathbb{R}^{([0,\infty))}$ denotes the space of real functions $\boldsymbol{\beta}$ defined on $[0,\infty)$ such that $\beta(t) = 0$ for all but finitely many t, and $\langle\boldsymbol{\beta},\boldsymbol{\alpha}\rangle = \Sigma_{t\in[0,\infty)}\beta(t)\alpha(t)$. Finally, $\tau(\boldsymbol{\alpha})(t) = \alpha(t)\mathbf{1}(|\alpha(t)| \leq 1)$.

Some examples of the measurable functionals $\phi : \mathbb{R}^{[0,\infty)} \to (-\infty,\infty]$ on $\mathbf{X}$ we consider are

$$\phi_{\sup}(\boldsymbol{\alpha}) = \sup_{t\geq 0}\,[\alpha(t)]_+\,, \quad \phi(\boldsymbol{\alpha}) = \sup\{t > 0 :\, \alpha(t) > 0\}\,, \quad \phi(\boldsymbol{\alpha}) = \int_0^\infty [\alpha(s)]_+^p\, ds\,, \tag{1.1}$$

where $y_+ = \max(0,y)$ and $p \in (0,1]$. The supremum functional $\phi_{\sup}$ has gained particular importance in the context of queuing and insurance, where one is interested in quantitative measures for the excesses of $\mathbf{X}$ over high level thresholds which event is interpreted as buffer overflow or ruin in the different contexts. The above functionals have in common that they are *subadditive*, i.e., for any $\boldsymbol{\alpha}_1,\, \boldsymbol{\alpha}_2 \in \mathbb{R}^{[0,\infty)}$,

$$\phi(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) \leq \phi(\boldsymbol{\alpha}_1) + \phi(\boldsymbol{\alpha}_2)\,.$$

If, with probability 1, $\phi(\mathbf{X} - \boldsymbol{\mu}) < \infty$ is finite, it makes sense to measure the thickness of the distributional tail $P(\phi(\mathbf{X} - \boldsymbol{\mu}) > u)$ for large u. Suppose this tail does not decay "too fast" as $u \to \infty$ and define

$$\psi(u) = \nu(\{\boldsymbol{\alpha} :\, \phi(\boldsymbol{\alpha} - \boldsymbol{\mu}) > u\})\,. \tag{1.2}$$

The subadditivity of the functional ϕ, the presence of heavy tails and the logic of large deviations saying that unlikely events happen in the most likely way, lead one to the conjecture that $\psi(u)$ and $P(\phi(\mathbf{X} - \boldsymbol{\mu}) > u)$ are *equivalent* in the following sense:

$$\lim_{u\to\infty} \frac{P(\phi(\mathbf{X} - \boldsymbol{\mu}) > u)}{\psi(u)} = 1\,. \tag{1.3}$$

Indeed, relations of type (1.3) were proved in the theory of laws with so-called subexponential tails. For example, [4] considered the overall supremum of Lévy processes, and [14] studied very general subadditive functionals.

The setup in the latter paper is, in fact, close to the present one. However, there is one crucial difference: the functionals in [14] were assumed to be bounded by an almost surely finite pseudonorm of the process. Hence these processes are, in a certain sense, bounded “from above and below”. This assumption is far away from the situation in the present paper. Our functionals are akin to the supremum of a negative drift random walk over the entire infinite horizon. In this sense, they are bounded “only from one side”.

The validity of relation (1.3) has been established for the overall supremum functional $\phi_{\sup}$ and some particular classes of processes with subexponential tails. Those include Lévy processes with a negative linear drift (see [6]) and symmetric α-stable processes, $\alpha \in (1,2)$, with stationary ergodic increments and negative linear drift. In general, the precise circumstances under which (1.3) is valid for subadditive functionals are not known, even in the particular case of Lévy processes. The presented results provide a further step in the process of understanding the tail equivalence relation (1.3) for heavy tailed processes.

In related work [7] considered the tail behavior of the supremum functional $\phi_{\sup}$ of certain Gaussian processes, including fractional Brownian motion, with negative (not necessarily linear) drift. The Gaussian nature of the underlying process causes exponential decay of the tails $P(\phi_{\sup}(\mathbf{X} - \boldsymbol{\mu}) > u)$.

2 Assumptions and notation

We denote by C a generic positive constant. Its value will be allowed to change from appearance to appearance, even if we do not mention it explicitly.

Let $\mathbf{X} = \{X(t),\, t \geq 0\}$ be a Lévy process, i.e., a real-valued process with stationary and independent increments, and Lévy measure ρ on $\mathbb{R}$. We refer the reader to [1] and [15] for encyclopedic treatments of Lévy processes. In particular, one can find detailed proofs of the properties we mention and use below.

Specifically, the marginal distributions of a Lévy process are determined by the characteristic function

$$Ee^{i\theta X(1)} = \exp\left\{\int_{-\infty}^{\infty} \left(e^{i\theta x} - 1 - i\theta x\mathbf{1}(|x| \leq 1)\right) \rho(dx)\right\}, \quad \theta \in \mathbb{R}. \tag{2.1}$$

We always take a version of $\mathbf{X}$ with all sample paths in the Skorokhod space $\mathbb{D}[0,\infty)$, i.e., with paths which are right-continuous at every $t \geq 0$ and have left limits at every $t > 0$. This version of $\mathbf{X}$ is automatically measurable; this feature will become useful as we will have many opportunities to integrate the sample paths of $\mathbf{X}$.

The Lévy measure ν of the process $\mathbf{X}$ has the form

$$\nu(A) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in A\right) \rho(dx)\, ds, \tag{2.2}$$

for any measurable set $A \subset \mathbb{R}^{[0,\infty)}$. Therefore the function ψ in (1.2) turns into

$$\psi(u) = \int_0^{\infty}\int_{-\infty}^{\infty} \mathbf{1}\left(\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > u\right) \rho(dx)\, ds, \quad u > 0. \tag{2.3}$$

We denote the right tail of the one dimensional Lévy measure ρ by

$$H(u) = \rho\big([u,\infty)\big), \quad u > 0.$$

A few comments on the conditions below. The reader should realize that the number of conditions we had to assume is due to our desire to cover the largest possible number of functionals and processes. The conditions simplify drastically in the special cases of Section 4.

Assumptions on the Lévy measure ρ

Dominance of the right tail of the Lévy measure
We assume that the right tail of the one dimensional Lévy measure ρ dominates its left tail in the sense that there is a constant $A_1 > 0$ such that

$$\rho((-\infty\,, -t]) \leq A_1\ \rho([t, \infty)) \quad \text{for all } t \geq 1\,. \tag{2.4}$$

Δ_2 condition
There is $a_1 > 0$ such that

$$H(2u) \geq a_1 H(u) \quad \text{for all } u \geq 1\,. \tag{2.5}$$

Notice that the Δ_2 condition on H yield a bound from below; it excludes exponential decay of $H(u)$.

Bound from above

There is $\beta_1 > 0$ such that

$$H(u) = o(u^{-\beta_1})\,, \quad u \to \infty\,. \tag{2.6}$$

Assumptions on the drift $\boldsymbol{\mu}$

Let $\boldsymbol{\mu} = \{\mu(t),\, t \geq 0\}$ be a nonnegative function satisfying the following assumptions.

Power law bound from below

There are $a_2 > 0$ and $\beta_2 > \max(\beta_1^{-1}, 0.5)$ such that

$$\mu(t) \geq a_2\, t^{\beta_2}, \quad t > 0\,. \tag{2.7}$$

Δ_2 condition

There is an $A_2 > 0$ and $t_0 \geq 0$ such that

$$\mu(2t) \leq\ A_2\, \mu(t) \quad \text{for all } t \geq t_0. \tag{2.8}$$

The Δ_2 condition on $\boldsymbol{\mu}$ excludes too fast (in particular exponential) growth of $\boldsymbol{\mu}$.

Quasi–monotonicity of $\boldsymbol{\mu}$

There is an $a_3 \in (0, 1]$ and $t_0 \geq 0$ such that

$$\inf_{s \geq t} \mu(s) \geq a_3\, \mu(t) \quad \text{for all } t \geq t_0. \tag{2.9}$$

Assumptions on the subadditive functional ϕ

Let $\phi : \mathbb{R}^{[0,\infty)} \to [0, \infty]$ be a measurable subadditive functional satisfying the following conditions.

The functional "lives off only positive values of its argument"

This means that

$$\phi(\mathbf{0}) = 0, \quad \text{and if} \quad \alpha(t) \leq 0 \quad \text{for all } t > t_0,\ \text{some } t_0,\ \text{then} \quad \phi(\boldsymbol{\alpha}) = \phi\left(\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]}\right)\,. \tag{2.10}$$

Here $\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]} = \{\alpha(t)\mathbf{1}_{[0,t_0]}(t),\, t \geq 0\}$.

The functional is finite on locally bounded functions that are eventually non-positive

This means that

$$\phi(\boldsymbol{\alpha}) = \phi\left(\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]}\right) < \infty \quad \text{if } \alpha(t) \leq 0 \text{ for all } t > t_0, \text{ some } t_0, \text{ and } \sup_{t \leq t_0} \alpha(t) < \infty\,. \tag{2.11}$$

Monotonicity

This means that

$$\text{if} \quad \alpha(t) \le \beta(t) \quad \text{for all } t \quad \text{then} \quad \phi(\boldsymbol{\alpha}) \le \phi(\boldsymbol{\beta}) \tag{2.12}$$

and

$$\phi\left(c\boldsymbol{\alpha}\right) \le \phi(\boldsymbol{\alpha}) \quad \text{for all } c \in [0,1] \text{ and } \boldsymbol{\alpha} \in \mathbb{R}^{[0,\infty)}\,. \tag{2.13}$$

Notice that (2.13) is implied by (2.12) if $\alpha(t) \ge 0$ for all $t \ge 0$.

Assumptions involving the triple $(\rho, \phi, \boldsymbol{\mu})$

One can easily give separate sufficient conditions for the assumptions below, i.e., conditions which involve ρ, ϕ and $\boldsymbol{\mu}$ separately. However, when doing so one gets into more restrictive situations. The assumptions we impose are easy to check in applications. Therefore we have chosen to formulate them in the present form.

For $s \ge 0$ and $u > 0$ define

$$T(s,u) = \inf\{x > 0 : \ \phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > u\}\,, \tag{2.14}$$

and denote

$$T(u) = T(0,u)\,.$$

Relation between $T(s,u)$ and $T(u)$

There is $A_3 > 0$ such that

$$T(s,u) \le A_3\left[\mu(s) + T(u)\right] \ \text{ for all } s,u > 0\,. \tag{2.15}$$

A scaling property

There are positive functions $g(\delta)$ and $h(\delta)$, $0 < \delta \le 1$, satisfying

$$h(\delta) \to 1 \ \text{ as } \delta \uparrow 1, \quad |\log(g(\delta))| \le O(\delta^{-1}) \ \text{ as } \delta \downarrow 0 \tag{2.16}$$

and such that for every $u > u(\delta)$ and $0 < \delta \le 1$

$$\int_0^\infty H\left(\delta T(s,\delta u)\right)\,ds \le h(\delta)\int_0^\infty H\left(T(s,u)\right)\,ds\,, \tag{2.17}$$

and for every $u \ge u_0$ and $0 < \delta \le 1$

$$\int_0^\infty H\left(\delta T(s,\delta u)\right)\,ds \le g(\delta)\int_0^\infty H\left(T(s,u)\right)\,ds\,. \tag{2.18}$$

The latter conditions are easily checked if one assumes appropriate regular variation conditions.

How to verify condition (2.15)?

Here is an easily verifiable sufficient condition for (2.15).

Proposition 2.1. *Assume that the following conditions hold:*

1. *The subadditive functional ϕ satisfies* (2.10) – (2.13).

2. *There exists $\gamma > 0$ such that for all* $0 < c < 1$

$$\phi\left(cx\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \le c^\gamma \phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right)\,, \tag{2.19}$$

3. *There exists $a > 0$ such that for every $s, x > 0$*

$$\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s\right) \geq \phi\left(ax\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right), \tag{2.20}$$

where $\mu_s(t) = \mu\left((t-s)_+\right)$.

4. *$\boldsymbol{\mu}$ is nondecreasing.*

Then (2.15) *holds.*

In fact, condition

$$\phi(c\boldsymbol{\alpha}) \leq c^\gamma \phi(\boldsymbol{\alpha}) \text{ for every } 0 < c < 1, \tag{2.21}$$

implies, and is more restrictive, than (2.13) and (2.19). Indeed, if (2.21) holds, monotonicity of ϕ implies

$$\phi\left(cx\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) = \phi\left(c\left(x\mathbf{1}_{[0,\infty)} - c^{-1}\boldsymbol{\mu}\right)\right) \leq c^\gamma \phi\left(x\mathbf{1}_{[0,\infty)} - c^{-1}\boldsymbol{\mu}\right) \leq c^\gamma \phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right).$$

Moreover, many of the functionals of interest have the property

$$\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s\right) = \phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right). \tag{2.22}$$

which implies (2.20).

3 The main theorem

Here we give our main result which was announced in Section 1. The proof is quite technical, and therefore is omitted.

First recall the definition of the quantity $\psi(u)$ from (2.3).

Theorem 3.2. *Let $\mathbf{X}$ be a Lévy process, $\boldsymbol{\mu}$ a deterministic function and ϕ a subadditive measurable functional satisfying the Assumptions of Section 2. If $\psi(u)$ is regularly varying (at infinity) with exponent $-\alpha < 0$, then $\psi(u)$ and $P(\phi(\mathbf{X} - \boldsymbol{\mu}) > u)$ are equivalent:*

$$\lim_{u\to\infty} \frac{P(\phi(\mathbf{X} - \boldsymbol{\mu}) > u)}{\psi(u)} = 1. \tag{3.1}$$

At this point, the large variety of assumptions on $\mathbf{X}$, ϕ, $\boldsymbol{\mu}$ and ψ may look quite restrictive and difficult to verify. We will, however, see in Section 4 that these assumptions hold under very natural conditions for various important subadditive functionals.

4 Some examples of subadditive functionals

In this section we consider several important and common subadditive functionals ϕ acting on Lévy processes. We apply Theorem 3.2 to characterize the tail behavior of the distribution of these functionals.

Throughout this section we assume that the following assumptions hold.

$$H \text{ is regularly varying with exponent } -\alpha \text{ for some } \alpha > 0, \tag{4.1}$$

there is a constant $C > 0$ such that

$$\rho((-\infty, -t]) \leq C\,\rho([t,\infty)) \text{ for all } t \geq 1 \tag{4.2}$$

and

$$\mu \text{ is regularly varying with exponent } \beta \text{ for some } \beta > \max(\alpha^{-1}, 0.5)\ . \tag{4.3}$$

Of course, the assumption (4.2) is the same as (2.4). Since it is our goal to collect all the relevant assumptions in this section together for easy reference, this assumption is repeated here. The following lemma collects several well known facts on regular varying functions. The reader is referred to [2]) for proofs and more information. Let

$$\mu^{\leftarrow}(u) = \sup\{t > 0 : \mu(t) \leq u\},\ u > 0 \tag{4.4}$$

be the generalized inverse of μ.

4.1 The overall supremum

One of the interesting subadditive functionals is the overall supremum

$$\phi_{\sup}(\boldsymbol{\alpha}) = \sup_{t\geq 0} \alpha(t)\,.$$

It has numerous applications. among them in insurance mathematics for describing eventual ruin (see [5]) or in queuing for the buffer overflow (see [11]).

Remark 4.3. Here we deal with "power–like" tails and, hence, the following theorem that describes the tail behavior of the distribution of the overall supremum of a Lévy process is stated under the assumptions of regular variation. We conjecture, however, that the first asymptotic equivalence in (4.5) below holds in greater generality, perhaps under the assumption of subexponentiality of the tail of H. In fact, if $\alpha > 1$ and $\mu(t) = \mu t$ for some $\mu > 0$, is a linear function, then the first asymptotic equivalence in (4.5) is just the classical result for the ruin probability as proved by [6]:

$$P\left(\sup_{t\geq 0}\left(X(t) - \mu(t)\right) > u\right) \sim \frac{1}{\mu}\int_u^\infty H(s)\,ds\,,$$

and the latter result is known to hold when H has a subexponential right tail.

In fact, it is quite possible that the curve $\boldsymbol{\mu}$ may be allowed to belong to a wider class of functions as well.

Theorem 4.4. *Assume* (4.1)–(4.3). *Then*

$$\begin{aligned} P\left(\phi_{\sup}(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\sup_{t\geq 0}\left(X(t) - \mu(t)\right) > u\right) && (4.5)\\ &\sim \int_0^\infty H(\mu(s) + u))\,ds \sim C(\alpha,\beta)\mu^{\leftarrow}(u)H(u) \end{aligned}$$

as $u \to \infty$. *Here* $C(\alpha,\beta) = \alpha\int_0^\infty z^{1/\beta}(1+z)^{-(1+\alpha)}\,dz$.

4.2 The time the process spends above zero

In this section we consider the sojourn time

$$\phi_{\text{sojourn}}(\boldsymbol{\alpha}) = \int_0^\infty \mathbf{1}\left(\alpha(t) > 0\right)\,dt\,,$$

which is easily seen to be a subadditive functional.

Theorem 4.5. *Assume* (4.1)–(4.3). *Then*

$$\begin{aligned} P\left(\phi_{\text{sojourn}}(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\int_0^\infty \mathbf{1}\left(X(t) - \mu(t) > 0\right)dt > u\right)\\ &\sim \int_u^\infty H(\mu(s))\,ds \sim C(\alpha,\beta)\;u\;H(\mu(u)) && (4.6) \end{aligned}$$

as $u \to \infty$. *Here* $C(\alpha,\beta) = (\alpha\beta - 1)^{-1}$.

4.3 The last hitting time of zero

In this section we consider the functional

$$\phi_{\text{last}}(\boldsymbol{\alpha}) = \sup\{t > 0 : \alpha(t) \geq 0\}\,.$$

It is not difficult to see that this functional is subadditive.

Theorem 4.6. *Assume* (4.1)–(4.3). *Then*

$$\begin{aligned} P\left(\phi_{\text{last}}(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\sup\{t > 0 : X(t) \geq \mu(t)\} > u\right)\\ &\sim uH(\mu(u)) + \int_u^\infty H(\mu(s))\,ds \sim C(\alpha,\beta)\;u\;H(\mu(u)) && (4.7) \end{aligned}$$

as $u \to \infty$. *Here* $C(\alpha,\beta) = 1 + (\alpha\beta - 1)^{-1}$.

4.4 Integral of a nonnegative subadditive function

The functional ϕ_{sojourn} of Theorem 4.5 is a particular case of a more general group of subadditive functionals obtained by appropriate space–dependent weighting of the positive values of a process. Consider a nondecreasing nonnegative function f such that $f(x) = 0$ for $x \le 0$ and

$$f(x_1 + x_2) \le f(x_1) + f(x_2) \quad \text{for } x_1, x_2 > 0,$$

and let

$$\phi_{I(f)}(\boldsymbol{\alpha}) = \int_0^\infty f(\alpha(t))\, dt\,. \tag{4.8}$$

It is clear that $\phi_{I(f)}$ is a subadditive functional. We will not address here the question what functionals $\phi_{I(f)}$ fit in the framework of the theory developed in the present paper. Instead, we will briefly consider the class of functionals corresponding to the power functions

$$f(x) = [x_+]^p,\ 0 \le p \le 1\,. \tag{4.9}$$

We will denote the corresponding functional by $\phi_p(\boldsymbol{\alpha})$. The case $p = 0$ corresponds to the functional ϕ_{sojourn}.

The tail behavior of the distribution of the functional $\phi_p(\boldsymbol{\alpha})$ is described in the following theorem.

Theorem 4.7. *Assume* (4.1)–(4.3)*. Then for every* $0 < p \le 1$

$$\begin{aligned} P\left(\phi_p(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\int_0^\infty [X(t) - \mu(t)]_+^p\, dt > u\right) \\ &\sim C(\alpha, \beta, p)\ u\ \left(F^{\leftarrow}(u)\right)^{-p} H\left(F^{\leftarrow}(u)\right) \end{aligned} \tag{4.10}$$

as $u \to \infty$*. Here*

$$F(x) = x^p \mu^{\leftarrow}(x),\ x > 0\,,$$

and $C(\alpha, \beta, p)$ *is a finite positive constant given by*

$$C(\alpha, \beta, p) = \int_0^\infty y(t)^{-\alpha} t^{-\alpha\beta}\, dt\,,$$

where $y(t) = h^{-1}(t^{-(1+p\beta)})$*,* $t > 0$*, and* h *is a strictly increasing continuous function on* $[1, \infty)$ *given by*

$$h(y) = py^p \int_{1/y}^1 \left((yz)^\beta - 1\right) (z-1)^{p-1}\, dt\,.$$

4.5 The supremum of the integral of the process

Here we consider the subadditive functional

$$\phi_{\text{supint}}(\boldsymbol{\alpha}) = \sup_{v>0} \int_0^v \alpha(t)\ dt\,.$$

Unlike other functionals considered in this section, this functional is affected by the negative values of the process. The tail behavior of this functional is described in the theorem below.

Theorem 4.8. *Assume* (4.1)–(4.3)*. Then*

$$\begin{aligned} P\left(\phi_{\text{supint}}(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\sup_{v\ge 0} \int_0^v (X(t) - \mu(t))\ dt > u\right) \\ &\sim C(\alpha, \beta)\ \mu_1^{\leftarrow}(u)\ H\left(\frac{u}{\mu_1^{\leftarrow}(u)}\right) \end{aligned} \tag{4.11}$$

as $u \to \infty$*. Here*

$$\mu_1(x) = \int_0^x \mu(y)\, dy, \quad x > 0\,,$$

and $C(\alpha,\beta)$ is a finite positive constant given by

$$C(\alpha,\beta) = \int_0^\infty y(t)^{-\alpha} t^{-\alpha}\, dt\,,$$

where $y(t) = h^{-1}(4^\beta(1+\beta)t^{1+\beta})$, $t > 0$, and h is a strictly increasing continuous function on $[0,\infty)$ given by

$$h(y) = \frac{y^{1+\beta}}{(1+y)^\beta}\,.$$

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LÉVY-DRIVEN CONTINUOUS-TIME ARMA PROCESSES WITH FINANCIAL APPLICATIONS

PETER J. BROCKWELL

1 Gaussian CARMA Processes

Gaussian continuous-time ARMA (CARMA) processes furnish a useful class of stationary time series models for dealing with irregularly spaced data (see Jones (1981), Jones and Ackerson (1990) and Doob (1944), where some of their properties can be found). Nonlinear versions have been found to give good models for returns on stock market indices (see e.g. Brockwell (2000)). A zero-mean Gaussian CARMA(p, q) process $\{Y(t)\}$ with $0 \leq q < p$ and coefficients $a_1, \ldots, a_p$, $b_0, \ldots, b_q$, is defined (see e.g. Brockwell and Davis (1996)) to be a stationary solution of the (suitably interpreted) pth order linear differential equation,

$$a(D)Y(t) = b(D)DW(t), \ t \geq 0, \tag{1.1}$$

where D denotes differentiation with respect to t, $\{W(t)\}$ is standard Brownian motion,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$
$$b(z) := b_0 + b_1 z + \cdots + b_{p-1} z^{p-1},$$

and the coefficients b_j satisfy $b_q \neq 0$ and $b_j = 0$ for $q < j < p$. Since the derivatives $D^j W(t)$ do not exist in the usual sense, we interpret (1.1) as being equivalent to the *observation* and *state* equations,

$$Y(t) = \mathbf{b}'\mathbf{X}(t), \tag{1.2}$$

and

$$d\mathbf{X}(t) - A\mathbf{X}(t)dt = \mathbf{e}\ dW(t), \tag{1.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}$$

and we assume that $\mathbf{X}(0)$ is a Gaussian random vector such that

$$\mathbf{X}(0) \text{ is independent of } \{W(t), t \geq 0\}. \tag{1.4}$$

The state equation (1.3) is an Ito differential equation for $\mathbf{X}(t)$. If $p = 1$, A is defined to be $-a_1$. Because of the linearity of (1.3), its solution has the simple form,

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-u)}\mathbf{e}\ dW(u), \tag{1.5}$$

where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums. The process $\{\mathbf{X}(u), u \geq 0\}$ also satisfies the relations,

$$\mathbf{X}(t) = e^{A(t-s)}\mathbf{X}(s) + \int_s^t e^{A(t-u)}\mathbf{e}\ dW(u), \quad \text{for all } t > s \geq 0, \tag{1.6}$$

which clearly show (by the independence of increments of $\{W(t)\}$) that $\{\mathbf{X}(u)\}$ is Markov.

It is well-known (see e.g. Brockwell (2001a)) that the equations (1.4) and (1.6) have a weakly stationary solution if and only if the eigenvalues $\lambda_1, \ldots, \lambda_p$ of A (which are the same as the zeroes of the autoregressive polynomial $z^p + a_1 z^{p-1} + \cdots + a_p$) all have negative real parts, i.e if and only if

$$\Re(\lambda_i) < 0, \; i = 1, \ldots, p. \tag{1.7}$$

If $\{\mathbf{X}(t)\}$ is such a solution then it is easy to show that

$$E(\mathbf{X}(0)) = \mathbf{0} \tag{1.8}$$

and

$$E(\mathbf{X}(0)\mathbf{X}'(0)) = \Sigma := \int_0^\infty e^{Ay}\mathbf{e}\,\mathbf{e}'e^{A'y}dy. \tag{1.9}$$

Conversely if (1.4), (1.7), (1.8) and (1.9) are satisfied, then the process $\{\mathbf{X}(t)\}$ defined by (1.5) is weakly stationary and satisfies the relations,

$$E[\mathbf{X}(t)] = \mathbf{0}, \; t \geq 0,$$

and

$$E[\mathbf{X}(t+h)\mathbf{X}(t)'] = e^{Ah}\Sigma, \; h \geq 0.$$

From (1.2) the mean and autocovariance function of the CARMA(p,q) process $\{Y(t)\}$ are then given by

$$E[Y(t)] = 0, \; t \geq 0$$

and

$$\gamma_Y(h) = E[Y(t+h)Y(t)] = \mathbf{b}'\, e^{A|h|}\Sigma\, \mathbf{b}. \tag{1.10}$$

If in addition the zeroes of the autoregressive polynomial are all distinct then the autocovariance function of $\{Y(t)\}$ has the simple form (see Brockwell (2001a)),

$$\gamma_Y(h) = \sum_{\lambda: a(\lambda)=0} \frac{e^{\lambda|h|}b(\lambda)b(-\lambda)}{a'(\lambda)a(-\lambda)}. \tag{1.11}$$

If $\mathbf{X}(0)$ satisfies (1.4), (1.8) and (1.9) and is also Gaussian, then $\{\mathbf{X}(t)\}$ and $\{Y(t)\}$ are strictly stationary and Gaussian.

For the process to be *minimum phase* the roots of $1 + b_1 z + \cdots + b_q z^q = 0$ must have negative real parts. (This corresponds to *invertibility* for discrete time ARMA processes.)

In order to define a Gaussian CARMA process indexed by $(-\infty, \infty)$, we introduce two independent standard Brownian motions $\{W^+(t), 0 \leq t < \infty\}$ and $\{W^-(t), 0 \leq t < \infty\}$ (with $W^+(0) = W^-(0) = 0$), letting

$$W(t) = W^+(t)I_{[0,\infty)}(t) - W^-(-t)I_{(-\infty,0]}(t), \; -\infty < t < \infty, \tag{1.12}$$

and defining, for functions f on $(-\infty, \infty)$ which are square integrable with respect to Lebesgue measure,

$$\int_{-\infty}^t f(u)dW(u) = \begin{cases} \int_0^\infty f(-u)dW^-(u) - \int_0^{-t} f(-u)dW^-(u), & \text{if } t < 0, \\ \int_0^\infty f(-u)dW^-(u) + \int_0^t f(u)dW^+(u), & \text{if } t \geq 0. \end{cases}$$

Provided the eigenvalues of A all have negative real parts, the process $\{\mathbf{X}(t)\}$ defined by

$$\mathbf{X}(t) = \int_{-\infty}^t e^{A(t-u)}\mathbf{e}\; dW(u),$$

is then a strictly stationary solution of (1.3) for $t \in (-\infty, \infty)$ with corresponding CARMA process,

$$Y(t) = \int_{-\infty}^t \mathbf{b}'e^{A(t-u)}\mathbf{e}\; dW(u), \; -\infty < t < \infty. \tag{1.13}$$

If the eigenvalues, $\lambda_1, \ldots, \lambda_p$, of A are distinct, the corresponding eigenvectors can be written down, and the spectral representation of the matrix A then gives the explicit expression,

$$\mathbf{b}' e^{Au}\mathbf{e} = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u}. \tag{1.14}$$

If we now define

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu\lambda} \frac{b(i\lambda)}{a(i\lambda)}\, d\lambda, \tag{1.15}$$

the change of variable $z = i\lambda$ and a simple contour integration with respect to z shows that

$$g(u) = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u} I_{(-\infty,0)}(\Re(\lambda_r u)),$$

so that the representation (1.12) of the CARMA process can be rewritten as

$$Y(t) = \int_{-\infty}^{\infty} g(t-u) dW(u). \tag{1.16}$$

The case of repeated eigenvalues can be obtained from that of distinct eigenvalues by letting a group of distinct eigenvalues approach a common limit. The continuity of $a(\cdot)$ as a function of its zeroes means that the representation defined by (1.15) and (1.16) remains valid when there are repeated eigenvalues (although (1.14) does not).

Note. The above calculations were all performed under the assumption that the real parts of the eigenvalues of A are all (strictly) negative. However the process defined by (1.15) and (1.16) is a strictly stationary solution of the CARMA equations for $t \in (-\infty, \infty)$, *even when one or more of the eigenvalues has strictly positive real part.* If all the eigenvalues of A have negative real part the process is **causal**, i.e. $g(u) = 0$ for $u < 0$. If all the eigenvalues have positive real part then $g(u) = 0$ for $u > 0$. If some have positive and some have negative real parts, then $g(u)$ is non-zero for all u. This classification is analogous to the classification of discrete-time ARMA processes as causal or otherwise, depending on whether or not the zeroes of the autoregressive polynomial lie outside the unit circle (see e.g. Brockwell and Davis (1996)).

2 Lévy-driven CARMA Processes

In order to generate processes with the heavy tailed marginal distributions frequently observed in time series data, the Gaussian CARMA processes defined in Section 1 can be extended to the class of Lévy-driven CARMA processes by replacing the standard Brownian motion $\{W(t)\}$ in Section 1 by a Lévy process. Lévy processes are processes continuous in probability with stationary independent increments. Brownian motion is a special case. For more information on Lévy processes see Ito (1969), Bertoin (1996) and Sato (1999). An excellent account of stochastic integration with respect to Lévy processes is contained in the book of Protter (1991). If $\{W(t)\}$ is a Lévy process with $W(0) = 0$, then the characteristic function of $W(t)$, $\phi_t(\theta) := E(\exp(i\theta W(t))$, necessarily has the form

$$\phi_t(\theta) = \exp(t\xi(\theta)),\ \theta \in \mathbb{R}, \tag{2.1}$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}_0} \Big(e^{i\theta x} - 1 - \frac{ix\theta}{1+x^2}\Big)\nu(dx), \tag{2.2}$$

for some $m \in \mathbb{R}$, $\sigma \geq 0$, and measure ν on the Borel subsets of $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. The measure ν is called the Lévy measure of the process W and has the property,

$$\int_{\mathbb{R}_0} \frac{u^2}{1+u^2}\nu(du) < \infty.$$

If ν is the zero measure then $\{W(t)\}$ is Brownian motion with $E(W(t)) = mt$ and $\text{Var}(W(t)) = \sigma^2 t$. If $m = \sigma^2 = 0$ and $\nu(\mathbb{R}_0) < \infty$, then $W(t) = at + P(t)$, where $\{P(t)\}$ is a compound Poisson process with jump-rate $\nu(\mathbb{R}_0)$, jump-size distribution $\nu/\nu(\mathbb{R}_0)$, and $a = -\int_{\mathbb{R}_0} \frac{u}{1+u^2}\nu(du)$. Another important example is the gamma process $\{W(t)\}$, for which

$$\xi(\theta) = \int_{\mathbb{R}_0} (e^{i\theta x} - 1)\nu(dx), \tag{2.3}$$

$\nu(du) = \alpha u^{-1}e^{-\beta u}du, u > 0$, and $W(t)$ has probability density function $\beta^{\alpha t}x^{\alpha t-1}e^{-\beta x}/\Gamma(\alpha t)$, $x > 0$. This is an example of a Lévy process whose sample-paths have (a.s.) infinitely many jumps in every interval of positive length. If $\{L_1(t)\}$ and $\{L_2(t)\}$ are two independent and identically distributed gamma processes then $L_1 - L_2$ is a symmetrized gamma process with Lévy measure, $\nu(du) = \frac{1}{2}\alpha|u|^{-1}e^{-\beta|u|}du$. For the non-decreasing stable process $X(t)$ with

$$E[\exp(i\theta X(t))] = \exp[-t\beta(-i\theta)^{\alpha}/\Gamma(1-\alpha)],\ \beta > 0,\ 0 < \alpha < 1,$$

ξ also has the form (2.3), but with

$$\nu(du) = \alpha\beta u^{-1-\alpha}du,\ u > 0.$$

This is another example of a Lévy process which in each finite interval has infinitely many jumps with probability 1. Moreover it has infinite moments of all orders greater than or equal to α.

Definition 2.1 If $\{W(t)\}$ is a Lévy process and p and q are integers such that $0 \le q < p$, then $\{Y(t), t \ge 0\}$ is a **Lévy-driven CARMA(p, q) process** with parameters $a_1, \dots, a_p, b_0, \dots, b_q$, if and only if $\{Y(t)\}$ satisfies (1.2) with $\{\mathbf{X}(t)\}$ a strictly stationary solution of the equations (1.4) and (1.6). The following existence result is established in Brockwell (2001b).

Theorem 2.1 *If $\{W(t)\}$ is a Lévy process with characteristic function (2.1) and $E|W(1)|^r < \infty$ for some $r > 0$, then the Lévy-driven CARMA process specified by Definition 1.2 exists if condition (1.7) is satisfied, in which case the cumulant generating function of $Y(t_1), Y(t_2), \dots, Y(t_n)$, $(0 \le t_1 < t_2 < \cdots < t_n)$ is*

$$\begin{aligned}
&\ln E[\exp(i\theta_1 Y(t_1) + \cdots + i\theta_n Y(t_n))] = \\
&\int_0^{\infty} \xi\left(\sum_{i=1}^{n} \theta_i \mathbf{b}' e^{A(t_i+u)}\right)\mathbf{e}du + \int_0^{t_1} \xi\left(\sum_{i=1}^{n} \theta_i \mathbf{b}' e^{A(t_i-u)}\right)\mathbf{e}du + \\
&\int_{t_1}^{t_2} \xi\left(\sum_{i=2}^{n} \theta_i \mathbf{b}' e^{A(t_i-u)}\right)\mathbf{e}du + \cdots + \int_{t_{n-1}}^{t_n} \xi\left(\sum_{i=2}^{n} \theta_i \mathbf{b}' e^{A(t_i-u)}\right)\mathbf{e}du.
\end{aligned} \tag{2.4}$$

In particular, the marginal distribution of $Y(t)$ has cgf,

$$\ln E[\exp(i\theta Y(t))] = \int_0^{\infty} \xi(\theta\mathbf{b}' e^{Au}\mathbf{e})du. \tag{2.5}$$

Example On the left side of Figure 1 are the histogram and sample autocorrelation function of the absolute daily returns $(100\ln(P(t)/P(t-1))$ on the Hang Seng Index for the period July 1st, 1997 - April 9th, 1999. It has been observed by Granger et al. (1999) that, as in this example, such absolute daily returns frequently follow an approximately exponential distribution with a slowly decaying positive autocorrelation function. The sample autocorrelation function can be well approximated by that of a CARMA(2,1) model with coefficients $a_1 = 2.86, a_2 = .30, b_0 = 1.0$ and $b_1 = 2.71$, estimated by maximization of the Gaussian likelihood. In an attempt to approximate the empirical marginal distribution, the two parameters of a gamma process $\{W(t)\}$ were adjusted so that the simulated marginal distribution of the corresponding gamma-driven CARMA(2,1) process had approximately the appropriate shape. A good match (shown on the right side of Figure 1) was obtained by choosing the distribution of $W(1)$ to have exponent .0.135 and scale parameter 4.08. A more systematic approach to maximum likelihood estimation for such models is described in the talk. The lack of evidence for long memory in the sample autocorrelation function of this data is very likely due to its relatively short length and is consistent with the suggestion of Granger et al. (1999) that the long memory observed in longer realizations of these series may be due to shifting levels.

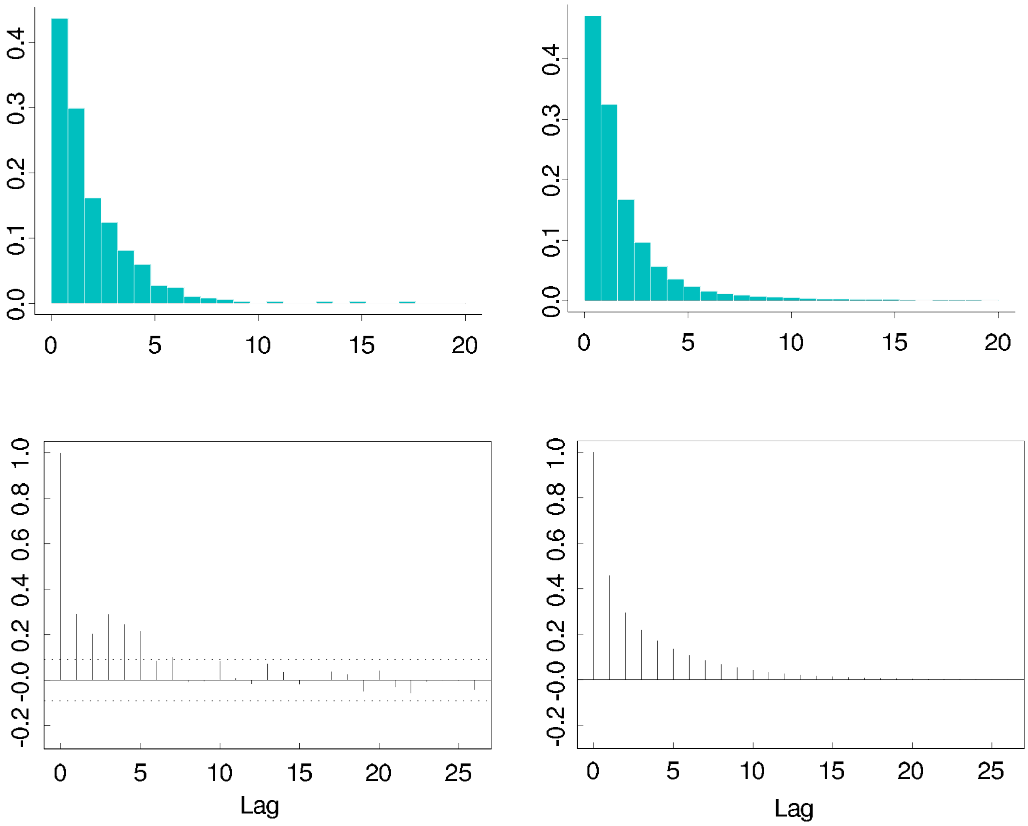

Figure 1: The figures on the left show the histogram (top) and sample autocorrelation function of the absolute daily returns on the Hang Seng Index, July 1, 1997 - April 9, 1999. The figures on the right are the corresponding graphs for the model defined in the example. The top right graph is based on 10,000 simulated values generated by the model.

Although the ad hoc procedure used in this example gives a good match between the model and empirical marginal distributions and autocorrelation functions, this does not necessarily mean that the model gives a good representation of the dynamics of the process. A more systematic approach to the fitting of such models is needed. A simulation-based method will be described for computing the likelihood under a Lévy-driven CARMA model when the state vector has a transition density.

Definition 2.2 If in Definition 2.1 we impose the restriction $E(W(1)^2) < \infty$ then $\{Y(t), t \geq 0\}$ is a **second-order Lévy-driven CARMA process**. For such a process the mean and variance of the increments $W(t+s) - W(s)$ of the driving Lévy process have the form mt and vt respectively and the autocorrelation function of the corresponding CARMA process is the same as if the driving process were Brownian motion.

The representation (1.15) and (1.16) remains valid for Lévy-driven CARMA processes. Thus $Y(t)$ can be expressed as

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)dW(u), \tag{2.6}$$

where $\{W(t), -\infty < t < \infty\}$ is defined as in (1.12), modified so as to be cadlag, and

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu\lambda} \frac{b(i\lambda)}{a(i\lambda)}\, d\lambda. \tag{2.7}$$

If the polynomial $a(\lambda)$ has distinct zeroes, $\lambda_1, \ldots, \lambda_p$, then $g(u)$ can be expressed more simply as,

$$g(u) = \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r u} I_{(-\infty,0)}(\Re(\lambda_r u)). \tag{2.8}$$

For second-order Lévy-driven CARMA processes, if $\{W(t)\}$ is scaled so that the increment in a time interval of length t has variance t and if the roots of $a(\lambda) = 0$ are distinct, then the autocovariance function of $\{Y(t)\}$ is again given by (1.11), i.e.

$$\gamma_Y(h) = \sum_{\lambda: a(\lambda)=0} \frac{e^{\lambda|h|} b(\lambda) b(-\lambda)}{a'(\lambda) a(-\lambda)}. \tag{2.9}$$

3 Applications

Stochastic differential equations driven by non-decreasing Lévy processes have been used in storage theory to represent the content of a dam whose cumulative input process is a non-decreasing Lévy process (see e.g. the papers of Cinlar and Pinsky (1972), Harrison and Resnick (1976) and Brockwell et al. (1982)).

More recently Barndorff-Nielsen and Shephard (2001) have introduced a very interesting and relatively tractable continuous-time stochastic volatility model in which the volatility process $\{\sigma^2(t)\}$ is an Ornstein-Uhlenbeck (or CARMA(1,0)) process driven by a non-decreasing Lévy process. Since the kernel function g in the representation (2.6) of the Ornstein-Uhlenbeck process is non-negative and the driving Lévy process has non-negative increments, the resulting process is non-negative as required. For such a process the autocorrelation function at lag h has the form $e^{-c|h|}$ for some $c > 0$. In order to extend the range of autocorrelation functions attainable by their model they considered the class of linear combinations of independent Lévy-driven Ornstein-Uhlenbeck processes with positive coefficients, for which the autocovariance functions have the form $\sum_{i=1}^{k} \alpha_i e^{-c_i h}$, where $c_i > 0$ and $\alpha_i \geq 0$ for each i. Such linear combinations are a subset of the class of CARMA processes.

This raises the question of the extent to which more general CARMA processes driven by non-decreasing Lévy processes (and restricted to have non-negative kernels) can increase the range of attainable autocorrelation functions. They do indeed extend the range (see Brockwell and Davis (2001)) and in fact it is possible to construct CARMA processes with complex autoregressive roots which at the same time have a non-negative kernel. These processes appear to have interesting potential for volatility modelling in conjunction with the model of Barndorff-Nielsen and Shephard.

Lévy-driven CARMA models in general constitute a class of continuous-time models closely analogous to the discrete-time ARMA processes and those of second order exhibit a comparable range of autocovariance functions. They are potentially useful for the modelling of irregularly spaced non-Gaussian time series (Gaussian CARMA processes have been used for ths purpose by Jones (1981)) and as possibly heavy-tailed models for financial time series. Some of the problems associated with inference for these processes will be discussed.

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Self Decomposability and Returns

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1 Introduction

The standard models for returns in portfolio allocation (Merton (1971)) and option pricing (Black and Scholes (1973)) both assume that continuously compounded returns are normally distributed. The central limit theorem is often invoked as a primary motivation for this assumption. By this theorem, the normal distribution arises as the limiting distribution for the sum of n independent random variables, when the sum is divided by $\sqrt{n}$. Hence, if returns are realized as the sum of a large number of independent influences, then one can anticipate that returns will in fact be normally distributed.

Unfortunately, it is well documented that the assumed normality of the return distribution is violated in both the time series data and in option prices. This has led many authors to consider jump-diffusion models (Merton (1976), Bates (1991)), stochastic volatility models (Heston (1993)), Lévy processes (Madan, Carr and Chang (1998), Barndorff-Nielsen (1998), Eberlein, Keller and Prause (1998)), and various combinations of these alternatives Bates (1996, 2000), Duffie, Pan and Singleton (2000), Barndorff-Nielsen and Shephard (2001), Carr, Geman, Madan and Yor (2001)).

This paper examines if there are alternatives to the Gaussian distribution as a limit law. This objective leads to the so-called laws of class L, which were defined by Khintchine (1938) and Lévy (1937) as limit laws for sums of n independent variables when centered and scaled by functions of n, not necessarily $\sqrt{n}$. These laws were subsequently found to be identical to the so-called class of self decomposable laws. Sato (1991) shows that the self-decomposable laws are associated with the unit time distribution of self-similar additive processes, whose increments are independent, but need not be stationary. Jeanblanc, Pitman, and Yor (2001) recently show how one may easily pass between these additive self similar representations and stationary solutions to OU equations driven by Lévy processes (Barndorff-Nielsen and Shephard (2001)). This paper presents a number of interesting candidate models in this class of what we call Sato processes.

The outline of the extended abstract is as follows. The next section introduces the laws of class L and the concept of self-decomposable laws, and outlines their association with self-similar processes and with stationary solutions to OU equations. We then go on to introduce 6 particular self decomposable laws in somewhat greater detail.

2 Self Decomposable Laws and Associated Processes

This section introduces the laws of class L and the self decomposable laws, which are known to be identical.

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2.1 Laws of class L and Self Decomposable Laws

Consider a sequence $(Z_k : k = 1, 2, \dots)$ of independent random variables and let $S_n = \sum_{k=1}^{n} Z_k$ denote their sum. Suppose that there exist centering constants c_n and scaling constants b_n such that the distribution of $b_n S_n + c_n$ converges to the distribution of some random variable X. Then the random variable X is said to have the class L property.

The distribution of a random variable X is said to be self decomposable (Sato (1999), page 90, Definition 15.1) if for any constant c, $0 < c < 1$ there exists an independent random variable say, $X^{(c)}$ such that

$$X \stackrel{law}{=} c\,X + X^{(c)}.$$

In other words, a random variable is self-decomposable if it has the same distribution as the sum of a scaled down version of itself and an independent residual random variable. Self decomposable laws have the property that the associated densities are unimodal (Yamazato (1978), Sato (1999), page 404).

The self decomposable laws are an important sub-class of the class of infinitely divisible laws. Lévy (1937) (see also Loève (1945)) showed that self decomposable laws are infinitely divisible with a special structure of their Lévy measure. Specifically, the characteristic function of these laws (Sato (1999), page 95, Corollary 15.11) has the form

$$E\left[e^{iuX}\right] = \exp\left[-\frac{1}{2}Au^2 + ibu + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}\right)\frac{h(x)}{|x|}dx\right]$$

where $A \geq 0$, b is a real constant, $h(x) \geq 0$, $\int_{-\infty}^{\infty}\left(|x|^2 \wedge 1\right)\frac{h(x)}{|x|}dx < \infty$, and $h(x)$ is increasing for negative x and decreasing for positive x. We call $h(x)$ the self decomposability characteristic (SDC) of the random variable X.

Sato (1999, page 91, Theorem 15.3) shows that a random variable has a distribution of class L if and only if the law of the random variable is self decomposable. Many compound Poisson processes employed in the finance literature do not enjoy the self decomposable property. In contrast, the recent Lévy models employed by Barndorff-Nielsen (1998), Eberlein, Keller and Prause(1998), Madan, Carr and Chang(1998), and Carr, Geman, Madan and Yor (2002)) all use Lévy densities associated with self decomposable laws.

2.2 Processes associated with Self Decomposable Laws

Sato (1991) establishes a connection between a self decomposable law holding at a fixed time and a stochastic process reigning over the time interval.

First note that a $\gamma-$self-similar process is defined as a stochastic process $(Y(t), t \geq 0)$ with the property that for any $\lambda > 0$ and all t,

$$Y(\lambda t) \stackrel{law}{=} \lambda^{\gamma} Y(t). \tag{2.1}$$

Sato (1991) defines additive processes as processes with inhomogeneous (in general) and independent increments. In the particular case when the increments are time homogeneous, the process is called a Lévy process. Sato (1991) showed that a law is self decomposable if and only if it is the law at unit time of an additive process, that is also a self similar process.

To relate these concepts in a simple setting, suppose that a self decomposable random variable X is the value at unit time of some pure jump Lévy process whose sample paths have bounded variation. We consider the case when the Lévy density integrates $|x|$ in the region $|x| < 1$ for which $b = \int_{|x|<1} x\frac{h(x)}{|x|}dx$. In this case the characteristic function of X has the form

$$E\left[e^{iuX}\right] = \exp\left[\int_{-\infty}^{\infty}(e^{iux} - 1)\frac{h(x)}{|x|}dx\right] \tag{2.2}$$

Let $Y(t)$ be the value at time t of a self-similar additive process with paths of bounded variation. The characteristic function for $Y(t)$ may be written as

$$E\left[e^{iuY(t)}\right] = \exp\left[\int_0^t\int_{-\infty}^{\infty}\left(e^{iuy} - 1\right)g(y, s)dyds\right], \tag{2.3}$$

for some time-dependent Lévy system $g(y,t)$. Suppose that we require that the law of the self-similar additive process at unit time be the self-decomposable law of the random variable X:

$$Y(1) \stackrel{law}{=} X. \tag{2.4}$$

Then the following theorem relates the time dependent Lévy system to the SDC, $h(x)$ of the self-decomposable law.

Theorem 2.1. *Given a self decomposable law for the time one distribution (2.4) with a characteristic function satisfying (2.2), then there exists a self similar process $Y(t)$ defined with respect to the increasing scaling function t^γ by (2.1) and which satisfies (2.3) when:*

$$g(y,t) = \begin{cases} -\frac{h'\left(\frac{y}{t^\gamma}\right)\gamma}{t^{1+\gamma}}, & y > 0 \\ \frac{h'\left(\frac{y}{t^\gamma}\right)\gamma}{t^{1+\gamma}}, & y < 0 \end{cases}$$

Observe that it is precisely the property of h that it be increasing on the left and decreasing on the right that yields g as a positive inhomogeneous Lévy density.

2.2.1 Some other processes associated with self similar processes

It is shown in Lamperti (1962) that one may associate with any γ -self *similar* process $Y(t)$ a stationary process Z_t defined by

$$\begin{aligned} Z_u &= e^{-\gamma u} Y(e^u) \\ Y(t) &= t^\gamma Z(\log(t)) \end{aligned}$$

and so we observe that our scaled self decomposable process $Y(t)$ is also a scaled and time changed stationary process.

It is further shown in Jeanblanc, Pitman and Yor (2001) that the stationary process $Z_u, u \geq 0$ is the solution to Ornstein-Uhlenbeck equation associated with a Background Driving Lévy Process (Barndorff-Nielsen and Shephard (2001)) $U(t)$

$$dZ = -\gamma Z dt + dU$$

with initial condition $Z(0) = X$.

The Lévy process may itself be constructed from the γ-self *similar* process $Y(t)$ in accordance with

$$U(t) = \int_1^{e^t} \frac{1}{s^\gamma} dY(s).$$

3 Some Specific Self Decomposable Processes

In this section we consider six examples of self decomposable laws with which we shall associate a γ-*self similar* additive process $Y(t)$.

The first three self decomposable laws are those for the unit time Variance Gamma (VG) model, of Madan, Carr, and Chang (1998), the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), and the Meixner process ($MXNR$) developed by Grigelionis (1999) and Schoutens (2001). In addition we develop three new processes based on laws related to the hyperbolic functions and studied by Pitman and Yor (2000): the three processes involved employ the hyperbolic cosine, sine and tangent functions in their analytical structure. The six processes are denoted $VGSSD$, $NIGSSD$, $MXNRSSD$, $VCSSD$, $VSSSD$, $VTSSD$ where the addition of the extension SSD signifies that the density varies with maturity by a scaled self decomposable law.

3.1 VGSSD

The variance gamma (VG) process is defined by time changing an arithmetic Brownian motion with drift θ and volatility σ by an independent gamma process with unit mean rate and variance rate ν. Let $G(t;\nu)$ be the gamma process, then the variance gamma process may be written as:

$$X_{VG}(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu))$$

where $W(t)$ is an independent standard Brownian motion. Madan, Carr, and Chang (1998) show that the VG process can also be expressed as the difference of two independent gamma processes. Carr, Geman, Madan and Yor (2002) show that the VG process is a Lévy process whose Lévy density has the form

$$k_{VG}(x) = \begin{cases} C\frac{\exp(Gx)}{|x|} & x < 0 \\ C\frac{\exp(-Mx)}{x} & x > 0 \end{cases}$$

where the parameters C, G, M are explicitly related to the original parameters by

$$\begin{aligned} C &= \frac{1}{\nu} \\ G &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}\right)^{-1} \\ M &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}\right)^{-1} \end{aligned}$$

We observe that the SDC for the VG process is

$$h_{VG}(x) = \begin{cases} C\exp(Gx) & x < 0 \\ C\exp(-Mx) & x > 0 \end{cases}$$

The exponential and negative exponential are classic examples of functions which are increasing and decreasing, when the domains are restricted to the negative and positive axis respectively.

3.2 NIGSSD

The NIG process also has a characteristic function defined by three parameters (see Barndorff-Nielsen (1998)). To obtain the characteristic function we follow the presentation in Carr, Geman, Madan and Yor (2002). From this perspective, we first define inverse Gaussian time I_t^ν as the time it takes an independent Brownian motion with drift ν to reach the level t. It is well known that the Laplace transform of this random time is:

$$E\left[\exp\left(-\lambda I_t^\nu\right)\right] = \exp\left(-t\left(\sqrt{2\lambda+\nu^2} - \nu\right)\right) \tag{3.1}$$

The process is well defined for $\nu > 0$, while for $\nu < 0$ it gets infinite almost surely; more precisely, $P(I_t^\nu < \infty) = \exp(2t\nu)$. Next we evaluate an independent arithmetic Brownian motion with drift θ and volatility σ at this inverse Gaussian time:

$$X_{NIG}(t;\sigma,\nu,\theta) = \theta I_t^\nu + \sigma W(I_t^\nu) \tag{3.2}$$

on the set $I_t^\nu < \infty$. The NIG Lévy density is given by

$$k_{NIG}(x) = \sqrt{\frac{2}{\pi}}\sigma\alpha^2 \frac{e^{\frac{\theta}{\sigma^2}x}K_1(|x|)}{|x|}. \tag{3.3}$$

It follows that the SDC for NIG process is given by

$$h_{NIG}(x) = \sqrt{\frac{2}{\pi}}\sigma\alpha^2 e^{\frac{\theta}{\sigma^2}x}K_1(|x|)$$

and hence the law is self-decomposable for θ/σ^2 sufficiently small.

3.3 MXNRSSD

The Meixner process has recently been proposed by Grigelionis (1999) and Schoutens (2001). The characteristic function for zero drift is

$$E\left[e^{iuX_{MXNR}(t)};a,b,d\right]=\left(\frac{\cos(\frac{b}{2})}{\cosh\left(\frac{au-ib}{2}\right)}\right)^{2dt}$$

The probability density of the Meixner distribution is given by:

$$f(x;a,b,d)=\frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)}\exp\left(\frac{b}{a}x\right)\left|\Gamma(d+i\frac{x}{a})\right|^2$$

where $\Gamma(z)$ is the gamma function with complex argument z.

The Lévy density is given by

$$k_{MXNR}(x)=d\frac{\exp\left(\frac{b}{a}x\right)}{x\sinh\left(\frac{\pi x}{a}\right)}.$$

Hence, the *SDC* of the Meixner process is given by

$$h_{MXNR}(x)=d\frac{\exp\left(\frac{b}{a}x\right)}{\left|\sinh\left(\frac{\pi x}{a}\right)\right|}.$$

This function also satisfies the self-decomposability condition for small enough values of b/a.

3.4 The Hyperbolic Processes VCSSD, VSSSD and VTSSD

We define two increasing additive processes denoted by C_t, S_t by their Laplace transforms:

$$E\left[e^{-\lambda C_t}\right]=\left(\frac{1}{\cosh\left(\sqrt{2\lambda t}\right)}\right)$$
$$E\left[e^{-\lambda S_t}\right]=\left(\frac{\sqrt{2\lambda t}}{\sinh\left(\sqrt{2\lambda t}\right)}\right).$$

These processes may be described by:

$$C_t=\inf\{s\,||B_s|=t\}$$
$$S_t=\inf\{s\,|BES(3,s)=t\},$$

where B_s is a standard Brownian motion and $BES(3,s)$ is the Bessel process of dimension 3, i.e. the norm of a 3 dimensional standard Brownian motion.

Using Lévy's theorem for C_t and the results of Pitman (1975) on three dimensional Brownian motion for S_t, we write alternative characterizations for these processes as

$$C_t\overset{(d)}{=}\inf\{s\,|M_s-B_s=t\}$$
$$S_t\overset{(d)}{=}\inf\{s\,|2M_s-B_s=t\}$$

where $M_t=\sup_{s\le t}B_s$.

We now allow for drift in the Brownian motion. Hence, let

$$B_t^{(\nu)}=\nu t+B_t$$

and define

$$C_t^{(\nu)}=\inf\left\{s\left|M_s^{(\nu)}-B_s^{(\nu)}=t\right.\right\}$$
$$S_t^{(\nu)}=\inf\left\{s\left|2M_s^{(\nu)}-B_s^{(\nu)}=t\right.\right\}$$

where $M_t^{(\nu)} = \sup_{s \le t} B_s^{(\nu)}$.

We also consider a one dimensional diffusion $Z_t^{(\nu)}$ with infinitesimal generator

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} + \nu \tanh(\nu x)\frac{\partial}{\partial x}$$

and define

$$T_t^{(\nu)} = \inf\left\{ s \,\middle|\, \left|Z_s^{(\nu)}\right| = t \right\}.$$

We note that $\left(\left|Z_t^{(\nu)}\right|, t \ge 0\right) \stackrel{(d)}{=} \left(\left|B_t^{(\nu)}\right|, t \ge 0\right)$ when both start at zero.

The resulting Laplace transforms are

$$\begin{aligned}
E\left[e^{-\lambda C_t^{(\nu)}}\right] &= \frac{\exp(-\nu t)\sqrt{\nu^2+2\lambda}}{\sqrt{\nu^2+2\lambda}\cosh\left(t\sqrt{\nu^2+2\lambda}\right) - \nu\sinh\left(t\sqrt{\nu^2+2\lambda}\right)} \\
E\left[e^{-\lambda S_t^{(\nu)}}\right] &= \frac{\sinh(\nu t)}{\nu}\frac{\sqrt{\nu^2+2\lambda}}{\sinh\left(t\sqrt{\nu^2+2\lambda}\right)} \\
E\left[e^{-\lambda T_t^{(\nu)}}\right] &= \frac{\cosh(\nu t)}{\cosh\left(t\sqrt{\nu^2+2\lambda}\right)}.
\end{aligned}$$

The processes VC, VS, VT are constructed by first evaluating an independent Brownian with volatility σ at the times $C_t^{(\nu)}$, $S_t^{(\nu)}$, and $T_t^{(\nu)}$ respectively, to obtain the characteristic functions:

$$E\left[e^{iu\sigma B(C_t^{(\nu)})}\right] = \frac{\exp(-\nu t)\sqrt{\nu^2+\sigma^2u^2}}{\sqrt{\nu^2+\sigma^2u^2}\cosh\left(t\sqrt{\nu^2+\sigma^2u^2}\right) - \nu\sinh\left(t\sqrt{\nu^2+\sigma^2u^2}\right)} \tag{3.4}$$

$$E\left[e^{iu\sigma B(S_t^{(\nu)})}\right] = \frac{\sinh(\nu t)}{\nu}\frac{\sqrt{\nu^2+\sigma^2u^2}}{\sinh\left(t\sqrt{\nu^2+\sigma^2u^2}\right)} \tag{3.5}$$

$$E\left[e^{iu\sigma B(T_t^{(\nu)})}\right] = \frac{\cosh(\nu t)}{\cosh\left(t\sqrt{\nu^2+\sigma^2u^2}\right)}. \tag{3.6}$$

To add asymmetry, we use the Esscher transform for a transform parameter θ. We define for $H_t \in \left\{C_t^{(\nu)}, S_t^{(\nu)}, T_t^{(\nu)}\right\}$

$$E^{(\theta)}\left[e^{iu\sigma B(H_t)}\right] = \frac{E[e^{i(u-i\theta)\sigma B(H_t)}]}{E\left[e^{i(-i\theta)\sigma B(H_t)}\right]} \tag{3.7}$$

The characteristic functions for $X_{VC}(t)$, $X_{VS}(t)$, $X_{VT}(t)$ may then be easily obtained.

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Boundary Crossing Problems for Lévy processes

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1 Introduction

There are now many results known about boundary crossing problems for random walks, usually involving the asymptotics of the exit time from a one-sided or two-sided interval, or the overshoot over the boundary. Some recent references are [6], [7], [8], [9], [10], [11], [12], [13] and [16]. Extensions to curved boundaries of power-law type have also been carried out, see [2], [14], and [15].

Recently, in [4] and [5] some of these questions have been studied for Lévy Processes, where there is the interesting possibility of studying the small-time, as well as the large-time, behaviour. In this talk, rather than describing all our results, I want to concentrate mainly on the following problem.

When is the overshoot small in comparison to the interval?

2 Random Walks

The random walk will be denoted by $S = (S_n, n \geq 0)$, where $S_n = \sum_1^n Y_i$, the $Y's$ being independent and identically distributed with distribution F. We assume throughout that the support of F is unbounded, and write

$$\begin{aligned} T(x) &= 1 - F(x) + F(-x), \\ D(x) &= 1 - F(x) - F(-x), \\ A(x) &= \int_0^x D(y)dy = E(Y;|Y| \leq x) + xD(x), \\ \text{and } U(x) &= \int_0^x 2yT(y)dy = E(Y^2;|Y| \leq x) + x^2T(x). \end{aligned}$$

2.1 One-sided Overshoots

To start at the beginning, suppose first that the $Y's$ are non-negative; so we are in the classical renewal theory situation. With $N^+(r) = \min\{n : S_n > r\}$, $O_r^+ = S_{N^+(r)} - r$ is the size of the overshoot when S first crosses the level r, but is also referred to as the unexpired lifetime. The answer to the question is well known, both for convergence in probability and a.s. convergence;

$$\frac{O_r^+}{r} \xrightarrow{a.s.} 0 \text{ as } r \to \infty; \Longleftrightarrow EY < \infty.$$

$$\frac{O_r^+}{r} \xrightarrow{P} 0 \text{ as } r \to \infty \Longleftrightarrow S \text{ is relatively stable} \Longleftrightarrow A \text{ is s.v. at } \infty.$$

(Here relative stability means the existence of a norming sequence b such that $S_n/b_n \xrightarrow{\dot{P}} 1$.)

In the general case, let $Z^{\pm}$ denote the first increasing ladder height in S and $-S$ respectively. Then O_r^+ coincides with the overshoot in the renewal process $H = (H_n; n \geq 0)$ of ladder heights and the following are equivalent

$$\frac{O_r^+}{r} \xrightarrow{a.s.} 0 \text{ as } r \to \infty;$$

$$EZ^+ < \infty;$$

$$0 < \mu = EY < \infty \text{ OR } \mu = 0 \text{ and } \int_1^\infty \frac{x(1 - F(x)dx}{\int_0^x dy \int_y^\infty F(-z)dz} < \infty.$$

The final result here is due to Chow[1], who completed an earlier result in [3]. Note that an analytic criterion in terms of F for H to be relatively stable is not yet known, so we don't have a completely satisfactory analogue for convergence in probability.

2.2 Two-sided Overshoots

With $N_r = \min\{n : |S_n| > r\}$,

$$O_r = |S_{N(r)}| - r$$

is the size of the overshoot when S first exits $[-r, r]$. Then clearly $N_r = N_r^+ \wedge N_r^-$, so O_r has to coincide with one of O_r^+, O_r^-, and these coincide with the overshoots in the corresponding ladder processes.

Suppose S drifts to $+\infty$; then $a.s.$ for sufficiently large r, $N_r = N_r^+$, and since $\mu = 0$ is not possible, to get $\frac{O_r}{r} \xrightarrow{a.s.} 0$ we must have $0 < \mu < \infty$ in order that $EZ^+ < \infty$.

Suppose S oscillates; then N_r coincides with each of N_r^+ and N_r^- for arbitrarily large r, so to get $\frac{O_r}{r} \xrightarrow{a.s.} 0$ we "must" have both of EZ^+ and EZ^- finite; but this can only happen if $\mu = 0$ and $EY^2 < \infty$.

This is NOT a proof, but nevertheless the following **are** equivalent

(i) $$\frac{O_r}{r} \xrightarrow{a.s.} 0 \text{ as } r \to \infty;$$

(ii) $$\int_1^\infty \frac{x^2|dT(x)|}{x|A(x)| + U(x)} < \infty$$

(iii) (a) EY^2 finite and $\mu = 0$ or (b) $E|Y| < \infty, \mu \neq 0$.

The proof of this works as follows; the basic fact, which goes back to Pruitt[17], is that

$$ET_r \approx \frac{1}{k(r)} \text{ where } k(r) = \frac{r|A(r)| + U(r)}{r^2}.$$

By considering the position from which S exits $[-r, r]$ we easily get

$$ET_r P(|Y| > (2 + \delta)r) \le P(O_r > \delta r) \le ET_r P(|Y| > \delta r).$$

A Borel-Cantelli argument shows the equivalence of (i) and (ii), and an analytic argument in [10] shows that (ii) is equivalent to (iii).

A key point in this last argument is that

$$\frac{P(|Y| > x)}{k(x)} = \frac{x^2 T(x)}{x|A(x)| + U(x)} \to 0$$

(which is immediate from (ii)) can happen in 2 distinct ways only. The first, in which U dominates, is that

$$\frac{U(x)}{x^2 T(x) + x|A(x)|} \to \infty \text{ as } x \to \infty, \tag{2.1}$$

which is a known NASC for $\exists\, b(n) \to \infty$ with

$$S_n / b(n) \xrightarrow{D} N(0, 1).$$

Say $S \in D_0(N)$ in this case.

(Note; (2.1) holds if $\mu = EY = 0$ and $\sigma^2 = EY^2$ is finite since then $x^2T(x) \to 0$ and $U(x) \to \sigma^2$. It is actually equivalent to $\mu = 0$ and U slowly varying at ∞.)

The second, in which A is the dominant term, is that

$$\frac{A(x)}{xT(x)} \to \pm\infty \text{ as } x \to \infty, \tag{2.2}$$

in which case S is relatively stable, which we denote by $S \in RS$. (Note; (2.2) holds if μ is finite and $\neq 0$, since then $A(x) \to \mu$ and $xT(x) \to 0$. But it can also hold with one or both of EY^+ and EY^- infinite, or with $\mu = 0$. If $Y \geq 0$, it is equivalent to A being slowly varying at ∞.)

This solves the "in probability problem", since similar arguments show that the following are equivalent

$$\frac{O_r}{r} \xrightarrow{P} 0 \text{ as } r \to \infty;$$
$$\frac{x|A(x)| + U(x)}{x^2T(x)} \to \infty \text{ as } x \to \infty,$$
$$\text{(2.1) or (2.2) holds, i.e. } S \in RS \cup D_0(N).$$

3 Lévy processes

The Lévy process $X = \{X_t, t \geq 0\}$ will be specified via its Lévy exponent $\Psi(\theta)$;

$$E\{e^{i\theta X_t}\} = \exp\{-t\Psi(\theta)\},\ t \geq 0,\ \theta \in \mathbb{R},$$

and this in turn is specified via the *characteristics of* X, which are the quantities a, σ, and Π in the famous Lévy-Khintchine formula;

$$\Psi(\theta) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{|x|\geq 1}\left(1 - e^{i\theta x}\right)\Pi(dx) + \int_{|x|<1}\left(1 - e^{i\theta x} - i\theta x\right)\Pi(dx). \tag{3.1}$$

Associated with any Lévy process X are two subordinators, H^+ and H^-, which correspond to the ladder height processes for S and $-S$ in the random walk context. In the following, we denote the drifts of these processes by δ^+, δ^-, respectively.

3.1 Behaviour of X at ∞

Since $\{X(nc), n \geq 0\}$ is a random walk for any fixed $c > 0$ it is easy to guess there will be results similar to those of section 2. However, it doesn't seem possible to deduce the Lévy process results from the random walk ones. Rather, we have to give a separate proof, but following the same lines. In fact, if we now set

$$\begin{aligned}
N(x) &= \Pi\{(x,\infty)\}, \quad M(x) = \Pi\{(-\infty,-x)\},\\
T(x) &= N(x) + M(x),\\
D(x) &= N(x) - M(x),\\
A(x) &= a + D(1) + \int_1^x D(y)dy,\\
\text{and } U(x) &= \sigma^2 + \int_0^x 2yT(y)dy,
\end{aligned}$$

then, in essence, *exactly the same results hold.* (n.b. If $\lim_{x\to\infty} A(x)$ exists, it equals EX_1.) In particular, the conditions for $X \in D_0(N)$ and $X \in RS$ at ∞ are precisely (2.1) and (2.2).

3.2 Behaviour at 0

Now, apparently the random walk results will have no relevance,..but in fact the class $\mathrm{D}_0(N)$ at zero is specified by

$\exists\, b(t) \downarrow 0$ with $X_t/b(t) \xrightarrow{D} N(0,1)$ as $t \to 0$ iff

$$\frac{U(x)}{x|A(x)| + x^2 T(x)} \to \infty \text{ as } x \downarrow 0, \tag{3.2}$$

and then b is rv with index $1/2$ and U is s.v. at 0, and the class RS at zero by

$\exists\, b(t) \downarrow 0$ with $X_t/b(t) \xrightarrow{p} \pm 1$ as $t \to 0$ iff

$$\sigma = 0, \text{ and } \frac{A(x)}{xT(x)} \to \pm\infty \text{ as } x \to 0, \tag{3.3}$$

and then b is rv at 0 with index 1.

Remark Suppose $\sigma = 0, \lim_{x\downarrow 0} A(x) = \delta \neq 0$, and.$\lim_{x\downarrow 0} xT(x) = 0$. Then (3.3) holds and $t^{-1}X_t \xrightarrow{p} \delta = a + D(1) - \int_0^1 D(y)dy$. So $\lim_{x\downarrow 0} A(x)$, when it exists, plays the rôle of a "mean" at zero. But, just as in the rw case, (3.3) can also hold with this being zero, or $\pm\infty$.

3.2.1 One-sided Overshoots

Again we see that these are the same for the subordinator H^+ as they are for X. But what plays the rôle of $EZ^+ < \infty$?

The answer is $\delta^+ > 0$, where δ^+ is the *drift* of H^+. In fact if $O_r^+ = X(T_r^+) - r$, where $T_r^+ = \inf(t : X_t > r)$, then the following are equivalent

$$\frac{O_r^+}{r} \xrightarrow{a.s.} 0 \text{ as } r \to 0;$$
$$P(O_r^+ = 0) \longrightarrow 1 \text{ as } r \to 0;$$
$$P(O_{r_o}^+ = 0) > 0 \text{ for some } r_0 > 0;$$
$$\delta^+ > 0.$$

Processes with the third of these properties are said to "creep upwards". A very recent result of V. Vigon [18] solves the problem, which had been around for some time, of finding an analytic condition in terms of the characteristics of X equivalent to these. His result is that $\delta^+ > 0$ occurs if and only if

$$\int_0^1 \frac{xN(x)dx}{\int_0^x dy \int_y^1 M(z)dz} < \infty.$$

Note how close this is to Chow's condition for $EZ^+ < \infty$.

3.2.2 Two-sided Overshoots

Our results again are surprisingly similar at 0 and ∞. Specifically, with $O_r = X(T_r) - r$, where $T_r = \inf(t : |X_t| > r)$,

(i) the following are equivalent

$$\frac{O_r}{r} \xrightarrow{a.s.} 0 \text{ as } r \to 0;$$
$$\int_0^1 \frac{x^2|dT(x)|}{x|A(x)| + U(x)} < \infty;$$

(a) $$\sigma^2 > 0 \text{ or}$$

(b) $$\sigma^2 = 0, \quad \int_0^1 T(x)dx < \infty, \quad \lim_{x\downarrow 0} A(x) \neq 0.$$

{In case (a) the drifts δ^+ and δ^- are both positive, and the probability that X exits the interval at the top $\to 1/2$. In case (b) X is bv with drift $\delta = \lim_{x\downarrow 0} A(x)$, so the probability that X exits the interval at the top $\to 1$ or 0.}

(ii) the following are equivalent

$$\begin{aligned} \frac{O_r}{r} \xrightarrow{P} 0 \text{ as } r &\to 0; \\ \frac{x|A(x)| + U(x)}{x^2 T(x)} &\to \infty \text{ as } x \to 0, \\ (3.2) \text{ or } (3.3) \text{ holds, i.e. } S &\in RS \cup D_0(N) \text{ at zero.} \end{aligned}$$

Again, it doesn't seem possible to see the equivalence of the first and third condition without establishing the analytic condition. Fortunately Pruitt's bound turns out to be just as useful for small r as for large r, and the Borel-Cantelli type arguments can also be adapted...

Postscript Not all such problems make sense and have such similar solutions at zero.

Another way of measuring "smallness" of the overshoot with respect to the boundary is to ask for $E\{O_r\}^p$ to be bounded as $r \to \infty$. (This and analogous questions are discussed in depth for random walks in [9], [10], and [11].) However at zero this condition is automatically satisfied.

Another question is when does

$$P\{S_{N_r} > 0\} \to 1 \text{ as } r \to \infty?$$

This was solved in [10], and results in [12] show that it is equivalent to both of

$$P(S_n > 0) \to 1 \text{ as } n \to \infty,$$

and

$$S_n \xrightarrow{p} \infty \text{ as } n \to \infty.$$

But no Lévy process can have

$$X_t \xrightarrow{p} \infty \text{ as } t \to 0,$$

and for this reason we don't yet know a NASC for either

$$P\{X_{T_r} > 0\} \to 1 \text{ as } r \to 0$$

or

$$P\{X_t > 0\} \to 1 \text{ as } t \to 0.$$

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The stable continuum random tree

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Abstract

We study the stable continuum random tree, which is a generalization of the Brownian continuum random tree introduced by Aldous. In particular, we give explicit formulas for the finite-dimensional marginals of this tree.

1 Introduction

The Brownian continuum random tree (or CRT) was introduced and studied by Aldous [1], [2]. It has been shown to appear in certain limit theorems for models of statistical mechanics, see in particular Derbez and Slade [3] and Hara and Slade [6]. The most concrete way to describe the Brownian continuum random tree is via its coding by a Brownian excursion. To understanding this coding, let us first consider the discrete Galton-Watson tree associated with an offspring distribution μ on the nonnegative integers. We assume that μ has mean one (critical case) and finite nonzero variance σ^2. The μ-Galton-Watson tree $\mathcal{T}$ is then the genealogical tree corresponding to the Galton-Watson process with offspring distribution μ, started with one ancestor. This tree, which is finite a.s. by the criticality of μ, can be viewed as a random subset of the set of labels

$$\bigcup_{n=0}^{\infty} \mathbb{N}^n \qquad \text{(by convention } \mathbb{N}^0 = \{\emptyset\}\text{)}$$

as shown in Fig. 1. One convenient way to code the tree is to draw the associated contour process $(C_t, t \geq 0)$, whose definition should be evident from Fig.1 below. By convention, $C_t = 0$ if $t \geq 2(N-1)$, where $N = |\mathcal{T}|$ denotes the total progeny of the tree.

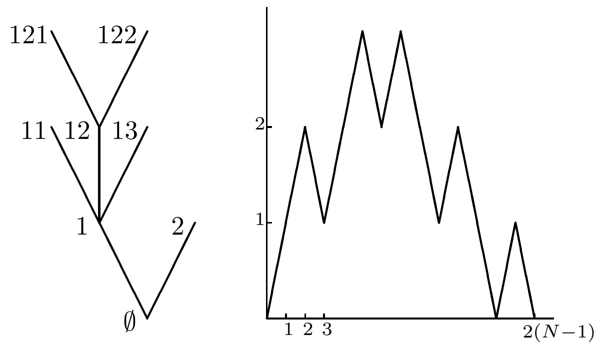

Figure 1

Suppose that μ is aperiodic, so that the event $\{N = n\}$ has positive probability for all n sufficiently large. Let $C^n = (C^n(t), t \geq 0)$ be a process which has the distribution of $(C_t, t \geq 0)$ conditioned on the event $\{N = n\}$. Aldous [2] proved that

$$\Big(\frac{1}{\sqrt{n}} C^n_{2nt}, 0 \leq t \leq 1\Big) \xrightarrow[n\to\infty]{\text{(d)}} \Big(\frac{2}{\sigma} e_t, 0 \leq t \leq 1\Big),$$

where $(e_t, 0 \leq t \leq 1)$ is a normalized Brownian excursion and the convergence holds in the sense of weak convergence of the laws on $C([0,1], \mathbb{R}_+)$. Furthermore the Brownian continuum random tree can be thought of as the tree coded by $(e_t, 0 \leq t \leq 1)$, in much the same way as the contour process codes a discrete Galton-Watson tree:

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- To every $s \in [0,1]$ corresponds a vertex of the tree at generation e_s.
- Vertex s is an ancestor of vertex s' if $e_s = \inf_{[s\wedge s', s\vee s']} e_r$. In general, $\inf_{[s\wedge s', s\vee s']} e_r$ is the generation of the last common ancestor to s and s'.
- The distance on the tree is $d(s,s') = e_s + e_{s'} - 2\inf_{[s\wedge s', s\vee s']} e_r$, and we identify vertices s and s' if $d(s,s') = 0$.

If instead of assuming that μ has finite variance we consider the case where μ is in the domain of attraction of a stable law, then an analogue of Aldous' result, presented in Section 3 below, leads to a stable CRT. The role of the normalized Brownian excursion is played by an auxiliary process constructed in Section 2 as a functional of a stable Lévy process. Explicit calculations of finite-dimensional distributions of the stable tree are given in Section 4.

2 The stable height process

Let $\alpha \in (1,2)$, and let $X = (X_t, t \geq 0)$ be a stable Lévy process without negative jumps and with index α: In particular, the Laplace transform of X_t is well defined and given by

$$E[\exp(-\lambda X_t)] = \exp(c\,t\,\lambda^\alpha)$$

for some positive constant c. For definiteness, we assume that $c = 1$ in what follows.

For every $0 \leq s \leq t$, we set

$$I_t = \inf_{r\in[0,t]} X_r \;, \qquad I_{s,t} = \inf_{r\in[s,t]} X_r \,.$$

Proposition 2.1. ([7],[5]) *For every $t \geq 0$, the limit*

$$H_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t 1_{\{X_s < I_{s,t}+\varepsilon\}}\, ds$$

exists a.s. The process $(H_s, s \geq 0)$ has a continuous modification, which is Hölder continuous with exponent β for every $\beta < 1 - 1/\alpha$.

The process $(H_s, s \geq 0)$ is called the *stable height process*. For our purposes, it will be important to define an analogous functional for the normalized excursion of the stable process X above its minimum I. This can be achieved via the following simple construction. Let

$$g_1 = \sup\{t \in [0,1) : X_t = I_t\} \;, \quad d_1 = \inf\{t \in (1,\infty) : X_t = I_t\}$$

be respectively the beginning and the end of the excursion of $X - I$ away from 0 that straddles 1. If $\zeta_1 = d_1 - g_1$, the process

$$\left(\zeta_1^{-1/\alpha}(X_{g_1+\zeta_1 t} - X_{g_1}), 0 \leq t \leq 1\right)$$

is distributed as a normalized excursion of $X - I$. The normalized excursion of the stable height process can then be defined by

$$H_t^0 = \zeta_1^{\frac{1}{\alpha}-1} H_{g_1+\zeta_1 t} \;, \qquad 0 \leq t \leq 1.$$

3 A limit theorem

As in Section 1, let μ be an aperiodic critical offspring distribution. We assume now that μ is in the domain of attraction of a stable law with index α. More precisely, if ν is the zero-mean probability measure on $\{-1,0,1,2,\ldots\}$ defined by $\nu(k) = \mu(k+1)$, and if W_n is a random variable distributed as the sum of n independent variables with distribution ν, we assume that there exists a sequence (a_n) of positive numbers converging to $+\infty$ such that

$$\frac{1}{a_n} W_n \xrightarrow[n\to\infty]{\text{(d)}} X_1$$

where X is as previously.

Following Section 1, we can for every integer n large enough consider the μ-Galton-Watson tree conditioned to have exactly n vertices. We denote this tree by $\mathcal{T}_n$ and we let $C^n = (C^n_t, t \geq 0)$ be its contour process. The next result is due to Duquesne [4].

Theorem 3.1. *Under the preceding assumptions on μ,*

$$\Big(\frac{a_n}{n} C^n_{2nt}, 0 \leq t \leq 1\Big) \xrightarrow[n\to\infty]{\text{(d)}} (H^0_t, 0 \leq t \leq 1),$$

where the limit process is the normalized excursion of the stable height process with index α.

In view of this result, it is natural to call the tree coded by $(H^0_t, 0 \leq t \leq 1)$ the stable continuum random tree with index α.

4 Marginal distributions

Consider the discrete tree $\mathcal{T}_n$ of the previous section. Suppose we choose uniformly at random p vertices on this tree. We can then consider the reduced genealogical structure consisting only of the ancestors of the p chosen vertices. In the limit $n \to \infty$ and after a suitable scaling, the law of this reduced tree will converge to the p-th marginal distribution of the stable tree.

More formally, for every choice of $t_1, \ldots, t_p \in [0,1]$, we define a marked tree $\theta(H^0; t_1, \ldots, t_p)$ that describes the genealogy of $t_1, \ldots, t_p$ in the tree structure coded by H^0 (see [5] Section 3.2 for a precise construction). The tree $\theta(H^0; t_1, \ldots, t_p)$ consists of a discrete skeleton $\mathcal{T}(H^0; t_1, \ldots, t_p)$, which is a discrete rooted ordered tree (as in Fig.1) with p leaves, and a collection $(h_v, v \in \mathcal{T})$ of marks, h_v representing the length of the branch v. The law of $\theta(H^0; t_1, \ldots, t_p)$ when $t_1, \ldots, t_p$ are uniformly distributed over $[0,1]$ is the p-th marginal of the tree. In the case of the Brownian CRT, these marginals were computed by Aldous [2]. The next result extends these calculations to the stable case.

If $\mathcal{T}$ is a (rooted ordered) tree, we let $\mathcal{N}_\mathcal{T}$ be the set of nodes of $\mathcal{T}$ (vertices that are not leaves). For every $v \in \mathcal{N}_\mathcal{T}$, $k_v = k_v(\mathcal{T})$ is the number of children of v in $\mathcal{T}$. We let $\mathbf{T}_p$ be the set of all (rooted ordered) trees with p leaves such that $k_v \geq 2$ for every $v \in \mathcal{N}_\mathcal{T}$. By construction, the tree $\mathcal{T}(H^0; t_1, \ldots, t_p)$ belongs to $\mathbf{T}_p$.

Theorem 4.1. [5] *The p-th marginal distribution of the stable continuum random tree can be described as follows.*
(i) *The probability of a given skeleton $\mathcal{T} \in \mathbf{T}_p$ is*

$$\frac{p!}{\prod_{v\in\mathcal{N}_\mathcal{T}} k_v!} \, \frac{\prod_{v\in\mathcal{N}_\mathcal{T}} \big|(\alpha-1)(\alpha-2)\ldots(\alpha-k_v+1)\big|}{(\alpha-1)(2\alpha-1)\ldots((p-1)\alpha-1)}.$$

(ii) *If $p \geq 2$, then conditionally on the skeleton $\mathcal{T}$, the marks $(h_v)_{v\in\mathcal{T}}$ have a density with respect to Lebesgue measure on $\mathbb{R}_+^\mathcal{T}$ given by*

$$\frac{\Gamma(p-\frac{1}{\alpha})}{\Gamma(\delta_\mathcal{T})} \alpha^{|\mathcal{T}|} \int_0^1 du\, u^{\delta_\mathcal{T}-1} q(\alpha \sum_{v\in\mathcal{T}} h_v, 1-u)$$

where $\delta_\mathcal{T} = p - (1-\frac{1}{\alpha})|\mathcal{T}| - \frac{1}{\alpha} > 0$, and $q(s,u)$ is the continuous density at time s of the stable subordinator with exponent $1-\frac{1}{\alpha}$. If $p = 1$, then $\mathcal{T} = \{\emptyset\}$ and the law of $h_\emptyset$ has density

$$\alpha\, \Gamma(1-\tfrac{1}{\alpha})\, q(\alpha h, 1).$$

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Modelling of Lévy term structures

Ernst Eberlein and Fehmi Özkan

A default-free zero-coupon bond is a financial security paying its owner one currency unit at a prespecified date T in the future. A defaultable bond, usually a corporate bond, is a financial security *promising* its owner to pay one currency unit at a prespecified maturity date T in the future. In contrast to the case of default-free bonds the issuer of a credit risky bond will default with a certain probability before or at time T. In case of default the holder of such a bond will receive only a fractional amount of the promised currency unit or nothing at all.

Credit risk models which have been studied in the literature so far are models driven by a Brownian motion, or are standard jump diffusions. See Zhou (1997), Schönbucher (1998), or Duffie and Singleton (1999) for examples, and the surveys on credit risk models in Lando (1997), Ammann (1999), and Schönbucher (2000).

A new approach to credit risk based on the methodology of Heath, Jarrow, and Morton (1992) was introduced in Bielecki and Rutkowski (1999, 2000). The Bielecki-Rutkowski model takes the information on rating migration and on credit spreads into account and yields an arbitrage-free model of defaultable bonds. We follow this approach to construct an intensity-based credit risk framework for term structure models driven by Lévy processes.

We start with the default free instantaneous forward rate $f(t,T)$ given by

$$df(t,T) \;=\; \partial_2 A(t,T)dt - \partial_2 \Sigma(t,T)^\top dL_t \tag{1}$$

where $A(t,T)$ and $\Sigma(t,T)$ are processes satisfying certain smoothness conditions and (L_t) is a d-dimensional Lévy process which in the canonical decomposition can be written as

$$L_t \;=\; bt + cW_t + \int_0^t \int_{\mathbb{R}^d} x\,(\mu^L - \nu^L)(ds,dx).$$

The corresponding price of a default-free bond is then given by

$$\begin{aligned} B(t,T) = B(0,T)\exp\Bigg(&\int_0^t (r(s) - A(s,T))\,ds + \int_0^t \Sigma(s,T)^\top c\,dW_s \\ &+ \int_0^t \int_{\mathbb{R}^d} \Sigma(s,T)^\top x(\mu^L - \nu^L)(ds,dx)\Bigg). \end{aligned} \tag{2}$$

Now assume that an internal rating system $\mathcal{K} = \{1,\dots,K\}$ is given. Class 1 corresponds to the best possible rating following default-freeness – which is denoted AAA in the Standard & Poor's rating – class K corresponds to default. The instantaneous forward rate for class $i \in \{1,\dots,K-1\}$ is assumed to satisfy the equation

$$dg_i(t,T) \;=\; \partial_2 A_i(t,T)\,dt - \partial_2 \Sigma_i(t,T)^\top\,dL_t^{(i)}, \tag{3}$$

where $(L_t^{(i)})$ is given by $L_t^{(i)} = b_i t + c_i W_t + \int_0^t \int_{\mathbb{R}^d} p_i x(\mu^L - \nu^L)\,(ds,dx)$. In order to ensure that risky corporate bonds have higher forward rates than less risky bonds, we assume

$$g_{K-1}(t,T) \;>\; g_{K-2}(t,T) > \cdots > g_1(t,T) > f(t,T). \tag{4}$$

The dynamics of the conditional bond price based on the forward rate g_i can then be derived in the form

$$\begin{aligned} dD_i(t,T) = D_i(t-,T)\Bigg(&\Big(a_i(t,T) + g_i(t,t)\Big)\,dt \\ &+ \int_{\mathbb{R}^d} \Sigma_i(t,T)^\top p_i x\left(\mu^L - \nu^L\right)(dt,dx) + \Sigma_i(t,T)^\top c_i\,dW_t \\ &+ \int_{\mathbb{R}^d}\Big(e^{\Sigma_i(t,T)^\top p_i x} - 1 - \Sigma_i(t,T)^\top p_i x\Big)\,\mu^L(dt,dx)\Bigg), \end{aligned} \tag{5}$$

where $a_i(t,T) = \frac{1}{2}\left|\Sigma_i(t,T)^\top c_i\right|^2 - A_i(t,T)$. The credit migration process is modelled by a conditional Markov process C on the space of rating classes $\mathcal{K}$. We define $H_i(t) = 1\!\!1_{\{s\geq 0 | C_s = i\}}(t)$ and write for $i \neq j$, $H_{ij}(t)$ for the number of transitions from rating i to rating j in the time interval $[0,t]$. Then the time t price $D_C(t,T)$ of a defaultable bond maturing at time T, which is currently rated at C_t, equals

$$D_C(t,T) = B(t,T)\sum_{i=1}^{K-1}\left(H_i(t)\exp\left(-\int_t^T \gamma_i(t,u)du\right) + \delta_i H_{i,K}(t)\right) \tag{6}$$

where $\gamma_i(t,u) = g_i(t,u) - f(t,u)$ is the i-th forward credit spread and $\delta_i \in [0,1)$ the corresponding recovery rate which is assumed to be constant. This defaultable bond price can also be expressed in terms of a risk-neutral valuation formula, i.e. as an expectation with respect to a martingale measure. Further extensions of the model to include reorganization of firms and multiple defaults are considered.

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Means of nonparametric priors based on Increasing Additive Processes

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Abstract

We provide a survey on some distributional results concerning means of a random probability measure constructed via suitable transformations of an increasing additive process (abbreviated as IAP), *i.e.* an increasing, not necessarily homogeneous, and purely discontinuous Lévy process. In particular, we deal with normalized random measures, having independent increments, and with neutral to the right (NTR) random probability measures. The former are obtained by normalizing IAPs and the exact distribution of a mean is found by resorting to a well-known inversion formula for characteristic functions. Moreover, expressions of the posterior distributions of those means, in the presence of exchangeable observations, are given. Also the latter may be characterized in terms of IAPs and we show the connection between a mean of a NTR prior and the so called exponential functional. We study finiteness and absolute continuity of these functionals and provide some *formulae* for computing their moments, provided they exist. All the results contained in the first section can be found in Regazzini, Lijoi and Prünster (2000), whereas those of Section 2 are based on Epifani, Lijoi and Prünster (2002).

1 Means of normalized RMI

1.1 Definition of normalized RMI

A first type of random probability measures we consider are those constructed by normalization of a IAP $\xi = \{\xi_t : t \geq 0\}$. For an exhaustive account about the theory of IAPs we refer to, *e.g.*, Sato (1999) and Skorohod (1991). Suppose α is a non-null finite measure on $\mathbb{R}$ with distribution function (abbreviated as d.f.) A and assume that $\xi_{\alpha(\mathbb{R})}$ is strictly positive and finite a.s.. In terms of the Lévy measure ν_t, this condition can be restated as

$$\nu_{\alpha(\mathbb{R})}((0, +\infty)) = +\infty.$$

Under this condition, $x \mapsto \xi_{A(x)}$ is an a.s. bounded d.f. on $\mathbb{R}$, and

$$x \mapsto \tilde{F}(x) = \xi_{A(x)}/\xi_{\alpha(\mathbb{R})}$$

is a random probability d.f. on $\mathbb{R}$ a.s.. By *random measure with independent increments* (RMI) we mean the random measure $\tilde{\xi}$ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ associated with ξ_A. Consistently, the random probability measure $\tilde{\varphi}$ associated with $\tilde{F}$ is said to be a *normalized* RMI.

Before proceeding, it is worth mentioning that the results of the following paragraphs carry over, with slight modifications, to normalized IAP driven random measures, *i.e.* normalized convoluted IAPs, which include the well-known, and widely used, mixture of Dirichlet process prior introduced by Lo (1984). See Nieto-Barajas, Prünster and Walker (2001) for details.

1.2 Distributional results for means of normalized RMI

We aim at studying the distribution of $\tilde{\varphi}(f) := \int_{\mathbb{R}} f \mathrm{d}\tilde{\varphi}$. To this end, one first needs to state conditions for the existence of such a mean. Let us denote by $\tilde{\nu}_\alpha$ the intensity measure of the reparameterized process $\xi_{A(\,\cdot\,)}$ and observe that

$$P\{\tilde{\varphi}(|f|) < +\infty\} = P\{\tilde{\xi}(|f|) < +\infty\} = 1. \tag{1}$$

It is possible to show that (1) is satisfied if and only if one of the following is fulfilled

(i) $\int_{\mathbb{R}\times(0,+\infty)}[1-\exp(-\lambda y|f(x)|)]\,\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}y) < +\infty$ holds for every $\lambda > 0$;

(ii) $\int_{\mathbb{R}\times(0,+\infty)}[1-\cos(yt|f(x)|)]\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}y) < +\infty$ and
$\int_{\mathbb{R}\times(0,+\infty)}|\sin(yt|f(x)|)|\,\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}y) < +\infty$ hold for every $t \in \mathbb{R}$.

Observe that the criterion for finiteness of $\tilde\varphi(f)$ depends only on the intensity measure $\tilde\nu_\alpha$.
In order to obtain the distribution of a mean of normalized RMI we need three elements.

(I) A trivial trick

$$P\Big\{\int_{\mathbb{R}} f(x)\tilde\varphi(\mathrm{d}x) \le \sigma\Big\} = P\Big\{\int_{\mathbb{R}} f(x)\tilde\xi(\mathrm{d}x) \le \sigma\tilde\xi(\mathbb{R})\Big\} = P\Big\{\int_{\mathbb{R}} (f(x)-\sigma)\tilde\xi(\mathrm{d}x) \le 0\Big\}$$

(II) The characteristic function of $\tilde\xi(f)$

$$\gamma_f(t) = \exp\Big[\int_{\mathbb{R}\times(0,+\infty)} (\mathrm{e}^{\mathrm{i}tvf(x)}-1)\,\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}v)\Big]$$

(III) An inversion formula provided by Gurland (1948)

$$\frac{1}{2}\big[P\{\tilde\varphi(f)\le\sigma\} + P\{\tilde\varphi(f)<\sigma\}\big] = \frac{1}{2} - \frac{1}{\pi}\lim_{\epsilon\downarrow 0, T\uparrow+\infty}\int_\epsilon^T \frac{1}{t}\,\mathrm{Im}\,E\big[\exp(\mathrm{i}t\tilde\xi(f-\sigma))\big]\mathrm{d}t$$

Hence, an expression of the probability d.f., $\mathbb{F}$, of a mean of a normalized RMI is given by

$$\frac{1}{2}\big[\mathbb{F}(\sigma)+\mathbb{F}(\sigma-0)\big] = \frac{1}{2} - \frac{1}{\pi}\lim_{T\uparrow+\infty}\int_0^T \frac{1}{t}\exp\Big\{\int_{\mathbb{R}\times(0,+\infty)}[\cos(tv(f(x)-\sigma))-1]\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}v)\Big\}$$
$$\times\sin\Big(\int_{\mathbb{R}\times(0,+\infty)}\sin(tv(f(x)-\sigma))\tilde\nu_\alpha(\mathrm{d}x\mathrm{d}v)\Big)\,\mathrm{d}t \qquad (\sigma\in\mathbb{R}).$$

1.3 Posterior means of normalized RMI

Here a technique for evaluating posterior distributions of means is briefly described. A remarkable feature is represented by the fact that the approach being undertaken does not depend on the particular structure of the underlying random probability measure: it only requires the knowledge of the prior distribution of the mean.

We restrict our attention to *exchangeable* observations, *i.e.* given a sequence $(X_n)_{n\ge1}$ of real-valued observations, we have

$$P(X_1\in A_1,\dots,X_n\in A_n|\tilde\varphi) = \tilde\varphi(A_1)\cdots\tilde\varphi(A_n) \qquad \text{a.s.},$$

for every family of measurable sets $A_1,\dots,A_n$ and $n\ge1$.

First, suppose that α has finite support, *e.g.* $\mathrm{supp}(\alpha)=\{s_1,\dots,s_N\}$ and (a,b) is an interval containing all the $f(s_j)$s. In such a case, let $\mathbb{F}(\sigma;t_1,\dots,t_N)$ denote the distribution function of $\tilde\varphi(f)$ and $x^{(n)}=(x_1,\dots,x_n)$ be a sample including $n_{i_r}>0$ terms equal to s_{i_r}, for $r=1,\dots,k$, with $\sum_r n_{i_r}=n$. Moreover, define $C(x^{(n)})^{-1} := \int\prod_{r=1}^k \varphi(s_{i_r})^{n_{i_r}}Q(\mathrm{d}\varphi)$, where Q is the prior distribution of $\tilde\varphi$, and set

$$I^n_{a^+}h(\sigma) := \int_a^\sigma \frac{(\sigma-u)^{n-1}}{(n-1)!}h(u)\mathrm{d}u$$

to be the *Liouville-Weyl fractional integral*, for $n\ge1$, whereas $I^0_{a^+}$ represents the identity operator.

Under suitable technical conditions, it is possible to prove that a posterior probability density function (with respect to the Lebesgue measure on $\mathbb{R}$) of $\tilde\varphi(f)$, given $X^{(n)}=x^{(n)}$, coincides with

$$(-1)^nC(x^{(n)})\frac{\partial^n}{\partial t_{i_1}^{n_{i_1}}\cdots\partial t_{i_k}^{n_{i_k}}}\,I^{n-1}_{a^+}\mathbb{F}(\sigma;t_1,\dots,t_N)\Big|_{(t_1,\dots,t_N)=(f(s_1),\dots,f(s_N))} \qquad (\sigma\in\mathbb{R}).$$

For arbitrary parameters α, one can benefit from a discretization procedure introduced by Regazzini and Sazonov (2000) in order to obtain the posterior d.f. of a mean as an almost sure weak limit of a sequence of posterior d.f.'s.

1.4 A generalization of the Dirichlet process

The Dirichlet process, introduced in Ferguson (1973), represents a cornerstone of Bayesian Nonparametrics. It is well-known that such a process can be viewed as a normalized RMI, since it also coincides with the normalization of a reparameterized Gamma process. A generalization of the Dirichlet process based on the following family of Lévy measures

$$\mathcal{N} = \left\{ \nu_t(\mathrm{d}v) = t\,\frac{(1-\mathrm{e}^{-\gamma v})}{(1-\mathrm{e}^{-v})}\,\frac{\mathrm{e}^{-v}}{v}\,\mathrm{d}v \qquad \gamma > 0, t \geq 0 \right\}$$

is studied in Regazzini *et al.* (2000). Notice that the corresponding IAP is a gamma process when $\gamma = 1$. If $\xi_{\alpha(B)} = \tilde{\xi}(B)$, for any $B \in \mathscr{B}(\mathbb{R})$, α being any non-null and finite measure on $\mathbb{R}$, and γ is a positive integer, a necessary and sufficient condition for $\tilde{\varphi}(|f|)$ to be finite is

$$\int_{\mathbb{R}} \log(\gamma + \lambda |f(x)|)_\gamma\, \alpha(\mathrm{d}x) < +\infty \tag{2}$$

with $(a)_n := a(a-1)\cdots(a-n+1)$. In the Dirichlet case (*i.e.* $\gamma = 1$), it reduces to a condition obtained by Feigin and Tweedie (1989) and by Cifarelli and Regazzini (1990).

If (2) holds true, the probability d.f. of $\int_{\mathbb{R}} f \mathrm{d}\tilde{\varphi}$ turns out to be

$$\mathbb{F}(\sigma) = \frac{1}{2} - \frac{(\gamma!)^{\alpha(\mathbb{R})}}{\pi} \int_0^{+\infty} \frac{1}{t} \mathrm{Im}\left(\exp\left\{ -\sum_{k=1}^{\gamma} \int_{\mathbb{R}} \log[k + it(\sigma - x)]\alpha(\mathrm{d}x) \right\} \right) \mathrm{d}t,$$

which, in the case $\gamma = 1$, coincides with the representation of the probability d.f. of a mean of the Dirichlet process provided in Cifarelli and Regazzini (1990) and in Regazzini, Guglielmi and Di Nunno (2001).

It is also possible to derive a representation for the posterior probability density function given $X^{(n)} = x^{(n)}$. In the Dirichlet case we obtain a useful alternative representation of the posterior probability density function of the mean.

2 Means of NTR random probability measures

2.1 The exponential functional as mean of a random probability measure

A well-known class of prior distributions in Bayesian nonparametric statistics, introduced by Doksum (1974), takes on the name of neutral to the right (NTR) random probability d.f.. A random probability d.f., F, is NTR if and only if it can be written in terms of a transient IAP as

$$F(t) = 1 - \mathrm{e}^{-\xi_t} \qquad (t \geq 0).$$

Notice that the mean of a random probability d.f. F over $(0, +\infty)$ can be written as $\int_0^{+\infty} (1 - F(t))\, \mathrm{d}t$. If F is NTR, the previous expression turns out to be the exponential functional of a IAP, *i.e.*

$$\int_0^{+\infty} \mathrm{e}^{-\xi_t}\, \mathrm{d}t =: I(\xi).$$

Hence, a trivial argument leads to establishing the connection between means of NTR priors and the exponential functional, $I(\xi)$, of IAPs. In recent years much attention has been paid to the study of the distributional properties of $I(\xi)$ when ξ is a Lévy process. See, *e.g.*, the deep and insightful works by Bertoin and Yor (2001), Carmona, Petit and Yor (1997,2001) and Urbanik (1992,1995).

Previous remarks suggest that one can study distributional properties of means of NTR priors by importing the theory of the exponential functional into Bayesian Nonparametrics. This leads us to state some results about means of NTR priors, which, to the authors' knowledge, have not been investigated in Bayesian nonparametric literature. Moreover, we would like to stress the additional insight provided by the interpretation of $I(\xi)$ as a mean of a random probability measure which should, in our opinion, lead to fruitful interactions. Here we can confine ourselves to studying prior distributions, since the posterior distribution of a NTR random probability measure is still NTR and can be explicitly obtained by updating the Lévy measure in a quite straightforward way and by adding fixed points of discontinuity for the exact observations. See, *e.g.*, Ferguson and Phadia (1979) and Walker and Muliere (1997).

2.2 Some results

In the present section, our treatment is mainly based on the results contained in Carmona *et al.* (1997).

Following the approach suggested by the above-mentioned authors, it is possible to establish a criterion for the existence of $I(\xi)$, which applies to some examples of IAPs.
Let ξ be a IAP whose Lévy measure admits representation $\nu_t(\mathrm{d}x) = \gamma(t)\nu(\mathrm{d}x)$, where γ is a non-negative, continuous and increasing function such that $\gamma(0) = 0$ and $\lim_{t\to+\infty}\gamma(t) = +\infty$. We will refer to such a process as *parameterized* IAP. In this case it is possible to prove that:
If there exist a and c in $[0,+\infty)$ such that $a < \int_0^{+\infty}\nu(x,+\infty)\mathrm{d}x < +\infty$ and $\lim_{t\to+\infty}\mathrm{e}^{-at}\gamma^{-1}(t) = c$, then $I(\xi) < +\infty$ a.s..
Moreover, by imposing further conditions on ν, it is possible to prove existence of moments of any order. Our interest in parameterized IAPs is mainly motivated by the fact that most NTR priors known in literature belong to this class.

When studying further distributional properties of a mean of a NTR prior, we prove that if γ is sufficiently smooth, then $I(\xi)$ has absolutely continuous probability distribution with respect to the Lebesgue measure on $\mathbb{R}$. Moreover, a sufficient condition for the absolute continuity of the probability distribution of $I(\xi)$, when ξ is any IAP, is provided.

An immediate condition for the existence of the moment of order n of the exponential functional can be given by resorting to its statistical interpretation, *i.e.*

$$\int_0^{+\infty} t^n \,\mathrm{d}F_0(t) < +\infty$$

where, from a Bayesian standpoint, F_0 is meant as the prior guess at the shape of the random d.f. F, *i.e.* $F_0 = E(F)$.

When the moment of order n exists, one can compute it by the following formula

$$n!\int_0^{+\infty}\cdots\int_{t_{n-1}}^{+\infty}\prod_{j=1}^{n}\exp\left\{-\int_0^{+\infty}(1-\mathrm{e}^{-x})\;\nu_{t_j}^{(n-j)}(\mathrm{d}x)\right\}\;\mathrm{d}t_n\ldots\mathrm{d}t_1,$$

where $\nu_{t_j}^{(n-j)} := \mathrm{e}^{-(n-j)x}\nu_{t_j}$. Notice that the previous formula reduces to the one provided in Carmona *et al* (1997) and Urbanik (1992), if ξ is a homogeneous IAP, and to the one provided by Cifarelli and Regazzini (1979), if ξ is the IAP which gives raise to the Dirichlet process. Finally, if $I(\xi)$ has bounded support $[a,b]$ and is absolutely continuous, we can implement some experiments in order to recover the density function of $I(\xi)$ by the knowledge of the first n moments. More precisely, we can calculate the maximum entropy estimator, *i.e.* the function which maximizes the value of the Boltzmann-Shannon entropy $H(f) = -\int f(x)\log(f(x))\mathrm{d}x$ for all densities f having the first n moments preassigned.

2.3 Examples

The considerations developed in the previous section can be applied successfully to all the NTR priors known in literature. With reference to the NTR priors introduced in Ferguson and Phadia (1979), they both belong to the class of parameterized IAPs and are based on the gamma process and on the so called simple homogeneous process, respectively. We are able to state conditions for existence of their means and moments. Moreover, these means turn out to be a.c. and the moment formula becomes much simpler for particular choices of γ. The other NTR prior to which we apply our results is given by the Beta Stacy process introduced by Walker and Muliere (1997). Also in this case the mean is a.c. and the moment formula tractable.

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Free Lévy Processes on Dual Groups

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Abstract

We give a short introduction to the theory of Lévy processes on dual groups. As examples we consider Lévy processes with additive increments and Lévy processes on the dual affine group.

1 Introduction

Lévy processes play a fundamental rôle in probability theory and have many important applications in other areas such as statistics, financial mathematics, functional analysis or mathematical physics, as well.

In quantum probability they first appeared in a model for the laser in [Wal84]. This lead to the theory of Lévy processes on involutive bialgebras, cf. [ASW88, Sch93, FS99]. The increments of these Lévy processes are independent in the sense of tensor independence, which is a straightforward generalisation of the notion of independence used in classical probability theory. However, in quantum probability there exist also other notions of independence like, e.g., freeness [VDN92], see Paragraph 2.2. In order to formulate a general theory of Lévy processes for these independences, the $*$-bialgebras or quantum groups have to be replaced by the dual groups introduced in [Voi87], see [Sch95b, BGS99, Fra01a, Fra01b].

In this paper we give an introduction to the theory of Lévy processes on dual groups, which avoids most of the algebraic prerequisites. In particular, we will not define dual groups, but only consider two examples, namely tensor and free Lévy processes with additive increments and tensor and free Lévy processes on the dual affine group. Our approach is similar to rewriting the definition of classical Lie group-valued Lévy processes in terms of a coordinate system, see Definitions 3.1, 3.2, 4.1, and 4.3.

Quantum Lévy processes play an important rôle in the theory of continuous measurement, cf. [Hol01], and in the theory of dilations, where they describe the evolution of a big system or heat bath, which is coupled to the small system whose evolution one wants to describe.

Additive free Lévy processes where first studied in [GSS92], and more recently in [Bia98, Ans01a, Ans01b, BNT01a, BNT01b].

2 Preliminaries

In this section we introduce the basic notions and definitions that we will use. For more detailed introductions to quantum probability see, e.g., [Par92, Bia93, Mey95, Hol01].

2.1 Quantum probability

Non-commutative probability or quantum probability is motivated by the statistical interpretation of quantum mechanics where an operator is interpreted as an analog of a random variable. The rôle of the classical probability space is played by a (pre-)Hilbert space $\mathcal{H}$ and the measure is replaced by a unit vector $\Omega \in \mathcal{H}$ called state vector.

In this paper we will mean by a (real) quantom random variable X on $(\mathcal{H}, \Omega)$ a (symmetric) linear operator on the pre-Hilbert space $\mathcal{H}$, which has an adjoint, i.e. for which there exists a linear operator X^*, such that

$$\langle u, Xv\rangle = \langle X^*u, v\rangle$$

for all $u, v \in \mathcal{H}$. Its law (w.r.t. the state vector Ω) is the functional $\phi_X : \mathbb{C}[x] \to \mathbb{C}$ on the algebra $\mathbb{C}[x]$ of polynomials in one variable defined by

$$\phi_X(x^k) = \langle \Omega, X^k\Omega\rangle,$$

for $k \in \mathbb{N}$. If X is symmetric, then there exists a (possibly non-unique) probability measure μ on $\mathbb{R}$ such that

$$\phi_X(x^k) = \int_{\mathbb{R}} x^k \mathrm{d}\mu.$$

Let X be a classical $\mathbb{R}$- or $\mathbb{C}$-valued random variable with finite moments on some probability space $(M, \mathcal{M}, P)$. It becomes a quantum random variable on $\mathcal{H} = L^\infty(M, \mathcal{M}, P)$, if we let it act on the bounded functions on M by multiplication, $L^\infty(M) \ni f \mapsto Xf \in L^\infty(M)$ with $Xf(m) = X(m)f(m)$ for $m \in M$. If we take the constant function $\Omega(m) = 1$ for all $m \in M$ for the state vector, then we recover the classical distribution of X, i.e.,

$$\langle \Omega, X^k \Omega \rangle = \int_M X^k \mathrm{d}P = \mathbb{E}(X^k),$$

for $k \in \mathbb{N}$. If X is $\mathbb{R}$-valued, then it is also real as quantum random variable, i.e. symmetric.

A (real) quantum random vector on $(\mathcal{H}, \Omega)$ is an n-tuple $X = (X_1, \ldots, X_n)$ of (real) quantum random variables on $(\mathcal{H}, \Omega)$. Its law is the functional $\phi_X : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}$ on the algebra of non-commutative polynomials $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ in n variables defined by

$$\phi_X(x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}) = \langle \Omega, X_{i_1}^{k_1} \cdots X_{i_r}^{k_r} \Omega \rangle,$$

for all $i_1, \ldots, i_r \in \{1, \ldots, n\}$, $k_1, \ldots, k_r \in \mathbb{N}$.

A (real) operator process is an indexed family $(X_\imath)_{\imath \in I}$ on $(\mathcal{H}, \Omega)$ of (real) quantum random variables or (real) quantum random vectors on $(\mathcal{H}, \Omega)$. We will call two operator processes $(X_\imath)_{\imath \in I}$ and $(Y_\imath)_{\imath \in I}$ equivalent, if they have the same joint moments, i.e. if

$$\langle \Omega, X_{\imath_1}^{k_1} \cdots X_{\imath_r}^{k_r} \Omega \rangle = \langle \Omega, Y_{\imath_1}^{k_1} \cdots Y_{\imath_r}^{k_r} \Omega \rangle$$

for all $\imath_1, \ldots, \imath_r \in I$ and all $k_1, \ldots, k_r \in \mathbb{N}$.

2.2 Freeness and Independence

Let now $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be algebras of adjointable linear operators on some pre-Hilbert space $\mathcal{H}$, closed under taking adjoints and containing the identity operator $\mathbf{1}$.

Definition 2.1. $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are called tensor independent (w.r.t. to the state vector Ω), if

(i) for all $1 \le i, j \le k$ and all $X \in \mathcal{A}_i$ and $Y \in \mathcal{A}_j$, we have

$$[X, Y] := XY - YX = 0,$$

(ii) and for all $X_1 \in \mathcal{A}_1, \ldots, X_k \in \mathcal{A}_k$ we have

$$\langle \Omega, X_1 \cdots X_k \Omega \rangle = \langle \Omega, X_1 \Omega \rangle \cdots \langle \Omega, X_k \Omega \rangle.$$

This definition is the natural analogue of the notion of independence in classical probability to our setting. It is also the one used in quantum physics when one speaks of independent observables. But in quantum probability there exist other, inequivalent notions of independence.

Definition 2.2. $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are called free, if for all $1 \le i_1, \ldots, i_r \le k$ with $i_1 \ne i_2 \ne \cdots \ne i_r$ (i.e. neighboring indices are different) and all $X_1 \in \mathcal{A}_{i_1}, \ldots, X_r \in \mathcal{A}_{i_r}$ with

$$\langle \Omega, X_1 \Omega \rangle = \cdots = \langle \Omega, X_r \Omega \rangle = 0,$$

we have

$$\langle \Omega, X_1 \cdots X_r \Omega \rangle = 0.$$

Quantum random variables or quantum random vectors $X, Y, Z, \ldots$ are called tensor independent or free, iff the unital $*$-algebras they generate are tensor independent or free.

Remark 2.3. These definitions allow to compute arbitrary joint moments of tensor independent or free random variables from their marginal distributions.

For the free case this computation can be done recursively on the order of the moment be expanding

$$0 = \varphi\Big(\big(X_1 - \varphi(X_1)\mathbf{1}\big) \cdots \big(X_r - \varphi(X_r)\mathbf{1}\big)\Big),$$

where we wrote $\varphi(\cdot)$ instead of $\langle \Omega, \cdot\, \Omega\rangle$ for the expectation.

Let X_1 and X_2 be two free quantum random variables, then one obtains in this way

$$0 = \varphi\Big(\big(X_1 - \varphi(X_1)\mathbf{1}\big)\big(X_2 - \varphi(X_2)\mathbf{1}\big)\Big) = \varphi(X_1X_2) - \varphi(X_1)\varphi(X_2),$$

and therefore

$$\varphi(X_1X_2) = \varphi(X_1)\varphi(X_2).$$

as for tensor independent quantum random variables or independent classical random variables. But for higher moments the formulas are different, one gets, e.g.,

$$\varphi(X_1X_2X_1X_2) = \varphi(X_1^2)\varphi(X_2)^2 + \varphi(X_1)^2\varphi(X_2^2) - \varphi(X_1)^2\varphi(X_2)^2.$$

This formula can also be used to show that there exist no non-trivial examples of commuting free quantum random variables. If X_1 and X_2 commute, then we would get

$$\varphi(X_1X_2X_1X_2) = \varphi(X_1^2X_2^2) = \varphi(X_1^2)\varphi(X_2)^2,$$

since X_1^2 and X_2^2 are also free. Therefore

$$\varphi\big(X_1 - \varphi(X_1)\mathbf{1}\big)\varphi\big(X_2 - \varphi(X_2)\mathbf{1}\big) = 0,$$

i.e. at least one of the two quantum random variables has a trivial distribution.

Remark 2.4. Besides tensor independence and freeness there exist other notions of independence that are used in quantum probability. In a series of papers [Sch95a, Spe97, BGS99, Mur01, Mur02] it was shown that there exist exactly five "universal" notions of independence satisfying a natural set of axioms. Besides tensor independence and freeness these are boolean, monotone, and anti-monotone independence. In [Fra01b] the boolean, monotone, and anti-monotone independence where reduced to tensor independence. If this is also possible for freeness is still an open problem.

3 Additive Lévy Processes

Definition 3.1. An operator process $(X_t)_{t\geq 0}$ on $(\mathcal{H}, \Omega)$ is called an additive tensor Lévy process (w.r.t. Ω), if the increments

$$X_{st} := X_t - X_s,$$

are

(i) tensor independent, i.e. the quantum random variables $X_{s_1t_1}, \ldots, X_{s_rt_r}$ are tensor independent for all $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_r \leq t_t$,

(ii) stationary, i.e. the law of an increment depends only on $t - s$, and

(iii) weakly continuous, i.e. $\lim_{t\searrow s}\langle \Omega, X_{st}^k \Omega\rangle = 0$ for $k = 1, 2, \ldots$

Replacing tensor independence by another universal notion of independence we can define the corresponding other classes of Lévy processes. E.g., for freeness we get the following definition.

Definition 3.2. An operator process $(X_t)_{t\geq 0}$ on $(\mathcal{H},\Omega)$ is called an additive free Lévy process (w.r.t. Ω), if the increments

$$X_{st} := X_t - X_s,$$

are

(i') free, i.e. the quantum random variables $X_{s_1t_1},\ldots,X_{s_rt_r}$ are free for all $0\leq s_1\leq t_1\leq s_2\leq \cdots \leq s_r\leq t_t$,

and satisfy conditions (ii) and (iii) of Definition 3.1.

For each of these notions of independence one can define a Fock space and creation, annihilation, and conservation or gauge operators on this Fock space.

For example the (algebraic) free Fock space $\mathcal{H}=\mathcal{F}_{\rm F}(\mathfrak{h})$ over a (pre-)Hilbert space $\mathfrak{h}$ is defined as

$$\mathcal{F}_{\rm F}(\mathfrak{h}) = \bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes n}$$

where $\mathfrak{h}^{\otimes 0}=\mathbb{C}$. The vector $\Omega = 1+0+\cdots$ is called the vacuum vector. For a vector $u\in\mathfrak{h}$ we can define the creation operator $a^+(u)$ and the annihilation operator $a^-(u)$ by

$$\begin{aligned} a^+(u)u_1\otimes\cdots u_k &= u\otimes u_1\otimes\cdots\otimes u_k, \\ a^-(u)u_1\otimes\cdots u_k &= \langle u,u_1\rangle u_2\otimes\cdots\otimes u_k. \end{aligned}$$

These operators are mutually adjoint, on the vacuum vector they act as $a^+(u)\Omega=u$ and $a^-(u)\Omega=0$.

The conservation operator $\Lambda(X)$ of some linear operator X on $\mathfrak{h}$ is defined by

$$\Lambda(X)u_1\otimes\cdots u_k = (Xu_1)\otimes u_2\otimes\cdots\otimes u_k$$

and $\Lambda(X)\Omega=0$. It satisfies $\Lambda(X)^*=\Lambda(X^*)$.

Glockner, Schürmann, and Speicher have shown that every additive free Lévy process can be realized as a linear combination of these three operators and time.

Theorem 3.3. *[GSS92] Let $(X_t)_{t\geq 0}$ be an additive free Lévy process. Then there exists a pre-Hilbert space $\mathfrak{k}$, a linear operator T on $\mathfrak{k}$, vectors $u,v\in\mathfrak{k}$, and a scalar $\lambda\in\mathbb{C}$ such that $(X_t)_{t\geq 0}$ is equivalent to the operator process $(X'_t)_{t\geq 0}$ on the free Fock space $\mathcal{F}_{\rm F}\big(L^2(\mathbb{R}_+,\mathfrak{k})\big)$ over $\mathfrak{h}=L^2(\mathbb{R}_+,\mathfrak{k})\cong L^2(\mathbb{R}_+)\otimes\mathfrak{k}$ defined by*

$$X'_t = \Lambda(\chi_{[0,t]}\otimes T) + a^+(\chi_{[0,t]}\otimes u) + a^-(\chi_{[0,t]}\otimes v) + t\lambda\mathbf{1}$$

for $t\geq 0$. Furthermore, if we require that $\mathfrak{k}$ is spanned by $\{T^ku, T^kv | k=0,1,\ldots\}$, then $\mathfrak{k}$, T, u, v, and λ are unique up to unitary equivalence.

$(X'_t)_{t\geq 0}$ is symmetric, if and only if $T^*=T$, $u=v$ and $\lambda\in\mathbb{R}$ in the unique minimal tuple.

Remark 3.4. Analogous results hold for the other universal independences. For tensor independence see [Sch91b], for the boolean [BG01], and [Fra01b] for the monotone case. Note that in the boolean and in the monotone case the time process has to be modified.

The five independences can also be used to define convolutions for compactly supported measures. Let μ_1 and μ_2 be two compactly supported probability measures on $\mathbb{R}$ and choose two independent real quantum random variables X_1 and X_2 on some pre-Hilbert space $\mathcal{H}$ such that

$$\langle\Omega, X_i^k\Omega\rangle = \int_{\mathbb{R}} x^k{\rm d}\mu_i$$

for all $k\in\mathbb{N}$ and $i=1,2$ and some unit vector $\Omega\in\mathcal{H}$. The operator X_1+X_2 is again symmetric and bounded, therefore there exists a unique compactly supported probability measure μ such that

$$\int_{\mathbb{R}} x^k{\rm d}\mu = \langle\Omega,(X_1+X_2)^k\Omega\rangle \qquad \text{for all } k\in\mathbb{N}.$$

It is always possible to construct such a pair and the law of $X_1 + X_2$ depends only on the laws of X_1 and X_2 and the notion of independence that has been chosen.

If X_1 and X_2 are tensor independent, then the measure μ obtained in this way is the usual additive convolution of μ_1 and μ_2. If X_1 and X_2 are free, then μ is the free additive convolution of μ_1 and μ_2.

These convolutions can actually be defined for arbitrary probability measures.

It is possible to show that infinitely divisible measures can be embedded into continuous convolution semigroups in all five cases and that furthermore there exists a Lévy process for every continuous convolution semigroup. This shows that in all five cases the infinitely divisible measures on $\mathbb{R}$ (which are characterized by their moments) can be classified by tuples $(\mathfrak{k}, T, u, \lambda)$ consisting of a pre-Hilbert space $\mathfrak{k}$, a symmetric operator T on $\mathfrak{k}$, a vector $u \in \mathfrak{k}$, and a real number λ.

Corollary 3.5. *There exist bijections (up to moment uniqueness) between the five classes of infinitely divisible measures with finite moments.*

Remark 3.6. The bijection between the usual infinitely divisible measures and the freely infinitely divisible measures is known under the name Pata-Bercovici bijection, cf. [BP99], it actually extends to all infinitely divisible measures, not just those characterized by their moments, and has many useful properties, cf. [BNT01a] and the references therein.

For example, the Bercovici-Pata bijection is a homomorphism between the usual infinitely divisible measures and the freely infinitely divisible measures and their respective convolutions. This is not the case for the bijection between usual infinitely divisible measures and the monotone infinitely divisible measures, because due to the non-commutativity of the monotone convolution this is impossible. For the Lévy-Khintchine formula for the boolean and monotone case, see [SW97, Mur00].

Definition 3.7. Let $(X_t)_{t\geq 0}$ be a real additive Lévy process for one of the five universal independences.

If there exists a tuple $(\mathfrak{k}, T, u, \lambda)$ for $(X_t)_{t\geq 0}$ with $T = 0$, then $(X_t)_{t\geq 0}$ is called Gaussian.

If there exists a tuple $(\mathfrak{k}, T, u, \lambda)$ for $(X_t)_{t\geq 0}$ and vector $\omega \in \mathfrak{k}$ such that $u = T\omega$ and $\lambda = \langle \omega, T\omega \rangle$, then $(X_t)_{t\geq 0}$ is called a compound Poisson process.

If $(X_t)_{t\geq 0}$ is Gaussian, then the unique minimal tuple associated to it by Theorem 3.3 has the form $(\mathbb{C}, 0, z, \lambda)$ and $(X_t)_{t\geq 0}$ can be realized as a sum of creation, annihilation and time only, with no conservation part.

Example 3.8. Let $(X_t)_{t\geq 0}$ be a classical compound Poisson process with Lévy measure μ, i.e. with characteric function

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left(t \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1) \mathrm{d}\mu(x)\right).$$

We assume that μ has finite moments, then $(X_t)_{t\geq 0}$ is an additive tensor Lévy process in the sense of Definition 3.1. We can define a tuple $(\mathfrak{k}, T, u, \lambda)$ for $(X_t)_{t\geq 0}$ by Theorem 3.3 as follows. For the pre-Hilbert space $\mathfrak{k}$ we take the space of polynomials

$$\mathfrak{k} = \mathrm{span}\,\{x^k; k = 0, 1, 2, \ldots\},$$

considered as a subspace of the Hilbert space $L^2(\mathbb{R}, \mu)$, i.e., with the inner product

$$\langle x^k, x^\ell \rangle = \int_{\mathbb{R}} x^{k+\ell} \mathrm{d}\mu(x),$$

and divided by the the nullspace of this inner product, if μ is finitely supported. The operator T is multiplication by x, i.e., $Tx^k = x^{k+1}$, the vector u is the function $f(x) = x$, and the scalar λ is the first moment of μ, i.e., $\lambda = \int_{\mathbb{R}} x \mathrm{d}\mu(x)$.

Taking for ω the constant function 1, we see that $(X_t)_{t>0}$ is also a compound Poisson process in sense of Definition 3.7.

Theorem 3.3 can also be used to give an Itô-Lévy-type decomposition of additive quantum Lévy processes.

Corollary 3.9. *Let $(X_t)_{t\geq 0}$ be a real additive Lévy process for one of the five universal independences. Then $(X_t)_{t\geq 0}$ can be realized as a sum of a Gaussian Lévy process $(X_t^{\mathrm{G}})_{t\geq 0}$ and a "jump" part $(X_t^{\mathrm{P}})_{t\geq 0}$, which can be approximated by (compensated) compound Poisson processes.*

Proof. We only briefly outline the proof.

Let $(\mathfrak{k}, T, u, \lambda)$ be a tuple for $(X_t)_{t\geq 0}$. Since T is symmetric, we can decompose the closure of $\mathfrak{k}$ into a direct sum of the closures of the kernel of T and the image of T. Let $u = u_0 + u_1$ with $u_0 \in \overline{\ker T}$ and $u_1 \in \overline{\operatorname{im} T}$.

If u_1 is actually in the image of T, then there exists a vector $\omega \in \mathfrak{k}$ with $T\omega = u_1$ and $(X_t)_{t\geq 0}$ is equivalent to the sum of the Gaussian Lévy process $(X_t^{\mathrm{G}})_{t\geq 0}$ with the tuple $(\mathbb{C}, 0, u_0, \lambda - \langle \omega, T\omega\rangle)$ and the compound Poisson process $(X_t^{\mathrm{P}})_{t\geq 0}$ with the tuple $(\mathfrak{k}, T, u_1, \langle \omega, T\omega\rangle)$.

If u_1 is not in the image of T, then we take a sequence $\omega_n \in \mathfrak{k}$ such that $\lim T\omega_n = u_1$ and define the "jump" part by

$$\begin{aligned} X_t^{\mathrm{P}} &= \Lambda(\chi_{[0,t]} \otimes T) + a^+(\chi_{[0,t]} \otimes u_1) + a^-(\chi_{[0,t]} \otimes u_1) \\ &= \lim_{n\to\infty} \Big(\chi_{[0,t]} \otimes \Lambda(T) + a^+\big(\chi_{[0,t]} \otimes (T\omega_n)\big) + a^-\big(\chi_{[0,t]} \otimes (T\omega_n)\big)\Big), \end{aligned}$$

i.e. as the limit of compensated compound Poisson processes. The Gaussian part is then determined by the tuple $(\mathbb{C}, 0, u_0, \lambda)$. □

Remark 3.10. Using the spectral representation $\overline{T} = \int_{\mathbb{R}} x \mathrm{d}P_x$ of the closure of T, the "jump" part can be written as an integral over the "jump" sizes.

However, note that the Itô-Lévy-type decomposition gives a decomposition into a continuous Gaussian part and a jump part only in the tensor case. In the other cases the classical processes that can be associated to the corresponding Gaussian processes do not have continuous paths, see, e.g., [Bia98].

Using different methods and not assuming the existence of moments, Barndorff-Nielsen and Thorbjørnson [BNT01b] have also obtained an Itô-Lévy decomposition for additive free Lévy processes.

4 Lévy Processes on the (Dual) Affine Group

Recall that the affine group can be defined as the group of matrices

$$\mathrm{Aff} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.$$

The calculation

$$\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix}$$

shows that the group multiplication takes the form

$$\begin{aligned} A(g_1 g_2) &= A(g_1)A(g_2), \\ B(g_1 g_2) &= A(g_1)B(g_2) + B(g_1), \end{aligned}$$

for the coordinates A, B defined by

$$A\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a, \qquad B\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = b.$$

We define tensor Lévy processes on the dual affine group in term of increments. Since the increments are tensor independent for disjoint time intervals, they commute, and we can write the products in the multiplication formula in any order we like.

Definition 4.1. An operator process $\big((A_{st}, B_{st})\big)_{0\le s\le t}$ on $(\mathcal{H}, \Omega)$ is called a (left) tensor Lévy process on the dual affine group (w.r.t. Ω), if the following four conditions are satisfied.

(i) (Increment property) For all $0 \le s \le t \le u$,

$$\begin{aligned} A_{su} &= A_{tu} A_{st}, \\ B_{su} &= A_{tu} B_{st} + B_{tu}. \end{aligned}$$

(ii) (Independence) The increments $(A_{s_1t_1}, B_{s_1t_1}), \ldots, (A_{s_rt_r}, B_{s_rt_r})$ are tensor independent for all $0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_r \le t_r$.

(iii) (Stationarity) The law of (A_{st}, B_{st}) depends only on $t - s$.

(iv) (Weak continuity) For all $k_1, \ldots, k_r, \ell_1, \ldots, l_r \in \mathbb{N}$, we have

$$\lim_{t\searrow s} \langle \Omega, A_{st}^{k_1} B_{st}^{\ell_1} \cdots A_{st}^{k_r} B_{st}^{\ell_r} \Omega \rangle = \begin{cases} 1 & \text{if } \ell_1 + \cdots \ell_r = 0, \\ 0 & \text{if } \ell_1 + \cdots \ell_r > 0. \end{cases}$$

Every Lévy process with values in the semi-group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

with finite moments gives an example of a tensor Lévy process on the dual affine group in the sense of our definition, since we didn't impose that the A_{st} are strictly positive or invertible.

Example 4.2. There are also examples in which A_{st} and B_{st} do not commute and which do not correspond to classical Lévy processes. E.g., the quantum Azéma martingale [Eme89, Par90, Sch91a] with paramater $q \in \mathbb{R}$ defined by the quantum stochastic differential equations

$$\begin{aligned} \mathrm{d}A_{st} &= A_{st} \mathrm{d}\Lambda_t(q-1), \\ \mathrm{d}B_{st} &= B_{st} \mathrm{d}\Lambda_t(q-1) + \mathrm{d}a_t^+(1) + \mathrm{d}a_t^-(1), \end{aligned}$$

on the Bose Fock space $\mathcal{F}_{\mathrm{T}}(L^2(\mathbb{R}_+)$, with initial conditions

$$\begin{aligned} A_{ss} &= \mathrm{id}, \\ B_{ss} &= 0, \end{aligned}$$

defines a tensor Lévy process on the dual affine group. For $q = 1$, we have $A_{st} = \mathrm{id}$ for all $0 \le s \le t$ and in the vacuum state (B_{st}) is equivalent to classical Brownian motion. For $q \ne 1$, A_{st} and B_{st} do not commute and (B_{st}) is equivalent to the classical Azéma martingale with parameter $c = q-1$.

When we want to define free Lévy processes, different orders of the products in the multiplication formula will lead to different classes of Lévy processes. The choice in the definition proposed here is motivated by the fact that if B_{st} and B_{tu} are symmetric, then B_{su} is also symmetric.

Definition 4.3. An operator process $\big((a_{st}, B_{st})\big)_{0\le s\le t}$ on $(\mathcal{H}, \Omega)$ is called a (left) free Lévy process on the dual affine group (w.r.t. Ω), if the conditions

(i') (Increment property) For all $0 \le s \le t \le u$,

$$\begin{aligned} a_{su} &= a_{tu} a_{st}, \\ B_{su} &= a_{tu} B_{st} a_{tu}^* + B_{st}. \end{aligned}$$

(ii') (Independence) The increments $(a_{s_1t_1}, B_{s_1t_1}), \ldots, (a_{s_rt_r}, B_{s_rt_r})$ are free for all $0 \le s_1 \le t_1 \le s_2 \le \cdots \le s_r \le t_r$.

and conditions (iii) and (iv) from the previous definition are satisfied (with $A_{st} = a_{st} a_{st}^*$).

Note that with this definition $A_{st} = a_{st} a_{st}^*$ is automatically positive.

Example 4.4. Let $\gamma \in \mathbb{C}$. The operator process $\big((a_{st}, B_{st})\big)_{0\le s\le t}$ defined by the quantum stochastic equations

$$\begin{aligned} \mathrm{d}a_{st} &= \mathrm{d}\Lambda_t(\gamma-1)a_{st}, \\ \mathrm{d}B_{st} &= \mathrm{d}\Lambda_t(\gamma-1)B_{st} + B_{st}\mathrm{d}\Lambda_t(\overline{\gamma}-1) + \mathrm{d}a_t^+(1) + \mathrm{d}a_t^-(1), \end{aligned}$$

on the free Fock space $\mathcal{F}_{\mathrm{F}}(L^2(\mathbb{R}_+))$, with initial conditions

$$\begin{aligned} a_{ss} &= \mathrm{id}, \\ B_{ss} &= 0, \end{aligned}$$

defines a free Lévy process on the dual affine group. For $\gamma = 1$, we get $a_{st} = \mathrm{id}$ and (B_{st}) is equal to the free Brownian motion,

$$B_{st} = a^+(\chi_{[s,t[}) + a^-(\chi_{[s,t[}).$$

For general $\gamma \in \mathbb{C}$, the process (B_{st}) can be considered as a free analog of the (quantum) Azéma martingale with parameter $q = |\gamma|^2$.

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EXPLICIT SOLUTIONS OF THE SEQUENTIAL ANALYSIS PROBLEMS FOR COMPOUND POISSON PROCESSES WITH EXPONENTIAL JUMPS

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We give the explicit solutions for the Bayesian problem of sequential testing and the Bayesian disorder problem for an observed compound Poisson process having exponentially distributed jumps. Also, we determine the explicit stopping boundaries for the Wald's sequential probability ratio test, which is optimal in the variational formulation of the sequential testing problem, and remark some facts about the solution of the variational disorder problem.

1 Introduction

Assume that at time $t = 0$ we begin to observe the process $X = (X_t)_{t \geq 0}$ given by

$$X_t = \sum_{j=0}^{N_t} \xi_j$$

for $t > 0$ and $X_0 = 0$, where $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $1/\lambda$, $\lambda > 0$, and $(\xi_j)_{j \in \mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter λ (N and $(\xi_j)_{j \in \mathbb{N}}$ are supposed to be mutually independent). We call the process $X = (X_t)_{t \geq 0}$ the compound Poisson process with exponential jumps (with parameter $\lambda > 0$). It is easily verified that the process $X = (X_t)_{t \geq 0}$ is a semimartingale with the triplet $(t/\lambda^2, 0, dtI(x > 0)e^{-\lambda x}dx)$ with respect to the function $h(x) = x$, $x \in \mathbb{R}$ (see, e.g., [8], Chapter II, Section 2).

We consider the following two problems of sequential analysis for a compound Poisson process with exponential jumps: the sequential hypotheses testing problem and the problem of detecting a "disorder".

In the first problem it is supposed that the parameter λ is unknown and takes on the values λ_0 and λ_1. The problem of sequential testing of two simple hypotheses about the parameter λ is to decide as soon as possible and with minimal error probabilities if the true value of λ is either λ_0 or λ_1. This problem admits two different formulations (see [15], [13]). In the Bayesian formulation it is assumed that at time $t = 0$ the parameter λ takes on the values λ_0 and λ_1 according to an a priori distribution, which is given to us. The variational formulation (also called a fixed error probability formulation) involves no probabilistic assumptions on the unknown parameter λ.

In the second problem it is assumed that the value of the parameter λ changes from λ_0 to λ_1 at some unknown time $\theta \geq 0$ called the time of "disorder", which can not be observed directly. The problem of disorder is to decide by observing the process X at which time instant one should give an "alarm signal" indicating the occurence of "disorder" as close as possible to the time θ in the sence, that both the probability of "false alarm" and the expectation of the time interval between the "alarm signal" and the occurence of "disorder" (when the "alarm signal" is given correctly) should be minimal. We suppose that the random time θ is exponentially distributed under $\theta > 0$.

By use of the Bayesian approach, Wald and Wolfowitz (see [16], [17]) proved the optimality of the Wald's sequential probability ratio test (SPRT) in the variational formulation of the sequential testing problem for the case of i.i.d. observations and under some special assumptions. The problem of disorder for the discrete time case was considered in several works of A.N. Shiryaev (see [13], p. 208, for historical notes and references). Shiryaev ([12]) gave the explicit solution to both problems of sequential analysis in the Bayesian and variational formulations in the case of Wiener process (see also [13], Chapter 4). Some particular cases of the Poisson disorder problem were considered in [4] and in [3]. Peskir and Shiryaev ([9]) obtained the explicit solution for the Bayesian problem of testing hypotheses about the intensity of an observed Poisson process and then applied their result to the proof of the optimality of the SPRT in the corresponding variational problem. They also presented the complete explicit solution of the Poisson disorder problem in the Bayesian formulation ([10]).

Our aim is to present the explicit solutions of the sequential testing problem and the disorder problem in the Bayesian and variational formulations for a compound Poisson process with exponential jumps. Actually, we present the next example of process (followed by the Wiener and Poisson cases) for which these problems of sequential analysis can be solved in the explicit form. Such (multivariate point) Lévy processes are used, for example, in several models of stochastic finance and insurance (see, e.g., [14]).

In Section 2, following the schema of arguments used in [13], Chapter 4, and in [9], we reduce the initial Bayesian problem to the relevant Stephan problem for an integro-differential operator, which can be solved explicitly by use of the smooth and continuous fit principles, and then we prove that its solution is a solution of the initial (optimal stopping) problem.

In Section 3 we construct the SPRT for the variational problem and, using the argumments in [9], obtain the explicit form of the optimal stopping boundaries. Also, we give a precise description of the set of all admissible error probabilities and find the explicit expressions for them and for the mean time of the observations.

In Section 4, following the schema of arguments in [10], we obtain the complete explicit solution of the Bayesian disorder problem for a compound Poisson process with exponential jumps. For proof we reduce the initial problem to a free-boundary integro-differential Stephan problem and then specify the cases of the breakdown of the smooth fit principle and its replacement by the principle of the continuous fit. These effects can be explained both by the examination of the sample-path properties of the a posteriori probability process and by the existence of a singularity point of the relevant integro-differential equation. Also, we remark some facts about the solutions of the Bayesian disorder problem for pure jump natural exponential families, for example, for an inverse Gaussian process (see, e.g., [1]) and for a Gamma process (see, e.g., [11] or [14]).

Section 5 is devoted to some notes about the solution of the variational disorder problem for a compound Poisson process with exponential jumps.

2. Bayesian sequential testing problem.

3. Variational sequential testing problem.

4. Bayesian disorder problem.

5. Variational disorder problem.

Remark. The results announced above are presented in [5] and [6].

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Non-Markovian random walk models, scaling and diffusion limits

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Abstract

A proper transition to the so-called diffusion limit is discussed for continuous time random walks. It turns out that the probability density function for the limit process obeys a space-time fractional diffusion equation.

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1 Introduction and basic equations

In this paper we show how by a properly scaled passage to the limit of compressed waiting times and jump widths the space-time fractional diffusion equation can be obtained from the master equation for a continuous time random walk or, equivalently, from the master equation describing a cumulative renewal process. For the basic principles of *continuous time random walk* (CTRW), that was introduced in Statistical Mechanics by Montroll and Weiss [22], see *e.g.* [1, 15, 20, 21, 23, 30, 31], of renewal processes, see *e.g.* [3, 4, 6, 16, 27, 28, 29].

The continuous random walk arises by a sequence of independently identically distributed (*iid*) positive random waiting times $T_1, T_2, \dots$, each having probability density function (*pdf*) $\psi(t)$, $t > 0$, and a sequence of *iid* random jumps $X_1, X_2, X_3, \dots$ in $\mathbb{R}$, each having *pdf* $w(x)$, $x \in \mathbb{R}$. Setting $t_0 = 0$, $t_n = T_1 + T_2 + \dots T_n$ for $n \in \mathbb{N}$, $0 < t_1 < t_2 < \dots$, the wandering particle starts at point $x = 0$ in instant $t = 0$ and makes a jump of length X_n in instant t_n, so that its position is $x = 0$ for $0 \le t < T_1 = t_1$, and

$$S_n = X_1 + X_2 + \cdots X_n \,, \quad \text{for} \quad t_n \le t < t_{n+1} \,.$$

An essential assumption is that the waiting time distribution and the jump width distribution are independent of each other. It is well known that this stochastic process is *Markovian* if and only if the waiting time *pdf* is of the form $\psi(t) = m\,\mathrm{e}^{-mt}$ with some positive constant m (*compound Poisson process*), see *e.g.* [6].

Then, by natural probabilistic arguments we arrive at the *master equation* for the probability density function $p(x,t)$ of the particle being in point x at instant t, see [11, 19, 26],

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \psi(t-t')\,\left[\int_{-\infty}^{+\infty} w(x-x')\,p(x',t')\,dx'\right]\,dt' \,, \tag{1.1}$$

in which $\delta(x)$ denotes the Dirac generalized function, and, for abbreviation, $\Psi(t) = \int_t^\infty \psi(t')\, dt'$, is the probability that at instant t the particle is still sitting in its starting position $x = 0$. Actually, $p(x,t)$ as containing a point measure, is a generalized *pdf*, but for ease of language we omit the qualification "generalized". Clearly, (1.1) satisfies the initial condition $p(x,0) = \delta(x)$.

Throughout this paper we will denote by $\widehat{f}(\kappa)$ and $\widetilde{g}(s)$ the transforms of Fourier and Laplace, respectively, of functions $f(x)$ and $g(t)$ according to

$$\mathcal{F}\{f(x);\kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{\infty} \mathrm{e}^{\,i\kappa x}\, f(x)\, dx\,, \quad \kappa \in \mathbb{R}\,,$$

$$\mathcal{L}\{g(t);s\} = \widetilde{g}(s) = \int_0^{\infty} \mathrm{e}^{\,-st}\, g(t)\, dt\,, \quad s > s_0{}^1\,,$$

and consistently, we will have for Dirac's delta function $\widehat{\delta}(\kappa) \equiv \widetilde{\delta}(s) \equiv 1$.

Then in the Fourier-Laplace domain the master equation (1.1) appears as

$$\widehat{\widetilde{p}}(\kappa,s) = \widetilde{\Psi}(s) + \widetilde{\psi}(s)\, \widehat{w}(\kappa)\, \widehat{\widetilde{p}}(\kappa,s)\,, \tag{1.2}$$

whence,

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{\widetilde{\Psi}(s)}{1 - \widehat{w}(\kappa)\, \widetilde{\psi}(s)} = \frac{1 - \widetilde{\psi}(s)}{s}\, \frac{1}{1 - \widehat{w}(\kappa)\, \widetilde{\psi}(s)}\,. \tag{1.3}$$

We will henceforth assume that in our continuous time random walk the jump width *pdf* $w(x)$ is an even function ($w(x) = w(-x)$) and has a finite second moment (variance) or exhibits the asymptotic behaviour $w(|x|) \sim b\, |x|^{-(\alpha+1)}$ with some α , $0 < \alpha < 2$, for $|x| \to \infty$, and the waiting time *pdf* $\psi(t)$ has a finite first moment (mean) or exhibits the asymptotic behaviour $\psi(t) \sim c\, t^{-(\beta+1)}$ with some β , $0 < \beta < 1$, for $t \to \infty$, where b and c are positive constants.

Our aim is to derive from the master equation (1.1), by properly rescaling the waiting times and the jump widths and passing to the diffusion limit, the *space-time fractional diffusion equation.* This is a partial pseudo-differential equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a fractional Riesz derivative of order $\alpha \in (0,2]$ and the first-order time derivative with a fractional Caputo derivative of order $\beta \in (0,1]$. Choosing the initial condition in analogy to that fulfilled for (1.1), we write

$$\begin{gathered} {}_tD_*^\beta\, u(x,t) = {}_xD_0^\alpha\, u(x,t)\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,, \\ u(x,0^+) = \delta(x)\,, \quad x \in \mathbb{R}\,, \quad t > 0\,. \end{gathered} \tag{1.4}$$

In view of the particular initial condition, the solution of this Cauchy problem is referred to as the *fundamental solution* or the *Green function.*

The fractional *Riesz* derivative ${}_xD_0^\alpha$ is defined as the pseudo-differential operator with symbol $-|\kappa|^\alpha$. This means that for a sufficiently well-behaved function $f(x)$ we have[2]

$$\mathcal{F}\{\,{}_xD_0^\alpha\, f(x);\kappa\} = -|\kappa|^\alpha\, \widehat{f}(\kappa)\,, \quad \kappa \in \mathbb{R}\,. \tag{1.5}$$

The symbol of the Riesz fractional derivative is nothing but the logarithm of the characteristic function of the generic symmetric *stable* (in the Lévy sense) probability density, see [5, 6, 25]. Noting $-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$, we recognize that

$${}_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\,. \tag{1.6}$$

In other words: the Riesz derivative is a symmetric fractional generalization of the second derivative.

[1] For our purposes we agree to take s real

[2] Let us recall that a generic linear pseudo-differential operator A, acting with respect to the variable $x \in \mathbb{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\, A\,[f(x)]\, dx = \widehat{A}(\kappa)\, \widehat{f}(\kappa)$, where $\widehat{A}(\kappa)$ is referred to as symbol of A , formally given as $\widehat{A}(\kappa) = \left(A\, \mathrm{e}^{\,-i\kappa x}\right)\, \mathrm{e}^{\,+i\kappa x}$.

The *Caputo* fractional derivative in time provides a fractional generalization of the first derivative through the following rule in the Laplace transform domain,

$$\mathcal{L}\left\{{}_tD_*^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - s^{\beta-1}\, f(0^+)\,, \quad 0 < \beta \le 1\,, \quad s > 0\,, \tag{1.7}$$

hence turns out to be defined as, see *e.g.* [9],

$$ {}_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \dfrac{f^{(1)}(\tau)}{(t-\tau)^\beta}\, d\tau\,, & 0 < \beta < 1, \\ \dfrac{d}{dt} f(t), & \beta = 1. \end{cases} \tag{1.8}$$

It can alternatively be written in the form

$$\begin{aligned} {}_tD_*^\beta\, f(t) &= \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, d\tau + \frac{t^{-\beta}}{\Gamma(1-\beta)}\, f(0^+) \\ &= \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \int_0^t \frac{f(\tau) - f(0^+)}{(t-\tau)^\beta}\, d\tau\,, \quad 0 < \beta < 1\,. \end{aligned} \tag{1.9}$$

The Caputo derivative has been indexed with $*$ in order to distinguish it from the classical Riemann-Liouville fractional derivative ${}_tD^\beta$, the first term at the R.H.S. of the first equality in (1.9). As it can be noted from the last equality in (1.9) the Caputo derivative provides a sort of regularization at $t = 0$ of the Riemann-Liouville derivative.

The *space-time fractional diffusion* equation (1.4) contains as particular cases the *space fractional diffusion* equation when $0 < \alpha < 2$ and $\beta = 1$, the *time fractional diffusion* equation when $\alpha = 2$ and $0 < \beta < 1$, and the *standard diffusion equation* when $\alpha = 2$ and $\beta = 1$. We note that for the fractional cases the word "diffusion" is also justified because the fundamental solution (or the *Green function*) in all cases can be interpreted as a space probability density evolving in time, see *e.g.* Gorenflo, Iskenderov & Luchko [8], Mainardi, Luchko & Pagnini [17] and Mainardi, Pagnini & Gorenflo [18].

In the Fourier-Laplace domain the Cauchy problem for the space-time fractional diffusion equation (1.4) appears in the form

$$s^\beta\, \widehat{\widetilde{u}}(\kappa, s) - s^{\beta-1} = -|\kappa|^\alpha\, \widehat{\widetilde{u}}(\kappa, s)\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,, \tag{1.10}$$

and we obtain immediately

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}\,, \quad s > 0\,, \quad \kappa \in \mathbb{R}\,. \tag{1.11}$$

By our derivation of (1.4) from (1.1) we de-mystify the often asked-for meaning of the time fractional derivative in the fractional diffusion equation. In plain words, the fractional derivatives in time as well in space are caused by asymptotic power laws and well-scaled passage to the diffusion limit.

The paper is divided as follows. In Section 2 we present two lemmata on asymptotics of characteristic functions and of Laplace transforms of *pdf*'s obeying asymptotic power laws. In Section 3 we carry out the promised transition to the diffusion limit. Finally, the main conclusions are drawn in Section 4.

2 Two Lemmata

The first Lemma is a modified specialisation of Gnedenko's theorem in [7], see also [2]. It was already used by us, but not formally called a Lemma, in [10]. The second Lemma can be obtained by aid of a corollary in Widder's book [32].

Lemma 1. *Assume* $w(x) \ge 0$, $w(x) = w(-x)$ *for* $x \in I\!\!R$, $\int_{-\infty}^{+\infty} w(x)\, dx = 1$, *and either*

$$\sigma^2 := \int_{-\infty}^{+\infty} x^2\, w(x)\, dx < \infty \tag{2.1}$$

(relevant in the case $\alpha = 2$*) or, with* $b > 0$ *and some* $\alpha \in (0, 2)$,

$$w(x) = (b + \epsilon(|x|))\, |x|^{-(\alpha+1)} . \tag{2.2}$$

In (2.2) assume $\epsilon(|x|)$ *bounded and* $O\left(|x|^{-\eta}\right)$ *with some* $\eta > 0$ *as* $|x| \to \infty$.

Then, with a positive scaling parameter h *and a scaling constant*

$$\mu = \begin{cases} \dfrac{\sigma^2}{2} , & \text{if} \quad \alpha = 2 , \\[2ex] \dfrac{b\, \pi}{\Gamma(\alpha + 1)\, \sin(\alpha\pi/2)} , & \text{if} \quad 0 < \alpha < 2 , \end{cases} \tag{2.3}$$

we have, for each fixed $\kappa \in I\!\!R$, *the asymptotic relation*

$$\widehat{w}(\kappa h) = 1 - \mu\, (|\kappa|\, h)^\alpha + o(h^\alpha) \quad \text{for} \quad h \to 0 . \tag{2.4}$$

Remark. Eq. (2.4) holds trivially if $\kappa = 0$ since $\widehat{w}(0) = 1$.

Lemma 2. *Assume* $\psi(t) \ge 0$ *for* $t > 0$, $\int_0^{\infty} \psi(t)\, dt = 1$, *and either*

$$\rho := \int_0^{\infty} t\, \psi(t)\, dt < \infty \tag{2.5}$$

(relevant in the case $\beta = 1$*), or, with* $c > 0$ *and some* $\beta \in (0, 1)$,

$$\psi(t) \sim c\, t^{-(\beta+1)} \quad \text{for} \quad t \to \infty . \tag{2.6}$$

Then, with a positive scaling parameter τ *and a scaling constant*

$$\lambda = \begin{cases} \rho , & \text{if} \quad \beta = 1 , \\[2ex] \dfrac{c\, \Gamma(1 - \beta)}{\beta} , & \text{if} \quad 0 < \beta < 1 , \end{cases} \tag{2.7}$$

we have, for each fixed $s > 0$, *the asymptotic relation*

$$\widetilde{\psi}(s\tau) = 1 - \lambda\, (s\tau)^\beta + o(\tau^\beta) \quad \text{for} \quad \tau \to 0 . \tag{2.8}$$

Remark. Eq. (2.8) holds trivially if $s = 0$ since $\widetilde{\psi}(0) = 1$.

Proof of Lemma 1. For convenience we abbreviate $\nu = \kappa\, h$. We observe that $\widehat{w}$, like w , is an even function.

In the case $\alpha = 2$ the well known fact $\sigma^2 = -\widehat{w}''(0)$ immediately implies

$$\widehat{w}(\nu) = 1 - \mu\, \nu^2 + o(\nu^2) \quad \text{as} \quad \nu \to 0 ,$$

hence (2.4) with $\mu = \sigma^2/2$.

In the case $0 < \alpha < 2$ we find for $\nu \neq 0$

$$\widehat{w}(\nu) - 1 = \int_0^\infty \left(e^{i\nu x} + e^{-i\nu x} - 2\right) w(x)\, dx = -4 \int_0^\infty \left(\sin(\nu x/2)\right)^2 w(x)\, dx\,,$$

hence, in view of (2.2),

$$\widehat{w}(\nu) = 1 - 2^{-\alpha+2}\, b\, \nu^\alpha \int_0^\infty \xi^{-\alpha-1} \left(\sin\xi\right)^2 d\xi - 4 \int_0^\infty \epsilon(x)\, x^{-\alpha-1} \left(\sin(\nu x/2)\right)^2 dx\,.$$

The first integral can be evaluated in terms of the gamma function. In fact, from Gradshteyn and Ryzhik [12], see (3.823), we take

$$\int_0^\infty \xi^{-\alpha-1} \left(\sin\xi\right)^2 d\xi = -\frac{\Gamma(-\alpha)\, \cos(\alpha\pi/2)}{2^{1-\alpha}} = \frac{\pi}{2^{2-\alpha}\, \Gamma(\alpha+1)\, \sin(\alpha\pi/2)}\,.$$

The latter equality follows by the reflection formula for the gamma function.

We estimate the second integral via decomposition $\int_0^\infty \cdots = \int_0^A \cdots + \int_A^\infty \cdots$, taking $A = \nu^{-(2\alpha+\eta)/(2\alpha+2\eta)}$, using $|\sin\xi| \leq \min\{\xi, 1\}$ for $\xi \geq 0$ and the condition on $\epsilon(|x|)$. By careful calculation we find that it behaves asymptotically as $o\left(\nu^\alpha\right) = |\kappa|^\alpha\, o\left(h^\alpha\right)$. Combining these results and observing that $\widehat{w}$ (like w) is an even function, we obtain

$$\widehat{w}(\kappa h) = 1 - |\kappa|^\alpha \frac{b\,\pi}{\Gamma(\alpha+1)\, \sin(\alpha\pi/2)}\, h^\alpha + |\kappa|^\alpha\, o\left(h^\alpha\right)\,, \quad h \to 0\,,$$

as valid for all $\kappa \in \mathbb{R}$. This is equivalent to (2.4), and the lemma is proved. □

Proof of Lemma 2. In the case $\beta = 1$ Eq. (2.8) is a consequence of $\lambda = \rho = -\widetilde{\psi}'(0)$. In the case $0 < \beta < 1$ we invoke Corollary 1a of Widder [32], see p. 182. It states (among other things) that if $\alpha(t) \sim A\, t^\gamma/\Gamma(\gamma+1)$ as $t \to \infty$ for some $A \neq 0$ and some non-negative γ, then

$$\int_0^\infty \mathrm{e}^{-st}\, d\alpha(t) \sim \frac{A}{s^\gamma}\,, \quad \text{for} \quad s \to 0^+\,. \tag{2.9}$$

Now assume the conditions of Lemma 2 met. Then, for $0 < \beta < 1$,

$$\widetilde{\psi}(s) = \int_0^\infty \psi(t)\, dt - \int_0^\infty \left(1 - e^{-st}\right) \psi(t)\, dt = 1 - \int_0^\infty \left(1 - e^{-st}\right) \psi(t)\, dt\,. \tag{2.10}$$

With $\Psi(t) = \int_t^\infty \psi(t')\, dt'$ as used in (1.1), then

$$\Psi(0) = 1\,, \quad \Psi(\infty) = 0\,, \quad \Psi(t) \sim \frac{c}{\beta}\, t^{-\beta}\,, \quad \text{as} \quad t \to \infty\,,$$

hence

$$\int_0^\infty \left(1 - e^{-st}\right) \psi(t)\, dt = s \int_0^\infty \mathrm{e}^{-st}\, \Psi(t)\, dt = s\, \widetilde{\Psi}(s)\,. \tag{2.11}$$

For application of Widder's corollary to $\widetilde{\Psi}(s)$ we set $\alpha'(t) = \Psi(t)$ and obtain (the constant of integration being irrelevant)

$$\alpha(t) \sim \frac{c}{\beta(1-\beta)}\, t^{-\beta+1} = A\, \frac{t^\gamma}{\Gamma(\gamma+1)}$$

with $\gamma = 1 - \beta > 0$, $A = \lambda$. Then (2.9) yields

$$\widetilde{\Psi}(s) \sim \lambda\, s^{\beta-1} \quad \text{for} \quad s \to 0^+\,.$$

Inserting this into (2.11) and using (2.10) we get

$$\widetilde{\psi}(s) \sim 1 - \lambda\, s^\beta + o(s^\beta) \quad \text{for} \quad s \to 0^+\,,$$

and, replacing here s by $s\tau$, finally (2.8). □

3 Well-scaled transition to the diffusion limit

Scaling is achieved by making smaller all waiting times by a positive factor τ, and all jumps by a positive factor h. So we get the jump instants

$$t_n(\tau) = \tau T_1 + \tau T_2 + \cdots + \tau T_n \quad \text{for} \quad n \in \mathbb{N}\,, \tag{3.1}$$

and the jump sums,

$$S_0(h) = 0\,, \quad S_n(h) = hX_1 + hX_2 + \cdots + hX_n \quad \text{for} \quad n \in \mathbb{N}\,, \tag{3.2}$$

The reduced waiting times τT_n all have the *pdf* $\psi_\tau(t) = \psi(t/\tau)/\tau$, $t > 0$, and analogously the reduced jumps hX_n all have the *pdf* $w_h(x) = w(x/h)/h$, $x \in \mathbb{R}$. Readily we see

$$\widetilde{\psi}_\tau(s) = \widetilde{\psi}(s\tau)\,, \quad \widehat{w}_h(\kappa) = \widehat{w}(\kappa h)\,. \tag{3.3}$$

Replacing in (1.1) $\psi(t)$ by $\psi_\tau(t)$, $\Psi(t)$ by $\Psi_\tau(t) = \int_t^\infty \psi_\tau(t')\,dt'$, $w(x)$ by $w_h(x)$, $p(x,t)$ by $p_{h,\tau}(x,t)$ we obtain the rescaled master equation which in the Fourier-Laplace domain reads as

$$\widehat{\widetilde{p_{h,\tau}}}(\kappa,s) = \left(1 - \widetilde{\psi}_\tau(s)\right)/s + \widetilde{\psi}_\tau(s)\,\widehat{w}_h(\kappa)\,\widehat{\widetilde{p_{h,\tau}}}(\kappa,s)\,, \tag{3.4}$$

whose solution is

$$\widehat{\widetilde{p_{h,\tau}}}(\kappa,s) = \frac{1 - \widetilde{\psi}_\tau(s)}{s}\,\frac{1}{1 - \widehat{w}_h(\kappa)\,\widetilde{\psi}_\tau(s)}\,. \tag{3.5}$$

To proceed further we assume the *pdf*'s $w(x)$ and $\psi(t)$ of the jumps X_n and the waiting times T_n to meet the conditions of Lemma 1 and Lemma 2, respectively. Eq. (3.5) then becomes asymptotically

$$\widehat{\widetilde{p_{h,\tau}}}(\kappa,s) \sim \frac{\lambda\,\tau^\beta\,s^{\beta-1}}{\lambda\,\tau^\beta\,s^\beta + \mu\,h^\alpha\,|\kappa|^\alpha}\,, \quad h,\tau \to 0\,. \tag{3.6}$$

By imposing the *scaling relation*

$$\lambda\,\tau^\beta = \mu\,h^\alpha\,, \tag{3.7}$$

the asymptotics (3.6) yields

$$\widehat{\widetilde{p_{h,\tau}}}(\kappa,s) \to \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}\,. \tag{3.8}$$

Hence, in view of (1.11),

$$\widehat{\widetilde{p_{h,\tau}}}(\kappa,s) \to \widehat{\widetilde{u}}(\kappa,s) \quad \text{for} \quad h,\tau \to 0\,, \tag{3.9}$$

under condition (3.7). In this kind of passage to the limit, (3.5) and (1.11), likewise (3.4) and (1.10) are asymptotically equivalent in the Fourier-Laplace domain. Then, the asymptotic equivalence in the space-time domain between the master equation (1.1) after rescaling and the fractional diffusion equation (1.4) is provided by the continuity theorem for sequences of characteristic functions after having applied the analogous theorem for sequences of Laplace transforms, see *e.g.* , [6]. Therefore we have "convergence in law" or "weak convergence" for the corresponding probability distributions. We can state this result as a theorem.

Theorem. *Assume the jumps X_n and the waiting times T_n on which the master equation (1.1) is based to fulfil the conditions of Lemma 1 and Lemma 2, respectively. Replace the jump widths X_n by $h\,X_n$, the waiting times T_n by $\tau\,T_n$ with $h > 0$, $\tau > 0$, and assume with the scaling constants μ as in (2.3) and λ as in (2.7) the scaling relation (3.7) to hold. Then, for $h \to 0$ (and consequently $\tau \to 0$) the master equation (1.1), after rescaling, goes over into the space-time fractional diffusion equation (1.4).*

4 Discussion and conclusions

We have shown how, by properly scaled compression of waiting times and jump widths, the (spatially symmetric) space-time fractional diffusion equation (1.4) in the cases $0 < \alpha < 2$ and $0 < \beta < 1$ can be obtained from the (spatially symmetric) master equation (1.1) of the continuous time random walk. For this it suffices to assume there an asymptotic power law of the form $c\,t^{-(\beta+1)}$ as $t \to \infty$ for the waiting time *pdf*, and of the form $b\,|x|^{-(\alpha+1)}$ as $|x| \to \infty$ for the jump width *pdf*. In the time-extremal case $\beta = 1$ the essential requirement is finite mean waiting time, in the space-extremal case $\alpha = 2$ we require finite variance of the jump width distribution.

Only in the case $\beta = 1$ the limiting equation (1.4) describes a Markov process while, however, the master equation still describes a non-Markovian process if the waiting time is not exponentially distributed.

By stating our two lemmata and displaying explicitly the scaling relation $\lambda\,\tau^\beta = \mu\,h^\alpha$ for the compression factors τ in time, h in space, and its consequence in the Fourier-Laplace domain we hope to have clarified what in the literature is often called *long-time large-space* behaviour and vaguely described by the asymptotics of the transforms of Fourier and Laplace near the origin. And we have related this behaviour to the space-time fractional diffusion equation which can be analyzed by techniques belonging to the theory of pseudo-differential equations and, in particular, of the so-called fractional calculus.

Needless to say that, for consideration of the long-time large-space behaviour of the continuous time random walk, our compression of waiting times and jump widths simply amounts to an inflation of the measurement units in space and time, so that medium size waiting times and jumps appear small.

For ease of presentation we have only treated the case of one space dimension and of symmetric jump *pdf*, but these restrictions are not essential. Likewise, in order to avoid more cumbersome calculations with Stieltjes integrals and measures, we have not treated the more general case of probability distribution functions obeying corresponding power laws (with adjusted exponents) given for the waiting times and jump widths. We leave these extensions to a future paper.

One of our intentions was to de-mystify the meaning of the time-fractional derivative in the space-time fractional diffusion equation (1.4). Some authors, see *e.g.* Hilfer and Anton [13], Hilfer [14], Saichev and Zaslavsky [24] and also we in [11, 19], have introduced already in the master equation for CTRW a fractional time derivative by requiring the waiting time to be given via a Mittag-Leffler type function. After doing so, already in [11] we carried out our scaled transition to the limit. However, if one is willing to make the scaled transition to the diffusion limit in time as well as in space directly in the master equation, one can replace, as we have shown here, the Mittag-Leffler waiting time *pdf* (which exhibits a special asymptotic power law) by a more general asymptotic power law.

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Tails of solutions of certain nonlinear stochastic differential equations driven by heavy tailed Lévy motions

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Extended abstract

Let $\big(\Omega, \mathcal{F}, P\big)$ be a probability space and $\big(\mathcal{F}_t,\, t \geq 0)$ a filtration on that space. We will always assume, without further special notice, that the filtration is complete and right continuous (the " usual hypothesis"; see [5]).

Recall that a Lévy motion $(L(t),\, t \geq 0)$ is a continuous in probability stochastic process adapted to the filtration $\big(\mathcal{F}_t,\, t \geq 0)$ with $L(0) = 0$ and stationary increments, such that for every $0 \leq s < t$ the increment $L(t) - L(s)$ is independent of $\mathcal{F}_s$. A Lévy motion has a version with sample paths in the space $D[0, \infty)$ of right continuous functions with left limits, and we will always assume that we are dealing with such a version. The law of a Lévy process is completely characterized by its one-dimensional distribution at time 1 (say) and

$$Ee^{i\theta L(1)} = \exp\left\{ \int_{-\infty}^{\infty} \big(e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leq 1)\big)\, \nu(dx) + i\theta a \right\}, \quad \theta \in \mathbb{R}\,, \tag{1}$$

with ν a σ-finite measure that does not charge the origin, such that $\int_{\mathbb{R}} \min(x^2, 1)\, \nu(dx) < \infty$ and $a \in \mathbb{R}$. The measure ν is the Lévy measure of the process. If the Lévy measure is infinite then, on an event of probability 1, the Lévy process has a dense set of discontinuities in every interval of positive length. On the other hand, if the Lévy measure ν is finite, then the jumps of the process L form a homogeneous Poisson process on the positive half line with intensity $\nu(\mathbb{R})$. The jump sizes are iid random variables independent of the jump times with common distribution $\nu/\nu(\mathbb{R})$. Between every two jumps the process is linear with the slope $a - \int_{|x| \leq 1} x\, \nu(dx)$. No matter whether the Lévy measure is finite or not, the Lévy process L is continuous with probability 1 at any fixed point.

A Lévy process is symmetric (i.e. L and $-L$ have the same finite dimensional distributions) if and only if in (1) $a = 0$ and the Lévy measure is symmetric. In that case the characteristic function (1) can be written in a simpler form

$$Ee^{i\theta L(1)} = \exp\left\{ \int_{-\infty}^{\infty} \big(e^{i\theta x} - 1\big)\, \nu(dx) \right\}, \quad \theta \in \mathbb{R}\,. \tag{2}$$

An important fact is that, in certain cases, the tails of the one dimensional distributions of the Lévy process and its Lévy measure are equivalent. Specifically, the equivalence

$$P(L(t) > u) \sim t\nu\Big((u, \infty)\Big) \text{ as } u \to \infty \tag{3}$$

for any $t > 0$ holds under the assumption of *subexponentiality* on the tail of either $L(1)$ or that of ν; see [2]. We will, actually, assume that the tails are regularly varying:

$$\nu\Big((u, \infty)\Big) = u^{-\alpha} l(u) \tag{4}$$

for some $\alpha > 0$, where l is a slowly varying at infinity function. Since regularly varying tails are subexponential, (3) holds in this case. An important example of Lévy motions satisfying (4) is that of α-stable motions in which case

$$\nu(dx) = \begin{cases} c_+ x^{-(\alpha+1)}\, dx & \text{if } x > 0 \\ c_- |x|^{-(\alpha+1)}\, dx & \text{if } x < 0 \end{cases} \tag{5}$$

for $0 < \alpha < 2$ and $c_+, c_- \geq 0$. A source of information on α-stable processes, of which α-stable motions is an example, is in [6].

We study a stochastic differential equation of the form

$$dX(t) = -f(X(t))\, dt + dL(t)\,, \tag{6}$$

where $(L(t),\, t \geq 0)$ is a symmetric Lévy motion with Lévy measure ν satisfying the regular variation assumption (4). In the main result of the next section, that describes the tail behaviour of the solution to the above equation at any fixed positive time t we will also assume that the Lévy motion is symmetric. This assumption is not needed for the arguments used in the present section. The following assumptions are imposed on the function f.

$$f \text{ is Lipschitz on compact intervals} \tag{7}$$

$$f(0) = 0, \text{ and } f \text{ is nondecreasing.} \tag{8}$$

$$f \text{ is regularly varying at infinity with exponent } \beta > 1, \tag{9}$$

and for some constants $A \in (0, \infty)$ and $\beta_1 > 1$,

$$-f(-x) \geq A x^{-\beta_1} \quad \text{for all } x \geq 1. \tag{10}$$

Note that a Lévy motion is a semimartingale and, hence, the standard theory of stochastic integration applies to stochastic differential equations with respect to Lévy motions. Our reference on stochastic integration is [5].

It follows from the standard theory of stochastic integration that for any $\mathcal{F}_0$-measurable $X(0)$ the equation (6) has a strongly unique solution $(X(t),\, t \geq 0)$, which is, then, automatically a semimartingale. Furthermore, this solution is strongly Markov.

The Markov property of the solution to our stochastic differential equation allows us, in particular, to use the usual Markovian notation P_x when we want to emphasize that we are working with a solution to that equation with $X(0) = x$. We will use this notation throughout the paper without further comments. We also note at this point that it is an immediate application of Theorem Theorem 5.4 in [3] that the Markov process $(X(t),\, t \geq 0)$ is a *Feller* process. That is, for any bounded and continuous function f on the real line, the function $y \to E_y f(X(t))$ is continuous for every $t \geq 0$.

Even though the equation (6) has a “nice” solution, direct understanding of many properties of this solution is not easy. For this reason our approach is to approximate that solution by “throwing away” the small jumps of the Lévy process L. Specifically, given a Lévy process satisfying (1) and a number $\sigma > 0$ we consider a Lévy motion L_σ satisfying

$$\begin{aligned} E e^{i\theta L(1)} &= \exp\left\{ \int_{|x|>\sigma} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leq 1) \right) \nu(dx) + i\theta a \right\} \\ &= \exp\left\{ \int_{-\infty}^{\infty} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}(|x| \leq 1) \right) \nu_\sigma(dx) + i\theta a \right\}, \quad \theta \in \mathbb{R}\,, \end{aligned} \tag{11}$$

where $\nu_\sigma(A) = \nu(A \cap \{|x| > \sigma\})$ for a Borel set A. Note that ν_σ is a finite measure, and we will consider the corresponding stochastic differential equation driven by L_σ

$$dX_\sigma(t) = -f(X_\sigma(t))\, dt + dL_\sigma(t)\,. \tag{12}$$

It is easy to construct the increments $L_\sigma(t) - L_\sigma(s)$ for $0 \leq s < t$ as measurable functions of $L(u) - L(v)$, $s \leq v < u \leq t$. Hence, one may assume that $(L_\sigma(t),\, t \geq 0)$ is a Lévy motion with respect to the same filtration $(\mathcal{F}_t,\, t \geq 0)$ as $(L(t),\, t \geq 0)$ is. For our purposes the specific filtration does not matter. What is important for our purposes that the solution to the equation (12) converges weakly to that of (6). More precisely, if $(X(t),\, t \geq 0)$ is the solution to (6) and for $\sigma > 0$, $(X_\sigma(t),\, t \geq 0)$ is the solution to (12), then $X_\sigma(0) \Rightarrow X(0)$ as $\sigma \to 0$ implies that $(X_\sigma(t),\, t \geq 0) \Rightarrow (X(t),\, t \geq 0)$ weakly in $D[0,\infty)$ as $\sigma \to 0$. See Theorem 5.4 in [3]. Moreover, since the set of discontinuities of $(X(t),\, t \geq 0)$ coincides with that of $(L(t),\, t \geq 0)$, this means that the process $(X(t),\, t \geq 0)$ is a.s. continuous at every fixed t, and so $X_\sigma(t) \Rightarrow X(t)$ as $\sigma \to 0$ for every $t \geq 0$. See Theorem 12.5 in [1].

Of course, the point of switching from the equation (6) to the equation (12) is that Lévy process $(L_\sigma(t),\, t \geq 0)$ has finitely many jumps in any interval of a finite length, and is linear between two successive jumps. Thus, between any two successive jumps the process $(X_\sigma(t),\, t \geq 0)$ satisfies a deterministic differential equation, which can be explicitly solved. This allows us to get a good "handle" on the process $(X_\sigma(t),\, t \geq 0)$, and the weak convergence of the latter to $(X(t),\, t \geq 0)$ allows us to derive, thus, conclusions about the solution to the equation (6). It is clear that implementation of this approach will require us to obtain "uniform in σ" results for the solution of the easier equation (12), so that the results will be preserved under the weak limit.

Our next result determines the tail behaviour of the solution to the stochastic differential equation (6) at any fixed positive time t. It turns out that, under our assumptions, this tail behaviour is independent of both t and of the initial value $X(0)$. Denote

$$h(u) = \int_u^\infty \frac{\nu\Big((y,\infty)\Big)}{f(y)}\, dy, \quad u \geq 0\,. \tag{13}$$

Note that by the assumptions (4) and (7) – (9) the function h is finite for large u and, moreover,

$$h \text{ is regularly varying at infinity with exponent } -(\alpha+\beta)+1\,. \tag{14}$$

Theorem 1. *Let $(X(t), t \geq 0)$ be the solution to the stochastic differential equation (6) with $X(0) = x$. We assume that the the Lévy motion $(L(t),\, t \geq 0)$ is symmetric and that the assumptions (4) and (7) – (10) hold. Then for every $t_0 > 0$*

$$\lim_{u\to\infty} \frac{P_x(X(t) > u)}{h(u)} = 1 \quad \textit{uniformly in } x \in \mathbb{R} \textit{ and } t \geq t_0. \tag{15}$$

Next we show that the Markov process $(X(t),\, t \geq 0)$ solving the stochastic differential equation (6) has a unique stationary distribution.

To this end let, once again, $\sigma > 0$ and let $(X_\sigma(t),\, t \geq 0)$ be the solution to the approximating equation (12). Tregenerative structure of the latter process shiws that, at least for small $\sigma > 0$, this process has a (unique) stationary distribution.

From here we are in a position to show that the process $(X(t),\, t \geq 0)$ solving the stochastic differential equation (6) has a stationary distribution. To this end, let μ_σ be the stationary distribution for $(X_\sigma(t),\, t \geq 0)$, $0 < \sigma \leq \sigma_0$. It follows from the arguments in Theorem 1 that the family $\big(\mu_\sigma,\, 0 < \sigma \leq \sigma_0\big)$ is tight, at least if we reduce σ_0. Let $\big(\mu_{\sigma_n},\, n \geq 1\big)$ be a weakly convergent sequence, for some $\sigma_n \downarrow 0$, $n \to \infty$. Then $\mu_{\sigma_n} \Rightarrow \mu$ as $n \to \infty$ for some probability measure μ.

In the stochastic differential equation (12) we choose the initial value distributed according to μ_{σ_n} for $n \geq 1$. We now apply, once again, Theorem 5.4 in [3] to conclude that the resulting sequence $\big((X_{\sigma_n}(t),\, t \geq 0),\, n \geq 1\big)$ of stationary processes converges weakly in $D[0,\infty)$ to the solution $(X(t),\, t \geq 0)$ of the equation (6) for which the initial has the distribution μ. Since the process $(X(t),\, t \geq 0)$ is a.s. continuous at each $t \geq 0$ we conclude that $X(t)$ has the law μ for each $t \geq 0$ and, hence, μ is a stationary distribution for the Markov process $(X(t),\, t \geq 0)$.

In order to show that a stationary distribution of the Markov process $(X(t),\, t \geq 0)$ solving the stochastic differential equation (6) is unique, we use a coupling argument. Coupling is a powerful technique of treating stationary distributions of Markov processes as well as other limit phenomena. One indication of its successes are the two recent books, [4] and [7]. We will use a simple version of approximate coupling, described in the lemma below. It is similar to the ϵ-coupling in [4], page 74. Note, however, that [7] uses the term ϵ-coupling in a different sense.

Lemma 1. *Let* $\big(X(t),\, t \geq 0\big)$ *and* $\big(Y(t),\, t \geq 0\big)$ *be two stochastic processes on a probability space* $\big(\Omega, \mathcal{F}, P\big)$ *such that* $X(t) \stackrel{\mathrm{d}}{=} X(0)$ *and* $Y(t) \stackrel{\mathrm{d}}{=} Y(0)$ *for all* $t \geq 0$*. Let* $\epsilon > 0$*, and suppose that there is an event* $\Omega_+ \in \mathcal{F}$ *with* $P\big(\Omega_+\big) = 1$ *and a random variable* $T_\epsilon = T_\epsilon(\omega) \in [0, \infty)$ *such that for all* $\omega \in \Omega_+$ *and* $t \geq T_\epsilon(\omega)$ *we have* $\big|X(t,\omega) - Y(t,\omega)\big| \leq \epsilon$*. Then*

$$P\Big(X(0) > x + \epsilon\Big) \leq P\Big(Y(0) > x\Big) \leq P\Big(X(0) > x - \epsilon\Big) \tag{16}$$

for all $x \in \mathbb{R}$*.*

This lemma helps us to show the uniqueness of the stationary distribution and, moreover, we have

Theorem 2. *Assume that the the Lévy motion* $(L(t),\, t \geq 0)$ *is symmetric and that the assumptions (4) and (7) – (10) hold. Then the Markov process* $(X(t),\, t \geq 0)$ *solving the stochastic differential equation (6) has a unique stationary distribution* μ*. This stationary distribution satisfies*

$$\lim_{u \to \infty} \frac{\mu\big\{(u, \infty)\big\}}{h(u)} = 1\,, \tag{17}$$

where the function h *is defined in (13).*

Proof. Existence and uniqueness of the stationary distribution has been established above, and (17) follows from Theorem 1. □

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On L^p-semigroups, capacities and balayage

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1 Introduction

Problems arising in L^p-potential theory are typically of nonlinear nature. In this article we present some results jointly obtained with N. Jacob [10] concerning a new approach to this problem. It is based on the theory of monotone operators due to Browder and Minty. We will show that this general technique gives a optimally suited frame for the nonlinear situation in L^p-potential theory.

Among different approaches to the construction of a stochastic process starting from a given operator the L^2-approach has turned out to be one of the most successful. In particular the theory of Dirichlet forms leads to comprehensive results in very general situations (see [6], [16]). However, a certain weakness of this apprach lies in the fact that one has to take into account exceptional sets and a process constructed by this method is determined only for starting points outside an exceptional set. The exceptional sets themselves are given by the sets of capacity zero, so the potential theory of the operator under consideration comes into play.

A possible remedy to this difficulty is to refine the potential theory and to replace the L^2-setting by an L^p-theory having in mind that an L^p-approach should give better regularity results. This led to the notion of (r,p)-capacities, see V.G Maz'ya, V.P. Havin [18] and D.R. Adams, L.I. Hedberg [1] as a standard reference. In the context of Dirichlet forms the concept of (r,p)-capacities was first introduced by P. Malliavin [17] and subsequently studied by M. Fukushima and H. Kaneko [4, 5, 14] and T. Kazumi, I. Shigekawa [15].

It turns out that by choosing the parameter p (or r) sufficiently large in many cases the exceptional sets disappear, i.e. every nonempty set has strictly positive (r,p)-capacity. Consequently, in this case it is possible to construct by Dirichlet form techniques an associated process starting at every point.

2 (r,p)-capacities

The classical capacity corresponding to the Laplace operator is the Newtonian capacity or as a slightly modified version the 1-capacity, which for an open set $G \subset \mathbb{R}^n$ is defined by the minimization problem

$$\operatorname{cap}(G) := \inf\Big\{ \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \int_{\mathbb{R}^n} |u|^2\,dx, \quad u \in H^{1,2}(\mathbb{R}^n),\ u \geq 1 \text{ on } G \text{ a.e.}\Big\}.$$

It is well-known that this problem has a unique minimizer u_G which is called the equilbrium potential of G.

In order to define analogous capacities in an L^p-setting one replaces for $1 \leq p < \infty$ and $r \geq 0$ the Sobolev space $H^{1,2}(\mathbb{R}^n)$ by the Bessel potential spaces

$$\begin{aligned} H^{r,p}(\mathbb{R}^n) &= (\mathrm{Id} - \Delta)^{-r/2}(L^p(\mathbb{R}^n)) \\ &= \big\{ f \in L^p(\mathbb{R}^n) : f = G_r * g,\ g \in L^p(\mathbb{R}^n) \big\} \end{aligned}$$

and the corresponding norm

$$\|f\|_{r,p} = \|g\|_{L^p}.$$

Here G_r is the Bessel potential kernel given by its Fourier transform $\widehat{G_r}(\xi) = (1 + |\xi|^2)^{-r/2}$. The (r,p)-capacity then is defined as above by

$$\mathrm{cap}_{r,p}(G) := \inf\{\|u\|_{r,p}^p \quad u \in H^{r,p}(\mathbb{R}^n),\ u \geq 1 \text{ on } G \text{ a.e.}\},$$

which of course reduces to the initial case for $p = 2$ and $r = 1$.

The idea can be carried over also to the investigation of Lévy processes with characteristic exponent Ψ, i.e. Ψ is a continuous negative definite function or to Lévy-type processes generated by pseudo differential operators

$$-p(x,D)u(x) = \int_{\mathbb{R}^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi)\, d\xi,$$

where the symbol $p(x,\xi)$ is assumed to satisfy certain estimates in terms of the fixed continuous negative definite reference function Ψ (see W. Hoh and N. Jacob [11, 7, 12, 8, 9]). In this case the correct function spaces are modified so-called Ψ-Bessel potential spaces

$$H_p^{\Psi,r}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \|u\|_{H_p^{\Psi,r}}, \infty\}$$

with norm $\|u\|_{H_p^{\Psi,r}} = \|F^{-1}((1+\Psi)^{r/2} \cdot Fu)\|_{L^p}$ (F denotes the Fourier transform). These spaces where studied in great detail by W. Farkas, N. Jacob, and R.L. Schilling [2, 3].

Our starting point will be as in the considerations of (r,p)-capacities for Dirichlet forms an L^p-semigroup. Let X be separable metric space equipped with a Radon measure μ and let for some $1 < p < \infty$

$$T_t^{(p)} : L^p(X) \to L^p(X), \quad t \geq 0,$$

be a strongly continuous, positivity preserving contraction semigroup on $L^p(X)$ with L^p-generator $A^{(p)}$. In particular we do not assume that $T_t^{(p)}$ is sub-Markovian. Even more important, since we are interested also in non-symmetric situations, we do neither assume that any symmetry is involved nor that the adjoint semigroup $T_t^{(p)*}$ on $L^{p'}(X)$, $\frac{1}{p} + \frac{1}{p'} = 1$, is sub-Markovian.

In order to define the appropriate function spaces we need the fractional power $(\mathrm{Id} - A^{(p)})^{-r/2}$ which can be defined directly in terms of the semigroup by the Gamma-transform

$$V_r^{(p)}u = \frac{1}{\Gamma(\frac{r}{2})} \int_0^\infty t^{\frac{r}{2}-1} e^{-t} T_t^{(p)} u\, dt.$$

Then $V_r^{(p)} : L^p(X) \to L^p(X)$ is injective, we denote its inverse by $T_{r,p}$. Define the function space

$$\mathcal{F}_{r,p} = V_r^{(p)}(L^p(X))$$

and the norm

$$\|u\|_{\mathcal{F}_{r,p}} = \|T_{r,p}u\|_{L^p}.$$

For an explicit investigation of the corresponding integral kernels in concrete situations we refer to N. Jacob and R.L. Schilling [13]

3 Monotone operators

Let Y be a reflexive separable Banach space with dual space Y^*.

Definition. Let $K \subset Y$ a be closed convex set and let $T : K \to Y^*$ be a (nonlinear) operator.

A. We call T monotone if $\langle Tu - Tv, u - v\rangle \geq 0$ for all $u, v \in K$.

B. The operator is called strictly monotone if $\langle Tu - Tv, u - v\rangle > 0$ for all $u, v \in K$ and $u \neq v$.

C. T is called uniformly monotone if there is a strictly increasing continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}$, $\gamma(0) = 0$ and $\lim_{t\to\infty} \gamma(t) = \infty$, such that for all $u, v \in K$

$$\langle Tu - Tv, u - v\rangle \geq \gamma(\|u - v\|_Y) \cdot \|u - v\|_Y$$

holds.

D. T is coercive with respect to K if there is an element $\varphi \in K$ such that

$$\lim_{\substack{\|u\|_Y \to\infty \\ u\in K}} \frac{\langle Tu - T\varphi, u - \varphi\rangle}{\|u - \varphi\|_Y} = \infty.$$

Moreover we need

Definition. Let $T : Y \to Y^*$ be an operator.

A. T is called hemicontinuous if for all $u, v \in Y$ and $h \in Y$ the function

$$s \mapsto \langle T(u + sv), h\rangle$$

is continuous on $[0, 1]$.

B. T is called demicontinuous if

$$u_n \to u \text{ in } Y \Rightarrow Tu_n \rightharpoonup Tu \text{ in } Y^*.$$

Now the main theorem on monotone operators states (see E. Zeidler [19]):

Theorem (Browder–Minty). Let $T : Y \to Y^*$ be a monotone, coercive, and hemicontinuous operator.

A. For every $f \in Y^*$ the set of solutions of

$$Tv = f$$

is non-empty, closed and convex.

B. If in addition T is strictly monotone, then the solution is unique and the inverse operator $T^{-1} : Y^* \to Y$ is strictly monotone, demicontinuous and bounded.

C. If T is even uniformly monotone, then T^{-1} is continuous.

4 Application to (r, p)-capacities

In order to define (r, p)-capacities in our general setting we have to consider a minimization problem for the functional

$$E_{r,p}(u) := \frac{1}{p}\|u\|^p_{\mathcal{F}_{r,p}} = \frac{1}{p}\int_X |T_{r,p}u|^p \mu(dx).$$

First note that the functional $E_{r,p} : \mathcal{F}_{r,p} \to \mathbb{R}$ is strictly convex and coercive, i.e. $\frac{E_{r,p}(u)}{\|u\|_{\mathcal{F}_{r,p}}} \to \infty$ as $\|u\|_{\mathcal{F}_{r,p}} \to \infty$. Moreover, $E_{r,p}$ is Gâteaux differentiable and we can explicitly calculate the Gâteaux derivative

$$\mathcal{A}^{(p)}_r : \mathcal{F}_{r,p} \to \mathcal{F}^*_{r,p}$$

at $u \in \mathcal{F}_{r,p}$:

$$\mathcal{A}^{(p)}_r u := T^*_{r,p}(|T_{r,p}u|^{p-2} \cdot T_{r,p}u).$$

Note that $\mathcal{A}^{(p)}_r$ is a nonlinear operator unless $p = 2$. We can prove the following inequality:

$$\langle \mathcal{A}^{(p)}_r u - \mathcal{A}^{(p)}_r v, u - v\rangle \geq 2^{2-p}\|u - v\|^p_{\mathcal{F}_{r,p}}$$

In particular this implies that on every closed convex subset of $\mathcal{F}_{r,p}$ the operator $\mathcal{A}_r^{(p)}$ is uniformly monotone and coercive.

Since $E_{r,p}$ is strictly convex, coercive and by definition continuous it is clear by the general theory of coercive functionals (see E. Zeidler [19] Theo. 25 D) that for every open subset $G \subset X$ $E_{r,p}$ attains a unique minimum on the closed convex subset $\{u \in \mathcal{F}_{r,p} : u \geq 1 \text{ on } G \text{ a.e.}\}$. Therefore, the unique minimizer e_G again defines an (r,p)-equilibrium potential and the (r,p)-capacity is given by

$$\mathrm{cap}_{r,p}(G) = E_{r,p}(e_G).$$

Analogously, for $h \in \mathcal{F}_{r,p}$ one can consider the closed convex set $\{u \in \mathcal{F}_{r,p} : u \geq h \text{ on } G \text{ a.e.}\}$ and obtains as the unique minimizer the balayaged function h_G.

In accordance with Dirichlet forms we introduce the notion of a mutual energy on $\mathcal{F}_{r,p} \times \mathcal{F}_{r,p}$:

$$\mathcal{E}_r^{(p)}(u,v) := \langle \mathcal{A}_r^{(p)} u, v \rangle,$$

which is again nonlinear with respect to the first argument. We now can find a better description of the minimizer in terms of a variational inequality which, as one would expect, involves the derivative $\mathcal{A}_r^{(p)}$ of the functional $E_{r,p}$:

Proposition. Let $K \subset \mathcal{F}_{r,p}$ be closed and convex. The unique minimizer of $E_{r,p}$ on K satisfies

$$\mathcal{E}_r^{(p)}(u, \varphi - u) = \langle \mathcal{A}_r^{(p)} u, \varphi - u \rangle \geq 0 \text{ for all } \varphi \in K.$$

This in particular implies that for an equilibrium potential e_G the variational inequality

$$\mathcal{E}_r^{(p)}(u_G, \psi) \geq 0 \text{ for all } \psi \in \mathcal{F}_{r,p},\ \psi|_G \geq 0$$

holds. In analogy to Dirichlet forms we call a function $u \in \mathcal{F}_{r,p}$ a potential if

$$\mathcal{E}_r^{(p)}(u, \psi) = \langle \mathcal{A}_r^{(p)} u, \psi \rangle \geq 0 \text{ for all } \psi \in \mathcal{F}_{r,p},\ \psi \geq 0.$$

Especially, equilibrium potentials are potentials in this sense.

In other words a potential is a function $u \in \mathcal{F}_{r,p}$ having the property that $\mathcal{A}_r^{(p)} u$ is a positive element in $\mathcal{F}_{r,p}^*$ (in a distributional sense). But, since $\mathcal{A}_r^{(p)}$ is an uniformly monotone operator that satisfies all assumptions of the Browder-Minty theorem, we know that it is invertible. We denote its inverse by

$$U_r^{(p)} = (\mathcal{A}_r^{(p)})^{-1} : \mathcal{F}_{r,p}^* \to \mathcal{F}_{r,p}$$

and thus have shown:

$u \in \mathcal{F}_{r,p}$ is a potential if and only if $u = U_r^{(p)} f$ for some positive $f \in \mathcal{F}_{r,p}^*$.

Again an explicit calculation is possible:

$$U_r^{(p)} f = V_r^{(p)}(|V_r^{(p)*} f|^{p'-2} \cdot V_r^{(p)*} f).$$

This operator $U_r^{(p)}$ is a well-known object called the nonlinear potential operator and has been investigated before for instance by V.G. Maz'ya, V.P. Havin [18] and D.R Adams, L.I. Hedberg [1]. Note that under reasonable assumptions the positive elements in $\mathcal{F}_{r,p}^*$ can be identified with measures on X (of finite energy), see T. Kazumi, I. Shigekawa [15]. In this sense we obtain a representation of the potentials which is completely analogous to the Riesz representation in classical potential theory.

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On a conjecture of Barndorff-Nielsen relating probability densities and Lévy densities

Friedrich Hubalek

1 Introduction

Consider a compound Poisson process $(X_t, t \geq 0)$ with intensity parameter c and jumps from a distribution with density $a(x)$. In this case the Lévy density is simply $u(x) = ca(x)$. Let $f(x,t)$ denote the density of the abolutely continous part of the distribution of X_t. Then we have the well-known formula

$$f(x,t) = \sum_{n \geq 1} e^{-ct} \frac{(ct)^n}{n!} a^{*n}(x).$$

For the purpose of this work it is useful to expand e^{-ct} in t and rearrange terms to obtain

$$f(x,t) = \sum_{n \geq 1} u_n(x) \frac{t^n}{n!},$$

where

$$u_n(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} c^{n-k} u^{*k}(x).$$

Barndorff-Nielsen studied in [BN00], how this formula can be generalized to the non-compound Poisson case, ie when the Lévy measure is not a finite measure, and convolution is a priori not defined.

Let $\mathcal{P}_+$ denote the infinitely divisible laws on $\mathbb{R}_+$ resp Lévy processes, such that the infimum of the support of the Lévy measure is 0.

Theorem 1.1 (Barndorff-Nielsen). *Let $u(x)$ the Lévy density for a distribution in $\mathcal{P}_+$ and $u_\varepsilon(x)$ Lévy densities, such that*

$$\int_0^\infty u(x)dx = \infty, \qquad \int_0^\infty u_\varepsilon(x)dx < \infty,$$

and

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u(x), \qquad \lim_{\varepsilon \to 0} \int_x^\infty u_\varepsilon(y)dy = \int_x^\infty u(y)dy,$$

pointwise for $x > 0$. Define

$$c(\varepsilon) = \int_0^\infty u_\varepsilon(x)dx$$

and

$$U_{n\varepsilon}^+(x) = \sum_{k=1} \binom{n}{k} (-1)^{n-k} c(\varepsilon)^{n-k} (U_\varepsilon^+)^{*k}(x).$$

Then

$$F^+(x,t) = \lim_{\varepsilon \to 0} \sum_{n \geq 1} U_{n\varepsilon}^+(x) \frac{t^n}{n!}.$$

Here we use the notation

$$F^+(x,t) = \int_x^\infty f(y,t)dy, \qquad U_{n\varepsilon}^+(x) = \int_x^\infty u_{n\varepsilon}(y)dy,$$

with

$$u_{n\varepsilon}(x) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} c(\varepsilon)^{n-k} u_{\varepsilon}^{*k}(x). \tag{1.1}$$

A family of truncated Lévy densities, that satisfies the assumption of the theorem is given, for example, by $u_\varepsilon(x) = u(x)1_{(\varepsilon,\infty}(x)$. Barndorff-Nielsen conjectured

$$\lim_{\varepsilon\to 0} U_{n\varepsilon}^{+}(x) = U_n^{+}(x), \qquad F^{+}(x,t) = \sum_{n\geq 1} U_n^{+}(x)\frac{t^n}{n!}$$

and

$$\lim_{\varepsilon\to 0} u_{n\varepsilon}(x) = u_n(x), \qquad f(x,t) = \sum_{n\geq 1} u_n(x)\frac{t^n}{n!}$$

for some limiting functions $U_n^+(x)$ and $u_n(x)$. He showed that, for differentiable $u(x)$, the sequence $u_{2\varepsilon}(x)$ does in fact converge and has the limit

$$u_2(x) = \frac{2}{x}\left(\int_0^x u(y)\left(\bar{u}(x-y) - \bar{u}(x)\right)dy - \bar{u}(x)U^{+}(x)\right),$$

where $\bar{u}(x) = xu(x)$, though it remained open, how to generalize that for $u_n(x)$ with $n \geq 3$.

Note that the difficulty is not mereley to justify an interchange of limits, but the fact that each term in the alternating sum (1.1) diverges to infinity, yet massive cancellation yields convergence of the sum in total.

2 Main results

Theorem 2.1. *Assume that a Lévy density $u(x)$ from $\mathcal{P}_+$ satisfies*

$$(i) \qquad \int_0^\infty u(x)dx = \infty \tag{2.1}$$

$$(ii) \qquad u \in C^\infty(\mathbb{R}_+) \tag{2.2}$$

$$(iii) \qquad \int_0^\infty e^{-\theta_n x} x^{n+1}|u^{(n)}|dx < \infty \tag{2.3}$$

for some sequence $\theta_n > 0$, $n \in \mathbb{N}$. For $\varepsilon > 0$ let $u_\varepsilon(x) = u(x)e^{-\varepsilon/x}$. Then $V_n = (U^+)^{*n}$ is $C^\infty(\mathbb{R}_+)$ and

$$\lim_{\varepsilon\downarrow 0} u_{n\varepsilon}(x) = u_n(x), \quad \lim_{\varepsilon\downarrow 0} U_{n\varepsilon}^{+}(x) = U_n^{+}(x), \qquad \forall x \in \mathbb{R}_+, n \in \mathbb{N}\setminus\{0\}. \tag{2.4}$$

with $u_n(x) = (-1)^n V_n^{(n)}(x)$ and $U_n^+(x) = (-1)^{n-1}V_n^{(n-1)}(x)$. As a consequence $f(x;t)$ and $F^+(x;t)$ are entire functions in t for fixed $x > 0$ and

$$f(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}u_n(x), \qquad F^{+}(x;t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}U_n^{+}(x). \tag{2.5}$$

The proof uses Laplace transforms and mimicks the continuity theorem for densities with integrable transforms. To obtain uniform estimates in ε, the classical relation of smoothness and integrability of derivatives of an original function and the decay of its Laplace transform in a vertical strip is employed.

An interesting class of distributions is the class T_2 of generalized convolutions of mixtures of exponential distributions [Bon81, Bon82, Rog83]. They are characterised by a completeley monotone Lévy density. All examples in [BN00] belong to this class.

Proposition 2.2. *If the Lévy density $u(x)$ is completely monotone and satisfies (2.1) then (2.2) and (2.3) hold with arbitrary $\theta_n > 0$.*

Remarks

- We use strong smoothness and integrability assumptions (yet we cover all examples given in [BN00]) to obtain a strong result, an entire function with rather explicit Taylor coefficients. We would like to mention the literature on the *asymptotic* behavior of $F(x,t)$ as $t \to 0$, that typically uses weaker (or even no) additional assuptions, eg [Lea87, Ish94, Pic97, RW00].
- An example where (2.1)–(2.3) are satisfied, but the corresponding distribution is not in T_2 is given by
$$u(x) = x^{-3/2} e^{\sin(x)}. \tag{2.6}$$
- An example where (2.1) and (2.2) are satisfied, but (2.3) fails for $n \geq 1$ is given by
$$u(x) = x^{-3/2} \sin(x^{-3})^2. \tag{2.7}$$
- The assertion, that the V_n are C^∞ functions in the theorem is not redundant, as the convolution of C^∞ functions is in general not necessarily C^∞. A striking counterexample is given in [Ulu98].

3 Examples

Let us recall some examples for the series representation
$$f(x,t) = \sum_{n\geq 1} u_n(x) \frac{t^n}{n!} \tag{3.1}$$
from [BN00].

- Positive α-stable laws $S(\alpha)$: In general ($\alpha \neq 1/2$) there is no closed form expression for $f(x,t)$, but we have the series expansion (3.1) with
$$u_n(x) = \frac{(-1)^{n-1}}{\pi} \Gamma(1+n\alpha) \sin(n\pi\alpha) x^{-1-\alpha}.$$
- Gamma distribution $\Gamma(\nu = 1, \alpha = 1)$ The density is
$$f(x,t) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x}$$
and the coefficients in (3.1) are given by
$$u_n(x) = x^{-1} e^{-x} \sum_{k=0}^{n-1} (n)_k c_k \ln^{n-k-1} x$$
where $(n)_k = n(n-1)\dots(n-k+1)$ and the coefficients c_k arise in the expansion
$$\frac{1}{\Gamma(1+z)} = \sum_{n\geq 0} c_n z^n.$$
They can be expressed explicitely as
$$c_n = \frac{1}{(n-1)!} Y_{n-1}\left(\gamma, -\zeta(2), 2\zeta(3), \dots, (-1)^{n-2}(n-2)!\zeta(n-1)\right)$$
with Y_n the complete exponential Bell polynomials [Com74], $\gamma = 0.57221\dots$ the Euler-Masceroni constant, and ζ the Riemann Zeta function.
- Inverse Gaussian distribution $IG(\delta = 1, \gamma = 1)$ The density is
$$f(x,t) = \frac{1}{\sqrt{2\pi}} t e^t x^{-3/2} e^{-(t^2 x^{-1} + x)/2}. \tag{3.2}$$
and we obtain
$$u_n(x) = \frac{n}{\sqrt{\pi}} 2^{-n/2} x^{-1-n/2} e^{-x/2} H_{n-1}\left(\sqrt{\frac{x}{2}}\right). \tag{3.3}$$
with H_n denoting the Hermite polynomials.

4 Further issues

- We currently investigate (with Ole Barndorff-Nielsen) multivariate versions ($\mathbb{R}_+^d$) of this result. It looks promising.
- More difficult is an extension to distributions on $\mathbb{R}$ as our method does not extend directly to that case. Examples suggest that similar expansion should hold, eg for the Meixner distribution [Sch02] with parameters $\mu = 0$, $\delta = 1$, $\alpha = 1$, and $\beta = 0$ we have the density

$$f(x,t) = \frac{1}{\pi} 2^{2t-1} \frac{\Gamma(t+ix)\Gamma(t-ix)}{\Gamma(2t)}.$$

  It can be expanded in a series

$$f(x,t) = \sum_{n\geq 1} u_n(x) \frac{t^n}{n!}, \qquad |t| < |x|$$

  with

$$u_n(x) = \frac{n}{x\sinh(\pi x)} Y_{n-1}\left(a_1(x), \ldots, a_{n-1}(x)\right)$$

  where

$$a_1(x) = \psi(ix) + \psi(-ix) + 2\ln 2 + 2\gamma$$

  and

$$a_n(x) = \psi^{(n)}(ix) + \psi^{(n)}(-ix) - (-1)^n 2^n (n-1)!\zeta(n) \quad (n \geq 2).$$

  Here Y denotes again the complete exponential Bell polynomials, γ is the Euler-Masceroni constant, ψ is the digamma function, and ζ the Riemann zeta function. Note, however, that here $f(x,t)$ is not an entire function in t, due to the poles of the gamma function. Thus we have to expect qualitative differences to the case $\mathcal{P}_+$.
- It is completely unknown whether one can weaken the assuptions or how to contstruct any counterexamples in the present context. Using $x^{-3/2}\sqrt{1-x}1_{(0,1)}(x)$ as a building block it seems possible to construct a counterexample with divergence on a countable dense set, but as we are concerned with densities only divergence on a set of positive measure should be considered as a proper counterexample.
- Applications: As discussed in [Con02] the small time asymptotics of the Lévy transition probabilities or densities can be useful to describe and characterize volatility surfaces derived from Lévy process based option pricing models.

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Portfolio selection in Lévy markets via Hellinger processes

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1 Overview

This talk was based on the joint work [2] with Tahir Choulli. In this paper we studied Merton's problem [5] of finding for a given utility function $U(x)$ a wealth process $\hat{X}_t$ which maximizes the expected utility of wealth at time $T > 0$, that is it solves the "primal problem"

$$u(x) = \sup_{X \in \mathcal{A}(x)} E(U(X_T)) \tag{1.1}$$

where $\mathcal{A}(x)$ is a class of admissible wealth portfolios with initial value x at $t = 0$. [4],[6] have proved fundamental results for markets modelled by general semimartingales by applying martingale techniques to derive the equivalent "first dual problem"

$$v(y) = \inf_{Y \in \mathcal{A}^*(y)} E(V(Y_T)) \tag{1.2}$$

where $\mathcal{A}^*(y)$ is an appropriately defined "dual" to $\mathcal{A}(x)$. In ideal cases, the solution $\hat{Y}$ defines a measure Q equivalent to the physical measure P which is interpreted as the "pricing" martingale measure, and which can be used for example in expectation pricing of derivative securities in the market. Counterexamples in the general semimartingale theory given in [4] show that the solution of the dual problem is sometimes a supermartingale, not a martingale. When this happens, the standard financial interpretation of martingale measures becomes obscured. This raises the important question of when (1.2) may be replaced by the easier (and financially natural) "second dual problem"

$$v(y) = \inf_{Y \in \mathcal{M}^a(y)} E(V(Y_T)) \tag{1.3}$$

where $\mathcal{M}^a(y)$ denotes a space of positive martingales (which therefore yields equivalent martingale measures).

In [2] we address this question by analyzing in complete detail three canonical utility functions $-e^{-x}, x^p/p$ and $\log x$ in a market of the jump–diffusion type modelled by an exponentiated Lévy process in which the log stock returns jumps may be unbounded. For each utility, we solve the primal problem, we solve the second dual problem (1.3), and then compare the results to give an explicit check on the dual correspondence which is the main result of [4]. Our findings show that $-e^{-x}$ leads to a picture free of any pathological counterexamples. However, for x^p/p and $\log x$, no borrowing from the bank account or shortselling of the stock will be admissible which leads to the consequence that for certain parameters the solution of (1.2) is a supermartingale not a martingale and thus (1.3) cannot give the correct solution. In exactly the same cases, we will also observe that the solution of (1.3) yields a martingale measure Q which is not equivalent to the physical measure P (in other words, Q assigns zero probability to some events with positive P–probability). These two pathologies seem to be directly related each other and to the presence of no borrowing/shortselling constraints in the problem. Our solution of the dual problem (1.3) makes use of a technique introduced in [1] which shows that the pricing measure arising from (1.3) is in each case identical to the equivalent martingale measure which minimizes a corresponding "generalized Hellinger process".

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2 The problem

The financial market consists of a risk-free asset (bank account) B given by $B_t = e^{rt}$ and a stock S (risky asset). For simplicity, we take the interest rate $r = 0$. The stock is given as the Doléans–Dade exponential $B^{-1}S = S_0\mathcal{E}(L)$ of the following Lévy process (stationary process with independent increments)

$$L_t = (b-r)t + \sigma W_t + \int_0^t \int z\ I_{\{|z|\le 1\}}\ \widetilde{N}(ds,dz) + \int_0^t \int z\ I_{\{|z|>1\}}\ N(ds,dz). \tag{2.1}$$

Here $\mathcal{E}(L)$ is the unique solution to the (SDE) $dK = K_- dL,\ \ K_0 = 1$ and $\sigma > 0, b$ are constants $\widetilde{N}$ is the compensated Poisson random measure given by $\widetilde{N}(dt,dz) = N(dt,dz) - dt\ \nu(dz)$. In addition to standard assumptions on the Lévy measure, we assume that jump sizes are unbounded

$$\mathrm{supp}(\nu) = [-1,\infty).$$

An ideal market without transaction costs and liquidity effects is assumed.

Consider an investor who wants to invest in their wealth in this market in an optimal way over the period $[0,T]$. Letting π_t be the wealth invested at time t in the stock and making the usual self–financing requirement, the wealth process $X_t^{\pi,x}$ which follows from an initial endowment $x = X_0$ is given by

$$\begin{aligned} X_t^{\pi,x} &= x + \int_0^t b\pi_s ds \\ &\quad + \int_0^t \pi_s \left[\sigma dW_s + \int (zI_{\{|z|\le 1\}}\widetilde{N}(ds,dz) + zI_{\{|z|>1\}}N(ds,dz))\right] \end{aligned} \tag{2.2}$$

A dynamical version of the primal Merton problem is defined by

$$u(t,x) = \sup_{\pi\in\mathcal{A}(t,x)} E\left(U(X_T^\pi)\mid X_t^\pi = x\right). \tag{2.3}$$

where U is a strictly increasing, strictly concave, twice continuously differentiable function. This leads to the study of the HJB equation for $u(t,x)$:

$$\begin{cases} \frac{\partial u}{\partial t} + \sup_{\pi\in\mathbb{R}}\Big[\pi b u_x + \frac{1}{2}\pi^2\sigma^2 u_{xx} + \int \big[u(x+\pi z) \\ \qquad -u(x) - \pi z I_{\{|z|\le 1\}} u_x\big]\nu(dz)\Big] = 0 & t\in[0,T) \\ u(T,x) = U(x) & x\in\mathbb{R}^+ \end{cases} \tag{2.4}$$

As is now well known, the primal problem (1.1) can also be addressed by focusing on the Legendre transform V of U defined by

$$V(y) = \sup_{x>\underline{x}}[U(x) - xy]. \tag{2.5}$$

which is a strictly convex, twice differentiable function on $[0,\infty)$. Now one studies the "first dual problem"

$$v(y) = \inf_{Y\in\mathcal{A}^*(0,y)} E\left(V(Y_T)\right). \tag{2.6}$$

where $\mathcal{A}^*(0,y)$ is dual to $\mathcal{A}(0,x)$ with $y = u'(x)$. Under some circumstances, [4],[6] have shown that the functions $u(x)$ and $v(y)$ can themselves be obtained from each other by using Legendre transform:

$$v(y) = \sup_{x\ge\underline{x}}[u(x) - xy],\quad u(x) = \inf_{y\ge 0}[v(y) + xy]\quad x,y\ge 0 \tag{2.7}$$

and the optimizers $\hat{X}(x), \hat{Y}(y)$ with $y = u'(x)$ are related by

$$\hat{X}(x) = -V'(\hat{Y}(y)),\qquad \hat{Y}(y) = U'(\hat{X}(x)) \tag{2.8}$$

If we replace $\mathcal{A}^*(0,y)$ by a space $\mathcal{M}^a(y)$ of local martingales Y such that SY is a local martingale, we obtain the "second dual problem"

$$v(y) = \inf_{Y \in \mathcal{M}^a(y)} E\left(V(Y_T)\right). \tag{2.9}$$

The general $Y \in \mathcal{M}^a(y)$ is given by $\mathcal{E}(M)$ for a (local) martingale of the form

$$M_t = \int_0^t \beta(\omega, s) dW_s + \int_0^t \int (Z(\omega, s^-, z) - 1)\widetilde{N}(ds, dz) \tag{2.10}$$

where β and Z satisfy

$$b + \sigma\beta_t + \int z\left[Z_t(z) - I_{\{|z|\le 1\}}\right]\nu(dz) = 0, \qquad \text{dt-a.e.} \tag{2.11}$$

In some cases, the first and second dual problems have the same solution $\hat{Y}$ which can taken to be the conditional density $\hat{Y}_t = E(d\hat{Q}/dP|\mathcal{F}_t)$ of an absolutely continuous martingale measure $\hat{Q} \ll P$ interpreted as the martingale measure (pricing measure) which captures the risk preferences coded into the utility function U.

Three important special cases of utility functions and their Legendre transforms are treated in [2]:

$$U^{(q)}(x) = x^p/p, \qquad V^{(q)}(y) = -y^q/q, \tag{2.12}$$
$$p = q/(q-1) \in (-\infty, 1) \setminus \{0\}.$$
$$U^L(x) = \log x, \qquad V^L(y) = -\log y - 1. \tag{2.13}$$
$$U^E(x) = -e^{-x}, \qquad V^E(y) = y(\log y - 1). \tag{2.14}$$

3 Results

The method of generalized Hellinger processes developed in [1] lead to the following result

Theorem 3.1. *Let M be a local martingale in the form of (2.10) such that $1 + \Delta M > 0$ P-almost surely and let $Y = \mathcal{E}(M)$. In the following, $h^{(q)}, h^L, h^E$ are predictable increasing processes and*

1. *for $0 < q < 1$, the process $V^{(q)} = -Y^q/q$ is a negative local submartingale which can be written*

$$\begin{aligned} V_t^{(q)} &= -\int_0^t V_{s-}^{(q)}\, dh_s^{(q)} + \text{ local martingale} \\ h_t^{(q)}(\beta, Z) &= \frac{1}{2}q(1-q)\int_0^t \beta_s^2 ds \\ &\quad - \int_0^t \int \left[Z_s(z)^q - 1 - q\left(Z_s(z) - 1\right)\right]\nu(dz)ds; \end{aligned} \tag{3.1}$$

2. *for $q < 0$, the process $V^{(q)} = -Y^q/q$ is a positive local submartingale which can be written*

$$\begin{aligned} V_t^{(q)} &= \int_0^t V_{s-}^{(q)}\, dh_s^{(q)} + \text{local martingale} \\ h_t^{(q)}(\beta, Z) &= \frac{1}{2}q(q-1)\int_0^t \beta_s^2 ds \\ &\quad + \int_0^t \int \left[Z_s(z)^q - 1 - q\left(Z_s(z) - 1\right)\right]\nu(dz)ds; \end{aligned} \tag{3.2}$$

3. The process $V^L = -\log(Y) - 1$ is a local submartingale which can be written

$$\begin{aligned} V_t^L &= h_t^L + \text{local martingale} \\ h_t^L(\beta, Z) &= \frac{1}{2}\int_0^t \beta_s^2 ds \\ &\quad + \int_0^t \int [-\log(Z_s(z)) + Z_s(z) - 1]\,\nu(dz)ds; \end{aligned} \tag{3.3}$$

4. The process $V^E = Y(\log(Y) - 1)$ is a local submartingale which can be written

$$\begin{aligned} V_t^E &= \int_0^t Y_{s-}\ dh_s^E + \text{local martingale} \\ h_t^E(\beta, Z) &= \frac{1}{2}\int_0^t \beta_s^2 ds \\ &\quad + \int_0^t \int [Z_s(z)\log(Z_s(z)) - Z_s(z) + 1]\,\nu(dz)ds. \end{aligned}$$

We then defined Hellinger–like integrals for the above functions V by

$$H_t = E(V(Y_t)) \tag{3.4}$$

for each $t \geq 0$. Denoting by $\mathcal{K}_t$ the space of $\mathcal{F}_{t-}$ random variables (β, Y), $Y \geq 0$ which satisfy (2.11), we then proved the result

Theorem 3.2. *Consider one of the four utilities identified in Theorem 3.1. Let (β_t^*, Z_t^*) solve the problem*

$$\inf_{(\beta,Z)\in\mathcal{K}_t} \frac{dh_t(\beta, Z)}{dt} \tag{3.5}$$

for all $t \in [0, T]$. Then (β_t^*, Z_t^*) can be taken as a deterministic process and $Y^* = y\mathcal{E}(M(\beta^*, Z^*))$ solves the second dual problem

$$\inf_{Y\in\mathcal{M}^a(y)} E(V(Y_T)) \tag{3.6}$$

This reformulation of the second dual problem leads to a complete solution of the stated problem. The exponential and power law utilities illustrate the features of the general problem.

Theorem 3.3. *Let the utility function be $U(x) = -e^{-x}$.*

1. The solution of the primal problem is the pair $(u, \hat{X}_T(x))$ where

$$u(t, x) = -e^{K^E(T-t)-x} \tag{3.7}$$

and $\hat{X}_T(x)$ is given by (2.2) with the constant trading strategy $\pi_t = \pi^E$ which is the unique minimizer of the convex function

$$G(\pi) = -b\pi + \frac{1}{2}\sigma^2\pi^2 + \int [e^{-\pi z} - 1 + \pi z I_{\{|z|\leq 1\}}]\nu(dz) \tag{3.8}$$

and $K^E = G(\pi^E)$.

2. The solution of the second dual problem is the pair $(v, \bar{Y}_T(y))$ where

$$v(t, y) = y(\log y - 1 + K^E(T - t)). \tag{3.9}$$

and $\bar{Y}_T(y) = y\mathcal{E}(M)_T$ where

$$M_t := -\int_0^t \pi^E \sigma dW_s + \int_0^t \int [e^{-\pi^E z} - 1]\widetilde{N}(ds, dz) \tag{3.10}$$

3. The solutions to the two dual problems (2.6) and (2.9) coincide.

Remarks:

1. Since the wealth process is unconstrained, the two dual problems have the same solution.
2. The quantity $E(Z_T \log Z_T)$ is the entropy of P relative to Q and we see that the solution of the resulting optimal problem gives the "minimal entropy martingale measure" put forward by Frittelli [3].

Theorem 3.4. *Let the utility function be $U(x) = x^p/p$ for $p \in (-\infty, 0) \cup (0, 1)$ and let $q = p/(p-1)$.*

1. The solution of the primal problem is the pair $(u, \hat{X}_T(x))$ where

$$u(t, x) = e^{K^{(p)}(T-t)}\ x^p/p \tag{3.11}$$

and $\hat{X}_T(x)$ is given by (2.2) with the trading strategy $\pi_t = \phi^{(p)}\hat{X}_t(x)$. The constants $K^{(p)}, \phi^{(p)}$ are determined by the concave function

$$\begin{aligned} F(\phi) &= b\phi + (p-1)\sigma^2\phi^2/2 \\ &\quad + p^{-1}\int [(1+\phi z)^p - 1 - p\phi z I_{\{|z|\le 1\}}]\nu(dz) \end{aligned} \tag{3.12}$$

$K^{(p)} = pF(\phi^{(p)})$ where

(a) If $F'(0) < 0$, $\phi^{(p)} = 0$;
(b) If $F'(1) > 0$, $\phi^{(p)} = 1$;
(c) If $F'(1) \le 0 \le F'(0)$, $\phi^{(p)} \in [0, 1]$ is the unique root of $F'(\phi) = 0$.

2. The solution of the first dual problem is the pair $(v, \hat{Y}_T(y))$ where

$$v(t, y) = -e^{-K^{(p)}(T-t)(q-1)}\ y^q/q \tag{3.13}$$

and $\hat{Y}_T(y) = y\mathcal{E}(M)_T$ where

$$M_t := \int_0^t (p-1)\phi^{(p)}\sigma dW_s + \int_0^t \int [(1+\phi^{(p)}z)^{p-1} - 1]\widetilde{N}(ds, dz) - \int_0^t \phi^{(p)}F'(\phi^{(p)})ds \tag{3.14}$$

3. The solution of the second dual problem (2.9) is the pair $(\tilde{v}, \bar{Y}_T(y))$ where

$$\tilde{v}(t, y) = -e^{K^{(q)}(t-T)(q-1)}\ y^q/q \tag{3.15}$$

and $\bar{Y}_T(y)) = y\mathcal{E}(\bar{M})_T$ with

$$\bar{M}_t := \int_0^t (p-1)\tilde{\phi}^{(p)}\sigma dW_s + \int_0^t \int [\max(1+\tilde{\phi}^{(p)}z, 0)^{p-1} - 1]\widetilde{N}(dtdz). \tag{3.16}$$

Here $\tilde{\phi}^{(p)}$ is the unique root of the equation

$$b + (p-1)\sigma^2\phi + \int [z(\max(1+\phi z, 0))^{p-1} - zI_{\le}]\nu(dz) = 0 \tag{3.17}$$

4. The solutions to the two dual problems (2.6) and (2.9) coincide if and only if $F'(1) \le 0 \le F'(0)$.

Remarks:

1. If $F'(0) \le 0$ the optimal strategy is the risk free strategy $\hat{X}_t = B_t$ because the mean rate of return of the stock is lower than the risk–free rate. In the presence of unbounded jumps short–selling the stock involves the risk of a negative portfolio value, and thus the optimal solution has zero investment in the risky asset.
2. If $F'(1) \ge 0$, then $(\phi^{(p)}, K^{(p)}) = (1, F(1))$, and the solution is $\hat{X}_t = S_t$, the maximally risky strategy which can be tolerated without violating the no–borrowing constraint.
3. We see from the equations for $(\phi^{(p)}, K^{(p)})$ that $\hat{Y}$ is a P–martingale if and only if $\phi^{(p)}F'(\phi^{(p)}) = 0$ (i.e. $F'(1) \le 0$). If $F'(1) > 0$, then $\hat{Y}$ is a supermartingale. One can also check that $S\hat{Y}$ is a P–martingale if and only if $F'(0) \ge 0$. If $F'(0) < 0$ then $S\hat{Y}$ is only a supermartingale.

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Small deviations property and its application

Yasushi ISHIKAWA

1 Introduction

Let X_t be an $\mathbf{R}^d$ valued, cádlàg Lévy process on the probability space $(D, \mathcal{F}, P)$, with the initial value $X_0 = \mathbf{o}$. Here $D = D([0,T], \mathbf{R}^d)$ denotes the space of cádlàg trajectries from $[0,T]$ to $\mathbf{R}^d$ attached with the Skorohod topology.

We say X has a small deviations property if for all $T > 0$ for all $\epsilon > 0$

$$P(\sup_{t \le T} |X_t| < \epsilon) > 0. \tag{1.1}$$

This property seems to have first appeared in [3], Lemma 8 in 1977. Before that this term seems to have been studied in connection with the random walk.

The process X has a representation

$$X_t = d.t + QB_t + \int_0^t \int_{|z| \le 1} z\tilde{\mu}(dsdz) + \int_0^t \int_{|z|>1} z\mu(dsdz). \tag{1.2}$$

Here B_t denotes an $\mathbf{R}^m$-valued Wiener process, Q a $d \times m$-matrix, μ a Poisson random measure on $\mathbf{R}^d$ such that $E[\mu(dsdz)] = ds.d\nu(z)$, where ν is a measure on $\mathbf{R}^m$ such that $\int \frac{|z|^2}{|z|^2+1} \nu(dz) < \infty$, and $\tilde{\mu} = \mu - E[\mu]$.

We define the subspace

$$L = \Big\{ x \in \mathbf{R}^d; \int_{|z| \le 1} | < x, z > | \nu(dz) < +\infty \Big\}. \tag{1.3}$$

This denotes the set (cone) consisting of directions of finite variations of ν.

We write π_L the orthonormal projection (in $\mathbf{R}^d$) to L, and put $\nu_L \equiv \pi_L^* \nu = \nu \circ \pi_L$, and $a_L \equiv \int_{|z| \le 1} z \nu_L(dz)$.

We further put $H = Q(\mathbf{R}^d)$ and denote by L' the supplementary orthogonal projection of $L \cap H$ in L, that is, $(L \cap H)^\perp \cap L$. We remark that if $Q \equiv 0$, then $L' = L$.

For $\eta \in (0,1]$, let $\nu^\eta \equiv \nu|_{\{|z| \le \eta\}}$ and let B^η be the convex cone generated by $\{0\}$ and supp ν^η. We write

$$\mathrm{B} = \bigcap\nolimits_{\eta > 0} \mathrm{B}^\eta \tag{1.4}$$

and

$$\mathrm{B}' = \bigcap\nolimits_{\eta > 0} \bar{\mathrm{B}}^\eta_{L'}, \tag{1.5}$$

where $\mathrm{B}^\eta_{L'} \equiv \pi_{L'} \mathrm{B}^\eta$. Here $^-$ means the closure in the euclidian topology.

Proposition 1.1 ([20]). *X has small deviations property iff*

$$d \in \pi_{L'}^{-1}(a_{L'} - B'). \tag{1.6}$$

Intuitively, this condition means that the effect of the infinitesimal drift a_L can be compensated by the small jump part B plus the apparent drift d.

Extension of the definition.
Let Y_t be an $\mathbf{R}^d$-valued strong Markov process given by the SDE : $Y_0(y) = y$,

$$Y_t(y) = y + \int_0^t a(Y_s(x))ds + \int_0^t \int_{|z|\le 1} b(Y_{s-}, z)\tilde{\mu}(dsdz) + \int_0^t \int_{|z|>1} b(Y_{s-}, z)\mu(dsdz). \tag{1.7}$$

Then Y_t is a strong Markov process, which has an infinitesimal generator A given by

$$Af(x) \ = < \text{ grad } f(x), a(x) > +$$

$$\int_0^t \int_{|z|\le 1} [f(x + b(x, z)) - f(x) - < \text{ grad } f(x), b(x, z) >]\nu(dz)$$

$$+ \int_0^t \int_{|z|>1} [f(x + b(x, z)) - f(x)]\nu(dz).$$

We assume the coefficient functions $a(x), b(x, z)$ satisfy that for each z, $y \mapsto a(y)$ and $y \mapsto b(y, z)$ are smooth,

$$b(y, z) = \tilde{b}(y).z + b'(y, z),$$

where $\tilde{b}$ is a $d \times m$-matrix, and $|b'(y, z)| \le C|z|^\alpha$ for some $\alpha \in (1, 2]$. This SDE has an unique solution Y_t for each y. Then Y_t can be decomposed by, given $\eta > 0$,

$$Y_t(y) = y + \int_0^t \tilde{a}(Y_s(x))ds +$$

$$\int_0^t \int_{\mathbf{R}^m\setminus\{0\}} b'(Y_{s-}, z)\tilde{\mu}(dsdz) + \int_0^t \int_{|z|\le\eta} \tilde{b}(Y_{s-}).z\tilde{\mu}(dsdz) + \int_0^t \int_{|z|>\eta} \tilde{b}(Y_{s-}).z\mu(dsdz).$$

Here

$$\tilde{a}(x) = a(x) - \int_{|z|\le\eta} b'(x, z)\nu(dz).$$

We say the Felller process Y_t has a *small deiations property* iff for all $T > 0$ for all $\epsilon > 0$

$$P(\sup_{t\le T} |Y_t - y| < \epsilon) > 0. \tag{1.8}$$

For a technical reason which is used in Sect. 3, we put $\tilde{Y}_t^\eta$ by

$$\tilde{Y}_t^\eta = \int_0^t \int_{|z|\le\eta} \tilde{b}(Y_{s-}).z\tilde{\mu}(dsdz) + \int_0^t \int_{|z|\le\eta} b'(Y_{s-}, z)\mu(dsdz),$$

$$\tilde{Y}_0^\eta = 0.$$

We give here a sufficient conditions for $\tilde{Y}_t^\eta$ to have the small deviations property, namely **(A.1)** and **(A.2)** below. (cf. [8], [9], [18])

To state **(A.1)**, we put for $0 < \rho < \eta$,

$$u_\rho^\eta \ = \int_{\rho\le|z|\le\eta} z\,\nu(dz).$$

We say that X is *quasi-symmetric* if for every $\eta > 0$, there exists a sequence $\{\eta_k\}$ decreasing to 0 such that

$$\left|u_{\eta_k}^\eta\right| \longrightarrow 0 \tag{1.9}$$

as $k \to +\infty$. This means that for every η the compensation involved in the martingale part of $\tilde{Y}^\eta$ is somehow negligible, and of course this is true when X is really symmetric.

We put the sets $(J_n)_{n\in\mathbf{N}}$ by

$$J_0 = \{0\}$$

$$J_1 = \text{supp } \nu$$

$$\dots$$

$$J_n = \sum_{i=1}^{n} J_1 \qquad \text{in the sense of vector sum .}$$

We put $J = \sum_{n \geq 1} J_n$. Now let α_ρ^η denote the smallest angle between the direction $u_\rho^\eta = \int_\rho^\eta z\nu(dz)$ and the set $J|_{\{|z|=\gamma\rho\}}$. We put the following condition:

(A.1) For every $\eta > 0$ such that (1.7) does not hold, there exist $\gamma = \gamma(\eta) > 1$ and a sequence $\{\rho_k\}$ tending to 0 such that

$$\alpha_{\rho_k}^\eta = o(\frac{1}{|u_{\rho_k}^\eta|})$$

as $k \to \infty$.

This condition is satisfied if supp ν contains a sequence of spheres whose radius tend to 0.

(A.2) There exists $\beta \in [1, 2)$ and positive constants C_1, C_2 such that for any $\rho \leq 1$

$$C_1 \rho^{2-\beta} I \;\leq\; \int_{|z| \leq \rho} zz^* \nu(dz) \;\leq\; C_2 \rho^{2-\beta} I. \tag{1.10}$$

If $\tilde{Y}_t^\eta$ has the small deviations property for each $\eta > 0$ then we have the support theorem. See [8].

2 Analytic approach

There are some attemps to measure the small ball probability $P\big(\sup_{t \leq T} |Y_t - y| < r\big)$ analytically. For example, [15], [16], [4], [5], [2].

We may regard the infinitesimal generator A of $Y_.$ as a pseudodifferential operator

$$Af(x) = P(x, D)f(x) \equiv (1/2\pi)^d \int e^{i<x,\xi>} p(x, \xi) \hat{f}(\xi) d\xi,$$

where

$$p(x, \xi) = -ia(x)\xi + \int \{e^{i<y,\xi>} - 1 - \frac{i < y, \xi >}{1 + |y|^2}\} K(x, dy). \tag{2.1}$$

Here the kernel K is given by

$$\int_B K(x, dy) = \int 1(b(x, z) \in B) \nu(dz), B \subset \mathbf{R}^d.$$

By the assumptions on the smoothness of $x \mapsto a(x)$ and $x \mapsto b(x, z)$, and by the definition of $\nu(dz)$, $x \mapsto p(x, \xi)$ is continuous for each ξ.

We put for $\epsilon > 0$,

$$H(x, \epsilon) \equiv \sup_{|x-y| \leq 2\epsilon} \sup_{|\eta| \leq 1} |p(y, \eta/\epsilon)|. \tag{2.2}$$

Then it holds ([15], Lemma 4.1) for all $T > 0$ and all $\epsilon > 0$

$$P\Big(\sup_{t \leq T} |Y_t - y| \geq \epsilon\Big) \leq c_d T H(y, \epsilon), \tag{2.3}$$

where c_d is an absolute constant. Hence

$$P\Big(\sup_{t \leq T} |Y_t - y| < \epsilon\Big) \geq 1 - c_d T H(y, \epsilon). \tag{2.4}$$

This implies that if R.H.S. is positive, then we have the small deviations property. Obviously this is not possible for large $T > 0$. But there are some techniques to this overcome, as will be mentioned below.

We state here a small result for the small deviations property.

Theorem 2.1. *Suppose that $\epsilon \mapsto H(x,\epsilon)$ is bounded below for each x, that is, for each x there exists $M = M_x > 0$ such that for all $\epsilon > 0$*

$$|H(x,\epsilon)| \leq M. \tag{2.5}$$

Then we have (1.6).

Proof. We put $T_0 = T_0(x,\epsilon) \equiv (1/2).\frac{1}{c_d.H(x,\epsilon)}$, so that for $t < T_0$

$$P\Big(\sup_{t\leq T}|Y_t - y| < \epsilon\Big) \geq 1-(1/2) = 1/2 > 0.$$

By the assumption (2.5), we have

$$T_0 \geq (1/2).(1/c_d).(1/M) > 0 \tag{2.6}$$

uniformly in ϵ.

Given $\epsilon > 0$, we devide the interval $[0,T]$ by $0 < T_0 < 2T_0 < \cdots < (n-1)T_0 < T \leq nT_0$, where

$$n = n(\epsilon) \ \sim \ \frac{T}{T_0} = 2Tc_dH(x,\epsilon) \quad (\leq 2Tc_dM) \tag{2.7}$$

as $T \to \infty$.

We will estimate the small ball probability for $t < T_0$

$$P\Big(\sup_{s\leq t}|Y_s - y| < \epsilon/n(\epsilon)\Big) \geq 1 - c_dtH(x,\epsilon/n(\epsilon)). \tag{2.8}$$

We remark

$$H(x,\epsilon/n(\epsilon)) = \sup_{|x-y|\leq 2\epsilon/n(\epsilon)}\ \sup_{|\eta|\leq 1}|p(y,\eta n(\epsilon)/\epsilon)| \leq M.$$

Hence if we choose $t < \frac{1}{2c_dM}$, then we have

$$P\Big(\sup_{s\leq t}|Y_s - y| < \epsilon/n(\epsilon)\Big) \geq 1 - \frac{c_dH(x,\frac{\epsilon}{n(\epsilon)})}{2c_dM} > 1-(1/2) = 1/2 > 0.$$

Hence for $t < T_0$ we have

$$P\Big(\sup_{s\leq t}|Y_s - y| < \epsilon/n(\epsilon)\Big) > 1/2 > 0 \tag{2.9}$$

by (2.6).

Now we construct processes $Y_s^0, Y_s^{T_0}, \ldots, Y_s^{(n-1)T_0}$ for $s \in [0,T_0)$ by

$$\begin{aligned} Y_s^0 &= Y_s, \\ Y_s^{T_0} &= Y_s \circ \theta_{T_0} = Y_{T_0+s} - Y_{T_0}, \\ Y_s^{2T_0} &= Y_s \circ \theta_{T_0}^2 = Y_{2T_0+s} - Y_{2T_0}, \\ &\cdots\cdots \\ Y_s^{(n-2)T_0} &= Y_s \circ \theta_{T_0}^{n-2} = Y_{(n-2)T_0+s} - Y_{(n-2)T_0}, \end{aligned}$$

and

$$Y_s^{(n-1)T_0} = Y_{s\wedge T-(n-1)T_0} \circ \theta_{T_0}^{n-1} = Y_{(n-1)T_0+s\wedge T} - Y_{(n-1)T_0}.$$

Here θ_t is the translation operator associated with the strong Markov process $Y_\cdot$. Then by the above estimate we have

$$\begin{aligned} P\Big(\sup_{s\leq T_0}|Y_s^0 - y| < \epsilon/n(\epsilon)\Big) &> 1/2, \\ P\Big(\sup_{s\leq T_0}|Y_s^{T_0}| < \epsilon/n(\epsilon)\Big) &> 1/2, \\ &\cdots\cdots \\ P\Big(\sup_{s\leq T_0}|Y_s^{(n-1)T_0}| < \epsilon/n(\epsilon)\Big) &> 1/2. \end{aligned} \tag{2.10}$$

By the Markov property of the original Y_s, the processes $Y_s^0, Y_s^{T_0}, \ldots, Y_s^{(n-1)T_0}$ $(s \geq 0)$ are conditionally independent. Hence

$$
\begin{aligned}
&P\Big(\sup_{s\leq T}|Y_s - y| < \epsilon\Big) \qquad (2.11)\\
&\geq P^{(x)}\Big(\sup_{s\leq T_0}\big|Y_s^0 - y\big| < \epsilon/n(\epsilon); P^{\left(Y_{T_0}^0\right)}\Big(\sup_{s\leq T_0}\big|Y_s^{T_0}\big| < \epsilon/n(\epsilon); P^{\left(Y_{T_0}^{T_0}\right)}\Big(\ldots\\
&\qquad \ldots\ P^{\left(Y_{T_0}^{(n-2)T_0}\right)}\Big(\sup_{s\leq T_0}\big|Y_s^{(n-1)T_0}\big| < \epsilon/n(\epsilon)\Big)\ldots\Big)\Big)\Big)\\
&= P\Big(\sup_{s\leq T_0}\big|Y_s^0 - y\big| < \epsilon/n(\epsilon); \sup_{s\leq T_0}\big|Y_s^{T_0}\big| < \epsilon/n(\epsilon); \ldots; \sup_{s\leq T_0}\big|Y_s^{(n-1)T_0}\big| < \epsilon/n(\epsilon)\Big)\\
&= P\Big(\sup_{s\leq T_0}\big|Y_s^0 - y\big| < \epsilon/n(\epsilon)\Big)\ldots P\Big(\sup_{s\leq T_0}\big|Y_s^{(n-1)T_0}\big| < \epsilon/n(\epsilon)\Big)\\
&\geq (1/2)^{n(\epsilon)} > 0
\end{aligned}
$$

which is positive for all $\epsilon > 0$ and all $T > 0$. Hence we have a small deviations property. q.e.d.

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Fractional derivatives, pseudo-differential operators and Lévy(-type) processes

Niels Jacob

Let $G \subset \mathbb{R}^n$ be open, $\lambda^{(n)}(G^c) > 0$ and ∂G smooth. Further let $q : \overline{G} \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $q(x, \cdot) : \mathbb{R}^n \to \mathbb{C}$ is negative definite for all $x \in \overline{G}$. A central problem is to find conditions in order that $(x, \xi) \mapsto q(x, \xi)$ is the symbol of a Markov process with state space $\overline{G}$, i.e., when does exist a Markov process $((X_t)_{t \geq 0}, P^x)_{x \in \overline{G}}$ such that for $(x, \xi) \in \overline{G} \times \mathbb{R}^n$

$$q(x, \xi) = - \lim_{t \to 0} \frac{E^x(e^{i(X_t - x)\cdot \xi}) - 1}{t}$$

holds. Clearly this leads to boundary value problem for the operator $-q(x, D)$, but if $q(x, D)$ does not have a dominating diffusion part, so far almost nothing is known about such type of (Wentzel) boundary value problems.

Certain fractional derivatives have on $C_0^\infty(G)$ for some special sets G a representation as pseudodifferential operators with negative definite symbols and some of their extensions allow to study the behaviour at the boundary even in situations when the transmission condition does not hold. Thus they may provide us with non-trivial examples for solving related Wentzel boundary problem.

In the talk we will discuss some of such examples.

Some limit theorems for the Euler scheme for Lévy driven stochastic differential equations

Jean JACOD *

We consider the following stochastic differential equation (SDE):

$$X_t = x_0 + \int_0^t f(X_{s-})dY_s \tag{1}$$

where f denotes a C^3 (three times differentiable) function and Y is a Lévy process with characteristics (b, c, F) with respect to the truncation function $h(x) = x1_{\{|x|\leq 1\}}$, that is

$$E\big(e^{iuY_t}\big) = \exp t\left(iub - \frac{cu^2}{2} + \int F(dx)\big(e^{iux} - 1 - iux1_{\{|x|\leq 1\}}\big)\right). \tag{2}$$

We also suppose that f is such that (1) admits a (necessarily unique) *non-exploding* solution (this is the case for example if f has at most linear growth).

A number of papers have been devoted to studying the rate of convergence of the Euler scheme for this equation. That is, the approximated solution is defined at the times i/n, by induction on the integer i, according to the formula:

$$X_0^n = x_0, \qquad X_{\frac{i}{n}} = X_{\frac{i-1}{n}} + f\big(X^n_{\frac{i-1}{n}}\big)\big(Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}\big). \tag{3}$$

This scheme allows for numerical computations, using Monte-Carlo techniques, provided one can simulate the increments $Y_t - Y_s$ of the Lévy process Y: A first problem consists in computing an approximation of the expected value $E(h(X_1))$ for smooth enough functions g, and we need to evaluate the error $a_n(h) = E(h(X_1^n)) - E(h(X_1))$. A second problem is to compute an approximation of the law of some functional of the path, like e.g. $\sup_{t\leq 1} X_t$, and for this we need to evaluate the (discretized) error process, which is defined as

$$U_t^n = X^n_{[nt]/n} - X_{[nt]/n}. \tag{4}$$

Problem 1 has been extensively studied when Y is continuous (i.e. $F = 0$) and $c > 0$: we can quote, with increasing order of generality as to the smoothness of f and h, the works of Talay & Tubaro [6] and Bally & Talay [1], [2], where it is proved that $a_n(h)$ is of order $1/n$ and where an expansion of $a_n(h)$ as increasing powers of $1/n$ is even exhibited. In Protter & Talay [5] the same problem is studied for discontinuous Y, but they only prove that $a_n(h) = O(1/n)$ and this rate is probably not optimal; see also a forthcoming paper by Kohatsu-Hida & Yoshida [4] for an equation driven by a Wiener process plus a Poisson random measure. The techniques are essentially analytical.

For problem 2 one uses stochastic calculus techniques, and the idea is to find a *rate* u_n, that is a sequence going to ∞ such that the sequence (u_nU^n) is tight; the rate is called *sharp* if further the sequence (u_nU^n) admits some limiting processes that are not identically 0. Even better is the case when the whole sequence (u_nU^n) converges to a non-degenerate limit. In [3] we have proved that (more precise results are recalled below):

(1) If $c > 0$ then a sharp rate is $u_n = \sqrt{n}$, and the sequence $(\sqrt{n}U^n)$ converges in law to a non-degenerate limit.

(2) If $c = 0$ and F is a finite measure, hence Y is a compound Poisson process plus a drift, then a sharp rate is $u_n = n$ if the drift b is not 0; when $b = 0$ the rate is "infinite", meaning that for any t we have $U_s^n = 0$ for all $s \leq t$ for n large enough.

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(3) If $c = 0$ and F is an infinite measure, then a rate is $u_n = \sqrt{n}$, but this rate is not sharp in the sense that $(\sqrt{n}U^n)$ goes in law to 0.

Although the implicit assumption that the increments of Y can be simulated is somewhat unrealistic except in particular situations, which however include the case where Y is a symmetric stable process plus a drift (see the discussion in Protter & Talay [5]), finding the exact rate of convergence is at least of much theoretical importance. Here we aim to find sharp rates for Problem 2, when $c = 0$ and $F(\mathbb{R}) = \infty$. The crucial factor is the behaviour of the Lévy measure F near 0 (that is, how many "small jumps" we have), which will be expressed through the following functions on $\mathbb{R}_+$:

$$\theta_+(\beta) = F((\beta,\infty)), \qquad \theta_-(\beta) = F((-\infty,-\beta)), \qquad \theta(\beta) = \theta_+(\beta) + \theta_-(\beta). \tag{5}$$

We introduce several assumptions, in which α denotes our basic index; here and below C denotes a constant which may change from line to line, and may depend on F just here, and also on b and f further below:

Hypothesis (H1-α): We have $\theta(\beta) \le \frac{C}{\beta^\alpha}$ for all $\beta \in (0,1]$. □

Hypothesis (H2-α): We have $\beta^\alpha\theta_+(\beta) \to \theta_+$ and $\beta^\alpha\theta_-(\beta) \to \theta_-$ as $\beta \to 0$ for some constants $\theta_+, \theta_- \ge 0$, and further $\theta := \theta_+ + \theta_- > 0$. We also set $\theta' = \theta_+ - \theta_-$, and we observe that $\theta(\beta) \sim \frac{\theta}{\beta^\alpha}$ as $\beta \to 0$. □

Hypothesis (H3): The measure F is symmetrical about 0. □

Hypothesis (H4): We have $b = 0$. □

Note that (H2-α) $\Rightarrow$ (H1-α), and that (H1-2) always holds because F integrates $x \mapsto |x|^2 \bigwedge 1$, and (H1-0) holds iff the measure F is finite, a case which we exclude. Under (H3) we have (H2-α) as soon as $\theta(\beta) \sim \frac{\theta}{\beta^\alpha}$ as $\beta \to 0$, and $\theta_+ = \theta_- = \theta/2$.

Unfortunately we cannot totally fulfill our aim. But we find rates u_n that are bigger than $\sqrt{n}$. And we prove that these rates are sharp and even that u_nU^n converges in some reasonably general circumstances. Let us single out five different cases:

- **Case 1:** We have (H1-α) for some $\alpha > 1$; then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha}$.
- **Case 2-a:** We have (H1-α) for $\alpha = 1$; then $u_n = \frac{n}{(\log n)^2}$.
- **Case 2-b:** We have (H1-α) for $\alpha = 1$ and (H3); then $u_n = \frac{n}{\log n}$.
- **Case 3-a:** We have (H1-α) for some $\alpha < 1$; then $u_n = n$.
- **Case 3-b:** We have (H1-α) for some $\alpha < 1$ and (H3) and (H4); then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha}$.

Clearly (H1-α) $\Rightarrow$ (H1-α') if $\alpha < \alpha'$, while the rate is better (i.e. bigger) when α decreases: one should take the smallest possible α for which (H1-α) holds, although of course there might not be such a minimal α. Observe also that the rate in Case 2-b (resp. 3-b) is strictly bigger than in Case 2-a (resp. 3-a): the symmetry of the driving process improves the quality of the Euler scheme under (H1-α) when $\alpha \le 1$, while it does not affect the rate when $\alpha > 1$.

Now we describe the results of this paper. The first one concerns tightness (the assumption of f is always that it is C^3 and that Equation (1) has a non-exploding solution; this is not repeated in the next theorems):

Theorem 1. *Assume that $c = 0$ and that (H1-α) holds for some $\alpha \in (0,2)$. Then with the above choice of u_n the sequence (u_nU^n) is tight.*

In Case 3-b, this result improves on Protter & Talay [5]: for every Lipschitz function h it gives $a_n(h) = O((\log n/n)^{1/\alpha})$ instead of $a_n(h) = O(1/n)$.

The results about limits necessitate the stronger (H2-α) instead of (H1-α), except in Case 3-a; in all cases except 2-a the description of the limit invloves another process or additional random variables which are independent of Y, so we might need to enlarge the probability space to accomodate these.

Theorem 2. *Assume that $c = 0$ and that (H1-α) holds for some $\alpha \in (0,2)$. Then in the following cases and with u_n as above the sequence $(\overline{Y}^n, nU^n)$ converges in law (for the Skorokhod topology) to (Y,U), where U is the unique solution U of the linear equation*

$$U_t = \int_0^t f'(X_{s-})U_{s-}dY_s - W_t, \tag{6}$$

and where the process W may be described as follows:

a) In Case 1, and if further (H2-α) holds, then

$$W_t = \int_0^t f(X_{s-})f'(X_{s-})dV_s, \tag{7}$$

where V is another Lévy process, independent of Y and characterized by

$$E(e^{iuV_t}) = \exp\ t\int \frac{\alpha}{2}\left((\theta_+^2+\theta_-^2)1_{\{x>0\}} + 2\theta_+\theta_- 1_{\{x<0\}}\right)\ \frac{1}{|x|^{1+\alpha}}(e^{iux}-1-iux) \tag{8}$$

(hence V is a stable process with index α).

b) In Case 2-a, and if further (H2-α) holds for $\alpha = 1$, then

$$W_t = -\frac{(\theta_+ - \theta_-)^2}{4}\int_0^t f(X_{s-})f'(X_{s-})ds, \tag{9}$$

and $w < e$ even have that u_nU^n converges to U in probability (locally uniformly in time).

c) In Cases 2-b and 3-b, and if further (H2-α) holds, then we have (7), where V is another Lévy process, independent of Y and characterized by

$$E(e^{iuV_t}) = \exp\ t\int \frac{\theta^2\alpha}{4}\ \frac{1}{|x|^{1+\alpha}}(e^{iux}-1-iux1_{\{|x|\le 1\}}) \tag{10}$$

(hence V is a symmetric stable process with index α).

d) In Case 3-a, then

$$\begin{aligned} W_t \;=\; & d\sum_{n:R_n\le t}\left([f(X_{R_n}) - f((X_{R_n-})]\,\xi_n\right. \\ & \left. + f'(X_{R_n-})\Delta X_{R_n}(1-\xi_n)\right) + \frac{d^2}{2}\int_0^t f(X_{s-})f'(X_{s-})ds. \end{aligned} \tag{11}$$

where $d = b - \int_{\{|x|\le 1\}} xF(dx)$ and $(\xi_n)_{n\ge1}$ is a sequence of i.i.d. variables, uniform on $[0,1]$ and independent of Y, and $(R_n)_{n\ge1}$ is an enumeration of the jump times of Y (or of X).

Remark 1: For comparison with the cases excluded here and studied in [3], let us mention that if $c = 0$ and F is a finite measure (i.e. (H1-0) holds), then (d) above holds without change. When $c > 0$ the sequence $(\overline{Y}^n, \sqrt{n}U^n)$ converges in law to (Y,U), where U solves (6) with

$$\begin{aligned} W_t = \sqrt{c}\sum_{n:R_n\le t} & \Big([f(X_{R_n}) - f((X_{R_n-})]\sqrt{\xi_n}\kappa_n \\ & + f'(X_{R_n-})\Delta X_{R_n}\sqrt{1-\xi_n}\ \kappa'_n\Big) + \frac{c}{\sqrt{2}}\int_0^t f(X_{s-})f'(X_{s-})dB_s \end{aligned}$$

and where B is a standard Brownian motion, and ξ_n is uniform over $[0,1]$, and κ_n and κ'_n are standard normal variables, all these being independent one from the other and of Y as well. □

Remark 2: When $\theta_+ = \theta_-$ (for example under (H3)) then (8) and (10) agree (but of course for different values of α). In (b) (resp. (d)), if $\theta' = \theta_+ - \theta_- = 0$ (resp. $d = 0$) the limiting process U

is identically 0. So these results are interesting only when $\theta' \neq 0$ (resp. $d \neq 0$), implying that Y is dissymmetric, and otherwise the rate is not sharp. $\square$

Remark 3: In (b) we have convergence in probability, so there ought to be an associated "central limit theorem": this suggests that we can improve the Euler scheme and simultaneously improve the rate, but probably not more than going to $n/\log n$ in view of (c): the improvement is thus negligible on the numerical point of view. When $d = 0$ in (d) it is also likely that there is a rate in between n and $\left(\frac{n}{\log n}\right)^{1/\alpha}$ for which $u_n U^n$ converges to a non-trivial process, but we have made no attempt towards this case. $\square$

Remark 4: It would be possible, at the price of even more complicated computations, to accomodate other forms for (H2-α): for example if $\theta_+(\beta)$ and $\theta_-(\beta)$ are of order $\beta^{-\alpha}\left(\log \frac{1}{\beta}\right)^\gamma$ as $\beta \to 0$ for some $\alpha \in (0,2)$ and $\gamma \in \mathbb{R}$. On the contrary, it seems rather difficult to express the rates u_n directly in terms of the two functions $\theta_+(\beta)$ and $\theta_-(\beta)$. $\square$

The assumption that f is C^3 is certainly too strong. It is used only in Case 1; for the other cases, that f is C^2 is enough. Finally let us mention that, for the sake of notational simplicity, we have considered only the 1-dimensional case, but everything goes through in the multi-dimensional case as well.

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Self-similar processes with independent increments associated with Lévy and Bessel processes.

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A particularly interesting subclass of infinitely divisible laws (on $\mathbb{R}$, or $\mathbb{R}^+$ to be concrete) is the so-called class (L) of *limit laws* which was studied by P. Lévy (1937) and M. Loeve (1945), to start with.

Limit laws are precisely the distributions of random variables X which satisfy : for every $u \in (0,1)$, $X \overset{(d)}{=} uX + X_u$ where X_u is a "residual" variable, independent of X. Such random variables, and their laws, are also called self-decomposable. Wolfe [6] and Jurek-Vervaat [2] have shown that X is self-decomposable if and only if

$$X \overset{(d)}{=} \int_0^\infty e^{-s} dY_s$$

for some Lévy process (Y_s) called the Background Driving Lévy Process of X, abbreviated as BGDL of X.
Later, Sato [5] showed that X is self-decomposable if and only if it is the value at time 1 of a self-similar additive process $(X_u, u \geq 0)$ of scaling exponent H, that is : for any $c > 0$, $(X_{cu}, u \geq 0) \overset{(d)}{=} (c^H X_u, u \geq 0)$. We show some explicit relationship between the BDLP Y and the Sato process (X_u) associated with X, precisely :

$$\left(Y_t^{(-)} = \int_{e^{-t}}^1 \frac{dX_r}{r^H}, t \geq 0 \right), \text{ and } \left(Y_t^{(+)} = \int_1^{e^t} \frac{dX_r}{r^H}, t \geq 0 \right),$$

are two independent and identically distributed Lévy processes from which $(X_r, r \geq 0)$ can be recovered by

$$X_r = \begin{cases} \displaystyle\int_{\ln(1/r)}^\infty e^{-tH} dY_t^{(-)} & \text{if } 0 \leq r \leq 1 \\ \displaystyle X_1 + \int_0^{\ln r} e^{tH} dY_t^{(+)} & \text{if } r \geq 1 \end{cases}$$

Two kinds of other processes may be naturally associated with a self-decomposable variable X; they are

(i) the stationary process $(Z_u, u \in \mathbb{R})$ defined from (X_t) via the stationary Lamperti transform:

$$X_r = r^H Z_{\ln r}, \text{ or equivalently } Z_u = e^{-uH} X_{e^u} \tag{1}$$

As observed by Lamperti [3], the formulae (1) set up a one to one correspondence between H-self similar processes $(X_r, r \geq 0)$ and stationary processes $(Z_u, u \in \mathbb{R})$.

(ii) The Ornstein-Uhlenbeck process driven by $(Y_t^{(+)}, t \geq 0)$ with initial state U_0 and parameter $c \in \mathbb{R}$, that is the solution of

$$\begin{aligned} U_t &= U_0 + Y_t^{(+)} - c \int_0^t U_s ds \\ &\equiv e^{-ct} \left(U_0 + \int_0^t e^{cs} dY_s^{(+)} \right) \end{aligned} \tag{2}$$

and similarly for $Y^{(-)}$. Moreover, these processes Z and U associated to X via (1) and (2) are related as follows : $(Z_u, u \geq 0)$ is the Ornstein-Uhlenbeck process driven by $(Y_t^{(+)}, t \geq 0)$ with initial state X_1 and parameter $c = H$.

We now give some explicit description of the preceding processes for the particular cases of $X = T_1$, or $X = \Lambda_1$, with

$$T_r = \inf\{t : R_t = r\} \quad \Lambda_r = \sup\{t : R_t = r\}, \quad r \geq 0,$$

where $(R_t, t \geq 0)$ is a Bessel process with dimension $d = 2(1+\nu) > 0$ starting from 0. The scaling proprety of R, together with the strong Markov property imply that T_1 and Λ_1 are self decomposable variables, and that $(T_r, r \geq 0)$ and $(\Lambda_r, r \geq 0)$ ar the Sato processes with scaling exponent $H = 2$ associated respectively to T_1 and Λ_1. With the help of the local time at level 1 associated to R

$$L_t = \lim_{\epsilon \to 0} \int_0^t ds 1\!\!1_{\{|R_s - 1| \leq \epsilon\}}$$

and its inverse process $\tau_\ell = \inf\{t : L_t > \ell\}$, we can exhibit a pathwise representation of the BDLP's of T_1 and Λ_1, which we denote respectively by $(Y_s^T, s \geq 0)$ and $(Y_s^\Lambda, s \geq 0)$; precisely, for each $\ell > 0$,

$$(Y_\lambda^T, 0 \leq \lambda \leq \ell) \stackrel{(d)}{=} \left(\left(\int_{T_1}^{\tau_\lambda} 1\!\!1_{\{R_t \leq 1\}} dt \right)_{0 \leq \lambda \leq \ell} | \tau_\ell < \infty \right) \tag{3}$$

and

$$(Y_\lambda^\Lambda, 0 \leq \lambda \leq \ell) \stackrel{(d)}{=} \left(\left(\int_0^{\tau_\lambda} 1\!\!1_{\{R_t > 1\}} dt \right)_{0 \leq \lambda \leq \ell} | \tau_\ell < \infty \right) \tag{4}$$

where the conditioning on $(\tau_\ell < \infty)$ is made necessary by the fact that for $d > 2$ (i.e., $\nu > 0$), the process $(R_t, t \geq 0)$ is transient, hence $L_\infty < \infty$ a.s.
To prove (3), one may use the following lemma, which is of interest in its own right (See [4] for developments)

Lemma. *Let $(S_t, t \geq 0)$ be a continuous semi-martingale and (A_t) a continuous adapted bounded variation process such that $(S_t \exp(-A_t), t \geq 0)$ is a local martingale. Then, for any $s \in \mathbb{R}$, denoting by $L_t^s(S)$ the local time of s at level s, the process*

$$(S_t \wedge s) \exp\left(\frac{1}{2s} L_t^s(S) - \int_0^t 1\!\!1_{\{S_u \leq s\}} dA_u \right)$$

is a local martingale.

Finally, we stress that the Sato processes associated with self-decomposable laws seem to be particularly well adapted to fit option prices across strikes and maturities (See Carr et al. [1]) due to their scaling properties and inhomogeneous independent increments.

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On Generalized Green Functions of Stable Jump Processes

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Abstract: We present a theorem of existence for a Green function related to an integro-differential operator of fractional order with measurable bounded coefficents. Moreover, we present certain function space inclusions that are known to be sharp.

1 Definitions and Main Results

$\Omega \subset \mathbb{R}^n$ shall be a bounded domain. Let $J(dx, dy)$ denote a positive Radon measure on $(\Omega \times \Omega) \setminus \operatorname{diag}$ satisfying:

$$\int\limits_{(K\times K)\setminus \operatorname{diag}} |x-y|^2 J(dx,dy) < \infty\,, \qquad J(K, \Omega - O) < \infty$$

for all compact sets K and open sets O with $K \subset O \subset \Omega$. Let $\tilde{k}(dx)$ be a positive Radon measure on Ω. We consider the following bilinear form:

$$a(u,\varphi) = \int\limits_{\Omega}\int\limits_{\Omega} \big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big) J(dx,dy) + \int\limits_{\Omega} u(x)\varphi(x)\tilde{k}(dx)\,. \tag{1.1}$$

$J(dx, dy)$ is called jumping measure, $\tilde{k}(dx)$ killing measure.

Definition 1.1. *A function $G(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$ is called* generalized Green function *of $a(\cdot,\cdot)$ if it satisfies for all $y \in \Omega$:*

$$a\big(G(\cdot,y)\,,\ \varphi\big) = \varphi(y) \qquad \forall \varphi \in C_0^\infty(\Omega)\,. \tag{1.2}$$

Assumption 1.2. *There exist positive constants λ and Λ and a measurable function k with:*

$$k(x,y) = k(y,x), \quad \lambda \le k(x,y) \le \Lambda \qquad \forall x,y \in \Omega\,,$$

such that

$$J(dx,dy) = |x-y|^{-n-\alpha} k(x,y)\,dx\,dy$$
$$\tilde{k}(dx) = \Big(2 \int\limits_{\mathbb{R}^n\setminus\Omega} |x-y|^{-n-\alpha} k(x,y)\,dy\Big)\,dx\,.$$

Theorem 1.3 (Kassmann/Steinhauer). *Under the above assumptions there exists a nonnegative generalized Green function $G(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\}$, such that for given $y \in \Omega$ equation (1.2) holds. Moreover one has:*

$$G(\cdot,y) \in H^{\alpha/2,2}(\Omega\setminus B_r(y)) \cap H_0^{\alpha/2,1}(\Omega) \quad \forall r > 0\,, \tag{1.3}$$

$$G(\cdot,y) \in L_{\text{weak}}^{\frac{n}{n-\alpha}}(\Omega) \text{ with } \|G(\cdot,y)\|_{L_{\text{weak}}^{\frac{n}{n-\alpha}}} \le C\,, \tag{1.4}$$

$$G(\cdot,y) \in H_0^{\alpha/2,s}(\Omega) \quad \forall s \in [1, \frac{n}{n-\frac{\alpha}{2}})\,, \tag{1.5}$$

2 Notation

For $p \in [1,\infty)$ we denote by $L^p_{\text{weak}}(\Omega)$ the Banach space of measurable functions $v:\Omega \to \mathbb{R}$ such that the expression

$$[v]_{L^p_{\text{weak}}} := \sup_{t>0} t|\{x \in \Omega : |v(x)| > t\}|^{\frac{1}{p}}$$

is finite. One has:

$$\begin{aligned} L^p(\Omega) \subsetneqq L^p_{\text{weak}}(\Omega) \ \text{ with } [f]_{L^p_{\text{weak}}(\Omega)} \le \|f\|_{L^p(\Omega)} \\ L^p_{\text{weak}}(\Omega) \subset L^{p-\varepsilon}(\Omega) \ \text{ with } \|f\|_{L^{p-\varepsilon}(\Omega)} \le (\tfrac{p}{\varepsilon})^{\frac{1}{p-\varepsilon}} |\Omega|^{\frac{\varepsilon}{p(p-\varepsilon)}} [f]_{L^p_{\text{weak}}(\Omega)} \end{aligned}$$

for $0 < \varepsilon \le p-1$. We also use Sobolev spaces of fractional order (or Slobodeckij spaces) defined as:

$$\begin{aligned} W^{\beta,p}(\Omega) &:= \{u \in L^p(\Omega) : \|u\|_{W^{\beta,p}(\Omega)} < \infty\} \\ \text{with} \quad \|u\|^p_{W^{\beta,p}(\Omega)} &:= \|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta p}}\, dxdy \end{aligned}$$

for $0<\beta<1$, $1 \le p < \infty$. These are Banach spaces respective Hilbert spaces for $p=2$. We write:

$$H^\beta(\Omega) = H^{\beta,2}(\Omega) = W^{\beta,2}(\Omega).$$

3 Related Works

Theorem 3.1 (Chen/Song, Kulczycki). *Let X be the symmetric α-stable process on $\mathbb{R}^n$. Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ with $n \ge 2$. Let G_D be the classical Green function on $D \times D$, i.e. G_D satisfying $\mathbb{E}_x \left[\int_0^{\tau_D} f(X_s)\, ds \right] = \int_D G_D(x,y) f(y)\, dy$ for $x \in D$. Define $\delta(x,\partial D) = \delta(x) =$ dist$(x,\partial D)$. Then there exists a constant $c = c(D,\alpha) > 1$ such that*

$$c^{-1} \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \le G_D(x,y) \le c \min\left\{ |x-y|^{\alpha-n}, \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n} \right\} \tag{3.1}$$

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4 Proof of Theorem 1.3

Theorem 1.3 is proven as follows. In a first step for $\rho > 0$ one proves existence of a regularized Green function $G_\rho(\cdot,\cdot)$ satisfying for $x_0 \in \Omega$

$$a\big(G_\rho(\cdot, x_0), \varphi\big) = \frac{1}{B_\rho(x_0)} \int\limits_{B_\rho(x_0)} \varphi(x)\, dx \qquad \forall \varphi \in C_0^\infty(\Omega)\,.$$

Next, one derives L^q-bounds on G_ρ that are uniform with respect to ρ. Also uniform bounds on G_ρ in $H_0^s(\Omega)$ for some $s > 0$ have to be proven. Compactness arguments finally assure the existence of a subsequence ρ_k such that G_{ρ_k} converges to G.

Proposition 4.1. *There exists a constant C independent of ρ, such that*

$$\|G_\rho\|_{L^{\frac{n}{n-\alpha}}_{\text{weak}}(\Omega)} \leq C\,, \tag{4.1}$$

$$\int\limits_\Omega \int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^2}{|x-y|^{n+\alpha}}\, dy\, dx \leq C\rho^{\alpha-n}\,. \tag{4.2}$$

Proposition 4.2. *There exists a constant C independent of ρ, such that for $p \in \big[1, \frac{n}{n-\alpha/2}\big)$ one has*

$$\int\limits_\Omega \int\limits_\Omega \frac{|G_\rho(x) - G_\rho(y)|^p}{|x-y|^{n+\frac{\alpha}{2}p}}\, dy\, dx \leq C\,. \tag{4.3}$$

Optimal portfolios with bounded Capital-at-Risk

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The market model

We consider a Black-Scholes type of market consisting in the simplest case of one *riskless bond* and one *risky stock*. Their price processes P_0 and P evolve according to the equations

$$P_0(t) = e^{rt}\,, \quad t \geq 0\,,$$
$$P(t) = p\exp(bt + L(t))\,, \quad t \geq 0\,,$$

where $r \in \mathbb{R}$ is the riskless rate, $p > 0$, $b \in \mathbb{R}$. The fluctuations of the risky asset are modelled by the Lévy process

$$L(t) = at + \beta W(t) + \sum_{0<s\leq t} \Delta L(s) 1_{\{|\Delta L(s)|>1\}} + \int_0^t \int_{\{|x|\leq 1\}} x(M(ds,dx) - ds\nu(dx))\,, \quad t \geq 0\,.$$

M defines a Poisson random measure on $[0,\infty)\times\mathbb{R}\backslash\{0\}$ with intensity $m(dt,dx) = dt\nu(dx)$. The process L has Lévy-Khintchine representation

$$E\exp(isL(t)) = \exp(t\Psi(s)),$$

where

$$\Psi(s) = ias - \beta^2\frac{s^2}{2} + \int_{-\infty}^{\infty}\left(e^{isx} - 1 - isx1_{\{|x|\leq 1\}}\right)\nu(dx)\,.$$

(a,β,ν) is called the characteristic triplet. The quantity $\beta^2 \geq 0$ denotes the variance of the Wiener component and the Lévy measure ν is defined by $\nu(\Lambda) = E\big[\sum_{0<s\leq 1} 1_{\{\Delta L(s)\in\Lambda\}}\big]$ for all $\Lambda \subset \mathbb{R}\setminus\{0\}$. It indicates that a jump of size x occurs at rate $\nu(dx)$. For background on Lévy processes see Sato [4].

Portfolio Optimization

Let $\pi(t) = \pi \in [0,1]$ for $t \in [0,T]$ (T denotes a fixed planning horizon) be the portfolio; i.e. the fraction of wealth, which is invested in the risky asset. Denoting X^π the *wealth process*, it follows the dynamic

$$X^\pi(t) = x\exp((r + \pi(b-r))t)\mathcal{E}(\pi\widehat{L}(t))\,, \quad t \geq 0\,,$$

where $\mathcal{E}(\widehat{L}) = \exp L$, i.e. $\ln\mathcal{E}(\pi\widehat{L})$ is again a Lévy process.

Whereas the classical mean-variance criterion of portfolio optimization consists in maximizing the expected terminal wealth under a constraint on the variance as a risk measure, we use the *mean-Capital-at-Risk criterion*, where the Capital-at-Risk is the excess risk above the riskless investment. More precisely,

$$\begin{aligned}\mathrm{CaR}(x,\pi,T) &= xe^{rT} - \alpha\text{-quantile of } X^\pi(T)\\ &= xe^{rT} - \mathrm{VaR}(x,\pi,T)\\ &= xe^{rT}\left(1 - z_\alpha e^{\pi(b-r)T}\right)\end{aligned}$$

where z_α is the α-quantile of $\mathcal{E}(\pi\widehat{L}(T))$; i.e.

$$z_\alpha = \inf\{z \in \mathbb{R} : P(\mathcal{E}(\pi\widehat{L}(T)) \leq z) \geq \alpha\}\,.$$

The optimization problem is then

$$\max_{\pi\in[0,1]} E[X^\pi(T)] \quad \text{subject to} \quad \mathrm{CaR}(x,\pi,T) \le C\,.$$

To solve this problem we need to calculate the mean wealth process and the CaR. The mean wealth process can be calculated immediately from the moment generating function of $L(1)$, provided it exists. Define $\widehat{f}(s) = E\exp(sL(1))$ then

$$E\left[X^\pi(t)\right] = x\exp\left[\Big(r+\pi\big[b-r+\ln\big(\widehat{f}(1)\big)\big]\Big)t\right], \quad t\ge 0\,.$$

Example [Geometric Brownian Motion (Emmer, Klüppelberg and Korn (2000))]. *It is not difficult to calculate for initial wealth x, portfolio π and planning horizon T*

$$\mathrm{CaR}(x,\pi,T) = xe^{rT}\left(1-z_\alpha e^{\pi(b-r)T}\right),$$

where ($\widehat{z}_\alpha$ is the α-quantile of the standard normal distribution)

$$z_\alpha = \exp\left\{-\frac{\sigma^2}{2}\pi^2 T + \widehat{z}_\alpha\sigma\pi\sqrt{T}\right\}.$$

The optimization problem *can be solved explicitly by*

$$\pi_{\text{opt}} = \frac{1}{\sigma}\left(\frac{b-r}{\sigma}+\frac{\widehat{z}_\alpha}{\sqrt{T}}+\sqrt{\left(\frac{b-r}{\sigma}+\frac{\widehat{z}_\alpha}{\sqrt{T}}\right)^2-2\frac{\kappa(C)}{T}}\right),$$

where $\kappa(C)$ is some deterministic function of the risk bound C.

In general the CaR or VaR cannot be calculated explictly. We invoke an idea of Asmussen and Rosinski (2000), which has been used for the simulation of Lévy processes:

$$\begin{aligned} L(t) &= \mu(\varepsilon)t+\beta W(t)+N^\varepsilon(t)+\int_0^t\int_{|x|<\varepsilon} x(M(ds,dx)-ds\nu(dx))\\ &\approx \mu(\varepsilon)t+(\beta^2+\sigma^2(\varepsilon))^{\frac12}\widetilde{W}(t)+N^\varepsilon(t)\,, \quad t\ge 0\,, \end{aligned}$$

where

$$\begin{aligned} \sigma^2(\varepsilon) &= \int_{|x|<\varepsilon} x^2\nu(dx)\,,\\ \mu(\varepsilon) &= a-\int_{\varepsilon\le|x|\le 1} x\nu(dx)\,,\\ N^\varepsilon(t) &= \sum_{s\le t}\Delta L(s)1_{\{|\Delta L(s)|\ge\varepsilon\}}\,. \end{aligned}$$

The approximation is a consequence of a functional central limit theorem which holds provided that for $\varepsilon\to 0$

$$\sigma(\varepsilon)^{-1}\int_0^t\int_{|x|<\varepsilon} x(M(ds,dx)-ds\nu(dx)) = \sigma(\varepsilon)^{-1}(L(t)-L_\varepsilon(t)) \xrightarrow{d} W'(t)\,,$$

where

$$L_\varepsilon(t) = \mu(\varepsilon)t+\beta W(t)+N_\varepsilon(t)\,. \tag{1}$$

It means that small jumps ($<\varepsilon$) are approximated by Brownian motion, large ones ($\ge\varepsilon$) constitute a compound Poisson process N^ε.

We extend Asmussen and Rosinski (2000) to an approximation of the VaR by the following theorem.

Theorem (Emmer and Klüppelberg (2001)).
Let L be a Lévy process with Lévy measure ν. Define L_ε as in (1) with the quantities in (1). Let furthermore $\mathcal{E}^{\leftarrow}(e^L) = \widehat{L}$ be such that $\mathcal{E}\widehat{L} = e^L$ and define $\widehat{L}_\varepsilon$ analogously to L_ε, Then the following are equivalent for $\varepsilon \downarrow 0$

(a) $\sigma(h\sigma(\varepsilon) \wedge \varepsilon) \sim \sigma(\varepsilon)$ *for each* $h > 0\,,$

(b) $\sigma(\varepsilon)^{-1}(L(t) - L_\varepsilon(t)) \stackrel{d}{\to} W'(t)\,, \quad t \geq 0\,,$

(c) $(\pi\sigma(\varepsilon))^{-1}\left(\ln \mathcal{E}(\pi\widehat{L}(t)) - \ln \mathcal{E}(\pi\widehat{L}_\varepsilon(t)\})\right) \stackrel{d}{\to} W'(t)\,, \quad t \geq 0\,.$

Here W' may be any stochastic process and $\stackrel{d}{\to}$ denotes weak convergence in $D[0,\infty)$ with the supremum norm, uniformly on compacta.

Based on this result, with $W' = W$ a standard Wiener process, we approximate the α-quantile of $\mathcal{E}(\pi\widehat{L}(T)$ by

$$z_\alpha \approx z_\alpha^\varepsilon(\pi) = \inf\{z \in \mathbb{R} : P(\gamma_\pi^\varepsilon T + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2} W(T) + M_\pi^\varepsilon(T) \leq \ln z) \geq \alpha\}\,,$$

where we have used the approximation

$$\begin{aligned} \ln \mathcal{E}(\pi\widehat{L}(t)) &\approx \gamma_\pi^\varepsilon t + \pi(\beta^2 + \sigma_L^2(\varepsilon))^{1/2} W(t) + M_\pi^\varepsilon(t)\,, \\ \gamma_\pi^\varepsilon &= \pi(\mu_\varepsilon + \frac{1}{2}\beta^2(1-\pi)), \\ M_\pi^\varepsilon(t) &= \sum_{s\leq t} \ln(1 + \pi(e^{\Delta L(s) 1(|\Delta L(s)| > \varepsilon)} - 1)). \end{aligned}$$

M_π^ε is a compound Poisson process with jump measure given for any Borel set $\Lambda \subset \mathbb{R}\backslash\{0\}$

$$\nu_{M_\pi^\varepsilon}(\Lambda) = \nu_L(\{x \in \mathbb{R} : \ln(1 + \pi(e^x - 1)) \in \Lambda\} \backslash (-\varepsilon, \varepsilon))\,.$$

From this we conclude

$$\begin{aligned} \mathrm{VaR}(x,\pi,T) &\approx x z_\alpha^\varepsilon(\pi) \exp((\pi(b-r) + r)T) \\ \mathrm{CaR}(x,\pi,T) &\approx x e^{rT}\left(1 - z_\alpha^\varepsilon(\pi) e^{\pi(b-r)T}\right) \end{aligned}$$

This result applies to various examples, which have been suggested as price processes.

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A Stable-Like Process over an Infinite Extension of a Local Field

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1. A remarkable feature of the contemporary probability theory is the tendency to study stochastic objects related to various structures appearing in other branches of mathematics such as algebra, geometry, or topology. The interaction of different trends of mathematical thought becomes even more fruitful under the influence of mathematical physics. The study of analytic, probabilistic and physical structures related to p-adic numbers and general local fields (non-discrete locally compact totally disconnected topological fields) is a good example of such an interaction.

The field $\mathbb{Q}_p$ of p-adic numbers and related structures (its ring of integers, group of units, finite and infinite extensions etc.) constitute a class of algebraic structures admitting rich harmonic analysis, which can be employed in the construction and study of stochastic objects. Recent activity in p-adic models of quantum mechanics and quantum field theory (see [10, 21]) has led to new results in the study of various operators in function spaces over $\mathbb{Q}_p$.

Since $\mathbb{Q}_p$ is totally disconnected, stochastic processes on $\mathbb{Q}_p$ (with real time) are of pure jump type. The simplest Markov process on $\mathbb{Q}_p$ is the p-adic analog $X_\alpha(t)$ of the symmetric stable process introduced independently by several authors [7, 8, 9, 11, 21]; its generator is the fractional differentiation operator D^α , $\alpha > 0$ [21, 17]. On the Schwartz-Bruhat test function space D^α can be defined as a pseudo-differential operator with the symbol $|\xi|_p^\alpha$. D^α admits a hyper-singular integral representation which makes it possible to extend the operator to wider classes of functions. Note that a probability distribution with the characteristic function $\exp(-a|\xi|_p^\alpha)$, $a > 0$, not only resembles the classical symmetric stable distribution. In fact it is a representative of a family of distributions on $\mathbb{Q}_p$ characterized as weak limits of normalized sums of independent identically distributed p-adic random variables [14, 17, 23]; for its geometric interpretation see [3].

In [11] a theory of parabolic equations based on the operator D^α was developed. This has led to analytic construction of a large class of Markov processes on $\mathbb{Q}_p$. On the other hand, in [12] (see also [17]) a theory of stochastic differential equations based on the process $X_\alpha(t)$ was initiated. The operator D^α can be used also to construct processes on p-adic balls and spheres; it is interesting that the corresponding process on the group of units (the "unit sphere") is connected to the multiplicative structure of the field though it is defined primarily in terms of its additive structure [13, 17].

Another approach to constructing and investigating Markov processes on local fields was proposed by Albeverio and Karwowski [1]. The idea was to cover $\mathbb{Q}_p$ by a family $\mathcal{K}$ of disjoint balls, to construct a suitable process on $\mathcal{K}$, and then to shrink balls to their centers obtaining a process on $\mathbb{Q}_p$. Later Yasuda [22] showed that any rotation-invariant process with independent increments can be obtained by means of this construction. A number of various generalizations and applications can be found in recent papers by Albeverio, Karwowski, Vilela Mendes, Zhao, and others (see [2] and references there).

In conlusion of this brief review of stochastic processes on local fields, it is appropriate to note that above we dealt only with processes with a real positive time parameter. However there exists also a theory of processes with both time and values non-Archimedean, initiated by Evans, and a theory of real-valued Gaussian processes with a p-adic time parameter (Evans, Bikulov and Volovich). For an introduction and references see [4, 6, 17].

2. Let us consider, in a little greater detail, the development of a version of infinite-dimensional non-Archimedean analysis initiated in [15, 16, 17, 18, 24].

All the results mentioned above, as well as their classical counterparts, use in a very strong way the local compactness of the underlying field; in particular they rely on existence of the Haar measure. On the other hand, the calculus over infinite-dimensional vector spaces where no invariant measure can exist is among well-established topics of real analysis. Some general principles of such a calculus for the p-adic situation were introduced by Evans (see [6]); the first constructive results were obtained by Mądrecki [19] and Satoh [20].

The well-known Minlos theorem states that every continuous positive-definite function on a real nuclear locally convex topological vector space is the Fourier transform of some Radon measure on the conjugate space equipped with the $*$-weak topology. It was shown by Mądrecki [19] that a similar result in the p-adic case holds without any nuclearity assumptions.

Satoh's paper [20] is devoted to a p-adic version of the theory of abstract Wiener spaces. This leads to a construction of Wiener-type measures on some non-Archimedean Banach spaces; in particular, a certain space of power series has been considered.

Note that while a large part of the real infinite-dimensional analysis is devoted to the study of measures, function spaces, and operators over a Hilbert space, there is no clear counterpart of a Hilbert space in the p-adic case. However there are other infinite-dimensional spaces over p-adics which are of purely arithmetical nature and constitute a natural arena for developing analysis. These are infinite extensions of local fields.

We consider an infinite extension K of a local field k, char $k = 0$, which is a union of an increasing sequence

$$k = K_1 \subset \ldots \subset K_n \subset \ldots \tag{1}$$

of finite extensions. The field K is a topological vector space over k with the inductive limit topology. Its conjugate $\overline{K}$ is a completion of K with respect to a certain topology defined in arithmetical terms. We construct a Radon measure μ on $\overline{K}$ which is Gaussian in the sense of Evans [6] and possesses some (partial) invariance properties. A version of the Fourier-Wiener transform is introduced over $\overline{K}$, and Fourier images of certain test functions are described. This allows to define and study a pseudo-differential operator over $\overline{K}$ similar to the fractional differentiation operator D^α over a local field. This operator is proved to be a generator of a Markov process $\overline{X}_\alpha(t)$ on $\overline{K}$. If we deal with Galois extensions then all these objects are invariant with respect to the Galois group of the extension K/k.

Just as for the process $X_\alpha(t)$ on a local field, this process is defined for any $\alpha > 0$. Some of its properties are similar to those of the classical stable processes or processes on local fields while others are different. In particular, the Gaussian measure μ is invariant for the process $\overline{X}_\alpha(t)$; the transition probabilities of $\overline{X}_\alpha(t)$ are not absolutely continuous with respect to μ.

Both μ and the convolution semigroup of measures $\pi(t, dx)$, which defines $\overline{X}_\alpha(t)$, are concentrated on a compact subgroup $S \subset \overline{K}$, and μ coincides with the normalized Haar measure on S. Thus an essential information on the process $\overline{X}_\alpha(t)$ is contained in the properties of its part $\overline{X}_{S,\alpha}(t)$ in S.

In order to study sample path properties of $\overline{X}_{S,\alpha}(t)$, we can use the results by Evans [5] who investigated Lévy processes on a general Vilenkin group (a non-discrete locally compact totally disconnected Abelian topological group). The topology in a Vilenkin group is determined by a descending chain of compact open subgroups. This chain is not unique, and as soon as we manage to write such a chain $\{S_n\}$ explicitly for our case and compute the Lévy measure of $S \setminus S_n$, the general theorems from [5] yield immediately the asymptotics of the first exit time $\pi(n)$ of $\overline{X}_{S,\alpha}(t)$ out of the subgroup S_n, and an information on the local behavior of sample paths. We also prove that both the Hausdorff and packing dimensions of a sample path of $\overline{X}_{S,\alpha}(t)$ equal 0 almost surely, which is quite different both from the classical case and the case of a local field.

The last result shows the importance of finding, for our situation, a correct Hausdorff measure. However this problem is more complicated. In order to use the appropriate theorem from [5], we have to know that

$$\liminf_{n\to\infty} Q(n, N) > 0 \tag{2}$$

where

$$Q(n, N) = \mathbf{P}\left\{\overline{X}_{S,\alpha}(t) \notin S_n \ \forall\, t \in [\pi(n), \pi(N))\right\}, \quad n > N.$$

Evans proved this property for processes with locally spherically symmetric Lévy measures. This condition is not fulfilled in our case, and we give a direct proof of (2), and construct the Hausdorff measure, under the assumption that all the extensions in (1) are tamely ramified. Such an assumption is often made in algebraic number theory because the algebraic structure of tamely ramified extensions is more or less transparent while general extensions may behave quite wildly.

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On Lévy processes, Malliavin calculus and market models with jumps

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Introduction

This paper has two objectives. The first one is to develop the initial steps of Malliavin Calculus for Lévy Processes. The Malliavin calculus in the Brownian setup has two approaches which turn out to be equivalent: one as a weak derivative in canonical space and the other one through Wiener chaos (see Nualart (1995) for a complete account of this theory). In general, a Lévy process has no chaotic decomposition property in the sense that Brownian motion, Poisson process, or so-called normal martingales have (see Ma et al. (1998)), but recent work (Nualart and Schoutens (2000)), where a kind of chaotic representation property for Lévy processes has been proved, has enabled us to define a Malliavin derivative using the chaotic approach. However, for present purposes, the Lévy processes studied by Nualart and Schoutens (2000) are too general and we will restrict ourselves to a brief investigation of their most relevant properties. We center our research on a very simple Lévy process –the sum of a Brownian motion and k independent Poisson processes– for which is possible to obtain a weak derivative interpretation and useful formulas. More general Lévy processes, such as the sum of a Brownian motion and a compound Poisson process can be approximated by these simple Lévy processes. In preparing this paper we came across a paper by Løkka (1999) which containes several extremely useful ideas.

The second goal of this paper is option hedging in a jump-diffusion model. Models with jumps for a market are an old topic in Mathematical Finance; The initial paper of Black–Scholes (1973) was rapidily followed by Merton (1976) where the first jump–diffusion model was proposed. Here we approximate a jump–diffusion model for a simple Lévy process of the type studied in the first part of the paper, and using a Malliavin Calculus approach (see Øksendal (1996) for this technique in the Black-Scholes case) we hedge a european call. In this connexion we would like to acknowledge the paper by Jensen (1999) dealing with the problem of pricing a european call in a jump–diffusion model. A different approach for extending Clark-Haussman-Ocone formula using white noise analysis and its applications to mathematical finance can be found in Aase et al. (2000).

The present paper is organized as follows. The first section deals with the theory of Malliavin derivatives for a general Lévy process and we answer some questions that naturally arise from the paper by Nualart and Schoutens (2000). In the second section we center the results of Section 1 on more tractable Lévy processes that we call *simple Lévy processes* and we obtain some interesting formulas to compute Malliavin derivatives. In Section three, we prove some results in order to approximate the Lévy process used as a jump–diffusion model by means of simple Lévy processes. Finally, in section four, we obtain formulas which enables an approximate hedge of a european call in a jump–diffusion model.

1 General theory

1.1 Notations

Let $X = \{X_t,\, t \geq 0\}$ be a Lévy process and henceforth we always assume that we are using the *cadlag* version) on a complete probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t,\, t \geq 0\}$ be the natural

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filtration of X completed with the null sets of $\mathcal{F}$. We also assume that the Lévy measure ν of X satisfies that there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_{(-\varepsilon,\varepsilon)^c} e^{\delta\,|x|}\,\nu(dx) < \infty. \tag{1.1}$$

This implies that X_t has moments of all orders and that the polynomials are dense in $L^2(\mathbb{R},\, P{\circ}X_1^{-1})$ (see Nualart and Schoutens (2000))

Define

$$X_t^{(1)} = X_t,$$
$$X_t^{(i)} = \sum_{0<s\le t} \big(\Delta X_s\big)^i, \quad i \ge 2.$$

We have:

- The processes $X^{(i)} = \{X_t^{(i)},\, t \ge 0\},\ i = 1, 2, \ldots$, are Lévy processes that jump at the same points as X.
- $E(X_t^{(i)}) = m_i t$, where $m_1 = E(X_1)$ and $m_i = \int_{-\infty}^{\infty} x^i\,\nu(dx)$, $i \ge 2$.

Now define the processes

$$Y_t^{(i)} = X_t^{(i)} - m_i\,t, \quad i \ge 1.$$

The processes $Y^{(i)} = \{Y_t^{(i)},\, t \ge 0\}$ are martingales.

Besides, we introduce the processes

$$H_t^{(i)} = \sum_{j=1}^{i} a_{ij} Y_t^{(j)}, \quad i \ge 1, \tag{1.2}$$

where the constants a_{ij} are chosen in such a way that $a_{i1} = 1$ and the martingales $H^{(i)},\ i = 1, 2, \ldots$ are pairwise strongly orthogonal, that means, for $i \ne j$, the process $H^{(i)}H^{(j)}$ is a martingale.

1.2 Iterated integrals

For Lévy processes, we use iterated integrals (instead of multiple ones) because of the chaotic representation given by Nualart and Schautens (2000), which involves the family $H^{(i)},\ i = 1, 2, \ldots$ as integrators.

Let $\Sigma_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n : 0 < t_1 < t_2 < \cdots < t_n\}$ be the positive simplex of $\mathbb{R}^n$. Given $f \in L^2(\mathbb{R}_+^n)$ we will denote by $J_n^{(i_1,\ldots,i_n)}(f)$ the iterated integral of f with respect to $H^{(i_1)}, \ldots, H^{(i_n)}$:

$$J_n^{(i_1,\ldots,i_n)}(f) = \int_0^\infty \Big(\int_0^{t_n-} \cdots \Big(\int_0^{t_2-} f(t_1, \ldots, t_n)\,dH^{(i_1)}(t_1)\Big) \cdots \\ \cdot dH^{(i_{n-1})}(t_{n-1})\Big) dH^{(i_n)}(t_n).$$

We remark that all these integrals are well defined since all the processes $H^{(i)},\ i = 1, 2, \ldots$ are square integrable martingales with respect to the filtration $\{\mathcal{F}_t,\, t \ge 0\}$.

In the remain of this paper, we use the notations of Ma et al. (1998) for the indices $(t_1, \ldots, t_n)$ in the simplex Σ_n, where $t_1 < \cdots < t_n$, instead of that of Nualart and Schoutens (2000) who write the indices in decreasing order.

The main results of the paper of Nualart and Schoutens (2000) are the chaotic and the predictable representation properties of the square integrable random variables:

Theorem 1.1 (Nualart and Schoutens). *Let $F \in L^2(\Omega, \mathcal{F}, P)$. Then F has a unique representation of the form*

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n \ge 1} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}),$$

where $f_{i_i,\ldots,i_j} \in L^2(\Sigma_j)$.

An immediate consequence of Theorem 1.1 is the predictable representation property of the square integrable random variables:

Theorem 1.2 (Nualart and Schoutens). *Let $F \in L^2(\Omega, \mathcal{F}, P)$. Then F has a representation of the form*

$$F = E[F] + \sum_{k=1}^{\infty} \int_0^{\infty} \phi_k(t)\, dH^{(k)}(t),$$

where $\{\phi_k(t),\, t \geq 0\}$, $k \geq 1$, are predictable processes.

1.3 Derivative operators

This section is devoted to the study of some properties of the derivatives in the context of calculus of variations.

In the remaining of this paper, write

$$\Sigma_n^{(k)}(t) = \{(t_1, \ldots, \widehat{t_k}, \ldots, t_n) \in \Sigma_{n-1} : \\ 0 < t_1 < \cdots < t_{k-1} < t \leq t_{k+1} < \cdots < t_n\}$$

and $\widehat{i}$ means that the i-th index is omitted. Observe that if $k \neq k'$ then $\Sigma_n^{(k)}(t) \cap \Sigma_n^{(k')}(t) = \emptyset$.

Definition 1.3. *Let $f \in L^2(\mathbb{R}_+^n)$ and $\ell \geq 1$. The process*

$$D_t^{(\ell)} J_n^{(i_1,\ldots,i_n)}(f) = \sum_{k=1}^{n} \mathbf{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\ldots,\widehat{i_k},\ldots,i_n)} \big(f(\underbrace{\ldots}_{k-1}, t, \ldots)\mathbf{1}_{\Sigma_n^{(k)}(t)}(\cdot)\big),$$

is called the derivative of $J_n^{(i_1,\ldots,i_n)}(f)$ in the ℓ–th direction.

Also we define the spaces of the random variables that are differentiable in the ℓ-th direction. For this, we define the following subset of $L^2(\Omega)$:

$$\mathbb{D}^{(\ell)} = \Big\{F \in L^2(\Omega),\, F = EF + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n \geq 1} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) : \\ \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n \geq 1} \sum_{k=1}^{n} \mathbf{1}_{\{i_k=\ell\}} q_{i_1} \cdots \widehat{q_{i_k}} \cdots q_{i_n} \\ \cdot \int_0^{\infty} \|f_{i_1,\ldots,i_n}(\ldots, t, \ldots)\mathbf{1}_{\Sigma_n^{(k)}(t)}\|^2_{L^2([0,\infty)^{n-1})}\, dt < \infty\Big\}$$

Definition 1.4. *Given $F \in \mathbb{D}^{(\ell)}$ such that*

$$F = EF + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n \geq 1} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}),$$

we define the derivative of F in the ℓ-th direction as the element of $L^2(\Omega \times \mathbb{R}_+)$ given by

$$D_t^{(\ell)} F = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} \sum_{k=1}^{n} \mathbf{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\ldots,\widehat{i_k},\ldots,i_n)} \big(f_{i_1,\ldots,i_n}(\ldots, t, \ldots)\mathbf{1}_{\Sigma_n^{(k)}(t)}(\cdot)\big)$$

Observe that, as in the classical situation for Gaussian processes, $\mathbb{D}^{(\ell)}$ is dense in $L^2(\Omega)$, since the elements of $L^2(\Omega)$ with a finite chaotic expansion are in $\mathbb{D}^{(\ell)}$. Then, also as in the Brownian case, we can define the adjoint operator of $D^{(\ell)}$, $\delta^{(\ell)}$ which will be called the Skorohod integral in the ℓ–th direction. We will not continue this development here because we will only deal with the derivative operators.

From the chaotic representation property (Theorem 1.1) we can prove the following formula:

Theorem 1.5 (Clark-Ocone formula). *Let $F \in \bigcap_{n=1}^{\infty} \mathbb{D}^{(n)}$. Then*

$$F = E[F] + \sum_{k=1}^{\infty} \int_0^{\infty} {}^{p}(D_t^{(k)} F)\, dH_t^{(k)},$$

where ${}^{p}(D_t^{(k)})F$ denotes the predictable projection $E\big[D_t^{(k)} F / \mathcal{F}_{t-}\big]$

1.4 Lévy processes with a finite number of jump sizes

In this section we consider a Lévy process for which the family $\{H^{(i)},\, i \geq 1\}$ has only a finite number of elements different than zero and we summarize here some results about this case.

Proposition 1.6. *Suppose that, for some $j \geq 1$, $H^{(j)} = 0$. Then the number of different jump sizes of X that are not zero is at most $j-1$.*

The following is the reciprocal of the last proposition.

Proposition 1.7. *If the process X has only j different jump sizes, then*

a. $H^{(k)} = 0$, $\forall k \geq j+1$, if X has no continuous part.

b. $H^{(k)} = 0$, $\forall k \geq j+2$, if X has continuous part.

A normal martingale X (see Ma et al (1998)) is a martingale such that $\langle X, X\rangle_t = t$ and that possesses the chaotic representation property (see also Dellacherie et al. (1992) pag. 199). As a consequence of the last two propositions we obtain the next corollary that says that only the brownian motion and the (compensated) Poisson processes are normal martingales and Lévy processes.

Corollary 1.8. *Let X be a Lévy process that satisfies condition (1.1) and is also a normal martingale. Then X is a brownian motion or a process of the form $\alpha N_t - t/\alpha$, where N_t is Poisson process of intensity λ and $\alpha = \pm 1/\sqrt{\lambda}$.*

2 Simple Lévy processes

In this section we deal with the simple Lévy process given by

$$X_t = \sigma W_t + \alpha_1 N_1(t) + \cdots + \alpha_k N_k(t), \quad t \geq 0. \tag{2.1}$$

where $\{W_t,\, t \geq 0\}$ is a standard Brownian motion, $\{N_j(t),\, t \geq 0\}$, $j = 1, \ldots, k$, are independent Poisson processes (and independent of Brownian motion) of parameter $\lambda_1, \ldots, \lambda_k$, respectively, $\sigma > 0$ and $\alpha_1, \ldots, \alpha_k$ are different non-null numbers. The Lévy measure of X is $\nu = \sum_{j=1}^{k} \lambda_j \delta_{\alpha_j}$ and satisfies the condition (1.1) of Nualart and Schoutens (2000) for the validity of the chaotic representation property.

However, in this context, in addition to the family $\{H^{(1)},\ldots,H^{(k+1)}\}$, we also have the set of martingales: $\{W_t,\, N_1(t)-\lambda_1 t,\ldots,N_k(t)-\lambda_k t\}$. It seems sensible to use the last family instead of the former.

It follows that $H^{(i)}$ are a linear combination of $W_t, N_1(t)-\lambda_1 t,\ldots,N_k(t)-\lambda_k t$. Therefore, we have unicity in the chaotic representation property in terms of the iterated integrals with respect $W_t,\, N_1(t)-\lambda_1 t,\ldots,N_k(t)-\lambda_k t$.

A predictable representation property of the following form also holds:

Proposition 2.1. *Let $F\in L^2(\Omega,\mathcal{F},P)$. Then F admits a representation of the form*

$$F=E[F]+\int_0^\infty \phi_0(t)\,dW_t+\sum_{j=1}^k\int_0^\infty \phi_j(t)d(N_j(t)-\lambda_j\, t),$$

where $\phi_0,\ldots,\phi_k$ are predictable processes such that $\int_0^\infty E[\phi_j^2(t)]\,dt<\infty$.

Further, we can define derivatives in the directions $W_t, N_1(t)-\lambda_1 t,\ldots, N_k(t)-\lambda_k t$ through the iterated integrals, mimicking the definition given in Section 1.3. We will denote by $D^{(0)},\ldots,D^{(k)}$ the derivatives in the directions $W_t, N_1(t)-\lambda_1 t,\ldots,N_k(t)-\lambda_k t$ respectively. So we have

Theorem 2.2. *Let $F\in\bigcap_{j=0}^k \mathbb{D}^{(j)}$. Then*

$$F=E[F]+\int_0^\infty {}^p(D_t^{(0)}F)\,dW_t+\sum_{j=1}^k\int_0^\infty {}^p(D_t^{(j)}F)\,d(N_j(t)-\lambda_j\, t).$$

2.1 Interpretation of the derivatives for a simple Lévy process

In this section we still work with the simple model (2.1). We interpret the operators $D^{(\ell)}$ appearing in Theorem 2.2 and show some formulas that have interesting applications. The main result is that in certain cases it is possible to compute the derivatives in the directions W, N_1, etc. following the classical rules for each case. We point out that this is a nice idea due to Løkka (1999) that we think is only true for certain types of Lévy processes, such as the ones that we are considering.

We will prove that $D^{(0)}F$ (respectively $D^{(1)}F$) can be interpreted as the usual brownian Malliavin derivative (resp., the Poisson Malliavin derivative). This useful property is summarized in the next proposition.

Proposition 2.3. *Consider a simple Lévy process*

$$X_t=\sigma W_t+\alpha_1 N_1(t)+\cdots+\alpha_k N_k(t),\quad t\geq 0.$$

(a) Let $F=f(Z,Z')\in L^2(\Omega)$, where Z only depends on the Brownian motion W, and Z' only depends on the Poisson processes $N_1,\ldots,N_k$. Assume that $f(x,y)$ is a continuosly differentiable function with bounded partial derivatives in the variable x, and that $Z\in\mathbb{D}^{(0)}$. Then $F\in\mathbb{D}^{(0)}$ and

$$D^{(0)}F=\frac{\partial f}{\partial x}(Z,Z')\,D^{(0)}Z.$$

(b) Let $F\in\mathbb{D}^{(j)}$ for some $j\in\{1,\ldots,k\}$. Consider a generic element $\omega=(\omega_0,\ldots,\omega_k)\in\Omega=\Omega_W\times\Omega_{N_1}\times\cdots\times\Omega_{N_k}$, $\omega_j\in\bigcup_{n=0}^\infty[0,T]^n$. Then

$$D_t^{(j)}F(\omega)=F(\omega_0,\ldots,\omega_{j-1},\omega_j+\delta_t,\omega_{j+1},\ldots,\omega_k)-F(\omega),$$

where

$$\omega_j+\delta_t=\begin{cases}(s_1,\ldots,s_r,t), & \text{if }\ \omega_j=(s_1,\ldots,s_r),\\ t, & \text{if }\ \omega_j=\{a\}.\end{cases}$$

3 Approximation by simple Lévy processes

In this section we will consider a Lévy process of the form

$$X_t = \sigma W_t + \sum_{j=1}^{N_t} Z_j, \qquad t \geq 0, \tag{3.1}$$

(with the convention that the sum is 0 when $N_t = 0$) where

1. $\{W_t,\, t \geq 0\}$ is a standard Brownian motion.
2. $\{N_t,\, t \geq 0\}$ is a Poisson process of parameter $\lambda > 0$.
3. $\{Z_n,\, n \geq 1\}$ is a sequence of i.i.d. square integrable random variables.
4. $\{W_t,\, t \geq 0\}$, $\{N_t,\, t \geq 0\}$ and $\{Z_n,\, n \geq 1\}$ are independent.

We get that, in the same way that every square integrable random variable can be approximated (in $L^2(\Omega)$) by simple random variables (taking only a finite number of different values), a Lévy process of type (3.1) can be approximated by simple Lévy processes in $L^2(\Omega \times [0,T])$.

Theorem 3.1. *Let $T > 0$. For each n, there exists a family of independent Poisson processes (and independent of $\{W_t,\, t \geq 0\}$), $\{N_1^n(t),\, t \geq 0\}, \ldots, \{N_{k_n}^n(t),\, t \geq 0\}$, and a family of constants $\alpha_1^n, \ldots, \alpha_{k_n}^n$ such that the Lévy process*

$$X^n(t) = \sigma W_t + \sum_{r=1}^{k_n} \alpha_r^n\, N_r^n(t), \qquad t \in [0,T], \tag{3.2}$$

converges to $\{X_t,\, t \geq [0,T]\}$ in $L^2\big(\Omega \times [0,T]\big)$.

4 Option hedging in a market with jumps

Here we consider a market with only one riskless asset, $A(t)$, determined by

$$\begin{aligned} dA(t) &= r\, A(t)\, dt \\ A(0) &= 1 \end{aligned}$$

where $r > 0$ is a constant. Also consider an asset with risk, $S(t)$, satisfying the equation

$$\begin{aligned} dS(t) &= S(t-)\, dX(t) \\ S(0) &= s_0 \end{aligned}$$

(s_0 is a constant) where $X(t)$ is the Lévy process defined in (3.1) with the condition that $Z_n > -1$, $n \geq 1$, which implies that the asset prices will be always non negative.

4.1 Approximative market

Our objective is to hedge an european call based on the asset S with maturity T and strike price $K > 0$; the final profit is

$$U = \big(S(T) - K\big)^+.$$

Since U is a square integrable random variable, by Theorem 1.2 it can be represented as a (possibly) infinite sum of stochastic integrals with respect to the family $\{H^{(i)},\, i \geq 1\}$ related to X. In order to carry out a realistic hedging, we need to do two things: to consider a finite sum and to be able to compute the integrands. Both things can be done approximating the Lévy process X by a simple process, which is possible thanks to Theorem 3.1. Therefore, we will consider the approximating process

$$X^n(t) = \mu t + \sigma W_t + \sum_{s=1}^{k_n} \alpha_s^n \big(N_s^n(t) - \lambda_s^n t\big),$$

where $\alpha_s^n > -1,\ s = 1, \ldots, k_n$. The market modeled by this process will be called approximative market. We can write the following results,

Lemma 4.1. *Let $\{S^n(t), t \in [0,T]\}$, be the solution of the equation*

$$\begin{aligned} dS^n(t) &= S^n(t-)\, dX^n(t) \\ S^n(0) &= s_0 \end{aligned}$$

Then

$$\lim_n S^n = S, \quad \textit{in } L^2(\Omega \times [0,T]).$$

Proposition 4.2. *Let $U_n = \big(S^n(T) - K\big)^+$. Then U_n converges to U in $L^2(\Omega)$.*

To simplify the notation, we will omit the index n in the expressions of $X^n(t)$, $S^n(t)$, U_n, α_r^n and λ_r^n. We will also denote by S_t^* and U^* the discounted price:

$$S_t^* = e^{-rt}\, S_t \quad \text{and} \quad U^* = e^{-rt}\, U.$$

4.2 Option pricing in the approximative market

It is well-known that the problem of pricing an option consists in finding a unique equivalent measure Q that makes S_t^* a martingale. In this context this problem has been solved by Jensen (1999) assuming the existence of k additional assets, defined by equations

$$dP_j(t) = P_j(t-)\, dX_j(t), \quad j = 1, \ldots, k,$$

where

$$X_j(t) = \mu_j t + \sigma_j W_t + \sum_{r=1}^{k} \alpha_{j,r} \big(N_r(t) - \lambda_r t\big).$$

This is a known idea: to hedge an option when there are $k+1$ sources of randomness $(W, N_1, \ldots, N_k)$, we need $k+1$ assets so that the market be complete. Using the Girsanov Theorem (see Sato (1999)), Jensen (1999) gives a condition for the existence of one and only one equivalent probability Q (that depens on k) such that the discounted prices $S^*, P_1^*, \ldots, P_k^*$ $(P_j^*(t) = e^{-rt} P_j(t))$ are Q–martingales.

From our point of view we can present another explanation of these results. The Clark-Ocone formula given in Theorem 2.2 can be applied to the approximated marked. In particular, this implies that every square integrable random variable of the type $f(S_T)$ can be represented as a sum of $k+1$ stochastic integrals (respect to $W, N^{(1)}, \ldots, N^{(k)}$). However, what we need is a representation using the assets S, $P_1, \ldots, P_k$ as integrators; in other words, that the Q-martingales (discounted) S^*, $P_1^*, \ldots, P_k^*$ be a base for the representation. As we will see, the condition we get for that is exactly Jensen's condition.

We keep the result (see Jensen (1999)) that under Jensen's condition there is one and only one equivalent probability Q, a Q–Brownian motion W^Q and k independent Q–Poisson processes of parameter $\widetilde{\lambda}_j = \lambda_j - L_j$, $N_1^Q, \ldots, N_k^Q$, independent of W^Q, such that we can write

$$\begin{aligned}\frac{dS(t)}{S(t-)} &= r\,dt + \sigma dW^Q(t) + \sum_{r=1}^{k} \alpha_r \big(dN_r^Q(t) - \widetilde{\lambda}_r dt\big) \\ \frac{dP_1(t)}{P_1(t-)} &= r\,dt + \sigma_1 dW^Q(t) + \sum_{r=1}^{k} \alpha_{1,r} \big(dN_r^Q(t) - \widetilde{\lambda}_r dt\big) \\ &\vdots \\ \frac{dP_k(t)}{P_k(t-)} &= r\,dt + \sigma_k dW^Q(t) + \sum_{r=1}^{k} \alpha_{k,r} \big(dN_r^Q(t) - \widetilde{\lambda}_r dt\big)\end{aligned} \tag{4.1}$$

The price of an european call $U = \big(S(T) - K\big)^+$ obtained by Jensen (1999) is

$$\begin{aligned}c_0 = \exp\Big\{T \sum_{j=1}^{k} \widetilde{\lambda}_j\Big\} & \sum_{n_1,\ldots,n_k=1}^{\infty} \prod_{j=1}^{k} \frac{\big[(1+\alpha_j)\widetilde{\lambda}_j T\big]^{n_j}}{n_j!} \\ & \cdot \Big(S_0 \exp\Big\{-T \sum_{j=1}^{k} \alpha_j \widetilde{\lambda}_j\Big\} N\big(d_1(n_1,\ldots,n_k;0;S_0)\big) \\ & - \frac{e^{-rT}K}{\prod_{j=1}^{k}(1+\alpha_j)^{n_j}} N\big(d_2(n_1,\ldots,n_k;0;S_0)\big)\Big)\end{aligned} \tag{4.2}$$

where $N(z)$ denotes the distribution function at point z of a standard normal random variable,

$$\begin{aligned}d_1(n_1,\ldots,n_k;t;x) = & \frac{1}{\sigma\sqrt{T-t}}\Big[\ln\frac{x}{K} + \big(r + \frac{1}{2}\sigma^2 - \sum_{j=1}^{k} \alpha_j \widetilde{\lambda}_j\big)(T-t)\Big] \\ & + \frac{1}{\sigma\sqrt{T-t}} \sum_{j=1}^{k} n_j \ln(1+\alpha_j)\end{aligned} \tag{4.3}$$

and

$$d_2(n_1,\ldots,n_k;t;x) = d_1(n_1,\ldots,n_k;t;x) - \sigma\sqrt{T-t}. \tag{4.4}$$

(Remember that $x > 0$ and $\alpha_j > -1,\ j = 1,\ldots,k$).

4.3 Option hedging in the approximative marked

By Clark-Ocone formula (Theorem 2.2) we have

$$U^* = E_Q[U^*] + \int_0^T \phi_0(t)\, dW^Q(t) + \sum_{j=1}^{k} \int_0^T \phi_j(t)\, d(N_j^Q(t)(t) - \widetilde{\lambda}_j t).$$

where

$$\phi_j(t) = E_Q\big[D_t^{(j)} U^* / \mathcal{F}_{t-}],\quad j = 0,\ldots,k,$$

that can be explicitelly calculated, and we obtain the formulas

$$\begin{aligned}\phi_0(t) = \sigma E[f(S_T)/\mathcal{F}_{t-}] = \sigma S_{t-} \exp\Big\{\big(r + \sum_{j=1}^{k}(\alpha_j - 1)\widetilde{\lambda}_j\big)(T-t)\Big\} \\ \cdot \sum_{n_1,\ldots,n_k=1}^{\infty} \prod_{j=1}^{k} \frac{\big[(1+\alpha_j)(\widetilde{\lambda}_j(T-t))\big]^{n_j}}{n_j!} N\big(d_1(n_1,\ldots,n_k;t;S_{t-})\big),\end{aligned}$$

and for $j = 1,\ldots,k$,

$$
\begin{aligned}
\phi_j(t) = E\Big[D_t^{(j)}(S(T)-K)^+/\mathcal{F}_{t-}\Big] = \\
&(1+\alpha_j)S_{t-}\exp\Big\{\big(r+\sum_{j=1}^k(\alpha_j-1)\widetilde{\lambda}_j\big)(T-t)\Big\} \\
&\cdot \sum_{n_1,\dots,n_k=1}^{\infty}\prod_{j=1}^k \frac{\Big[(1+\alpha_j)(\widetilde{\lambda}_j(T-t))\Big]^{n_j}}{n_j!} N\big(d_1(n_1,\dots,n_k;t;(1+\alpha_j)S_{t-})\big) \\
&- e^{-\sum_{j=1}^k\widetilde{\lambda}_j(T-t)} \sum_{n_1,\dots,n_k=1}^{\infty}\prod_{j=1}^k \frac{\Big[(\widetilde{\lambda}_j(T-t))\Big]^{n_j}}{n_j!} \\
&\cdot N\big(d_2(n_1,\dots,n_k;t;(1+\alpha_j)S_{t-})\big) \\
&- S_{t-}\exp\Big\{\big(r+\sum_{j=1}^k(\alpha_j-1)\widetilde{\lambda}_j\big)(T-t)\Big\} \\
&\cdot \sum_{n_1,\dots,n_k=1}^{\infty}\prod_{j=1}^k \frac{\Big[(1+\alpha_j)(\widetilde{\lambda}_j(T-t))\Big]^{n_j}}{n_j!} N\big(d_1(n_1,\dots,n_k;t;S_{t-})\big) \\
&+ e^{-\sum_{j=1}^k\widetilde{\lambda}_j(T-t)} \sum_{n_1,\dots,n_k=1}^{\infty}\prod_{j=1}^k \frac{\Big[(\widetilde{\lambda}_j(T-t))\Big]^{n_j}}{n_j!} N\big(d_2(n_1,\dots,n_k;t;S_{t-})\big),
\end{aligned}
$$

where $d_1(n_1,\dots,n_k;t;x)$ and $d_2(n_1,\dots,n_k;t;x)$ are given in (4.3) and (4.4) respectively.

The last step is to write U^* as a sum of integrals with respect the assets S^*, $P_1^*,\dots,P_k^*$. Denote by B the matrix

$$
B = \begin{pmatrix} \sigma & \alpha_1 & \cdots & \alpha_k \\ \sigma_1 & \alpha_{11} & \cdots & \alpha_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_k & \alpha_{k1} & \cdots & \alpha_{kk} \end{pmatrix}
$$

The system (4.1) is

$$
\begin{pmatrix} \frac{dS^*(t)}{S^*(t-)} \\ \frac{dP_1^*(t)}{P_1^*(t-)} \\ \vdots \\ \frac{dP_k^*(t)}{P_k^*(t-)} \end{pmatrix} = B \begin{pmatrix} dW_t^Q \\ d(N_1(t)^Q-\widetilde{\lambda}_1 t) \\ \vdots \\ d(N_k(t)^Q-\widetilde{\lambda}_k t) \end{pmatrix}
$$

It follows that if B is invertible (which is exactly the Jensen's condition) we can define

$$
\big(\psi_0(t),\dots,\psi_k(t)\big) = \big(\phi_0(t),\dots,\phi_k(t)\big)B^{-1} \begin{pmatrix} \frac{1}{S^*(t-)} & 0 & \cdots & 0 \\ 0 & \frac{1}{P_1^*(t-)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{P_k^*(t-)} \end{pmatrix}
$$

and we obtain the hedging

$$
U^* = E_Q[U^*] + \int_0^T \psi_0(t)\,dS^*(t) + \sum_{j=1}^k \int_0^T \psi_j(t)\,dP_j^*(t). \tag{4.5}
$$

Note that the Black-Scholes model is a special case assuming that all the jump sizes are zero: $\alpha_1=\cdots=\alpha_k=0$. Then $\phi_1=\cdots=\phi_k=0$ and formula (4.5) becomes a Black-Scholes hedging.

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The use of Lévy processes in the classification of the exponential families

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Abstract

This lecture explains how Bayesian theory leads to the search of new exponential families, how Lévy processes and random walks are used in some proofs of existence and comments on the mysteries of the Zolotarev formula.

I Natural exponential families

Given a finite dimensional real linear space E with dual E^*, denote by $\mathcal{M}(E)$ the set of positive measures μ on E not concentrated on an affine hyperplane and such that the set of $\theta \in E^*$ with

$$L_\mu(\theta) = \int_E e^{\langle \theta, x \rangle} \mu(dx) < \infty$$

has a non empty interior $\Theta(\mu)$. Denote $k_\mu = \log L_\mu$. The natural exponential family (NEF) $F = F(\mu)$ generated by $\mu \in \mathcal{M}(E)$ is the set of the probabilities $P(\theta, \mu)(dx) = \exp(\langle \theta, x \rangle - k_\mu(\theta))\mu(dx)$ for $\theta \in \Theta(\mu)$. It is easily seen that

$$\Theta(\mu) \to E: \quad \theta \mapsto k'_\mu(\theta) = \int_E xP(\theta, \mu)(dx)$$

is one-to-one. Its image M_F is called the domain of the means of F. We denote by $\psi_\mu : M_F \to \Theta(\mu)$ the inverse of k'_μ and we write $P(m, F) = P(\psi_\mu(m), \mu)$. The map $m \mapsto P(m, F)$ on M_F is the parametrization of F by the mean. The covariance of $P(m, F)$ is denoted by $V_F(m)$ and the map $m \mapsto V_F(m)$ on M_F is the variance function of F. This is a nice exercise to check that V_F characterizes F, like a Fourier transform characterizes a measure.

We shall also need the concept of the *Jorgensen set* $\Lambda(\mu)$ of a measure $\mu \in \mathcal{M}(E)$: this is the set of $t > 0$ such that there exists $\mu_t \in \mathcal{M}(E)$ with the two properties

1. $\Theta(\mu_t) = \Theta(\mu)$
2. $L_{\mu_t} = (L_\mu)^t$.

In other terms, $\Lambda(\mu)$ is the set of acceptable t such that the convolution power $\mu_t = \mu^{*t}$ does exist. Clearly $\overline{\Lambda}(\mu) = \Lambda(\mu) \cup \{0\}$ is a closed additive semi group containing the set $\mathbf{N}$ of non negative integers. If μ is a probability and $\Lambda(\mu) = (0, \infty)$ then μ is infinitely divisible and a Lévy process is associated to it. This Jorgensen set can be quite complicated, even for something as simple as distribution of the sum of two Bernoulli and negative binomial independent random variables.

II The simplest variance functions

For $E = \mathbb{R}$ the simplest variance functions have been investigated by Morris (1982) who points out that the only variance functions which are the restriction to an interval of some quadratic polynomial belong to one of the six following types (up to an affine transformation and up to power of convolution):

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1. The normal type: $M_F = \mathbb{R}$ and $V_F(m) = 1 > 0$.
2. The Poisson type: $M_F = (0, \infty)$ and $V_F(m) = m$.
3. The Bernoulli type: $M_F = (0, 1)$ and $V_F = m - m^2$
4. The geometric type: $M_F = (0, \infty)$ and $V_F = m + m^2$
5. The exponential type $M_F = (0, \infty)$ and $V_F = m^2$
6. The hyperbolic type $M_F = (0, \infty)$ and $V_F = 1 + m^2$.

All these types are infinitely divisible, except the Bernoulli one, whose Jorgensen set is $\{1, 2, \ldots\}$.

In higher dimensions the simplest variance functions are still the quadratic ones. To be specific, let us say that a function Q on E and valued in some linear space is homogeneous quadratic if $B_Q(x, y) = Q(x + y) - Q(x) - Q(y)$ is bilinear, and let us say that a function P is quadratic if $P = Q + L + C$, where Q is homogeneous quadratic, L is linear and C is constant. We say that F is of quadratic type if its variance function is the restriction to M_F of some quadratic function P (valued in the space of symmetric bilinear forms on E^*) It is of simple quadratic type if furthermore the Q part of P is such that $B_Q(x, y)(\alpha, \beta) = c\langle\alpha, x\rangle\langle\beta, y\rangle$ for some real constant c (we write $B_Q(x, y) = cx \otimes y$ in this case). Muriel Casalis (1996) has made a classification of these simple quadratic NEF nad has shown that there are $2d + 4$ types of them for $d = \dim E$. Up to affinities and powers of convolution, and for $E = \mathbb{R}^d$ their variance functions are

1. The $d + 1$ Poisson normal types: for $k = 0, 1, \ldots, d$
$$M_F = (0, \infty)^k \times \mathbb{R}^{d-k},\ V_F(m) = \mathrm{diag}(m_1, \ldots, m_k, 1, \ldots, 1).$$
2. The $d + 1$ negative multinomial types: for $k = 0, 1, \ldots, d$
$$M_F = (0, \infty)^{k+1} \times \mathbb{R}^{d-k-1},\ V_F(m) = m \otimes m + \mathrm{diag}(m_1, \ldots, m_k, 0, m_{k+1}, \ldots, m_{k+1}).$$
3. The Bernoulli type:
$$M_F = \{m \in \mathbb{R}^d; m_j > 0\ \forall j,\ \sum_{j=1}^{d} m_j < 1\},\ V_F(m) = -m \otimes m + \mathrm{diag}(m_1, \ldots, m_d).$$
4. The hyperbolic type:
$$M_F = (0, \infty)^{d-1} \times \mathbb{R},\ V_F(m) = m \otimes m + \mathrm{diag}(m_1, \ldots, m_{d-1}, 1 + \sum_{j=1}^{d-1} m_j).$$

The only other known types of quadratic variance functions are the Wishart distributions on the symmetric cones Ω associated to each of the five types of Euclidean simple Jordan algebras V. Casalis (1991) shows that they are the only homogeneous quadratic variance fonctions. If P is the quadratic map of V and if t is in the Gyndikin set of V (which is also the Jorgensen set of the Wishart distribution) they have the form
$$M_F = \Omega,\ V_F(m) = P(m)/t.$$

III Bayesian theory needs new exponential families

General exponential families: Given a measured space $(\Lambda, \mathcal{A}, \nu)$, a finite dimensional real space E, and a map u from Λ to E such that the image $\mu = u_*\nu$ is in $\mathcal{M}(E)$, the general exponential family (GEF) generated by the pair (ν, u) is the set $F = F(\nu, u)$ of probabilities on Λ defined for $\theta \in \Theta(\mu)$ by
$$P_\theta(d\lambda) = P(\theta, \nu, u)(d\lambda) = e^{\langle\theta, u(\lambda)\rangle - k_\mu(\theta)}\nu(d\lambda).$$

Example: If $\Lambda = (0,1)$, $\nu(d\lambda) = \frac{d\lambda}{\lambda(1-\lambda)}$, $E = \mathbb{R}^2$ and $u(\lambda) = (\log\lambda, \log(1-\lambda)$, then $\Theta(\mu) = (0,\infty)^2$ and

$$P_\theta(d\lambda) = \frac{1}{B(\theta_1,\theta_2)}\lambda^{\theta_1-1}(1-\lambda)^{\theta_2-1}d\lambda.$$

Thus F is the familiar family of beta distributions.

GEF as conjugate families: Consider now a GEF $F(\nu,u)$, or rather a part of it: $F = (P_\theta)_{\theta\in D}$ on $(\Lambda,\mathcal{A})$ where $D \subset \Theta(\mu)$, and D is either closed or open. Consider a Markov kernel K_λ from Λ to some measurable space $(X,\mathcal{B})$. Thus $\eta_\theta(d\lambda,dx) = P_\theta(d\lambda)K_\lambda(dx)$ is a probability on $\Lambda\times X$ which can be desintegrated with respect to x rather that λ :

$$\eta_\theta(d\lambda,dx) = \gamma_\theta(dx)J_{\theta,x}(d\lambda).$$

We shall say that F is conjugate with respect to the kernel $(K_\lambda)_{\lambda\in\Lambda}$ if for all θ in D then $J_{\theta,x}$ is in F for γ_θ almost all x. (For fixed θ, in Bayesian theory P_θ is the prior probability on the parameter λ and X is the sample space. Observing x provides the posterior probability $J_{\theta,x}(d\lambda)$. Statisticians find desirable to deal with not only one prior probability, but with a whole family F of them such that all the corresponding posterior probabilities still belong to F).

The basic convex set $\mathcal{H}(S,G)$: Recall that the previous GEF $F(\nu,u)$ was associated to the linear space E where u was taking its values. We select in E^* a closed additive semi group G and a closed subset S of E. We consider the set $\mathcal{H}(S,G)$ of measures q on G such that for all $v\in S$ one has

$$\int_G e^{\langle\theta,v\rangle}q(d\theta) = 1.$$

This is obviously a convex set which is closed for convolution. It is not difficult to see that it is reduced to δ_0 in many circumstances, in particular when $S\cap \mathrm{intconv}(S)$ is not empty.

Recall that the previous F was not the whole GEF $F(\nu,u)$ but a part of it defined by a closed or open set $D\subset\Theta(\mu)$. We shall consider the above $\mathcal{H}(S,G)$ for S equal to the support of μ and for $G = G(D) = \{\theta\in E^*; \theta + D\subset D\}$. We leave as an exercise to show that $G(D)$ is indeed a closed additive semigroup. To simplify the notations, we make this choice for S and G in the sequel.

Which kernels make F conjugate? We have now the following statement:
Theorem 1: With the notations above, F is conjugate with respect to $(K_\lambda)_{\lambda\in\Lambda}$ if and only if there exists a measure Q on $(X,\mathcal{B})$ and a map $h: X\to G$ with $h_*Q = q$ in $\mathcal{H}(S,G)$ such that

$$K_\lambda(dx) = e^{\langle h(x),u(\lambda)\rangle}Q(dx).$$

Note that $(K_\lambda)_{\lambda\in\Lambda}$ is itself a subset of an exponential family. A surprising feature is the fact that the acceptable kernels do not depend much on μ except by the support S of μ itself. Since the statement of the theorem can look strange, let us give the proof of $\Leftarrow$. Without loss of generality we may assume that $\Lambda = S\subset E$, that $\mu=\nu$, that $X = E^*$, that $Q = q$ and that u and h are the identity maps. Thus

$$K_\theta(dv) = e^{\langle\theta,v\rangle}q(d\theta)$$

is a probability on $G\subset E^*$ for all $v\in S$. Now let us choose an arbitrary $\theta_0\in D$ and let us take $P(\theta_0,\mu)$ as the a priori distribution on S. The product distribution on $S\times E^*$ is

$$\eta_{\theta_0}(dv,d\theta) = e^{\langle\theta,v\rangle+\langle\theta_0,v\rangle-k_\mu(\theta_0)}q(d\theta)\mu(dv).$$

Since $\theta\in G$ the a posteriori distribution after conditioning by θ is

$$e^{\langle\theta+\theta_0,v\rangle-k_\mu(\theta+\theta_0)}\mu(dv).$$

But the definition of $G = G(D)$ implies that $\theta+\theta_0\in D$. Therefore F is conjugate with respect to the kernel $(K_\theta)_{\theta\in S}$.

Example (continued): Let us work on the example of the beginning of the section, taking $D = \Theta(\mu)$ for simplication. Then $E^* = \mathbb{R}^2$, $G = G(D) = [0,\infty)^2$, the set S is the curve parametrized by $\lambda \in (0,1) \mapsto (\log\lambda, \log(1-\lambda) \in \mathbb{R}^2$, and $\mathcal{H}(S,G)$ is the set of measures q on $[0,\infty)^2$ such that for all $\lambda \in (0,1)$ one has

$$\int_0^\infty \int_0^\infty \lambda^{h_1}(1-\lambda)^{h_2} q(dh_1, dh_2) = 1.$$

Two examples of $q \in \mathcal{H}(S,G)$ are (N is a fixed integer, $p > 0$ is a fixed number):

$$q(dh_1, dh_2) = \sum_{k=0}^{N} C_N^k \delta_k(dh_1)\delta_{N-k}(dh_2), \tag{3.1}$$

$$q(dh_1, dh_2) = \sum_{k=0}^{\infty} \frac{p(p+1)\dots(p+k-1)}{k!} \delta_p(dh_1)\delta_k(dh_2). \tag{3.2}$$

This example calls for several remarks:

1. The same GEF (here the beta distributions) can serve as a conjugate family for several models (here for (3.1) the binomial model $\{B(N,\lambda); 0 < \lambda < 1\}$ and for (3.2) the negative binomial model $\{NB(p,\lambda); 0 < \lambda < 1\}$).

2. These exponential families are specially simple.

3. The measures q appearing in (3.1) and (3.2) are *concentrated on linear affine subspaces* ($h_1 + h_2 = N$ for the binomial case; $h_1 = p$ for the negative binomial case).

As we shall see in the next theorem, there is a link between 2) and 3).

The affine case:

We now assume that S is an analytic manifold contained in the linear space E with dimension $d < \dim E$. We assume a technical condition of local non degeneracy that we do not state here, which prevents for instance S from containing linear affine segments. The curve $\lambda \in (0,1) \mapsto (\log\lambda, \log(1-\lambda)) \in \mathbb{R}^2$, is an example of such S with $d = 1$. We also decompose E^* into a direct sum $E_0^* \oplus E_1^*$ such that $\dim E_1^* = d$ the dimension of S. This induces a natural decomposition of $E = E_0 \oplus E_1$ with $\dim E_1 = d$. Finally we denote by $u_0(\lambda)$ and $u_1(\lambda)$ the coordinates of $\lambda \in \Lambda \mapsto u(\lambda) \in S \subset E_0 \oplus E_1$.

Theorem 2: We keep the notations and hypothesis as above. Suppose that $q \in \mathcal{H}(S,G)$ is concentrated on the affine subpace $e_0 + E_1^* \neq E_1^*$. Then q is unique. Denote by $q_1 = q * \delta_{-e_0}$. A corresponding kernel $(K_\lambda)_{\lambda\in\Lambda}$ for which F is conjugated is for $X = E_1^*$

$$K_\lambda(dh) = e^{\langle h, u_1(\lambda)\rangle + \langle e_0, u_0(\lambda)\rangle} q_1(dh).$$

Furthermore, the cumulant transform of q_1 statisfies for all $\lambda \in \Lambda$ the equality

$$k_{q_1}(u_1(\lambda)) = -\langle e_0, u_0(\lambda)\rangle \tag{3.3}$$

Up to the proof of uniqueness, the proof is not difficult. We do not state the converse of this theorem (see Bar-Lev *et al* (1994), Th. 5.1). The present Theorem 2 is already surprising. We are given an analytic manifold $S \in E$ and an affine subspace $e_0 + E_1^*$ of the proper dimension. This may or may not generate a measure q on the affine space, but the remarkable thing is that q is unique and in principle computable by the equation (3.3). Therefore the problem which is in front to us is the following: when does (3.3) define a measure q_1? This problem has not been much studied beyond the cases $d = 1$ and $\dim E \leq 3$ (thus S is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$) or the case $\dim E - d = 1$ (that is S is part of a convex surface).

IV The case where S is a curve

In this section we specialize Theorem 2 to the case $1 = \dim S = \dim E_1^*$. Denote by e_1 a basis of E_1^* and write $\theta_0(\lambda) = \langle e_0, u_0(\lambda)\rangle$ and $\theta_1(\lambda) = \langle e_1, u_0(\lambda)\rangle$. Since q_1 is concentrated on the one dimensional space Thus E_1^* let us denote by q_0 the corresponding distribution on $\mathbb{R}$. Thus (3.3) becomes

$$k_{q_0}(\theta_1(\lambda)) = -\theta_0(\lambda). \tag{4.1}$$

We have now the following theorem, which enables us to write explicitely the variance function of the NEF $F(q_0)$:

Theorem 3: With the notations above, the map $\lambda \mapsto m(\lambda) = -\theta_0'(\lambda)/\theta_1'(\lambda)$ is analytic on $\tilde{\Lambda} = \{\lambda \in \Lambda; \theta_1 \in \Theta(q_0).\}$. Denoting $\tilde{M} = m(\tilde{\Lambda})$, then the variance function of $F(q_0)$ is defined on $\tilde{M}$ by

$$V_{F(q_0)}(m(\lambda)) = k''_{\theta_0}(\theta_1(\lambda)) = \frac{1}{\theta_1'(\lambda)^3}\begin{vmatrix} \theta_0'(\lambda) & \theta_0''(\lambda) \\ \theta_1'(\lambda) & \theta_1''(\lambda)\end{vmatrix}. \tag{4.2}$$

We now specialize furthermore to the case where there exists a function T on Λ such that $T(\lambda)u'(\lambda)$ is a polynomial with degree $< \dim E.$: this is indeed the case in the beta example with $T(\lambda) = \lambda(1-\lambda)$. There are many other examples which can be found in Letac (1992) and Barlev *et al* (1994). We therefore write $T(\lambda)\theta' i(\lambda) = P_i(\lambda)$ thus $m(\lambda) = -P_0(\lambda)/P_1(\lambda)$ and the variance fonction becomes

$$\frac{T(\lambda)}{P_1(\lambda)^3}\begin{vmatrix} P_0(\lambda) & P_0'(\lambda) \\ P_1(\lambda) & P_1'(\lambda)\end{vmatrix}. \tag{4.3}$$

In order to compute $V(m)$ explicitely with respect to m we have to solve the equation $P_0(\lambda) + mP_1(\lambda) = 0$ with respect to λ and to carry it in (4.3). Let us carry this program for $\dim E = 2$ and $\deg T \leq 3$. Thus $P_0(\lambda) = a_0 - b_0\lambda$ and $P_1(\lambda) = a_1 - b_1\lambda$. Thus λ is a Moebius function of m. Thus we obtain from (4.3) that the variance function

$$V(m) = \frac{(b_0 + b_1 m)^3}{(a_0 b_1 - a_1 b_0)^2} T\Big(\frac{a_0 + a_1 m}{b_0 + b_1 m}\Big) \tag{4.4}$$

is a polynomial with degree ≤ 3 on the interval $\tilde{M}$, thus extending the Morris families described in section 2. This is sometimes called the Morris-Mora class. Their classification is in Letac-Mora (1990).

The case where $\dim E = 3$ is even more interesting. This case does occur in concrete cases. For instance if we take as the conjugate family F the family of generalized inverse Gaussian distributions on $\Lambda = (0, \infty)$

$$C\lambda^{b-1}e^{-a\lambda - c\lambda^{-1}}d\lambda$$

where a, b, c are positive parameters then $u'(\lambda) = (1, \lambda^{-1}, \lambda^{-2})$ and we can take $T(\lambda) = \lambda^2$. We have now the following result:

Theorem 4: If $\dim E = 3$ and $\deg T \leq 4$ then

$$V_{F(q_0)(m)} = P(m)\Delta(m) + Q(m)\sqrt{\Delta(m)} \tag{4.5}$$

where the polynomials P, Q, Δ have degrees not greater than 1,2,2.

We have given a proof of this in Bar-Lev *et al.* (1994), Th. 5.3, but I do not really understand the magic role played in this proof by the quadratic extension by $\sqrt{\Delta(m)}$ of the field of rational functions of m. For $\deg P \leq 0$ and $\deg Q \leq 1$, a complete classification is given in Letac (1992).

V The case where S is a convex surface.

We come back to the equation (3.3) when S has dimension $\dim E - 1$. In this case S is essentially the graph of the cumulant function $u \mapsto k_{q_1}(u)$ defined on a part of $\Theta(q_1) \subset E_1$ and valued in

$\mathbb{R}$. I say "essentially", since $E_1 \times \mathbb{R}$ is linearly isomorphic to E and S is obtained from the above graph by a linear affine transformation. We have here a very powerful way to build new measures: we take a measure q_1 on $E_1^* \sim \mathbb{R}^d$ such that its Laplace transform exists, we draw the graph of the convex function $u \mapsto k_{q_1}(u)$ in $E_1 \times \mathbb{R}$. We perform an affine transformation on this graph, obtaining a convex surface S_1 in $E_1 \times \mathbb{R}$. Its projection D_1 on E_1 parallel to $\mathbb{R}$ is a convex set. In general, there is a natural convex function on D_1 whose graph is inbedded in S_1. If we are lucky, this natural convex function is the cumulant function of a new measure q_2, which depend on q_1 and the affine transformation of $E_1 \times \mathbb{R}$. One can prove that the variance function V_2 of the NEF generated by q_2 is related to the variance function V_1 by a formula of the form

$$V_2(m) = \frac{1}{\langle c, m\rangle + d}(h'(m))^{-1}V_1(h(m))(h'(m)^*)^{-1} \tag{5.1}$$

where $h(m) = \frac{a(m)+b}{\langle c,m\rangle+d}$ is a Moebius transform of $E_1 \times \mathbb{R}$ easily related to the affine transformation of $E_1 \times \mathbb{R}$.

Let us illustrate this phenomena for $E_1 = \mathbb{R}$, q_1 a normal distribution and q_2 a stable distribution with parameter $1/2$. We draw the parabola $u \mapsto u^2/2$, which is the cumulant function of a normal distribution. We rotate it by 90 degrees, obtaining the graphs of the two functions on $D_1 = (-\infty, 0)$ defined by $u \mapsto \sqrt{-2u}$ and $u \mapsto -\sqrt{-2u}$. We do not keep the first one, which is concave. The second one is the cumulant transform of q_2, a stable distribution with parameter $1/2$. In the formula (5.1) this rotation corresponds to $h(m) = 1/m$ and

$$V_2(m) = m^3 V_1\left(\tfrac{1}{m}\right). \tag{5.2}$$

If one apply this process to the members of Morris class, one gets exactly the Morris Mora class, since the action of the affinities in $\mathbb{R}^2$ is exactly mirrored on variance functions by formula (4.3). Replacing the Morris-Mora class in $\mathbb{R}$ by the simple quadratic NEF in $\mathbb{R}^d$ of Muriel Casalis described in section 2 leads to the Hassaïri class of cubic variance functions in $\mathbb{R}^d$ which have been completely classified in Hassaïri's thesis (1994). See also Hassaïri (1992) for a sample of his results.

VI Lévy processes, reciprocity and the Zolotarev formula.

As we have seen in the previous section, the Bayesian theory leads us to consider a large number of potential exponential families through their variance functions. However, proving the existence of an NEF with a given variance function V defined on an open subset M of the linear space E can be a difficult problem. The steps are

1. Finding $\psi : M \to E^*$ such that $(V(m))^{-1} = \psi(m)$ (assuming the necessary condition on V that the bilinear map on E defined by $(u, v) \mapsto V(m)(V(m)u)(v)$ is symmetric in (u, v)).

2. Inverting the map $m \mapsto \psi(m)$ in order to get the differential of the cumulant function k'.

3. Computing the cumulant function k and $L = \exp k$.

4. Checking that L is the Laplace transform of some positive measure.

The hard parts are steps 2 and 4. The Lagrange formula is often helpful for step 2. But the best tool for step 4 is the finding of a probabilistic interpretation.

Let us formalize in a definition the link between two exponential families on $\mathbb{R}$ appearing in (5.2). Suppose that the NEF F_1 on $\mathbb{R}$ with domain of the means M_{F_1} is such that $\tilde{M}_{F_1} = M_{F_1} \cap (0, \infty)$ is not empty. Then the NEF F_2 on $\mathbb{R}$ is called the *reciprocal* NEF of F_1 if $V_2(m) = m^3 V_1(\frac{1}{m})$ for all $m \in \tilde{M}_{F_2}$.

Not all NEF have reciprocal: the NEF generated by a positive stable distribution whose parameter is in $(0, 1)$ has no reciprocal. But suppose that we want to prove the existence of a NEF with variance function $m^3 + m^2$. By translation this is equivalent to the existence of a NEF F_2 with variance function $m(m-1)^2$ which would be the reciprocal of F_1 with variance function $(1-m)^2$ with F_1 concentrated on $(-\infty, 1)$. The NEF F_1 exists, this is nothing but the NEF generated by the Lebesgue measure restricted to $(-\infty, 1)$. Actually we have the following result (Letac-Mora (1990)):

Theorem 5: Let $(X(t)_{t\geq 0}$ be a Lévy process with Lévy measure concentrated on the negative line. Let $T(x)$ be the hitting time of $x>0$. Then the exponential families F_1 and F_2 respectively generated by the distribution of $X(1)$ and the distribution of $T(1)$ (restricted to $(0,\infty)$) are reciprocal. Furthermore the distributions of $X(t)$ (restricted to the positive line) and T_x are related by the following Zolotarev's formula, which indicates the coincidence of two measures on $(0,\infty)^2$

$$xP(X(t)\in dx)dt = tP(T(x)\in dt)dx. \tag{6.1}$$

This magic formula (6.1) has actually been given by Zolotarev (1964). Borovkov (1964) and Dozzi and Vallois (1997) give other proofs. I have learned the above elegant formulation in Bertoin (1999). Let us insist on the fact that (6.1) is not an absolutely continuous measure on $(0,\infty)^2$: only the margins have densities. For instance, if $X(t)=at-bN(t)$, where $a>0$ and $b>0$ and $N(t)$ is a Poisson process with intensity λ then the measure (6.1) is concentrated on the lines $x=at-bn$ where $n\in\mathbf{N}$.

As an example, we apply the theorem to $X(t)=t-Y(t)$ where Y is the standard gamma process ($\mathbf{E}(e^{-sY(t)})=(1+s)^{-t}$). The variance for $X(1)$ being $(1-m)^2$, then the existence of the variance function $m(m-1)^2$ follows. A result similar to Theorem 5 can be obtained with the right continuous random walks in the integers (see Letac-Mora (1990)), providing a relatively explicit generating measure for F_2 (the Ressel Kendall distribution in our example). The same is true for the random walk case, where Lagrange replaces Zolotarev.

However, it is false to think that any reciprocal pair has a similar probabilistic interpretation. Actually, this conference on Lévy processes is an excellent place to attract attention of experts on the following intriguing problem:

Why do we always have a Zolotarev formula in case of reciprocity?

To be more specific, let us say that two measures μ_1 and μ_2 in $\mathcal{M}(\mathbb{R})$ are reciprocal if the sets

$$\tilde{\Theta}(\mu_i)=\{\theta\in\Theta(\mu_i);k_{\mu_i}(\theta)>0\}$$

are not empty and such that the map $\theta\mapsto -k_{\mu_i}(\theta)$ is a one to one map from $\tilde{\Theta}(\mu_i)$ onto $\tilde{\Theta}(\mu_{3-i})$ whose inverse is $\theta\mapsto -k_{\mu_{3-i}}(\theta)$. Needless to say, under these circumstances, the NEF's $F(\mu_1)$ and $F(\mu_2)$ are reciprocal. A tentative of clarification is offered by the following conjecture (which is even not quite correct, see example 3 below).

Conjecture: Let $\mu(dx)$ and $\nu(dt)$ in $\mathcal{M}(\mathbb{R})$ be reciprocal. Denote by $\lambda(dx)$ and $\eta(dt)$ the measures on $[0,\infty)$ of the form $\sum_{n=0}^{\infty}\delta_{an+b}$ or $\mathbf{1}_{[0,\infty)}(x)dx$ such that $\mathbf{1}_{[0,\infty)}(x)\mu(dx)$ and $\mathbf{1}_{[0,\infty)}(t)\mu(dt)$ are absolutely continuous with respect to $\lambda(dx)$ and $\eta(dt)$ respectively (assuming the existence of $\lambda(dx)$ and $\eta(dt)$). Then the following equality between measures on $\overline{\Lambda}(\nu)\times\overline{\Lambda}(\mu)$ holds:

$$x\mu_t(dx)\eta(dt)=t\nu_x(dt)\lambda(dx). \tag{6.2}$$

Let us give now examples of reciprocity where neither the conditions of Theorem 5 nor the conditions of its right continuous random walks analog are fullfiled.

Example 1: We take $\mu(dx)=\eta(dx)=\mathbf{1}_{(0,\infty)}dx$. Thus the Jorgensen set $\Lambda(\mu)$ is $(0,\infty)$, and we have $\mu_t(dx)=\frac{x^{t-1}}{\Gamma(t)}\eta(dx)$. The reciprocal measure of μ is $\nu(dt)=\frac{1}{\Gamma(t+1)}\lambda(dt)$ where $\lambda(dt)=\sum_{n=0}^{\infty}\delta_n(dt)$. The Jorgensen set $\Lambda(\nu)$ is $(0,\infty)$, and we have $\nu_x(dt)=\frac{x^t}{\Gamma(t+1)}\lambda(dt)$. Clearly (6.2) is satisfied, and the reciprocity is the reciprocity of the Poisson NEF and the exponential distributions NEF with respective variance fonctions $V_\nu(m)=m$ and $V_\mu(m)=m^2$. We are not in the conditions of application of Theorem 5.

Example 2: We take

$$\mu(dx)=\frac{1}{2\cosh\frac{\pi x}{2}}dx.$$

Thus $\eta(dx) = \mathbf{1}_{(0,\infty)}dx$. One can prove (see Morris (1982)) that $\Lambda(\mu) = (0,\infty)$ and that

$$\mu_t(dx) = \frac{2^{t-2}}{\Gamma(t)} \left| \frac{\Gamma((t+ix)/2)}{\Gamma(t/2} \right|^2 .$$

Here $V_\mu(m) = m^2 + 1$. One can prove (see Letac-Mora (1990)) that the reciprocal measure ν of μ is concentrated on nonegative integers, its Jorgensen set is $(0,\infty)$ and the measures $\nu_x(dt)$ are explicitely given by

$$\nu_x(dt) = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!}\delta_n(dt)$$

where the $p_n(x)$ are the following polynomials:

$$p_{2n}(x) = \prod_{k=0}^{n-1}(x^2+4k^2),\ p_{2n+1}(x) = x\prod_{k=0}^{n-1}(x^2+(2k+1)^2).$$

Here $V_\mu(m) = m(m^2+1)$.

However, taking $\lambda(dt) = \sum_{n=0}^{\infty}\delta_n(dt)$ one checks again that (6.2) is satisfied, without any probabilistic explanation.

Example 3: We take $\mu(dx) = \sum_{n=0}^{\infty}\frac{1}{n!}\delta_{n-1}(dx)$. Thus $M_\mu = (-1,\infty)$ and $V_\mu(m) = m+1$: this is a shifted Poisson family. The Jorgensen set is $(0,\infty)$ and

$$\mu_t(dx) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\delta_{n-t}(dx).$$

The reciprocal family has therefore a variance function equal to $V_\nu(m) = m^2(1+m)$ which is a Ressel Kendall family.

One can prove (see Letac-Mora (1990)) that the reciprocal measure ν of μ has a Jorgensen set equal to $(0,\infty)$ and that the measures $\nu_x(dt)$ are explicitely given by

$$\nu_x(dt) = \frac{xt^{x+t-1}}{\Gamma(x+t+1)}\eta(dt)$$

where $\eta(dt) = \mathbf{1}_{(0,\infty)}(t)dt$. In this case the conjecture is not quite satisfied, since the reference measure $\lambda(dx)$ is the restriction to the positive line of

$$\sum_{a\in\mathbf{N}-t}\delta_a(dx),$$

which depends a little bit of t. Up to this (6.2) is satisfied.

Example 4: We take μ as the distribution of the difference of two independent Poisson random variables with means 1/2. Thus $M_\mu = (-1,\infty)\mathbb{R}$ and $V_\mu(m) = (m^2+1)^{1/2}$, $\lambda(dx) = \sum_{n=0}^{\infty}\delta_n(dx)$, μ is infinitely divisible and since we have

$$e^{t\cosh\theta - t} = \sum_{n\in\mathbf{Z}}\mu_t(n)e^{n\theta},$$

thus $\mu_t(n) = e^{-t}I_{|n|}(t)$ where

$$I_x(t) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+x+1)}\left(\frac{t}{2}\right)^{2n+x}.$$

The reciprocal family does exist and has variance function $V_\nu(m) = m^2(m^2+1)^{1/2}$. It is generated by the reciprocal measure $\nu(dt) = e^{-t}\frac{1}{t}I_1(t)\eta(dt)$ where $\eta(dt) = \mathbf{1}_{(0,\infty}(t)dt$. See Feller (1966) pages 414, formula (3.8) and page 427, example (d). This is also infinitely divisible and (6.2) holds.

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Stable Processes and Metric Entropy

Werner Linde (Jena)

Let $X = (X(t))_{t\in T}$ be a stochastic process over an index set $T \neq \emptyset$ and defined on a probability space $(\Omega, \mathbb{P})$. Suppose that for some $r > 0$ and all $t \in T$ we have

$$\mathbb{E}\,|X(t)|^r < \infty\,.$$

Then the set of paths

$$s(X) := \{X(t) : t \in T\} \tag{1}$$

may be regarded as subset of $L_r(\Omega, \mathbb{P})$. A classical problem is to relate geometric properties of $s(X)$ (as a subset of L_r) with probabilistic properties of the process X. Yet, since

$$\sup_{t\in T} |X(t)| = \sup \left\{ \Big| \sum_{j=1}^{n} \lambda_j X(t_j) \Big| : \sum_{j=1}^{n} |\lambda_j| \le 1\,,\ t_j \in T \right\}, \tag{2}$$

in many cases the symmetric convex hull

$$c(X) := \Big\{ \sum_{j=1}^{n} \lambda_j X(t_j) : \sum_{j=1}^{n} |\lambda_j| \le 1\,,\ t_j \in T \Big\} \tag{3}$$

should be more adequate than $s(X)$ for describing probabilistic properties of X.

First results due to R. M. Dudley [6] and V. N. Sudakov [18] relate in the Gaussian case the size of $s(X) \subseteq L_2(\Omega, \mathbb{P})$ with the existence of bounded or continuous versions of X. For the description of the size of $s(X)$ we need the following definition: Given a metric space (E, d) and a subset $B \subseteq E$ the n–th (dyadic) entropy number of B is defined by

$$e_n(B) := \inf \Big\{ \varepsilon > 0 : \exists\, C \subseteq E,\ |C| \le 2^{n-1} \text{ s.t. } \sup_{x\in B} d(x, C) < \varepsilon \Big\}\,.$$

Recall that $e_n(B) \to 0$ iff B is precompact in E.

In this language the above mentioned results of R. M. Dudley and V. N. Sudakov may be formulated as follow:

Proposition 1. *Let X be a centered Gaussian process over an index set T and regard $s(X)$ as subset of $L_2(\Omega, \mathbb{P})$.*

(a) *If*

$$\sum_{n=1}^{\infty} n^{-1/2} e_n(s(X)) < \infty\,,$$

then there exists an a.s. bounded version of X.

(b) *Whenever X has an a.s. bounded version, then*

$$\sup_{n\ge 1} n^{1/2} e_n(s(X)) < \infty\,.$$

Remark: In view of (2) it is a bit surprising that both conditions are valid for $s(X)$ (and not only for $c(X)$). This phenomenon is tightly related with the problem of the size of convex hulls of sets in Hilbert spaces (cf. [4], [10]).

The assertions of Proposition 1 suggest that a faster decay of $e_n(s(X))$ (better $e_n(c(X))$) implies more regularity for X. Indeed, in the Gaussian case this is so (cf. [7] and [9]):

Proposition 2. *Let X be a centered Gaussian process over T and let $\delta \in (0,2)$, $\beta \in \mathbb{R}$ be given. Then the following are equivalent:*

(1) *There is a $c > 0$ such that*
$$e_n(c(X)) \leq c \cdot n^{-1/\delta} (\log n)^\beta \ . \tag{4}$$

(2) *For some $c' > 0$ it follows that*
$$-\log \mathbb{P}\Big(\sup_{t\in T} |X(t)| < \varepsilon\Big) \leq c' \cdot \varepsilon^{-1/(1/\delta - 1/2)} \ (\log(1/\varepsilon))^{\beta/(1/\delta-1/2)} \ . \tag{5}$$

Remark: The equivalence of (4) and (5) remains true for two–sided estimates, i.e. we have
$$e_n(c(X)) \approx n^{-1/\delta} (\log n)^\beta$$
iff
$$-\log \mathbb{P}\Big(\sup_{t\in T} |X(t)| < \varepsilon\Big) \approx \varepsilon^{-1/(1/\delta - 1/2)} \ (\log(1/\varepsilon))^{\beta(1/\delta-1/2)} \ .$$

Yet the following interesting problem remains open:

Problem: Does
$$e_n(c(X)) \geq c \cdot n^{-1/\delta} (\log n)^\beta$$
always imply
$$-\log \mathbb{P}\Big(\sup_{t\in T} |X(t)| < \varepsilon\Big) \geq c' \cdot \varepsilon^{-1/(1/\delta - 1/2)} \ (\log(1/\varepsilon))^{\beta(1/\delta-1/2)} \ ?$$
What about the converse implication ?

Let us answer a question tightly related to this problem. Suppose there is a measure μ on T and a number $p \geq 1$ such that
$$\mathbb{P}\Big(\int_T |X(t)|^p \ d\mu(t) < \infty\Big) = 1 \ . \tag{6}$$
Then we define the set
$$c_p(X) := \Big\{\int_T X(t)g(t)\, d\mu(t) : \int_T |g(t)|^{p'} \ d\mu(t) \leq 1\Big\} \tag{7}$$
where p' is as usual given by $1/p + 1/p' = 1$.

With this notation the following is true.

Proposition 3. *Let X be centered Gaussian with* (6) *for some $p \in [1,2]$. Then, if $0 < \delta < 2$ and $\beta \in \mathbb{R}$,*
$$e_n(c_p(X)) \geq c \cdot n^{-1/\delta} (\log n)^\beta$$
implies
$$-\log \mathbb{P}\Big(\int_T |X(t)|^p \ d\mu(t) < \varepsilon^p\Big) \geq c' \cdot \varepsilon^{-1/(1/\delta - 1/2)} \ (\log(1/\varepsilon))^{\beta(1/\delta-1/2)} \ .$$

We turn now to the symmetric α–stable ($S\alpha S$) case with $0 < \alpha < 2$. If $0 < r < \alpha$ and X is $S\alpha S$ over T, then we have $s(X) \subseteq c(X) \subseteq L_r(\Omega, \mathbb{P})$. The following result similar to Proposition 1 may be found in [8] or [17] :

Proposition 4. *Let X be $S\alpha S$ and regard $s(X)$ as subset of L_r for some $r < \alpha$.*

(a) *If $1 < \alpha < 2$ and*

$$\sum_{n=1}^{\infty} 2^{n/\alpha} e_n(s(X)) < \infty , \tag{8}$$

then X has a version with a.s. bounded paths.

(b) *Conversely, if a version with a.s. bounded paths exists, then*

$$e_n(s(X)) \le c \cdot n^{-1/\alpha'} \tag{9}$$

for $1 < \alpha < 2$ while for $\alpha = 1$

$$e_n(s(X)) \le c \cdot (\log n)^{-1} . \tag{10}$$

Remark: We do not know of any sufficient condition for the boundedness of X in terms of $e_n(c(X))$. It is very likely that such a condition would narrow the huge gap between (8) and (9) or (10), respectively.

For the small ball behaviour of $S\alpha S$–processes the following analogue of Proposition 3 is true. Note that this is the first general small ball result for stable non–Gaussian processes.

Proposition 5. *Let X be an $S\alpha S$–process over T and suppose that there is a measure μ on T such that $c_p(X) \subset L_r(\Omega, \mathbb{P})$ for a certain $p \in [0, 2]$. Here $c_p(X)$ is defined by (7). Then, if $\delta > 0$ with $1/\delta > 1 - 1/\alpha$ and $\beta \in \mathbb{R}$, the estimate*

$$e_n(c_p(X)) \ge c \cdot n^{-1/\delta} (\log n)^{\beta}$$

implies

$$-\log \mathbb{P}\Big(\int_T |X(t)|^p \, d\mu(t) < \varepsilon^p \Big) \ge c' \cdot \varepsilon^{-1/(1/\delta + 1/\alpha - 1)} \, (\log(1/\varepsilon))^{\beta/(1/\delta+1/\alpha-1)} . \tag{11}$$

Remarks:

1. The proof of Proposition 5 follows the ideas developed in [12] by applying Proposition 3.

2. In view of Proposition 4 for $1 < \alpha < 2$ the condition $1/\delta > 1 - 1/\alpha$ is natural. Moreover, for $0 < \alpha < 1$ estimate (11) fits together with the general lower estimate proved in [15].

Question: It is very likely that Proposition 5 remains true for all $p \ge 1$ and for the sup–norm as well (here w.r.t. $c(X)$). The answer to this conjecture depends heavily on the so–called duality problem for entropy numbers (cf. [19], [3] and [13]).

Examples:
1. Let Z_α be an α–Levy Motion on $[0, 1]$. By estimates for the entropy numbers of Volterra integral operators (cf. [11]) it follows that

$$e_n(c_p(Z_\alpha)) \approx n^{-1} .$$

Hence, if $1 \le p \le 2$, by Proposition 5 we obtain

$$-\log \mathbb{P}\Big(\int_0^1 |Z_\alpha(t)|^p \, dt < \varepsilon^p \Big) \ge c \cdot \varepsilon^{-1/(1+1/\alpha-1)} = c \cdot \varepsilon^{-\alpha} . \tag{12}$$

By monotonicity of the L_p-norms the lower order in (12) is $\varepsilon^{-\alpha}$ for all $p \ge 1$ as well as for the supremum. It is the correct one by the results of [14] and [5] .

2. Let us regard the Linear Fractional Stable Motion of the form

$$X_{H,\alpha}(t) := \int_{-\infty}^{\infty} \Big[(t-x)_+^{H-1/\alpha} - ((-x)_+)^{H-1/\alpha} \Big] \, dM(x) , \quad t \in [0, 1],$$

where M denotes an $S\alpha S$ random measure with Lebesque control measure and $0 < H < 1$, $H \neq 1/\alpha$. For $1 < \alpha < 2$ and $1/\alpha < H < 1$ it follows from [1] and [2] that

$$e_n(c_p(X_{H,\alpha})) \approx n^{-(H-1/\alpha+1)} .$$

Here the underlying measure on $[0,1]$ is the Lebesque measure. Of course, $1/\delta > 1 - 1/\alpha$ with $1/\delta := H - 1/\alpha + 1$. Hence, Proposition 5 applies and leads to

$$-\log \mathbb{P}\Big(\int_0^1 |X_{H,\alpha}(t)|^p \, dt < \varepsilon^p \Big) \geq c' \cdot \varepsilon^{-1/H} . \tag{13}$$

Again by monotonicity the lower order in (13) is true for all $p \geq 1$ as well as for $\sup_{t\in[0,1]} |X_{H,\alpha}(t)|$. In the latter case this was proved in [16] by different methods. It is open whether or not this is optimal.

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Probability distributions as solutions to fractional diffusion equations

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Abstract

The fundamental solutions (Green functions) for the Cauchy problem of the space-time fractional diffusion equation are investigated with respect to their scaling and similarity properties, starting from their composite Fourier-Laplace representation. By using the Mellin transform, a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane is presented, that allows us to obtain their computational form in the space-time domain and to analyse their probability interpretation.

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1 Introduction

We consider the Cauchy problem for the *space-time fractional* partial differential equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0,2]$ and skewness θ $(|\theta| \leq \min\{\alpha, 2-\alpha\})$, and the first-order time derivative with a Caputo derivative of order $\beta \in (0,2]$. The fundamental solutions (Green functions) for the Cauchy problem are investigated with respect to their scaling and similarity properties, starting from their combined Fourier-Laplace representation.

In the cases $\{0 < \alpha \leq 2\,, \beta = 1\}$ and $\{\alpha = 2\,, 0 < \beta \leq 1\}$ the fundamental solutions are known to be interpreted as a *spatial probability density functions evolving in time*, so we talk of *space-fractional diffusion* and *time-fractional diffusion*, respectively. Then, by using the Mellin transform, we provide a general representation of the Green functions in terms of Mellin-Barnes integrals in the complex plane, which allows us *to extend the probability interpretation to the ranges* $\{0 < \alpha \leq 2\,, 0 < \beta \leq 1\}$ *and* $\{1 < \beta \leq \alpha \leq 2\}$. Furthermore, from this representation it is possible to derive explicit formulae (convergent series and asymptotic expansions), which enable us to plot the spatial probability densities for different values of the relevant parameters α, θ, β .

2 The space-time fractional diffusion equation

By replacing in the standard diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{2.1}$$

where $u = u(x,t)$ is the (real) field variable, the second-order space derivative and the first-order time derivative by suitable *integro-differential* operators, which can be interpreted as a space and time derivative of fractional order, we obtain a sort of "generalized diffusion" equation. Such equation may be referred to as the *space-time fractional diffusion* equation when its fundamental solution (see below) can be interpreted as a probability density. We write

$$_tD_*^\beta\, u(x,t) \;=\; _xD_\theta^\alpha\, u(x,t)\,, \quad -\infty < x < +\infty\,, \quad t \geq 0\,, \tag{2.2}$$

where the α, θ, β are real parameters restricted as follows

$$0 < \alpha \leq 2\,, \quad |\theta| \leq \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \leq 2\,. \tag{2.3}$$

In (2.2) $_xD_\theta^\alpha$ is the *Riesz-Feller fractional derivative* (in space) of order α and skewness θ, and $_tD_*^\beta$ is the *Caputo fractional derivative* (in time) of order β. The definitions of these fractional derivatives are more easily understood if given in terms of Fourier transform and Laplace transform, respectively.

For the *Riesz-Feller fractional derivative* we have

$$\mathcal{F}\{\,_xD_\theta^\alpha\, f(x);\kappa\} = -\psi_\alpha^\theta(\kappa)\;\widehat{f}(\kappa)\,, \quad \psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,, \tag{2.4}$$

where $\kappa \in \mathbb{R}$ and $\widehat{f}(\kappa) = \mathcal{F}\{f(x);\kappa\} = \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x} f(x)\,dx$. In other words the symbol of the pseudo-differential operator[1] $_xD_\theta^\alpha$ is required to be the logarithm of the characteristic function of the generic *stable* (in the Lévy sense) probability density, according to the Feller parameterization [6], [7].

For $\alpha = 2$ (hence $\theta = 0$) we have $\widehat{_xD_0^2}(\kappa) = -\kappa^2 = (-i\kappa)^2$, so we recover the standard second derivative.

For $0 < \alpha < 2$ and $\theta = 0$ we have $\widehat{_xD_0^\alpha}(\kappa) = -|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$ so

$$_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\,. \tag{2.5}$$

In this case we call the LHS of (2.5) simply the *Riesz fractional derivative* of order α. For the explicit expressions in integral form of the general *Riesz-Feller fractional derivative* we refer the interested reader *e.g.* to [13], [15], [25], [34].

Let us now consider the *Caputo fractional derivative.* Following the original idea by Caputo [2], see also [3], [12], [32], a proper time fractional derivative of order $\beta \in (m-1, m]$ with $m \in \mathbb{N}$, useful for physical applications, may be defined in terms of the following rule for the Laplace transform:

$$\mathcal{L}\left\{_tD_*^\beta\, f(t); s\right\} = s^\beta\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\, f^{(k)}(0^+)\,, \quad m-1 < \beta \leq m\,, \tag{2.6}$$

where $s \in \mathbb{C}$ and $\widetilde{f}(s) = \mathcal{L}\{f(t);s\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt$. Then the *Caputo fractional derivative* of $f(t)$ turns out to be

$$_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\beta+1-m}}\,, & m-1 < \beta < m\,, \\[2ex] \dfrac{d^m}{dt^m} f(t)\,, & \beta = m\,, \end{cases} \qquad t \geq 0\,. \tag{2.7}$$

[1] Let us recall that a generic linear pseudo-differential operator A, acting with respect to the variable $x \in \mathbb{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, A\,[f(x)]\, dx = \widehat{A}(\kappa)\, \widehat{f}(\kappa)$, where $\widehat{A}(\kappa)$ is referred to as symbol of A, given as $\widehat{A}(\kappa) = \left(A\, \mathrm{e}^{-i\kappa x}\right)\, \mathrm{e}^{+i\kappa x}$.

In order to formulate and solve the Cauchy problems for (2.2) we have to select explicit initial conditions concerning $u(x,0^+)$ if $0<\beta\le 1$ and $u(x,0^+)$, $u_t(x,0^+)$ if $1<\beta\le 2$. If $\phi_1(x)$ and $\phi_2(x)$ denote two given real functions of $x\in\mathbb{R}$, the Cauchy problems consist in finding the solution of (2.2) subjected to the additional conditions:

$$u(x,0^+)=\phi_1(x)\,,\qquad x\in\mathbb{R}\,,\quad \text{if}\quad 0<\beta\le 1\,; \tag{2.8a}$$

$$\begin{cases} u(x,0^+)=\phi_1(x)\,,\\ u_t(x,0^+)=\phi_2(x)\,,\end{cases}\qquad x\in\mathbb{R}\,,\quad \text{if}\quad 1<\beta\le 2\,. \tag{2.8b}$$

3 Representations of the Green functions

The Cauchy problems can be conveniently treated by making use of the most common integral transforms, *i.e.* the Fourier transform (in space) and the Laplace transform (in time). The composite Fourier-Laplace transforms of the solutions of the two Cauchy problems turn out to be, by using (2.4) and (2.6) with $m=1,2$,

$$\widehat{\widetilde{u}}(\kappa,s)=\frac{s^{\beta-1}}{s^\beta+\psi_\alpha^\theta(\kappa)}\,\widehat{\phi_1}(\kappa)\,,\quad 0<\beta\le 1\,, \tag{3.1a}$$

$$\widehat{\widetilde{u}}(\kappa,s)=\frac{s^{\beta-1}}{s^\beta+\psi_\alpha^\theta(\kappa)}\,\widehat{\phi_1}(\kappa)+\frac{s^{\beta-2}}{s^\beta+\psi_\alpha^\theta(\kappa)}\,\widehat{\phi_2}(\kappa)\,,\quad 1<\beta\le 2\,. \tag{3.1b}$$

By fundamental solutions (or Green functions) of the above Cauchy problems we mean the (generalized) solutions corresponding to the initial conditions:

$$G^{\theta\,(1)}_{\alpha,\beta}(x,0^+)=\delta(x)\,,\quad 0<\beta\le 1\,; \tag{3.2a}$$

$$\begin{cases} G^{\theta\,(1)}_{\alpha,\beta}(x,0^+)=\delta(x)\,,\\[4pt] \dfrac{\partial}{\partial t}\,G^{\theta\,(1)}_{\alpha,\beta}(x,0^+)=0\,,\end{cases}\qquad \begin{cases} G^{\theta\,(2)}_{\alpha,\beta}(x,0^+)=0\,,\\[4pt] \dfrac{\partial}{\partial t}\,G^{\theta\,(2)}_{\alpha,\beta}(x,0^+)=\delta(x)\,,\end{cases}\qquad 1<\beta\le 2\,. \tag{3.2b}$$

We have denoted by $\delta(x)$ the delta-Dirac generalized function, whose (generalized) Fourier transform is known to be 1, and we have distinguished by the apices (1) and (2) the two types of Green functions. From Eqs (3.1a)-(3.1b) the composite Fourier-Laplace transforms of these Green functions turn out to be

$$\widehat{\widetilde{G^{\theta\,(j)}_{\alpha,\beta}}}(\kappa,s)=\frac{s^{\beta-j}}{s^\beta+\psi_\alpha^\theta(\kappa)}\,,\quad 0<\beta\le 2\,,\quad j=1,2\,. \tag{3.3}$$

Furthermore, by recalling the Fourier convolution property, we note that the Green functions allow us to represent the solutions of the above two Cauchy problems through the relevant integral formulas:

$$u(x,t)=\int_{-\infty}^{+\infty}G^{\theta\,(1)}_{\alpha,\beta}(\xi,t)\,\phi_1(x-\xi)\,d\xi\,,\quad 0<\beta\le 1\,; \tag{3.4a}$$

$$u(x,t)=\int_{-\infty}^{+\infty}\left[G^{\theta\,(1)}_{\alpha,\beta}(\xi,t)\phi_1(x-\xi)+G^{\theta\,(2)}_{\alpha,\beta}(\xi,t)\phi_2(x-\xi)\right]d\xi,\ 1<\beta\le 2. \tag{3.4b}$$

We recognize from (3.3) that the function $G^{\theta\,(2)}_{\alpha,\beta}(x,t)$ along with its Fourier-Laplace transform is well defined also for $0<\beta\le 1$ even if it loses its meaning of being a fundamental solution of (3.2), resulting

$$G^{\theta\,(2)}_{\alpha,\beta}(x,t)=\int_0^t G^{\theta\,(1)}_{\alpha,\beta}(x,\tau)\,d\tau\,,\quad 0<\beta\le 2\,. \tag{3.5}$$

By using the known scaling rules for the Fourier and Laplace transforms, and introducing the *similarity variable* $x/t^{\beta/\alpha}$, we infer from (3.3) (thus without inverting the two transforms) the *scaling properties* of the Green functions,

$$G_{\alpha,\beta}^{\theta\,(j)}(x,t) = t^{-\beta/\alpha+j-1}\, K_{\alpha,\beta}^{\theta\,(j)}\left(x/t^{\beta/\alpha}\right)\,, \quad j=1,2\,, \tag{3.6}$$

where the one-variable functions $K_{\alpha,\beta}^{\theta\,(j)}(x)$, obtained by setting $t=1$, are called the *reduced Green functions*. We also note the *symmetry relation*:

$$G_{\alpha,\beta}^{\theta\,(j)}(-x,t) = G_{\alpha,\beta}^{-\theta\,(j)}(x,t)\,, \quad j=1,2\,, \tag{3.7}$$

so for the determination of the Green functions we can restrict our attention to $x>0$. Extending the method illustrated in [9], [25], where only the Green function of type (1) was determined, we first invert the Laplace transforms in (3.3) getting

$$\widehat{G_{\alpha,\beta}^{\theta\,(j)}}(\kappa,t) = t^{j-1}\, E_{\beta,j}[-\psi_\alpha^\theta(\kappa)t^\beta],\ \widehat{K_{\alpha,\beta}^{\theta\,(j)}}(\kappa) = E_{\beta,j}[-\psi_\alpha^\theta(\kappa)],\ j=1,2, \tag{3.8}$$

where $E_{\beta,j}$ denotes the two-parameter Mittag-Leffler function[2]. We note the normalization property satisfied by both reduced Green functions: $\int_{-\infty}^{+\infty} K_{\alpha,\beta}^{\theta\,(j)}(x)\,dx = E_{\beta,j}(0) = 1/\Gamma(j) = 1$ for $j=1,2$. However, the normalization property holds true for all times only for the first complete Green function as we can note from the first equality in (3.8). Following [25] we invert the Fourier transforms of $K_{\alpha,\beta}^{\theta\,(j)}(x)$ by using the convolution theorem of the Mellin transforms arriving at the Mellin-Barnes integral representation

$$K_{\alpha,\beta}^{\theta\,(j)}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})\,\Gamma(1-s)}{\Gamma(j-\frac{\beta}{\alpha}s)\,\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\, ds\,, \tag{3.9}$$

where $0<\gamma<\min\{\alpha,1\}$, and $\rho = (\alpha-\theta)/(2\,\alpha)$.

For later use we recall the main formulas concerning the Mellin transform. For more details, see *e.g.* [27]. If

$$\mathcal{M}\{f(r);s\} = f^*(s) = \int_0^{+\infty} f(r)\, r^{s-1}\, dr, \quad \gamma_1 < \Re(s) < \gamma_2 \tag{3.10}$$

denotes the Mellin transform of $f(r)$ with $r\in\mathbb{R}^+$, the inversion is provided by

$$\mathcal{M}^{-1}\{f^*(s);r\} = f(r) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\, r^{-s}\, ds\,, \tag{3.11}$$

where $r>0$, $\gamma=\Re(s)$, $\gamma_1<\gamma<\gamma_2$. The Mellin convolution formula reads

$$h(r) = \int_0^\infty \frac{1}{\rho}\, f(\rho)\, g(r/\rho)\, d\rho \overset{\mathcal{M}}{\leftrightarrow} h^*(s) = f^*(s)\, g^*(s)\,. \tag{3.12}$$

We note that the Mellin-Barnes integral representation (3.9)[3] allows us to construct computationally the fundamental solutions of Eq. (3.2) for any triplet $\{\alpha,\beta,\theta\}$ by matching their convergent

[2]The Mittag-Leffler function $E_{\beta,\mu}(z)$ with $\beta,\mu>0$ is an entire transcendental function of order $\rho=1/\beta$, defined in the complex plane by the power series

$$E_{\beta,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\, n+\mu)}\,, \quad \beta,\mu>0\,, \quad z\in\mathbb{C}\,.$$

For information on the Mittag-Leffler-type functions the reader may consult *e.g.* [5], [12], [32].

[3]The names refer to the two authors, who in the beginning of the past century developed the theory of these integrals using them for a complete integration of the hypergeometric differential equation. However, as revisited in [26], these integrals were first introduced in 1888 by S. Pincherle (Professor of Mathematics at the University of Bologna from 1880 to 1928).
As a matter of fact this type of integrals turns out to be useful in inverting the Mellin transforms.
Readers acquainted with Fox H functions can recognize in (3.9) the representation of a certain function of this class, see *e.g.* [28], [37]. Unfortunately, as far as we know, computing routines for this general class of special functions are not yet available.

and asymptotic expansions, as shown in [25] for the first Green function. The interested reader may take vision of several plots of the *reduced* Green functions in [25] in a number of cases where these functions, being non-negative and normalized, can be interpreted as probability densities. In order to give the reader a better impression about the behaviours of the tails, the logarithmic scale was adopted.

We also note that the space-time fractional diffusion equation has been analysed (but without numerical computations) in several papers, see *e.g.* Anh and Leonenko [1] and references therein.

4 Probability interpretation of the Green functions

For the following cases that allow simplifications in the integrand of Eq. (3.9), we obtain relevant expressions of the corresponding Green functions that can be interpreted as probability densities.
(a) For $j = 1$ and $\{0 < \alpha < 2,\ \beta = 1\}$ (*strictly space fractional diffusion*) we have $K_{\alpha,1}^{\theta\,(1)}(x) = L_\alpha^\theta(x)$, *i.e.* the class of the strictly stable (non-Gaussian) probability densities [7][4]exhibiting fat tails (with the algebraic decay $\propto |x|^{-(\alpha+1)}$) and infinite variance. Their Mellin-Barnes integral representation reads

$$K_{\alpha,1}^{\theta\,(1)}(x) = L_\alpha^\theta(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/\alpha)\,\Gamma(1-s)}{\Gamma(\rho\, s)\,\Gamma(1-\rho s)}\, x^{\,s}\, ds\,, \tag{4.1}$$

where $0 < \gamma < \min\{\alpha, 1\}$.

(b) For $j = 1, 2$ and $\{\alpha = 2,\ 0 < \beta < 2\}$ (*time fractional diffusion* including *standard diffusion*), we have $K_{2,\beta}^{0\,(j)}(x) = M_{\beta/2}^{(j)}(x)/2$, *i.e.* the class of the Wright type[5] probability densities exhibiting stretched exponential tails. Their Mellin-Barnes integral representation reads

$$K_{2,\beta}^{0\,(j)}(x) = \frac{1}{2} M_{\beta/2}^{(j)}(x) = \frac{1}{2x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1-s)}{\Gamma(j-\beta s/2)}\, x^{\,s}\, ds\,, \tag{4.2}$$

where $0 < \gamma < 1$.

(c) For $j = 1$ and $\{0 < \alpha = \beta < 2\}$ (*neutral fractional diffusion*), we have $K_{\alpha,\alpha}^{\theta\,(1)}(x) = N_\alpha^\theta(x)$, *i.e.* the class of the Cauchy type probability densities [25]. Indeed, in this special case, the Mellin-Barnes integral representation provides an explicit expression which generalizes the Cauchy density,

$$\begin{aligned} K_{\alpha,\alpha}^{\theta\,(1)}(x) = N_\alpha^\theta(x) &= \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})}{\Gamma(\rho\, s)\,\Gamma(1-\rho\, s)}\, x^{\,s}\, ds \\ &= \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\sin(\pi\,\rho\, s)}{\sin(\pi\, s/\alpha)}\, x^{\,s}\, ds = \frac{1}{\pi}\,\frac{x^{\alpha-1}\sin[\frac{\pi}{2}(\alpha-\theta)]}{1+2x^\alpha\cos[\frac{\pi}{2}(\alpha-\theta)]+x^{2\alpha}}\,. \end{aligned} \tag{4.3}$$

[4]For recent treatises on Lévy stable distributions see *e.g.* [20], [35], [36], [38].

[5]The function $M_\nu^{(j)}(z)$ is defined for any order $\nu \in (0,1)$ and $\forall z \in \mathbb{C}$ by

$$M_\nu^{(j)}(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (j-\nu)]}\,, \quad 0 < \nu < 1\,, \quad z \in \mathbb{C}\,.$$

It turns out that $M_\nu^{(j)}(z)$ is an entire function of order $\rho = 1/(1-\nu)$. For $\nu = 1/2$ we obtain

$$M_{1/2}^{(1)}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-z^2/4\right)\,, \quad M_{1/2}^{(2)}(z) = \frac{1}{\sqrt{\pi}}\exp\left(-z^2/4\right) - \frac{z}{2}\mathrm{erfc}\left(\frac{z}{2}\right)\,.$$

The M functions are special cases of the Wright function defined by the series representation, valid in the whole complex plane,

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n+\mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbb{C}\,, \quad z \in \mathbb{C}\,.$$

Indeed, we recognize $M_\nu^{(j)}(z) = \Phi_{-\nu,j-\nu}(-z)$, $0 < \nu < 1$. Originally, Wright introduced and investigated this function with the restriction $\lambda \ge 0$ in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions. Only later, in 1940, he considered the case $-1 < \lambda < 0$. For detailed information on the Wright-type functions the interested reader may consult, *e.g.* [5] (where, presumably for a misprint, λ is restricted to be non-negative), [10], [11], [21].

where $0 < \gamma < \alpha$.

Based on the arguments outlined in [25], we extend the meaning of probability density to the cases $\{0 < \alpha < 2\,,\, 0 < \beta < 1\}$ and $\{1 < \beta \le \alpha < 2\}$ by proving the following composition rules of the Mellin convolution type:

$$K_{\alpha,\beta}^{\theta\,(j)}(x) = \begin{cases} \alpha \displaystyle\int_0^\infty \Big[\xi^{\alpha-1}\, M_\beta^{(j)}\,(\xi^\alpha)\Big]\; L_\alpha^\theta\,(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta < 1\,, \\ \displaystyle\int_0^\infty M_{\beta/\alpha}^{(j)}(\xi)\, N_\alpha^\theta(x/\xi)\, \frac{d\xi}{\xi}\,, & 0 < \beta/\alpha < 1\,. \end{cases} \tag{4.4}$$

The absolute moments of $K_{\alpha,\beta}^{\theta\,(j)}(x)$ can be obtained by considering the Mellin transform of $x\, K_{\alpha,\beta}^{\theta\,(j)}(x)$ which reads, by using (3.9) and (3.10)-(3.11),

$$\int_0^{+\infty} K_{\alpha,\beta}^{\theta\,(j)}(x)\, x^{\,s}\, dx = \rho\, \frac{\Gamma(1-s/\alpha)\,\Gamma(1+s/\alpha)\,\Gamma(1+s)}{\Gamma(1-\rho\, s)\,\Gamma(1+\rho\, s)\,\Gamma(j+\beta\, s/\alpha)}\,, \tag{4.5}$$

where $-\min\{\alpha, 1\} < \Re(s) < \alpha$. In particular we find $\int_0^{+\infty} K_{\alpha,\beta}^{\theta\,(j)}(x)\, dx = \rho$ (with $\rho = 1/2$ if $\theta = 0$). We note that Eq. (4.5) is strictly valid as soon as cancellations in the "gamma fraction" at the RHS are not possible. Then this equation allows us to evaluate (in $\mathbb{R}_0^+$) the (absolute) moments of order δ for the Green function if $-\min\{\alpha, 1\} < \delta < \alpha$. In other words, it states that $K_{\alpha,\beta}^{\theta\,(j)}(x) = \mathcal{O}\left(x^{-(\alpha+1)}\right)$ as $x \to +\infty$. However, cancellations occur in the following cases where the restriction $\delta < \alpha$ is expected to disappear:

a) $\{\alpha = 2,\, \theta = 0,\, 0 < \beta < 2\}$ (*time fractional diffusion* including *standard diffusion*), for which $\rho = 1/2$;

b) $\{1 < \alpha < 2\,,\, \theta = \alpha - 2,\, 0 < \beta < \alpha\}$ (*extremal diffusion*), for which $\rho = 1/\alpha$. We note that this may happen only for one tail of the extremal density.

We recognize that case a) is included in case b) in the limit $\alpha = 2$. In the above cases Eq. (4.5) reduces to

$$\int_0^{+\infty} K_{\alpha,\beta}^{2-\alpha\,(j)}(x)\, x^{\,s}\, dx = \frac{1}{2}\, \frac{\Gamma(1+s)}{\Gamma[j+\beta\, s/\alpha]}\,, \quad \Re(s) > -1\,, \tag{4.6}$$

and consequently any absolute moment of order $\delta > -1$ is finite. We can show that the corresponding Green functions result of the Wright type[6] and exhibit a stretched exponential decay according to the asymptotic representation

$$\begin{aligned} K_{\alpha,\beta}^{2-\alpha\,(j)}(x) \sim \alpha^{-1}\, &[2\pi(1-\beta/\alpha)]^{-1/2} (x\beta/\alpha)^{(1/2-j+\beta/\alpha)/(1-\beta/\alpha)} \\ &\cdot \exp\left[-(\alpha/\beta - 1)\,(x\beta/\alpha)^{1/(1-\beta/\alpha)}\right]\,, \quad x \to +\infty\,. \end{aligned} \tag{4.7}$$

Then, due to the previous discussion, in the cases $\{0 < \alpha < 2\,,\, 0 < \beta < 1\}$ and $\{1 < \beta \le \alpha < 2\}$ (*i.e. strictly space-time-fractional diffusion*) we obtain a class of probability densities (symmetric or non-symmetric according to $\theta = 0$ or $\theta \ne 0$) which exhibit fat tails (only one fat tail in the extremal cases) with an algebraic decay $\propto |x|^{-(\alpha+1)}$. Thus, they belong to the domain of attraction of the Lévy stable densities of index α and can be referred to as *fractional stable densities*.

When the time variable is considered, in all above cases the first Green function evolves in time as a probability density because it keeps the normalization. The integral over all of $\mathbb{R}$ of the first Green function is independent of time whereas that of the second Green function increases linearly with time.

[6] We have

$$K_{\alpha,\beta}^{2-\alpha\,(j)}(x) = \frac{1}{\alpha}\, \Phi_{-\beta/\alpha, j-\beta/\alpha}(-x) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma[-n\,\beta/\alpha + (j-\beta/\alpha)]}\,.$$

5 Conclusions and outlook

In this paper we have summarized our approach to obtain the fundamental solutions of fractional diffusion equations and have shown how they can be interpreted as probability densities evolving in time.

In recent years evolution equations containing fractional derivatives have gained revived interest in that they are expected to provide suitable mathematical models for describing phenomena of anomalous diffusion and transport dynamics in complex systems, see *e.g.* [4], [19], [23], [24], [29], [31], [33], [38]. and references therein. We point out the fact that all these fractional evolution equations can be considered as master equations for random walk models that turn out to be beyond the classical Brownian motion, see *e.g.* Klafter *et al.* [22]. For a recent review we refer the reader to Metzler and Klafter [30]. Gorenflo and collaborators, see *e.g.* [8], [13], [14], [15], [17], [18], have recently proposed a variety of models of random walk, discrete or continuous in space and time, suitable for simulating fractional diffusion processes.

In [16] Gorenflo and Mainardi have shown how to obtain the space-time fractional diffusion equation (2.1), in the case $0 < \beta \le 1\,,\ \theta = 0\,$, by a properly scaled transition to the limit from a general master equation.

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On Recent Results on Type G Multivariate Laws

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Abstract

We review recent results on multivariate extensions of type G laws and present new results on type G random matrices.

1 Introduction

Marcus (1987) introduced the concept of type G random variables and processes (see also Rosinski (1991) and Maejima and Samorodnitsky (1999)). We here review several of the recent attempts to extend this concept to multivariate situations, including examples and corresponding characterizations. In Section 2 we recall some of the properties and examples of one dimensional type G laws. In Section 3 we review three possible multivariate extensions studied by Maejima and Rosinski (2001, 2002) and Barndorff-Nielsen and Pérez-Abreu (2000, 2002). Finally, Section 4 presents several extensions to matrix laws obtained recently by the authors.

We denote by $\mathbf{M}_m$ the space of symmetric $m \times m$ matrices and by $\mathbf{M}_m^+$ the cone of nonnegative definite matrices in $\mathbf{M}_m$.

2 One dimensional type G laws

Marcus (1987) introduced the concept of type G random variables as those infinitely divisible random variables whose Lévy measure is given by

$$U(A) = E\left[U_0(Z^{-1}A)\right], \tag{2.1}$$

for Z a standard Gaussian random variable, U_0 a Borel measure on $\mathbf{B}(\mathbf{R})$, and for all $A \in \mathbf{B}(\mathbf{R}\backslash\{0\})$.

It can be shown (see for example Rosinski, 1991) that a random variable x is of type G if in law x is of the form $z\sqrt{s}$ where z and $s > 0$ are independent random variables with s being infinitely divisible and z having the standard normal distribution (we refer to $z\sqrt{s}$ as a *G-representation* of x). Those, type G random variables are variance mixtures of Gaussian random variables, where the random variance is infinitely divisible.

2.1 Examples

Many important examples of one dimensional distributions are of type G:

a) Symmetric stable distributions (called subgaussian by Samorodnitsky and Taqqu, 1994), where s has the law of a positive $\alpha/2-$stable random variable.

b) The *Student$-t$* (see Steutel, 1979), where s has the law of the reciprocal chi-square distribution (which is infinitely divisible).

c) The normal inverse Gaussian (Barndorff-Nielsen, 1996), where s has the inverse Gaussian law.

2.2 Characterization

An important characterization of a type G distribution is that its Lévy measure is the Gaussian mixture with respect to the Lévy measure v of the random variance s (see Rosinski, 1991).

Proposition 2.1. Let x be one-dimensional type G with G-representation $zs^{1/2}$, and suppose that the infinitely divisible nonnegative random variable s has Lévy measure v. Then x is infinitely divisible with Lévy density u given by

$$u(x) = \int_0^\infty \varphi(x;\sigma)v(\mathrm{d}\sigma), \tag{2.2}$$

where $\varphi(x;\sigma)$ denotes the density function of the one-dimensional normal distribution with mean 0 and variance σ.

Proposition 2.2. A one-dimensional infinitely divisible random variable x is of type G if and only if its Lévy measure U is of form

$$U(A) = \int_A g(r^2)v(\mathrm{d}r), \quad A \in \mathbf{B}(\mathbf{R}\backslash\{\mathbf{0}\}), \tag{2.3}$$

where $g(r)$ is a completely monotone function on $(0,\infty)$ and v is a finite measure on $\mathbf{B}(\mathbf{R}\backslash\{0\})$ such that

$$\int_0^\infty \min(1,r^2)g(r^2)v(\mathrm{d}r) < \infty.$$

2.3 Relation to subordination

There is an interesting relation between one dimensional type G and the notion of subordination. Namely, if $s(t)$ is a one-dimensional subordinator, independent of the Brownian motion $B(t)$, then $x(t) = B(s(t))$ has the *G-representation* $B(1)\sqrt{s(t)}$.

3 Multivariate type G laws

As one can expect, there is no single extension of type G to higher dimensions. In this section we review three possible extensions given recently by Maejima and Rosinski (2001, 2002) and Barndorff-Nielsen and Pérez-Abreu (2002). We emphasize the multivariate versions of the examples, characterizations and relation to subordination presented in Section 2 for the one-dimensional laws.

3.1 Extension via Marcus' definition

Following the above idea of M. Marcus, Maejima and Rosinski (2001, 2002) propose the concept of type G random vectors as those infinitely divisible random vectors whose Lévy measure is given by (2.1), for z a standard Gaussian random variable, U_0 a Borel measure on $\mathbf{B}(\mathbf{R}^m\backslash\{\mathbf{0}\})$ and for all $A \in \mathbf{B}_0(\mathbf{R}^m\}$, the class of all Borel sets such that $A \subset \{|x| > \varepsilon\}$ for some $\varepsilon > 0$. The latter authors give, in particular, the following multivariate extension of Proposition 2 in terms, of radial components.

Proposition 3.1. A symmetric probability measure P on $\mathbf{B}(\mathbf{R}^m)$ is of type G if and only if it is infinitely divisible and its Lévy measure U is either zero or represented as

$$U(EB) = \int_B \lambda(dx)\int_E g_x(r^2)v(\mathrm{d}r), \text{ for } E \in \mathbf{B}(\mathbf{R}_+), B \in \mathbf{B}(\mathbf{S}), \tag{3.1}$$

where λ is a probability measure on the unit sphere $\mathbf{S}$ of $\mathbf{R}^m$ and $g_x(r)$ is a jointly measurable function which, for any fixed x, is a completely monotone function on $(0,\infty)$ and satisfies

$$0 < \int_0^\infty \min(1,r^2)g_x(r^2)v(\mathrm{d}r) = c < \infty$$

with c independent of x.

3.2 The mult$G$ class

Barndorff-Nielsen and Pérez-Abreu (2000, 2002) present two possible multivariate extensions of the one dimensional concept of type G, called mult$G$ and ext$G$. The first extension is to random vectors x with the representation

$$x \overset{d}{=} zS^{1/2} \tag{3.2}$$

where z is a standard m-dimensional normal vector independent of the infinitely divisible nonnegative definite random $m \times m$ matrix S. The following result extends Proposition 1 above.

Proposition 3.2. Let x be a mult$G$ random vector with G-representation (3.2) and suppose that the infinitely divisible nonnegative definite random matrix S has Lévy measure V. Then x is an infinitely divisible random vector with Lévy density u given by

$$u(x) = \int_{M_+^m} \varphi_m(x; \Sigma) V(\mathrm{d}\Sigma), \tag{3.3}$$

where $\varphi_m(x; \Sigma)$ denotes the density function of the m-dimensional normal distribution with zero mean and covariance Σ.

As examples of mult$G$ distributions, Barndorff-Nielsen and Pérez-Abreu (2002) show that, as in the one dimensional case, any multivariate symmetric $\alpha-$stable law is a random covariance mixture of a Gaussian vector. They also introduce a matrix extension of the inverse Gaussian distribution and construct the corresponding multivariate normal inverse Gaussian law.

The relation of mult$G$ to subordination is not straightforward as in the one dimensional case and leads to the study of cone parameter Lévy processes and multivariate subordination (see Barndorff-Nielsen, Pedersen and Sato, 2001). Recently, Pedersen and Sato (2001) have shown the relation of mult$G$ to subordination of cone-parameter convolution semigroups.

3.3 The ext$G$ class

A third, more general, extension, called ext$G$, consists of those random vectors whose one dimensional marginals are type G. In this direction we introduce the concept of marginal infinite divisibility.

Definition 3.3. Let E be a Euclidean space and let Ψ be a collection of linear maps, each element ψ of Ψ mapping E into some Euclidean space. A random variate x in E is said to be marginal infinitely divisible relative to Ψ if the law of $\psi(x)$ is infinitely divisible for all $\psi \in \Psi$.

Trivially, infinite divisibility implies marginal infinite divisibility whatever the class Ψ.

In particular, the above concept is applied to marginal infinite divisibility of random $m \times m$ symmetric matrices M with Ψ being the class of mappings of the form

$$\psi : M \to \mathrm{tr}(CM^T)$$

for C an $m \times m$ symmetric matrix. Depending on the rank d of the class of matrices C under consideration, we obtain the corresponding concepts of marginal infinite divisibility of order d, $0 < d \leq m$. Thus we say that a random symmetric matrix is marginal infinitely divisible of order d if for all $m \times m$ matrix C of rank d, $\mathrm{tr}(SC)$ is a one-dimensional infinitely divisible random variable. The case $d = 1$ was considered in Barndorff-Nielsen and Pérez-Abreu (2002), the Wishart and the inverse Wishart providing examples of such random matrices, which are not themselves infinitely divisible.

The above concept allows us to introduce an important class of multivariate ext$G$ laws. Let S be a nonnegative definite $m \times m$ random matrix marginal infinitely divisible of order one. Then the vector $x = zS^{1/2}$ has an ext$G$ law. Several distributions commonly used in the multivariate statistics literature are of this type, including a multivariate extension of the *Student$-t$* distribution (see Barndorff-Nielsen and Pérez-Abreu (2002)), where S has the inverse Wishart law.

A stochastic processes $x(t)$ is said to be of type ext$G$, if all finite dimensional laws of $x(t)$ are of type ext$G$. A wide class of strictly stationary stochastic processes of type ext$G$ is constructed in Barndorff-Nielsen and Pérez-Abreu (1999).

4 Matrix type G laws

The mult$G$ concept can be easily extended to matrix distributions. A random matrix x is said to be mat$G$, if its law is of the form $ZS^{1/2}$, where Z is a standard normal matrix independent of the infinitely divisible nonnegative definite random $m \times m$ matrix S. In this case we are able to prove the corresponding versions of Propositions 1 and 2.

4.1 Characterization

By $\mathbf{M}_m$ we denote the set of $m \times m$ symmetric matrices and $\mathbf{M}_m^+$ stands for the subset consisting of the nonnegative definite elements of $\mathbf{M}_m$.

Proposition 4.1. Let x be a mat$G$ random matrix with G-representation $ZS^{1/2}$ and suppose that the infinitely divisible nonnegative random matrix S has Lévy measure V. Then x is an infinitely divisible random matrix with Lévy density u given by

$$u(x) = \int_{\mathbf{M}_m^+} \varphi_{mm}(x; I_m \bigotimes \Sigma) V(\mathrm{d}\Sigma), \tag{4.1}$$

where $\varphi_{mm}(x; I_m \bigotimes \Sigma)$ denotes the density function of the $m \times m$ matrix normal distribution with zero matrix mean and covariance $I_m \bigotimes \Sigma$ with I_m the identity matrix in $\mathbf{R}^m$ and where $\bigotimes$ denotes tensor product of matrices.

Proposition 4.2. An $m \times m$ symmetric random matrix x is mat$G$ if and only if its Lévy measure U is of the form

$$U(A) = \int_A g(x^2)\mathrm{d}x, \quad A \in \mathbf{B}(\mathbf{R}^{m\times m}), \tag{4.2}$$

where $g : \mathbf{M}_m^+ \to \mathbf{R}_+$ is a completely monotone function.

4.2 Examples

Similarly to the construction of mult$G$ examples in Barndorff-Nielsen and Pérez-Abreu (2002), we can construct important examples of mat$G$ random matrices.

a) *Matrix symmetric $\alpha-$stable distributions.* The matrix symmetric $\alpha-$stable law is mat$G$ with log characteristic function

$$\mathrm{C}\{\zeta \ddagger x\} = -\frac{1}{2^{\alpha/2}} \int_{\mathbf{SM}_m^+} \left[\mathrm{tr}(\zeta^2\Theta)\right]^{\alpha/2} \Gamma(\mathrm{d}\Theta), \quad \zeta \in \mathbf{M}_m, \tag{4.3}$$

where Γ is a measure on $\mathbf{SM}_m^+ = \mathbf{S} \cap \mathbf{M}_m^+$ and $\mathbf{S}$ is the unit sphere of $\mathbf{M}_m$.□

b) *Matrix extension of the symmetric normal inverse Gaussian law.* Let be Γ as in the last example and $\Sigma \in \mathbf{M}_m^+$. The infinitely divisible random matrix x with log characteristic function

$$\mathrm{C}\{\zeta \ddagger x\} = \int_{\mathbf{SM}_m^+} \left[\mathrm{tr}(\Sigma\Theta)\right]^{\alpha/2}\Gamma(\mathrm{d}\Theta) - \int_{\mathbf{SM}_m^+} [\mathrm{tr}((\Sigma + \frac{1}{2}\zeta^2)\Theta)]^{\alpha/2}\Gamma(\mathrm{d}\Theta), \quad \zeta \in \mathbf{M}_m \tag{4.4}$$

and Lévy density

$$u(x; \Sigma, \Gamma) = \int_{\mathbf{SM}_m^+} \int_0^\infty \varphi_{mm}(x; I_m \bigotimes r\Theta) \frac{e^{-r\mathrm{tr}(\Sigma\Theta)}}{r^{1+\alpha/2}} \mathrm{d}r\Gamma(\mathrm{d}\Theta), \quad x \in \mathbf{M}_m. \tag{4.5}$$

is mat$G$ and it is a matrix generalization of the symmetric normal inverse Gaussian distribution.

The proofs of the above results as well as a discussion of the relation of mat$G$ to subordination will be given elsewhere.

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Smoothness of harmonic functions for Markov processes with jumps

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1 The case of continuous diffusions

Consider a second-order differential operator L on $\mathbb{R}^d$ with C_b^∞ coefficients, and its associated semigroup P_t. There exists a continuous diffusion X_t defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$, such that $\mathbb{P}^x[X_0 = x] = 1$ and

$$P_t f(x) = \mathbb{E}^x\Big[f(X_t)\exp\int_0^t g(X_s)ds\Big]$$

with $g = L1$. One wants to study the hypoellipticity of L with probabilistic methods. Under Hörmander's condition, one can prove with the help of Malliavin's calculus that

$$P_t f(x) = \int f(y)p(t,x,y)dy$$

for a C^∞ density $y \mapsto p(t,x,y)$. This density is solution of the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = L^\star p, \tag{1.1}$$

so Malliavin's calculus says that solutions of this equation are smooth. Now consider a bounded function h which is harmonic on a domain $D \subset \mathbb{R}^d$, so that $Lh = 0$ on D; this means that

$$M_t = h(X_t)\exp\int_0^t g(X_s)ds \tag{1.2}$$

is a $\mathbb{P}^x$ local martingale up to the first exit of D. One wants to prove that h is C^∞ on D. Firstly the equation $Lh = 0$ involves the operator L, whereas the Fokker-Planck equation (1.1) involves the adjoint $L^\star$; thus it would be better to consider the diffusion associated to $L^\star$. Secondly, the Fokker-Planck is written on the whole space $\mathbb{R}^d$, whereas our equation is only satisfied on D; thus we have to apply a localisation method. These techniques were worked out in [8, 4]. Let us explain how this can be done, in a way which can be generalised to processes with jumps.

Let D_0 be a relatively compact open subset of D and ϕ be a C^∞ function with compact support in D, such that $\phi = 1$ on D_0. Consider the distribution

$$u = L(h\phi) = h\,L\phi - hg\phi + \Gamma(h,\phi) \tag{1.3}$$

where Γ is the "carré du champ" associated to $L - g$. By noticing that $\Gamma(h,h)$ is related to the quadratic variation of the martingale M_t of (1.2), one can check that $\Gamma(h,h)$ is in $L^1_{loc}(D)$, so u is a function in $L^2(D)$ and its support is included in $D \setminus D_0$. Moreover the equation $u = L(h\phi)$ implies

$$\begin{aligned} h(x)\phi(x) &= \int h(y)\phi(y)p(1,x,y)dy - \int_0^1\int u(y)p(t,x,y)dt\,dy \\ &= \int h(y)\phi(y)p^\star(1,y,x)dy - \int_0^1\int u(y)p^\star(t,y,x)dt\,dy \end{aligned} \tag{1.4}$$

for the density $p^\star$ of the adjoint semigroup $P_t^\star$. We know from Malliavin's calculus that $x \mapsto p^\star(t,y,x)$ is smooth for $t > 0$, so the first integral of (1.4) is C^∞. We also have good small time estimates for $p^\star$ and its derivatives outside the diagonal $\{y = x\}$. In the second integral, we take x in D_0 and y is in the support of u, so is outside D_0; thus we can integrate the estimates and deduce that h is C^∞ on D_0.

2 A class of Markov processes with jumps

We now describe the class of non local operators L to which the previous procedure can be extended. Details can be found in [14]. Let μ be a measure on $\mathbb{R}^m \setminus \{0\}$ such that

$$\int (|\lambda|^2 \wedge 1)\mu(d\lambda) < \infty.$$

Then μ is the Lévy measure of a Lévy process Λ_t with characteristic function

$$\mathbb{E}e^{i\,w.\Lambda_1} = \exp \int \Big(e^{i\,w.\lambda} - 1 - i\,w.\lambda 1_{\{|\lambda|\le 1\}}\Big)\mu(d\lambda).$$

Let $\gamma : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be a function such that

$$\gamma(x,\lambda) = \gamma_0(x)\lambda + O(|\lambda|^2)$$

as $\lambda \to 0$ for a matrix-valued function γ_0; we suppose that for μ almost any λ, the map $x \mapsto x + \gamma(x,\lambda)$ is invertible (and its Jacobian determinant is bounded below in absolute value). We also consider a real-valued function

$$\psi(x,\lambda) = 1 + \psi_0(x)\lambda + O(|\lambda^2|),$$

a real-valued function $g(x)$, and a $\mathbb{R}^d$-valued function $b(x)$. We suppose that γ, ψ, γ_0, ψ_0, g, b are C_b^∞ with respect to x, uniformly in λ. Then our operator will be

$$\begin{aligned} Lf(x) =& f'(x)b(x) + g(x)f(x) \\ &+ \int \Big(f(x+\gamma(x,\lambda)) - f(x) - f'(x)\gamma(x,\lambda)\Big)\psi(x,\lambda)\mu(d\lambda). \end{aligned}$$

This expression is well defined if f is C_b^2.

The advantage of this class of operators is twofold. Firstly, the adjoint $L^\star$ is in the same class; the associated function $\gamma^\star$ is such that $x \mapsto x+\gamma^\star(x,\lambda)$ is the inverse of $x \mapsto x+\gamma(x,\lambda)$. Secondly, it has a probabilistic representation to which it will be possible to apply an extension of Malliavin's calculus. More precisely, there exist functions b_0, g_1, ψ_1, a Markov process X_t solution of

$$dX_t = b_0(X_t)dt + \gamma(X_{t-}, d\Lambda_t)$$

and a multiplicative functional Γ_t solution of

$$d\Gamma_t = \Gamma_{t-}\Big(g_1(X_t)dt + \psi_1(X_{t-}, d\Lambda_t)\Big), \quad \Gamma_0 = 1,$$

so that the semigroup P_t of L can be written as

$$P_t f(x) = \mathbb{E}^x\big[\Gamma_t f(X_t)\big]. \tag{2.1}$$

Then the equation $Lh = 0$ in D means that $M_t = \Gamma_t h(X_t)$ is a $\mathbb{P}^x$ local martingale up to the first exit of D.

3 Existence of smooth densities

The absolute continuity of the law of Lévy processes has been studied for a long time, see [15] and the references therein. On the other hand, the study of Markov processes which are solutions of Lévy driven equations (like our process X_t) was initiated in the early 80's, see [2, 9, 11, 1]. The idea was to construct a differential calculus based on infinitesimal perturbations on the sizes of the jumps $\Delta\Lambda_t$, and, by means of an integration by parts formula, to deduce the smoothness of the law as in Malliavin's calculus. Another method consists in perturbing the times of the jumps, but this can be applied only to a particular class of equations, see [3, 5].

One can also perturb the process by adding jumps. This is not an infinitesimal perturbation, so this does not lead to a differential calculus. However, one can construct a finite difference calculus, obtain a duality formula similar to the integration by parts formula, and again deduce the existence of a smooth density ([12]). The advantage with respect to the jump size perturbation method is that it does not require any smoothness on the Lévy measure μ; for instance, μ can be supported by a countable set.

In this technique, the basic assumption concerns the "number" of small jumps. We suppose that there exists a $0 < \beta < 2$ such that

$$c\,\rho^{2-\beta}|v|^2 \leq \int_{\{|\lambda|\leq\rho\}} (\lambda.v)^2\mu(d\lambda) \leq C\,\rho^{2-\beta}|v|^2$$

for $0 < \rho \leq 1$. This assumption is satisfied in the case of a β-stable non degenerate Lévy process, so it says that our process has approximately the same number of small jumps than a stable process; in particular, it has finite variation if and only if $\beta < 1$. Then under the ellipticity condition $\gamma_0\gamma_0^\star \geq c\,I$, one proves in [12] that X_t has a C^∞ density for $t > 0$. The same result can be deduced for the density of the semigroup P_t of (2.1).

As in the continuous case, we also need small time estimates. This problem was studied in [6, 10], and in [13, 7] in our finite difference framework. In the continuous case, the density $p(t, x, y)$ and its derivatives decrease exponentially as $1/t \to \infty$ for $y \neq x$. Here, the estimates are not so good because the diffusion can go faster when it jumps. The density and its derivatives generally behave like a (positive or negative) power of t as $t \to 0$; this power depends on the number of jumps needed by X_t in order to go from x to y (it also depends on the "concentration" of these jumps). The larger the number of needed jumps is, the smaller the density will be. These types of estimates have been proved in the above framework when $\beta > 1$; if $\beta = 1$, we need an additional assumption and if $\beta < 1$, we want X_t to be a pure jump process (it should be the sum of its jumps without drift).

4 Smoothness of harmonic functions

We now consider a bounded solution of $Lh = 0$ on D, and apply the method which was described for continuous processes. We again consider the function u of (1.3) and the first integral of (1.4) is again C^∞; however, since L is not local, the function u is no more supported by $D \setminus D_0$. The support is compact because ϕ has compact support and the jumps are assumed to be bounded; moreover, since $\phi = 1$ on D_0, one has $u = 0$ on the subset D_1 of D_0 consisting of points from which the process cannot jump out of D_0,

$$D_1 = \Big\{x \in D_0;\ \mu\big\{\lambda; x + \gamma(x,\lambda) \notin D_0\big\} = 0\Big\}.$$

Thus, in the second integral of (1.4), we have $y \notin D_1$, so we take $x \in D_1$. However, this is not sufficient. As it has been said, the density $p^\star(t, y, x)$ and its derivatives behave like some power of t as $t \to 0$, so if we want to deduce an estimate for the derivatives of h, this power has to be larger than -1. The small time study shows that this is the case when the adjoint process $X_t^\star$ corresponding to $p^\star$ needs a large enough number of jumps to go from y to x, or equivalently, if the original process X_t needs a large enough number of jumps to go from x to y. Thus we have to assume that X_t needs a large enough number of jumps to go from x to D_1^c; moreover, this number depends on the order of differentiability that we study.

With this technique, we prove under the above assumptions that for any j, there exists a n such that if the process X_t cannot quit D from x with n jumps, then h is C^j in a neighbourhood of x. A complete proof is given in [14].

Under additional smoothness assumptions (if μ has a smooth density and γ is smooth with respect to λ), then the condition on the number of jumps can be dropped; roughly speaking, this is because the jumps can be separated into small jumps (a large number of them is needed to quit D) and large jumps which are regularising from our additional assumption.

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Tempered Stable Processes

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1 Introduction

A probability distribution on the real line is said to be tilted stable if is infinitely divisible with Lévy measure of the form $Q(dx) = \sigma_1|x|^{-\alpha-1}\exp(-\lambda_1|x|)I(x<0)\,dx + \sigma_2 x^{-\alpha-1}\exp(-\lambda_2 x)I(x>0)\,dx$ with $\sigma_1, \sigma_2 \geq 0, \lambda_1, \lambda_2 > 0$, and without Gaussian component. Tilted stable distributions were proposed in statistical physics by Mantegna and Stanley [8], Koponen [7], and Novikov [9] to study models exhibiting local spatiotemporal fractality and global aggregational Gaussianity. Such distributions were also used in financial mathematics by Carr, Geman, Madan, and Yor [5], Barndorff-Nielsen and Shephard [2], [3], and others.

In this work we define and study a more general and robust class of *tempered stable* distributions and processes. An interesting feature of tempered stable Lévy processes is that they behave like stable processes in a short period of time while in a long time frame they are approximately Gaussian. Unlike stable processes, the tempered ones may have all moments finite, including moments of exponential order. Still, their distributions are absolutely continuous with respect to the distributions of stable processes and we evaluate the corresponding Radon-Nikodym derivatives. Then we provide series representations of tempered stable processes which give further insight into their structure and can be used for simulation. Finally, since tempered stable distributions are selfdecomposable, we give series representations of their background driving Lévy processes.

2 Tempered stable distributions

Definition 2.1. *A probability measure μ on $\mathbb{R}^d$ is called tempered stable if is infinitely divisible without Gaussian part and with Lévy measure Q of the form*

$$Q(A) = \int_{\mathbb{R}_0^d}\int_0^\infty I_A(sx)s^{-\alpha-1}e^{-s}\,ds\,\nu(dx), \quad A \subset \mathbb{R}_0^d \tag{2.1}$$

where $\alpha \in (0,2)$ and ν is a σ–finite Borel measure on $\mathbb{R}_0^d \stackrel{def}{=} \mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}_0^d} \|x\|^\alpha\,\nu(dx) < \infty. \tag{2.2}$$

Notice that (2.1) can be viewed as a mixture of gamma densities with shape parameter $-\alpha$ and scale 1. Tilted stable distributions, which were defined in the Introduction, are the special case of (2.1) corresponding to $\nu = \sigma_1\lambda_1^\alpha\delta_{-\lambda_1^{-1}} + \sigma_2\lambda_2^\alpha\delta_{\lambda_2^{-1}}$ and $d=1$. The following proposition gives an alternative characterization of Lévy measures of tempered stable distributions.

Proposition 2.2. *Q is the Lévy measure of a tempered stable distribution on $\mathbb{R}^d$ if and only if in polar coordinates it has the form*

$$Q(dr,du) = r^{-\alpha-1}q(r,u)\,dr\sigma(du), \qquad r>0,\, u \in S^{d-1} \tag{2.3}$$

where $q : (0,\infty)\times S^{d-1} \mapsto (0,\infty)$ is a Borel function such that $q(\cdot,u)$ is completely monotone with $q(\infty,u)=0$ for every $u \in S^{d-1}$, and σ is a probability measure on S^{d-1} such that

$$\int_{S^{d-1}} q(0+,u)\,\sigma(du) < \infty. \tag{2.4}$$

From Proposition 2.2 it also follows that tempered stable distributions are selfdecomposable. Moreover, they constitute a proper subclass of Bondesson's class of extended generalized gamma convolutions when $d = 1$, see [4].

Proposition 2.3. *The characteristic function $\hat{\mu}$ of a tempered stable distribution μ with Lévy measure (2.1) is given by*

$$\hat{\mu}(y) = \exp\left\{k_\alpha \int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, \nu(dx) + i\langle y, b\rangle\right\} \tag{2.5}$$

where

$$\psi_\alpha(s) = \begin{cases} 1-(1-is)^\alpha, & 0<\alpha<1 \\ (1-is)\log(1-is)-1+i, & \alpha=1 \\ (1-is)^\alpha - 1 + i\alpha s, & 1<\alpha<2. \end{cases} \tag{2.6}$$

and $k_\alpha = |\Gamma(-\alpha)|$ if $\alpha \neq 1$, $k_1 = 1$.

Theorem 2.4. *Suppose that*

$$\int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, \nu_1(dx) = \int_{\mathbb{R}_0^d} \psi_\alpha(\langle y, x\rangle)\, \nu_2(dx), \quad y \in \mathbb{R}^d,$$

where ν_i are Borel measures on $\mathbb{R}_0^d$ with $\int_{\mathbb{R}_0^d} \|x\|^\alpha\, \nu_i(dx) < \infty$, $i = 1, 2$. Then $\nu_1 = \nu_2$.

We will write

$$\mu \sim TS(\alpha, \nu; b) \tag{2.7}$$

when (2.5) holds. Theorem 2.4 implies that parametrization (2.7) is indistinguishable. The next result addresses the question of moments of tempered stable distributions.

Proposition 2.5. *Let μ be a tempered stable distribution μ with Lévy measure (2.1). Then*

(i) $\int \|x\|^p\, \mu(dx) < \infty$ *for every* $p \in (0, \alpha)$*;*

(ii) $\int \|x\|^\alpha\, \mu(dx) < \infty$ *if and only if*

$$\int_{\|x\|>1} \|x\|^\alpha \log\|x\|\, \nu(dx) < \infty;$$

(iii) if $\nu(\{x : \|x\| > \epsilon\}) = 0$ for some $\epsilon > 0$, then for every $\theta \in (0, \epsilon^{-1})$

$$\int \exp(\theta\|x\|)\, \mu(dx) < \infty.$$

3 Short and long time behavior of tempered stable Lévy processes

Theorem 3.1. *Let $X(t)$, $t \geq 0$ be a tempered stable Lévy process in $\mathbb{R}^d$ with $\mathcal{L}(X(1)) \sim TS(\alpha, \nu; 0)$. Define*

$$X_h(t) = X(ht), \qquad t \geq 0.$$

(i) Short time behavior. If $\alpha \neq 1$, then

$$h^{-1/\alpha} X_h \xrightarrow{d} Y \qquad as \;\; h \to 0,$$

where $Y(t)$, $t \geq 0$ is a strictly α–stable Lévy process with characteristic function

$$Ee^{i\langle y, Y(t)\rangle} = \exp\Big\{ - tc_\alpha \int_{\mathbb{R}^d_0} |\langle y, x\rangle|^\alpha \big(1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn}\langle y, x\rangle\big)\, \nu(dx)\Big\} \tag{3.1}$$

If $\alpha = 1$, then

$$h^{-1} X_h - k_h \xrightarrow{d} Y \qquad as \;\; h \to 0,$$

where

$$k_h(t) = t \log(eh) \int_{\mathbb{R}^d_0} x\, \nu(dx),$$

and $Y(t)$, $t \geq 0$ is a 1–stable Lévy process with characteristic function

$$Ee^{i\langle y, Y(t)\rangle} = \exp\Big\{ - tc_1 \int_{\mathbb{R}^d_0} \big(|\langle y, x\rangle| + i\frac{2}{\pi}\langle y, x\rangle \log |\langle y, x\rangle|\big)\, \nu(dx)\Big\}. \tag{3.2}$$

Here c_α is a positive constant depending only on α.

(ii) Long time behavior. Assume that

$$\int_{\mathbb{R}^d_0} \|x\|^2\, \nu(dx) < \infty.$$

If $1 \leq \alpha < 2$, then

$$h^{-1/2} X_h \xrightarrow{d} B \qquad as \;\; h \to \infty,$$

where $B(t)$, $t \geq 0$ is a Brownian motion with the characteristic function

$$Ee^{i\langle y, B(t)\rangle} = \exp\Big\{ - \frac{t}{2}\Gamma(2-\alpha) \int_{\mathbb{R}^d_0} \langle y, x\rangle^2\, \nu(dx)\Big\}.$$

If $0 < \alpha < 1$ and $\int_{\mathbb{R}^d_0} \|x\|\, \nu(dx) < \infty$, then

$$h^{-1/2}(X_h - b_h) \xrightarrow{d} B \qquad as \;\; h \to \infty,$$

where B is as above and

$$b_h(t) = \Gamma(1-\alpha) th \int_{\mathbb{R}^d_0} x\, \nu(dx).$$

4 Absolute continuity with respect to stable processes

Theorem 3.1 says that a tempered stable Lévy process looks locally as a stable process. There is actually a deeper connection between these two processes.

Theorem 4.1. *Let $X(t)$, $t \in [0, \tau]$ be a tempered stable Lévy process in $\mathbb{R}^d$ with $\mathcal{L}(X(\tau)) \sim TS(\alpha, \tau\nu; 0)$. Then its distribution on $D[0, \tau]$ is absolutely continuous with respect to the distribution of the stable process $Y(t)$, $t \in [0, \tau]$ given by (3.1)–(3.2). Moreover,*

$$\log \frac{d\mathcal{L}(X)}{d\mathcal{L}(Y)} = \int_0^\tau \int_{g(x) \geq \frac{1}{2}} \log g(x)\, \bar{N}(dt, dx) + \int_0^\tau \int_{g(x) < \frac{1}{2}} \log g(x)\, N(dt, dx) + a \tag{4.1}$$

where

$$g(x) = \begin{cases} \dfrac{q\big(\|x\|, \frac{x}{\|x\|}\big)}{q\big(0+, \frac{x}{\|x\|}\big)} & \textit{if } q\big(0+, \frac{x}{\|x\|}\big) > 0 \\ 1 & \textit{otherwise} \end{cases} \tag{4.2}$$

and q is given by (2.3), while N is the process of jumps of Y, $\bar{N}$ is the compensated version of N, and a is a normalizing constant.

5 Series representations

The inverse of the tail of the Lévy measure of a tempered stable distribution does not have a closed form even in the simplest case of $\nu = \delta_1$. This makes the method of Inverse Lévy Measure practically difficult to apply. Therefore, we give another series representation in the framework of a generalized shot noise, see [11].

Theorem 5.1. *Let $X(t)$, $t \in [0, 1]$ be a tempered stable Lévy process in $\mathbb{R}^d$ with $\mathcal{L}(X(1)) \sim TS(\alpha, \nu; 0)$. If $\alpha \in (0, 1)$, or if ν is symmetric and $\alpha \in (0, 2)$, then*

$$X(t) \stackrel{d}{=} \sum_{j=1}^{\infty} V_j \Big[\frac{m(\nu)}{\|V_j\|} (\alpha\Gamma_j)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} \Big] I_{[T_j, 1]}(t) \tag{5.1}$$

where the equality holds for finite dimensional distributions and the convergence on the right hand side holds a.s. uniformly in $t \in [0, 1]$. Here T_j, U_j are i.i.d. uniform on $[0, 1]$ random variables and $\Gamma_j - \Gamma_{j-1}, E_j$ are i.i.d. exponential random variables with mean 1, $j \geq 1$, $\Gamma_0 = 0$. V_j are i.i.d. random vectors in $\mathbb{R}_0^d$ with the common distribution ν_1 given by

$$\nu_1(dx) = \frac{1}{m(\nu)^\alpha} \|x\|^\alpha \, \nu(dx)$$

where

$$m(\nu) = \Big(\int_{\mathbb{R}_0^d} \|x\|^\alpha \, \nu(dx) \Big)^{1/\alpha}.$$

Furthermore, the sequences $\{T_j\}$, $\{U_j\}$, $\{\Gamma_j\}$, $\{E_j\}$, and $\{V_j\}$ are independent of each other.

Terms $E_j U_j^{1/\alpha}$ in (5.1) produce exponential tempering; without these terms the series converges to a stable Lévy process given by (3.1)–(3.2) of Theorem 3.1. Series representations in a general nonsymmetric case and $\alpha \in [1, 2)$ involve centering in (5.1) that we do not consider here for the sake of simplicity. One can also extend (5.1) to represent general infinitely divisible processes whose finite dimensional marginal distributions are tempered stable. Finally, (5.1) can also be used for simulation of tempered stable random vectors and processes; in addition, Gaussian approximation of the small jumps part is applicable along the lines of Asmussen and Rosiński [1].

6 BDLP

Let $\mu \sim TS(\alpha, \nu; 0)$. Since μ is self-decomposable, there exists a Lévy process $Y(t)$, $t \geq 0$ in $\mathbb{R}^d$, called a *background driving Lévy process* (BDLP) for μ, such that

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-t}\, dY(t) \right) \tag{6.1}$$

(see Jurek and Vervaat [6]). By [6], Lévy measure Q_0 of $Y(1)$ is related to Lévy measure Q of μ by

$$Q(A) = \int_0^\infty Q_0(e^t A)\, dt, \quad A \subset \mathbb{R}_0^d.$$

It is easy to verify that

$$Q_0(A) = \int_{\mathbb{R}_0^d} \int_0^\infty I_A(sx)(\alpha s^{-\alpha-1} + s^{-\alpha}) e^{-s}\, ds\, \nu(dx).$$

The following result extends a series representation of the BDLP given in [10] for tilted stable positive random variables.

Proposition 6.1. *Under assumptions of Theorem 5.1,*

$$Y(t) \stackrel{d}{=} \sum_{j=1}^\infty V_j \left[\frac{m(\nu)}{\|V_j\|} (\alpha U_j \Gamma_j)^{-1/\alpha} \wedge E_j \right] I_{[\frac{-1}{\alpha} \ln U_j, 1]}(t), \quad t \geq 0 \tag{6.2}$$

where the equality holds for finite dimensional distributions and the convergence on the right hand side holds a.s. uniformly in t on each bounded subinterval of $[0, \infty)$.

Using series (6.2) in the development of $\int_0^\infty e^{-t}\, dY(t)$ we get

$$\begin{aligned} \int_0^\infty e^{-t}\, dY(t) &\stackrel{d}{=} \sum_{j=1}^\infty U_j^{1/\alpha} V_j \left[\frac{m(\nu)}{\|V_j\|} (\alpha U_j \Gamma_j)^{-1/\alpha} \wedge E_j \right] \\ &= \sum_{j=1}^\infty V_j \left[\frac{m(\nu)}{\|V_j\|} (\alpha \Gamma_j)^{-1/\alpha} \wedge E_j U_j^{1/\alpha} \right]. \end{aligned}$$

The latter series is exactly (5.1) for $t = 1$, which confirms (6.1).

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Lévy processes and convolution semigroups with parameter in a cone

Ken-iti Sato (joint work with Jan Pedersen)

This is a report of our work [13]. Proofs of all assertions are given there.

Usual Lévy processes and convolution semigroups have $\mathbb{R}_+ = [0, \infty)$ as domain of the parameter. The basic correspondences among them are formulated as follows.

(i) The class of convolution semigroups $\{\mu_t : t \geqslant 0\}$ on $\mathbb{R}^d$ corresponds to the class of infinitely divisible distributions μ through $\mu = \mu_1$, as the characteristic function $\widehat{\mu}_t(z)$ of μ_t satisfies $\widehat{\mu}_t(z) = \widehat{\mu}_1(z)^t$.

(ii) The class of Lévy processes in law $\{X_t : t \geqslant 0\}$ on $\mathbb{R}^d$ corresponds to the class of convolution semigroups on $\mathbb{R}^d$ through $\mu_t = \mathcal{L}(X_t)$, the distribution of X_t. This correspondence is one-to-one if processes with the same law are identified. Here we recall that a Lévy process in law is continuous in probability, but, unlike Lévy processes, the sample functions are not assumed to be cadlag.

(iii) Every Lévy process in law has a modification which is a Lévy process.

A natural generalization of $\mathbb{R}_+$ is a cone K in a Euclidean space. We study K-parameter convolution semigroups, Lévy processes, and Lévy processes in law, and investigate whether the correspondences above are generalized. The study of the case $K = \mathbb{R}_+^N$ was initiated by Barndorff-Nielsen, Pedersen, and Sato [1]. Here we study the case where K is a general cone.

Our main results are summarized in the following.

Definition 1. A subset K of $\mathbb{R}^M$ is a *cone* if it is a non-empty closed convex set closed under multiplication by nonnegative reals and containing no straight line through 0 and if $K \neq \{0\}$.

Definition 2. Write $s^1 \leqslant_K s^2$ (or $s^2 \geqslant_K s^1$) if $s^2 - s^1 \in K$. A sequence $s^1 \leqslant_K s^2 \leqslant_K \dots$ is *K-increasing.* A sequence $s^1 \geqslant_K s^2 \geqslant_K \dots$ is *K-decreasing.* A mapping $f\colon K \to \mathbb{R}^d$ is *K-right continuous* at $s^0 \in K$ if, for every K-decreasing sequence $\{s^n\}_{n=1,2,\dots}$ in K with $|s^n - s^0| \to 0$, we have $|f(s^n) - f(s^0)| \to 0$; f has *K-left limits* at $s^0 \in K \setminus \{0\}$ if, for every K-increasing sequence $\{s^n\}_{n=1,2,\dots}$ in $K \setminus \{s^0\}$ satisfying $|s^n - s^0| \to 0$, $\lim_{n\to\infty} f(s^n)$ exists in $\mathbb{R}^d$; f is *K-cadlag* if it is K-right continuous at each $s^0 \in K$ and has K-left limits at each $s^0 \in K \setminus \{0\}$.

Definition 3. A family $\{X_s : s \in K\}$ of random variables on $\mathbb{R}^d$ is a *K-parameter Lévy process* on $\mathbb{R}^d$ if

(i) $X_{s^{j+1}} - X_{s^j}$, $1 \leqslant j \leqslant n-1$, are independent for every K-increasing sequence $\{s^j\}_{1\leqslant j\leqslant n}$,

(ii) $\mathcal{L}(X_{s^2} - X_{s^1}) = \mathcal{L}(X_{s^4} - X_{s^3})$ for $s^2 - s^1 = s^4 - s^3 \in K$,

(iii) $X_0 = 0$ a. s.,

(iv) X_s is K-cadlag in s a. s.,

(v) $X_{s^n} \to X_{s^0}$ in probability if $|s^n - s^0| \to 0$.

$\{X_s : s \in K\}$ is called a *K-parameter Lévy process in law* if it satisfies (i)–(iii) and (v).
(We can prove that (v) follows from (i)–(iv).)

Definition 4. A family $\{\mu_s\colon s\in K\}$ of probability measures on $\mathbb{R}^d$ is a K*-parameter convolution semigroup* on $\mathbb{R}^d$ if

(i) $\mu_{s^1}*\mu_{s^2}=\mu_{s^1+s^2}$ for all $s^1,s^2\in K$,

(ii) $\mu_{ts}\to\delta_0$ for $s\in K$ as $t\downarrow 0$. (It follows from (i)–(ii) that, if $|s^n-s^0|\to 0$, then $\mu_{s^n}\to\mu_{s^0}$.)

Definition 5. A system $\{e^1,\dots,e^N\}$ is a *weak basis* of K if it is a basis of the linear subspace L generated by K and if $e^j\in K$ for $j=1,\dots,N$. A system $\{e^1,\dots,e^N\}$ is a *strong basis* of K if it is a weak basis of K and if every $s\in K$ is expressible as $s=s_1e^1+\cdots+s_Ne^N$ with $s_1,\dots,s_N\in\mathbb{R}_+$. (Every cone K has a weak basis, which is not unique. A strong basis of K is essentially unique, if it exists.)

Example 1. Every 2-dimensional cone has a strong basis. A 3-dimensional cone has a strong basis if and only if it is a triangular cone. The cone $\mathbb{R}_+^N$ has a strong basis.

Example 2. Let K have a strong basis $\{e^1,\dots,e^N\}$. The following three constructions of X_s for $s=s_1e^1+\cdots+s_Ne^N\in K$ give K-parameter Lévy processes on $\mathbb{R}^d$.

(i) Let $\{V_t\colon t\geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$. Fix $c_j\in\mathbb{R}_+$, $1\leqslant j\leqslant N$, and define $X_s=V_{c_1s_1+\cdots+c_Ns_N}$.

(ii) Let $\{V_t^j\colon t\geqslant 0\}$, $1\leqslant j\leqslant N$, be independent Lévy processes on $\mathbb{R}^d$. Define $X_s=V^1_{s_1}+\cdots+V^N_{s_N}$.

(iii) Let $\{U_t^j\colon t\geqslant 0\}$, $1\leqslant j\leqslant N$, be independent Lévy processes on $\mathbb{R}^{d_j}$, where $d_1+\cdots+d_N=d$. Define $X_s=(U^1_{s_1},\dots,U^N_{s_N})^\top$. (This notation gives a stacked vector. We understand that $\mathbb{R}^d$ is the set of column d-vectors.)

Example 3. Let $K=M^+_{d\times d}$ be the set of symmetric nonnegative-definite $d\times d$ matrices. This is a nondegenerate cone in $\mathbb{R}^{d(d+1)/2}$ and does not have a strong basis. For $s\in K$ let $\mu_s=N_d(0,s)$, the Gaussian distribution on $\mathbb{R}^d$ with mean 0 and covariance matrix s. Then, $\{\mu_s\colon s\in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$, which we call the *canonical* $M^+_{d\times d}$*-parameter convolution semigroup.*

In the following let K be an N-dimensional cone and let $\{e^1,\dots,e^N\}$ be a weak basis of K.

Theorem 1. *If $\{\mu_s\colon s\in K\}$ is a K-parameter convolution semigroup on $\mathbb{R}^d$, then μ_s is infinitely divisible and determined by $\mu_{e^1},\dots,\mu_{e^N}$ as*

$$\widehat{\mu}_s(z)=\widehat{\mu}_{e^1}(z)^{s_1}\dots\widehat{\mu}_{e^N}(z)^{s_N}\quad \textit{for } s=s_1e^1+\cdots+s_Ne^N\in K,$$

where $s_1,\dots,s_N$ are not necessarily nonnegative.

Definition 6. A set of infinitely divisible distributions $\{\rho_1,\dots,\rho_N\}$ on $\mathbb{R}^d$ is *admissible* with respect to $\{e^1,\dots,e^N\}$ if there is a K-parameter convolution semigroup $\{\mu_s\}$ such that $\mu_{e^j}=\rho_j$ for $j=1,\dots,N$.

In Theorem 2, Examples 4, and 5 below, ρ_j is an infinitely divisible distribution on $\mathbb{R}^d$ with generating triplet (A_j,ν_j,γ_j) in the sense of Sato [14] for each j. We denote by $\mathcal{B}_0(\mathbb{R}^d)$ the class of Borel sets $B\subset\mathbb{R}^d$ satisfying $\inf_{x\in B}|x|>0$.

Theorem 2. *If $\{e^1,\dots,e^N\}$ is a strong basis, then any set $\{\rho_1,\dots,\rho_N\}$ on $\mathbb{R}^d$ is admissible. If $\{e^1,\dots,e^N\}$ is not a strong basis, then, for every d, there exists a set $\{\rho_1,\dots,\rho_N\}$ on $\mathbb{R}^d$ which is not admissible. A necessary and sufficient condition for admissibility is that, if $s_1e^1+\cdots+s_Ne^N\in K$, then $s_1A_1+\cdots+s_NA_N$ is nonnegative-definite and $s_1\nu_1+\cdots+s_N\nu_N$ is nonnegative on $\mathcal{B}_0(\mathbb{R}^d)$.*

Example 4. Let K be the circular cone in $\mathbb{R}^3$ defined by $x_1^2+x_2^2=x_3^2$, $x_3\geqslant 0$, and let e^1,e^2,e^3 be points on the circle $x_1^2+x_2^2=1=x_3$ which have equal distances from each other. Then a set $\{\rho_1,\rho_2,\rho_3\}$ is admissible with respect to $\{e^1,e^2,e^3\}$ if and only if, for $(j,k,l)=(1,2,3)$, $(2,3,1)$, and $(3,1,2)$, $\langle A_jz,z\rangle^{1/2}\leqslant\langle A_kz,z\rangle^{1/2}+\langle A_lz,z\rangle^{1/2}$ for all z and $\nu_j(B)^{1/2}\leqslant\nu_k(B)^{1/2}+\nu_l(B)^{1/2}$ for all $B\in\mathcal{B}_0(\mathbb{R}^d)$. This cone K is isomorphic to the cone $M^+_{2\times 2}$.

Example 5. Let K be the square cone generated by $e^1 = (0,0,1)^\top$, $e^2 = (1,1,1)^\top$, $e^3 = (1,0,1)^\top$, $e^4 = (0,1,1)^\top$. Then $\{e^1, e^2, e^3\}$ is a weak basis. A set $\{\rho_1, \rho_2, \rho_3\}$ is admissible with respect to $\{e^1, e^2, e^3\}$ if and only if $A_1 + A_2 - A_3$ is nonnegative-definite and $\nu_1 + \nu_2 - \nu_3$ is nonnegative on $\mathcal{B}_0(\mathbb{R}^d)$.

Theorem 3. *Any K-parameter Lévy process in law $\{X_s\}$ on $\mathbb{R}^d$ induces a K-parameter convolution semigroup $\{\mu_s\}$ by $\mu_s = \mathcal{L}(X_s)$. In the converse direction, for a given K-parameter convolution semigroup $\{\mu_s\}$ on $\mathbb{R}^d$, two cases can occur:*

(a) there exists a K-parameter Lévy process in law which induces $\{\mu_s\}$;

(b) no K-parameter Lévy process in law induces $\{\mu_s\}$.

In the case (a) two subcases can occur:

(a_1) all K-parameter Lévy processes in law which induce $\{\mu_s\}$ are identical in law (that is, have an identical system of marginal distributions),

(a_2) there are two K-parameter Lévy processes in law which induce $\{\mu_s\}$ and which are not identical in law.

Definition 7. A K-parameter convolution semigroup $\{\mu_s\}$ is called *generative* or *non-generative* if it is in the case (a) or (b), respectively. It is called *unique-generative* or *multiple-generative* if it is in the case (a_1) or (a_2), respectively. A K-parameter Lévy process in law which induces $\{\mu_s\}$ is said to be *associated* with $\{\mu_s\}$.

Example 6. Let $K = \mathbb{R}^2_+$, $e^1 = (1,0)^\top$, and $e^2 = (0,1)^\top$. Let $\{\mu_s : s \in K\}$ be the convolution semigroup on $\mathbb{R}$ given by $\mu_s = N(0, s_1 + s_2)$ for $s = s_1 e^1 + s_2 e^2$. Then $\{\mu_s\}$ is multiple-generative. Indeed, let $\{V^j_t : t \geqslant 0\}$, $j = 1, 2$, be independent Brownian motions on $\mathbb{R}$ and define, for $s = s_1 e^1 + s_2 e^2$, $X^0_s = V^1_{s_1} + V^2_{s_2}$ and $X^1_s = V^1_{s_1+s_2}$. Then both $\{X^0_s\}$ and $\{X^1_s\}$ are K-parameter Lévy processes associated with $\{\mu_s\}$ but they are not identical in law.

Definition 8. Let $\{X_s\}$ be a K-parameter Lévy process in law on $\mathbb{R}^d$. When $\{s^j\}_{1 \leqslant j \leqslant n}$ is a K-increasing sequence, the distribution of $(X_{s^1}, \ldots, X_{s^n})^\top$ is called a *K-increasing marginal* of $\{X_s\}$.

Theorem 4. *If $\{X_s\}$ is a K-parameter Lévy process in law on $\mathbb{R}^d$ associated with $\{\mu_s\}$, then all K-increasing marginals of $\{X_s\}$ are determined by $\{\mu_s\}$ and are infinitely divisible. If, moreover, $\{\mu_s\}$ is unique-generative and K has a strong basis, then all marginal distributions of $\{X_s\}$ are infinitely divisible. But there are examples of K-parameter Lévy processes in law of which some marginal distributions are not infinitely divisible (see Example 7).*

Theorem 5. *Let $K = M^+_{d\times d}$ with $d \geqslant 2$. Let $\{\mu_s : s \in K\}$ be a nontrivial K-parameter convolution semigroup on $\mathbb{R}^d$ such that $\int |x|^2 \mu_s(dx) < \infty$ and the covariance matrix v_s of μ_s satisfies $v_s \leqslant_K s$ for all $s \in K$. Then $\{\mu_s\}$ is non-generative. In particular, the canonical $M^+_{d\times d}$-parameter convolution semigroup is non-generative.*

Theorem 6. *Let $\{\mu_s\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ such that each μ_s is purely non-Gaussian with triplet $(0, \nu_s, \gamma_s)$. Then $\{\mu_s\}$ is generative. To construct an associated K-parameter Lévy process in law $\{X_s\}$, let $\nu = \nu_{e^1} + \cdots + \nu_{e^N}$ and let $\{J(A) : A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)\}$ be the Poisson random measure with intensity measure $\lambda(d(t,x)) = dt\, \nu(dx)$. Choose an appropriate version $\phi_s(x)$ of $\nu_s(dx)/\nu(dx)$ which is measurable in (s,x) and let*

$$X_s = \lim_{\varepsilon \downarrow 0} \int_{D_s} x 1_{\{\varepsilon < |x| \leqslant 1\}}(x) (J(d(t,x)) - \lambda(d(t,x))) + \int_{D_s} x 1_{\{|x|>1\}} J(d(t,x)) + \gamma_s,$$

where $D_s = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d : 0 \leqslant t \leqslant \phi_s(x)\}$ and the limit is in probability.

Theorem 7. *Any K-parameter convolution semigroup $\{\mu_s\}$ on $\mathbb{R}$ is generative.*

Theorem 8. *If K has a strong basis, then any K-parameter convolution semigroup $\{\mu_s\}$ on $\mathbb{R}^d$ is generative and an associated Lévy process can be given in the form of (ii) of Example 2.*

Theorem 9. *Let $\{\mu_s\}$ be a generative K-parameter convolution semigroup on $\mathbb{R}^d$. Let $\mathbb{L}$ be the set of probability measures, on the path space $(\mathbb{R}^d)^K$, induced by K-parameter Lévy processes in law associated with $\{\mu_s\}$. Then $\mathbb{L}$ is a convex set.*

Example 7. Let $K = \mathbb{R}^2_+$ and let $\{\mu_s\}$, $\{X^0_s\}$, and $\{X^1_s\}$ be as in Example 6. Denote by P^0 and P^1 the probability measures on the path space $\mathbb{R}^K$ induced by $\{X^0_s\}$ and $\{X^1_s\}$, respectively, and let $P^p = (1-p)P^0 + pP^1$ for $0 < p < 1$. Then, for any p, P^p determines a K-parameter Lévy process in law $\{X^p_s\}$ associated with $\{\mu_s\}$. If $p \neq 0, 1$, then the joint distribution of $X^p_{e^1}$ and $X^p_{e^2}$ is not infinitely divisible.

Theorem 10. . *Let K have a strong basis $\{e^1, \ldots, e^N\}$. Let $\{\mu_s\}$ be a K-parameter convolution semigroup on $\mathbb{R}^d$ and let (A_s, ν_s, γ_s) be the generating triplet of μ_s.*

(i) In order for $\{\mu_s\}$ to be unique-generative, it is necessary and sufficient that, for every K-parameter Lévy process in law $\{X_s\}$ associated with $\{\mu_s\}$, $X_s = X_{s_1 e^1} + \cdots + X_{s_N e^N}$ a. s. for $s = s_1 e^1 + \cdots + s_N e^N \in K$.

(ii) If $\{\mu_s\}$ is unique-generative and $\{X_s\}$ is the associated K-parameter Lévy process in law, then $\{X_{te^j} : t \geqslant 0\}$, $1 \leqslant j \leqslant N$, are independent.

(iii) Let L_j, $1 \leqslant j \leqslant N$, be additive subgroups of $\mathbb{R}^d$ such that each L_j is a Borel set and $L_j \cap L_k = \{0\}$ for $j \neq k$. If $\mu_{te^j}(L_j) = 1$ for all t and j, then $\{\mu_s\}$ is unique-generative.

(iv) If $A_{e^j}(\mathbb{R}^d) \cap A_{e^k}(\mathbb{R}^d) \neq \{0\}$ for some $j \neq k$ or if ν_{e^j} and ν_{e^k} are not mutually singular for some $j \neq k$, then $\{\mu_s\}$ is multiple-generative.

Theorem 11. *If K has a strong basis and if $\{X_s\}$ is a K-parameter Lévy process in law associated with a unique-generative K-parameter convolution semigroup, then $\{X_s\}$ has a modification which is a K-parameter Lévy process.*

In the cone-parameter case, study of subordination of convolution semigroups is important, as they do not always correspond to Lévy processes in law. For $j = 1, 2$ let K_j be an N_j-dimensional cone in $\mathbb{R}^{M_j}$. For a probability measure μ and a bounded continouous function f we denote $\mu(f) = \int f(x)\mu(dx)$.

Theorem 12 (Subordination of semigroups). *Let $\{\mu_u \colon u \in K_2\}$ be a K_2-parameter convolution semigroup on $\mathbb{R}^d$ and $\{\rho_s \colon s \in K_1\}$ a K_1-parameter convolution semigroup with* $\operatorname{Supp}(\rho_s) \subset K_2$. *For each $s \in K_1$ a probability measure σ_s on $\mathbb{R}^d$ is defined by $\sigma_s(f) = \int_{K_2} \mu_u(f)\rho_s(du)$ and $\{\sigma_s \colon s \in K_1\}$ forms a K_1-parameter convolution semigroup on $\mathbb{R}^d$. (We can determine their generating triplets. It is an extension of Theorem 30.1 of [14] and Theorem 4.7 of [1].)*

Theorem 13 (Subordination of processes). *Let $\{Z_s \colon s \in K_1\}$ be a measurable K_1-parameter Lévy process in law on $\mathbb{R}^{M_2}$ such that $Z_s \in K_2$ a. s. for each $s \in K_1$ and let $\{X_u \colon u \in K_2\}$ be a measurable K_2-parameter Lévy process in law on $\mathbb{R}^d$. Suppose that they are independent. Define $Y_s = X_{Z'_s}$, where $Z'_s = Z_s 1_{K_2}(Z_s)$. Then $\{Y_s \colon s \in K_1\}$ is a measurable K_1-parameter Lévy process in law on $\mathbb{R}^d$.*

Theorem 14. *Let $\{e^1, \ldots, e^{N_1}\}$ be a weak basis of K_1. Let $\{\rho_s\}$ be a K_1-parameter convolution semigroup on $\mathbb{R}^{M_2}$. Let (A_s, ν_s, γ_s) be the triplet of ρ_s. Then* $\operatorname{Supp}(\rho_s) \subset K_2$ *for all $s \in K_1$ if and only if the following two conditions are satisfied:*

(i) $A_{e^j} = 0$, $\nu_{e^j}(\mathbb{R}^{M_2} \setminus K_2) = 0$, and $\int_{K_2 \cap \{|s| \leqslant 1\}} |s|\nu_{e^j}(ds) < \infty$ for $1 \leqslant j \leqslant N_1$,

(ii) if $s_1 e^1 + \cdots + s_{N_1} e^{N_1} \in K_1$, then $s_1 \gamma^0_{e^1} + \cdots + s_{N_1} \gamma^0_{e^{N_1}} \in K_2$, where $\gamma^0_{e^j}$ is the drift of ρ_{e^j}.

Application 1. Barndorff-Nielsen and Pérez-Abreu [2] introduce the class of distributions of type mult$G$, which is a generalization on $\mathbb{R}^d$ of the type G on $\mathbb{R}^1$. This is studied also by Maejima and Rosiński [11]. A necessary and sufficient condition for a distribution on $\mathbb{R}^d$ to be of type mult$G$ can be given by using subordination of the canonical $M^+_{d\times d}$-parameter convolution semigroup.

Application 2. Preservation of selfdecomposability, stability, and the L_m property in cone-parameter subordination can be studied.

As to works earlier than [1], we mention that Bochner, [3] pp. 106–108, made a heuristic discussion of cone-parameter convolution semigroups, and that there exist several studies of $\mathbb{R}_+^N$-parameter Lévy processes of the form (ii) or (iii) of Example 2. Hirsch [8] and Khoshnevisan, Xiao and Zhong [9] and others studied the process of the form (ii). Dynkin [4], Evans [6], Fitzsimmons and Salisbury [7] and others worked on processes which generalize the process of the form (iii). The Brownian sheet discussed in many papers (e. g. Orey and Pruitt [12]), the multiparameter stable processes of Ehm [5], and the two-parameter Lévy processes of Lagaize [10] and Vares [15] do not satisfy the condition (ii) in Definition 3 of K-parameter Lévy processes with $K = \mathbb{R}_+^N$, although they satisfy the condition (i).

In the following we give a proof of Theorem 5. Since it uses Theorem 10 (ii) and (iii) and since Theorem 10 (ii) and (iii) are proved from Theorem 10 (i), we also give their proofs.

Proof of Theorem 5. We assume that $K = M_{2\times 2}^+$ and that μ_s has mean 0. We can show that it is enough to give a proof in this case. The covariance matrices v_s satisfy $v_{s^1+s^2} = v_{s^1} + v_{s^2}$ and $v_{ts} = tv_s$. Suppose that there is $\{X_s \colon s \in K\}$, a K-parameter Lévy process in law on $\mathbb{R}^2$ associated with $\{\mu_s\}$. Let us show that we are led to a contradiction. Let

$$e^1 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$$

and let K_0 be the cone generated by e^1 and e^2. Let L_1 and L_2 be the straight lines through 0 generated by $(1, \sqrt{2})^\top$ and $(\sqrt{2}, 1)^\top$, respectively. Since $v_s \leqslant_K s$ and since e^1 and e^2 are of rank 1, $v_{e^1} = t_1 e^1$ and $v_{e^2} = t_2 e^2$ with some $t_1, t_2 \geqslant 0$. Thus $\mathrm{Supp}(\mu_{te^1}) \subset L_1$ and $\mathrm{Supp}(\mu_{te^2}) \subset L_2$ for any $t \geqslant 0$. Hence, applying Theorem 10 (ii) and (iii) to the K_0-parameter Lévy process in law $\{X_s \colon s \in K_0\}$, we see that X_{e^1} and X_{e^2} are independent. Since

$$X_{e^3} - X_{e^1} \stackrel{\mathrm{d}}{=} X_{e^3 - e^1} \quad \text{and} \quad v_{e^3 - e^1} \leqslant_K e^3 - e^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have $(X_{e^3} - X_{e^1})_2 = 0$ a. s. Similarly $(X_{e^3} - X_{e^2})_1 = 0$ a. s. Here $(\cdot)_j$ denotes the jth component. It follows that $(X_{e^3})_2 = (X_{e^1})_2$ a. s. and $(X_{e^3})_1 = (X_{e^2})_1$ a. s. Thus $(X_{e^3})_1$ and $(X_{e^3})_2$ are independent. This means that

$$v_{e^3} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad \text{with some } a_1, a_2 \geqslant 0.$$

Notice that there are $t_3, t_4 \geqslant 0$ such that

$$t_1 \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} = v_{e^1} = v_{e^3} - v_{e^3 - e^1} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - t_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$t_2 \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} = v_{e^2} = v_{e^3} - v_{e^3 - e^2} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - t_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $t_1 = t_2 = a_1 = a_2 = 0$. It follows that $v_{e^1} = v_{e^2} = v_{e^3} = 0$, that is, $\mu_{e^1} = \mu_{e^2} = \mu_{e^3} = \delta_0$. Since $\{e^1, e^2, e^3\}$ is a weak basis of K, we get $\mu_s = \delta_0$ for all $s \in K$ by Theorem 1. This contradicts our assumption of nontriviality. □

Proof of Theorem 10 (i) and (ii). We are given a cone K with a strong basis $\{e^1, \ldots, e^N\}$ and a K-parameter convolution semigroup $\{\mu_s\}$. Let $\{V_t^j \colon t \geqslant 0\}$, $1 \leqslant j \leqslant N$, be independent Lévy processes with $\mathcal{L}(V_1^j) = \mu_{e^j}$ and let $Y_s = V_{s_1}^1 + \cdots + V_{s_N}^N$ for $s = s_1 e^1 + \cdots + s_N e^N$. Then $\{Y_s\}$ is a K-parameter Lévy process associated with $\{\mu_s\}$.

Assume that $\{\mu_s\}$ is unique-generative and let $\{X_s\}$ be a K-parameter Lévy process associated with $\{\mu_s\}$. Then $\{X_s\} \stackrel{\mathrm{d}}{=} \{Y_s\}$. Hence

$$\begin{aligned} &P\,[X_{s_1 e^1 + \cdots + s_N e^N} = X_{s_1 e^1} + \cdots + X_{s_N e^N}] \\ &\qquad = P\,[Y_{s_1 e^1 + \cdots + s_N e^N} = Y_{s_1 e^1} + \cdots + Y_{s_N e^N}] = 1. \end{aligned}$$

Conversely assume that, for every K-parameter Lévy process in law $\{X_s\}$ associated with $\{\mu_s\}$, $X_{s_1e^1+\cdots+s_Ne^N} = X_{s_1e^1} + \cdots + X_{s_Ne^N}$ a. s. Given $0 = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_n$, define

$$Z_{j,k} = X_{s_ne^1+\cdots+s_ne^{j-1}+s_ke^j} \quad \text{for } 1 \leqslant j \leqslant N \text{ and } 0 \leqslant k \leqslant n.$$

Thus $Z_{1,0} = 0$ and $Z_{j,0} = Z_{j-1,n}$ for $j \geqslant 1$. By Definition 3 (i), $Z_{j,k} - Z_{j,k-1}$ with $1 \leqslant j \leqslant N$ and $1 \leqslant k \leqslant n$ are independent. Since, by the assumption,

$$Z_{j,k} = X_{s_ne^1} + \cdots + X_{s_ne^{j-1}} + X_{s_ke^j} \quad \text{a. s.},$$

$X_{s_ke^j} - X_{s_{k-1}e^j}$ with $1 \leqslant j \leqslant N$ and $1 \leqslant k \leqslant n$ are independent. Since n and $0 = s_0 \leqslant s_1 \leqslant \ldots \leqslant s_n$ are arbitrary, we see that $\{X_{te^j} : t \geqslant 0\}$, $1 \leqslant j \leqslant N$, are independent. It follows that $\{X_s\} \stackrel{\mathrm{d}}{=} \{Y_s\}$, which implies that $\{\mu_s\}$ is unique-generative. This shows (i). A proof of (ii) is contained in the argument above. □

Proof of Theorem 10 (iii). Induction in N. If $N = 1$, the assertion is trivially true. Assume that the assertion is true for $N-1$ in place of N. Let $\{\mu_s : s \in K\}$ be such that $\mu_{te^j}(L_j) = 1$ for $1 \leqslant j \leqslant N$ and $t \geqslant 0$. In order to show that $\{\mu_s\}$ is unique-generative, we use Theorem 10 (i). Let $\{X_s : s \in K\}$ be a Lévy process in law associated with $\{\mu_s\}$. Denote by K_1 and K_2 the cones generated by $\{e^2, e^3, \ldots, e^N\}$ and by $\{e^1, e^3, \ldots, e^N\}$, respectively. For $s = s_1e^1 + \cdots + s_Ne^N$, denote $s^1 = s - s_1e^1$ and $s^2 = s - s_2e^2$. Then, for $j = 1, 2$, $\{\mu_s : s \in K_j\}$ is unique-generative by the induction hypothesis and $\{X_s : s \in K_j\}$ is associated with it. Hence, by Theorem 10 (i),

$$X_s = X_{s^1} + (X_s - X_{s^1}) = X_{s_2e^2} + X_{s_3e^3} + \cdots + X_{s_Ne^N} + (X_s - X_{s^1}) \quad \text{a. s.},$$
$$X_s = X_{s^2} + (X_s - X_{s^2}) = X_{s_1e^1} + X_{s_3e^3} + \cdots + X_{s_Ne^N} + (X_s - X_{s^2}) \quad \text{a. s.}$$

Thus

$$(X_s - X_{s^1}) - X_{s_1e^1} = (X_s - X_{s^2}) - X_{s_2e^2} \quad \text{a. s.}$$

The left-hand side is in L_1 a. s. since $X_{s_1e^1} \in L_1$ a. s. and since $X_s - X_{s^1} \stackrel{\mathrm{d}}{=} X_{s-s^1} = X_{s_1e^1}$; similarly the right-hand side is in L_2 a. s. Since $L_1 \cap L_2 = \{0\}$, this implies that $(X_s - X_{s^1}) - X_{s_1e^1} = 0$ a. s. It follows that

$$X_s = X_{s_1e^1} + X_{s_2e^2} + \cdots + X_{s_Ne^N} \quad \text{a. s.}$$

Thus $\{\mu_s\}$ is unique-generative by Theorem 10 (i). □

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Realised power variation and stochastic volatility models

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Abstract

Limit distribution results on realised power variation, that is sums of absolute powers of increments of a process, are derived for certain types of semimartingale with continuous local martingale component, in particular for a class of flexible stochastic volatility models. The theory covers, for example, the cases of realised volatility and realised absolute variation. Such results should be helpful in, for example, the analysis of volatility models using high frequency information.

Some key words: Absolute returns; Mixed asymptotic normality; Realised volatility; p-variation; Quadratic variation; Semimartingale.

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1 Introduction

Stochastic volatility processes play an important role in financial economics, generalising Brownian motion to allow the scale of the increments (or returns in economics) to change through time in a stochastic manner. We show such intermittency can be coherently measured using sums of absolute powers of increments, which we name realised power variation. This paper derives limit theorems for these measures, over a fixed interval of time, as the number of high frequency increments goes off to infinity.

2 Models, notation and regularity conditions

We first introduce some notation for realised power variation quantities of an arbitrary semimartingale x. Let δ be positive real and, for any $t \geq 0$, define

$$x_\delta(t) = x(\lfloor t/\delta \rfloor \, \delta),$$

where $\lfloor a \rfloor$ for any real number a denotes the largest integer less than or equal to a. The process $x_\delta(t)$ is a discrete approximation to $x(t)$. Further, for r positive real we define the *realised power* *variation of order* r[1] or *realised* r*-tic variation* of $x_\delta(t)$ as

$$\begin{aligned} [x_\delta]^{[r]}(t) &= \sum_{j=1}^{M} \left| x_\delta\left(j\delta\right) - x_\delta((j-1)\delta)\right|^r \\ &= \sum_{j=1}^{M} \left| x\left(j\delta\right) - x((j-1)\delta)\right|^r \end{aligned} \tag{2.1}$$

where $M = M(t) = \lfloor t/\delta \rfloor$. Then, in particular, for $M \to \infty$, *realised quadratic variation*

$$[x_\delta]^{[2]}(t) \xrightarrow{p} [x](t),$$

where $[x]$ is the *quadratic variation* process of the semimartingale x. Note also that,

$$[x_\delta]^{[2]} = [x_\delta].$$

Henceforth, for simplicity of exposition, we fix t and take δ so that $M = \lfloor t/\delta \rfloor$ is an integer (and then $\delta M = t$).

Our detailed results will be established for the stochastic volatility (SV) model where basic Brownian motion is generalised to allow the volatility term to vary over time and there to be a rather general drift. Then the y^* follows

$$y^*(t) = \alpha(t) + \int_0^t \sigma(s)\mathrm{d}w(s), \qquad t \geq 0, \tag{2.2}$$

where $\sigma > 0$ and α are assumed to be stochastically independent of the standard Brownian motion w. Throughout this paper we will assume that the processes $\tau = \sigma^2$ and α are of locally bounded variation. This implies that τ and α are locally bounded Riemann integrable functions and that y^* is a semimartingale with a continuous local martingale component. We call σ the *spot volatility* process and α the *mean* or *risk premium* process. (For some general information on processes y^* of this type, see for example [20] and [9]). By allowing the spot volatility to be random and serially dependent, this model will imply its increments will exhibit volatility clustering and have unconditional distributions which are fat tailed. This allows it to be used in finance and econometrics as a model for log-prices. In turn, this provides the basis for option pricing models which overcome some of the major failings in the Black-Scholes option pricing approach. Leading references in this regard include [25], [23] and [35]. See also the recent work of [33].

For the price process (2.2) the *realised power variation of order* r of y^* is, at time t and discretisation δ, $[y^*_\delta]^{[r]}(t)$. Letting

$$y_j(t) = y^*(j\delta) - y^*((j-1)\delta)$$

we have that

$$[y^*_\delta]^{[r]}(t) = \sum_{j=1}^{M} |y_j(t)|^r \,.$$

We use the notation $\tau(t) = \sigma^2(t)$ and

$$\tau^*(t) = \int_0^t \tau(s)\mathrm{d}s$$

[1]The similarly named p-variation, $0 < p < \infty$, of a real-valued function f on $[a,b]$ is defined as

$$\sup_\kappa \sum |f(x_i) - f(x_{i-1})|^p \,,$$

where the supremum is taken over all subdivisions κ of $[a,b]$. If this function is finite then f is said to have bounded p-variation on $[a,b]$. The case of $p=1$ gives the usual definition of bounded variation.

This condition has been studied recently in the probability literature. See the work of, for example, [28] and [30].

and, more generally, we consider the *integrated power volatility of order r*

$$\tau^{r*}(t) = \int_0^t \tau^r(s)\mathrm{d}s.$$

That τ^r is Riemann integrable for every $r > 0$ follows from the assumed locally bounded variation of τ and the fact, due to Lebesgue, that a bounded function f on a finite interval I is Riemann integrable on I if and only if the Lebesgue measure of the set of discontinuity points of f is equal to 0 (see [24, pp. 465–466], [32, p. 174, Theorem 24.4] or [27]). In our case the latter property follows immediately from the bounded variation of τ (any function of bounded variation is the difference between an increasing and a decreasing function and any monotone function has at most countably many discontinuities).

Throughout the following, r denotes a positive number. Moreover we shall refer to the following conditions on the volatility and mean processes:

(V) The volatility process $\tau = \sigma^2$ is (pathwise) locally bounded away from 0 and has, moreover, the property

$$p\text{-}\lim_{\delta\downarrow 0} \delta^{1/2} \sum_{j=1}^{M} |\tau^r(\eta_j) - \tau^r(\xi_j)| = 0 \tag{2.3}$$

for some $r > 0$ (equivalently for all $r > 0$)[2] and for any ξ_j and η_j such that

$$0 \le \xi_1 \le \eta_1 \le \delta \le \xi_2 \le \eta_2 \le 2\delta \le \cdots \le \xi_j \le \eta_j \le M\delta = t.$$

(M) The mean process α satisfies (pathwise)[3]

$$\overline{\lim_{\delta\downarrow 0}} \max_{1\le j\le M} \delta^{-1}|\alpha(j\delta) - \alpha((j-1)\delta)| < \infty. \tag{2.4}$$

These regularity conditions are quite mild.[4] Of some special interest are cases where α is of the form

$$\alpha(t) = \int_0^t g(\sigma(s))\mathrm{d}s,$$

for g a smooth function. Then regularity of τ will imply regularity of α.

Note that the assumptions allow the spot volatility to have, for example, deterministic diurnal effects, jumps, long memory, no unconditional mean or to be non-stationary.

3 Results

Our main theoretical result is

Theorem 1 For $\delta \downarrow 0$ and $r \ge 1/2$, under conditions **(V)** and **(M)**,

$$\mu_r^{-1}\delta^{1-r/2}[y_\delta^*]^{[r]}(t) \xrightarrow{p} \tau^{r/2*}(t) \tag{3.1}$$

and

$$\frac{\mu_r^{-1}\delta^{1-r/2}[y_\delta^*]^{[r]}(t) - \tau^{r/2*}(t)}{\mu_r^{-1}\delta^{1-r/2}\sqrt{\mu_{2r}^{-1}v_r[y_\delta^*]^{[2r]}(t)}} \xrightarrow{\mathcal{L}} N(0,1), \tag{3.2}$$

[2]The equivalence follows on noting that for each j there exists an ω_j with

$$\inf_{(j-1)\delta\le s\le j\delta} \tau(s) \le \omega_j \le \sup_{(j-1)\delta\le s\le j\delta}$$

such that

$$|\tau^r(\eta_j) - \tau^r(\xi_j)| = r\omega_j^{r-1}|\tau(\eta_j) - \tau(\xi_j)|$$

and then using that τ is pointwise bounded away from 0 and ∞.

[3]This condition is implied by Lipschitz continuity and itself implies continuity of α.

[4]Condition **(V)** is satisfied in particular if τ is of OU type, cf. Example 1 below, and condition **(M)** is valid if, for instance, α is the intOU process plus drift, cf. Example 2.

where $\mu_r = \mathrm{E}\{|u|^r\}$ and $v_r = \mathrm{Var}\{|u|^r\}$, with $u \sim N(0,1)$. □

In the proof of this theorem, to be given in the next section, the only place where the assumption $r \geq 1/2$ is needed is where Lemma 3 is invoked.

This theorem tells us that, for $\delta \downarrow 0$, scaled realised power variation converges in probability to integrated power volatility and follows asymptotically a normal variance mixture distribution with variance distributed as

$$\delta \mu_r^{-2} v_r \tau^{r*}(t),$$

which is consistently estimated by the square of the denominator in (3.2). Hence the limit theory is statistically feasible and does not depend upon knowledge of α or σ^2.

Leading cases are *realised quadratic variation*, which is usually called *realised volatility* in the finance and econometrics literature,

$$[y_\delta^*]^{[2]}(t) = \sum_{j=1}^{M} y_j^2(t),$$

in which case

$$\frac{\sum_{j=1}^{M} y_j^2(t) - \tau^*(t)}{\sqrt{\frac{2}{3}\sum_{j=1}^{M} y_j^4(t)}} \xrightarrow{\mathcal{L}} N(0,1), \tag{3.3}$$

and *realised absolute variation*

$$[y_\delta^*]^{[1]}(t) = \sum_{j=1}^{M} |y_j(t)|,$$

when

$$\frac{\sqrt{\pi/2}\sqrt{\delta}\sum_{j=1}^{M} |y_j(t)| - \sigma^*(t)}{\sqrt{(\pi/2-1)\,\delta\sum_{j=1}^{M} y_j^2(t)}} \xrightarrow{\mathcal{L}} N(0,1). \tag{3.4}$$

In the case of $r = 2$ the result considerably strengthens the well known quadratic variation result that realised quadratic variation converges in probability to integrated volatility $\int_0^t \sigma^2(s)\mathrm{d}s$ — which was highlighted in concurrent and independent work by [2] and [9]. The asymptotic distribution of realised quadratic variation was discussed by [10] in the special case where $\alpha(t) = \mu t + \beta \int_0^t \sigma^2(s)\mathrm{d}s$. To our knowledge the probability limit of (normalised) realised absolute variation has not been previously derived, let alone its asymptotic distribution.

Taking sums of squares of increments of log-prices has a very long tradition in financial economics — see, for example, [34], [36], [39], [14], [16], [5] and [4]. However, for a long time no theory was known for the behaviour of such sums outside the Brownian motion case. Since the link to quadratic variation has been made there has been a remarkably fast development in this field. Contributions include [15], [5], [4], [10], [6], [8], [29], [7], [19], [12] and [11].

[3] and [1] empirically studied the properties of $\sum_{j=1}^{M} |y_j(t)|$ computed using sums of absolute values of intra-day returns on speculative assets (many authors in finance have based their empirical analysis on absolute values of returns — see, for example, [38, Ch. 2], [13], [18], [40], [21], [26], [37, Ch. IV] and [22]). This was empirically attractive, for using absolute values is less sensitive to possible large movements in high frequency data. There is evidence that if returns do not possess fourth moments then using absolute values rather than squares would be more reliable (see, for example, the work on the distributional behaviour of the correlogram of squared returns by [17] and [31]). However, the approach was abandoned in their subsequent work reported in [2], [4] and [5] due to the lack of appropriate theory for the sum of absolute returns as $\delta \downarrow 0$, although recently [6] have performed some interesting Monte Carlo studies in this context, while [37, pp. 349–350] mention interests in the limit of sums of absolute returns. Our work provides a theory for the use of sums of absolute returns.

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On the Construction of Feller and Lévy-Type Processes Starting with the Symbol of the Process

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A **Feller process** $\{X_t\}_{t\geq 0}$ is a process with values in $\mathbb{R}^n$ such that the associated operator semigroup

$$T_t u(x) = \mathbb{E}^x u(X_t), \qquad u \in C_\infty(\mathbb{R}^n),$$

on the continuous functions vanishing at infinity, $C_\infty(\mathbb{R}^n)$, is a **Feller semigroup**. This means that $\{T_t\}_{t\geq 0}$ is a strongly continuous family of sub-Markovian contraction operators acting on (in particular: preserving) the space $C_\infty(\mathbb{R}^n)$; examples comprise convolution semigroups which are just the operator semigroups associated with Lévy processes.

Denote by $(A, \mathcal{D}(A))$ the **infinitesimal generator** of the semigroup $\{T_t\}_{t\geq 0}$, that is $Au = \frac{d}{dt} T_t u\big|_{t=0}$ (norm-sense) on its domain $\mathcal{D}(A) \subset C_\infty(\mathbb{R}^n)$. Under the additional assumption that the test functions $C_c^\infty(\mathbb{R}^n) \subset \mathcal{D}(A)$, the following astonishing structure result was proved by Ph. Courrège:

Theorem 1 (Courrège). *Let $(A, \mathcal{D}(A))$ be the generator of a Feller semigroup (or Feller process) such that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{D}(A)$, then the generator, restricted to $C_c^\infty(\mathbb{R}^n)$, is a **pseudo-differential operator**, i.e.,*

$$Au(x) = -p(x, D)u(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\, p(x,\xi)\widehat{u}(\xi)\, d\xi \quad \forall u \in C_c^\infty(\mathbb{R}^n) \qquad (*)$$

*($\widehat{u}(\xi)$ stands for the Fourier transform of $u(x)$) with **symbol** $p(x,\xi)$, where $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. The symbol is locally bounded in both variables and **continuous** and **negative definite** as a function of ξ whenever x is fixed.*

The condition that $\xi \mapsto p(x,\xi)$ is continuous and negative definite is equivalent to saying that we have, for every fixed x, a Lévy-Khinchine represenation:

$$p(x,\xi) = a(x) - i\ell(x)\cdot\xi + \xi\cdot Q(x)\xi + \int\limits_{y\neq 0} \left(1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+|y|^2}\right) N(x, dy) \qquad (**)$$

where, for each fixed $x \in \mathbb{R}^n$, the tuple $(a(x), \ell(x), Q(x), N(x, \bullet))$ is a Lévy tuple.

Using (**) and Fourier inversion we can easily see that the representation (*) takes a slightly more familiar shape (for probabilists...),

$$\begin{aligned} Au(x) = -a(x)u(x) + \ell(x)\nabla u(x) + \sum_{j,k=1}^n q_{jk}(x)\partial_j\partial_k u(x) \\ + \int\limits_{y\neq 0} \left(u(x+y) - u(x) - \frac{y\cdot\nabla u(x)}{1+|y|^2}\right) N(x, dy). \end{aligned}$$

Both representations of the generator, as integro-differential and as pseudo-differential operator, do have (dis-)advantages. It should be mentioned, however, that there is a vast and extremely powerful theory of pseudo-differential operators which can be (partly) used in studying Feller processes.

These are all **necessary** conditions, i.e., we assume that the process or the semigroup are **given**. Of course, in the case of a Lévy process and a convolution semigroup, we just have $p(x,\xi) = \psi(\xi)$—the latter being the characteristic exponent of the Lévy process—and $(a(x), \ell(x), Q(x), N(x, dy))$ also shows no x-dependence, i.e., becomes (a, ℓ, Q, ν). This is the case of **constant coefficient** generators and building on long-established facts it is easy to see that there is a one-to-one correspondence between

- constant-coefficient continuous negative definite functions ψ (i.e., constant-coefficient **symbols**);
- constant-coefficient **pseudo-differential operators** generating Feller semigroups;
- **Lévy processes**.

Not so in the variable coefficient case. Here the situation is much more complicated and only the necessary condition (Courrège's Theorem) is clear. Some 10 years ago, N. Jacob started to investigate this question in detail. He, and subsequently W. Hoh, M. Tsuchiya, A. Negoro, F. Baldus and others found conditions in terms of the symbol $p(x,\xi)$ that were **sufficient** to guarantee the existence (and uniqueness) of Feller and Lévy-type processes. Typically, their conditions involve some smoothness assumptions for $x \mapsto p(x,\xi)$ as well as growth and "ellipticity" assumptions for $\xi \to p(x,\xi)$ as $\xi \to \infty$. The latter are expressed through two-sided comparison estimates of $1 + p(x,\xi)$ against a fixed Lévy-symbol $1 + \psi(\xi)$. This approach is very natural if one knows how to deal with second-order elliptic differential operators with variable coefficients which are in a similar way compared with the Laplacian!

Here is a slight generalization of their criteria.

Theorem 2. *Let $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $(x,\xi) \mapsto p(x,\xi)$, be locally bounded in (x,ξ) and continuous and negative definite as a function of ξ. Assume that*

1. $\sup_{x\in\mathbb{R}^n} |p(x,\xi)| \leq \kappa_p(1 + |\xi|^2) \qquad \forall \xi \in \mathbb{R}^n$;
2. *$\xi \mapsto p(x,\xi)$ is uniformly (in $x \in \mathbb{R}^n$) continuous at $\xi = 0$;*
3. *$x \mapsto p(x,\xi)$ is continuous for all $\xi \in \mathbb{R}^n$.*

Then the operator $(-p(x,D), C_c^\infty(\mathbb{R}^n))$ has an extension which generates a Feller process.

The conditions of the theorem, apart from (1) which is essentially saying that the coefficients of the generator are bounded, are rather close to the conditions known to be necessary for the existence of a Feller process. The proof is essentially a combination of methods of pseudo-differential operators and the Yosida's technique to prove the famous Hille-Yosida Theorem. Notice that Theorem 2 only asserts existence but not uniqueness of the semigroup resp. the process.

Once the symbol of a process is known, we can use it to derive various stochastic properties (Hausdorff dimension, asymptotics of the paths, smoothness of the paths etc.) from it. On the other hand, the above theorem now allows us to guarantee existence of processes with certain properties, a fact, that could be interesting in mathematical modelling.

For Feller processes (and, in fact, for a greater class of Markov processes) the symbol has an interesting **stochastic interpretation**. For Feller processes this can be stated in the following little

Lemma. *Let $\{X_t\}_{t\geq 0}$ be a Feller process such that the test functions $C_c^\infty(\mathbb{R}^n)$ are in the domain of its infinitesimal generator. Then*

$$\frac{d}{dt}\mathbb{E}^x\left(e^{i\xi\cdot(X_t - x)}\right)\bigg|_{t=0} = -p(x,\xi).$$

Note that this formula nicely encapsules the fact that the derivative of the characteristic function of a Lévy process at $t = 0$ is the characteristic exponent. In the general case the proof is far from being trivial since $p(x,\xi)$ is not at all an characteristic exponent; it is, however, the leading term in an **(asymptotic) expansion** of the logarithm of the characteristic function of the process.

A comprehensive bibliography and more information, in particular on stochastic properties and constructions using pseudo-differential methods (not Theorem 2, though...) is given in the survey paper

N. Jacob, R. L. Schilling: *Lévy-Type Processes and Pseudodifferential Operators*. In O. E. Barndorff-Nielsen, T. Mikosch, S. I. Resnick (eds.): *Lévy Processes—Theory and Applications*, Birkhäuser, Boston 2001, 139–168.

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The Meixner Process: Theory and Applications in Finance

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Financial mathematics has recently enjoyed considerable prestige on account of its impact on the finance industry. In parallel, the theory of Lévy processes has also seen exciting developments in recent years [2] [4] [16]. The fusion of these two fields of mathematics has provided new applied modeling perspectives within the context of finance and further stimulus for deep and intrinsically interesting problems within the context of Lévy processes.

We will focus on one particular Lévy process: The Meixner process. The Meixner process originates from the theory of orthogonal polynomials. Its underlying distribution, the Meixner distribution, is the measure of orthogonality of the Meixner-Pollaczek polynomials. The Meixner process was introduced in [17] (see also [18]) and was proposed to serve as a model of financial data in [10] (see also [19]).

The density of the Meixner distribution (Meixner(a, b, d, m)) is given by

$$f(x; a, b, m, d) = \frac{(2\cos(b/2))^{2d}}{2a\pi\Gamma(2d)} \exp\left(\frac{b(x-m)}{a}\right) \left|\Gamma\left(d + \frac{\mathrm{i}(x-m)}{a}\right)\right|^2,$$

where $a > 0, -\pi < b < \pi, d > 0$, and $m \in \mathbb{R}$.

The characteristic function of the Meixner(a, b, d, m) distribution is given by

$$E\left[\exp(\mathrm{i}uM_1)\right] = \left(\frac{\cos(b/2)}{\cosh\frac{au-\mathrm{i}b}{2}}\right)^{2d} \exp(\mathrm{i}mu)$$

Clearly, the Meixner(a, b, d, m) distribution is infinitely divisible and we can associate with it a Lévy process which we call the Meixner process. More precisely, a Meixner process $\{M_t, t \geq 0\}$ is a stochastic process which starts at zero, i.e. $M_0 = 0$, has independent and stationary increments, and where the distribution of M_t is given by the Meixner distribution Meixner(a, b, dt, mt). It is easy to show that our Meixner process $\{M_t, t \geq 0\}$ has no Brownian part and a pure jump part governed by the Lévy measure

$$\nu(\mathrm{d}x) = d\frac{\exp(bx/a)}{x\sinh(\pi x/a)}\mathrm{d}x.$$

Because $\int_{-\infty}^{+\infty} |x|\nu(\mathrm{d}x) = \infty$ it follows from standard Lévy process theory [4] [16], that our process is of infinite variation.

Moments of all order of the Meixner(a, b, d, m) distribution exist. Next, we give some relevant quantities; similar, but more involved, expressions exist for the moments and the skewness.

	Meixner(a, b, d, m)	**Normal**(μ, σ^2)
mean	$m + ad\tan(b/2)$	μ
variance	$\frac{a^2d}{2}(\cos^{-2}(b/2))$	σ^2
kurtosis	$3 + \frac{3-2\cos^2(b/2)}{d}$	3

One can clearly see that the kurtosis of the Meixner distribution is always greater than the Normal kurtosis.

A number of stylized features of observational series from finance are discussed in [3]. One of this features is the semihaviness of the tails. Our Meixner(a,b,d,m) distribution has semiheavy tails [11]. This means that the tails of the density function behave as

$$\begin{aligned} f(x,a,b,d,m) &\sim C_-|x|^{\rho_-}\exp(-\sigma_-|x|) \quad \text{as} \quad x\to-\infty \\ f(x,a,b,d,m) &\sim C_+|x|^{\rho_+}\exp(-\sigma_+|x|) \quad \text{as} \quad x\to+\infty, \end{aligned}$$

for some $\rho_-,\rho_+\in\mathbb{R}$ and $C_-,C_+,\sigma_-,\sigma_+\geq 0$. In case of the Meixner$(a,b,d,m)$,

$$\rho_-=\rho_+=2d-1, \quad \sigma_-=(\pi-b)/a, \quad \sigma_+=(\pi+b)/a.$$

The Meixner process ($a=1$, $m=0$, $d=1$, $\zeta=(b+\pi)/2$) is also connected with the monic Meixner-Pollaczek polynomials $\{\tilde{P}_m, m=0,1,\dots\}$ [12] by a martingale relation:

$$E[\tilde{P}_m\,(M_t;t,\zeta)\mid M_s]=\tilde{P}_m\,(M_s;s,\zeta)$$

Note the similarity with the classical martingale relation between standard Brownian motion $\{W_t,\geq 0\}$ and the Hermite Polynomials $\{H_m(x;\sigma), m=0,1,\dots\}$ [18]:

$$E\left[\tilde{H}_m(W_t;t)\mid W_s\right]=\tilde{H}_m(W_s;s)$$

The Meixner distribution can be seen as a special case of the Generalized z-distributions: The Generalized z-distribution (GZ) [11] is defined through the characteristic function:

$$\phi_{GZ}(z;a,b_1,b_2,d,m)=\left(\frac{B(b_1+\frac{\mathrm{i}az}{2\pi},b_2-\frac{\mathrm{i}az}{2\pi})}{B(b_1,b_2)}\right)^{2d}\exp(\mathrm{i}mz),$$

where $a,b_1,b_2,d>0$ and $m\in\mathbb{R}$.

For

$$b_1=\frac{1}{2}+\frac{b}{2\pi} \text{ and } b_2=\frac{1}{2}-\frac{b}{2\pi},$$

we obtain the Meixner Process. Note that the Generalized z-distributions and the Generalized Hyperbolic distribution [9] [15] are non-intersecting sets.

The Meixner Process is also related to the process studied by Biane, Pitman and Yor [5] (see also [14]):

$$C_t=\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\Gamma_{n,t}}{(n-\frac{1}{2})^2},$$

for a sequence of independent Gamma Processes $\Gamma_{n,t}$, i.e. Lévy process with $E[\exp(\mathrm{i}\theta\Gamma_{n,t})]=(1-\mathrm{i}\theta)^{-t}$.

In [5] one shows that C_t has Laplace transform

$$E[\exp(-uC_t)]=\left(\frac{1}{\cosh\sqrt{2u}}\right)^t$$

This means that the Brownian time change B_{C_t} has characteristic function

$$E[\exp(\mathrm{i}uB_{C_t})]=\left(\frac{1}{\cosh u}\right)^t,$$

or equivalently B_{C_t} follows a Meixner$(2,0,t,0)$ distribution.

We will apply the Meixner distribution and the Meixner process in the context of mathematical finance. More precisely, we will use the process to model the stochastic behaviour of financial assets like stocks or indices. The most famous continuous-time model for stock prices or indices is the celebrated Black-Scholes model [6]. It uses the Normal distribution to fit the log-returns of the underlying: the price process of the underlying is given by the geometric Brownian Motion

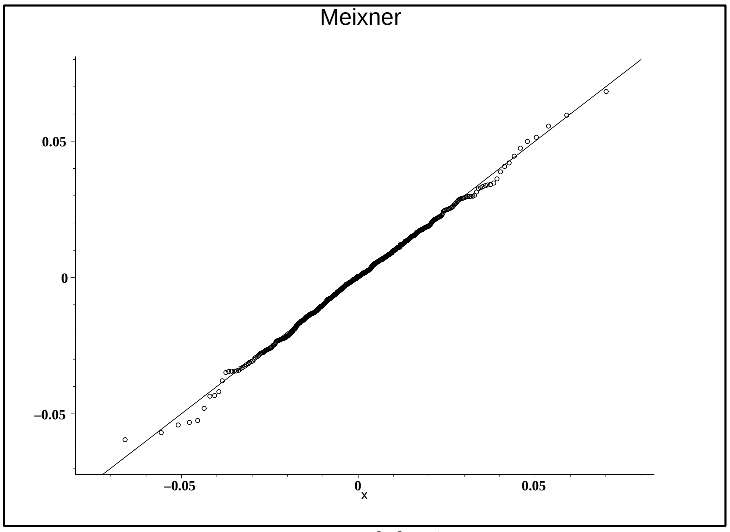
$$S_t=S_0\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B_t\right),$$

where $\{B_t, t \geq 0\}$ is standard Brownian motion, i.e. B_t follows a Normal distribution with mean 0 and variance t. It is well known however that the log-returns of most financial assets have an actual kurtosis that is higher than that of the Normal distribution. In this paper we therefore propose another model which is based on the Meixner distribution.

In the late 1980s and in the 1990s several other similar process models where proposed. Madan and Seneta [13] have proposed a Lévy process with Variance Gamma distributed increments. We mention also the Hyperbolic Model [9] proposed by Eberlein and Keller and their generalizations [15]. In the same year Barndorff-Nielsen proposed the Normal Inverse Gaussian Lévy process [1]. Recently the CMGY model was introduced [7]. All models give a much better fit to the data and lead to an improvement with respect to the Black-Scholes model. We provide statistical evidence that the Meixner model performs also significantly better then the Black-Scholes Model.

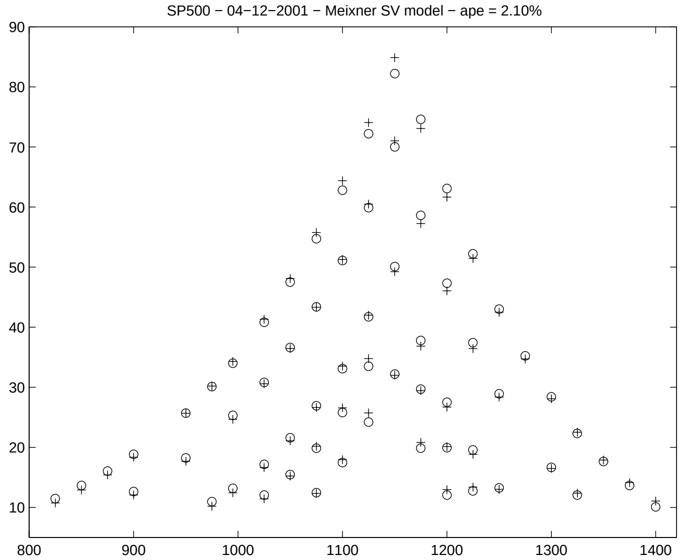
A second application can be found in the same context: the pricing of financial derivatives. First we will try to price derivatives using a model where the Brownian motion of the BS-model is just replace by a Lévy process. Although there is a significant improvement in accuracy with respect to the BS-model, there still is a discrepantion between model prices and market prices. The main feature which these Lévy models are missing, is the fact that the volatility or more general the environment is changing stochastically over time. In order to deal with this problem, we make (business) time stochastic as proposed in [8]. We show that by following the procedure of [8], we can almost perfectly calibrate model prices of the Meixner model with stochastic business time, also called the Meixner Stochastic Volatility model (Meixner-SV model), to market prices.

To illustrate the applications, we make use of two data sets. The first data set consist of the log-returns of the Nikkei-225 Index during a period of three years. We show that the Meixner distribution can be fitted much more accurate to this set than the Normal distribution using χ^2-test and QQ-plots.



Meixner QQ-plot

A second data set consists of the mid-prices of a set of European call and put options on the SP500-index at the close of the market on the 4th of December 2001. We will calibrate models based on the Meixner process to our set of option prices. We will show that the Meixner-SV model leads to option prices which can be calibrated almost perfectly to the market prices:



Meixner SV option prices (o's are market prices and +'s are model prices)

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Fock Space Decomposition of Lévy Processes

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1 Cyclic representations of groups

Let G be a group and $g \mapsto U_g$ a multiplier cyclic representation of G on a Hilbert space $\mathcal{H}$, with multiplier $\sigma : G \times G \to \mathbf{C}$ and cyclic vector Ψ. This means that

- $U_g U_h = \sigma(g,h) U_{gh}$ for all $g, h \in G$.
- $U(e) = I$ where e is the identity of the group and I is the identity operator on $\mathcal{H}$.
- Span $\{U_g \Psi : g \in G\}$ is dense in $\mathcal{H}$.

If $\sigma = 1$ we say that U is a true representation.

Recall that a multiplier of a group G is a measurable two-cocycle in $Z^2(G, U(1))$; so σ is a map $G \times G \to U(1)$ such that $\sigma(e,g) = \sigma(g,e) = 1$ and

$$\sigma(g,h)\sigma(g,hk)^{-1}\sigma(gh,k)\sigma(h,k)^{-1} = 1. \tag{1.1}$$

Here, $U(1)$ denotes the unit circle in the complex plane, which is a group under multiplication. We note that (1.1) expresses the associativity of operator multiplication of the $U(g)$. From this, or directly from (1.1), we discover that the set of two-cocycles form an abelian group denoted $Z^2(G, U(1))$. We say that σ is a two-coboundary if there is a map $b : G \to U(1)$ with $b(e) = 1$ and

$$\sigma(g,h) = b(gh)/(b(g)b(h)). \tag{1.2}$$

The product of coboundaries is also a coboundary, and they form a group $B^2(G, U(1))$.

We also need the concept of a one cocycle ψ in a Hilbert space $\mathcal{K}$ carrying a unitary representation V. A cocycle ψ is a map $G \to \mathcal{K}$ such that

$$V(g)\psi(h) = \psi(gh) - \psi(g) \qquad \text{for } g, h \in G. \tag{1.3}$$

The cocycles form an abelian group under addition, using the vector structure of $\mathcal{K}$; this group is denoted $Z^1(\mathcal{K}, V)$. We say that ψ is a coboundary if there is a vector $\psi_0 \in \mathcal{K}$ such that

$$\psi(g) = (V(g) - I)\psi_0. \tag{1.4}$$

Again, the coboundaries form an abelian group, here denoted $B^1(\mathcal{K}, V)$. We call V the action of G on the space $\mathcal{K}$. In these terms, for multipliers the action of G on $U(1)$ is trivial. Coboundaries are always cocycles, so the coboundary group is an invariant subgroup of the cocycle group. We say that, in (1.2) and (1.4), σ is the coboundary of b and ψ is the coboundary of ψ_0. The *cohomology group* is the quotient group $H = Z/B$, and we say that a group has non-trivial cohomology (of a given action V and degree) if H consists of more than the identity element.

We say that two cyclic σ-representations $\{\mathcal{H}, U, \Psi\}$ and $\{\mathcal{K}, V, \Phi\}$ are *cyclically equivalent* if there exists a unitary operator $W : \mathcal{H} \to \mathcal{K}$ such that $V_g = W U_g W^{-1}$ for all $g \in G$, and $W\Psi = \Phi$. Any cyclic multiplier representation $\{\mathcal{H}, U, \Psi\}$ defines a function F on the group by

$$F(g) := \langle \Psi, U_g \Psi \rangle, \tag{1.5}$$

which satisfies σ-positivity:

$$F(e) = 1 \tag{1.6}$$

$$\sum_{ij} \overline{\lambda}_i \lambda_j \sigma(g_i^{-1}, g_j) F(g_i^{-1} g_j) \geq 0. \tag{1.7}$$

F is called the characteristic function of the representation, because

- Two cyclic multiplier representations of G are cyclically equivalent if and only if they have the same characteristic function;
- Given a function on G satisfying σ-positivity, then there exists a cyclic σ-representation of which it is the characteristic function.

If $G = \{s \in \mathbf{R}\}$, $\sigma = 1$ and U_s is continuous, then F obeys the hypotheses of Bochner's theorem and defines a probability measure μ on $\mathbf{R}$. More generally, we can apply Bochner's theorem (if $\sigma = 1$) to any one-parameter subgroup $s \mapsto g(s) \in G_0 \subseteq\in G$. Then $U_{g(s)},\ s \in \mathbf{R}$ is a one-parameter unitary group; its infinitesimal generator is a self-adjoint operator X on $\mathcal{H}$. The relation to μ is given as follows: let $X = \int \lambda dE(\lambda)$ be the spectral resolution of X. Then

$$\mu(\lambda_1, \lambda_2] = \langle \Psi, (E(\lambda_2) - E(\lambda_1))\, \Psi \rangle. \tag{1.8}$$

Conversely, given any random variable X on a probability space (Ω, μ), we can define the cyclic unitary representation of the group $\mathbf{R}$ by the multiplication operator

$$U(s) = \exp\{isX\} \tag{1.9}$$

and use the cyclic vector $\Psi(\omega) = 1$ on the Hilbert space $L^2(\Omega, d\mu)$. In this way, probability theory is reduced to the study of cyclic representations of abelian groups, and quantum probability to that cyclic σ-representations of non-abelian groups.

2 Processes as Tensor Products

Given a cyclic σ-representation $\{\mathcal{H}, U, \Phi\}$ of a group G, we can get a multiplier representation of the product group $G^n := G \times G \times \ldots \times G$ (n factors) on $\mathcal{H} \otimes \mathcal{H} \ldots \otimes \mathcal{H}$, by acting on the vector $\Psi \otimes \Psi \ldots \otimes \Psi$ by the unitary operators $U(g_1, \ldots, g_n) := U(g_1) \otimes \ldots U(g_n)$, as each g_j runs over the group G. The resulting cyclic $\sigma^{\otimes n}$-representation is denoted

$$\left\{\mathcal{H}^{\otimes n}, U^{\otimes n}, \Psi^{\otimes n}\right\}. \tag{2.1}$$

The twisted positive function on G^n defined by this cyclic representation is easily computed to be

$$F^{\otimes n}(g_1, \ldots, g_n) = F(g_1)F(g_2) \ldots F(g_n). \tag{2.2}$$

If G has a one-parameter subgroup G_0, then the infinitesimal generators X_j of this subgroup in the j^{th} place define random variables ($j = 1, \ldots, n$) that are all independent in the measure $\mu^{\otimes n}$ on $\mathbf{R}^n$ defined by $F^{\otimes n}$, and are all identically distributed. They can thus be taken as the increments of a process in discrete time $t = 1, \ldots, n$. To get a process with time going to infinity, we can embed each tensor product $\mathcal{H}^{\otimes n}$ in the "incomplete infinite tensor product" of von Neumann, denoted

$$\bigotimes_{j=1}^{\infty} {}^{\Psi}\mathcal{H}_j \qquad \text{where } \mathcal{H}_j = \mathcal{H} \text{ for all } j. \tag{2.3}$$

It is harder to construct processes in continuous time. We made [12] the following definition:

Definition 2.1. A cyclic G-representation $\{\mathcal{H}, U, \Psi\}$ is said to be *infinitely divisible* if for each positive integer n there exists another cyclic G-representation $\{\mathcal{K}, V, \Phi\}$ such that $\{\mathcal{H}, U, \Psi\}$ is cyclically equivalent to $\{\mathcal{K}^{\otimes n}, V^{\otimes n}, \Phi^{\otimes n}\}$.

The picturesque notation $\left\{\mathcal{H}^{\otimes\frac{1}{n}}, U^{\otimes\frac{1}{n}}, \Psi^{\otimes\frac{1}{n}}\right\}$ can be used for $\{\mathcal{K}, V, \Phi\}$.

If $G = \mathbf{R}$ then $\{\mathcal{H}, U, \Psi\}$ is infinitely divisible if and only if the corresponding measure μ given by Bochner's theorem is infinitely divisible [12]. It is clear that $\{\mathcal{H}, U, \Psi\}$ is infinitely divisible if and only if there exists a branch of $F(g)^{\frac{1}{n}}$ which is positive semi-definite on G.

This criterion was extended in [10] to σ-representations. In that case, for each n, there should exist an n^{th} root $\sigma(g,h)^{\frac{1}{n}}$ which is also a multiplier. One can then consider cyclic representations such that for each n, $F(g)^{\frac{1}{n}}$ has a branch which is $\sigma^{\frac{1}{n}}$-positive semi-definite.

If $\{\mathcal{H}, U, \Psi\}$ is an infinitely divisible G-representation, then we may construct a continuous tensor product of the Hilbert spaces $\mathcal{H}_t$, where $t \in \mathbf{R}$ and all the Hilbert spaces are the same. This gives us, in the non-abelian case, quantum stochastic processes with independent increments. The possible constructions are classified in terms of cocycles of the group G. Here we shall limit discussion to the analysis of the Lévy formula in these terms.

3 The cocycle

Let $F : G \to \mathbf{C}$ and $F(e) = 1$. It is a classical result for $G = \mathbf{R}$ that a function $F^{\frac{1}{n}}$ has a branch that is positive semidefinite for all $n > 0$ if and only if $\log F$ has a branch f such that $f(0) = 0$ and f is *conditionally* positive semidefinite. This is equivalent to $f(x-y) - f(x) - f(-y)$ being positive semidefinite. This result is easily extended to groups [12] and σ-representations [10, 11]. Let us consider the case where $\sigma = 1$. It follows that an infinitely divisible true cyclic representation $\{\mathcal{H}, U, \Psi\}$ of G defines a conditionally positive semidefinite function $f(g) = \log\langle\Psi, U(g)\Psi\rangle$, so that

$$\sum_{j,k} \overline{\alpha}_j \alpha_k \left(f\left(g_j^{-1} g_k\right) - f(g_j)^{-1} - f(g_k)\right) \geq 0. \tag{3.1}$$

We can use this positive semidefinite form to make $\operatorname{Span} G$ into a pre-scalar product space, by defining

$$\langle\psi(g), \psi(h)\rangle := f(g^{-1}h) - f(g^{-1}) - f(h), \qquad g, h \in G. \tag{3.2}$$

Let $\mathcal{K}$ be the Hilbert space, that is the separated and completed space got this way. There is a natural injection $\psi : G \to \mathcal{K}$, namely, $g \mapsto [g]$, the equivalence class of g given by the relation $g \sim h$ if the seminorm defined by (3.1) vanishes on $g - h$. The left action of the group G on this function is not quite unitary; in fact the following is a unitary representation [1]:

$$V(h)\psi(g) := \psi(hg) - \psi(h). \tag{3.3}$$

One just has to check from (3.2) that the group law $V(g)V(h) = V(gh)$ holds, and that

$$\langle V(h)\psi(g_1), V(h)\psi(g_2)\rangle = \langle\psi(g_1), \psi(g_2)\rangle. \tag{3.4}$$

Thus we see that $\psi(g)$ is a one-cocycle relative to the G-representation V [1].

4 The embedding theorem

Given a Hilbert space $\mathcal{K}$, the *Fock space* defined by $\mathcal{K}$ is the direct sum of all symmetric tensor products of $\mathcal{K}$,

$$EXP\,\mathcal{K} := \mathbf{C} \bigoplus \mathcal{K} \bigoplus (\mathcal{K} \otimes \mathcal{K})_s \bigoplus \ldots. \tag{4.1}$$

The element $1 \in \mathbf{C}$ is called the Fock vacuum. The following *coherent states* form a total set in $EXP\,\mathcal{K}$:

$$EXP\,\psi := 1 + \psi + (1/2!)\psi \otimes \psi + \ldots, \qquad \psi \in \mathcal{K}. \tag{4.2}$$

The notation is natural, in view of the easy identity

$$\langle EXP\,\psi(g), EXP\,\psi(h)h\rangle = \exp\{\langle\psi(g), \psi(h)\rangle\}. \tag{4.3}$$

Fock space also has the functorial property

$$EXP\,(\mathcal{H}\oplus\mathcal{K}) = EXP\,\mathcal{H}\otimes EXP\,\mathcal{K}, \tag{4.4}$$

the equality being given by an isomorphism that intertwines the number operators. In particular, it maps the vacua to each other.

Then the embedding theorem [12] says that if $\{\mathcal{H}, U, \Psi\}$ is an infinitely divisible cyclic representation, then it is cyclically equivalent to the cyclic representation W on $EXP\,\mathcal{K}$, with the Fock vacuum as the cyclic vector, with the unitary representation $W(h)$ defined on the total set of coherent states by

$$W(h)EXP\,\psi(g) = F(hg)/F(g)EXP\,\psi(hg). \tag{4.5}$$

The proof is simply a verification. An immediate consequence is the occurrence of orthogonal polynomials associated with any infinitely divisible random variable X: the n^{th} symmetric tensor product in the Fock space is a polynomial in X orthogonal to the m^{th} tensor product, if $n \neq m$. This construction has been called [12, 10] the Araki-Woods embedding theorem, but it does not actually appear in [2]; more properly this name belongs to the embedding [12] of the *process* that one constructs from $\{\mathcal{H}, U, \Phi\}$, which is reminiscent of a deep result in [2]. The coherent vectors in the continuous tensor product lead to exponential martingales [7]. For the group $\mathbf{R}$ with Gaussian cocycle the expansion of a martingale into its n-particle components amounts to its decomposition into Wiener chaos.

5 The Lévy Formula

Every multiplier for the group $\mathbf{R}$ is a coboundary. This has the consequence that every projective representation of $\mathbf{R}$ can be implemented by a true representation, which is multiplicity free if it is cyclic. By reduction theory, it is then determined by a measure on the dual group, here $\mathbf{R}$. Araki [1] showed that a one-cocycle can be algebraic or topological. The topological cocycles are of the form

$$\psi(g) = (U(g) - I)\psi_0 \tag{5.1}$$

where ψ_0 need not be in the Hilbert space, but $\psi(g)$ is. For the group $\mathbf{R}$, the algebraic cocycles are all of the form $f(x-y) - f(x) - f(-y) = axy$. This is satisfied by the Gaussian term $\log F(x) = -\frac{a}{2}x^2 + ibx$, and this is the only possibility. The Poisson(λ) is an example of a coboundary, when $\log F(t) = c\lambda(e^{ipt} - 1)$ for some p, the increment of the jumps. The weighted mixture of these coboundaries gives di Finetti's formula [3]:

$$\log F(t) = \lambda\left\{ibt - \frac{a^2t^2}{2} + c\int\left(e^{ipt} - 1\right)dP(p)\right\}. \tag{5.2}$$

That this is not the most general infinitely divisible measure was recognised by Kolmogorov [8]. In our terms, this is the statement that not all topological cocycles are coboundaries (the topological cohomology is non-trivial). Kolmogorov considered random variables with finite variance relative to the measure dP. This is equivalent in our terms to $dP = |\widehat{\psi}(p)|^2 dp$ and the cocycle ψ being of the form $\psi(x) = (V(x) - I)\psi_0$, where $i\partial_x\psi_0$ is square integrable over the group $\mathbf{R}$, but ψ_0 might not be. This can be expressed by saying that ψ is a cocycle for the Lie algebra of the group, a case treated in general in [14]. This gives us Kolmogorov's formula

$$\log F(t) = \lambda\left\{ibt - a^2t^2/2 + \int\left(e^{ipt} - 1 - itp\right)|\psi(p)|^2 dp\right\}. \tag{5.3}$$

The term $\int(-itp)|\psi(p)|^2 dp$ is possibly divergent near $p = 0$ but is not required to exist on its own near $p = 0$, since the function $M = e^{ipt} - 1 - ipt$ behaves as p^2 near the origin. But to retain a meaning, Kolmogorov's formula does need $p|\widehat{\psi}(p)|^2$ to be integrable at infinity. This is not needed for the general cocycle, so the formula is not the most general. Lévy gave the answer [9] by replacing M by

$$e^{ipt} - 1 - ipt/(1 + p^2), \tag{5.4}$$

so that $\widehat{\psi}$ has no constraint at infinity other than being L^2. Lévy, in effect, constructed the most general cocycle of the group $\mathbf{R}$.

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The Lévy-Itô Decomposition in Free Probability

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1 About Free Probability

In classical probability, the basic objects of study are random variables X defined on a probability space $(\Omega, \mathcal{F}, P)$. Any such random variable X gives rise to a probability measure μ_X on $\mathbb{R}$ determined by the expression:

$$\int_{\mathbb{R}} f(t)\ \mu_X(dt) = \mathbb{E}(f(X)), \qquad (f \in \mathcal{B}_b(\mathbb{R})),$$

where $\mathcal{B}_b(\mathbb{R})$ is the space of bounded Borel functions $f\colon \mathbb{R} \to \mathbb{R}$, and where $\mathbb{E}$ denotes expectation (or integration) w.r.t. P. We call μ_X the distribution of X and denote it also by $L\{X\}$.

In non-commutative probability, the random variables are replaced by operators on a Hilbert space. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the vector space of continuous linear mappings $T\colon \mathcal{H} \to \mathcal{H}$. Recall that there is also a (non-commutative) multiplication on $\mathcal{B}(\mathcal{H})$ given by $ab = a \circ b$ (composition of mappings).

The expectation of an operator a in $\mathcal{B}(\mathcal{H})$ is the value at a of a specified state on $\mathcal{B}(\mathcal{H})$. More concretely, let ξ_0 be a unit vector in $\mathcal{H}$, and then consider the vector state $\tau\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ given by

$$\tau(a) = \langle a\xi, \xi\rangle, \quad (a \in \mathcal{B}(\mathcal{H})).$$

Then we consider $\tau(a)$ as the expectation of a w.r.t. τ. Note that $\tau(a) \geq 0$ if $a \geq 0$ and that $\tau(\mathbf{1}) = 1$. Having specified a (vector) state on $\mathcal{B}(\mathcal{H})$, we can associate a corresponding distribution to any selfadjoint (or hermitian) operator in $\mathcal{B}(\mathcal{H})$: Let a be a selfadjoint operator in $\mathcal{B}(\mathcal{H})$, i.e.

$$\langle a\xi, \eta\rangle = \langle \xi, a\eta\rangle, \qquad (\xi, \eta \in \mathcal{H}).$$

Then there exists a unique probability measure μ_a on $(\mathbb{R}, \mathcal{B})$, such that

$$\int_{\mathbb{R}} f(t)\ \mu_a(dt) = \tau(f(a)), \qquad (f \in \mathcal{B}_b(\mathbb{R})),$$

where $f(a)$ is defined in terms of spectral theory (in particular, $f(a)$ has the obvious meaning if f is a polynomial). We call μ_a the distribution of a w.r.t. τ, and denote it also by $L\{a\}$. Since a is bounded, μ_a is compactly supported (in fact, $\mathrm{supp}(\mu_a)$ is contained in the spectrum of a). If one wants to consider, in the non-commutative setting, distributions with unbounded support, one has to allow for a to be unbounded (i.e. non-continuous). In that case, the equation

$$\int_{\mathbb{R}} f(t)\ \mu_a(dt) = \tau(f(a)), \qquad (f \in \mathcal{B}_b(\mathbb{R})),$$

determines, again, a unique probability measure $\mu_a = L\{a\}$ on $\mathbb{R}$,which, in general, has unbounded support. Any probability measure on $\mathbb{R}$ can be realized as the distribution of a (possibly unbounded) selfadjoint operator.

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Example 1.1. Consider the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and recall that $\mathcal{B}(\mathcal{H}) \simeq M_n(\mathbb{C})$ (the $n \times n$ matrices over $\mathbb{C}$). Consider the state[1] $\mathrm{tr}_n \colon M_n(\mathbb{C}) \to \mathbb{C}$ defined by:

$$\mathrm{tr}_n[(a_{ij})] = \frac{1}{n}\sum_{i=1}^{n} a_{ii}, \qquad ((a_{ij}) \in M_n(\mathbb{C})).$$

Let a be a selfadjoint matrix in $M_n(\mathbb{C})$ (i.e. $a = a^* = (\overline{a})^t$). Then the distribution μ_a of a w.r.t. tr_n is given by:

$$\mu_a = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the (real) eigenvalues of a repeated according to multiplicity.

Recall next that two (classical) random variables X and Y on $(\Omega, \mathcal{F}, P)$ are independent if and only if

$$\mathbb{E}\big\{[f(X) - \mathbb{E}\{f(X)\}] \cdot [g(Y) - \mathbb{E}\{g(Y)\}]\big\} = 0, \qquad (f, g \in \mathcal{B}_b(\mathbb{R})). \tag{1.1}$$

The notion of free independence was introduced by D.V. Voiculescu in the early 1980's.

Definition 1.2. Let a_1, a_2 be selfadjoint operators in $\mathcal{B}(\mathcal{H})$ and let $\tau \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a (vector) state. Then a_1, a_2 are *freely independent* w.r.t. τ if the following condition is satisfied:

$$\begin{gathered}\begin{cases} k \in \mathbb{N},\ i_1, i_2, \dots, i_k \in \{1, 2\}, \\ i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k, \\ f_1, f_2, \dots, f_k \in \mathcal{B}_b(\mathbb{R}), \end{cases} \\ \Downarrow \\ \tau\big\{[f_1(a_{i_1}) - \tau(f_1(a_{i_1}))] \cdot [f_2(a_{i_2}) - \tau(f_2(a_{i_2}))] \cdots [f_k(a_{i_k}) - \tau(f_k(a_{i_k}))]\big\} = 0.\end{gathered} \tag{1.2}$$

Although condition (1.2) may seem rather similar to (1.1), the non-commutativity of the multiplication among the appearing operators makes free independence a quite different notion compared to classical independence. In particular, it is crucial that condition (1.2) involves products of arbitrary length k.

Example 1.3 (Classical random variables are "never" freely independent). Let X and Y be random variables on $(\Omega, \mathcal{F}, P)$ and assume, for simplicity, that X and Y are (essentially) bounded. If X and Y are freely independent w.r.t. $\mathbb{E}$, then one of them has to be a constant (almost surely). Indeed, assume that X and Y are freely independent w.r.t. $\mathbb{E}$ and put $m_X = \mathbb{E}(X)$ and $m_Y = \mathbb{E}(Y)$. Then by free independence we have:

$$\begin{aligned} 0 &= \mathbb{E}\big\{(X - m_X)(Y - m_Y)(X - m_X)(Y - m_Y)\big\} \\ &= \mathbb{E}\big\{(X - m_X)^2 (Y - m_Y)^2\big\} \\ &= \mathbb{E}\{(X - m_X)^2\}\mathbb{E}\{(Y - m_Y)^2\}, \end{aligned}$$

where the last equality is another consequence of the free independence. Hence, either X or Y has to be a constant (almost surely).

2 Voiculescu's Random Matrix Model.

One motivation for studying free independence is its interpretation as the asymptotic appearance of classical independence among large random matrices with complex entries.

[1] This state is not a vector state, but it is a convex combination of vector states.

Definition 2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. By SGRM(n, σ^2) we denote the set of random $n \times n$ matrices $X = (x_{ij})_{1 \le i,j \le n}$, defined on $(\Omega, \mathcal{F}, P)$, which satisfies the following conditions:

- $\forall i \geq j\colon x_{ij} = \overline{x_{ji}}$.
- the random variables x_{ij}, $1 \le i \le j \le n$, are independent.
- $\forall i < j\colon \mathrm{Re}(x_{ij}), \mathrm{Im}(x_{ij}) \sim$ i.i.d. $N(0, \frac{1}{2}\sigma^2)$.
- $\forall i\colon x_{ii} \sim N(0, \sigma^2)$.

Apart from providing the above mentioned interpretation of free independence, the following theorem, due to D.V. Voiculescu, generalizes E.P Wigner's famous semi-circle law.

Theorem 2.2 ([6]). *For each n in $\mathbb{N}$, let $X_1^{(n)}, \dots, X_r^{(n)}$ be independent random matrices in* SGRM$(n, \frac{1}{n})$*. Then for any p in $\mathbb{N}$ and $i_1, i_2, \dots, i_p$ in $\{1, 2, \dots, r\}$,*

$$\mathbb{E} \circ \mathrm{tr}_n \big[X_{i_1}^{(n)} X_{i_2}^{(n)} \cdots X_{i_p}^{(n)}\big] \xrightarrow[n \to \infty]{} \tau(x_{i_1} x_{i_2} \cdots x_{i_p}),$$

where

- *$x_1, x_2, \dots, x_r$ are selfadjoint, freely independent operators in $(\mathcal{B}(\mathcal{H}), \tau)$.*
- *$L\{x_i\} = \frac{1}{2\pi}\sqrt{4 - t^2} \cdot 1_{[-2,2]}(t)\ dt, \quad (i = 1, 2, \dots, r)$.*

3 Free convolution and infinite divisibility

Let μ_1 and μ_2 be probability measures on $\mathbb{R}$. Recall then that the (classical) convolution $\mu_1 * \mu_2$ of μ_1 and μ_2 is defined as follows: Consider *independent* (classical) random variables X_1 and X_2, such that $L\{X_1\} = \mu_1$ and $L\{X_2\} = \mu_2$. Then $\mu_1 * \mu_2 = L\{X_1 + X_2\}$.

Definition 3.1. Let μ_1 and μ_2 be probability measures on $\mathbb{R}$. Then their *free (additive) convolution* $\mu_1 \boxplus \mu_2$ is defined as follows: Choose *freely independent* selfadjoint operators x_1 and x_2, such that[2] $L\{x_1\} = \mu_1$ and $L\{x_2\} = \mu_2$. Then

$$\mu_1 \boxplus \mu_2 = L\{x_2 + x_2\}.$$

As in the classical case, one can verify that the above definition of $\mu_1 \boxplus \mu_2$ does not depend on the specific choice of the operators x_1 and x_2.

Having introduced free convolution $\boxplus$, we can define free infinite divisibility just as in classical probability.

Definition 3.2. A probability measure μ on $\mathbb{R}$ is infinitely divisible w.r.t. free additive convolution (or just $\boxplus$-infinitely divisible), if there exists, for each positive integer n, a probability measure μ_n on $\mathbb{R}$, such that:

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ terms}}.$$

We denote by $\mathcal{ID}(\boxplus)$ the class of $\boxplus$-infinitely divisible probability measures on $\mathbb{R}$.

In classical probability, the infinitely divisible probability measures are characterized by their Lévy-Khintchine representation.

Theorem 3.3 (Lévy-Khintchine). *Let μ be a probability measure on $\mathbb{R}$ with characteristic function (or Fourier transform) f_μ. Then μ is infinitely divisible w.r.t. classical convolution $*$, if and only if f_μ has a representation in the form:*

$$\log f_\mu(u) = i\gamma u - \tfrac{1}{2} a u^2 + \int_{\mathbb{R}} \big(e^{iut} - 1 - iut 1_{[-1,1]}(t)\big)\ \rho(dt),$$

[2] This situation can always be realized on a suitable Hilbert space.

where $\gamma \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$, i.e.

$$\rho(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} \min\{1, t^2\}\ \rho(dt) < \infty.$$

In that case the generating triplet (a, ρ, γ) is uniquely determined.

For a probability measure μ on $\mathbb{R}$, the free analog of $\log f_\mu$ is the free cumulant transform $\mathcal{C}_\mu \colon \Gamma_\mu \subseteq \mathbb{C} \to \mathbb{C}$, defined on a certain region Γ_μ of the complex plain. The key property of the free cumulant transform (proved in [3], [4] and [5]) is that

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z),$$

for any probability measures μ_1, μ_2 on $\mathbb{R}$. In terms of the free cumulant transform, the $\boxplus$-infinitely divisible probability measures are characterized by the following Lévy-Khintchine type theorem:

Theorem 3.4 ([3]). *Let μ be a probability measure on $\mathbb{R}$ with free cumulant transform $\mathcal{C}_\mu$. Then μ is $\boxplus$-infinitely divisible if and only if $\mathcal{C}_\mu$ has a representation in the form:*

$$\mathcal{C}_\mu(z) = \gamma z + a z^2 + \int_{\mathbb{R}} \Big(\frac{1}{1 - tz} - 1 - tz 1_{[-1,1]}(t) \Big)\ \rho(dt),$$

where $\gamma \in \mathbb{R}$, $a \geq 0$ and ρ is a Lévy measure on $\mathbb{R}$. In that case, the free generating triplet (a, ρ, γ) is uniquely determined.

4 The Bercovici-Pata bijection

From the two Lévy-Khintchine representations (Theorem 3.3 and Theorem 3.4), it follows immediately, that there is a bijection between the class $\mathcal{ID}(*)$ of classically infinitely divisible probability measures and the class $\mathcal{ID}(\boxplus)$ defined above.

Definition 4.1. The Bercovici-Pata bijection $\Lambda \colon \mathcal{ID}(*) \to \mathcal{ID}(\boxplus)$ is defined as follows: For a measure μ in $\mathcal{ID}(*)$ with generating triplet (a, ρ, γ), $\Lambda(\mu)$ is the measure in $\mathcal{ID}(\boxplus)$ with free generating triplet (a, ρ, γ).

Although the bijection Λ may seem, at a first glance, as a very formal correspondence, it turns out that it has some very useful algebraic properties. If μ is the distribution of a (classical) random variable X and $c \in \mathbb{R}$, then we denote by $D_c\mu$ the distribution of cX. Furthermore, δ_c denotes the Dirac measure at c.

Theorem 4.2 ([1]). *The Bercovici-Pata bijection Λ has the following algebraic properties:*

(i) *If $\mu_1, \mu_2 \in \mathcal{ID}(*)$, then $\Lambda(\mu_1 * \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.*

(ii) *If $\mu \in \mathcal{ID}(*)$ and $c \in \mathbb{R}$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$.*

(iii) *For any c in $\mathbb{R}$, $\Lambda(\delta_c) = \delta_c$.*

From a topological point of view too, the Bercovici-Pata bijection behaves very nicely.

Theorem 4.3 ([1]). *The Percovici-Pata bijection is a homeomorphism w.r.t. weak convergence. More precisely, for measures $\mu, \mu_1, \mu_2, \mu_3, \dots$ in $\mathcal{ID}(*)$, we have*

$$\mu_n \xrightarrow{\ \mathrm{w}\ } \mu \iff \Lambda(\mu_n) \xrightarrow{\ \mathrm{w}\ } \Lambda(\mu).$$

Examples 4.4.

(1) Let μ be the standard Gaussian distribution, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\ dx.$$

Then $\Lambda(\mu)$ is the semi-circle distribution, i.e.

$$\Lambda(\mu)(dx) = \frac{1}{2\pi}\sqrt{4 - x^2} \cdot 1_{[-2,2]}(x)\ dx.$$

(2) Let μ be the Poisson distribution with parameter $\lambda > 0$, i.e.

$$\mu(\{n\}) = e^{-\lambda}\frac{\lambda^n}{n!}, \qquad (n \in \mathbb{N}_0).$$

Then $\Lambda(\mu)$ is the free Poisson distribution (also known as the Marchenko-Pastur distribution) with parameter λ, i.e.

$$\Lambda(\mu)(dx) = \begin{cases} (1-\lambda)\delta_0 + \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\ dx, & \text{if } 0 \le \lambda \le 1, \\ \frac{1}{2\pi x}\sqrt{(x-a)(b-x)} \cdot 1_{[a,b]}(x)\ dx, & \text{if } \lambda > 1, \end{cases}$$

where $a = (1-\sqrt{\lambda})^2$ and $b = (1+\sqrt{\lambda})^2$.

5 Lévy Processes in free probability.

The following definition of Lévy processes in free probability corresponds exactly to the definition of classical Lévy processes, when classical independence is replaced by free independence.

Definition 5.1 ([1]). Let $\mathcal{H}$ be a Hilbert space and let τ be a state on $\mathcal{B}(\mathcal{H})$. A free Lévy process (in law) on $\mathcal{H}$ is a family $(Z_t)_{t\ge 0}$ of selfadjoint operators on $\mathcal{H}$, which satisfies the following conditions:

(i) whenever $n \in \mathbb{N}$ and $0 \le t_0 < t_1 < \cdots < t_n$, the increments

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \dots, Z_{t_n} - Z_{t_{n-1}},$$

are freely independent selfadjoint operators.

(ii) $Z_0 = 0$.

(iii) for any s, t in $[0, \infty[$, $L\{Z_{s+t} - Z_s\}$ does not depend on s.

(iv) for any s in $[0, \infty[$, $L\{Z_{s+t} - Z_s\} \xrightarrow{\text{w}} \delta_0$, as $t \to 0$.

If (Z_t) is a free Lévy process, then, just as in the classical case, $L\{Z_t\} \in \mathcal{ID}(\boxplus)$ for all t. Indeed, for any t in $[0, \infty[$ and n in $\mathbb{N}$,

$$Z_t = Z_{t/n} + (Z_{2t/n} - Z_{t/n}) + \cdots + (Z_t - Z_{(n-1)t/n}).$$

In particular, one may apply the Bercovici-Pata bijetion to each $L\{Z_t\}$, and it follows then, by virtue of the algebraic and topological properties of Λ, that Λ gives rise to a one-to-one correspondance (in law) between classical and free Lévy processes (in law).

Theorem 5.2 ([1]). *Let (X_t) be a classical Lévy process (in law) defined on some probability space $(\Omega, \mathcal{F}, P)$. Then there exists a free Lévy process (in law) (Z_t) on some Hilbert space $\mathcal{H}$, such that*

$$\Lambda(L\{X_t\}) = L\{Z_t\}, \quad (t \ge 0). \tag{5.1}$$

Conversely, to any free Lévy process (in law) (Z_t) there correponds a classical Lévy process (in law) (X_t), such that (5.1) *holds.*

6 The Lévy-Itô decomposition in free probability

In classical probability, the Lévy-Khinchine representation has a counterpart for Lévy-processes, the Lévy-Itô decomposition, which was actually derived, by P. Lévy, before the Lévy-Khintchine representation.

Theorem 6.1 (Lévy-Itô). *Let (X_t) be a classical Lévy process and let ρ be the Lévy measure appearing in the generating triplet for $L\{X_1\}$. Assume, for simplicity, that $\int_{-1}^{1} |t|\ \rho(dt) < \infty$. Then (X_t) has a representation in the form*[3]*:*

$$X_t \overset{\text{a.s.}}{=} at + bB_t + \int_{]0,t]\times\mathbb{R}} x\ N(ds, dx), \quad (t \geq 0), \tag{6.1}$$

where

- *$a \in \mathbb{R}$, $b \geq 0$ and (B_t) is a Brownian motion,*
- *N is a Poisson random measure on $]0, \infty[\times\mathbb{R}$ with intensity measure* Leb $\otimes \rho$ *(see Definition 6.2 below),*
- *the processes appearing in the right hand side of* (6.1) *are independent.*

Poisson random measures are defined as follows:

Definition 6.2. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space. A Poisson random measure on Θ with intensity measure ν is a family $\{N(E) \mid E \in \mathcal{E}\}$ of random variables (defined on some $(\Omega, \mathcal{F}, P)$) with the following properties:

(i) for all E in $\mathcal{E}$, $L\{N(E)\}$ is the Poisson distribution with parameter $\nu(E)$,

(ii) $E_1, \ldots, E_r$ disjoint sets from $\mathcal{E} \Longrightarrow N(E_1), \ldots, N(E_r)$ are independent,

(iii) for all ω in Ω, $N(\cdot, \omega)$ is a measure on $\mathcal{E}$.

The free version of the Lévy-Itô decomposition decomposes any free Lévy process (in law) into the sum of a drift term, a free Brownian motion (i.e. the free Lévy process corresponding to the classical Brownian motion as in Theorem 5.2) and an integral w.r.t. a free Poisson random measure. The latter notion is defined as follows:

Definition 6.3. Let $(\Theta, \mathcal{E}, \nu)$ be a σ-finite measure space, and put

$$\mathcal{E}_f = \{E \in \mathcal{E} \mid \nu(E) < \infty\}.$$

A *free* Poisson random measure on Θ with intensity measure ν is a family $\{M(E) \mid E \in \mathcal{E}_f\}$ of selfadjoint operators (on some Hilbert space $\mathcal{H}$) with the following properties:

(i) for all E in $\mathcal{E}_f$, $L\{M(E)\}$ is the free Poisson distribution with parameter $\nu(E)$ (cf. Example 4.4),

(ii) $E_1, \ldots, E_r$ disjoint sets from $\mathcal{E}_f \Longrightarrow M(E_1), \ldots, M(E_r)$ are freely independent,

(iii) $E_1, \ldots, E_r$ disjoint sets from $\mathcal{E}_f \Longrightarrow M(E_1 \cup \cdots \cup E_r) = M(E_1) + \cdots + M(E_r)$.

Although the definition of a free Poisson random measure may seem a little "poor" compared to that of a classial one, Definition 6.3 is sufficient to prove the following free version of the Lévy-Itô decomposition.

Theorem 6.4 ([2]). *Let (Z_t) be a free Lévy-process on a Hilbert space $\mathcal{H}$ and let ρ be the Lévy measure appearing in the free generating triplet for $L\{Z_1\}$. Assume, for simplicity, that $\int_{-1}^{1} |t|\ \rho(dt) < \infty$. Then (Z_t) has a representation in the form*[4]*:*

$$Z_t \overset{\text{d}}{=} at + bW_t + \int_{]0,t]\times\mathbb{R}} x\ M(ds, dx), \quad (t \geq 0), \tag{6.2}$$

where

- *$a \in \mathbb{R}$, $b \geq 0$ and (W_t) is a free Brownian motion,*
- *M is a free Poisson random measure on $]0, \infty[\times\mathbb{R}$ with intensity measure* Leb $\otimes \rho$*,*
- *the processes appearing in the right hand side of* (6.2) *are freely independent.*

[3] Here, a.s. stands for "almost surely".

[4] Here, $\overset{\text{d}}{=}$ means "equal in distribution".

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Limit theorems for selfsimilar additive processes

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1 Selfsimilar additive processes

An $\mathbf{R}^d$-valued stochastic process $\{Y(t), t \geq 0\}$ is said to be a *selfsimilar additive process* with exponent H if the following two conditions are satisfied :

(i) The process $\{Y(t), t \geq 0\}$ is selfsimilar with exponent H, that is, there exists $H > 0$ such that, for any $c > 0$,
$$\{Y(ct), t \geq 0\} \stackrel{\mathrm{d}}{=} \{c^H Y(t), t \geq 0\},$$
where the symbol $\stackrel{\mathrm{d}}{=}$ stands for the equality in finite-dimensional distributions.

(ii) The process $\{Y(t), t \geq 0\}$ has independent increments.

Let $a > 1$. An $\mathbf{R}^d$-valued random sequence $\{X(n), n \in \mathbf{Z}\}$ is called a *shift a-selfsimilar additive random sequence* if the following two conditions are satisfied :

(i) The sequence $\{X(n), n \in \mathbf{Z}\}$ has shift a-selfsimilarity, that is,
$$\{X(n+1), n \in \mathbf{Z}\} \stackrel{\mathrm{d}}{=} \{aX(n), n \in \mathbf{Z}\}.$$

(ii) The sequence $\{X(n), n \in \mathbf{Z}\}$ has independent increments, that is, for every $n \in \mathbf{Z}$, $\{X(k), k \leq n\}$ and $X(n+1) - X(n)$ are independent.

Note that shift selfsimilarity does not imply the usual selfsimilarity. For an $\mathbf{R}^d$-valued random variable Y, denote by $\mathcal{L}(Y)$ the distribution of Y. We use the words "increase" and "decrease" in the wide sense allowing flatness. Let $0 < b < 1$. A probability distribution μ on $\mathbf{R}^d$ is said to be b-decomposable if there exists a probability distribution ρ on $\mathbf{R}^d$ such that
$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z). \tag{1.1}$$
The probability distribution μ in (1.1) on $\mathbf{R}^d$ is called semi-selfdecomposable if it is b-decomposable for some b and if the distribution ρ in (1.1) is an infinitely divisible distribution on $\mathbf{R}^d$. A probability distribution μ on $\mathbf{R}^d$ is said to be selfdecomposable if, for every $b \in (0,1)$, there exists a probability distribution ρ_b on $\mathbf{R}^d$ such that
$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}_b(z).$$
The marginal distributions of stochastically continuous selfsimilar additive processes are selfdecomposable. Conversely, for any selfdecomposable distribution μ on $\mathbf{R}^d$, there exists a unique in law stochastically continuous selfsimilar additive process $\{Y(t), t \geq 0\}$ with exponent $H > 0$ and $\mathcal{L}(Y(1)) = \mu$. The marginal distributions of shift a-selfsimilar additive random sequences are a^{-1}-decomposable. Conversely, for any b-decomposable distribution μ on $\mathbf{R}^d$, there exists a (not necessarily unique in law) shift a-selfsimilar additive random sequence $\{X(n), n \in \mathbf{Z}\}$ on $\mathbf{R}^d$ with $a = b^{-1}$ and $\mathcal{L}(X(0)) = \mu$. In the case where the support of μ is contained in $\mathbf{R}^d_+$, the sequence $\{X(n), n \in \mathbf{Z}\}$ is determined uniquely in law.

Let $\{Z_n, n \in \mathbf{Z}_+\}$ be a supercritical Galton-Watson branching process with $Z_0 = 1$ and $a = E(Z_1) > 1$. Let $f(s)$ be the probability generating function of the offspring distribution $\mathcal{L}(Z_1)$. Then there exists an increasing sequence $\{c_n\}_{n=0}^{\infty}$ and an $\mathbf{R}_+$-valued random variable W such that $c_{n+1}/c_n \to a$ as $n \to \infty$ and $Z_n/c_n \to W$ almost surely as $n \to \infty$. In the case where $f(0) = 0$, there exists a unique in law increasing shift a-selfsimilar additive random sequence $\{W(n), n \in \mathbf{Z}\}$ with $\mathcal{L}(W(0)) = \mathcal{L}(W)$. We say that the sequence $\{W(n), n \in \mathbf{Z}\}$ is associated with the process $\{Z_n, n \in \mathbf{Z}_+\}$.

We obtained in [21,22,23,24] general limit theorems of "limitinf" type and "limitsup" type completely for increasing selfsimilar additive processes and increasing shift selfsimilar additive random sequences. We explain those limit theorems by applying them to three important examples.

2 Laws of the iterated logarithm for Bessel processes

Let $\{X(t), t \geq 0\}$ be a Bessel process on $\mathbf{R}_+$ starting at the origin with a real dimension $d = 2(1+\nu) > 0$. Define the first hitting time and the last exit time as $T_r = \inf\{t \geq 0 : X(t) = r\}$ and $L_r = \sup\{t \geq 0 : X(t) = r\}$ for $r \geq 0$. Then $\{T_r, r \geq 0\}$ and $\{L_r, r \geq 0\}$ are stochastically continuous increasing selfsimilar additive processes with exponent $H = 2$.

Theorem 2.1. *We have*

$$\liminf \frac{\log|\log r|}{r^2} T_r = \frac{1}{2} \qquad a.s.$$

and

$$\limsup \frac{T_r}{r^2 \log|\log r|} = \frac{2}{j_\nu^2} \qquad a.s.$$

both as $r \to 0+$ and as $r \to \infty$ where j_ν is the first positive zero of the Bessel function $J_\nu(x)$ of the first kind.

Corollary 2.2. *We have*

$$\limsup \frac{1}{\sqrt{t \log|\log t|}} \sup_{0 \leq s \leq t} X(s) = \sqrt{2} \qquad a.s.$$

and

$$\liminf \sqrt{\frac{\log|\log t|}{t}} \sup_{0 \leq s \leq t} X(s) = \frac{j_\nu}{\sqrt{2}} \qquad a.s.$$

both as $t \to 0+$ and as $t \to \infty$.

Theorem 2.3. *Let $\nu > 0$. We have*

$$\liminf \frac{\log|\log r|}{r^2} L_r = \frac{1}{2} \qquad a.s.$$

and

$$\limsup \frac{L_r}{r^2 |\log r|^\delta} = \begin{cases} \infty & a.s. \text{ for } 0 < \delta \leq 1/\nu \\ 0 & a.s. \text{ for } \delta > 1/\nu. \end{cases}$$

both as $r \to 0+$ and as $r \to \infty$.

Corollary 2.4. *Let $\nu > 0$. We have*

$$\limsup \frac{1}{\sqrt{t \log|\log t|}} \inf_{t \leq s} X(s) = \sqrt{2} \qquad a.s.$$

and

$$\liminf \frac{|\log t|^\delta}{\sqrt{t}} \inf_{t \leq s} X(s) = \begin{cases} 0 & a.s. \text{ for } 0 < \delta \leq 1/(2\nu) \\ \infty & a.s. \text{ for } \delta > 1/(2\nu). \end{cases}$$

both as $t \to 0+$ and as $t \to \infty$.

Remark 2.1 (1) In the second equalities of Theorem 2.3 and Corollary 2.4, there are no exact laws of the iterated logarithm. Moreover we have integral tests in those cases.

(2) As $x \to 0+$ and as $y \to \infty$ with $c > 0$,

$$-\log P(T_1 \leq x) \sim \frac{1}{2x}, \quad -\log P(L_1 \leq x) \sim \frac{1}{2x},$$

$$-\log P(T_1 > y) \sim \frac{j_\nu^2 y}{2}, \quad P(L_1 > y) \sim c y^{-\nu}.$$

3 Laws of the iterated logarithm for BM on SG

Let G be the Sierpinski gasket in $\mathbf{R}^2$ and let $\widehat{G} = \bigcup_{n=0}^{\infty} 2^n G$, $F = \{x \in \widehat{G} : |x| = 1\}$ and $F_n = 2^n F$ for $n \in \mathbf{Z}$. Let $\{B(t), t \geq 0\}$ be a Brownian motion on $\widehat{G}$. We assume that $B(0)$ is the origin. Then it is semi-selfsimilar, that is,

$$\{B(5t), t \geq 0\} \stackrel{\mathrm{d}}{=} \{2B(t), t \geq 0\}.$$

Let $W(n)$, $n \in \mathbf{Z}$, be the first hitting time of the set F_n for the process $\{B(t)\}$, namely,

$$W(n) = \inf\{t > 0 : B(t) \in F_n\}.$$

Proposition 3.1. *The sequence $\{W(n), n \in \mathbf{Z}\}$ is an increasing shift 5-selfsimilar additive random sequence associated with a supercritical branching process $\{Z_n, n \in \mathbf{Z}_+\}$ with $f(s) = s^2/(4-3s)$ and $E(W) = 1$.*

Remark 3.1 Barlow and Perkins [3] raised a question whether $\mathcal{L}(W(0))$ is unimodal or not. It is known by Yamazato [26] that all selfdecomposable distributions on $\mathbf{R}^1$ are unimodal. Yamazato [25] remarked that if a supercritical branching process $\{Z_n, n \in \mathbf{Z}_+\}$ with $f(0) = 0$ is embeddable in a continuous time branching process, then $\mathcal{L}(W)$ is selfdecomposable. However, we find from Karlin and McGregor [13] that the associated branching process to this $\{W(n), n \in \mathbf{Z}\}$ is not embeddable. Thus we know that $\mathcal{L}(W(0))$ is semi-selfdecomposable but do not know whether it is selfdecomposable.

Theorem 3.2. *Let $\beta = \log 2 / \log 5$.*

(i) We have

$$\liminf_{n \to \pm\infty} \frac{W(n)}{5^n (\log |n|)^{-(1-\beta)/\beta}} = \delta_0^{(1-\beta)/\beta} \qquad a.s.$$

where δ_0 is a positive constant determined by

$$E \exp(\delta W^{-\beta/(1-\beta)}) \begin{cases} < \infty & \text{for } 0 < \delta < \delta_0 \\ = \infty & \text{for } \delta > \delta_0. \end{cases}$$

(ii) We have

$$\limsup_{n \to \pm\infty} \frac{W(n)}{5^n \log |n|} = \sigma^{-1} \qquad a.s.$$

where σ is a positive constant given by

$$\sigma = \lim_{n \to \infty} 5^{n+1}((f_n)^{-1}(4/3) - 1)$$

with $f(s) = s^2/(4-3s)$ and $f_n(s)$ being the n-fold iteration of $f(s)$.

Corollary 3.3.

(i) We have

$$\limsup_{t \to \infty} \frac{1}{t^\beta (\log\log t)^{1-\beta}} \sup_{0 \leq s \leq t} |B(s)| = C_1 \qquad a.s.$$

$$\limsup_{t \downarrow 0} \frac{1}{t^\beta (\log|\log t|)^{1-\beta}} \sup_{0 \leq s \leq t} |B(s)| = C_2 \qquad a.s.$$

$$\liminf_{t \to \infty} \frac{1}{t^\beta (\log\log t)^{-\beta}} \sup_{0 \leq s \leq t} |B(s)| = C_3 \qquad a.s.$$

and

$$\liminf_{t \downarrow 0} \frac{1}{t^\beta (\log|\log t|)^{-\beta}} \sup_{0 \leq s \leq t} |B(s)| = C_4 \qquad a.s.$$

(ii) The constants C_j $(1 \le j \le 4)$ are bounded as follows :

$$\delta_0^{\beta-1} \le C_1, C_2 \le 2\delta_0^{\beta-1}$$

and

$$2^{-1}\sigma^{\beta} \le C_3, C_4 \le \sigma^{\beta}.$$

Remark 3.2 The equalities in (i) of Corollary 3.3 were already proved by Ben Arous and Kumagai [6], Barlow and Perkins [3] and Fukushima et al [10]. We do not know whether $C_1 = C_2 = \delta_0^{\beta-1}$ and $C_3 = C_4 = \sigma^{\beta}$. It is difficult to identify the constants C_j $(1 \le j \le 4)$ and the explicit values are not known. But σ is computed numerically as $\sigma = 1.318\cdots$. A roughly approximate value of δ_0 is known as 1.26 which is due to Bingham [8] based on the numerical calculation by Barlow and Perkins [3]. By virtue of large deviation principles, the equalities in (i) were proved for Brownian motions on other nested fractals by Bass and Kumagai [5] and Fukushima et al [10].

4 Exact packing measure for C_∞

Let $\{C_k, k \in \mathbf{Z}_+\}$ be a fractal percolation on $[0,1]^d$ with base $M \ge 2$ and probability p. Denote the limiting set and the branching set in a Galton-Watson tree by $C_\infty := \cap_{k=0}^{\infty} C_k$ and $\widehat{C}_\infty$, respectively. It is known that $P(C_\infty \ne \emptyset) > 0$ if and only if $pM^d > 1$. Let $q := P(C_\infty = \emptyset)$. The constant q is the first positive solution of the following equation : $(1 - p + pq)^{M^d} = q$.
We denote by ϕ-$H(C)$ and ϕ-$P(C)$ the ϕ-Hausdorff measure and ϕ-paking measure of the set C.

Theorem 4.1. *Let $pM^d > 1$ and $\alpha := \log(pM^d)/\log M$.*

(1) **(Graf-Mauldin-Williams [11])**
We have, a.s. on $\{C_\infty \ne \emptyset\}$,

$$0 < \phi\text{-}H(C_\infty) < \infty,$$

where $\phi(t) := t^{\alpha}(\log|\log t|)^{1-\alpha/d}$.

(2) **(Watanabe[24])**
A.s. on $\{C_\infty \ne \emptyset\}$, there is no exact paking measure for $\widehat{C}_\infty$. We have

$$\phi_1\text{-}P(\widehat{C}_\infty) = \begin{cases} 0 & \text{for } \beta > 1/\gamma \\ \infty & \text{for } 0 < \beta \le 1/\gamma, \end{cases}$$

where $\phi_1(t) := t^{\alpha}|\log t|^{-\beta}$ and $\gamma := \log(\frac{1-p+pq}{q})/\log(pM^d)$.

We finish this article by posing a problem :

Problem Is there an exact packing measure for C_∞ on $\{C_\infty \ne \emptyset\}$ a.s.?

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Approximation of subordinated Lévy processes with infinite jump rate and some related stochastic integrals

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1 Approximation of subordinated Lévy processes

We consider approximations of subordinated Lévy processes with application to simulation. Let $\{Y(t)\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued Lévy process, $d \geq 1$ and let $\{V(t)\}_{0\leq t\leq 1}$ be a subordinator. We then consider the process $\{X(t)\}_{0\leq t\leq 1} := \{Y(V(t))\}_{0\leq t\leq 1}$. We suppose that that the subordinand Y can be fairly easily simulated and further that it has at least two finite moments. We further suppose that the the subordinator has infinite jump rate and that it cannot easily be exactly simulated. We can decompose $\{X(t)\}_{0\leq t\leq 1}$ as

$$\{X(t)\} \stackrel{d}{=} \{Y_1(V_T(t))\} + \{Y_2(\varepsilon_V^T(t))\} := \{X^T(t)\} + \{\varepsilon_X^T(t)\}$$

where Y_1 and Y_2 are independent copies of Y, V_T is a compound Poisson process consisting of the large jumps of V and $\varepsilon_V^T(t) = V(t) - V_T(t)$ the process consisting of the remaining jumps of V. The parameter T determines the level of truncation for the jumps in $\varepsilon_V^T(t)$ with respect to some series representation of V. More precisely we have

$$V(t) = V_T(t) + \varepsilon_V^T(t) := \sum_{k:T_k\leq T} H(T_k)I(U_k \leq t) + \sum_{k:T_k>T} H(T_k)I(U_k \leq t)$$

where $\{T_k\}$ are the points in a homogeneous Poisson process with intensity λ and $\{U_k\}$ is a sequence of i.i.d. random variables uniformly distributed on $(0,1)$, $\{H(s)\}$ is a family of independent random variables such that for $x > 0$, $\lambda\int_0^\infty P(H(s) > x)\,\mathrm{d}s = \int_x^\infty M(\mathrm{d}y)$, M being the Lévy measure of $V(1)$ and that $H(s,\omega)$ is non-increasing in s. Further we have that the sequences $\{H(s)\}$, $\{U_k\}$ and $\{T_k\}$ are independent. There always exist a family $\{H(s)\}$ with the above properties. We can choose $H(s) = g(s)$, where $g(s) = \inf\{u > 0 : \int_u^\infty M(\mathrm{d}x) < s\}$. For more details on series representations we refer to [6].

Let $T > 0$ be arbitrary but fixed. We now propose the approximation

$$\overline{X}_T(t) := Y_1(V_T(t)) + Y_2(t\mathbb{E}\varepsilon_V^T(1)) := X_T(t) + \overline{\varepsilon}_X^T(t).$$

Theorem 1.1. MISE *For any fixed $T > 0$ we have that*

$$\begin{aligned}\int_0^t \mathbb{E}\big|X(t) - \overline{X}_T(t)\big|^2\,\mathrm{d}t &= \left(\sum_{k=1}^d \mathbb{V}Y_k(1)\right)\int_0^t \mathbb{E}\Big|\varepsilon_V^T(t) - t\mathbb{E}\varepsilon_V^T(1)\Big|\,\mathrm{d}t + \frac{1}{2}\big|\mathbb{E}Y(1)\big|^2\mathbb{V}\varepsilon_V^T(1)\\ &\leq \frac{2}{3}\left(\sum_{k=1}^d \mathbb{V}Y_k(1)\right)(\mathbb{V}\varepsilon_V^T(1))^{1/2} + \frac{1}{2}\big|\mathbb{E}Y(1)\big|^2\mathbb{V}\varepsilon_V^T(1).\end{aligned}$$

This should be compared to the error obtained if just use the truncated series representation $(\sum_{k=1}^d \mathbb{V}Y_k(1))\mathbb{E}\varepsilon_V^T(1)/2$.

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Theorem 1.2. Efficiency *We have that*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon} xM(\mathrm{d}x)=\infty$$

is a sufficient condition for the above proposed approximation to have an asymptotically better rate than the truncated series representation as $T\to\infty$. Moreover it assures that

$$\{\varepsilon_V^T(t)/\mathbb{E}\varepsilon_V^T(1)\}_{0\le t\le 1}\xrightarrow{L^2}\{t\}_{0\le t\le 1}\ \textit{as}\ T\to\infty$$

This condition approximately says that there must not be too few small jumps. In the one dimensional case [1] proposed a slightly different approach of approximating Lévy processes, where they used that in many cases the centred process of small jumps rescaled by its standard deviation converge weakly to a Brownian motion. This weak convergence is in fact guaranteed by a condition of the same type as in Th. 1.2.

We will now utilise the approximation proposed above to obtain approximations of stochastic integrals driven by Lévy processes.

2 Approximation of stochastic integrals driven by Lévy processes

We study stochastic integrals of the form

$$Z(t)=h(t)+\int_0^t f(t,s-)\ \mathrm{d}X(s) \tag{2.1}$$

where $\{X(s)\}$ is a subordinated Lévy process as above and $\{f(t,s)\}$ is adapted in s for each $t\in[0,1]$ with càdlag̀ (RCLL) paths, g is adapted in t and that f, h and X have compatible dimensions. We further suppose that we can simulate f_1 and f_2 exactly. We assume that h is independent of X but f and X may be dependent. For simplicity we from now on suppose that $\mathbb{E}Y(1)=0$ and that we use the series representation with $H(s)=g(s)$. The main reason for this assumption is that it makes $\{Y(\varepsilon_V^T(t))\}$ an L^2-martingale for each $T>0$. We will also denote $\int_0^t f(t,s-)\ \mathrm{d}X(s)$ by $I_X(f)_t$.

Example 2.1. *Suppose that we want to approximate the solution of the SDE* $\mathrm{d}Z(t)=A(t)Z(t)\ \mathrm{d}t+B(t)\ \mathrm{d}X(t), Z(0)=Z_0\in\mathcal{F}_0$ *where A and B are deterministic $d\times d$ matrix-valued functions. This equation has the explicit solution* $Z(t)=\exp(\int_0^t A(s)\ \mathrm{d}s)Z_0+\int_0^t\exp(\int_s^t A(u)\ \mathrm{d}u)B(s)\ \mathrm{d}X(s)$. *This is a stochastic integral of the above type where* $g(t)=\exp(\int_0^t A(s)\ \mathrm{d}s)Z_0$ *and* $f(t,s)=\exp(\int_s^t A(u)\ \mathrm{d}u)B(s)$.

2.1 Series representations of the stochastic integrals

For the special case of X being a real-valued type G process with no Gaussian component (i.e. a subordinated Wiener process) [5] suggested

$$\{Z(t)\}_{0\le t\le 1}\stackrel{d}{=}\Big\{\sum_k G_kH(T_k)^{1/2}f(t,U_k-)\Big\}_{0\le t\le 1} \tag{2.2}$$

as a series representation of the corresponding stochastic integrals, where $\{U_k\}$, $\{H(s)\}$ and $\{T_k\}$ are as defined above and $\{G_k\}$ is a sequence of i.i.d. standard Gaussian variables independent of the other sequences. This series representation can in principle be generalised to an arbitrary Lévy process if we add some centring terms and impose further restrictions on f. We basically need to replace $\{G_kH(T_k)\}$ by a more general random sequence $\{\tilde{H}(T_k,V_k)\}$. We will however, not proceed further in this direction.

2.2 The approximation procedure

To avoid measurability technicalities for the approximation in the case where f and X are dependent we by partial integration we rewrite the stochastic integral (2.1) as

$$Z(t) := \int_0^t f(t,s-)\ \mathrm{d}X_T(s) + f(t,t)\varepsilon_X^T(t) - \left(\int_0^t (\varepsilon_X^T(t))'\ \mathrm{d}_s f(t,s)\right)'$$

The measurability problem arise from that $Y_2(\varepsilon_V^T(t))$ and $Y_2(t\mathbb{E}\varepsilon_V^T(1))$ are not L^2-martingales with respect to the same filtration. We propose the approximation

$$\overline{Z}_T(t) := \int_0^t f(t,s-)\ \mathrm{d}X_T(s) + f(t,t)\overline{\varepsilon}_X^T(t) - \left(\int_0^t (\overline{\varepsilon}_X^T(t))'\ \mathrm{d}_s f(t,s)\right)'$$

where X' denote X transposed. In order to guarantee that the last integral is well defined we need, unless f is independent of X, that f is of finite variation.

Theorem 2.2. MSE *If f has finite variation and four finite moments or if f is independent of X and has two finite moments then there exist a constant C_f, depending on f such that*

$$\mathbb{E}\left|Z(t) - \overline{Z}^T(t)\right|^2 \leq \left(\mathbb{V}\varepsilon_V^T(1)\right)^{1/2}\left(\sum_{k=1}^d \mathbb{V}Y_k(1)\right)C_f.$$

In the next section we will propose another approximation technique based on a stochastic time change representation. This approach only works in the one dimensional case and for subordinated Wiener processes it is however of independent theoretical interest.

2.3 Time change representations of stochastic integrals driven by type G Lévy processes

Stochastic time change representations of stochastic integrals with respect to symmetric stable Lévy processes were first studied by [4], who also gave a necessary and sufficient condition for the existence of these stochastic integrals. Let $\{X(t)\}$ be a symmetric α-stable Lévy process with $0 < \alpha \leq 2$. We then have that

$$Z(t) = \int_0^t f(s)\ \mathrm{d}X(t) = \widetilde{X}\left(\int_0^t |f(s)|^\alpha\ \mathrm{d}s\right),$$

where $\{\widetilde{X}(t)\} \stackrel{d}{=} \{X(t)\}$, provided that f satisfies the condition $\int_0^t |f(s)|^\alpha\ \mathrm{d}s < \infty$ a.s. for any finite t. Moreover, the process $\{\widetilde{X}(t)\}$ can explicitly be constructed as

$$\widetilde{X}(t) = Z(\tau(t)),$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s |f(u)|^\alpha\ \mathrm{d}u) > t\right\}.$$

[2] generalised these results to asymmetric stable Lévy processes and indicated possible multi-dimensional extensions. [3] showed that this time change property is valid only for the class of α-stable Lévy processes. We will, however, show that a modification of the time change property is valid for type G Lévy processes in finite-dimensional distribution sense, provided that the integrand f and the integrator X are independent.

Proposition 2.3. *If $\{X(t)\} \stackrel{d}{=} \{W(V(t))\}$ is a type G Lévy process, the process $\{Z(t)\} = \{\int_0^t f(s)\ \mathrm{d}X(s)\}_{0\leq t\leq 1}$ can be represented as*

$$\{Z(t)\} \stackrel{d}{=} \{\widetilde{Z}(t)\} = \left\{\widetilde{W}\left(\int_0^t f(s)^2\ \mathrm{d}V(s)\right)\right\},$$

where $\{\widetilde{W}(t)\}$ *is a Wiener process independent of* $\{V(t)\}$ *and* $\{f(t)\}$*, provided that* $\{f(t)\}$ *is independent of* $\{X(t)\}$ *and satisfies*

$$\int_0^t f(s)^2 \, \mathrm{d}V(s) < \infty \text{ a.s. for } 0 \leq t \leq 1.$$

In order to obtain approximations of the stochastic integral we first split the integral into a sum of two terms,

$$Z(t) = Z_T(t) + \varepsilon_Z^T(t) = I_{X_T}(f)_t + I_{\varepsilon_X^T}(f)_t,$$

where $I_{X_T}(f)_t$ and $I_{\varepsilon_X^T}(f)_t$ are conditionally independent given f. Using the weak time change property of Proposition 2.3 we can represent the stochastic integral $\{Z(t)\}$ by

$$\{Z(t)\} \stackrel{d}{=} \{W_1(I_{V_T}(f^2)_t)\} + \{W_2(I_{\varepsilon_V^T}(f^2)_t)\},$$

where $\{W_1(t)\}$ and $\{W_2(t)\}$ are independent standard Wiener processes. We now propose an approximation $\{\overline{Z}_T(t)\}$ of $\{Z(t)\}$. For $0 \leq t \leq 1$ define $\{\overline{Z}_T(t)\}$ by

$$\overline{Z}_T(t) = W_1(I_{V_T}(f^2)_t) + W_2\left(\mathbb{E}\varepsilon_V^T(1) \int_0^t f(s)^2 \, \mathrm{d}s\right).$$

The difference $\Delta(T)_t$ between $Z(t)$ and its approximation $\overline{Z}_T(t)$ is thus given by

$$\Delta(T)_t = Z(t) - \overline{Z}_T(t) = W_2(I_{\varepsilon_V^T}(f^2)_t) - W_2\left(\mathbb{E}\varepsilon_V^T(1) \int_0^t f(s)^2 \, \mathrm{d}s\right).$$

Theorem 2.4.

(i) If f has two finite moments then the MSE of the approximation is given by

$$\mathbb{E}|\Delta(T)_t|^2 = \mathbb{E}\left|\int_0^t f(s)^2 \, \mathrm{d}\big(\varepsilon_V^T(s) - \mathbb{E}\varepsilon_V^T(s)\big)\right|.$$

(ii) If f has four finite moments then

$$\mathbb{E}|\Delta(T)_t|^2 \leq \mathbb{V}\big(\varepsilon_V^T(1)\big)^{1/2} \left(\int_0^t \mathbb{E}f(s)^4 \, \mathrm{d}s\right)^{1/2}.$$

Corollary 2.5. (MISE) *If f has four finite moments then*

$$\int_0^1 \mathbb{E}|\Delta(T)_t|^2 \, \mathrm{d}t \leq 2_{\bar{}}g(\mathbb{V}\varepsilon_V^T(1))^{1/2} \left(\int_0^1 \mathbb{E}f(s)^4 \, \mathrm{d}s\right)^{1/2}.$$

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Some aspects of subordinators and subordination*

Matthias Winkel†
MaPhySto‡

Extended abstract. The work [7] presented here was initiated by a video "The Foreign Exchange Market" [3] that Neil Shephard drew my attention to. Two curves move on the screen representing current offers to buy and sell a foreign currency:

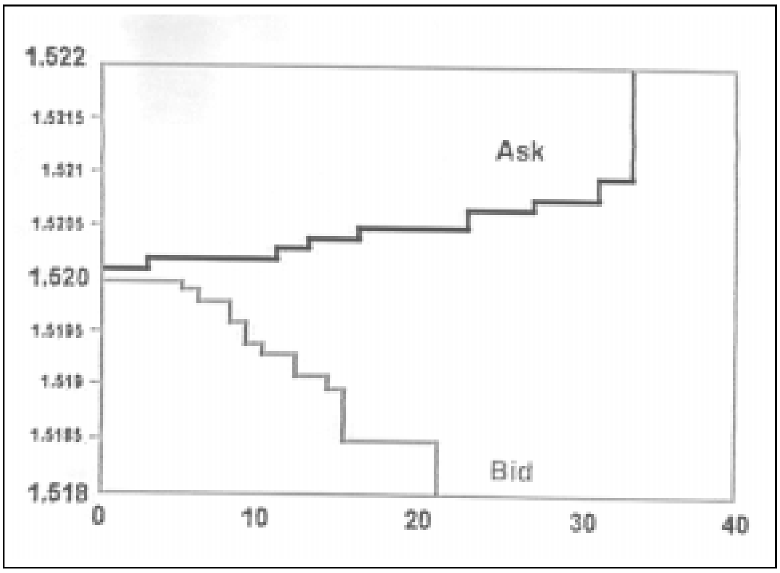

Figure 1: Snapshot of the DEM-USD electronic foreign exchange market (1997), taken from the London School of Economics video [3] by Charles Goodhart

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Figure 2 is meant to be an illustrated version of Figure 1. Each horizontal line of the two curves corresponds to an offer: its length $x_2 - x_1$ represents the amount of currency offered, its height p_0 is the price. Sellers' prices are above buyers' prices, since otherwise transaction would take place and the offers removed from the market. Sellers' offers are ordered by increasing price, buyers' offers by decreasing price so that the most interesting offers are on the left hand side. A mathematical model is proposed suggesting to add realised offers as in Figure 3. The picture is now considered as two paths of independent (inverse) subordinators, the first increasing from zero, the second decreasing from a positive height. The interest then lies in their passage event that provides the collective market price P and the total quantity Q traded at this price.

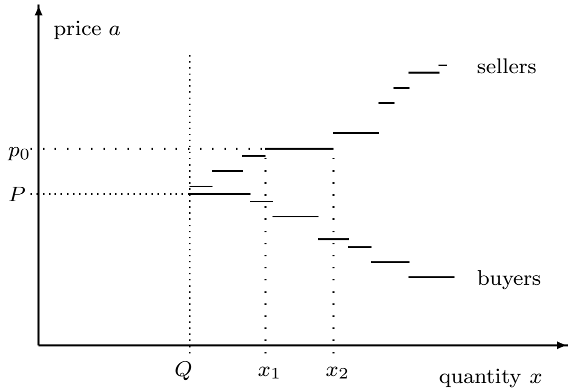

Figure 2: Offers on the market

The level passage event of stochastic processes is a classical problem. Motivations can be found in insurance, dams, finance etc. It concerns for some $h > 0$ the laws of times

$$T_h = \inf\{a \geq 0 : X_a > h\}$$
$$\text{or} \quad T_h^* = \inf\{a \geq 0 : X_a < -h\}$$

and the height $X(T_h) \geq h$ or $X(T_h^*) \leq -h$ attained by X at the passage. They are often at the center of interest in applications, e.g. the ruin of an insurance, the overflow of a reservoir, several events related to the price of the underlying or other financial assets. Reasonable process classes for X vary from one application to another. When X is a subordinator, the study was initiated by Gusak [4] and Kesten [5] in the 60s carrying out cumbersome approximations. Today, more elegant techniques are available based on Poisson point processes, cf. e.g. Bertoin [2] Section III.2. On this basis, [7] refines the study and gives two multivariate extensions formulated in the sequel.

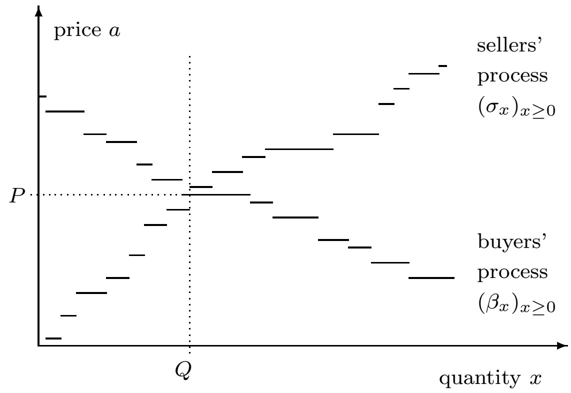

Figure 3: + realised transactions

Firstly, let $X = X^{(0)} + X^{(1)} + \ldots + X^{(m)}$ be the sum of a deterministic drift and m independent pure jump subordinators $X^{(j)}$ whose Laplace exponents are denoted by $\Phi^{(j)}(\xi) = -\log E(\exp\{-\xi X_1^{(j)}\})$, respectively. The study then concerns the individual heights and identifies the subordinator $X^{(j)}$ whose jump Δ_j performs the passage of X across level h, an event denoted by $A^{(j)}$.

Theorem 1. *Let $\tau \sim Exp(q)$ be an independent exponentially distributed level. Then for all $j = 0, \ldots, m$*

$$E\left(\exp\left\{-\kappa\tau - \alpha T_\tau - \xi_1 X^{(1)}(T_\tau-) - \ldots - \xi_m X^{(m)}(T_\tau-) - \nu\Delta_\tau\right\} 1_{A^{(j)}}\right)$$
$$= \frac{q\left(\Phi^{(j)}(q+\kappa+\nu) - \Phi^{(j)}(\nu)\right)}{(q+\kappa)\left(\alpha + \Phi^{(1)}(q+\kappa+\xi_1) + \ldots + \Phi^{(m)}(q+\kappa+\xi_m)\right)}.$$

One deduces in particular that $(T_\tau, X^{(1)}(T_\tau-), \ldots, X^{(m)}(T_\tau-))$ and $(\Delta_\tau, A^{(0)}, \ldots, A^{(m)})$ are independent. This extends in fact to $(X_a^{(0)}, \ldots, X_a^{(m)})_{a<T_\tau}$ on the one hand, and Δ_τ decomposed into undershoot $u_\tau = \tau - X(T_\tau-)$ and overshoot $o_\tau = X(T_\tau) - \tau$ on the other hand.

Secondly, let $Z = \left(Z^{(1)}, \ldots, Z^{(n)}\right)$ be any multivariate subordinator, or $Z = (X, Y)$ for $n = 2$. Note the joint Laplace exponent by $\Phi_{(X,Y)}(\xi, \eta) = -\log E(\exp\{-\xi X_1 - \eta Y_1\})$. One can then study the level passage of individual components, i.e. times

$$T_h^X = \inf\{a \geq 0 : X_a > h\} \qquad \text{and} \qquad T_k^Y = \inf\{a \geq 0 : Y_a > k\},$$

and associated heights and jumps. In the two-dimensional setting one obtains e.g.

Theorem 2. *Let $\tau \sim Exp(q)$ and $\sigma \sim Exp(p)$ be two independent exponential levels. Then*

$$\begin{aligned} &E\left(\exp\left\{-\alpha T_\tau^X - \xi X_{T_\tau^X -} - \eta Y_{T_\sigma^Y -} - \beta \Delta X_{T_\tau^X} - \gamma \Delta Y_{T_\sigma^Y}\right\} 1_{\{T_\tau^X = T_\sigma^Y\}}\right) \\ &\quad = \frac{\Phi_{(X,Y)}(\beta, p+\gamma) + \Phi_{(X,Y)}(q+\beta,\gamma) - \Phi_{(X,Y)}(\beta,\gamma) - \Phi_{(X,Y)}(q+\beta, p+\gamma)}{\alpha + \Phi_{(X,Y)}(q+\xi, p+\eta)} \\ &E\left(\exp\left\{-\alpha T_\tau^X - \xi X_{T_\tau^X -} - \beta \Delta X_{T_\tau^X} - \tilde{\alpha} T_\sigma^Y - \tilde{\eta} Y_{T_\sigma^Y -} - \tilde{\gamma} \Delta Y_{T_\sigma^Y}\right\} 1_{\{T_\tau^X < T_\sigma^Y\}}\right) \\ &\quad = \frac{\Phi_{(X,Y)}(q+\beta, p+\tilde{\eta}) - \Phi_{(X,Y)}(\beta, p+\tilde{\eta})}{\alpha + \tilde{\alpha} + \Phi_{(X,Y)}(q+\xi, p+\tilde{\eta})} \times \frac{\Phi_Y(p+\tilde{\gamma}) - \Phi_Y(\tilde{\gamma})}{\tilde{\alpha} + \Phi_Y(p+\tilde{\eta})}. \end{aligned}$$

The theorems give double Laplace transforms, i.e. the random variables in question are transformed and the level h is transformed as well which yields in fact an independent exponential level. The explicit non-transformed joint laws are established as well, in terms of the laws of the subordinators and their Lévy measures.

Theorem 1 for $m = 1$ completes the one-dimensional study in that it makes explicit the joint law of the four random variables T_τ, $X(T_\tau -)$, Δ_τ and τ, that are related to the level passage event. It seems that this had not been done before. Of course, other quantities like overshoot, undershoot and passage height can be deduced by linear transformations.

The situation in the model of an electronic foreign exchange market (Figure 3) is close to Theorem 1. Roughly, the passage time of the two processes is the passage in zero of their difference, which is a subordinator with a negative starting point. After some linear transformations one obtains

$$\begin{aligned} &E\left(\exp\left\{-\alpha P - \xi Q - \eta Q' - \nu \Delta\right\}\right) \\ &\qquad = \frac{q\left(\Phi_B(q+\nu) - \Phi_B(\nu-\eta) + \Phi_S(q+\nu+\eta+\xi) - \Phi_S(\nu+\xi)\right)}{(q+\eta)\left(\alpha + \Phi_B(q) + \Phi_S(q+\eta+\xi)\right)} \end{aligned}$$

where P is the current market price, Q and Q' are the quantities offered to buy and sell at a price at most P and at least P, respectively. As $Q \neq Q'$ a.s., the total quantity Δ offered precisely at the price P bears some additional interest and has been included.

Theorem 2 was established to introduce a dynamic feature in the exchange market model and then study the evolution of the market price and associated quantities. Unfortunately, the subordinator property required for each fixed t restricts the dynamic behaviour considerably. More precisely, the aim would be a field $(Z(a,t))_{a \geq 0, t \geq 0}$ which is a subordinator for each t and Markovian for each a. Continuous space branching processes are such a class (cf. Le Gall [6]), and one can construct the same for positive Ornstein-Uhlenbeck type processes. One can consider this in a wider framework of multivariate subordinators constructed by subordination or superposition, cf. Barndorff-Nielsen *et al.* [1]. Although the dynamics obtained do not seem suitable to model foreign exchange markets, Theorem 2 may be applied to study the evolution of the level passage event in these families of subordinators. For branching processes, this corresponds to studying the initial population size producing h children at time t, as t evolves.

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Lévy measures of the hitting time distributions for skip-free Lévy processes

Makoto Yamazato

1 Introduction

We say that a Lévy process $\{X_t\}$ on R is a Lévy process skip-free to the right if its moment generating function

$$Ee^{wX_t} = e^{t\Psi(w)} \quad \text{for } w \geq 0$$

satisfies one of the following three cases.
Case 1:

$$\Psi(w) = \frac{A}{2}w^2 + Bw + \int_{-\infty}^{0}(e^{wx} - 1 - 1_{[-1,0)}(x)wx)\nu(dx), \tag{1.1}$$

$$A, B \geq 0 \quad \text{and,} \quad A > 0 \quad \text{or} \quad \int_{-1}^{0}|x|\nu(dx) = \infty.$$

Case 2:

$$\Psi(w) = Bw + \int_{-\infty}^{0}(e^{wx} - 1)\nu(dx), \tag{1.2}$$

$$B > 0 \quad \text{and} \quad 0 < \int_{-1}^{0}|x|\nu(dx) < \infty.$$

Case 3:

$$\Psi(w) = \int_{R}(e^{wx} - 1)\nu(dx), \tag{1.3}$$

$$\text{supp } \nu \subset \{\ldots, -2, -1, 1\}, \quad 0 < \nu(R) < \infty.$$

Let w_0 be the biggest root of $\Psi(w) = 0$. Then there is an inverse function of Ψ on $[w_0, \infty)$. We write the inverse function as Ψ^{-1}. Let τ_x be the hitting time of $\{x\}$, $x > 0$, for $\{X_t\}$, i.e.

$$\tau_x = \begin{cases} \inf\{t > 0 : X_t = x\} & \text{if the set } \{\} \text{ is not empty,} \\ \infty & \text{if otherwise.} \end{cases}$$

Then, in all the cases, τ_x is a subordinator by the skip-freeness and its Laplace transform is given by

$$Ee^{-\theta\tau_x} = e^{-x\Psi^{-1}(\theta)} \quad \text{for } x > 0. \tag{1.4}$$

In Case 3, while the left hand side of the above equality has a meaning only for $x = 1, 2, \ldots$, the right hand side is the Laplace transform of a (sub) probability distribution for every $x > 0$. For $x \neq 1, 2, \ldots$, we use the same symbol in the left hand side as for $x = 1, 2, \ldots$ in the sequel.

Although the Laplace transform of τ_x is completely determined by (1.4), its Lévy measure is known partially [2]. In this talk, we give a unified representation of the Lévy measure of τ_x in terms of the local time at level 0 of the Lévy process $\{X_t\}$ skip-free to the right. We give other representations in terms of Lévy measure of the original process in compound Poisson with drift case and in terms of transition density if it exists in section 3. We also give a criterion whether the total mass of the Lévy measure of τ_x is finite or not in terms of the Lévy measure of $\{X_t\}$ in Section 3. Section 4 is devoted to exmaples. While we can also define Lévy process skip-free to the left, the result is symmetric. So, we restrict ourselves to the skip-free to the right case.

In the sequel, we denote by $\{X_t\}$ Lévy process skip free to the right and we use the notations $\Psi, \Psi^{-1}, A, B, \nu, \tau_x$ as above for $\{X_t\}$.

2 A representation of the Lévy measure of τ_x

In Cases 1 and 2, one-point set is not essentially polar. Then, it is known that q-potential ($q > 0$) of $\{X_t\}$ has a bounded density and its q-co-excessive version is continuous except at 0. We denote it by $u^q(x)$ in the sequel. Further, in Case 1, 0 is regular for itself. Hence u^q is continuous on R. Refer [4].

Lemma 2.1. *In Cases 1 and 2, $u^q(x)$ is represented as*

$$u^q(x) = (\Psi^{-1})'(q)e^{-\Psi^{-1}(q)x} \quad \text{for } x > 0, \tag{2.1}$$

$u^q(x)$ is integrable with respect to q on $(0,\infty)$ and

$$\int_0^\theta u^q(x)dq = \frac{e^{-\Psi^{-1}(\theta)x} - e^{-\Psi^{-1}(0)x}}{x}.$$

This lemma is given in [2] as an exercise (7.5 Exercise 2).

Theorem 2.1. *In Cases 1 and 2,*

$$E(e^{-\theta\tau_x}) \quad = \quad \exp[-x\{\Psi^{-1}(0) + \int_0^\theta u^q(0+)dq\}] \quad \text{for} \quad x > 0.$$

Proof. The conclusion is immediate by Lemma 2.1. □

In Case 3, according to Bertoin [1] we set $L(x,t) = \int_0^t 1_{\{x\}}(X_s)ds$.
In Case 2, 0 is irregular for itself and hence $P(\tau_0 > 0) = 1$. We define

$$L(x,t) \quad = \sum_{0\le s<t, X_s=x} \frac{1}{B} \quad \text{for } x \in R, t \ge 0.$$

This quantity $L(x,t)$ is nondecreasing and left continuous in t and is denoted $L_t^{(2)}(x)$ in [5]. We denote by $1_A(x)$ the indicator function of A.

Proposition 2.2. *In Case 2, for each $x \ge 0$ and $t > 0$,*

$$\frac{1}{\epsilon}\int_0^t 1_{(x,x+\epsilon]}(X_s)ds \to L(x,t) \tag{2.2}$$

as $\epsilon \to 0$ for a.e. ω.

Convergence in L^2 also holds ([5]).
In Case 1, 0 is regular for itself and for every $x \in R$

$$\frac{1}{2\epsilon}\int_0^t 1_{(x-\epsilon,x+\epsilon)}(X_s)ds$$

converges uniformly on compact time intervals as $\epsilon \to 0+$, in L^2. Define $L(x,t)$ by this limit. Then $L(x,t)$ is continuous in t a.s. See Bertoin [1].
In all Cases $1 \sim 3$, we call $L(x,t)$ the local time of $\{X_t\}$ at level x and time t.

Lemma 2.2. *Let $q > 0$. It holds that*

$$E\int_{[0,\infty)} e^{-qt}dL(x,t) = u^q(x+) \quad \text{for} \quad x \ge 0$$

in Cases 1 and 2, and

$$L(0,0+) = \frac{1}{B} \quad a.e.\ \omega$$

in Case 2.

Proof. The formula for Case 1 is seen in [1]. The formula for Case 2 is seen in [5] Lemma 1. Note that in [5] Lemma 1, compound Poisson process with drift is excluded, but the proof also works in this case. □

Theorem 2.3. *In Cases 1 and 3,*

$$E(e^{-\theta\tau_x}) = \exp[x\{-\Psi^{-1}(0) + \int_{(0,\infty)} (e^{-\theta t} - 1)\frac{1}{t} dE(L(0,t))\}]$$

holds. In Case 2, it holds that

$$E(e^{-\theta\tau_x}) = \exp[x\{-\Psi^{-1}(0) - \frac{\theta}{B} + \int_{(0,\infty)} (e^{-\theta t} - 1)\frac{1}{t} dE(L(0,t))\}]$$

for $x > 0$ *and*

$$E(e^{-q\tau_0}) = \frac{\int_0^\infty e^{-qt} dE(L(0,t))}{1 + \int_0^\infty e^{-qt} dE(L(0,t))}.$$

Here $L(0,t)$ *is the local time of* $\{X_t\}$ *at level* 0 *and time* t.

Proof. Case 3: An argument parallel to the part (iv) of Borovkov-Burq [3] yields the conclusion. Cases 1 and 2: Lemma 2.2 and Theorem 2.1 yield the conclusion. □

3 Other representations of the Lévy measures

Theorem 3.1. *Assume Case 2. If* $\lambda = \nu((-\infty, 0)) < \infty$*, then*

$$E(e^{-\theta\tau_x}) = \exp[x\{-\Psi^{-1}(0) - \frac{\theta}{B} + \frac{1}{B}\int_{(0,\infty)} (e^{-\theta t} - 1)\sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} e^{-\lambda t} G^{n*}(dt)\}]$$

and

$$\Psi^{-1}(0) = \frac{\lambda}{B} - \frac{1}{B}\sum_{n=1}^{\infty} \frac{1}{n!}\frac{d^{n-1}}{d\lambda^{n-1}}\{\hat{\nu}(\frac{\lambda}{B})^n\}$$

where $G^{n*}((0,x]) = \nu^{n*}([-Bx, 0))$ *and* $\hat{\nu}(\lambda) = \int_{(-\infty,0)} e^{\lambda x}\nu(dx)$.

Theorem 3.2. *Suppose that* $P(X_t \in dx)$*,* $t > 0$ *is absolutly continuous. Let* $p_t(x)$ *be its canonical density (Hawks [6]). Then in Case 1,*

$$E(e^{-\theta\tau_x}) = \exp[x\{-\Psi^{-1}(0) + \int_{(0,\infty)} (e^{-\theta t} - 1)\frac{1}{t} p_t(0) dt\}]$$

holds and in Case 2,

$$E(e^{-\theta\tau_x}) = \exp[x\{-\Psi^{-1}(0) - \frac{\theta}{B} + \int_{(0,\infty)} (e^{-\theta t} - 1)\frac{1}{t} p_t(0) dt\}]$$

holds.

Proof. In Hawks [6], it is shown that $u^q(0) = \int_0^\infty e^{-qt} p_t(0) dt$ for q-co-excessive version u^q. It is known that $u^q(0+) = u^q(0) + \frac{1}{B}$ in Case 2 (Port [7]). We get the conclusion by Theorem 2.3. □

Remark 3.1. *Bertoin [2] shows the following : If the transition probability* $P(X_t \in dx)$ *has a density* $p_t(x)$ *continuous at the origin, then the Lévy measure of* τ_x *has the density* $\frac{1}{t} p_t(0)$.

Now, we give a criterion for the finiteness of the total mass of the Lévy measure of a hitting time. Note that $\Psi^{-1}(0) = -\frac{1}{x} \log P(\tau_x < \infty) \geq 0$.

Theorem 3.3. *Let* $\lambda = \nu((-\infty, 0))$.

(1) *If* $\lambda < \infty$ *in Case 2, then* $\int_0^\infty \frac{1}{t} dEL(0,t) < \infty$ *and*

$$\Psi^{-1}(0) = \frac{\lambda}{B} - \int_0^\infty \frac{1}{t} dEL(0,t) \geq 0.$$

(2) *If* $\lambda = \infty$ *in Case 2, then* $\int_0^\infty \frac{1}{t} dEL(0,t) = \infty$.

(3) *In Case 1,* $\int_0^\infty \frac{1}{t} dEL(0,t) = \infty$ *holds.*

(4) *In Case 3, it holds that* $\int_0^\infty \frac{1}{t} dEL(0,t) = \infty$ *and*

$$\Psi^{-1}(0) = \int_0^\infty \{e^{-t} - P(X_t = 0)\} \frac{1}{t} dt - \log \nu(\{1\}) \geq 0.$$

4 Examples

In some cases, Lévy measures of the hitting times can be written explicitly. We list some of them.

Example 4.1. *Let* $B, \lambda > 0$. *Let* $\Psi(w) = Bw + \lambda(e^w - 1)$ *for* $w \geq 0$. *Then, by Theorem 3.1,*

$$Ee^{-\theta\tau_x} = e^{-x\Psi^{-1}(\theta)} = \exp[x\{-(\frac{\theta}{B} + \frac{\lambda}{B}) + \sum_{n=1}^\infty \frac{n^{n-1}}{n!} (\frac{\lambda}{B})^n e^{-n\frac{\theta+\lambda}{B}}\}].$$

Example 4.2. *(Gamma process with drift) Let* $a > 0$ *and*

$$\Psi(w) = B + \int_{-\infty}^0 (e^{wx} - 1) \frac{e^{-a|x|}}{|x|} dx$$

Then the density $p_t(x)$ *of transition probability of* $\{X_t\}$ *is given by*

$$p_t(x) = \frac{a^t}{\Gamma(t)} |x - Bt|^{t-1} e^{-a|x-Bt|} \quad \text{for} \quad x < Bt.$$

By Theorem 3.2, the Lévy measure of the hitting time τ_x *is given by*

$$\frac{a}{t\Gamma(t)} (aBt)^{t-1} e^{-aBt} dt \quad \text{for} \quad t > 0$$

Example 4.3. *(One sided* $\frac{1}{2}$*-stable process with drift) Let* $c > 0$ *and let*

$$\Psi(w) = -cw^{1/2} + Bw.$$

Then the density of the transition probability of $\{X_t\}$ *is*

$$(2\sqrt{\pi})^{-1} ct|x - Bt|^{-\frac{3}{2}} \exp\{-\frac{c^2t^2}{4|x - Bt|}\} \quad \text{for} \quad x < Bt.$$

By Theorem 3.2, the Lévy measure of the hitting time τ_x *is given by*

$$(2\sqrt{\pi})^{-1} c(Bt)^{-\frac{3}{2}} \exp\{-\frac{c^2 t}{4B}\} dt \quad \text{for} \quad t > 0.$$

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